November 29, 2022

Question 3 : Pricing errors in FM (3 points),compulsory

Solve the exercise in slide n. 100 of Chapter 3 "SUR, GRS, CSR, Fama-MacBeth". Note : the question in slide n. 100 of Chapter 3 has two sub-questions (i), (ii) : answer to both of them.

Set-up and Definitions

- The data generating process (DGP) for the returns in excess of the risk-free rate with κ tradable factors is:

$$r_t = \beta f_t + u_t \quad t = 1, ..., T \tag{1}$$

- r_t is the (N x 1) vector containing the excess return of each security at time t: $r_{i,t}$ for i = 1,...,N, with N corresponding to the total number of securities available in the investable universe.
- u_t is the (N x 1) vector containing the errors of the DGP and is assumed: $\stackrel{iid}{\sim} (0, \Omega)$ over t with finite matrix Ω.
- β is the (N x K)[1] matrix reporting on each line the factor loadings for i = 1,...,N[2].

  Rank(β) = K therefore, the inverse of $\beta'\beta$ exists and is definite positive.
- f_t is the (K x 1) vector of the excess returns of the tradable risk factors in t, distributed as per $\stackrel{iid}{\sim} (\lambda, \Omega_f)$ over t. λ represents the (K x 1) vector of the true risk premia: $E(f_t) = \lambda$.[3]
- u_s is independent from f_t $\forall(s, t)$ hence, $E[u_s f_t] = 0$ (hypothesis of strict exogeneity of the regressors).

[1] N is deemed to be larger than K.

[2] We are including the vector of ones: $i_N = [1, 1, ..., 1]'$ as the first column of β in order to add the intercept to the DGP introduced above.

[3] As we are including the intercept in the DGP and r_t is the vector of the excess returns, we would expect $\lambda_{1,t}$ to be equal to 0.

- We assume that the true factor loadings are known, hence we dont face a problem of error in variables (EIV) for this DPG[4].
- $\hat{\lambda}$ is estimated via OLS and corresponds to: $(\beta'\beta)^{-1}\beta' r_t$.

(i) Show $\hat{\epsilon} \xrightarrow{p} 0$

Demonstrate that $\hat{\epsilon} \xrightarrow{p} 0$ as $T \to +\infty$ Given the OLS formula for the pricing errors of 1 is $\hat{\epsilon}_t = r_t - \beta\hat{\lambda}_t$ and replacing $\hat{\lambda}$ with the corresponding OLS result, we find that $\hat{\epsilon}_t = r_t - \beta(\beta'\beta)^{-1}\beta' r_t = M_\beta r_t$[5].
Therefore we observe that:

$$\hat{\epsilon} = \frac{1}{T}\sum_{t=1}^{T}\epsilon_t = \frac{1}{T}\sum_{t=1}^{T} M_b r_t \tag{2}$$

Looking at 2 and replacing r_t with 1, we derive:

$$\begin{aligned}
\hat{\epsilon} &= \frac{1}{T}\sum_{t=1}^{T} M_b r_t \\
&= \frac{1}{T}\sum_{t=1}^{T} M_b(\beta f_t + u_t) \\
&= \frac{1}{T}\sum_{t=1}^{T} M_b(\beta f_t) + \frac{1}{T}\sum_{t=1}^{T} M_b u_t
\end{aligned} \tag{3}$$

The first component of 3 is equal to 0 as it becomes: $\frac{1}{T}\sum_{t=1}^{T}(\beta f_t - \beta f_t)$, while for the second term we see:

$$\begin{aligned}
\hat{\epsilon} &= \frac{1}{T}\sum_{t=1}^{T} M_b u_t \\
&= M_b(\frac{1}{T}\sum_{t=1}^{T} u_t)
\end{aligned}$$

As $u_t \overset{iid}{\sim} (0, \Omega)$ and Ω is finite, we can apply the Law of Large Numbers (LLN) and obtain that $(\frac{1}{T}\sum_{t=1}^{T} u_t) \xrightarrow{p} 0$, it follows that: $\hat{\epsilon} \xrightarrow{p} 0$.

[4] If β had not been known, we would have estimated the vector from the N timeseries regressions covering the entire sample of lenght: T.

[5] M_β corresponds to $I_N - \beta(\beta'\beta)^{-1}\beta$ which is symmetric and idempotent (I_N is the identity matrix (N x N)).

(ii) Show $\frac{1}{T}\sum_{t=1}^{T}(\hat{\epsilon}_t - \hat{\epsilon})(\hat{\epsilon}_t - \hat{\epsilon})' \xrightarrow{p} M_\beta \Omega M_\beta$

Demonstrate that $\frac{1}{T}\sum_{t=1}^{T}(\hat{\epsilon}_t - \hat{\epsilon})(\hat{\epsilon}_t - \hat{\epsilon})' \xrightarrow{p} M_\beta \Omega M_\beta$ as $T \to +\infty$ Following the same setup and results of 1.2 and recalling that $(M_\beta)' = M_\beta$, we derive that:

$$
\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}(\hat{\epsilon}_t - \hat{\epsilon})(\hat{\epsilon}_t - \hat{\epsilon})' &= \frac{1}{T}\sum_{t=1}^{T}(M_\beta r_t)(M_\beta r_t)' \\
&= \frac{1}{T}\sum_{t=1}^{T} M_\beta(\beta f_t + u_t)(\beta f_t + u_t)' M_\beta \\
&= \frac{1}{T}\sum_{t=1}^{T} M_\beta(\beta f_t f_t' \beta' + u_t u_t' + 2\beta f_t u_t') M_\beta \qquad (4)
\end{aligned}
$$

Additionally, we can decompose 4 into the following components:

- $\frac{1}{T}\sum_{t=1}^{T} M_\beta(\beta f_t f_t' \beta') M_\beta = \frac{1}{T}\sum_{t=1}^{T}(\beta f_t f_t' \beta' - \beta f_t f_t' \beta') M_\beta = 0$
- $\frac{1}{T}\sum_{t=1}^{T} M_\beta(2\beta f_t u_t') M_\beta = \frac{2}{T}\sum_{t=1}^{T}(\beta f_t u_t' - \beta f_t u_t') M_\beta = 0$
- $\frac{1}{T}\sum_{t=1}^{T} M_\beta u_t u_t' M_\beta$

Assuming: i) the fourth moment of u_t exists and is finite and ii) recalling that $u_t \overset{iid}{\sim} (0, \Omega)$, we can apply the LLN on $\frac{1}{T}\sum_{t=1}^{T} u_t u_t'$ and observe that this quantity $\xrightarrow{p} \Omega$. It follows that the last member of 4 $\xrightarrow{p} M_\beta \Omega M_\beta$, therefore:

$$
\frac{1}{T}\sum_{t=1}^{T}(\hat{\epsilon}_t - \hat{\epsilon})(\hat{\epsilon}_t - \hat{\epsilon})' \xrightarrow{p} M_\beta \Omega M_\beta \qquad (5)
$$