Foundation of Econometrics - Homework

Giovanni Maffei, Timo Predöhl

November 17, 2022

Contents

1 Exercise 2: Univariate Time Series Regression and Jensen's alpha

1.1 Set-up and Definitions

- We are assuming a timeseries regression for one asset: $i = 1, ..., N$, with N corresponding to the total number of securities available in the investable universe:

$$r_{it} = \alpha_i + \beta_{im} z_{mt} + u_{it} \quad t = 1, ..., T \tag{1}$$
$$u_i | z_{mt} \overset{iid}{\sim} (0, w_i^2)$$

- X is the matrix of the regressors.
- $E[x_i x_i']$ exists, and is definite positive.
- $Rank(X) = 2$ (no redunant regressors), hence $(X'X)^{-1}$ exists.
- $\hat{\alpha}_i$ and $\hat{\beta}_{im}$ are the OLS estimators of α and β in 1.
- z_{mt} corresponds to the return of a market index (used as proxy of the market portfolio) at time t.

1.2 Part 1: Demonstrate that $\hat{\alpha}_i = \overline{r}_i - \hat{\beta}_i \overline{z}_m$ and $\hat{\beta}_i = \frac{\hat{\sigma}_{im}}{\sigma_m^{\hat{2}}}$

Noting that if we substitute the intercept by adding 1_T as the first column of the regressors matrix: X, we can rewrite 1 in the following matricial representation: $r_{i,(1\times T)} = X_{i,(T\times 2)} \beta_{i,(2\times 1)} + u_{i,(T\times 1)}$, with X defined as a (1 x 2) matrix: $[1_{(Tx1)}, z_{m,(Tx1)}]$[1].
Additionally, recalling that: i) the OLS estimator of α_i and β_{im} corresponds to: $(X'X)^{-1}X'r_i$, ii) $1_T' 1_T = T$ and iii) the inverse of $(X'X)^{-1}$ is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$(X'X)^{-1} = \frac{1}{T \sum_{t=1}^{T} z_{mt}^2 - (\sum_{t=1}^{T} z_{mt})^2} \begin{bmatrix} \sum_{t=1}^{T} z_{mt}^2 & -\sum_{t=1}^{T} z_{mt} \\ -\sum_{t=1}^{T} z_{mt} & T \end{bmatrix} \tag{2}$$

[1]The second term of each matrix subscript is meant to represent the dimension of the corresponding metric for the avoidance of doubt, we will drop such notation for tractability reasons.

And observing $X'r_i = \begin{bmatrix} \sum_{t=1}^{T} r_{it} \\ \sum_{t=1}^{T} z_{mt} r_{it} \end{bmatrix}$, we derive that:

$$\begin{aligned} \hat{\beta}_i &= (X'X)^{-1}X'r_i \\ &= \begin{bmatrix} \frac{T(\frac{1}{T}\sum_{t=1}^{T} z_{mt}^2)T(\frac{1}{T}\sum_{t=1}^{T} r_{it}) - T(\frac{1}{T}\sum_{t=1}^{T} z_{mt})T(\frac{1}{T}\sum_{t=1}^{T} z_{mt}r_{it})}{T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt}^2) - T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt})^2} \\ \frac{T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt}r_{it}) - T(\frac{1}{T}\sum_{t=1}^{T} z_{mt})T(\frac{1}{T}\sum_{t=1}^{T} r_{it})}{T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt}^2) - T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt})^2} \end{bmatrix} \end{aligned} \tag{3}$$

Besides, we can further refine this expression by noting:

- $\frac{1}{T}\sum_{t=1}^{T} z_{mt} = \overline{z}_m$
- $\frac{1}{T}\sum_{t=1}^{T} r_{it} = \overline{r}_i$
- $T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt}^2) - T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt})^2 = T^2\hat{\sigma}_m^2$, with $\hat{\sigma}_m^2$ as the variance of the market index returns.
- $T^2(\frac{1}{T}\sum_{t=1}^{T} z_{mt}r_{it}) - T(\frac{1}{T}\sum_{t=1}^{T} z_{mt})T(\frac{1}{T}\sum_{t=1}^{T} r_{it}) = T^2\hat{\sigma}_{z,i}$, with $\hat{\sigma}_{z,i}$ as the covariance between the returns of the single asset and the market index.
- $T(\frac{1}{T}\sum_{t=1}^{T} z_{mt}^2)T(\frac{1}{T}\sum_{t=1}^{T} r_{it}) - T(\frac{1}{T}\sum_{t=1}^{T} z_{mt})T(\frac{1}{T}\sum_{t=1}^{T} z_{mt}r_{it}) =$
  $= T^2\overline{r}_i(\frac{1}{T}(\sum_{t=1}^{T} z_{mt}^2 - \overline{z}_m{}^2) - T^2\overline{z}_m(\frac{1}{T}(\sum_{t=1}^{T} z_{mt}r_{it} - \overline{z}_m\overline{r}_i)$
  $= T^2(\overline{r}_i\hat{\sigma}_m^2 - \overline{z}_m\hat{\sigma}_{z,i})$

By substituting the formulae above into 3, we derive:

$$\begin{aligned} \hat{\beta}_i &= \begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_{im} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\overline{r}_i\hat{\sigma}_m^2 - \overline{z}_m\hat{\sigma}_{z,i}}{\hat{\sigma}_m^2} \\ \frac{\hat{\sigma}_{z,i}}{\hat{\sigma}_m^2} \end{bmatrix} \\ &= \begin{bmatrix} \overline{r}_i - \overline{z}_m\hat{\beta}_{im} \\ \frac{\hat{\sigma}_{z,i}}{\hat{\sigma}_m^2} \end{bmatrix} \end{aligned} \tag{4}$$

1.3 Part 2: Demonstrate the asymptotic distribution of $\hat{\beta}_i|z_m$

As we are not assuming any specific form for $u_i|z_{mt}$, we will study the distribution of $\hat{\beta}_i|z_m$ asymptotically ($T \to \infty$). Therefore, by rearranging the definition of $\hat{\beta}_i$ we derive:

$$\begin{aligned}
\hat{\beta}_i|z_{mt} &= (X'X)^{-1}X'r_i = \beta_i + (X'X)^{-1}X'u_i \\
\sqrt{T}(\hat{\beta}_i - \beta_i) &= \sqrt{T}(X'X)^{-1}X'u_i \\
&= (\frac{X'X}{T})^{-1}\sqrt{T}X'u_i
\end{aligned}$$

Further noting that (under $T \to +\infty$):

- $(\frac{X'X}{T}) \xrightarrow{p} E[x_i x_i']$ due to the application of the Law of Large Numbers (LLN) because $[x_{it}]_{t=1}^T$ is a sequence of IID observations with finite expected value and variance. Additionally, by using the Slutsky's theorem we obtain: $(\frac{X'X}{T})^{-1} \xrightarrow{p} E[x_i x_i']^{-1}$.
- $X'u_i \xrightarrow{d} X'N(0, w_i^2 I_T)X$ due to the application of the Central Limit Theorem (CLT) in light of: i) IID observations, ii) errors uncorrelated with the regressors and iii) finite moments[2].

By applying the properties of the multivariate Normal distribution, we obtain:

$$\hat{\beta}_i \xrightarrow{d} N(\beta_i, (X'X)^{-1}X'w_i^2 I_T X(X'X)^{-1})$$

Simplifying the variance term of the distribution of $\hat{\beta}_i$:

$$\begin{aligned}
V(\hat{\beta}_i) &= (X'X)^{-1}X'w_i^2 I_T X(X'X)^{-1} \\
&= w_i^2(X'X)^{-1}
\end{aligned}$$

Additionally, we can rewrite 2 as per below:

$$\begin{aligned}
(X'X)^{-1} &= \frac{1}{T^2\hat{\sigma}_m^2}\begin{bmatrix} (\sum_{t=1}^T z_{mt}^2) - T\overline{z}_m^2 + T\overline{z}_m^2 & -T\overline{z}_m \\ -T\overline{z}_m & T \end{bmatrix} \\
&= \frac{1}{T^2\hat{\sigma}_m^2}\begin{bmatrix} (T\hat{\sigma}_m^2 + T\overline{z}_m^2) & -T\overline{z}_m \\ -T\overline{z}_m & T \end{bmatrix} \\
&= \frac{1}{T}\begin{bmatrix} (1 + \frac{\overline{z}_m^2}{\hat{\sigma}_m^2}) & \frac{-\overline{z}_m}{\hat{\sigma}_m^2} \\ \frac{-\overline{z}_m}{\hat{\sigma}_m^2} & \frac{1}{\hat{\sigma}_m^2} \end{bmatrix} \qquad (5)
\end{aligned}$$

[2]As mentioned in the *Set-up and Definitions* part of this exercise.

Eventually, we conclude this demonstration by expanding $\hat{\beta}_i$ into its components and by replacing 5 into $\hat{\beta}_i \xrightarrow{d} N(\beta_i, V(\hat{\beta}_i)$:

$$\begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_{im} \end{bmatrix} \sim N(\begin{bmatrix} \alpha_i \\ \beta_{im} \end{bmatrix}, \frac{w_i^2}{T} \begin{bmatrix} (1 + \frac{\bar{z}_m^2}{\hat{\sigma}_m^2}) & \frac{-\bar{z}_m}{\hat{\sigma}_m^2} \\ \frac{-\bar{z}_m}{\hat{\sigma}_m^2} & \frac{1}{\hat{\sigma}_m^2} \end{bmatrix}) \tag{6}$$

1.4 Part 3: Demonstrate the t-statistics to test H_0: $\alpha_i = 0$

We add below a few considerations additional to what presented in *Set-up and Definitions* to support this proof:

- We are studying the asymptotic distribution of this test under $T \rightarrow +\infty$[3].
- This is a linear hypothesis test, with $m = 1$ linear restrictions tested[4].
- The generic matrix R (m x K), formalizing the linear restrictions on β, corresponds to the vector $\begin{bmatrix} 1 & 0 \end{bmatrix}$.
- The generic vector q (m x 1) represents the restricted values imposed on β. As we are imposing only one restriction, q corresponds to a scalar metric: 0.
- As we don't know the actual variance of $u_i|z_{mt}$ but we estimate the corresponding OLS residuals, we utilize $\frac{1}{T}\sum_{t=1}^{T} \hat{u}_{it}^2$ as an asymptotically unbiased estimator by applying the LLN[5].

Under the null hypothesis H_0: $R\beta_i = q$:

$$R\hat{\beta}_i - q = R(\hat{\beta}_i - \beta_i)$$

Furthermore, as proved in the previous exercise and by applying the properties of the multivariate Gaussian distribution, we derive:

$$R(\hat{\beta}_i - \beta_i) \sim N(0, RV(\beta_i)R')$$
$$[RV(\beta_i)R']^{-\frac{1}{2}} R(\hat{\beta}_i - \beta_i) \xrightarrow{d} N(0, I_m) \tag{7}$$

[3] Indeed, no distribution is provided for the errors in 1.
[4] The only restriction we are imposing on β_i is: $\alpha_i = 0$.
[5] $\overline{u}_i = 0$ because of the inclusion of the intercept in X.

By substituting for $R = \begin{bmatrix} 1 & 0 \end{bmatrix}$ in $RV(\beta_i)R'$ and rearranging the terms:

$$\begin{aligned} RV(\beta_i)R' &= \frac{{w_i}^2}{T} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} (1 + \frac{\overline{z}_m^2}{\hat{\sigma}_m^2}) & \frac{-\overline{z}_m}{\hat{\sigma}_m^2} \\ \frac{-\overline{z}_m}{\hat{\sigma}_m^2} & \frac{1}{\hat{\sigma}_m^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{{w_i}^2}{T}(1 + \frac{\overline{z}_m^2}{\hat{\sigma}_m^2}) \end{aligned}$$

Therefore 7 becomes:

$$\begin{aligned} [RV(\beta_i)R']^{-\frac{1}{2}} R(\hat{\beta}_i - \beta_i) &= \frac{R(\hat{\beta}_i - \beta_i)}{w_i \sqrt{\frac{(1+\frac{\overline{z}_m^2}{\hat{\sigma}_m^2})}{T}}} \\ &= \frac{\hat{\alpha}_i - \alpha | H_0}{w_i \sqrt{\frac{(1+\frac{\overline{z}_m^2}{\hat{\sigma}_m^2})}{T}}} \\ &= \frac{\hat{\alpha}_i}{w_i \sqrt{\frac{(1+\frac{\overline{z}_m^2}{\hat{\sigma}_m^2})}{T}}} \end{aligned}$$

Eventually by using the unbiased estimator for w_i^2, as mentioned at the beginning of this proof, we derive[6]:

$$\frac{\hat{\alpha}_i}{\hat{w}_i \sqrt{\frac{(1+\frac{\overline{z}_m^2}{\hat{\sigma}_m^2})}{T}}} \xrightarrow{d} N(0,1) \tag{8}$$

2 Exercise 3: Pricing errors in FM

2.1 Set-up and Definitions

- The data generating process (DGP) for the returns in excess of the risk-free rate with κ tradable factors is:

$$r_t = \beta f_t + u_t \quad t = 1, ..., T \tag{9}$$

- r_t is the (N x 1) vector containing the excess return of each security at time t: $r_{i,t}$ for i = 1,...,N, with N corresponding to the total number of securities available in the investable universe.

[6] I_m is the scalar 1, hence $N(0, I_m) = N(0, 1)$.

- u_t is the (N x 1) vector containing the errors of the DGP and is assumed: $\overset{iid}{\sim}$ $(0, \Omega)$ over t with finite matrix Ω.
- β is the (N x K)[7] matrix reporting on each line the factor loadings for i = 1,...,N[8].

  Rank(β) = K therefore, the inverse of $\beta'\beta$ exists and is definite positive.
- f_t is the (K x 1) vector of the excess returns of the tradable risk factors in t, distributed as per $\overset{iid}{\sim}$ (λ, Ω_f) over t. λ represents the (K x 1) vector of the true risk premia: $E(f_t) = \lambda$.[9]
- u_s is independent from f_t $\forall(s,t)$ hence, $E[u_s f_t] = 0$ (hypothesis of strict exogeneity of the regressors).
- We assume that the true factor loadings are known, hence we dont face a problem of error in variables (EIV) for this DPG[10].
- $\hat{\lambda}$ is estimated via OLS and corresponds to: $(\beta'\beta)^{-1}\beta' r_t$.

2.2 Part 1

Demonstrate that $\hat{\epsilon} \xrightarrow{p} 0$ as $T \to +\infty$ Given the OLS formula for the pricing errors of 9 is $\hat{\epsilon}_t = r_t - \beta\hat{\lambda}_t$ and replacing $\hat{\lambda}$ with the corresponding OLS result, we find that $\hat{\epsilon}_t = r_t - \beta(\beta'\beta)^{-1}\beta' r_t = M_\beta r_t$[11].
Therefore we observe that:

$$\hat{\epsilon} = \frac{1}{T}\sum_{t=1}^{T}\epsilon_t = \frac{1}{T}\sum_{t=1}^{T} M_b r_t \tag{10}$$

[7]N is deemed to be larger than K.

[8]We are including the vector of ones: $i_N = [1, 1, ..., 1]'$ as the first column of β in order to add the intercept to the DGP introduced above.

[9]As we are including the intercept in the DGP and r_t is the vector of the excess returns, we would expect $\lambda_{1,t}$ to be equal to 0.

[10]If β had not been known, we would have estimated the vector from the N timeseries regressions covering the entire sample of lenght: T.

[11]M_β corresponds to $I_N - \beta(\beta'\beta)^{-1}\beta$ which is symmetric and idempotent (I_N is the identity matrix (N x N)).

Looking at 10 and replacing r_t with 9, we derive:

$$\begin{aligned}\hat{\epsilon} &= \frac{1}{T}\sum_{t=1}^{T} M_b r_t \\ &= \frac{1}{T}\sum_{t=1}^{T} M_b(\beta f_t + u_t) \\ &= \frac{1}{T}\sum_{t=1}^{T} M_b(\beta f_t) + \frac{1}{T}\sum_{t=1}^{T} M_b u_t \qquad (11)\end{aligned}$$

The first component of 11 is equal to 0 as it becomes: $\frac{1}{T}\sum_{t=1}^{T}(\beta f_t - \beta f_t)$, while for the second term we see:

$$\begin{aligned}\hat{\epsilon} &= \frac{1}{T}\sum_{t=1}^{T} M_b u_t \\ &= M_b(\frac{1}{T}\sum_{t=1}^{T} u_t)\end{aligned}$$

As $u_t \overset{iid}{\sim} (0, \Omega)$ and Ω is finite, we can apply the Law of Large Numbers (LLN) and obtain that $(\frac{1}{T}\sum_{t=1}^{T} u_t) \xrightarrow{p} 0$, it follows that: $\hat{\epsilon} \xrightarrow{p} 0$.

2.3 Part 2

Demonstrate that $\frac{1}{T}\sum_{t=1}^{T}(\hat{\epsilon}_t - \hat{\epsilon})(\hat{\epsilon}_t - \hat{\epsilon})' \xrightarrow{p} M_\beta \Omega M_\beta$ as $T \to +\infty$ Following the same setup and results of 1.2 and recalling that $(M_\beta)' = M_\beta$, we derive that:

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}(\hat{\epsilon}_t - \hat{\epsilon})(\hat{\epsilon}_t - \hat{\epsilon})' &= \frac{1}{T}\sum_{t=1}^{T}(M_\beta r_t)(M_\beta r_t)' \\ &= \frac{1}{T}\sum_{t=1}^{T} M_\beta(\beta f_t + u_t)(\beta f_t + u_t)' M_\beta \\ &= \frac{1}{T}\sum_{t=1}^{T} M_\beta(\beta f_t f_t' \beta' + u_t u_t' + 2\beta f_t u_t') M_\beta \qquad (12)\end{aligned}$$

Additionally, we can decompose 12 into the following components:

- $\frac{1}{T}\sum_{t=1}^{T} M_\beta(\beta f_t f_t'\beta')M_\beta = \frac{1}{T}\sum_{t=1}^{T}(\beta f_t f_t'\beta' - \beta f_t f_t'\beta')M_\beta = 0$
- $\frac{1}{T}\sum_{t=1}^{T} M_\beta(2\beta f_t u_t')M_\beta = \frac{2}{T}\sum_{t=1}^{T}(\beta f_t u_t' - \beta f_t u_t')M_\beta = 0$
- $\frac{1}{T}\sum_{t=1}^{T} M_\beta u_t u_t' M_\beta$

Assuming: i) the fourth moment of u_t exists and is finite and ii) recalling that $u_t \overset{iid}{\sim} (0, \Omega)$, we can apply the LLN on $\frac{1}{T}\sum_{t=1}^{T} u_t u_t'$ and observe that this quantity $\xrightarrow{p} \Omega$. It follows that the last member of 12 $\xrightarrow{p} M_\beta \Omega M_\beta$, therefore:

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{\epsilon}_t - \hat{\epsilon})(\hat{\epsilon}_t - \hat{\epsilon})' \xrightarrow{p} M_\beta \Omega M_\beta \tag{13}$$