EDHEC PhD Finance 2022 - Econometrics Homework

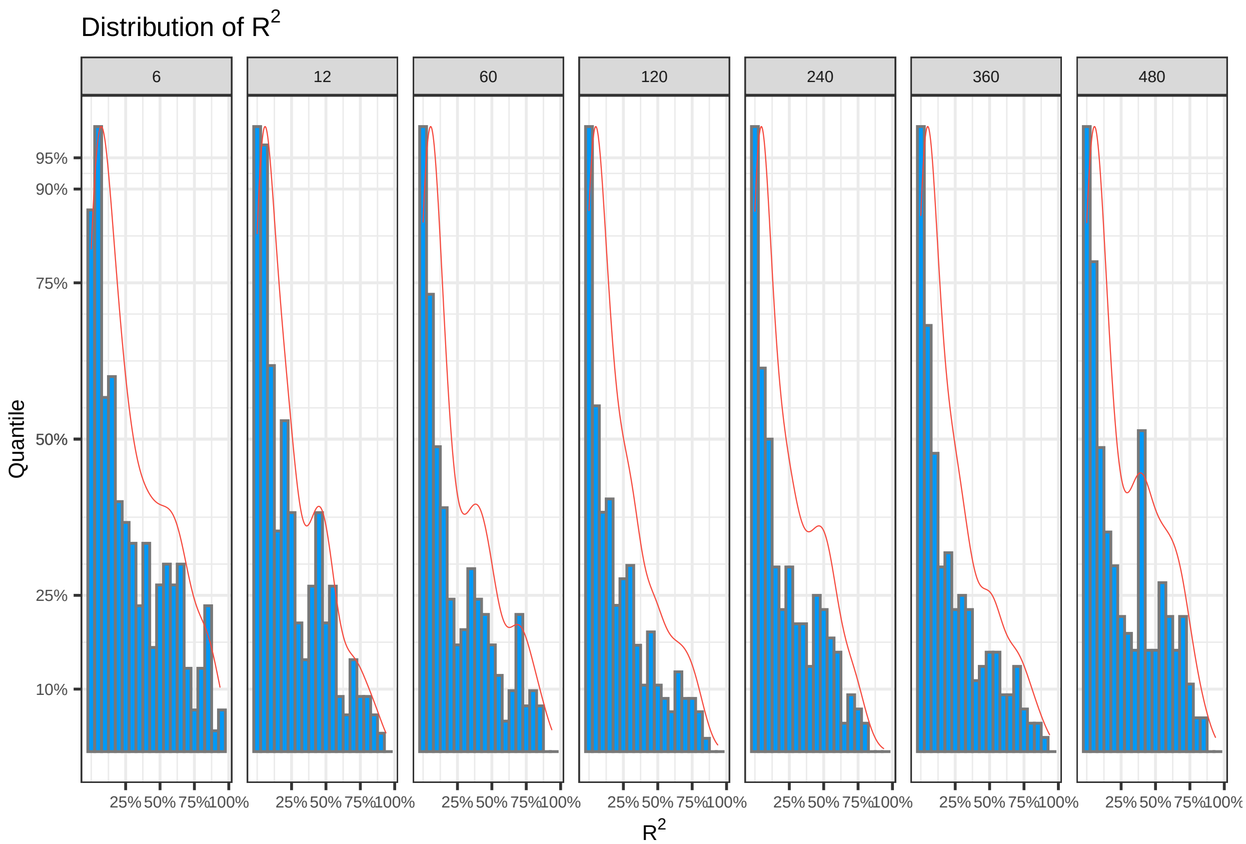
Giovanni Maffei, Timo Predoehl

Question 1: Spurious Regressions (2 points)

1.a. [1 Point] Replicate the analysis leading to Figure 14.1 in Davidson MacKinnon (2005, book)

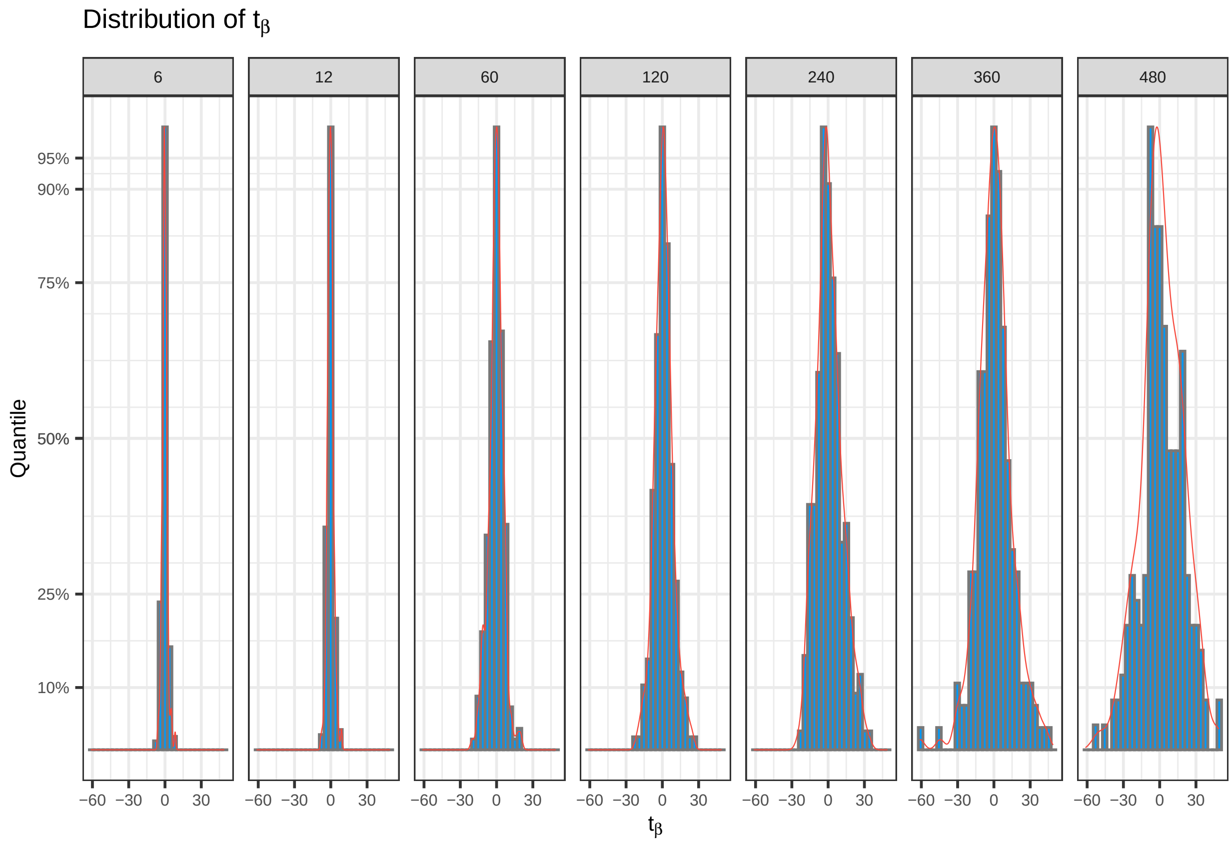
1.a.i

Compute also for each sample size T the distribution of the R^2 of the MC simulations with either 7 separate histograms, or one unique figure where you report on the y-axis the 5%, 10% 25%, 50%, 75%, 90% and 95% quantiles of the distributions of the simulated R^2, and on the x-axis you have T = 6, 12, 60, 120, 240, 360, 480.



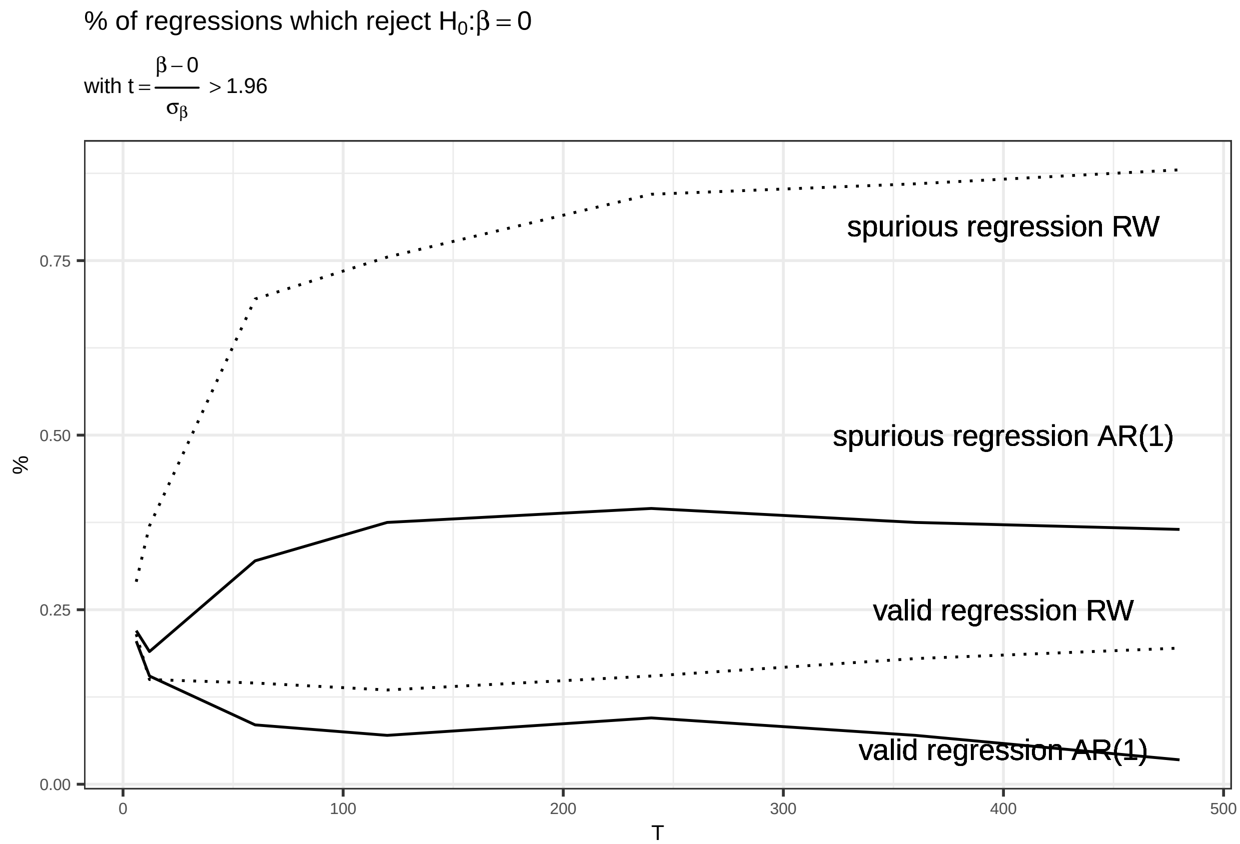
1.a.ii

Similarly (either with histograms, or with one plot of the quantiles) report the distributions of the estimates t-statistics for the test of the null $H0: \beta_2 = 0$



1.a.iii

Compute the empirical rejection frequencies (that is the empirical size of the tests), which is exactly the figure 14.1 in Davidson MacKinnon (2005, book).



1.b [1 Point] Based on the results obtained by answering to point (a) summarize the problems of spurious regressions in econometrics.

Spurious regression as outlined in Davidson MacKinnon occurs for two reasons:

1. incorrectly specified H_0 and
2. standard asymptotic results do not hold whenever at least one of the regressors is I(1), even when a model is correctly specified

The $H_0 : \beta_2 = 0$ tested with the model (14.12)

$$y_t = \beta_1 + \beta_2 y_{t-1} + v_t$$

implies a DGP': $y_t = \beta_1 + v_t$, when y_t is actually generated using DGP (14.01) $y_t = y_{t-1} + v_t, y_0 = 0, v_t \sim iidN(0, 1)$.

The wrongly specified H_0 is rejected with increasing frequency in n. This merely confirms that y_t is not generated by the implied DGP'. Correctly specifying the model as (14.13)

$$y_t = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + v_t$$

and testing $H_0 : \beta_2 = 0$, implying $\beta_3 = 1$ reduces the model to the actual DGP (14.01). This treatment, however, does not completely eliminate the problem i.e. leaves the rejection rate still significantly above 0.

For $\hat{\beta}$ to converge to β_0 asymtotically, the bias $(\hat{\beta} - \beta_0)$ must be $O_p(1)$:

$$
\begin{aligned}
(\hat{\beta} - \beta_0) &= (X'X)^{-1}X'u, with \\
(X'X)^{-1} &\in O_p(n^{-1}) and \\
X'u &\in O_p(n^{.5}). consequently: \\
n^{.5}(\hat{\beta} - \beta_0) &= n^{.5}(X'X)^{-1}X'u = n^{.5}O_p(n^{-1})O_p(n^{.5}) = O_p(1)
\end{aligned}
$$

The relevant assumption to be tested is therefore is $(X'X) \in n^{.5}O_p(n^{-1})$.

The random walk (14.01) is I(1), due to:

$$
\begin{aligned}
w_t &= w_{t-1} + \epsilon_t \\
w_t - w_{t-1} &= \epsilon_t \\
(1 - L)w_t &= \epsilon_t \\
(1 - \phi(z))w_t &= \epsilon_t \\
\phi(z) &= 1
\end{aligned}
$$

Consequently, both x_t and y_t are I(1).

Further, (14.01) reduces to $w_t = \sum_{s=1}^{t} \epsilon_s$, which enters as X'X or:

$$
\begin{aligned}
\sum_{t=1}^{n} \left(\sum_{r=1}^{t} \sum_{s=1}^{t} \right) \epsilon_r \epsilon_s &= \sum_{t=1}^{n} \sum_{r=1}^{t} E(\epsilon_r^2), \epsilon_r \epsilon_s = 0 \; \forall r \neq s \\
\sum_{t=1}^{n} \sum_{r=1}^{t} \sigma^2 &= \sum_{t=1}^{n} \sum_{r=1}^{t} 1 \\
\sum_{t=1}^{n} t &= \frac{1}{2} n(n+1)
\end{aligned}
$$

Consequently, given (14.01) being I(1), $X'X \in O_p(n^2)$ and therefore cannot possibly converge to a finite probability limit. The bias $(\hat{\beta} - \beta_0)$ therefore does not converge asymptotically in probability.

Question 2: Time Series regression test of CAPM for one asset i (3 points)

Black, Jensen and Scholes (1972) suggest to test the empirical validity of the CAPM by estimating the following LRM:

$$
\begin{aligned}
r_{it} - r_{ft} &= \alpha_i + \beta_i(r_{mt} - r_{ft}) + u_{it} \\
z_{it} &= \alpha_i + \beta_i z_{mt} + u_{it} \\
u_{it}|z_{mt} &\sim iid(0, \omega_i^2)
\end{aligned}
$$

(i) Estimate $\hat{\alpha}_i$ and $\hat{\beta}_i$

NOTE : $\hat{\alpha}_i$ is the Jensen's alpha, and represents an estimate of the expected return not justified by its exposure to market risk, the only one that matters according to the CAPM.

$$
\begin{aligned}
x_t &= (1, z'_{mt}) \\
X &= [x_1, ..., x_T]' \\
X'X &= TSxx \\
Sxx &= \frac{1}{T}X'X = \begin{bmatrix} 1 & \bar{z}_m \\ \bar{z}_m & \frac{1}{T}\sum z_m^2 \end{bmatrix} \\
Sxx^{-1} &= \begin{bmatrix} \frac{1}{T}\sum z_m^2 & -\bar{z}_m \\ -\bar{z}_m & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\sum z_m^2 - \bar{z}_m^2 + \bar{z}_m^2 & -\bar{z}_m \\ -\bar{z}_m & 1 \end{bmatrix} \\
&= \begin{bmatrix} \hat{\sigma}_m^2 + \bar{z}_m^2 & -\bar{z}_m \\ -\bar{z}_m & 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\bar{z}_m^2}{\hat{\sigma}_m^2} & \frac{-\bar{z}_m}{\hat{\sigma}_m^2} \\ \frac{-\bar{z}_m}{\hat{\sigma}_m^2} & \frac{1}{\hat{\sigma}^2} \end{bmatrix} \\
Sxz_i &= X'z_i = \begin{bmatrix} \bar{z}_i \\ \frac{1}{T}\sum z_m z_i \end{bmatrix} = \begin{bmatrix} \bar{z}_i \\ \frac{1}{T}\sum z_m z_i - \bar{z}_m\bar{z}_i + \bar{z}_m\bar{z}_i \end{bmatrix} = \begin{bmatrix} \bar{z}_i \\ \hat{\sigma}_{im} + \bar{z}_m\bar{z}_i \end{bmatrix}
\end{aligned}
$$

Solving for $\hat{\alpha}_i$ and $\hat{\beta}_i$:

$$
Sxx \begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{bmatrix} = Sxz_i
$$

$$
\begin{bmatrix} 1 & \bar{z}_m \\ \bar{z}_m & \frac{1}{n}\sum z_m^2 \end{bmatrix} \begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{bmatrix} = \begin{bmatrix} \bar{z}_i \\ \hat{\sigma}_{im} + \bar{z}_m\bar{z}_i \end{bmatrix}
$$

$$
\begin{matrix} (1) \\ (2) \end{matrix} \begin{bmatrix} \hat{\alpha}_i + \hat{\beta}_i\bar{z}_m \\ \hat{\alpha}_i\bar{z}_m + \hat{\beta}_i\frac{1}{n}\sum z_m^2 \end{bmatrix} = \begin{bmatrix} \bar{z}_i \\ \hat{\sigma}_{im} + \bar{z}_m\bar{z}_i \end{bmatrix}
$$

From (1) follows directly: $\hat{\alpha}_i = \bar{z}_i - \hat{\beta}_i\bar{z}_m$. Inserting this into (2) yields:

$$
\begin{aligned}
\hat{\alpha}_i\bar{z}_m + \hat{\beta}_i\frac{1}{n}\sum z_m^2 &= \hat{\sigma}_{im} + \bar{z}_m\bar{z}_i \\
(\bar{z}_i - \hat{\beta}_i\bar{z}_m)\bar{z}_m + \hat{\beta}_i\frac{1}{n}\sum z_m^2 &= \hat{\sigma}_{im} + \bar{z}_m\bar{z}_i \\
\bar{z}_i\bar{z}_m - \hat{\beta}_i\bar{z}_m^2 + \hat{\beta}_i\frac{1}{n}\sum z_m^2 &= \hat{\sigma}_{im} + \bar{z}_m\bar{z}_i \\
-\hat{\beta}_i\bar{z}_m^2 + \hat{\beta}_i\frac{1}{n}\sum z_m^2 &= \hat{\sigma}_{im} \\
\hat{\beta}_i(\frac{1}{n}\sum z_m^2 - \bar{z}_m^2) &= \hat{\sigma}_{im} \\
\hat{\beta}_i\hat{\sigma}_m^2 &= \hat{\sigma}_{im} \\
\hat{\beta}_i &= \frac{\hat{\sigma}_{im}}{\hat{\sigma}_m^2}
\end{aligned}
$$

(ii) Distribution of $\hat{\alpha}_i$ and $\hat{\beta}_i$

Expected value of $\hat{\alpha}_i$ and $\hat{\beta}_i$

$$\begin{aligned}
\hat{\beta} = \begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{bmatrix} &= (X'X)^{-1}X'z_i = (X'X)^{-1}X'(X\beta + u_{it}) \\
E[\hat{\beta}|z_m] &= (X'X)^{-1}X'X\beta + (X'X)^{-1}X'E[u_{it}|z_m] \\
E[\hat{\beta}|z_m] &= \beta + (X'X)^{-1}X'0, \text{ by assumption } u_i|z_m t \sim iid(0, \omega_i^2) \\
E[\hat{\beta}|z_m] &= \beta \\
E\begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{bmatrix} |z_m \Bigg] &= \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}
\end{aligned}$$

Variance $\hat{\alpha}_i$ and $\hat{\beta}_i$

$$\begin{aligned}
\hat{\beta} &= \begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{bmatrix} \\
V(\hat{\beta}|z_m) &= V((\hat{\beta} - \beta)|z_m) \text{ (given } \beta \text{ non-random)} \\
&= V((X'X)^{-1}X'u_i|z_m) = AV(u_i|z_m)A' \text{ (with } A = (X'X)^{-1}X') \\
&= AE[u_i^2|z_m]A' = A\omega_i^2 A' \text{ by assumption } u_i|z_m t \sim iid(0, \omega_i^2) \\
&= \omega_i^2 AA' = \omega_i^2 (X'X)^{-1}X'((X'X)^{-1}X')' = \omega_i^2 (X'X)^{-1}X'X(X'X)^{-1} \\
&= \omega_i^2 (X'X)^{-1} = \omega_i^2 \frac{1}{T} Sxx^{-1} \\
&= \frac{\omega_i^2}{T} Sxx^{-1} = \frac{\omega_i^2}{T} \begin{bmatrix} 1 + \frac{\bar{z}_m^2}{\hat{\sigma}_m^2} & \frac{-\bar{z}_m}{\hat{\sigma}_m^2} \\ \frac{-\bar{z}_m}{\hat{\sigma}_m^2} & \frac{1}{\hat{\sigma}^2} \end{bmatrix}
\end{aligned}$$

Therefore

$$\begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{bmatrix} |z_m \sim N\left(\begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}, \frac{\omega_i^2}{T} \begin{bmatrix} 1 + \frac{\bar{z}_m^2}{\hat{\sigma}_m^2} & \frac{-\bar{z}_m}{\hat{\sigma}_m^2} \\ \frac{-\bar{z}_m}{\hat{\sigma}_m^2} & \frac{1}{\hat{\sigma}^2} \end{bmatrix} \right) \square$$

(iii) Determine the t-statistics to test $H_0 : \beta_{k=1} = \alpha_i = 0$

Analog to (ii) and replacing the unknown true ω_i with the sample $\hat{\omega}_i$ obtained from the OLS residuals

$$\begin{aligned}
V((\hat{\beta} - \beta)|z_m) &= \hat{\omega}_i^2 (X'X)^{-1} = \hat{\omega}_i^2 \frac{1}{T} Sxx^{-1} \\
V((\hat{\beta}_{k=1} - 0)|z_m) &= \omega_i^2 (X'X)^{-1}_{kk} = \hat{\omega}_i^2 \frac{1}{T} Sxx^{-1}_{kk} \\
&= \frac{\hat{\omega}_i^2}{T} (1 + \frac{\bar{z}_m^2}{\hat{\sigma}_m^2}) \\
SE((\hat{\beta} - \beta)|z_m) &= \hat{\omega}_i [(1 + \frac{\bar{z}_m^2}{\hat{\sigma}_m^2})/T]^{1/2}
\end{aligned}$$

The t-statistic to test $H_0 : \beta_{k=1} = \alpha_i = 0$ therefore is :

$$\xi_i = \frac{\hat{\alpha}_i - 0}{\hat{\omega}_i [(1 + \frac{\bar{z}_m^2}{\hat{\sigma}_m^2})/T]^{1/2}} \xrightarrow{d} N(0, 1)$$