Topic Proposal
Beilinson-Bernstein Localisation

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Let G be a semisimple connected linear algebraic group over $k = \mathbb{C}$, $\mathfrak{g}$ its Lie algebra. Beilinson-Bernstein Localisation theory sets up a equivalence between the category of $\mathfrak{g}$ Modules with central character associated to a regular anti-dominant weight and the category of twisted $\mathcal{D}$ modules on the flag variety.

1 Borel-Weil Theory

Let B be a Borel subgroup of G, $X = G/B$ the flag variety, H a split maximal torus contained in B, U the unipotent radical of B, $\mathfrak{h}$ the Cartan subalgebra, Δ the root system, Δ^+ the set of positive roots, S the set of simple roots, W the Weyl group. For $\alpha \in S$, denote $s_\alpha \in W$ the simple reflection associated to α. For α a root, let $u_\alpha : G_a \to U_\alpha$ be the corresponding root subgroup, where G_a is the 1 dimensional additive linear algebraic group.

There is an equivalence between G equivariant vector bundles on X and finite dimensional representations of B. Specifically, if $\mathcal{V}$ is a G equivariant vector bundle over X, the fiber of $\mathcal{V}$ at coset B is a B representation. Conversely, if we have $\rho : B \to Aut_k(V)$, then $G \times_B V = G \times B/\sim$, where $(gb, v) \sim (g, \rho(b)v)$ for $b \in B$ $g \in G$, is a G equivariant vector bundle on X with G acts by left multiplication. There is a equivalence between 1 dimensional representations of B and characters $H \to k^\times$. If $\lambda \in \mathfrak{h}^*$ is integral, we can exponentiate λ to get $\chi : H \to k^\times$. Denote $\mathcal{L}_\lambda$ the line bundle associated to χ.

Theorem 1.1. *If λ is integral dominant, then $H^0(G/B, \mathcal{L}_{-\lambda}) = V(\lambda)^*$, where $V(\lambda)$ is the irreducible G representation of highest weight λ.*

Proof. Let $V = H^0(G/B, \mathcal{L}_{-\lambda}) = \{f : G \to k | f(gb) = \chi(b)f(g), \forall b \in B, g \in G\}$. The action of G on V is given by $g \cdot f(-) \to f(g^{-1} \cdot -)$. X is projective, so V is finite dimensional.

For every $w \in W$, fix $\dot{w} \in N_G(H)$ a representative of w. Let $C(w) = B\dot{w}B$ be the Bruhat cell associated to w. Then $U_{w^{-1}} \times B \to C(w)$, defined as $(u, b) \to u\dot{w}b$, is an isomorphism of varieties, where $U_{w^{-1}} = \Pi_{\alpha \in \{\alpha \in \Delta^+ | w^{-1}.\alpha \in -\Delta^+\}} U_\alpha$. Let w_0 be the longest element in W, so $C(w_0) = U\dot{w}_0B$ is open dense in G. Consider $V_0 \subseteq V$ the subspace invariant under the action of U, i.e. V_0 is the highest weight space. $dim V_0 \leq 1$ since if $f \in V_0$, on $C(w_0)$ its value is uniquely determined by χ. We are reduced to showing that $dim V_0 = 1$ if and only if χ is dominant.

On $C(w_0)$, let us define a function f as $f(u\dot{w}_0b) = f(\dot{w}_0)\chi(b)$, for $u \in U$ and $b \in B$. We want to show that such f can be extend to G if and only if χ is dominant. It suffices to find a formula for f on cells of codimension 1, i.e. cells of the form $C(s_\alpha w_0)$ for $\alpha \in S$. Define $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$, a representative of s_α, then $C(s_\alpha w_0) \subseteq n_\alpha C(w_0)$. Notice that if $u \in U = \Pi_{\beta\in\Delta^+}U_\beta$, we can write $u = \Pi_{\beta\in\Delta^+}u_\beta(c_\beta)$, for some scalar $c_\beta \in G_a$. When α is simple, c_α is independent of the ordering we chose for the product, so denote the scalar $c_\alpha(u)$. By

$$n_\alpha^2 = \alpha^\vee(-1)$$

$$u_\alpha(x)u_{-\alpha}(-x^{-1})u_\alpha(x) = \alpha^\vee(x)n_\alpha, x \in k^\times$$

direct calculation shows $C(w_0) \cap n_\alpha C(w_0) = \{x \in C(w_0) | x = u\dot{w}_0b, c_\alpha(u) \neq 0\}$.

On $n_\alpha C(w_0)$,

$$f(n_\alpha u\dot{w}_0b) = \chi(b)(-c_\alpha(u))^{-<\chi, w_0.\alpha^\vee>}f(\dot{w}_0).$$

When $c_\alpha(u) \to 0$, f is well defined if and only if $< \chi, w_0.\alpha^\vee >$ is negative. $\square$

This theory gives a geometric meaning of highest weight representation. Later Bott generalized the theory by showing $H^n(G/B, \mathcal{L}_{-\lambda}) = 0$ for positive n. In the next section we shall see that Beilinson-Bernstein Localisation gives a generalization of Borel-Weil-Bott theorem as $\mathcal{L}_{-\lambda}$ is naturally a twisted $\mathcal{D}$ module.

2 An Introduction to Beilinson-Bernstein Localisation

Let $\mathcal{U}\mathfrak{g}$ be the universal enveloping algebra with center Z, $\mathfrak{b}$ a Borel subalgebra, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ half sum of the positive roots, P weight lattice, $\mathcal{D}_X$ the sheaf of differential operators on X, $\mathcal{T}_X$ the tangent sheaf, ω_X the sheaf of differentials, $\Omega^\cdot_X$ the de Rham complex, T^*X the cotangent bundle over X.

Definition 2.1. A twisted ring of differential operators (or t.d.o) $\mathcal{A}$ on X is a $\mathcal{O}_X$ ring with an increasing filtration $F.\mathcal{A}$ that satisfies the following:

$$\mathcal{A} = \cup_{p=0}^\infty F_p(\mathcal{A})$$

$$\mathcal{O}_X \simeq F_0(\mathcal{A})$$
$$F_p(\mathcal{A}) = 0, p < 0$$
$$F_{p_1}(\mathcal{A})F_{p_2}(\mathcal{A}) \subseteq F_{p_1+p_2}(\mathcal{A})$$
$$[F_{p_1}(\mathcal{A}), F_{p_2}(\mathcal{A})] \subseteq F_{p_1+p_2-1}(\mathcal{A})$$
$$gr_1^F(\mathcal{A}) \simeq \mathcal{T}_X$$
$$F_p(\mathcal{A}) = F_1(\mathcal{A})F_{p-1}(\mathcal{A}), p \geq 1$$

The set of isomorphism classes of t.d.o is in bijection with $H^2(X, \tau_{\geq 1}\Omega^{\cdot}_X)$, and $\mathcal{A}ut_{\mathcal{O}_X}(\mathcal{A})$ is in bijection with $H^1(X, \tau_{\geq 1}\Omega^{\cdot}_X)$. In particular, if $\mathcal{L}$ is a line bundle on X, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee$ is a t.d.o. It has a very concrete description as a subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{L})$, with the filtration given by

$$F_p(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee) = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{L}) | [P, f] \in F_{p-1}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee), \forall f \in \mathcal{O}_X\}.$$

Let Φ be the Harish-Chandra isomorphism: $Z \to \mathcal{U}\mathfrak{h}^W$. For $\lambda \in \mathfrak{h}^*$, define a central character χ_λ as $\chi_\lambda(z) = \Phi(z)(\lambda)$, $\forall z \in Z$.

G acts on X by left translation, so it acts on $\mathcal{O}_X$. Differentiating this action we get a Lie algebra homomorphism $\alpha : \mathfrak{g} \to \mathcal{T}_X$. Define an $\mathcal{O}_X$ ring $\underline{\mathcal{U}\mathfrak{g}} = \mathcal{O}_X \otimes_k \mathcal{U}\mathfrak{g}$ with the multiplication rule: $[A, f] = \alpha(A)f$, for $f \in \mathcal{O}_X$ and $A \in \mathfrak{g}$. Let $\underline{\mathfrak{b}}$ be the kernel of $\mathcal{O}_X \otimes_k \mathfrak{g} \to \mathcal{T}_X$. λ determines $\underline{\lambda} : \underline{\mathfrak{b}} \to \mathcal{O}_X$. Let $\mathcal{D}_{\lambda-\rho} = \underline{\mathcal{U}\mathfrak{g}} / \sum_{\xi \in \underline{\mathfrak{b}}} \underline{\mathcal{U}\mathfrak{g}}(\xi - \underline{\lambda}(\xi))$. We can verify that $\mathcal{D}_\lambda$ is a t.d.o, since $\underline{\mathfrak{b}}$ is the G equivariant vector bundle associated to $\mathfrak{b}$, where $\mathfrak{b}$ is viewed as a B representation with adjoint action. Moreover, if λ is integral, $\mathcal{D}_\lambda \simeq \mathcal{L}_{\lambda+\rho} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda+\rho}{}^\vee$.

Consider the localization functor from the category of $\mathfrak{g}$ modules with central character χ_λ to the category of sheaves of $\mathcal{D}_\lambda$ modules on X, quasicoherent as $\mathcal{O}_X$ modules,

$$\Delta : \mathcal{M}od(\mathfrak{g}, \chi_\lambda) \to \mathcal{M}od_X(\mathcal{D}_\lambda)$$
$$M \to \mathcal{D}_\lambda \otimes_{\mathcal{U}\mathfrak{g}} M$$

and the global section functor

$$\Gamma : \Gamma(X, \mathcal{M}) \leftarrow \mathcal{M}$$

Δ is left adjoint to Γ.

To demonstrate the idea and have some pictures in mind, from now on let us assume λ is integral. By differentiating the action of G on $\mathcal{L}_{\lambda+\rho}$, we get an algebra homomorphism $\sigma(\lambda) : \mathcal{U}\mathfrak{g} \to \Gamma(X, \mathcal{D}_\lambda)$.

Theorem 2.2. *If λ is regular anti-dominant, Δ and Γ induce equivalence between categories.*

Lemma 2.3. *$\sigma(\lambda)$ is surjective, with $ker\sigma(\lambda) = \mathcal{U}\mathfrak{g}(ker\chi_\lambda)$.*

Proof. (sketch)

First observe that $\sigma(\lambda)|_Z = \chi_\lambda$. Since $\mathcal{L}_{\lambda+\rho}$ is generated by global sections by Borel-Weil theory, it suffices to check that $\sigma(\lambda)|_Z$ acts on $\Gamma(X, \mathcal{L}_{\lambda+\rho})$ as χ_λ. Let $z \in Z$, notice that $\sigma(\lambda)(z)$ commutes with the G action on $\Gamma(X, \mathcal{L}_{\lambda+\rho})$, so it suffices to check that $\sigma(\lambda)(z)$ acts on the lowest weight vector by $\chi_\lambda(z)$, which can be proved by a direct calculation.

Let $F.\mathcal{U}\mathfrak{g}$ be the PBW filtration of $\mathcal{U}\mathfrak{g}$, so $gr^F\mathcal{U}\mathfrak{g} \simeq S\mathfrak{g} \simeq \mathbb{C}[\mathfrak{g}^*]$, where $S\mathfrak{g}$ is the symmetric algebra of $\mathfrak{g}$. Because $\sigma(\lambda)$ respects the grading, it induces a homomorphism $gr\sigma(\lambda) : gr^F\mathcal{U}\mathfrak{g} \to \Gamma(X, gr^F\mathcal{D}_\lambda)$. Let $\pi : T^*X \to X$ be the natural projection, then $gr^F\mathcal{D}_\lambda \simeq \pi_*\mathcal{O}_{T^*X}$. Thus, we can view $gr\sigma(\lambda)$ as a homomorphism $\mathbb{C}[\mathfrak{g}^*] \to \Gamma(X, \mathcal{O}_{T^*X})$.

By the natural G equivariant isomorphism: $T^*X \simeq G \times_B (\mathfrak{g}/\mathfrak{b})^*$, we shall see that $gr\sigma(\lambda)$ is precisely the pull back of $\mu : G \times_B (\mathfrak{g}/\mathfrak{b})^* \to \mathfrak{g}^*$, for $\mu(g, \alpha) = g\alpha g^{-1}$. If $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ by the Killing form, the morphism μ factors through

$$T^*X \xrightarrow{\pi} \mathcal{N} \xrightarrow[immersion]{closed} \mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}^*,$$

where $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$. The morphism π is proper and birational, and $\mathcal{N}$ is a closed irreducible normal subvariety of $\mathfrak{g}$. Let $\mathbb{C}[\mathfrak{g}]_+^G$ be the G invariant polynomials on $\mathfrak{g}$ without constant terms. By a result due to Kostant, an element $x \in \mathfrak{g}$ is nilpotent if and only if x vanishes at $\mathbb{C}[\mathfrak{g}]_+^G$. Thus, $\Gamma(X, \mathcal{O}_{T^*X}) \simeq S\mathfrak{g}/S\mathfrak{g}S\mathfrak{g}_+^G$, where $S\mathfrak{g}S\mathfrak{g}_+^G = (\mathfrak{g}S\mathfrak{g})^G$.

Let $\mathcal{U}_\lambda = \mathcal{U}\mathfrak{g}/\mathcal{U}\mathfrak{g}(ker\chi_\lambda)$, and let $F.\mathcal{U}_\lambda$ be the filtration induced from $F.\mathcal{U}\mathfrak{g}$. Then $gr^F\mathcal{U}_\lambda \simeq S\mathfrak{g}/S\mathfrak{g}S\mathfrak{g}_+^G$. Consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_{m-1}\mathcal{U}_\lambda \arrow[r] \arrow[d, "\sigma(\lambda)"] & F_m\mathcal{U}_\lambda \arrow[r] \arrow[d, "\sigma(\lambda)"] & gr^F_m\mathcal{U}_\lambda \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \Gamma(X, F_{m-1}\mathcal{D}_\lambda) \arrow[r] & \Gamma(X, F_m\mathcal{D}_\lambda) \arrow[r] & \Gamma(X, gr^F_m\mathcal{D}_\lambda) &
\end{tikzcd}$$

We can complete the proof by induction and diagram chasing. □

Lemma 2.4. *Let $\mathcal{M}$ be a $\mathcal{D}_\lambda$ module, quasicoherent as $\mathcal{O}_X$ module, then $H^n(X, \mathcal{M}) = 0$ for $n \neq 0$, and $\mathcal{M}$ is generated by global sections.*

Proof. Since $H^n(X, \mathcal{M}) = \lim_\to H^n(X, \mathcal{F})$, where $\mathcal{F}$ ranges over coherent $\mathcal{O}_X$ submodules of $\mathcal{M}$, it suffices to show that $H^n(X, \mathcal{F}) \to H^n(X, \mathcal{M})$ is the zero map.

Let μ be an integral anti-dominant weight, $V^-(\mu)$ the irreducible finite dimensional representation of lowest weight μ. $V^-(\mu)$ has a filtration by B modules:

$$V^-(\mu) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N \supseteq V_{N+1} = 0$$

such that $V_i/V_{i+1} \simeq \mathbb{C}_{\mu_i}$, where $\mathbb{C}_{\mu_i}$ is the 1 dimensional B representation coming from weight μ_i. We have $\mu_i \geq \mu$, and $\mu_i = \mu$ if and only if $i = 0$.

Consequently, $\mathcal{O}_X \otimes_{\mathbb{C}} V^-(\mu)$ has a filtration by locally free G equivariant sheaves:

$$\mathcal{O}_X \otimes_{\mathbb{C}} V^-(\mu) = \mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \cdots \supseteq \mathcal{V}_N \supseteq \mathcal{V}_{N+1} = 0$$

such that $\mathcal{V}_i/\mathcal{V}_{i+1} \simeq \mathcal{L}_{\mu_i}$

Apply $- \otimes_{\mathcal{O}_X} \mathcal{M}$, we get a filtration as $\mathcal{U}\mathfrak{g}$ modules for $V^-(\mu) \otimes_{\mathbb{C}} \mathcal{M}$ with subquotients isomorphic to $\mathcal{L}_{\mu_i} \otimes_{\mathcal{O}_X} \mathcal{M}$. Notice $\mathcal{V}_0 \to \mathcal{V}_0/\mathcal{V}_1$ induce a surjection:

$$p_{\mathcal{M}} : V^-(\mu) \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{L}_\mu \otimes_{\mathcal{O}_X} \mathcal{M}$$

The key fact is that $p_{\mathcal{M}}$ splits as a morphism between sheaves of abelian groups. Since $\mathcal{L}_{\mu_i} \otimes_{\mathcal{O}_X} \mathcal{M}$ is a $\mathcal{D}_{\lambda+\mu_i}$ module, the center Z acts by $\chi_{\lambda+\mu_i}$. The Z action on $V^-(\mu) \otimes_{\mathbb{C}} \mathcal{M}$ is locally finite, so $V^-(\mu) \otimes_{\mathbb{C}} \mathcal{M}$ decompose into a direct sum of generalized Z eigenspaces. When λ is anti-dominant, the property of central character implies that $\chi_{\lambda+\mu_i} = \chi_{\lambda+\mu}$ if and only if $i = 0$. Thus $(V^-(\mu) \otimes_{\mathbb{C}} \mathcal{M})^{\chi_{\lambda+\mu}} \simeq \mathcal{L}_\mu \otimes_{\mathcal{O}_X} \mathcal{M}$.

Similarly, $V(-\mu)$ has a filtration by B module:

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_M = V(-\mu)$$

such that $W_j/W_{j-1} \simeq \mathbb{C}_{\xi_j}$. We have $\xi_j \leq -\mu$, and $\xi_j = -\mu$ if and only if $j = 0$.

Thus, we get a filtration of $\mathcal{L}_\mu \otimes_{\mathbb{C}} V(-\mu)$ by locally free G equivariant sheaves:

$$0 = \mathcal{W}_{-1} \subseteq \mathcal{W}_0 = \mathcal{O}_X \subseteq \cdots \subseteq \mathcal{W}_M = \mathcal{L}_\mu \otimes_{\mathbb{C}} V(-\mu)$$

Apply $- \otimes_{\mathcal{O}_X} \mathcal{M}$, we get a $\mathcal{U}\mathfrak{g}$ filtration of $\mathcal{L}_\mu \otimes_{\mathbb{C}} V(-\mu) \otimes_{\mathcal{O}_X} \mathcal{M}$ with subquotients isomorphic to $\mathcal{L}_{\mu+\xi_j} \otimes_{\mathcal{O}_X} \mathcal{M}$. Notice $\mathcal{W}_0 \to \mathcal{W}_M$ gives an injection:

$$i_{\mathcal{M}} : \mathcal{M} \to \mathcal{L}_\mu \otimes_{\mathbb{C}} V(-\mu) \otimes_{\mathcal{O}_X} \mathcal{M}$$

$i_{\mathcal{M}}$ also splits as a morphism between sheaves of abelian groups, by checking the Z action and noticing that $\chi_{\lambda+\mu+\xi_j} = \chi_\lambda$ if and only if $j = 0$.

We have the following commutative diagram:

$$\begin{array}{ccc} H^n(X,\mathcal{F}) & \longrightarrow & H^n(X,\mathcal{M}) \\ \downarrow & & \downarrow{\scriptstyle i_{\mathcal{M}*}} \\ H^n(X,\mathcal{L}_\mu \otimes_{\mathbb{C}} V(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}) & \longrightarrow & H^n(X,\mathcal{L}_\mu \otimes_{\mathbb{C}} V(-\mu) \otimes_{\mathcal{O}_X} \mathcal{M}) \end{array}$$

By Serre's theorem, we can assume that $H^n(X, \mathcal{L}_\mu \otimes_{\mathbb{C}} V(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}) = 0$. Because $i_{\mathcal{M}*}$ is injective, we conclude that $H^n(X, \mathcal{M}) = 0$.

To finish the proof, we need to show that $\mathcal{D}_\lambda \otimes_{\mathcal{U}\mathfrak{g}} \Gamma(X, \mathcal{M}) \to \mathcal{M}$ is surjective. It basically follows from that $p_{\mathcal{M}*}$ induces surjection on H^0 and $\mathcal{L}_\mu$ is ample. □

Proof of Theorem 2.2 From the vanishing of higher cohomology, we know Γ is an exact functor and $id \simeq \Gamma\Delta$. From modules in $\mathcal{M}od_X(\mathcal{D}_\lambda)$ are generated by global sections we know $\Delta\Gamma \simeq id$.

3 $(\mathfrak{g}, K)$ Modules and Holonomic $\mathcal{D}$ Modules

Let K be a closed subgroup of G, with $\mathfrak{k}$ its Lie algebra. G acts on its flag variety X by left multiplication. In this section we follow the notation for $\mathcal{D}$ modules given by Bernstein[3]. If f is a morphism between smooth varieties, denote $f^!$ the inverse image functor with shift, and f_* the direct image functor.

Recall that a coherent $\mathcal{D}_X$ module $\mathcal{M}$ is holonomic if its singular support has dimension $dimX$. Holonomic $\mathcal{D}$ modules have nice visible shape because the following are equivalent:

1. $\mathcal{M}$ is holonomic.

2. $\exists$ a sequence of closed subsets of X: $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = \varnothing$ such that $X_r \setminus X_{r+1}$ is smooth, and if we denote $i_r : X_r \setminus X_{r+1} \to X$ locally closed imbedding, $H^k(i_r^! \mathcal{M})$ is locally free of finite rank.

3. $\forall x \in X$, $i_x : \{x\} \to X$, $H^k(i_x^! \mathcal{M})$ is finite dimensional.

Definition 3.1. A $(\mathfrak{g}, K)$ module M is a locally finite K module which is also $\mathfrak{g}$ module such that $k.(A.u) = (Ad(k)A).(k.u)$, and two $\mathfrak{k}$ actions, differentiating the action of K and restricting $\mathfrak{g}$, agree.

By Beilinson-Bernstein Localisation, in the language of $\mathcal{D}$ modules, a $(\mathfrak{g}, K)$ module M with central character $\chi_{-\rho}$ corresponds to a K equivariant $\mathcal{D}_X$ module $\mathcal{M}$ which is quasi coherent as $\mathcal{O}_X$ module. Moreover, M is finitely generated as a $\mathcal{U}\mathfrak{g}$ module if and only if $\mathcal{M}$ is a coherent $\mathcal{D}_X$ module.

Proposition 3.2. *If K has finite many orbits on X, a K equivariant coherent $\mathcal{D}_X$ module $\mathcal{M}$ is holonomic.*

Proof. We will prove the proposition by induction on the number of K-orbits on $supp\mathcal{M}$.

Suppose there is only one orbit. Let $x_0 \in supp\mathcal{M}$, and define $p : K \to X$ by $p(k) = kx_0$. Let $i_{x_0} : \{x_0\} \to X$ be the natural embedding. By the K equivariance of $\mathcal{M}$, $p^!\mathcal{M} = \mathcal{O}_K \otimes_{\mathbb{C}} (i^!_{x_0}\mathcal{M})$. Since p restricting to the orbit of x_0 is a smooth morphism, the module $p^!\mathcal{M}$ is coherent, which implies that $(i^!_{x_0}\mathcal{M})$ must be finite dimensional. As a result, $\mathcal{M}$ is holonomic.

In general, we know there is a closed orbit Y. Denote $i : Y \to X$ the closed embedding and $j : X \setminus Y \to X$ the open embedding. By Kashiwari's theorem, we have an exact triangle :

$$i_*i^!\mathcal{M} \to \mathcal{M} \to j_*(\mathcal{M}|_{X\setminus Y})$$

By induction hypothesis $i^!\mathcal{M}$ and $\mathcal{M}|_{X\setminus Y}$ are holonomic, and the direct image functor preserves holonomicity, so $\mathcal{M}$ is holonomic. □

We want to work with the category of holonomic $\mathcal{D}$ modules because it is Artinian. Consequently, as an application of Beilinson-Bernstein Localisation, we see that Harish-Chandra modules with central character $\chi_{-\rho}$ have finite length.

4 The Story of SL_2

In this section we shall see some explicit calculation for SL_2. Let $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, the standard basis for $\mathfrak{sl}_2$. Take upper triangular Borel, and identify the flag variety X as $\mathbb{P}^1$. Set $U_0 = \mathbb{P}^1 \setminus \{\infty\}$ with coordinate z, and $U_1 = \mathbb{P}^1 \setminus \{0\}$ with coordinate $w = \frac{1}{z}$. G act on $\mathbb{P}^1$ by linear fractional transformation. The Casimir is $c = \frac{1}{2}h^2 + ef + fe$.

The invertible sheaf $\mathcal{O}(-n)$ is isomorphic to $\mathcal{L}_{n\rho}$. We can describe this line bundle by gluing $U_0 \times \mathbb{C}$ and $U_1 \times \mathbb{C}$, with the identification: $(z, u) = (w, v)$ if and only if $w = \frac{1}{z}$ and $v = w^{-n}u$.

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2$, on $U_0 \times \mathbb{C}$, the SL_2 equivaraint structure is given by:

$$(z, u) \to (\frac{az + b}{cz + d}, (cz + d)^n u).$$

The t.d.o $\mathcal{D}_{(n-1)\rho}$ when restrict to U_0 (resp. U_1) is isomorphic to $\mathcal{D}_{U_0}$ (resp. $\mathcal{D}_{U_1}$), and the gluing information is given by: $\partial_w = -z^2\partial_z - nz$.

To see what is $\sigma((n-1)\rho) : \mathcal{U}\mathfrak{sl}_2 \to \Gamma(X, \mathcal{D}_{n\rho})$, let us take $A \in \mathfrak{sl}_2$, then $(\sigma((n-1)\rho)(A))s = \frac{d}{dt}|_{t=0}(exp(tA)s)$ for $s \in \mathcal{L}_{n\rho}$. We have

$$h \to -2z\partial_z - n,$$

$$e \to -\partial_z,$$

$$f \to z^2\partial_z + nz.$$

The Casimir c acts by the scalar $\frac{n^2}{2} - n$.

Finally, set $n = 0$. Let $\mathcal{D}_X\delta$ be the $\mathcal{D}_X$ module generated by the δ-function supported at ∞. Then $\Gamma(X, \mathcal{D}_X\delta) \simeq \mathbb{C}[f]\delta$, with the relations $h\delta = -2\delta$ and $e\delta = 0$. Thus, $\Gamma(X, \mathcal{D}_X\delta)$ is isomorphic to the Verma module with highest weight -2.

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