有限元方法 2025 秋冬作业三

曾申昊 3220100701 *

电子科学与技术 *2202*，浙江大学

更新时间: 2025 年 10 月 22 日

1. 二阶 Sobolev 空间 $H^2(0,1)$

(a) 依次验证内积的三条准则：

- $(u+w,v)_{H^2}=(u,v)_{H^2}+(w,v)_{H^2}$ 且 $(\lambda u,v)_{H^2}=\lambda(u,v)_{H^2}$
- $(u,v)_{H^2}=\int_0^1(uv+u'v'+u''v'')\mathrm{d}x=(v,u)_{H^2}$
- $(u,u)_{H^2}=\int_0^1\left(u^2+(u')^2+(u'')^2\right)\mathrm{d}x\geq 0$，当且仅当 $u=0$ 时等号成立

因此 $(\bullet,\bullet)_{H^2}$ 是一个内积。

(b) 由内积 $(\bullet,\bullet)_{H^2}$ 诱导的范数为

$$\|u\|_{H^2}=\sqrt{(u,u)_{H^2}}=\sqrt{\int_0^1(u^2+(u')^2+(u'')^2)\,\mathrm{d}x}=\sqrt{\|u\|_{L^2}^2+\|u'\|_{L^2}^2+\|u''\|_{L^2}^2} \tag{1}$$

对于 $H^2(0,1)$ 中的任意 Cauchy 列 $\{u_n\}$ 有，对 $\forall\varepsilon>0$，$\exists N_0\in\mathbb{N}$，当 $m,n>N_0$ 时有

$$\|u_n-u_m\|_{H^2}<\varepsilon \tag{2}$$

也就是

$$\|u_n-u_m\|_{L^2}^2+\|u_n'-u_m'\|_{L^2}^2+\|u_n''-u_m''\|_{L^2}^2<\varepsilon^2 \tag{3}$$

由 $L^2(0,1)$ 的完备性可知 $\{u_n\}$、$\{u_n'\}$ 和 $\{u_n''\}$ 在 L^2 范数意义下分别收敛于 u、v 和 w。我们接下来需要证明 $u'\overset{a.e.}{=}v$ 以及 $v'\overset{a.e.}{=}w$。

$$\begin{aligned}\int_0^1|u_n'-v|\mathrm{d}x&\leq\sqrt{\int_0^1(u_n'-v)^2\mathrm{d}x}\sqrt{\int_0^1 1^2\mathrm{d}x}\\&=\|u_n'-v\|_{L^2}\to 0\end{aligned} \tag{4}$$

由 Fatou 引理可知

$$\begin{aligned}\int_0^1|u'-v|\mathrm{d}x&=\int_0^1\lim_{n\to\infty}|u_n'-v|\mathrm{d}x\\&\leq\liminf_{n\to\infty}\int_0^1|u_n'-v|\mathrm{d}x=0\end{aligned} \tag{5}$$

由此可得 $u'\overset{a.e.}{=}v$，同理可得 $v'\overset{a.e.}{=}w$，因此 $u\in H^2(0,1)$。最后易证 $\|u_n-u\|_{H^2}\to 0$，综上我们证明了 $H^2(0,1)$ 是一个 Hilbert 空间。

*邮箱: 3097714673@qq.com 或 3220100701@zju.edu.cn

(c) 由于 $v' \in \mathrm{AC}[0,1]$，我们有

$$u'(x) = u'(a) + \int_a^x u''(t)\mathrm{d}t \tag{6}$$

令 $F(s) = \int_a^s u''(t)\mathrm{d}t$，等式两边积分可得

$$\int_a^x u'(s)\mathrm{d}s = \int_a^x u'(a)\mathrm{d}s + \int_a^x F(s)\mathrm{d}s \tag{7}$$

$$u(x) - u(a) = u'(a)(x-a) + \int_a^x F(s)\mathrm{d}s \tag{8}$$

$$u(x) - u(a) = u'(a)(x-a) + sF(s)\Big|_{s=a}^{s=x} - \int_a^x sF'(s)\mathrm{d}s \tag{9}$$

$$u(x) - u(a) = u'(a)(x-a) + x\int_a^x u''(s)\mathrm{d}s - \int_a^x su''(s)\mathrm{d}s \tag{10}$$

整理后可得结论。

2. 区间 $[0,1]$ 上的分段 Lagrange 插值

(a) 由第 1 题的 (c) 小问可知

$$u(x) = u(x_i) + u'(x_i)(x-x_i) + r(x) \qquad x \in [x_i, x_{i+1}] \tag{11}$$

其中 $r(x) = \int_{x_i}^x u''(t)(x-t)\mathrm{d}t$。

$$\begin{aligned}(I_h u)(x) &= u(x_i) + u'(x_i)(x-x_i) + (I_h r)(x) \\ &= u(x_i) + u'(x_i)(x-x_i) + r(x_{i+1})\frac{x-x_i}{h_i}\end{aligned} \qquad x \in [x_i, x_{i+1}] \tag{12}$$

其中 $h_i = x_{i+1} - x_i$，然后可以进行误差估计

$$\begin{aligned}\|u - I_h u\|_{L^\infty(x_i,x_{i+1})} &= \max_{x\in[x_i,x_{i+1}]} |r(x) - (I_h r)(x)| \\ &= \max_{x\in[x_i,x_{i+1}]} \left| \int_{x_i}^x u''(t)(x-t)\mathrm{d}t - \frac{x-x_i}{h_i}\int_{x_i}^{x_{i+1}} u''(t)(x_{i+1}-t)\mathrm{d} \right| \\ &\le \max_{x\in[x_i,x_{i+1}]} \left\{ \left| \int_{x_i}^x u''(t)(x-t)\mathrm{d}t \right| + \left| \frac{x-x_i}{h_i}\int_{x_i}^{x_{i+1}} u''(t)(x_{i+1}-t)\mathrm{d}t \right| \right\} \\ &\le \|u''\|_{L^\infty(x_i,x_{i+1})} \max_{x\in[x_i,x_{i+1}]} \left\{ \left| \int_{x_i}^x (x-t)\mathrm{d}t \right| + \left| \frac{x-x_i}{h_i}\int_{x_i}^{x_{i+1}} (x_{i+1}-t)\mathrm{d}t \right| \right\} \\ &= \|u''\|_{L^\infty(x_i,x_{i+1})} \max_{x\in[x_i,x_{i+1}]} \left\{ \left| \frac{(x-x_i)^2}{2} \right| + \left| \frac{x-x_i}{h_i}\frac{h_i^2}{2} \right| \right\} \\ &\le h_i^2 \|u''\|_{L^\infty(x_i,x_{i+1})}\end{aligned} \tag{13}$$

$$\begin{aligned}\|u' - (I_h u)'\|_{L^\infty(x_i,x_{i+1})} &= \max_{x\in[x_i,x_{i+1}]} |r'(x) - (I_h r)'(x)| \\ &= \max_{x\in[x_i,x_{i+1}]} \left| \int_{x_i}^x u''(t)\mathrm{d}t - \frac{r(x_{i+1})}{h_i} \right| \\ &\le \max_{x\in[x_i,x_{i+1}]} \left\{ \left| \int_{x_i}^x u''(t)\mathrm{d}t \right| + \left| \frac{1}{h_i}\int_{x_i}^{x_{i+1}} u''(t)(x_{i+1}-t)\mathrm{d}t \right| \right\} \\ &\le \|u''\|_{L^\infty(x_i,x_{i+1})} \max_{x\in[x_i,x_{i+1}]} \left\{ \left| \int_{x_i}^x 1\mathrm{d}t \right| + \left| \frac{1}{h_i}\int_{x_i}^{x_{i+1}} (x_{i+1}-t)\mathrm{d}t \right| \right\} \\ &= \frac{3h_i}{2}\|u''\|_{L^\infty(x_i,x_{i+1})}\end{aligned} \tag{14}$$

上述两式控制了区间 $[x_i, x_{i+1}]$ 上的误差，取各子区间最大值即得整体误差估计。

(b) 将每个子区间 $[x_i, x_{i+1}]$ 划分成两个等长的子区间，令其中点为

$$m_i = \frac{x_i + x_{i+1}}{2} \qquad i = 0, 1, \dots, N \tag{15}$$

在每组相邻节点 $\{x_i, m_i, x_{i+1}\}$ 上构造二次插值多项式

$$(I_h f)(x) = f(x_i)L_{i,0}(x) + f(m_i)L_{i,1}(x) + f(x_{i+1})L_{i,2}(x) \qquad x \in [x_i, x_{i+1}] \tag{16}$$

其中

$$L_{i,0}(x) = \frac{(x-m_i)(x-x_{i+1})}{(x_i-m_i)(x_i-x_{i+1})} \tag{17}$$

$$L_{i,1}(x) = \frac{(x-x_i)(x-x_{i+1})}{(m_i-x_i)(m_i-x_{i+1})} \tag{18}$$

$$L_{i,2}(x) = \frac{(x-x_i)(x-m_i)}{(x_{i+1}-x_i)(x_{i+1}-m_i)} \tag{19}$$

(c) 类似 (a) 小问，作 $u(x)$ 的泰勒展开

$$u(x) = u(x_i) + u'(x_i)(x-x_i) + \frac{u''(x_i)}{2}(x-x_i)^2 + r(x) \qquad x \in [x_i, x_{i+1}] \tag{20}$$

其中 $r(x) = \frac{1}{2}\int_{x_i}^{x} u'''(t)(x-t)^2 \mathrm{d}t$。

$$\begin{aligned}(I_h u)(x) &= u(x_i) + u'(x_i)(x-x_i) + \frac{u''(x_i)}{2}(x-x_i)^2 + (I_h r)(x) \\ &= u(x_i) + u'(x_i)(x-x_i) + \frac{u''(x_i)}{2}(x-x_i)^2 \\ &\quad + r(m_i)\frac{(x-x_i)(x-x_{i+1})}{(m_i-x_i)(m_i-x_{i+1})} + r(x_{i+1})\frac{(x-x_i)(x-m_i)}{(x_{i+1}-x_i)(x_{i+1}-m_i)}\end{aligned} \qquad x \in [x_i, x_{i+1}] \tag{21}$$

然后可以进行误差估计

$$\begin{aligned}|r(x)| &\le \frac{1}{2}\int_{x_i}^{x} |u'''(t)|(x-t)^2 \mathrm{d}t \\ &\le \frac{1}{2}\sqrt{\int_{x_i}^{x} |u'''(t)|^2 \mathrm{d}t \int_{x_i}^{x} (x-t)^4 \mathrm{d}t} \\ &\le \frac{1}{2}\|u'''\|_{L^2(x_i,x_{i+1})}\sqrt{\int_{x_i}^{x} (x-t)^4 \mathrm{d}t} \\ &\le \frac{1}{2\sqrt{5}} h_i^{\frac{5}{2}} \|u'''\|_{L^2(x_i,x_{i+1})}\end{aligned} \tag{22}$$

$$\begin{aligned}\|u - I_h u\|_{L^2(x_i,x_{i+1})} &= \sqrt{\int_{x_i}^{x_{i+1}} |r(x) - (I_h r)(x)|^2 \mathrm{d}x} \\ &\le \sqrt{\int_{x_i}^{x_{i+1}} \big(|r(x)| + |r(m_i)L_{i,1}(x)| + |r(x_{i+1})L_{i,2}(x)|\big)^2 \mathrm{d}x} \\ &\le \|u'''\|_{L^2(x_i,x_{i+1})}\sqrt{\int_{x_i}^{x_{i+1}} \frac{1}{20}h_i^5\big(1 + |L_{i,1}(x)| + |L_{i,2}(x)|\big)^2 \mathrm{d}x} \\ &\le C_1 h_i^3 \|u'''\|_{L^2(x_i,x_{i+1})}\end{aligned} \tag{23}$$

$$
\begin{aligned}
|r'(x)| &\le \int_{x_i}^{x} |u'''(t)||(x-t)|\mathrm{d}t \\
&\le \sqrt{\int_{x_i}^{x} |u'''(t)|^2\mathrm{d}t \int_{x_i}^{x} (x-t)^2\mathrm{d}t} \\
&\le \|u'''\|_{L^2(x_i,x_{i+1})}\sqrt{\int_{x_i}^{x} (x-t)^2\mathrm{d}t} \\
&\le \frac{1}{2\sqrt{3}} h_i^{\frac{3}{2}} \|u'''\|_{L^2(x_i,x_{i+1})}
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\|u'-(I_hu)'\|_{L^2(x_i,x_{i+1})} &= \sqrt{\int_{x_i}^{x_{i+1}} |r'(x)-(I_hr)'(x)|^2\mathrm{d}x} \\
&\le \sqrt{\int_{x_i}^{x_{i+1}} \big(|r'(x)|+|r(m_i)L'_{i,1}(x)|+|r(x_{i+1})L'_{i,2}(x)|\big)^2\mathrm{d}x} \\
&\le \|u'''\|_{L^2(x_i,x_{i+1})}\sqrt{\int_{x_i}^{x_{i+1}} \frac{1}{12}h_i^3\big(1+|L'_{i,1}(x)|+|L'_{i,2}(x)|\big)^2\mathrm{d}x} \\
&\le C_2 h_i^2 \|u'''\|_{L^2(x_i,x_{i+1})}
\end{aligned} \tag{25}
$$

类似的，取各子区间最大值即得整体误差估计。

3. 区间 $[0,1]$ 上两点边值问题的有限元方法

(a) 范数 $\|\bullet\|_{H^1(0,1)}$, $\|\bullet\|_{H^2(0,1)}$ 和半范数 $|\bullet|_{H^2(0,1)}$, $|\bullet|_{H^3(0,1)}$ 的定义如下

$$
\|u\|_{H^1(0,1)} = \left(\sum_{|\alpha|\le 1} \|\partial^\alpha u\|_{L^2(0,1)}^2\right)^{\frac{1}{2}} = \sqrt{\|u\|_{L^2(0,1)}^2 + \|u'\|_{L^2(0,1)}^2} \tag{26}
$$

$$
\|u\|_{H^2(0,1)} = \left(\sum_{|\alpha|\le 2} \|\partial^\alpha u\|_{L^2(0,1)}^2\right)^{\frac{1}{2}} = \sqrt{\|u\|_{L^2(0,1)}^2 + \|u'\|_{L^2(0,1)}^2 + \|u''\|_{L^2(0,1)}^2} \tag{27}
$$

$$
|u|_{H^2(0,1)} = \left(\sum_{|\alpha|= 2} \|\partial^\alpha u\|_{L^2(0,1)}^2\right)^{\frac{1}{2}} = \|u''\|_{L^2(0,1)} \tag{28}
$$

$$
|u|_{H^3(0,1)} = \left(\sum_{|\alpha|= 3} \|\partial^\alpha u\|_{L^2(0,1)}^2\right)^{\frac{1}{2}} = \|u'''\|_{L^2(0,1)} \tag{29}
$$

(b) 该问题的有限元方法可以由以下推导得到

$$
\begin{aligned}
-\int_0^1 (au')'v\mathrm{d}x + 2\int_0^1 uv\mathrm{d}x &= \int_0^1 fv\mathrm{d}x \\
-au'v\Big|_0^1 + \int_0^1 au'v'\mathrm{d}x + 2\int_0^1 uv\mathrm{d}x &= \int_0^1 fv\mathrm{d}x \\
\int_0^1 au'v'\mathrm{d}x + 2\int_0^1 uv\mathrm{d}x &= \int_0^1 fv\mathrm{d}x
\end{aligned} \tag{30}
$$

取 $v_h \in V_h$，分段一次有限元空间定义如下

$$
\begin{aligned}
V_h &= \big\{v_h \in C[0,1] : v_h|_{[x_i,x_{i+1}]} \in P_1(x_i,x_{i+1}), v_h(0)=0\big\} \\
&= span\{\phi_1,\phi_2,\dots,\phi_N\}
\end{aligned} \tag{31}
$$

其中 ϕ_i 是节点 x_i 处的分段一次基函数：

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \tag{32}$$

误差估计由第 2 题的 (a) 小问易得

$$\|u - u_h\|_{L^2(0,1)} \le C_1 h^2 \|u''\|_{L^2(0,1)} = C_1 h^2 |u|_{H^2(0,1)} \tag{33}$$

$$\|u - u_h\|_{H^1(0,1)} = \|u' - u_h'\|^2_{L^2(0,1)} \le C_2 h \|u''\|_{L^2(0,1)} = C_2 h |u|_{H^2(0,1)} \tag{34}$$

(c) 该问题的有限元方法由 (b) 小问已经得到，其分段二次有限元空间定义如下

$$\begin{aligned} V_h &= \left\{ v_h \in C[0,1] : v_h|_{[x_i, x_{i+1}]} \in P_2(x_i, x_{i+1}), v_h(0) = 0 \right\} \\ &= span\{\phi_1, \phi_2, \dots, \phi_{2N+1}\} \end{aligned} \tag{35}$$

其中 ϕ_i 是节点 x_i 和中点 m_i 处的分段二次基函数：

$$\phi_{2i}(x) = \begin{cases} \dfrac{(x - x_{i-1})(x - m_{i-1})}{(x_i - x_{i-1})(x_i - m_{i-1})} & x \in [x_{i-1}, x_i] \\ \dfrac{(x - x_{i+1})(x - m_i)}{(x_i - x_{i+1})(x_i - m_i)} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \qquad i = 1, 2, \dots, N \tag{36}$$

$$\phi_{2i+1}(x) = \begin{cases} \dfrac{(x - x_i)(x - x_{i+1})}{(m_i - x_i)(m_i - x_{i+1})} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \qquad i = 0, 1, \dots, N \tag{37}$$

其中 $m_i = \frac{x_i + x_{i+1}}{2}$。误差估计由第 2 题的 (c) 小问易得

$$\|u - u_h\|_{L^2(0,1)} \le C_1 h^3 \|u'''\|_{L^2(0,1)} = C_1 h^3 |u|_{H^3(0,1)} \tag{38}$$

$$\|u - u_h\|_{H^1(0,1)} = \|u' - u_h'\|^2_{L^2(0,1)} \le C_2 h^2 \|u'''\|_{L^2(0,1)} = C_2 h^2 |u|_{H^3(0,1)} \tag{39}$$

4. 区间 $[0,1]$ 上两点边值问题的数值方法

(a) 记 $h = \frac{1}{N+1}$，该问题的分段一次有限元方法为

$$\int_0^1 u_h' v_h' \mathrm{d}x = \int_0^1 f v_h \mathrm{d}x \qquad \forall v_h \in H_0^1(0,1) \tag{40}$$

设 $u_h = \sum_{i=1}^N u_i \phi_i$，其中 ϕ_i 是节点 x_i 处的分段一次基函数：

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \tag{41}$$

将其代入式 (??) 计算可得有限元矩阵 $A_{\mathrm{FEM}} \in \mathbb{R}^{N\times N}$

$$A_{\mathrm{FEM}} = \begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \cdots & \cdots & 0 \\ 0 & -\frac{1}{h} & \frac{2}{h} & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ 0 & 0 & 0 & \cdots & -\frac{1}{h} & \frac{2}{h} \end{pmatrix} \tag{42}$$

该问题的有限差分方法为

$$-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = f(x_i) \qquad i=1,2,\dots,N \tag{43}$$

计算可得有限差分矩阵 $A_{\mathrm{FDM}} \in \mathbb{R}^{N\times N}$

$$A_{\mathrm{FDM}} = \begin{pmatrix} \frac{2}{h^2} & -\frac{1}{h^2} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & \cdots & \cdots & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2} & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ 0 & 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2} \end{pmatrix} \tag{44}$$

可以发现这两个仅仅差一个常数倍。

(b) 设 $u_h = \sum_{i=1}^{2N+1} u_i\phi_i$，其中 ϕ_i 是节点 x_i 和中点 m_i 处的分段二次基函数：

$$\phi_{2i}(x) = \begin{cases} \dfrac{2(x-x_{i-1})(x-m_{i-1})}{h^2} & x\in[x_{i-1},x_i] \\ \dfrac{2(x-x_{i+1})(x-m_i)}{h^2} & x\in[x_i,x_{i+1}] \\ 0 & \text{else} \end{cases} \qquad i=1,2,\dots,N \tag{45}$$

$$\phi_{2i+1}(x) = \begin{cases} \dfrac{4(x-x_i)(x-x_{i+1})}{h^2} & x\in[x_i,x_{i+1}] \\ 0 & \text{else} \end{cases} \qquad i=0,1,\dots,N \tag{46}$$

将其代入式 (??) 计算可得有限元矩阵 $A_{\mathrm{FEM}} \in \mathbb{R}^{(2N+1)\times(2N+1)}$

$$A_{\mathrm{FEM}} = \begin{pmatrix} \frac{16}{3h} & \frac{8}{3h} & 0 & 0 & \cdots & \cdots & 0 \\ \frac{8}{3h} & \frac{14}{3h} & \frac{8}{3h} & \frac{1}{3h} & \cdots & \cdots & 0 \\ 0 & \frac{8}{3h} & \frac{16}{3h} & \frac{8}{3h} & \cdots & \cdots & 0 \\ 0 & \frac{1}{3h} & \frac{8}{3h} & \frac{14}{3h} & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{8}{3h} & \frac{14}{3h} & \frac{8}{3h} \\ 0 & 0 & 0 & 0 & \cdots & \frac{8}{3h} & \frac{16}{3h} \end{pmatrix} \tag{47}$$

在分段二次有限元问题中，我们添加中点加细了网格。在高阶有限差分中，或许可以利用中点构造具有更高收敛阶的差分格式，来拟合二阶导数。

5. 格林函数 $G(\bullet, y) \in H_0^1(0,1)$

(a) 将 $\frac{\partial G}{\partial x}(x,y)$ 代入方程可得

$$\frac{\partial G}{\partial x}(x,y) = \begin{cases} 1-y, & x<y \\ -y, & x>y \end{cases} \tag{48}$$

$$\begin{aligned} \int_0^1 v'(x)\frac{\partial G}{\partial x}(x,y)\mathrm{d}x &= \int_0^y v'(x)\frac{\partial G}{\partial x}(x,y)\mathrm{d}x + \int_y^1 v'(x)\frac{\partial G}{\partial x}(x,y)\mathrm{d}x \\ &= \int_0^y v'(x)(1-y)\mathrm{d}x + \int_y^1 v'(x)(-y)\mathrm{d}x \\ &= \left(v(y)-v(0)\right)(1-y) + \left(v(1)-v(y)\right)(-y) \\ &= v(y) \end{aligned} \tag{49}$$

$G(x,y)$ 是该问题的解。

(b) 交换 x 和 y 可得

$$\begin{aligned} G(y,x) &= \begin{cases} (1-x)y, & y<x \\ x(1-y), & y>x \end{cases} \\ &= \begin{cases} (1-y)x, & x<y \\ y(1-x), & x>y \end{cases} = G(x,y) \end{aligned} \tag{50}$$

要证明该函数是 $-u''=f$ 的格林函数，即证明 $-\frac{\partial^2 G}{\partial x^2}(x,y) = \delta(x-y)$。

$$\begin{cases} \displaystyle\int_0^1 -\frac{\partial^2 G}{\partial x^2}(x,y)\mathrm{d}x = -\frac{\partial G}{\partial x}(x,y)\Big|_{x=0}^{x=1} = (-y)-(1-y) = 1 \\ \displaystyle\int_0^1 \delta(x-y)\mathrm{d}x = 1 \end{cases} \tag{51}$$

(c) 通过分部积分可证明结论

$$\begin{aligned} \int_0^1 G(x,y)f(y)\mathrm{d}y &= -\int_0^1 G(x,y)u''(y)\mathrm{d}y \\ &= -G(x,y)u'(y)\Big|_{y=0}^{y=1} + \int_0^1 \frac{\partial G}{\partial y}(x,y)u'(y)\mathrm{d}y \\ &= -G(x,1)u'(1) + G(x,0)u'(0) + u(x) \\ &= u(x) \end{aligned} \tag{52}$$

倒数第二个等号成立由 (a) 结论可得；最后一个等号成立是由于 $G(x,1)=G(x,0)=0$。

6. 分片一次的有限元方法编程