有限元方法 2025 秋冬作业六

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1. Morley 有限元的分析与编程

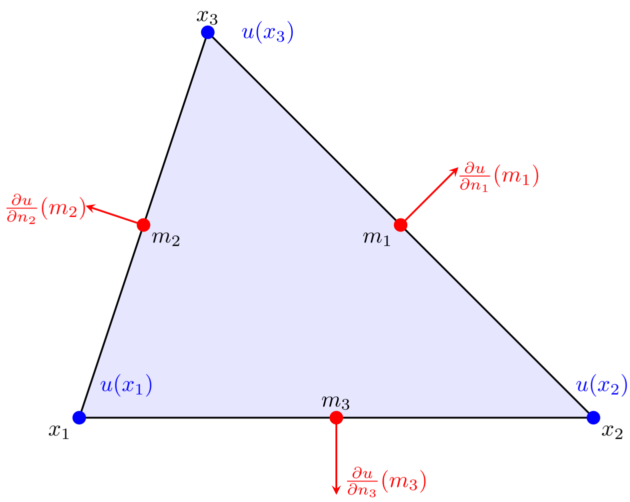

图 1: Morley 单位元示意图

(a) Morley 元 $(\mathcal{T},\mathcal{P}_2,\mathcal{N}_M)$ 的自由度为 6 个，包括三角形单元的三个顶点处的函数值 $u(\boldsymbol{x_i})$ 以及三个边中点处的法向导数 $\frac{\partial u}{\partial n_i}(\boldsymbol{m_i})$，如图 ?? 所示。定义 Morley 元的基函数 $\{\phi_i\}_{i=1}^6$ 满足

$$\begin{cases}\phi_i(\boldsymbol{x_j})=\delta_{ij} & j=1,2,3\\ \dfrac{\partial\phi_i}{\partial n_j}(\boldsymbol{m_j})=0 & j=1,2,3\end{cases}\qquad i=1,2,3 \tag{1}$$

$$\begin{cases}\phi_{i+3}(\boldsymbol{x_j})=0 & j=1,2,3\\ \dfrac{\partial\phi_{i+3}}{\partial n_j}(\boldsymbol{m_j})=\delta_{ij} & j=1,2,3\end{cases}\qquad i=1,2,3 \tag{2}$$

于是 Morley 元插值 I_T 可以表示为

$$I_T(u)=\sum_{i=1}^{3}u(\boldsymbol{x_i})\phi_i+\sum_{i=1}^{3}\frac{\partial u}{\partial n_i}(\boldsymbol{m_i})\phi_{i+3} \tag{3}$$

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(b) 下面的证明中我们先观察单位元上的 $u(x)$

$$\begin{cases} u(\boldsymbol{x_i}) = u(\boldsymbol{x}) + (\boldsymbol{x_i}-\boldsymbol{x})^T\nabla u(\boldsymbol{x}) + \dfrac{1}{2}(\boldsymbol{x_i}-\boldsymbol{x})^T\nabla^2 u(\boldsymbol{x})(\boldsymbol{x_i}-\boldsymbol{x}) + R(\boldsymbol{x_i}) \\ R(\boldsymbol{x_i}) = \dfrac{1}{2}\displaystyle\int_0^1 g'''(t;\boldsymbol{x_i})(1-t)^2\mathrm{d}t \\ g'''(t;\boldsymbol{x_i}) = \displaystyle\sum_{|\alpha|=3}\frac{\partial^3 u}{\partial x^\alpha}u(\boldsymbol{x}+t(\boldsymbol{x_i}-\boldsymbol{x}))(\boldsymbol{x_i}-\boldsymbol{x})^\alpha \end{cases} \tag{4}$$

$$\begin{cases} \nabla u(\boldsymbol{m_i}) = \nabla u(\boldsymbol{x}) + \nabla^2 u(\boldsymbol{x})(\boldsymbol{m_i}-\boldsymbol{x}) + \nabla R(\boldsymbol{m_i}) \\ \dfrac{\partial u}{\partial n_i}(\boldsymbol{m_i}) = \boldsymbol{n_i}^T\nabla u(\boldsymbol{x}) + \boldsymbol{n_i}^T\nabla^2 u(\boldsymbol{x})(\boldsymbol{m_i}-\boldsymbol{x}) + \boldsymbol{n_i}^T\nabla R(\boldsymbol{m_i}) \end{cases} \tag{5}$$

代入插值算子表达式，整理得到

$$\begin{aligned} I_T(u) = & \sum_{i=1}^3 u(\boldsymbol{x_i})\phi_i + \sum_{i=1}^3 \frac{\partial u}{\partial n_i}(\boldsymbol{m_i})\phi_{i+3} \\ = & \; u(x)\left(\sum_{i=1}^3\phi_i\right) + \nabla u(\boldsymbol{x})\cdot\left(\sum_{i=1}^3(\boldsymbol{x_i}-\boldsymbol{x})\phi_i + \sum_{i=1}^3\boldsymbol{n_i}\phi_{i+3}\right) \\ & + \frac{1}{2}\nabla^2 u(\boldsymbol{x}) : \left(\sum_{i=1}^3(\boldsymbol{x_i}-\boldsymbol{x})(\boldsymbol{x_i}-\boldsymbol{x})^T\phi_i + \sum_{i=1}^3\boldsymbol{n_i}(\boldsymbol{m_i}-\boldsymbol{x})^T\phi_{i+3}\right) \\ & + \sum_{i=1}^3 R(\boldsymbol{x_i})\phi_i + \sum_{i=1}^3\boldsymbol{n_i}^T\nabla R(\boldsymbol{m_i})\phi_{i+3} \\ = & \; u(x) + \sum_{i=1}^3 R(\boldsymbol{x_i})\phi_i + \sum_{i=1}^3\boldsymbol{n_i}^T\nabla R(\boldsymbol{m_i})\phi_{i+3} \end{aligned} \tag{6}$$

下面可以计算 $|u - I_T u|_{H^2(T)}$ 我们可以验证等式

$$\begin{aligned} \left|\nabla^2 u - \nabla^2(I_T u)\right| &= \left|\nabla^2\left(\sum_{i=1}^3 R(\boldsymbol{x_i})\phi_i + \sum_{i=1}^3\boldsymbol{n_i}^T\nabla R(\boldsymbol{m_i})\phi_{i+3}\right)\right| \\ &= \left|\sum_{i=1}^3 R(\boldsymbol{x_i})\nabla^2\phi_i + \sum_{i=1}^3\boldsymbol{n_i}^T\nabla R(\boldsymbol{m_i})\nabla^2\phi_{i+3}\right| \end{aligned} \tag{7}$$

然后我们可以对误差进行估计

$$\begin{aligned} \left|\nabla^2 u - \nabla^2(I_T u)\right| \le & \sum_{i=1}^3\left|R(\boldsymbol{x_i})\nabla^2\phi_i\right| + \sum_{i=1}^3\left|\boldsymbol{n_i}^T\nabla R(\boldsymbol{m_i})\nabla^2\phi_{i+3}\right| \\ \le & \tilde{C}_1\sum_{i=1}^3\left|h_T^3\nabla^3 u\cdot\frac{1}{h_T^2}\right| + \tilde{C}_2\sum_{i=1}^3\left|h_T\cdot h_T^2\nabla^3 u\cdot\frac{1}{h_T^2}\right| \\ \le & \tilde{C}h_T|\nabla^3 u| \end{aligned} \tag{8}$$

两边平方并积分可得结论。

(c) 令 $\phi_i = a\lambda_1^2 + b\lambda_2^2 + c\lambda_3^2 + d\lambda_1\lambda_2 + e\lambda_1\lambda_3 + f\lambda_2\lambda_3$。以 ϕ_1 为例进行计算，ϕ_2 和 ϕ_3 类似。

$$\begin{cases} \phi_1(\boldsymbol{x_j}) = \delta_{1j} & j=1,2,3 \\ \dfrac{\partial\phi_1}{\partial n_j}(\boldsymbol{m_j}) = 0 & j=1,2,3 \end{cases} \tag{9}$$

由前三个方程可设 $\phi_1 = \lambda_1^2 + d\lambda_1\lambda_2 + e\lambda_1\lambda_3 + f\lambda_2\lambda_3$，并且要记得

$$\begin{cases} \nabla\phi_1 = 2\lambda_1\nabla\lambda_1 + d\left(\lambda_1\nabla\lambda_2 + \lambda_2\nabla\lambda_1\right) + e\left(\lambda_1\nabla\lambda_3 + \lambda_3\nabla\lambda_1\right) + f\left(\lambda_2\nabla\lambda_3 + \lambda_3\nabla\lambda_2\right) \\ \dfrac{\partial\phi_1}{\partial n_j}(\boldsymbol{m_j}) = \boldsymbol{n_j}\cdot\nabla\phi_1(\boldsymbol{m_j}) \qquad j=1,2,3 \end{cases} \tag{10}$$

$$
\begin{cases}
\nabla\lambda_i = -\dfrac{\boldsymbol{n_i}}{d_i} & i=1,2,3 \\
\nabla\lambda_1 + \nabla\lambda_2 + \nabla\lambda_3 = 0
\end{cases}
\tag{11}
$$

代入后三个方程进行计算

$$
\begin{cases}
\boldsymbol{n_1}\cdot\left[d\left(0.5\nabla\lambda_1\right)+e\left(0.5\nabla\lambda_1\right)+f\left(0.5\nabla\lambda_3+0.5\nabla\lambda_2\right)\right]=0 \\
\boldsymbol{n_2}\cdot\left[\nabla\lambda_1+d\left(0.5\nabla\lambda_2\right)+e\left(0.5\nabla\lambda_3+0.5\nabla\lambda_1\right)+f\left(0.5\nabla\lambda_2\right)\right]=0 \\
\boldsymbol{n_3}\cdot\left[\nabla\lambda_1+d\left(0.5\nabla\lambda_2+0.5\nabla\lambda_1\right)+e\left(0.5\nabla\lambda_3\right)+f\left(0.5\nabla\lambda_3\right)\right]=0
\end{cases}
\tag{12}
$$

解得

$$
\begin{cases}
d = -\dfrac{d_2}{d_1}\boldsymbol{n_2}\cdot\boldsymbol{n_1} = -d_2^2\nabla\lambda_2\cdot\nabla\lambda_1 \\
e = -\dfrac{d_3}{d_1}\boldsymbol{n_3}\cdot\boldsymbol{n_1} = -d_3^2\nabla\lambda_3\cdot\nabla\lambda_1 \\
f = -\left(\dfrac{d_2}{d_1}\boldsymbol{n_2}+\dfrac{d_3}{d_1}\boldsymbol{n_3}\right)\cdot\boldsymbol{n_1} = -\left(d_2^2\nabla\lambda_2+d_3^2\nabla\lambda_3\right)\cdot\nabla\lambda_1
\end{cases}
\tag{13}
$$

以 ϕ_4 为例进行计算，ϕ_5 和 ϕ_6 类似。

$$
\begin{cases}
\phi_4(\boldsymbol{x_j}) = 0 & j=1,2,3 \\
\dfrac{\partial\phi_4}{\partial n_j}(\boldsymbol{m_j}) = \delta_{1j} & j=1,2,3
\end{cases}
\tag{14}
$$

由前三个方程可设 $\phi_4 = d\lambda_1\lambda_2 + e\lambda_1\lambda_3 + f\lambda_2\lambda_3$ 代入后三个方程进行计算

$$
\begin{cases}
\boldsymbol{n_1}\cdot\left[d\left(0.5\nabla\lambda_1\right)+e\left(0.5\nabla\lambda_1\right)+f\left(0.5\nabla\lambda_3+0.5\nabla\lambda_2\right)\right]=1 \\
\boldsymbol{n_2}\cdot\left[d\left(0.5\nabla\lambda_2\right)+e\left(0.5\nabla\lambda_3+0.5\nabla\lambda_1\right)+f\left(0.5\nabla\lambda_2\right)\right]=0 \\
\boldsymbol{n_3}\cdot\left[d\left(0.5\nabla\lambda_2+0.5\nabla\lambda_1\right)+e\left(0.5\nabla\lambda_3\right)+f\left(0.5\nabla\lambda_3\right)\right]=0
\end{cases}
\tag{15}
$$

解得 $d = e = -d_1$ 和 $f = 0$。

(d) 这里选取如下的 f 和对应的解析解

$$
\begin{cases}
f(x,y) = 24\left(x^2(1-x)^2+y^2(1-y)^2\right)+8(1-6x+6x^2)(1-6y+6y^2) \\
u(x,y) = x^2(1-x)^2y^2(1-y)^2
\end{cases}
\tag{16}
$$

数值解图像如图所示（不正确，误差估计没有做）

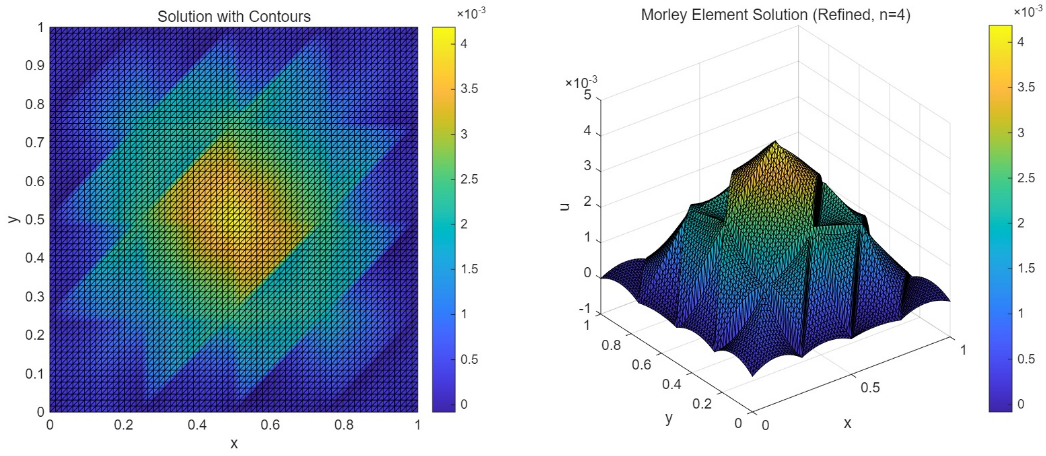
2. Nabla 算子的相关结论

(a) 用 Einstein 求和约定容易得到

$$
\begin{aligned}
\nabla\cdot(\nabla\times\vec{u}) &= \partial_i(\epsilon_{ijk}\partial_j u_k) \\
&= \partial_j(\epsilon_{ijk}\partial_i u_k) \\
&= -\partial_i(\epsilon_{ijk}\partial_j u_k) = -\nabla\cdot(\nabla\times\vec{u})
\end{aligned}
\tag{17}
$$

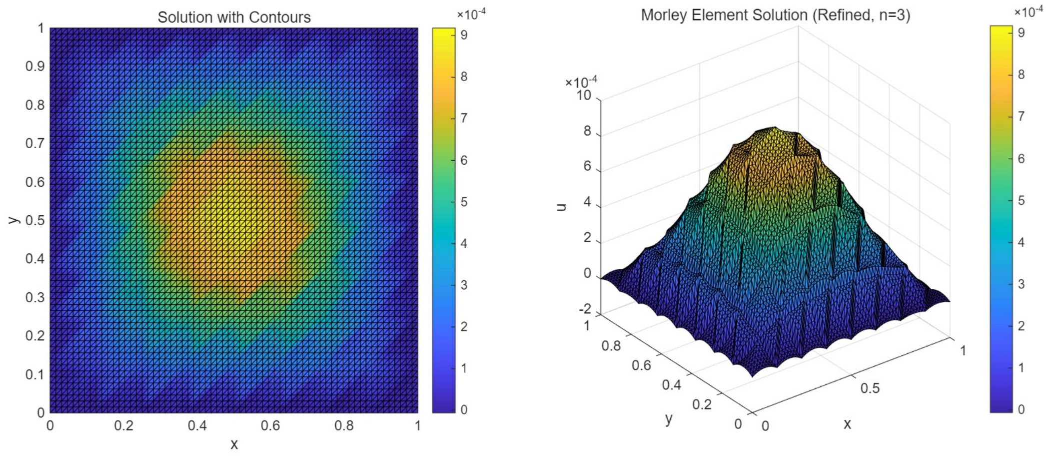
$$
\begin{aligned}
\nabla\times(\nabla v) &= \epsilon_{ijk}\partial_j(\partial_k v)e_i \\
&= \epsilon_{ijk}\partial_k(\partial_j v)e_i \\
&= -\epsilon_{ijk}\partial_j(\partial_k v)e_i = -\nabla\times(\nabla v)
\end{aligned}
\tag{18}
$$

可得恒等式成立。



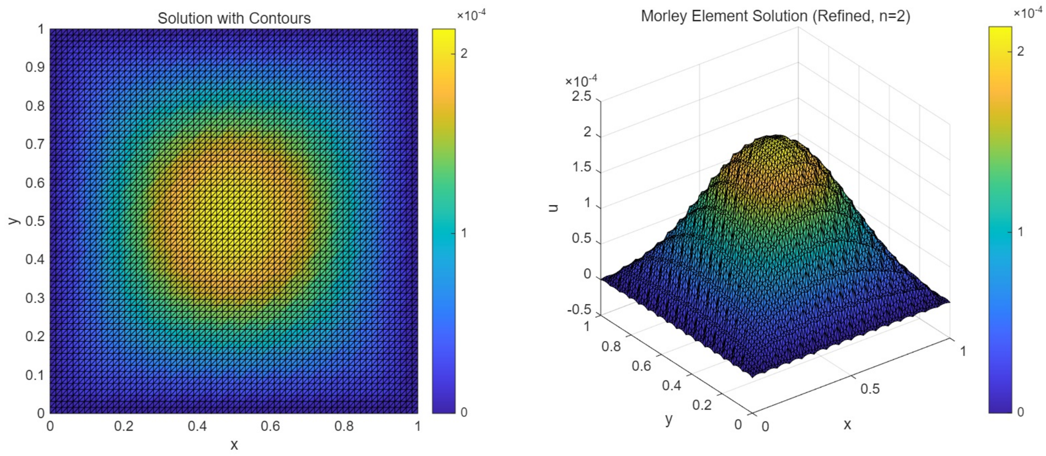
(a) 云图　　(b) 三维图

图 2: $h = 0.25$ 时有限元解的图像



(a) 云图　　(b) 三维图

图 3: $h = 0.125$ 时有限元解的图像



(a) 云图　　(b) 三维图

图 4: $h = 0.0625$ 时有限元解的图像

(b) 由 Stokes 定理可得

$$\int_{\partial\Omega} \vec{u} \cdot \mathrm{d}l = \int_{\Omega} \nabla \times \vec{u} \mathrm{d}S = 0 \tag{19}$$

因此我们可以定义

$$\phi(\vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{u} \cdot \mathrm{d}l \tag{20}$$

积分路径可以取从 $\vec{r}_0$ 到 $\vec{r}$ 的任意路径。则有 $\nabla\phi = \vec{u}$。
另一方面，根据 Helmholtz 分解定理，任何向量场 $\vec{u}$ 都可以分解为

$$\vec{u} = \nabla\phi + \nabla \times \vec{\varphi} \tag{21}$$

两边取散度可以得到

$$\begin{aligned} \nabla \cdot \vec{u} &= \nabla \cdot (\nabla\phi) + \nabla \cdot (\nabla \times \vec{\varphi}) \\ 0 &= \nabla \cdot \nabla\phi \end{aligned} \tag{22}$$

显然 ϕ 满足一个调和方程，因此在要求有界性的条件下，ϕ 是一个确定的常数。于是就有 $\vec{u} = \nabla \times \vec{\varphi}$。