TECHNISCHE UNIVERSITÄT WIEN
Vienna | Austria

Institut für Computertechnik
Institute of Computer Technology

Logic Synthesis

Digitale Integrierte Schaltungen, 2018 WS

Axel Jantsch

November 6, 2018

Outline

1. Data Structures
2. Two-Level Minimization
3. Multi-Level Minimization
4. Technology Mapping

Logic Synthesis

1. Data Structures
   - Boolean Formulas
   - Representation
   - Conversion
2. Two-Level Minimization
3. Multi-Level Minimization
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Some Terms

Boolean literal: $\mathbb{B} = \{0, 1\}$, {false,true}

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Boolean Variable:	$x, y, a, \ldots$
Boolean n-space $\mathbb{B}^n$:	n-dimensional Boolean space
cube	$a \in \mathbb{B}^n$, "x,y,z", "xyz", "110"

Don't Cares

Completely specified Boolean function $f : \mathbb{B}^n \rightarrow \mathbb{B} = \{0, 1\}$

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$$f(x, y) = \begin{cases} 1 & \text{if } xy = 01 \\ 0 & \text{if } xy = 11 \\ - & \text{otherwise} \end{cases}$$

Onsets and Offsets

For a function $f : \mathbb{B}^n \to \mathbb{B}_+$

onset: $f^{\text{on}} = \{a \in \mathbb{B}^n | f(a) = 1\}$

offset: $f^{\text{off}} = \{a \in \mathbb{B}^n | f(a) = 0\}$

dcset: $f^{\text{dc}} = \{a \in \mathbb{B}^n | f(a) = -\}$

Onsets and Offsets

x	y	z	f
0	0	0	1
0	0	1	0
0	1	0	–
0	1	1	1
1	0	0	1
1	0	1	1
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Onsets and Offsets

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onset: $f^{\text{on}} = \{000, 011, 100, 101, 110\}$
offset: $f^{\text{off}} = \{001, 111\}$
dcset: $f^{\text{dc}} = \{010\}$

Logic Synthesis

Truth Tables

x	y	$+$
0	0	0
0	1	1
1	0	1
1	1	1

x	y	$\cdot$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$\oplus$, XOR
0	0	0
0	1	1
1	0	1
1	1	0

x	y	$\leftrightarrow$, XNOR
0	0	1
0	1	0
1	0	0
1	1	1

- Canonical
- Effective for few input variables
- **Cubes** are the product terms in the truth table; cubes of the onset of XOR $=$ 01,10

Sum of Products

Sum of Products (SOP), Disjunctive Normal Form (DNF)

$$f = ab\overline{c} + a\overline{b}c + \overline{a}bc + \overline{a}\overline{b}\overline{c}$$

a	b	c	f
0	0	0	1
0	0	1	0
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- Disjunction of cubes
- The set of cubes of the DNF is a **cover**
- Canonical representation

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Complexity of Operations

Given: m variables,
n terms,

How many terms have to be processed by:

conjunction?

disjunction?

negation?

SAT?

TAUTOLOGY?

Sum of Products - Complexity of Operations

Given:

$$f_1(a, b, c) = ab\overline{c} + a\overline{b}c$$
$$f_2(a, b, c) = \overline{a}bc + \overline{a}\overline{b}\overline{c}$$

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Conjunction:

$$\begin{aligned} f_1 \cdot f_2 &= (ab\overline{c} + a\overline{b}c)(\overline{a}bc + \overline{a}\overline{b}\overline{c}) \\ &= ab\overline{c}\,\overline{a}bc + ab\overline{c}\,\overline{a}\overline{b}\overline{c} + a\overline{b}c\overline{a}bc + a\overline{b}c\overline{a}\overline{b}\overline{c} \\ &= 0 + 0 + 0 + 0 \end{aligned}$$

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Conjunction is of quadratic complexity: n^2, for n terms.

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Disjunction is of constant complexity.

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Complement:

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\overline{f} &= \overline{(ab\overline{c} + a\overline{b}c)} \\
&= \overline{(ab\overline{c})}\,\overline{(a\overline{b}c)} \\
&= (\overline{a} + \overline{b} + c)(\overline{a} + b + \overline{c}) \\
&= \overline{a}\overline{a} + \overline{a}b + \overline{a}\overline{c} + \overline{b}\overline{a} + \overline{b}b + \overline{b}\overline{c} + c\overline{a} + cb + c\overline{c}
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Complementing function

$$f = x_{1,1}x_{1,2}\cdots x_{1,m} + \cdots + x_{n,1}x_{n,2}\cdots x_{n,m}$$

will result in m^n product terms in the SOP representation.
Complement is of exponential complexity.

SAT - Satisfiability of a Function

A function $f(x_1, \cdots, x_m)$ is satisfiable if there exists a consistent variable assignment, such that $f = 1$

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$$f_1(x, y) = \overline{x}y \quad \cdots \quad \text{satisfiable since } f_1(0, 1) = 1$$

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For SOP formulas SAT can be determined in constant time.

Tautology Checking

A function $f(x_1, \cdots, x_m)$ is a tautology if $f = 1$ for every consistent variable assignment.

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For SOP formulas we need to test 2^m assignments; hence, TAUTOLOGY is determined in exponential time.

Sum of Products - Complexity of Operations

Operation	Complexity
Conjunction	Quadratic: n^2
Disjunction	Constant
Complement	**Exponential**: m^n
SAT	Constant
TAUTOLOGY	**Exponential**: 2^m

Product of Sums

Product of Sums (POS), Conjunctive Normal Form (CNF, KNF)

$$f = (a + b + \overline{c})(a + \overline{b} + c)(\overline{a} + b + c)(\overline{a} + \overline{b} + \overline{c})$$

a	b	c	f
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- Canonical representation
- POS is the dual of SOP
- Less widely used than SOP

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Products of Sums - Complexity of Operations

Given:

$$f_1(a, b, c) = (a + b + \overline{c})(a + \overline{b} + c)$$
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Conjunction is of constant complexity.

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Disjunction:

$$\begin{aligned} f_1 + f_2 &= (a+b+\overline{c})(a+\overline{b}+c) + (\overline{a}+b+c)(\overline{a}+\overline{b}+\overline{c}) \\ &= x_1x_2 + x_3x_4 \\ &= (x_1+x_3)(x_2+x_3)(x_1+x_4)(x_2+x_4) \qquad \text{(distributive law)} \end{aligned}$$

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&= x_1x_2 + x_3x_4 \\
&= (x_1+x_3)(x_2+x_3)(x_1+x_4)(x_2+x_4) \qquad \text{(distributive law)} \\
&= ((a+b+\overline{c})+(\overline{a}+b+c)) \\
&\quad ((a+\overline{b}+c)+(\overline{a}+b+c)) \\
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Disjunction is of quadratic complexity: n^2 for n terms.

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Complement:

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\overline{f} &= \overline{(a+b+\overline{c})} + \overline{(a+\overline{b}+c)} \\
&= (\overline{a}\,\overline{b}c) + (\overline{a}b\overline{c}) \\
&= (\overline{a}+\overline{a})(\overline{a}+b)(\overline{a}+\overline{c})(\overline{b}+\overline{a})(\overline{b}+b)(\overline{b}+\overline{c}) \\
&\quad (c+\overline{a})(c+b)(c+\overline{c})
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Complementing function

$$f = (x_{1,1} + x_{1,2} + \cdots + x_{1,m}) \cdots (x_{n,1} + x_{n,2} + \cdots + x_{n,m})$$

will result in m^n sum terms in the POS representation.
Complement is of exponential complexity.

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Complement	**Exponential**: m^n	**Exponential**: m^n
SAT	Constant	**Exponential**: 2^m
TAUTOLOGY	**Exponential**: 2^m	Constant

Binary Decision Diagram

x_1	x_2	x_3	f
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0	1	1	1
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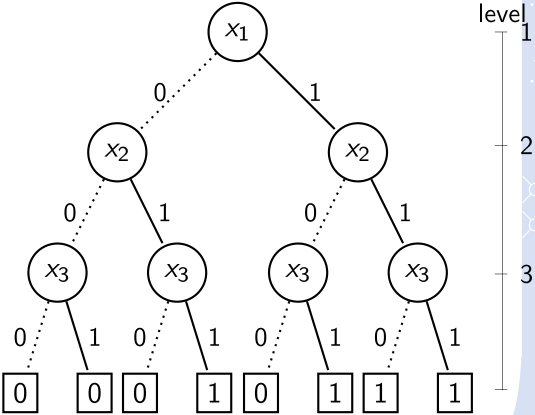

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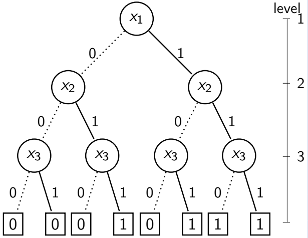
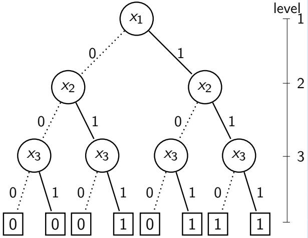


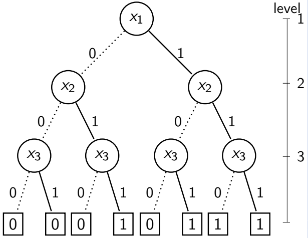
Binary Decision Diagram

- An n-level binary tree represents an n-variable Boolean function.



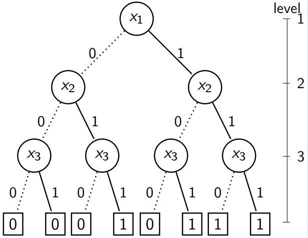
Binary Decision Diagram

- An n-level binary tree represents an n-variable Boolean function.
- A **terminal node** v has attribute value$(v) \in \{0, 1\}$.



Binary Decision Diagram

- An n-level binary tree represents an n-variable Boolean function.
- A **terminal node** v has attribute value$(v) \in \{0, 1\}$.
- A **non-terminal node** v has an index$(v) \in \{1..n\}$, and two children: the 0-child (else(v)) and the 1-child (then(v)).



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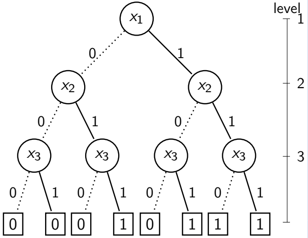

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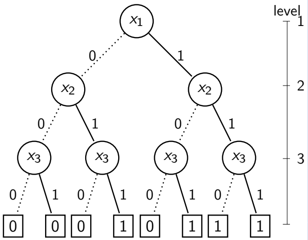
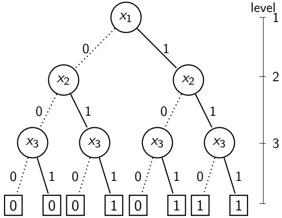


- The variable $x_{\text{index}(v)}$ is the decision variable for node v.
- Every node v corresponds to a Boolean function f_v recursively defined:
  1. If v is a terminal node, $f_v = \text{value}(v)$;
  2. If v is not a terminal node:

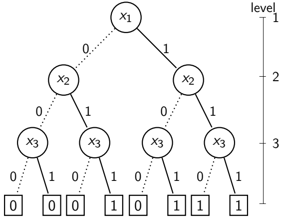
$$f_v(x_1, \ldots, x_n) = \overline{x_{\text{index}(v)}} f_{\text{else}(v)}(x_1, \ldots, x_n) + x_{\text{index}(v)} f_{\text{then}(v)}(x_1, \ldots, x_n)$$

Ordered Binary Decision Diagram



Ordered Binary Decision Diagram

- A BDD is **ordered** (it is an OBDD) if the nodes on every path from the root to a terminal node follow the same variable ordering.



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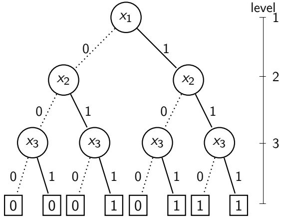

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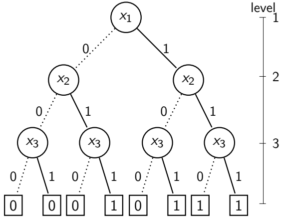
- Each Boolean function is represented by an OBDD.



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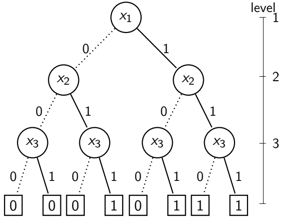

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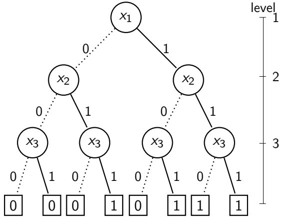
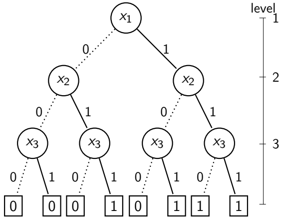
- A BDD is **ordered** (it is an OBDD) if the nodes on every path from the root to a terminal node follow the same variable ordering.
- Each OBDD represents a Boolean function.
- Each Boolean function is represented by an OBDD.
- Two OBDDs are isomorphic if they represent the same Boolean function.
- The isomorphism of two OBDDs can be checked in linear time with the size of the tree.

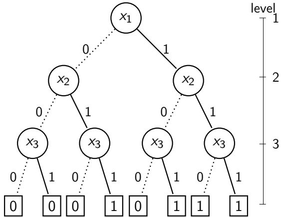


Reduced Ordered Binary Decision Diagram



Reduced Ordered Binary Decision Diagram

A **reduced OBDD (ROBDD)** is constructed from an OBDD as follows:



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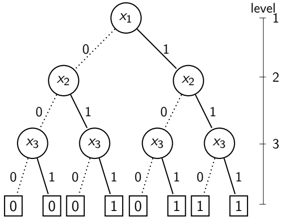
1. Two terminal nodes with the same value attribute are merged.



Reduced Ordered Binary Decision Diagram

A **reduced OBDD (ROBDD)** is constructed from an OBDD as follows:

1. Two terminal nodes with the same value attribute are merged.
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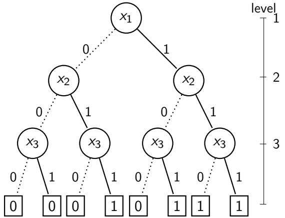

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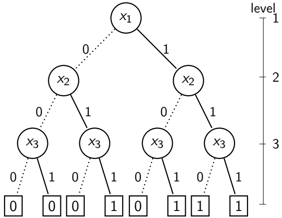
3. A non-terminal node v with $\text{then}(v) = \text{else}(u)$ is removed and its incident edges are redirected to its child node.



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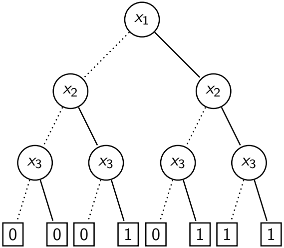

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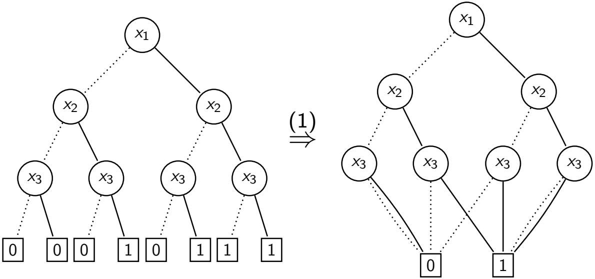

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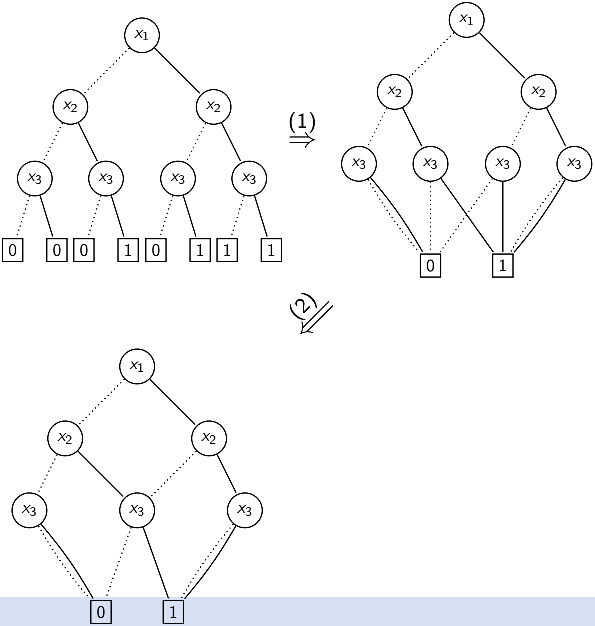

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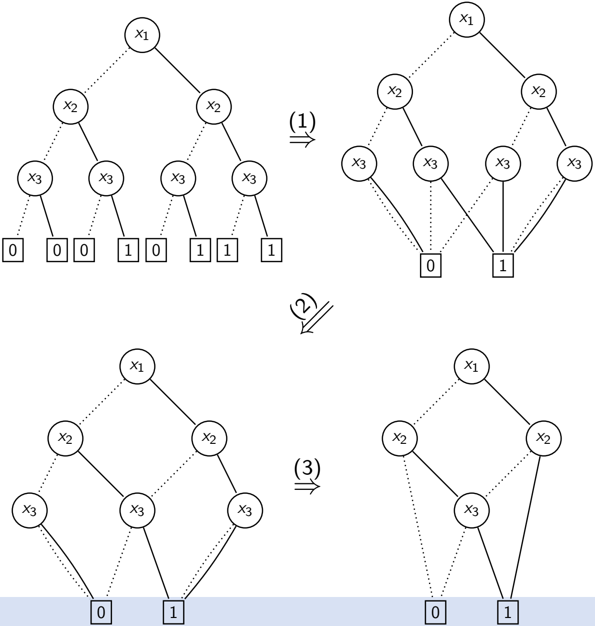
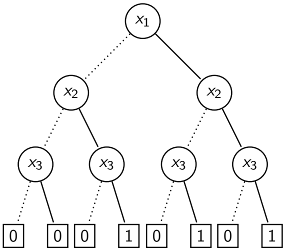


Reduced Ordered Binary Decision Diagram

- ROBBD has smallest number of nodes under a given variable ordering.
- ROBBD is a canonical representation of a Boolean function for a given variable ordering.

ROBDD Generation Example



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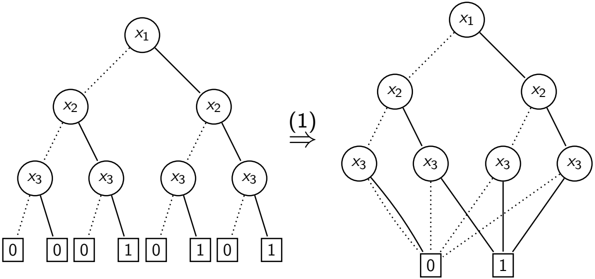
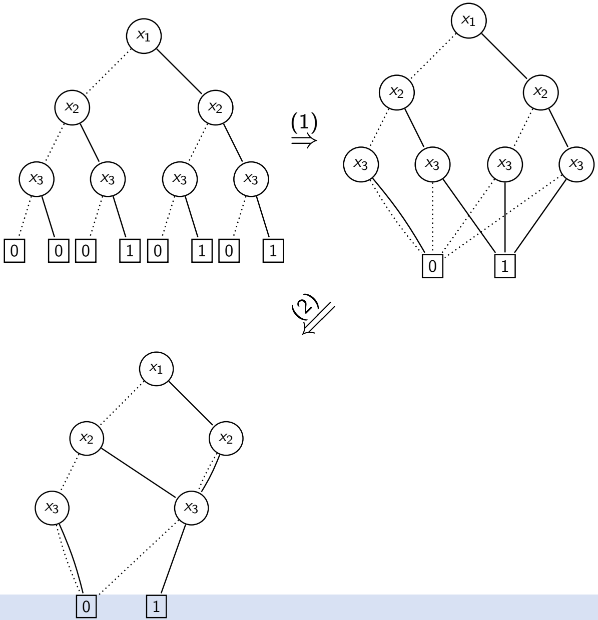


ROBDD Generation Example



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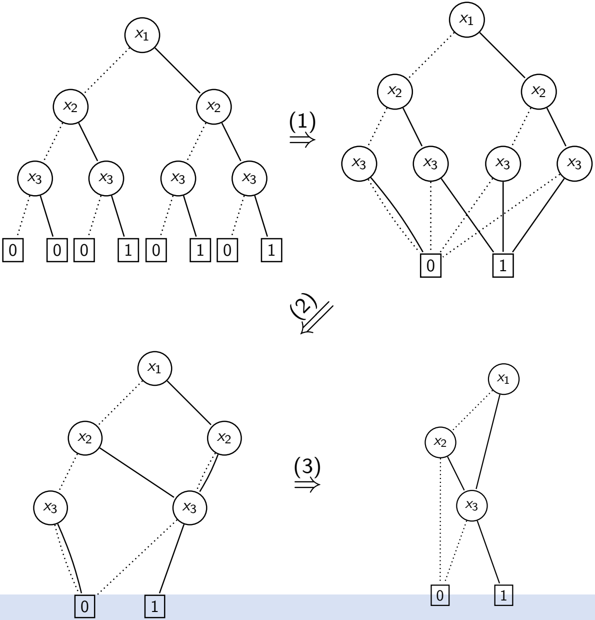
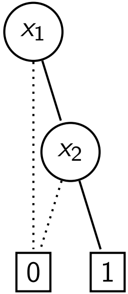


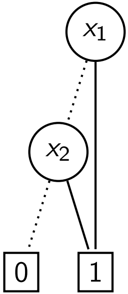
ROBDDs represent onsets and offsets: AND



x_1	x_2	$\cdot$
0	0	0
0	1	0
1	0	0
1	1	1

onset: $\{x_1 x_2\}$
offset: $\{\overline{x_1}, x_1\overline{x_2}\}$

ROBDDs represent onsets and offsets: OR

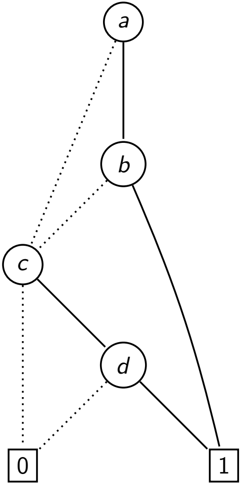


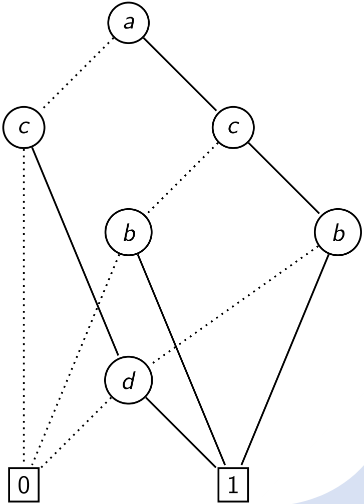
x_1	x_2	$+$
0	0	0
0	1	1
1	0	1
1	1	1

onset: $\{x_1, \overline{x_1}x_2\}$
offset: $\{\overline{x_1 x_2}\}$

ROBDD and Variable Ordering

Function: $f = ab + cd$





ROBDD and Variable Ordering

- Size of the ROBDD varies strongly with the ordering of variables.
- Finding the optimal variable ordering is of exponential complexity.
- There are good heuristics for finding a good ordering.
- There exist functions with ROBDD size growing exponential with the size of the Boolean formula **for all orderings**, e.g. multiplication.

Computational Efficiency of ROBBDs

Complement is done by complementing the terminal nodes.

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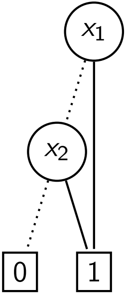

x_1	x_2	$+$
0	0	0
0	1	1
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1	1	1

Computational Efficiency of ROBBDs

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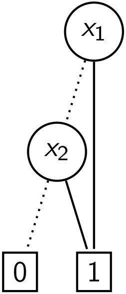

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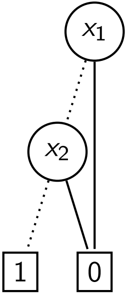

x_1	x_2	NOR
0	0	1
0	1	0
1	0	0
1	1	0

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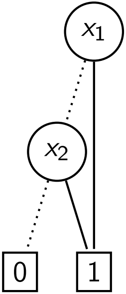

x_1	x_2	$+$
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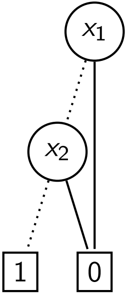

x_1	x_2	NOR
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0	1	0
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Complement is of constant complexity.

Computational Efficiency of ROBBDs

Binary operations are done based on the Shannon Expansion.

$$
\begin{aligned}
f(x_1, \ldots, x_i, \ldots, x_n) &= (x_i \cdot f_{x_i}) + (\overline{x_i} \cdot f_{\overline{x_i}}) \\
&= (x_i \cdot f(x_1, \ldots, 1, \ldots, x_n)) \\
&\quad + \ (\overline{x_i} \cdot f(x_1, \ldots, 0, \ldots, x_n))
\end{aligned}
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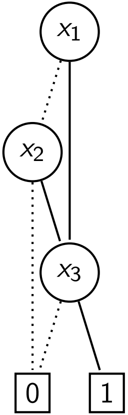
$$
f\,\langle \mathrm{op}\rangle g = (x_i \cdot (f_{x_i}\langle \mathrm{op}\rangle g_{x_i})) + (\overline{x_i} \cdot (f_{\overline{x_i}}\langle \mathrm{op}\rangle g_{\overline{x_i}}))
$$

Shannon Expansion with ROBBDs

$f = \overline{x_1}x_2x_3 + x_1x_3$



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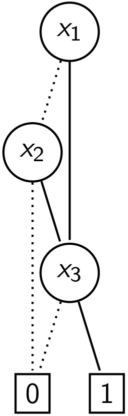
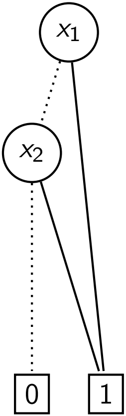
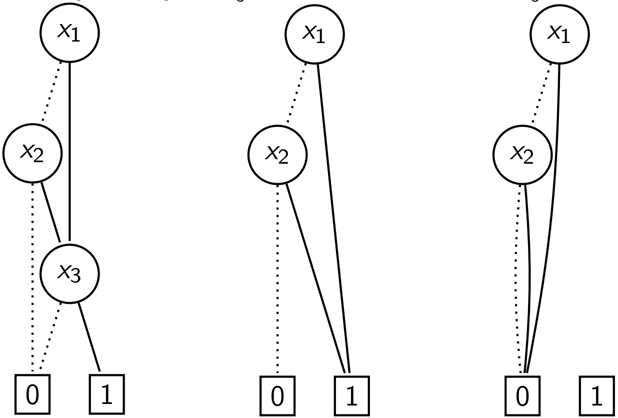


$f_{x_3} = \overline{x_1} x_2 + x_1$



Shannon Expansion with ROBBDs

$f = \overline{x_1}x_2x_3 + x_1x_3$ $\quad$ $f_{x_3} = \overline{x_1}x_2 + x_1$ $\quad$ $f_{\overline{x_3}} = 0$



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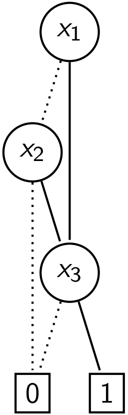

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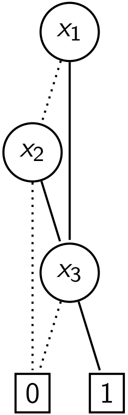
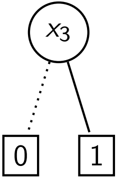


$f_{x_1} = x_3$



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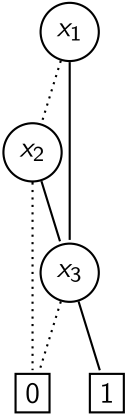

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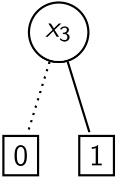
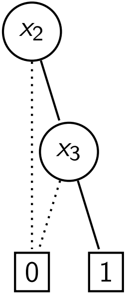


$f_{\overline{x_1}} = x_2x_3$



Binary Operation by Way of Shannon Expansion

$$\begin{aligned} f(a,b) &= a \cdot b \\ g(a,b) &= a + b \end{aligned}$$

Binary Operation by Way of Shannon Expansion

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&\quad + (\overline{a} \cdot ((b \cdot (f_{\overline{a},b} + g_{\overline{a},b})) + (\overline{b} \cdot (f_{\overline{a},\overline{b}} + g_{\overline{a},\overline{b}}))))
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&\quad + (\overline{a} \cdot ((b \cdot (f_{\overline{a},b} + g_{\overline{a},b})) + (\overline{b} \cdot (f_{\overline{a},\overline{b}} + g_{\overline{a},\overline{b}})))) \\
&= (a \cdot ((b \cdot (1 + 1)) + (\overline{b} \cdot (0 + 1)))) \\
&\quad + (\overline{a} \cdot ((b \cdot (0 + 1)) + (\overline{b} \cdot (0 + 0))))
\end{aligned}
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&= (a \cdot ((b \cdot (1+1)) + (\overline{b} \cdot (0+1)))) \\
&\quad +(\overline{a} \cdot ((b \cdot (0+1)) + (\overline{b} \cdot (0+0)))) \\
&= (a \cdot ((b \cdot 1) + (\overline{b} \cdot 1))) + (\overline{a} \cdot ((b \cdot 1) + (\overline{b} \cdot 0)))
\end{aligned}
$$

Binary Operation by Way of Shannon Expansion

$$
\begin{aligned}
f(a,b) &= a \cdot b \\
g(a,b) &= a + b \\
f + g &= (a \cdot b) + (a + b) = (a \cdot b) + a + b = a + b \\
f + g &= (a \cdot (f_a + g_a)) + (\overline{a} \cdot (f_{\overline{a}} + g_{\overline{a}})) \\
&= (a \cdot ((b \cdot (f_{a,b} + g_{a,b})) + (\overline{b} \cdot (f_{a,\overline{b}} + g_{a,\overline{b}})))) \\
&\quad +(\overline{a} \cdot ((b \cdot (f_{\overline{a},b} + g_{\overline{a},b})) + (\overline{b} \cdot (f_{\overline{a},\overline{b}} + g_{\overline{a},\overline{b}})))) \\
&= (a \cdot ((b \cdot (1 + 1)) + (\overline{b} \cdot (0 + 1)))) \\
&\quad +(\overline{a} \cdot ((b \cdot (0 + 1)) + (\overline{b} \cdot (0 + 0)))) \\
&= (a \cdot ((b \cdot 1) + (\overline{b} \cdot 1))) + (\overline{a} \cdot ((b \cdot 1) + (\overline{b} \cdot 0))) \\
&= (a \cdot (b + \overline{b})) + (\overline{a} \cdot (b + 0))
\end{aligned}
$$

Binary Operation by Way of Shannon Expansion

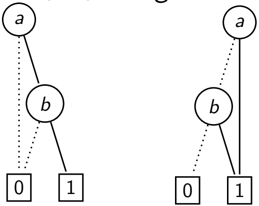
$$
\begin{aligned}
f(a,b) &= a \cdot b \\
g(a,b) &= a + b \\
f + g &= (a \cdot b) + (a + b) = (a \cdot b) + a + b = a + b \\
f + g &= (a \cdot (f_a + g_a)) + (\overline{a} \cdot (f_{\overline{a}} + g_{\overline{a}})) \\
&= (a \cdot ((b \cdot (f_{a,b} + g_{a,b})) + (\overline{b} \cdot (f_{a,\overline{b}} + g_{a,\overline{b}})))) \\
&\quad +(\overline{a} \cdot ((b \cdot (f_{\overline{a},b} + g_{\overline{a},b})) + (\overline{b} \cdot (f_{\overline{a},\overline{b}} + g_{\overline{a},\overline{b}})))) \\
&= (a \cdot ((b \cdot (1 + 1)) + (\overline{b} \cdot (0 + 1)))) \\
&\quad +(\overline{a} \cdot ((b \cdot (0 + 1)) + (\overline{b} \cdot (0 + 0)))) \\
&= (a \cdot ((b \cdot 1) + (\overline{b} \cdot 1))) + (\overline{a} \cdot ((b \cdot 1) + (\overline{b} \cdot 0))) \\
&= (a \cdot (b + \overline{b})) + (\overline{a} \cdot (b + 0)) \\
&= (a \cdot 1) + (\overline{a} \cdot b)
\end{aligned}
$$

Binary Operation by Way of Shannon Expansion

$$
\begin{aligned}
f(a,b) &= a \cdot b \\
g(a,b) &= a + b \\
f + g &= (a \cdot b) + (a + b) = (a \cdot b) + a + b = a + b \\
f + g &= (a \cdot (f_a + g_a)) + (\overline{a} \cdot (f_{\overline{a}} + g_{\overline{a}})) \\
&= (a \cdot ((b \cdot (f_{a,b} + g_{a,b})) + (\overline{b} \cdot (f_{a,\overline{b}} + g_{a,\overline{b}})))) \\
&\quad + (\overline{a} \cdot ((b \cdot (f_{\overline{a},b} + g_{\overline{a},b})) + (\overline{b} \cdot (f_{\overline{a},\overline{b}} + g_{\overline{a},\overline{b}})))) \\
&= (a \cdot ((b \cdot (1 + 1)) + (\overline{b} \cdot (0 + 1)))) \\
&\quad + (\overline{a} \cdot ((b \cdot (0 + 1)) + (\overline{b} \cdot (0 + 0)))) \\
&= (a \cdot ((b \cdot 1) + (\overline{b} \cdot 1))) + (\overline{a} \cdot ((b \cdot 1) + (\overline{b} \cdot 0))) \\
&= (a \cdot (b + \overline{b})) + (\overline{a} \cdot (b + 0)) \\
&= (a \cdot 1) + (\overline{a} \cdot b) \\
&= a + (\overline{a} \cdot b) = a + b
\end{aligned}
$$

Binary Operation by Way of Shannon Expansion

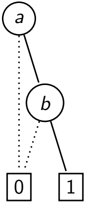
$f = a \cdot b$ $\qquad$ $g = a + b$



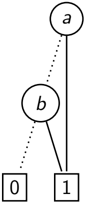
Binary Operation by Way of Shannon Expansion

$f + g$

$f = a \cdot b$

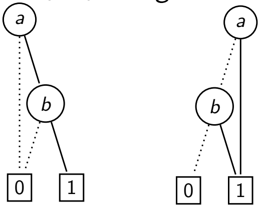


$g = a + b$

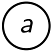


Binary Operation by Way of Shannon Expansion

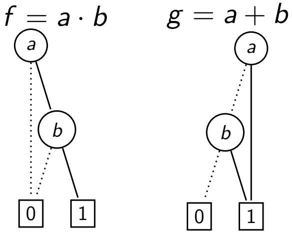
$f = a \cdot b$ $\quad$ $g = a + b$

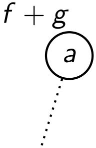


$f + g$



Binary Operation by Way of Shannon Expansion

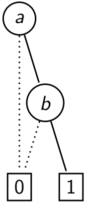




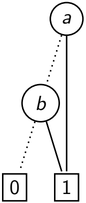
Shannon expansion by $\overline{a}$: $0 + b$

Binary Operation by Way of Shannon Expansion

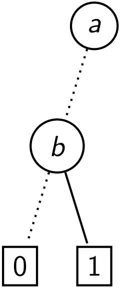
$f = a \cdot b$



$g = a + b$

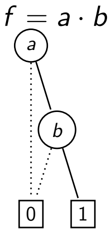


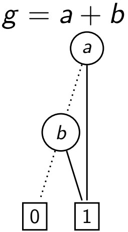
$f + g$

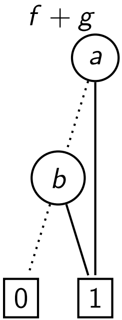


Shannon expansion by $\overline{a}$: $0 + b$

Binary Operation by Way of Shannon Expansion



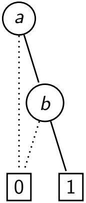




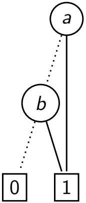
Shannon expansion by a: $b + 1$

Binary Operation by Way of Shannon Expansion

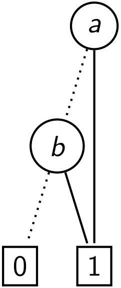
$f = a \cdot b$



$g = a + b$



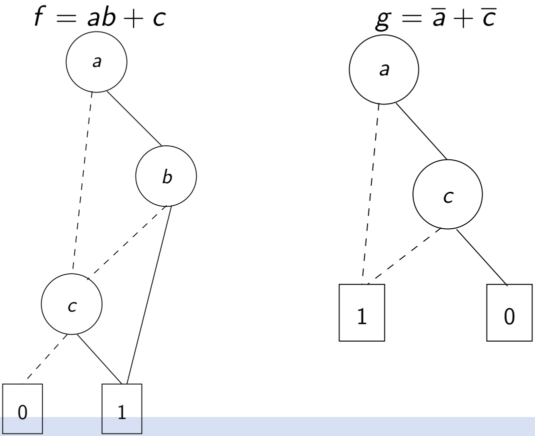
$f + g$



Binary operations are of quadratic complexity in the number of variables.

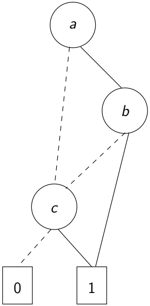
Binary Operation by Way of Shannon Expansion

$$
\begin{aligned}
f &= ab + c \\
g &= \overline{a} + \overline{c} \\
f \cdot g &= (ab + c) \cdot (\overline{a} + \overline{c}) = \overline{a}c + ab\overline{c}
\end{aligned}
$$

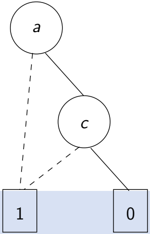


Binary Operation by Way of Shannon Expansion

$$f = ab + c$$



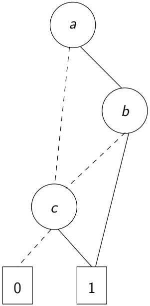
$$g = \overline{a} + \overline{c}$$



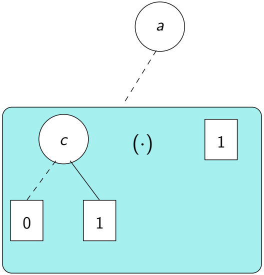
$$f \cdot g = \overline{a}c + ab\overline{c}$$

Binary Operation by Way of Shannon Expansion

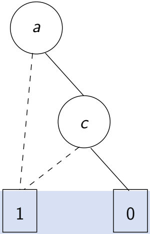
$$f = ab + c$$



$$f \cdot g = \overline{a}c + ab\overline{c}$$

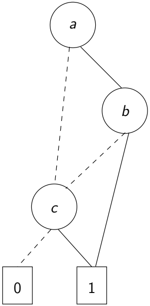


$$g = \overline{a} + \overline{c}$$

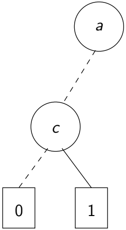


Binary Operation by Way of Shannon Expansion

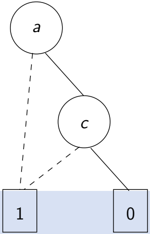
$f = ab + c$



$f \cdot g = \overline{a}c + ab\overline{c}$

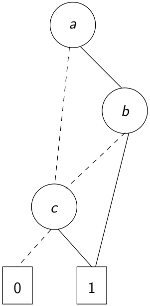


$g = \overline{a} + \overline{c}$

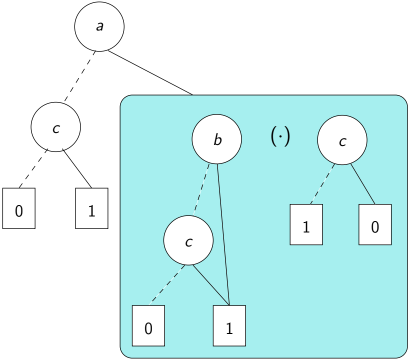


Binary Operation by Way of Shannon Expansion

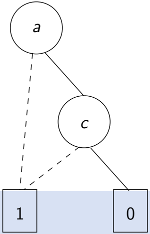
$f = ab + c$



$f \cdot g = \overline{a}c + ab\overline{c}$

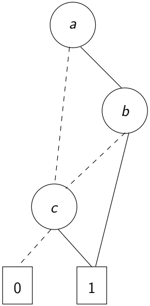


$g = \overline{a} + \overline{c}$

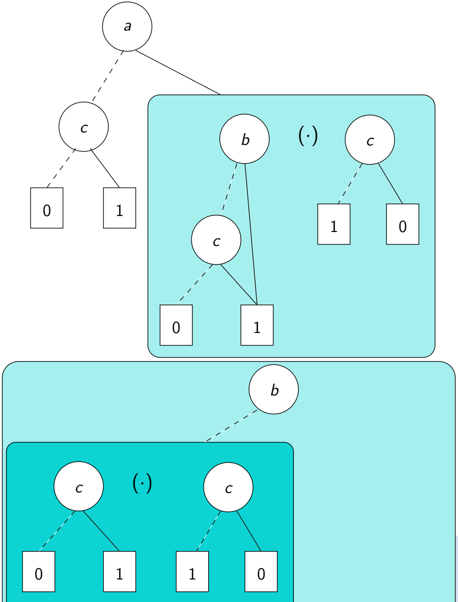


Binary Operation by Way of Shannon Expansion

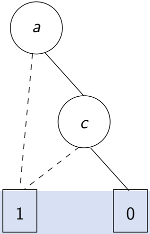
$f = ab + c$



$f \cdot g = \overline{a}c + ab\overline{c}$

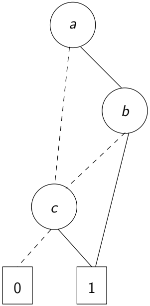


$g = \overline{a} + \overline{c}$

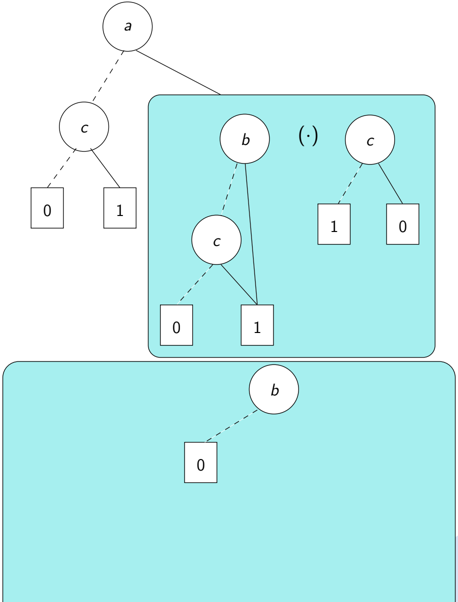


Binary Operation by Way of Shannon Expansion

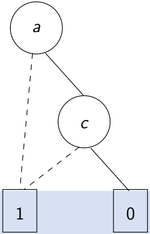
$f = ab + c$



$f \cdot g = \overline{a}c + ab\overline{c}$

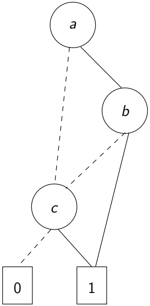


$g = \overline{a} + \overline{c}$

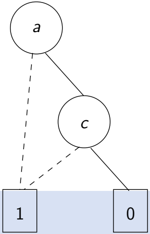


Binary Operation by Way of Shannon Expansion

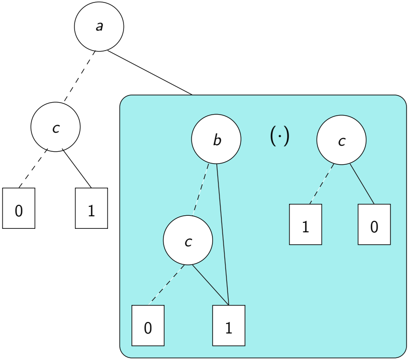
$$f = ab + c$$

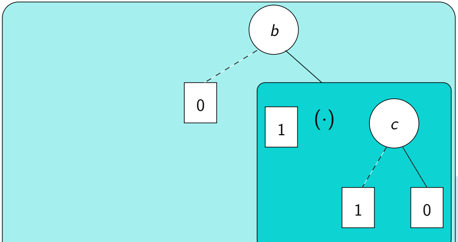


$$g = \overline{a} + \overline{c}$$



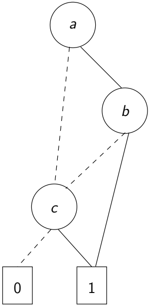
$$f \cdot g = \overline{a}c + ab\overline{c}$$



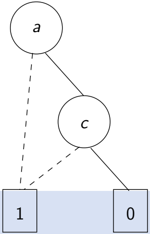


Binary Operation by Way of Shannon Expansion

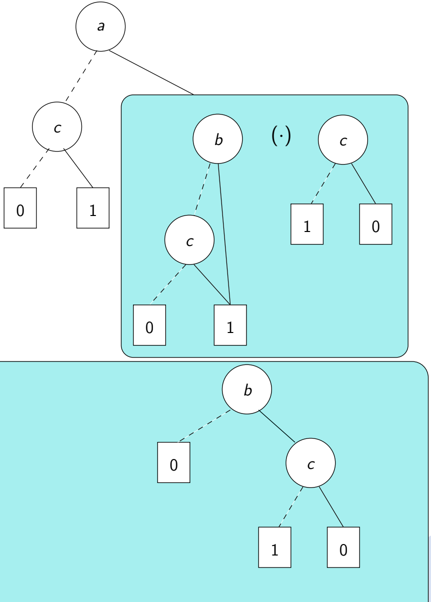
$f = ab + c$



$g = \overline{a} + \overline{c}$

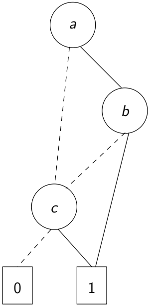


$f \cdot g = \overline{a}c + ab\overline{c}$

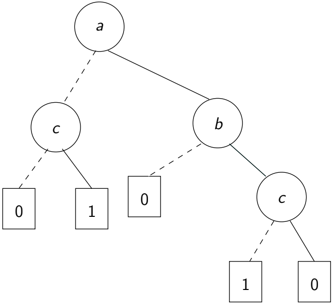


Binary Operation by Way of Shannon Expansion

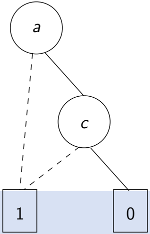
$$f = ab + c$$



$$f \cdot g = \overline{a}c + ab\overline{c}$$

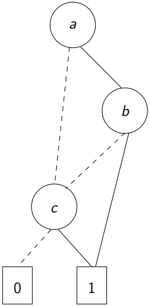


$$g = \overline{a} + \overline{c}$$

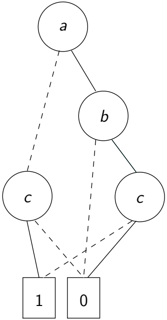


Binary Operation by Way of Shannon Expansion

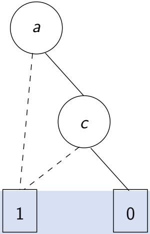
$f = ab + c$



$f \cdot g = \overline{a}c + ab\overline{c}$



$g = \overline{a} + \overline{c}$



ROBDD - Complexity of Operations

Operation	SOP	POS	ROBDD
Conjunction	Quadratic: n^2	Constant	Quadratic: m^2
Disjunction	Constant	Quadratic: n^2	Quadratic: m^2
Complement	**Exponential**: m^n	**Exponential**: m^n	Constant
SAT	Constant	**Exponential**: 2^m	Constant
TAUTOLOGY	**Exponential**: 2^m	Constant	Constant

Logic Synthesis

SoP $\leftrightarrow$ PoS Conversion

$$f_{\mathsf{SoP}} \xRightarrow{\text{Double Complement}} \xRightarrow{\text{DeMorgan}} \xRightarrow{\text{Distributive Law}} \xRightarrow{\text{DeMorgan}} f_{\mathsf{PoS}}$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\mathrm{SoP}} \overset{\text{Double Complement}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} \overset{\text{Distributive Law}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} f_{\mathrm{PoS}}$$

$$f(a,b,c,d) = (a \cdot b) + (c \cdot d)$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\text{SoP}} \overset{\text{Double Complement}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} \overset{\text{Distributive Law}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} f_{\text{PoS}}$$

$$\begin{aligned} f(a,b,c,d) &= (a \cdot b) + (c \cdot d) \\ &= \overline{\overline{a \cdot b + c \cdot d}} \end{aligned}$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\text{SoP}} \overset{\text{Double Complement}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} \overset{\text{Distributive Law}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} f_{\text{PoS}}$$

$$\begin{aligned}
f(a,b,c,d) &= (a\cdot b)+(c\cdot d)\\
&= \overline{\overline{a\cdot b+c\cdot d}}\\
&= \overline{\overline{a\cdot b}\cdot\overline{c\cdot d}}
\end{aligned}$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\text{SoP}} \xRightarrow{\text{Double Complement}} \xRightarrow{\text{DeMorgan}} \xRightarrow{\text{Distributive Law}} \xRightarrow{\text{DeMorgan}} f_{\text{PoS}}$$

$$\begin{aligned}
f(a,b,c,d) &= (a \cdot b) + (c \cdot d) \\
&= \overline{\overline{a \cdot b + c \cdot d}} \\
&= \overline{\overline{a \cdot b} \cdot \overline{c \cdot d}} \\
&= \overline{(\overline{a} + \overline{b}) \cdot (\overline{c} + \overline{d})}
\end{aligned}$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\text{SoP}} \overset{\text{Double Complement}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} \overset{\text{Distributive Law}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} f_{\text{PoS}}$$

$$\begin{aligned}
f(a,b,c,d) &= (a \cdot b) + (c \cdot d) \\
&= \overline{\overline{a \cdot b + c \cdot d}} \\
&= \overline{\overline{a \cdot b} \cdot \overline{c \cdot d}} \\
&= \overline{(\overline{a} + \overline{b}) \cdot (\overline{c} + \overline{d})} \\
&= \overline{(\overline{a} \cdot \overline{c}) + (\overline{a} \cdot \overline{d}) + (\overline{b} \cdot \overline{c}) + (\overline{b} \cdot \overline{d})}
\end{aligned}$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\text{SoP}} \xRightarrow{\text{Double Complement}} \xRightarrow{\text{DeMorgan}} \xRightarrow{\text{Distributive Law}} \xRightarrow{\text{DeMorgan}} f_{\text{PoS}}$$

$$\begin{aligned}
f(a,b,c,d) &= (a\cdot b)+(c\cdot d)\\
&= \overline{\overline{a\cdot b + c\cdot d}}\\
&= \overline{\overline{a\cdot b}\cdot\overline{c\cdot d}}\\
&= \overline{(\overline{a}+\overline{b})\cdot(\overline{c}+\overline{d})}\\
&= \overline{(\overline{a}\cdot\overline{c})+(\overline{a}\cdot\overline{d})+(\overline{b}\cdot\overline{c})+(\overline{b}\cdot\overline{d})}\\
&= \overline{(\overline{a}\cdot\overline{c})}\cdot\overline{(\overline{a}\cdot\overline{d})}\cdot\overline{(\overline{b}\cdot\overline{c})}\cdot\overline{(\overline{b}\cdot\overline{d})}
\end{aligned}$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\text{SoP}} \overset{\text{Double Complement}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} \overset{\text{Distributive Law}}{\Longrightarrow} \overset{\text{DeMorgan}}{\Longrightarrow} f_{\text{PoS}}$$

$$\begin{aligned}
f(a,b,c,d) &= (a \cdot b) + (c \cdot d) \\
&= \overline{\overline{a \cdot b + c \cdot d}} \\
&= \overline{\overline{a \cdot b} \cdot \overline{c \cdot d}} \\
&= \overline{(\overline{a} + \overline{b}) \cdot (\overline{c} + \overline{d})} \\
&= \overline{(\overline{a} \cdot \overline{c}) + (\overline{a} \cdot \overline{d}) + (\overline{b} \cdot \overline{c}) + (\overline{b} \cdot \overline{d})} \\
&= \overline{(\overline{a} \cdot \overline{c})} \cdot \overline{(\overline{a} \cdot \overline{d})} \cdot \overline{(\overline{b} \cdot \overline{c})} \cdot \overline{(\overline{b} \cdot \overline{d})} \\
&= (a + c) \cdot (a + d) \cdot (b + c) \cdot (b + d)
\end{aligned}$$

SoP $\leftrightarrow$ PoS Conversion

$$f_{\mathsf{SoP}} \xRightarrow{\text{Double Complement}} \xRightarrow{\text{DeMorgan}} \xRightarrow{\text{Distributive Law}} \xRightarrow{\text{DeMorgan}} f_{\mathsf{PoS}}$$

$$\begin{aligned}
f(a,b,c,d) &= (a\cdot b)+(c\cdot d) \\
&= \overline{\overline{a\cdot b + c\cdot d}} \\
&= \overline{\overline{a\cdot b}\cdot\overline{c\cdot d}} \\
&= \overline{(\overline{a}+\overline{b})\cdot(\overline{c}+\overline{d})} \\
&= \overline{(\overline{a}\cdot\overline{c})+(\overline{a}\cdot\overline{d})+(\overline{b}\cdot\overline{c})+(\overline{b}\cdot\overline{d})} \\
&= \overline{(\overline{a}\cdot\overline{c})}\cdot\overline{(\overline{a}\cdot\overline{d})}\cdot\overline{(\overline{b}\cdot\overline{c})}\cdot\overline{(\overline{b}\cdot\overline{d})} \\
&= (a+c)\cdot(a+d)\cdot(b+c)\cdot(b+d)
\end{aligned}$$

SoP $\leftrightarrow$ PoS Conversion is of exponential complexity: m^n.

Boolean formula $\leftrightarrow$ BDD Conversion

A Boolean function is converted to a ROBDD by starting with a variable representation and recursively applying the Boolean operators.

Some Boolean functions have ROBDD of exponential size.

Boolean formula $\leftrightarrow$ BDD Conversion is of exponential complexity.

Logic Synthesis

Two-level Minimization

- Karnaugh-Veitch Diagrams: Limited to 4 variables;
- Quine-McCluskey

Logic Synthesis

Multi-level Minimization

- Two-Level Minimization is limited to relatively small functions.
- Fan-in limitations of gates, depending on technology

Multi-level Minimization

- Two-Level Minimization is limited to relatively small functions.
- Fan-in limitations of gates, depending on technology

Multi-level minimization uses

- Factoring
- Decomposition
- Extraction
- Substitution
- Elimination

Factoring

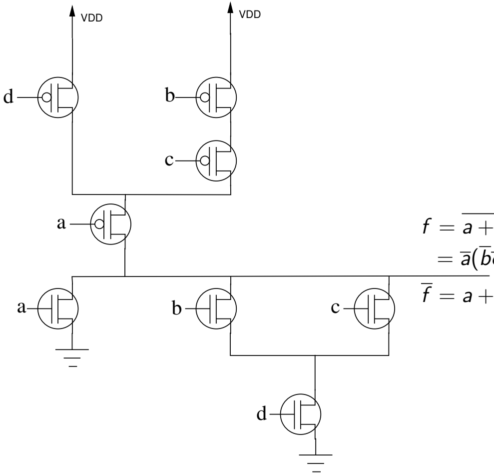
- A factored form of a Boolean formula is a tree representation with the nodes being AND-operations, OR-operations or literals.
- Factoring reformulates a Boolean function in a form with minimum number of literals, because an implementation with complex CMOS gates in general requires $2n$ transistors for a factored form with n literals

$$F = ac + ad + bc + bd + e$$

can be factored into

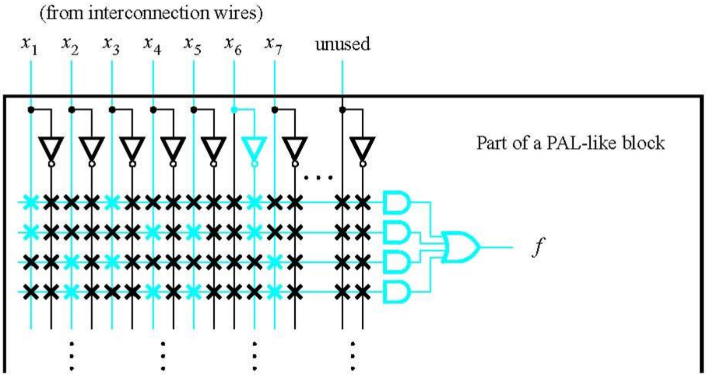
$$F = (a + b)(c + d) + e$$

CMOS Gate of a Factored Form



$$f = \overline{a + (b+c)d}$$
$$= \overline{a}(\overline{b}\,\overline{c} + \overline{d})$$
$$\overline{f} = a + (b+c)d$$

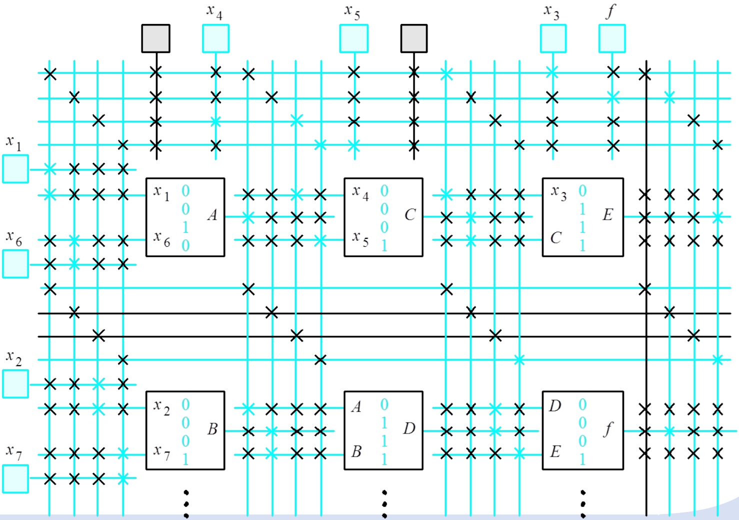
Consider $f = x_1 x_3 \overline{x_6} + x_1 x_4 x_5 \overline{x_6} + x_2 x_3 x_7 + x_2 x_4 x_5 x_7$



Implementation in an FPGA with 2-input LUTs is not suitable.
Transform f as follows:

$$\begin{aligned} f &= x_1 x_3 \overline{x_6} + x_1 x_4 x_5 \overline{x_6} + x_2 x_3 x_7 + x_2 x_4 x_5 x_7 \\ &= x_1 \overline{x_6}(x_3 + x_4 x_5) + x_2 x_7 (x_3 + x_4 x_5) \\ &= (x_3 + x_4 x_5)(x_1 \overline{x_6} + x_2 x_7) \end{aligned}$$

$$f = (x_3 + x_4 x_5)(x_1 \overline{x_6} + x_2 x_7)$$

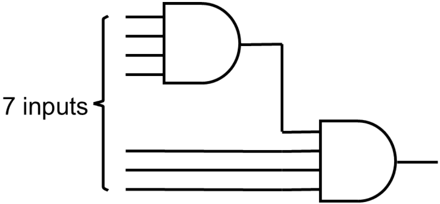


Fan-in Constraints

In all implementation technologies:

- CPLDs
- FPGAs
- ASICs

Large gates can be implemented with smaller gates:



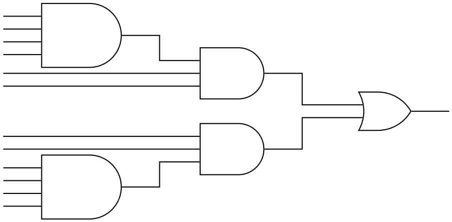
Consider

$$f = x_1\overline{x_2}x_3\overline{x_4}x_5x_6 + x_1x_2\overline{x_3}\,\overline{x_4}\,\overline{x_5}x_6$$

Consider

$$f = x_1\overline{x_2}x_3\overline{x_4}x_5x_6 + x_1x_2\overline{x_3}\,\overline{x_4}\,\overline{x_5}x_6$$

Can be implemented as



with cost=21.

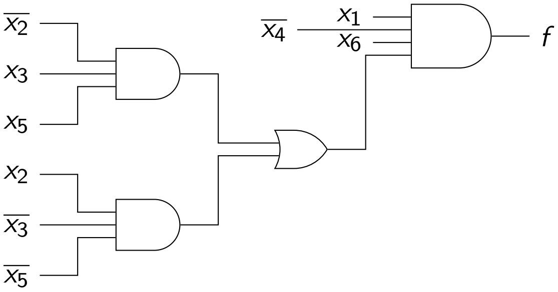
Or by factoring as follows:

$$\begin{aligned} f &= x_1\overline{x_2}x_3\overline{x_4}x_5x_6 + x_1x_2\overline{x_3}\,\overline{x_4}\,\overline{x_5}x_6 \\ &= x_1\overline{x_4}x_6\left(\overline{x_2}x_3x_5 + x_2\overline{x_3}\,\overline{x_5}\right) \end{aligned}$$

Or by factoring as follows:

$$
\begin{aligned}
f &= x_1\overline{x_2}x_3\overline{x_4}x_5x_6 + x_1x_2\overline{x_3}\,\overline{x_4}\,\overline{x_5}x_6 \\
&= x_1\overline{x_4}x_6\left(\overline{x_2}x_3x_5 + x_2\overline{x_3}\,\overline{x_5}\right)
\end{aligned}
$$

which can be implemented as



with cost=16.

Decomposition

A function is decomposed into other, simpler functions.

$$F(a,b,c,d) = abc + abd + \overline{a}\overline{c}\overline{d} + \overline{b}\overline{c}\overline{d}$$

$$\begin{aligned} F &= X \cdot Y + \overline{X} \cdot \overline{Y} \\ X &= ab \\ Y &= c + d \end{aligned}$$

Decomposition Example

$$f = \overline{x_1}x_2x_3 + x_1\overline{x_2}x_3 + x_1x_2x_4 + \overline{x_1}\,\overline{x_2}x_4$$

Decomposition Example

$$
\begin{aligned}
f &= \overline{x_1}x_2x_3 + x_1\overline{x_2}x_3 + x_1x_2x_4 + \overline{x_1}\,\overline{x_2}x_4 \\
&= (\overline{x_1}x_2 + x_1\overline{x_2})x_3 + (x_1x_2 + \overline{x_1}\,\overline{x_2})x_4
\end{aligned}
$$

Decomposition Example

$$
\begin{aligned}
f &= \overline{x_1}x_2x_3 + x_1\overline{x_2}x_3 + x_1x_2x_4 + \overline{x_1}\,\overline{x_2}x_4 \\
&= (\overline{x_1}x_2 + x_1\overline{x_2})x_3 + (x_1x_2 + \overline{x_1}\,\overline{x_2})x_4
\end{aligned}
$$

$$
g = \overline{x_1}x_2 + x_1\overline{x_2}
$$

Decomposition Example

$$
\begin{aligned}
f &= \overline{x_1}x_2x_3 + x_1\overline{x_2}x_3 + x_1x_2x_4 + \overline{x_1}\,\overline{x_2}x_4 \\
&= (\overline{x_1}x_2 + x_1\overline{x_2})x_3 + (x_1x_2 + \overline{x_1}\,\overline{x_2})x_4
\end{aligned}
$$

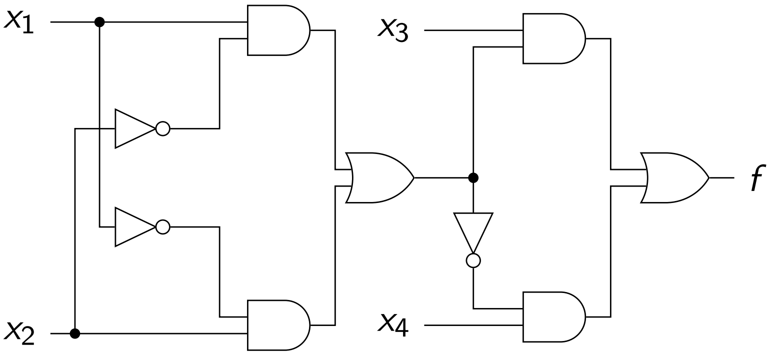
$$
\begin{aligned}
g &= \overline{x_1}x_2 + x_1\overline{x_2} \\
\overline{g} &= x_1x_2 + \overline{x_1}\,\overline{x_2}
\end{aligned}
$$

Decomposition Example

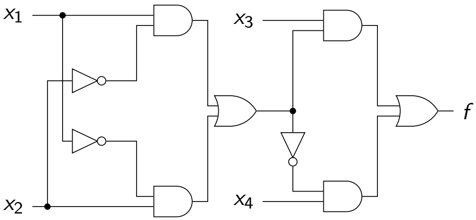
$$
\begin{aligned}
f &= \overline{x_1}x_2x_3 + x_1\overline{x_2}x_3 + x_1x_2x_4 + \overline{x_1}\,\overline{x_2}x_4 \\
&= (\overline{x_1}x_2 + x_1\overline{x_2})x_3 + (x_1x_2 + \overline{x_1}\,\overline{x_2})x_4
\end{aligned}
$$

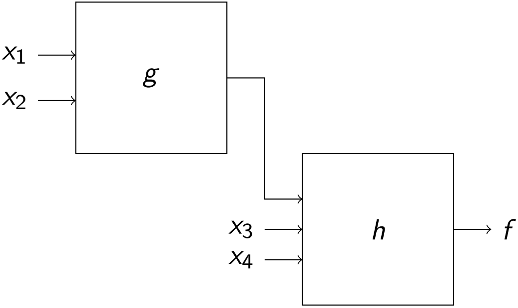
$$
\begin{aligned}
g &= \overline{x_1}x_2 + x_1\overline{x_2} \\
\overline{g} &= x_1x_2 + \overline{x_1}\,\overline{x_2}
\end{aligned}
$$

$$
f = gx_3 + \overline{g}x_4
$$



Logic circuit for $f = gx_3 + \overline{g}x_4$.





$$f = gx_3 + \overline{g}x_4 = h(g(x_1, x_2), x_3, x_4).$$

Logic Transformations

- Logic transformations change the structure of the Boolean network (restructuring).
- Heuristics and cost functions guide the restructuring process.

Logic Synthesis

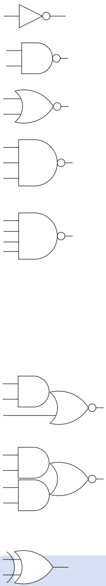
Technology Mapping

- Given a netlist of abstract gates, and a library of technology specific gates, technology mapping creates a technology specific netlist of gates.
- Technology mapping is different for standard cells, gate arrays and FPGAs.
- Only after this step, technology specific information such as gate delay, area, and power consumption is available.

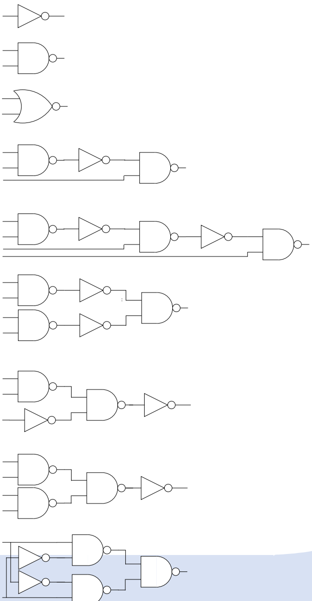
Technology Libraries

Gate	Cost
INV	2
NAND2	3
NOR2	4
NAND3	4
NAND4	5
AIO21	4
AOI22	5
XOR	4

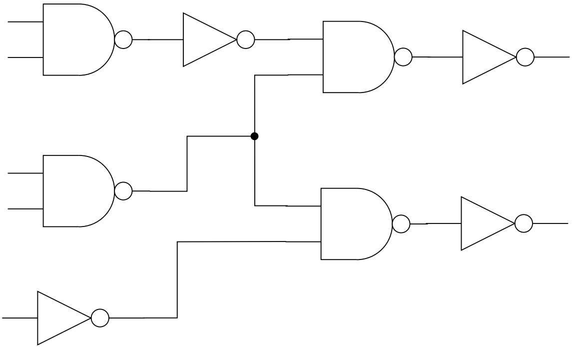
Symbol



Pattern Graph



Subject Graph



The **subject graph** is the standardized netlist representation (e.g. consisting only of NAND2 and INV) of the target design.

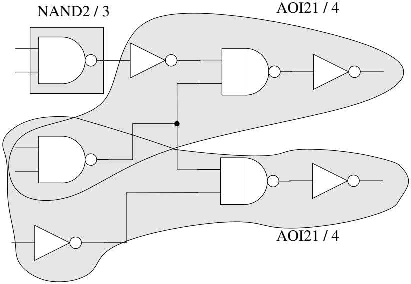
Graph Covering

Graph covering finds a cover of library elements (patterns DAG (Directed Acyclic Graph)) of the subject graph such that

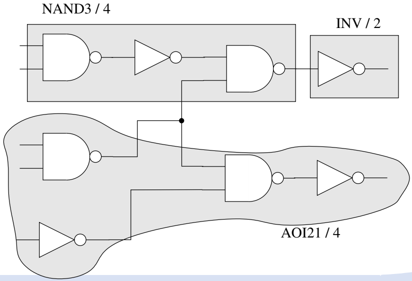
- All gates of the subject graph are covered;
- No internal node of a Pattern DAG is the input of another pattern graph;
- The cost function (area, power, delay, ...) is minimized;
- All design constraints are met.

Graph Covering

Legal covering:



Illegal covering:

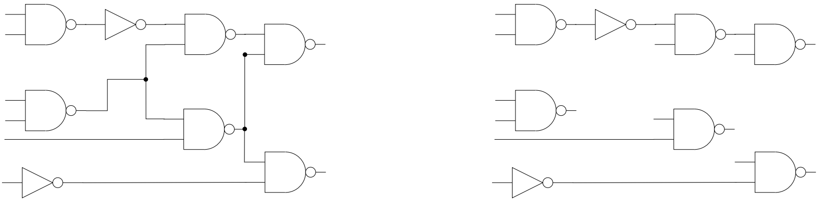


Atomic Pattern Set

Empirically the choice of NAND2 and INV as the only atomic patterns is a good solution.

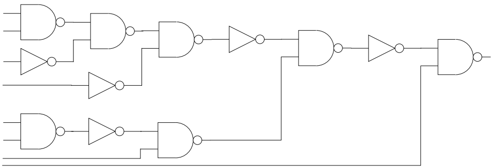
Additional patterns (e.g. AND2, NOR2, NAND3, ...) lead rarely to better solutions, but make the graph covering problem harder.

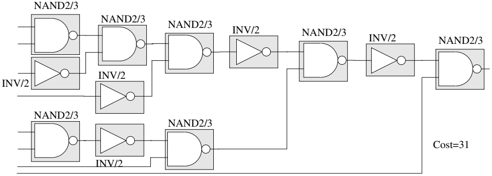
Tree Covering

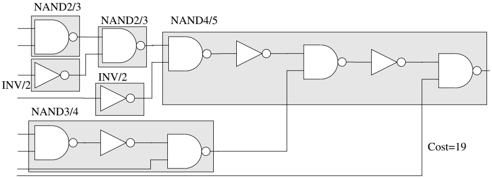


1. Convert the Boolean network into a NAND2-INV netlist to create the subject graph;
2. Partition the subject graph into a forest of trees;
3. Each tree is optimally covered separately:
   1. Generate a complete set of matches for the tree;
   2. Select the best match with a dynamic programming algorithm.

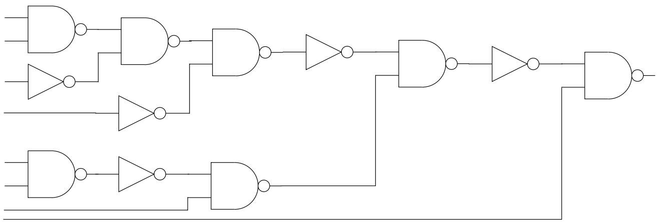
Tree Covering

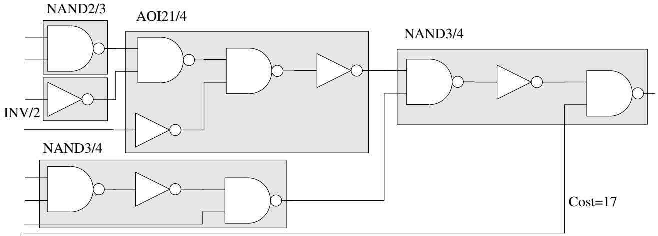




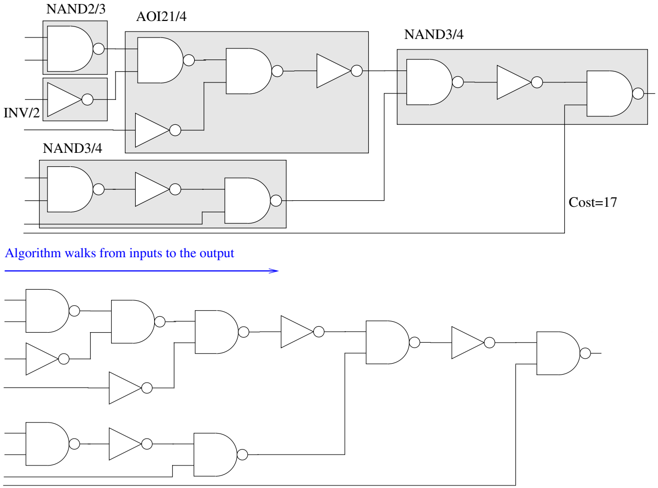


Tree Covering



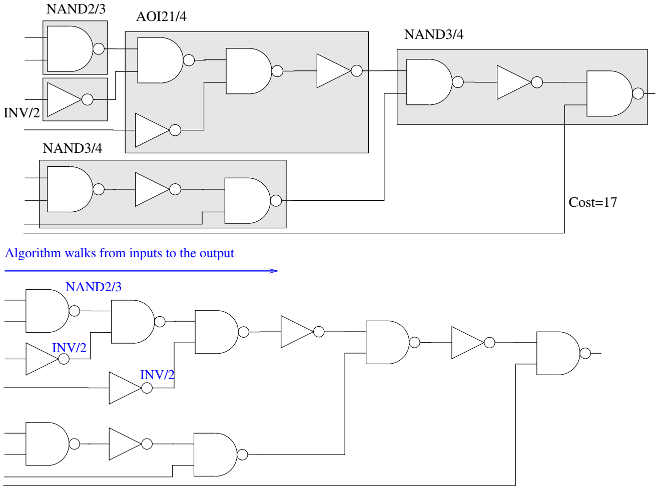


Tree Covering



Optimal tree covering based on recursive, optimal sub-tree covering.

Tree Covering



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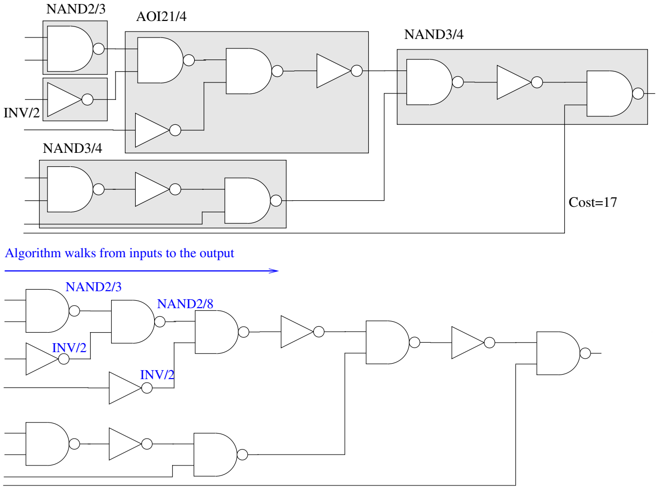

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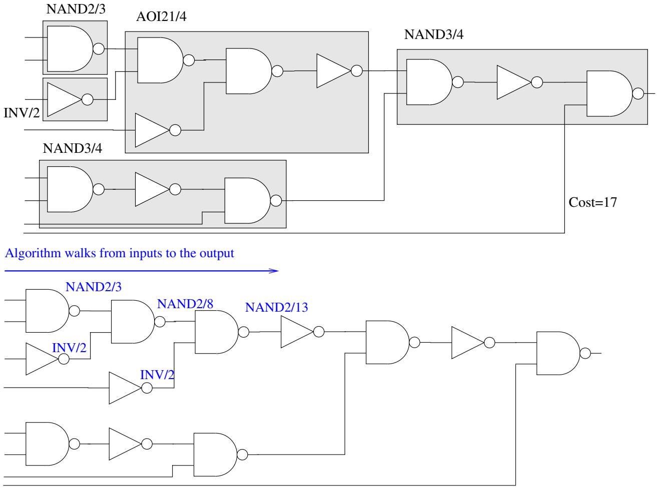

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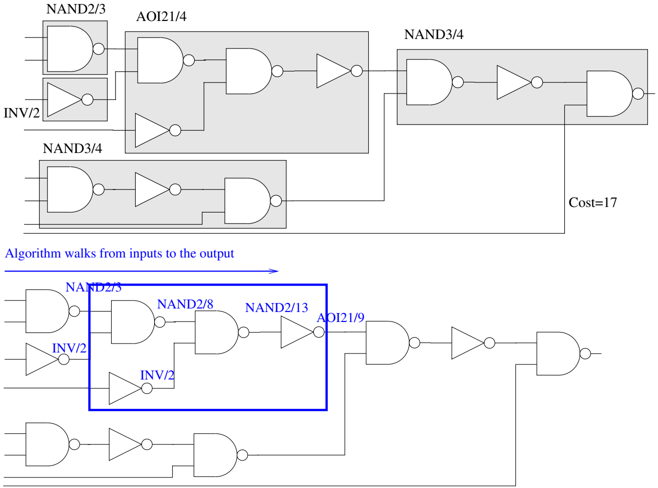

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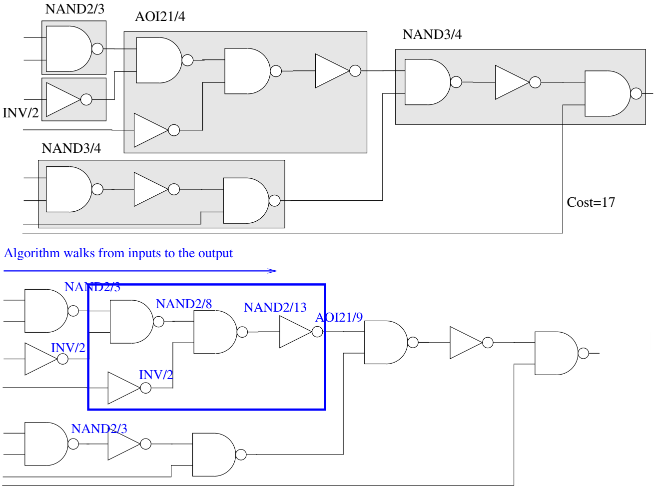

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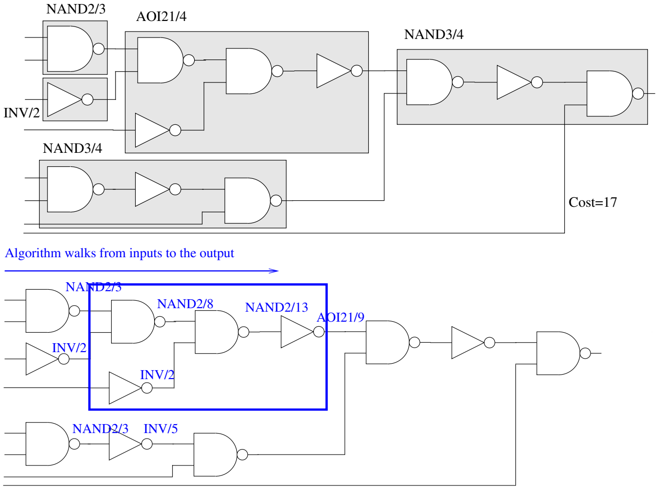

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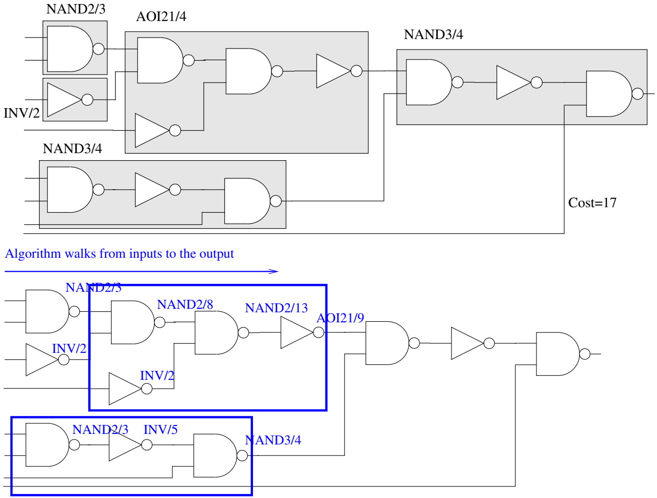

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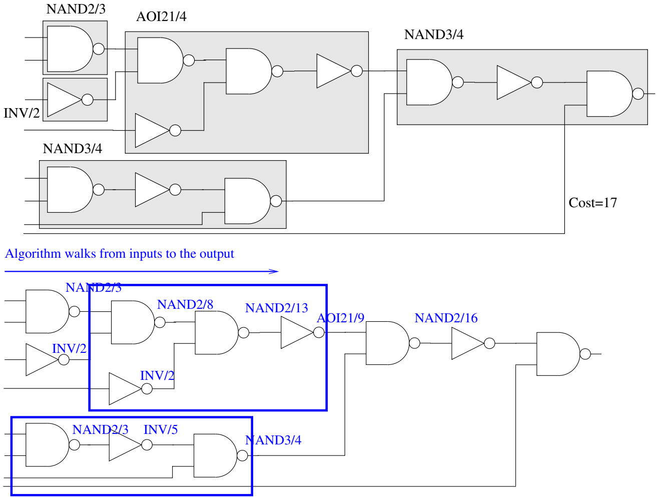

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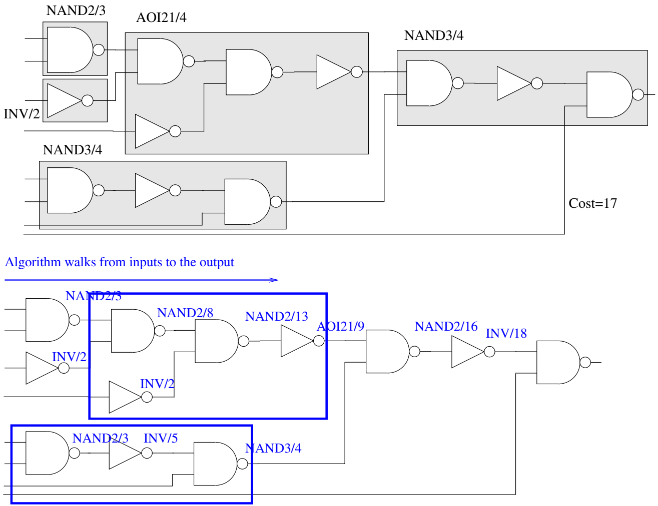

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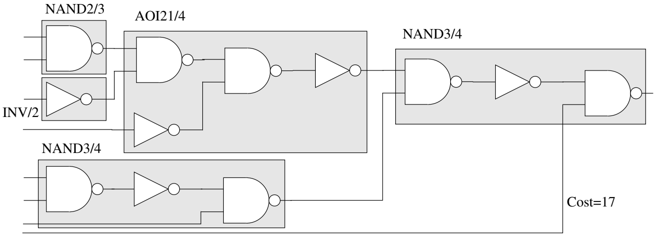

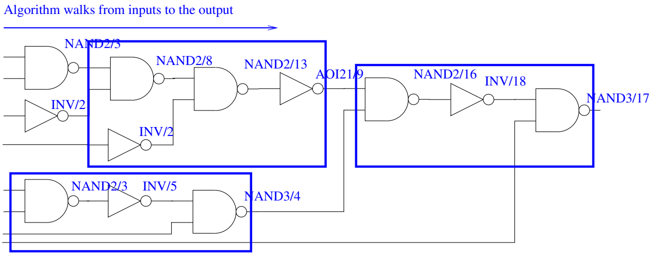

Optimal tree covering based on recursive, optimal sub-tree covering.

Summary

- Representation of Boolean functions
- Two-level optimization
- Multi-level optimization
- Technology mapping