# Quine – McCluskey Logic Optimization

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1908 - 2000

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1929 - 2016



# Quine – McCluskey Logic Optimization

$$A \cdot B + A \cdot \overline{B} = A \cdot (B + \overline{B}) = A$$

## Terminology

- Literals: variables, uncomplemented and complemented
- Minterm, Maxterm
- Implicant: A product term for which f=1
- Prime implicant: An implicant, that cannot be reduced
- Essential prime implicant: A prime implicant, which is the only prime implicant that covers a specific minterm
- Cover: A set of implicants that account for all valuations for which f=1
- Cost: Cost(gate) = 1+ # of inputs



## QM – Example 1

| min term **m** | A | B | Y |
|---|---|---|---|
| $m0$ | 0 | 0 | 0 |
| $m1$ | 0 | 1 | 0 |
| $m2$ | 1 | 0 | 1 |
| $m3$ | 1 | 1 | 1 |

| Logic Expression | min-term | binary representation |
|---|---|---|
| $A \cdot \overline{B}$ | m2 | 10 |
| $A \cdot B$ | m3 | 11 |

| Logic Expression | min-term | binary representation |
|---|---|---|
| $A \cdot \overline{B}$ | $m2$ | 10 |
| $A \cdot B$ | $m3$ | 11 |
| $A$ | $(m2, m3)$ | 1- |

## QM – Example 2

$$Y = A \cdot B \cdot C + A \cdot \overline{B} \cdot C$$

$$Y = m7 + m5$$

| Logic Expression | min-term | binary representation |
|---|---|---|
| $A \cdot B \cdot C$ | m7 | 111 |
| $A \cdot \overline{B} \cdot C$ | m5 | 101 |
| $A \cdot C$ | (m5,m7) | 1-1 |

$$Y = A \cdot C$$



## QM – Example 2

$$Y = A \cdot B \cdot C + A \cdot \overline{B} \cdot C$$

$$Y = A \cdot C$$

| ¬ *C* | | *C* | | |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | ¬ *B* |
| 0 | 0 | 1 | 0 | *B* |
| ¬ *A* | *A* | | ¬ *A* | |

# QM – Example 3

$$Y = \overline{A} \cdot \overline{B} \cdot \overline{C} \cdot \overline{D} + \overline{A} \cdot \overline{B} \cdot \overline{C} \cdot D + \overline{A} \cdot B \cdot \overline{C} \cdot \overline{D} + \overline{A} \cdot B \cdot \overline{C} \cdot D$$

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot \overline{D}$ | m0 | 0000 | 0 |
| $\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot D$ | m1 | 0001 | 1 |
| $\overline{A} \cdot B \cdot \overline{C} \cdot \overline{D}$ | m4 | 0100 | 1 |
| $\overline{A} \cdot B \cdot \overline{C} \cdot D$ | m5 | 0101 | 2 |

| | ¬ D | | D | | |
|---|---|---|---|---|---|
| ¬ A | 1 | 1 | 1 | 1 | ¬ C |
| A | 0 | 0 | 0 | 0 | |
| | 0 | 0 | 0 | 0 | C |
| ¬ A | 0 | 0 | 0 | 0 | |
| | ¬ B | B | | ¬ B | |



# QM – Example 3

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot \overline{D}$ | m0 | 0000 | 0 |
| $\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot D$ | m1 | 0001 | 1 |
| $\overline{A} \cdot B \cdot \overline{C} \cdot \overline{D}$ | m4 | 0100 | 1 |
| $\overline{A} \cdot B \cdot \overline{C} \cdot D$ | m5 | 0101 | 2 |

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A} \cdot \overline{B} \cdot \overline{C}$ | (m0,m1) | 000- | 0 |
| $\overline{A} \cdot \overline{C} \cdot \overline{D}$ | (m0,m4) | 0-00 | 0 |
| $\overline{A} \cdot \overline{C} \cdot D$ | (m1,m5) | 0-01 | 1 |
| $\overline{A} \cdot B \cdot \overline{C}$ | (m4,m5) | 010- | 1 |

## QM – Example 3

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot\overline{D}$ | m0 | 0000 | 0 |
| $\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot D$ | m1 | 0001 | 1 |
| $\overline{A}\cdot B\cdot\overline{C}\cdot\overline{D}$ | m4 | 0100 | 1 |
| $\overline{A}\cdot B\cdot\overline{C}\cdot D$ | m5 | 0101 | 2 |

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A}\cdot\overline{B}\cdot\overline{C}$ | (m0,m1) | 000- | 0 |
| $\overline{A}\cdot\overline{C}\cdot\overline{D}$ | (m0,m4) | 0-00 | 0 |
| $\overline{A}\cdot\overline{C}\cdot D$ | (m1,m5) | 0-01 | 1 |
| $\overline{A}\cdot B\cdot\overline{C}$ | (m4,m5) | 010- | 1 |
| $\overline{A}\cdot\overline{C}$ | $(m0, m1, m4, m5)$ | 0-0- | 0 |
| $\overline{A}\cdot\overline{C}$ | $(m0, m4, m1, m5)$ | 0-0- | 0 |



## QM – Merging of Terms

Terms that can be combined must meet these criteria:

- They differ by **only one** bit
- They contain the **same domain** of variables

# QM – Merging of Terms Procedure

- Count the number of 1's and sort the rows accordingly
- Terms with *n* number of 1's can only be combined with terms with *n*+1 number of 1's
  - Resulting terms will have *n* number of 1's
- Merged terms must have the '-' aligned (Same domain)





# QM – Example 4

$$Y = \overline{A}\cdot\overline{B}\cdot\overline{C}\cdot\overline{D}+\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot D+\overline{A}\cdot B\cdot\overline{C}\cdot\overline{D}+\overline{A}\cdot B\cdot\overline{C}\cdot D+\overline{A}\cdot B\cdot C\cdot D+A\cdot\overline{B}\cdot C\cdot D$$

$$Y = m0 + m1 + m4 + m5 + m7 + m11$$

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot\overline{D}$ | m0 | 0000 | 0 |
| $\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot D$ | m1 | 0001 | 1 |
| $\overline{A}\cdot B\cdot\overline{C}\cdot\overline{D}$ | m4 | 0100 | 1 |
| $\overline{A}\cdot B\cdot\overline{C}\cdot D$ | m5 | 0101 | 2 |
| $\overline{A}\cdot B\cdot C\cdot D$ | $m7$ | 0111 | 3 |
| $A\cdot\overline{B}\cdot C\cdot D$ | $m11$ | 1011 | 3 |

# QM – Example 4

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot \overline{D}$ | m0 | 0000 | 0 |
| $\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot D$ | m1 | 0001 | 1 |
| $\overline{A} \cdot B \cdot \overline{C} \cdot \overline{D}$ | m4 | 0100 | 1 |
| $\overline{A} \cdot B \cdot \overline{C} \cdot D$ | m5 | 0101 | 2 |
| $\overline{A} \cdot B \cdot C \cdot D$ | $m7$ | 0111 | 3 |
| $A \cdot \overline{B} \cdot C \cdot D^*$ | $m11^*$ | 1011 | 3 |
| $\overline{A} \cdot \overline{B} \cdot \overline{C}$ | $(m0, m1)$ | 000- | 0 |
| $\overline{A} \cdot \overline{C} \cdot \overline{D}$ | $(m0, m4)$ | 0-00 | 0 |
| $\overline{A} \cdot \overline{C} \cdot D$ | $(m1, m5)$ | 0-01 | 1 |
| $\overline{A} \cdot B \cdot \overline{C}$ | $(m4, m5)$ | 010- | 1 |
| $\overline{A} \cdot B \cdot D$ | $(m5, m7)$ | 01-1 | 2 |





# QM – Example 4

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A} \cdot \overline{B} \cdot \overline{C}$ | $(m0, m1)$ | 000- | 0 |
| $\overline{A} \cdot \overline{C} \cdot \overline{D}$ | $(m0, m4)$ | 0-00 | 0 |
| $\overline{A} \cdot \overline{C} \cdot D$ | $(m1, m5)$ | 0-01 | 1 |
| $\overline{A} \cdot B \cdot \overline{C}$ | $(m4, m5)$ | 010- | 1 |
| $\overline{A} \cdot B \cdot D^*$ | $(m5, m7)^*$ | 01-1 | 2 |
| $\overline{A} \cdot \overline{C}$ | $(m0, m1, m4, m5)$ | 0-0- | 0 |
| $\overline{A} \cdot \overline{C}$ | $(m0, m4, m1, m5)$ | 0-0- | 0 |

# QM – Example 4

| | $m0$ | $m1$ | $m4$ | $m5$ | $m7$ | $m11$ |
|---|---|---|---|---|---|---|
| $A \cdot \overline{B} \cdot C \cdot D^*$ | | | | | | X |
| $\overline{A} \cdot B \cdot D^*$ | | | | X | X | |
| $\overline{A} \cdot \overline{C}$ | X | X | X | X | | |

Optimized: $Y = \overline{A} \cdot \overline{C} + \overline{A} \cdot B \cdot D + A \cdot \overline{B} \cdot C \cdot D$

Original:

$$Y = \overline{A} \cdot \overline{B} \cdot \overline{C} \cdot \overline{D} + \overline{A} \cdot \overline{B} \cdot \overline{C} \cdot D + \overline{A} \cdot B \cdot \overline{C} \cdot \overline{D} + \overline{A} \cdot B \cdot \overline{C} \cdot D + \overline{A} \cdot B \cdot C \cdot D + A \cdot \overline{B} \cdot C \cdot D$$



# QM – Example 5: With Don't cares

| | $A$ | $B$ | $C$ | $Y$ |
|---|---|---|---|---|
| $d0$ | 0 | 0 | 0 | X |
| $m1$ | 0 | 0 | 1 | 0 |
| $m2$ | 0 | 1 | 0 | 0 |
| $m3$ | 0 | 1 | 1 | 0 |
| $m4$ | 1 | 0 | 0 | 1 |
| $m5$ | 1 | 0 | 1 | 1 |
| $d6$ | 1 | 1 | 0 | X |
| $m7$ | 1 | 1 | 1 | 1 |

| ¬ *C* | | *C* | | |
|---|---|---|---|---|
| X | 1 | 1 | 0 | ¬ *B* |
| 0 | X | 1 | 0 | *B* |
| ¬ *A* | *A* | | ¬ *A* | |

## QM – Example 5: With Don‘t cares

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A} \cdot \overline{B} \cdot \overline{C}$ | $d0$ | 000 | 0 |
| $A \cdot \overline{B} \cdot \overline{C}$ | $m4$ | 100 | 1 |
| $A \cdot \overline{B} \cdot C$ | $m5$ | 101 | 2 |
| $A \cdot B \cdot \overline{C}$ | $d6$ | 110 | 2 |
| $A \cdot B \cdot C$ | $m7$ | 111 | 3 |
| $\overline{B} \cdot \overline{C}$ | $(d0, m4)$ | -00 | 0 |
| $A \cdot \overline{B}$ | $(m4, m5)$ | 10- | 1 |
| $A \cdot \overline{C}$ | $(m4, d6)$ | 1-0 | 1 |
| $A \cdot C$ | $(m5, m7)$ | 1-1 | 2 |
| $A \cdot B$ | $(d6, m7)$ | 11- | 2 |



## QM – Example 5: With Don‘t cares

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{B} \cdot \overline{C}^*$ | $(d0, m4)^*$ | -00 | 0 |
| $A \cdot \overline{B}$ | $(m4, m5)$ | 10- | 1 |
| $A \cdot \overline{C}$ | $(m4, d6)$ | 1-0 | 1 |
| $A \cdot C$ | $(m5, m7)$ | 1-1 | 2 |
| $A \cdot B$ | $(d6, m7)$ | 11- | 2 |
| $A$ | $(m4, m5, d6, m7)$ | 1– | 1 |
| $A$ | $(m4, d6, m5, m7)$ | 1– | 1 |

# QM – Example 5: With Don´t cares

Table of essential terms:

| | $m4$ | $m5$ | $m7$ |
|---|---|---|---|
| $\overline{B} \cdot \overline{C}$ | X | | |
| $A$ | X | X | X |

- Only onset terms are included, not don´t cares;
- *m*5 and *m*7 are covered only by the term *A*, which means that *A* is compulsory;
- All essential minterms are covered by *A* (*m*4,*m*5, *d*6,*m*7) so no other terms are needed;
- *d*6 is used, but not *d*0, because *B* ∧ *C* (*d*0,*m*4) is not needed



# QM – Example 6: Multiple Solutions

| | A | B | C | D | Y |
|---|---|---|---|---|---|
| m0 | 0 | 0 | 0 | 0 | 0 |
| m1 | 0 | 0 | 0 | 1 | 0 |
| m2 | 0 | 0 | 1 | 0 | 0 |
| m3 | 0 | 0 | 1 | 1 | 0 |
| m4 | 0 | 1 | 0 | 0 | 1 |
| m5 | 0 | 1 | 0 | 1 | 0 |
| m6 | 0 | 1 | 1 | 0 | 0 |
| m7 | 0 | 1 | 1 | 1 | 0 |
| m8 | 1 | 0 | 0 | 0 | 1 |
| m9 | 1 | 0 | 0 | 1 | x |
| m10 | 1 | 0 | 1 | 0 | 1 |
| m11 | 1 | 0 | 1 | 1 | 1 |
| m12 | 1 | 1 | 0 | 0 | 1 |
| m13 | 1 | 1 | 0 | 1 | 0 |
| m14 | 1 | 1 | 1 | 0 | x |
| m15 | 1 | 1 | 1 | 1 | 1 |

# QM – Example 6: Multiple Solutions

$$f_{A,B,C,D} = A'BC'D' + AB'C'D' + AB'CD' + AB'CD + ABC'D' + ABCD$$

| Number of 1s | Minterm | Binary Representation |
|---|---|---|
| 1 | m4 | 0100 |
| | m8 | 1000 |
| 2 | m9 | 1001 |
| | m10 | 1010 |
| | m12 | 1100 |
| 3 | m11 | 1011 |
| | m14 | 1110 |
| 4 | m15 | 1111 |



# QM – Example 6: Multiple Solutions

| Number of 1s | Minterm | 0-Cube | Size 2 Implicants | Size 4 Implicants |
|---|---|---|---|---|
| 1 | m4 | 0100 | m(4,12) -100* | m(8,9,10,11) 10--* |
| | m8 | 1000 | m(8,9) 100- | m(8,10,12,14) 1--0* |
| | -- | -- | m(8,10) 10-0 | -- |
| | -- | -- | m(8,12) 1-00 | -- |
| 2 | m9 | 1001 | m(9,11) 10-1 | m(10,11,14,15) 1-1-* |
| | m10 | 1010 | m(10,11) 101- | -- |
| | -- | -- | m(10,14) 1-10 | -- |
| | m12 | 1100 | m(12,14) 11-0 | -- |
| 3 | m11 | 1011 | m(11,15) 1-11 | -- |
| | m14 | 1110 | m(14,15) 111- | -- |
| 4 | m15 | 1111 | -- | -- |

# QM – Example 6: Multiple Solutions

Prime Implicant Cover Table

| | m4 | m8 | m10 | m11 | m12 | m15 |
|---|---|---|---|---|---|---|
| m(4,12) | X | | | | X | |
| m(8,9,10,11) | | X | X | X | | |
| m(8,10,12,14) | | X | X | | X | |
| m(10,11,14,15) | | | X | X | | X |

Rows: Prime Implicants
Columns: On-Set terms
X: A term is covered by a prime implicant





# QM – Example 6: Multiple Solutions

Prime Implicant Cover Table

| | m4 | m8 | m10 | m11 | m12 | m15 |
|---|---|---|---|---|---|---|
| m(4,12) | X | | | | X | |
| m(8,9,10,11) | | X | X | X | | |
| m(8,10,12,14) | | X | X | | X | |
| m(10,11,14,15) | | | X | X | | X |

Only covered by 1 implicant

Essential
Prime
Implicants

If there is a single X In a column,
the corresponding prime implicant
is an **essential prime implicant**

# QM – Example 6: Multiple Solutions

Prime Implicant Cover Table

| | m4 | m8 | m10 | m11 | m12 | m15 |
|---|---|---|---|---|---|---|
| m(4,12) * | X | | | | X | |
| m(8,9,10,11) | | X | X | X | | |
| m(8,10,12,14) | | X | X | | X | |
| m(10,11,14,15) * | | | X | X | | X |

Eliminate all columns covered by essential primes.



# QM – Example 6: Multiple Solutions

Prime Implicant Cover Table

| | m4 | m8 | m10 | m11 | m12 | m15 |
|---|---|---|---|---|---|---|
| m(4,12) * | X | | | | X | |
| m(8,9,10,11) | | X | X | X | | |
| m(8,10,12,14) | | X | X | | X | |
| m(10,11,14,15) * | | | X | X | | X |

Find minimum set of rows that cover the remaining columns.

# QM – Example 6: Multiple Solutions

Prime Implicant Cover Table

| | m4 | m8 | m10 | m11 | m12 | m15 |
|---|---|---|---|---|---|---|
| m(4,12) * | X | | | | X | |
| m(8,9,10,11) | | X | X | X | | |
| m(8,10,12,14) | | X | X | | X | |
| m(10,11,14,15) * | | | X | X | | X |

Find minimum set of rows that cover the remaining columns.





# QM – Example 6: Multiple Solutions

Prime Implicant Cover Table

| | m4 | m8 | m10 | m11 | m12 | m15 |
|---|---|---|---|---|---|---|
| m(4,12) * | X | | | | X | |
| m(8,9,10,11) | | X | X | X | | |
| m(8,10,12,14) | | X | X | | X | |
| m(10,11,14,15) * | | | X | X | | X |

Optimized: $f_{A,B,C,D} = BC'D' + AB' + AC$

Original:

$f_{A,B,C,D} = A'BC'D' + AB'C'D' + AB'CD' + AB'CD + ABC'D' + ABCD$

# QM – Example 7: Multiple Solutions

| A | B | C | Y |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |





# QM – Example 7: Multiple Solutions

| ¬ *C* | | *C* | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | ¬ *B* |
| 1 | 0 | 1 | 1 | *B* |
| ¬ *A* | *A* | | ¬ *A* | |

| ¬ *C* | | *C* | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | ¬ *B* |
| 1 | 0 | 1 | 1 | *B* |
| ¬ *A* | *A* | | ¬ *A* | |

## QM – Example 7: Multiple Solutions

| Logic Term | min term | binary representation | Numbers of '1's |
|---|---|---|---|
| $\overline{A}\cdot\overline{B}\cdot\overline{C}$ | $m0$ | 000 | 0 |
| $\overline{A}\cdot B\cdot\overline{C}$ | $m2$ | 010 | 1 |
| $A\cdot\overline{B}\cdot\overline{C}$ | $m4$ | 100 | 1 |
| $\overline{A}\cdot B\cdot C$ | $m3$ | 011 | 2 |
| $A\cdot\overline{B}\cdot C$ | $m5$ | 101 | 2 |
| $A\cdot B\cdot C$ | $m7$ | 111 | 3 |
| $\overline{A}\cdot\overline{C}$ | $(m0, m2)$ | 0-0 | 0 |
| $\overline{B}\cdot\overline{C}$ | $(m0, m4)$ | -00 | 0 |
| $\overline{A}\cdot B$ | $(m2, m3)$ | 01- | 1 |
| $A\cdot\overline{B}$ | $(m4, m5)$ | 10- | 1 |
| $B\cdot C$ | $(m3, m7)$ | -11 | 2 |
| $A\cdot C$ | $(m5, m7)$ | 1-1 | 2 |





## QM – Example 7: Multiple Solutions

| | $m0$ | $m2$ | $m3$ | $m4$ | $m5$ | $m7$ |
|---|---|---|---|---|---|---|
| $\overline{A}\cdot\overline{C}$ | X | X | | | | |
| $\overline{B}\cdot\overline{C}$ | X | | | X | | |
| $\overline{A}\cdot B$ | | X | X | | | |
| $A\cdot\overline{B}$ | | | | X | X | |
| $B\cdot C$ | | | X | | | X |
| $A\cdot C$ | | | | | X | X |

Some possible solutions: $Y = \overline{B}\cdot\overline{C} + A\cdot C + \overline{A}\cdot B$

$Y = \overline{A}\cdot\overline{C} + B\cdot C + A\cdot\overline{B}$

$Y = \overline{A}\cdot\overline{C} + \overline{A}\cdot B + A\cdot\overline{B} + A\cdot C$

# Minimal Cover

1. Identification of Essential Primes
2. Removing essential primes and covered minterms
3. a) Elimination of dominated rows
   b) Elimination of dominating columns
4. Selection of remaining primes for a minimal cover





## Example 8

$$\begin{aligned} f = & \overline{x_1}\,\overline{x_2}\,\overline{x_3}\,\overline{x_4} \\ & + \overline{x_1}x_2\overline{x_3}\,\overline{x_4} + x_1\overline{x_2}\,\overline{x_3}\,\overline{x_4} \\ & + x_1\overline{x_2}x_3\overline{x_4} + x_1\overline{x_2}x_3x_4 \\ & + x_1x_2\overline{x_3}\,\overline{x_4} + x_1x_2\overline{x_3}x_4 \\ & + x_1x_2x_3x_4 \end{aligned}$$

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | f | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | m0 |
| 0 | 0 | 0 | 1 | 0 | m1 |
| 0 | 0 | 1 | 0 | 0 | m2 |
| 0 | 0 | 1 | 1 | 0 | m3 |
| 0 | 1 | 0 | 0 | 1 | m4 |
| 0 | 1 | 0 | 1 | 0 | m5 |
| 0 | 1 | 1 | 0 | 0 | m6 |
| 0 | 1 | 1 | 1 | 0 | m7 |
| 1 | 0 | 0 | 0 | 1 | m8 |
| 1 | 0 | 0 | 1 | 0 | m9 |
| 1 | 0 | 1 | 0 | 1 | m10 |
| 1 | 0 | 1 | 1 | 1 | m11 |
| 1 | 1 | 0 | 0 | 1 | m12 |
| 1 | 1 | 0 | 1 | 1 | m13 |
| 1 | 1 | 1 | 0 | 0 | m14 |
| 1 | 1 | 1 | 1 | 1 | m15 |

Sort lists by number of 1s

List 1

| | | |
|---|---|---|
| 0 | 0 0 0 0 | ✓ |
| 4<br>8 | 0 1 0 0<br>1 0 0 0 | ✓<br>✓ |
| 10<br>12 | 1 0 1 0<br>1 1 0 0 | ✓<br>✓ |
| 11<br>13 | 1 0 1 1<br>1 1 0 1 | ✓<br>✓ |
| 15 | 1 1 1 1 | ✓ |

0-cubes

List 2

| | | |
|---|---|---|
| 0,4<br>0,8 | 0 x 0 0<br>x 0 0 0 | ✓<br>✓ |
| 8,10<br>4,12<br>8,12 | 1 0 x 0<br>x 1 0 0<br>1 x 0 0 | <br>✓<br>✓ |
| 10,11<br>12,13 | 1 0 1 x<br>1 1 0 x | |
| 11,15<br>13,15 | 1 x 1 1<br>1 1 x 1 | |

1-cubes

List 3

| | |
|---|---|
| 0,4,8,12 | x x 0 0 |

2-cubes

Prime implicants P = {10x0, 101x,110x, 1x11, 11x1, xx00}

$= \{p_1, p_2, p_3, p_4, p_5, p_6\}$





p6 is essential

| Prime implicant | 0 | 4 | 8 | 10 | 11 | 12 | 13 | 15 |
|---|---|---|---|---|---|---|---|---|
| $p_1$ = 1 0 x 0 | | | ✓ | ✓ | | | | |
| $p_2$ = 1 0 1 x | | | | ✓ | ✓ | | | |
| $p_3$ = 1 1 0 x | | | | | | ✓ | ✓ | |
| $p_4$ = 1 x 1 1 | | | | | ✓ | | | ✓ |
| $p_5$ = 1 1 x 1 | | | | | | | ✓ | ✓ |
| $p_6$ = x x 0 0 | ✓ | ✓ | ✓ | | | ✓ | | |

(Column header printed as "Minterm" above the minterm numbers.)

Initial prime implicant cover table

p2 dominates p1

p5 dominates p3

| Prime implicant | 10 | 11 | 13 | 15 |
|---|---|---|---|---|
| $p_1$ | ✓ | | | |
| $p_2$ | ✓ | ✓ | | |
| $p_3$ | | | ✓ | |
| $p_4$ | | ✓ | | ✓ |
| $p_5$ | | | ✓ | ✓ |

After the removal of essential prime implicants

| Prime implicant | 10 | 11 | 13 | 15 |
|---|---|---|---|---|
| $p_2$ | ✓ | ✓ | | |
| $p_4$ | | ✓ | | ✓ |
| $p_5$ | | | ✓ | ✓ |

After the removal of dominated rows

Cover: C = {$p_2$, $p_5$, $p_6$} = {101x, 11x1, xx00}

$$f = x_1\overline{x_2}x_3 + x_1x_2x_4 + \overline{x_3}\,\overline{x_4}$$

original function:

$$f = \overline{x_1x_2x_3x_4} + \overline{x_1}x_2\overline{x_3x_4} + x_1\overline{x_2x_3x_4} + x_1\overline{x_2}x_3\overline{x_4} + x_1\overline{x_2}x_3x_4 + x_1x_2\overline{x_3x_4} + x_1x_2\overline{x_3}x_4 + x_1x_2x_3x_4$$



# Example 9 with Don't Cares

$$f = \sum m(0,2,5,6,7,8,9,13) + D(1,12,15)$$

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | f | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | m0 |
| 0 | 0 | 0 | 1 | d | m1 |
| 0 | 0 | 1 | 0 | 1 | m2 |
| 0 | 0 | 1 | 1 | 0 | m3 |
| 0 | 1 | 0 | 0 | 0 | m4 |
| 0 | 1 | 0 | 1 | 1 | m5 |
| 0 | 1 | 1 | 0 | 1 | m6 |
| 0 | 1 | 1 | 1 | 1 | m7 |
| 1 | 0 | 0 | 0 | 1 | m8 |
| 1 | 0 | 0 | 1 | 1 | m9 |
| 1 | 0 | 1 | 0 | 0 | m10 |
| 1 | 0 | 1 | 1 | 0 | m11 |
| 1 | 1 | 0 | 0 | d | m12 |
| 1 | 1 | 0 | 1 | 1 | m13 |
| 1 | 1 | 1 | 0 | 0 | m14 |
| 1 | 1 | 1 | 1 | d | m15 |

List 1

| | | |
|---|---|---|
| 0 | 0 0 0 0 | ✓ |
| 1 | 0 0 0 1 | ✓ |
| 2 | 0 0 1 0 | ✓ |
| 8 | 1 0 0 0 | ✓ |
| 5 | 0 1 0 1 | ✓ |
| 6 | 0 1 1 0 | ✓ |
| 9 | 1 0 0 1 | ✓ |
| 12 | 1 1 0 0 | ✓ |
| 7 | 0 1 1 1 | ✓ |
| 13 | 1 1 0 1 | ✓ |
| 15 | 1 1 1 1 | ✓ |

List 2

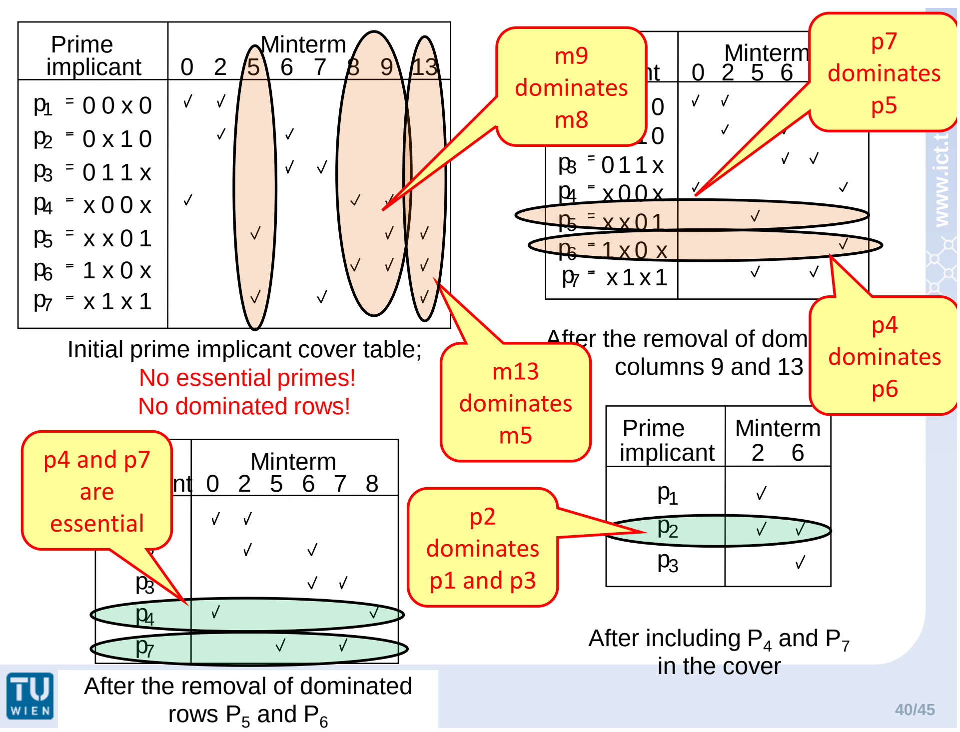
| | | |
|---|---|---|
| 0,1 | 0 0 0 x | ✓ |
| 0,2 | 0 0 x 0 | |
| 0,8 | x 0 0 0 | ✓ |
| 1,5 | 0 x 0 1 | ✓ |
| 2,6 | 0 x 1 0 | |
| 1,9 | x 0 0 1 | ✓ |
| 8,9 | 1 0 0 x | ✓ |
| 8,12 | 1 x 0 0 | ✓ |
| 5,7 | 0 1 x 1 | ✓ |
| 6,7 | 0 1 1 x | |
| 5,13 | x 1 0 1 | ✓ |
| 9,13 | 1 x 0 1 | ✓ |
| 12,13 | 1 1 0 x | ✓ |
| 7,15 | x 1 1 1 | ✓ |
| 13,15 | 1 1 x 1 | ✓ |

List 3

| | |
|---|---|
| 0,1,8,9 | x 0 0 x |
| 1,5,9,13 | x x 0 1 |
| 8,9,12,13 | 1 x 0 x |
| 5,7,13,15 | x 1 x 1 |

Prime implicants P = {00x0, 0x10, 011x, x00x, xx01, 1x0x, x1x1}
$= \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$





Cover: C = {$p_2$, $p_4$, $p_7$} = {0x10, x00x, x1x1}

$$f = \overline{x_1} x_3 \overline{x_4} + \overline{x_2}\ \overline{x_3} + x_2 x_4$$





# Example 10

$$f = \sum m(0,3,10,15) + D(1,2,7,8,11,14)$$

Prime implicants
P = {00xx, x0x0, x01x, xx11, 1x1x}
= {$p_1$, $p_2$, $p_3$, $p_4$, $p_5$}

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | f | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | m0 |
| 0 | 0 | 0 | 1 | d | m1 |
| 0 | 0 | 1 | 0 | d | m2 |
| 0 | 0 | 1 | 1 | 1 | m3 |
| 0 | 1 | 0 | 0 | 0 | m4 |
| 0 | 1 | 0 | 1 | 0 | m5 |
| 0 | 1 | 1 | 0 | 0 | m6 |
| 0 | 1 | 1 | 1 | d | m7 |
| 1 | 0 | 0 | 0 | d | m8 |
| 1 | 0 | 0 | 1 | 0 | m9 |
| 1 | 0 | 1 | 0 | 1 | m10 |
| 1 | 0 | 1 | 1 | d | m11 |
| 1 | 1 | 0 | 0 | 0 | m12 |
| 1 | 1 | 0 | 1 | 0 | m13 |
| 1 | 1 | 1 | 0 | d | m14 |
| 1 | 1 | 1 | 1 | 1 | m15 |

Branching with $P_3$

| Prime implicant | Minterm 0 | 15 |
|---|---|---|
| $p_1$ | √ | |
| $p_2$ | √ | |
| $p_4$ | | √ |
| $p_5$ | | √ |

After including $p_3$

| Prime implicant | | Minterm 0 | 3 | 10 | 15 |
|---|---|---|---|---|---|
| $p_1$ | = 0 0 x x | √ | √ | | |
| $p_2$ | = x 0 x 0 | √ | | √ | |
| $p_3$ | = x 0 1 x | | √ | √ | |
| $p_4$ | = x x 1 1 | | √ | | √ |
| $p_5$ | = 1 x 1 x | | | √ | √ |

Initial implicant cover table

No essential primes;
No dominating rows;
No dominating columns;

| Prime implicant | Minterm 0 | 3 | 10 | 15 |
|---|---|---|---|---|
| $p_1$ | √ | √ | | |
| $p_2$ | √ | | √ | |
| $p_4$ | | √ | | √ |
| $p_5$ | | | √ | √ |

After excluding $p_3$



Alternatives:

- $A_1 = \{P_1, P_3, P_4\}$ ; cost = 3+3+3+4 = 13
- $A_2 = \{P_1, P_3, P_5\}$ ; cost = 3+3+3+4 = 13
- $A_3 = \{P_2, P_3, P_4\}$ ; cost = 3+3+3+4 = 13
- $A_4 = \{P_2, P_3, P_5\}$ ; cost = 3+3+3+4 = 13
- $A_5 = \{P_1, P_5\}$ ; cost = 3+3+3 = 9
- $A_6 = \{P_2, P_4\}$ ; cost = 3+3+3 = 9

We choose the cover with minimal cost

# Summary of Quine-McCluskey

1. Start with a list of minterms for which f=1 or f=x;
2. Generate prime implicants by pairwise comparison of the cubes:
3. Derive a cover table;
4. Include the essential prime implicants;
5. Use the concept of row and column dominance;
6. Repeat steps 4 and 5 until table is empty or no further reduction is possible;
7. If table is not empty use a branching approach to determine the remaining prime implicants at minimal cost.