Universität Stuttgart

A new viscoelastic constitutive model for continuous media at finite thermomechanical changes.

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COMMAS

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Motivation and overview

Motivation

Figure 1: Rubber-like materials.

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Motivation

Figure 1: Rubber-like materials.

e.g. Melting point of Polypropylene (PP) varies from $130 - 171^o C$.
Thermo-visco-elastic model is essential to model rubber-like materials.

Overview (1/2)

Main points from the article:

- Thermomechanical visco-elastic model,
- Highly deformable with viscous dissipation,
- Concept: internal state variable and rational thermodynamics,
- Goal: constitutive eqns. for stress, entropy and internal variables,
- 2^{nd} law of thermodynamics satisfied,
- Decomposition of deformation into dilatational and deviatoric part,
- Formulated entirely in reference configuration,
- Numerically easy to handle.

Overview (2/2)

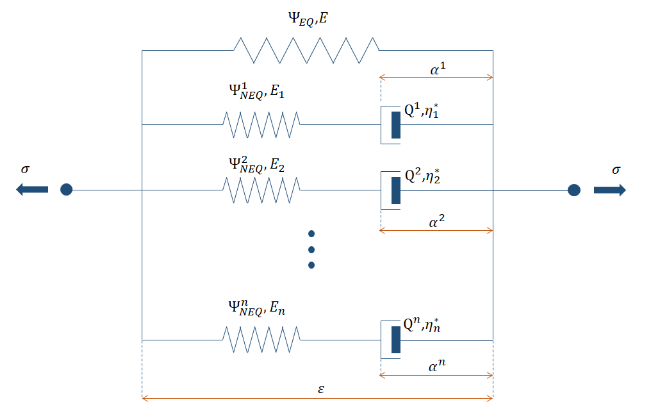

Figure 2: Generalized Maxwell model - 1D rheological interpretation.

where α^i, η^*_i, Q^i is inelastic strain, viscous coefficient and stress of dashpot i^{th}, respectively; E^i is spring stiffness. $(i = 1, 2, ..n)$

Mathematical Model

Preliminary

2nd law of thermodynamics in terms of ref. config. reads:

$$-\rho_0(\dot{\Psi} + \dot{\Theta}\eta) + \mathbf{S} \cdot \dot{\mathbf{E}} - \frac{1}{\Theta} q_0 \, Grad\Theta \geq 0 \tag{1}$$

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Time derivative of free energy function $\Psi(\mathbf{E}, \Theta, Grad\Theta)$ reads:

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial \Psi}{\partial \Theta} \cdot \dot{\Theta} + \frac{\partial \Psi}{\partial Grad\Theta} \cdot Grad\dot{\Theta} \tag{2}$$

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Substitution equation (1) into inequality (2) yields:

$$-\rho_0(\eta + \frac{\partial \Psi}{\partial \Theta})\dot{\Theta} + (\mathbf{S} - \rho_0 \frac{\partial \Psi}{\partial \mathbf{E}}) \cdot \dot{\mathbf{E}} - \rho_0 \frac{\partial \Psi}{\partial Grad\Theta} \cdot Grad\dot{\Theta} - \frac{1}{\Theta} \mathbf{q}_0 \, Grad\Theta \geq 0 \tag{3}$$

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Since 2nd law must hold for any admissible process, we conclude:

$$\boxed{\begin{aligned}\mathbf{S}&=\partial_{\mathbf{E}}\Psi && (4)\\ \eta&=-\partial_{\Theta}\Psi && (5)\end{aligned}}$$

Mathematical Model (1/5)

Thermoviscoelastic model defined by Helmholtz-free energy function:

$$\boxed{\Psi = \underbrace{\Psi_{EQ}(\mathbf{C}, \Theta)}_{\text{purely thermoelastic}} + \underbrace{\Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i)}_{\text{non-equilibrium part}}} \tag{6}$$

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Coleman's exploitation yields:

$$\mathbf{S} = \partial_{\mathbf{C}}\Psi = 2\partial_{\mathbf{C}}\Psi_{EQ}(\mathbf{C}, \Theta) + 2\partial_{\mathbf{C}}\Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i) \tag{7}$$

$$\eta = -\partial_{\Theta}\Psi = -\partial_{\Theta}\Psi_{EQ}(\mathbf{C}, \Theta) - \partial_{\Theta}\Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i) \tag{8}$$

$$\mathbf{Q}^i = -\partial_{\boldsymbol{\alpha}_i}\Psi = -\partial_{\boldsymbol{\alpha}_i}\Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i) \tag{9}$$

Reduced local dissipation expression: $\mathcal{D}_{loc}^{red} = \sum_{i=1}^{n} \mathbf{Q}^i . \dot{\boldsymbol{\alpha}}_i \geq 0$ (10)

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Reduced local dissipation expression: $\mathcal{D}^{red}_{loc} = \sum_{i=1}^{n} \mathbf{Q}^i . \dot{\boldsymbol{\alpha}}_i \geq 0$ (10)

Propose $\dot{\boldsymbol{\alpha}}_i = \frac{1}{\eta^*}\mathbf{Q}^i$, then inequality (10) becomes:

$$\mathcal{D}^{red}_{loc} = \sum_{i=1}^{n} \frac{1}{\eta^*}\left|\mathbf{Q}^i\right|^2 \geq 0 \tag{11}$$

which satisfies 2nd law of thermodynamics $\forall \eta^* > 0$.

Mathematical Model (2/5)

Recall evolution equation of driving forces $\mathbf{Q}^i$ in viscoelastic theory:

$$\dot{\mathbf{Q}}^i + \frac{\mathbf{Q}^i}{\tau} = E_i \; \dot{\varepsilon} \tag{12}$$

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$$E_i \dot{\varepsilon} \equiv \frac{E_i}{E_0} \frac{d}{dt}[\partial_\varepsilon \Psi_{EQ}(\varepsilon)] \equiv \gamma_i \frac{d}{dt}[\partial_\varepsilon \Psi_{EQ}(\varepsilon)] \equiv \frac{d}{dt}[\partial_\varepsilon \Psi^{\Gamma}_{EQ}(\varepsilon)] \tag{13}$$

where $\gamma_i \in (0, \infty)$.

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Here, $\Psi_{EQ}(\varepsilon) \rightarrow \Psi_{EQ}(\mathbf{C}, \Theta)$, hence:

$$\boxed{\dot{\mathbf{Q}}^i + \frac{\mathbf{Q}^i}{\tau} = \frac{d}{dt} [2 \partial_{\mathbf{C}} \Psi^\Gamma_{EQ}(\mathbf{C}, \Theta)] - \mathbf{Q}^i_{coupling}} \tag{15}$$

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The term $\mathbf{Q}^i_{coupling}$ is subtracted because:

$$\underbrace{\Psi^\Gamma_{EQ}(\mathbf{C}, \Theta)}_{\text{purely thermoelastic}} = \underbrace{\Psi^\Gamma_{EQ}(\mathbf{C})}_{\text{elastic}} + \underbrace{\Psi^\Gamma_{EQ}(\mathbf{C}, \Theta)}_{\text{thermoelastic coupling}} + \underbrace{\Psi^\Gamma_{EQ}(\Theta)}_{\text{rigid heat conduction}} \tag{16}$$

Mathematical Model (3/5)

Therefore, we have:

$$\boxed{\begin{aligned} \dot{\mathbf{Q}}^i + \frac{\mathbf{Q}^i}{\tau} &= \frac{d}{dt}[2\partial_{\mathbf{C}}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta)] - \mathbf{Q}^i_{coupling} && (17) \\ \mathbf{Q}^i|_{t=0} &= 2\partial_{\mathbf{C}}\Psi^{\Gamma}_{EQ}(\mathbf{C}_0,\Theta_0) && (18) \end{aligned}}$$

Mathematical Model (3/5)

Therefore, we have:

$$\boxed{\begin{aligned} \dot{\mathbf{Q}}^i + \frac{\mathbf{Q}^i}{\tau} &= \frac{d}{dt}[2\partial_{\mathbf{C}} \Psi^{\Gamma}_{EQ}(\mathbf{C}, \Theta)] - \mathbf{Q}^i_{coupling} && (17) \\ \mathbf{Q}^i|_{t=0} &= 2\partial_{\mathbf{C}} \Psi^{\Gamma}_{EQ}(\mathbf{C}_0, \Theta_0) && (18) \end{aligned}}$$

which can be rewritten into convolution representation by using integration:

$$\boxed{\mathbf{Q}^i = e^{\frac{-t}{\tau}} \mathbf{Q}^i_0 + \int_{0^+}^{t} e^{\frac{-(t-s)}{\tau}} \left\{ \frac{d}{ds}[2\partial_{\mathbf{C}} \Psi^{\Gamma}_{EQ}(\mathbf{C}, \Theta)] - \mathbf{Q}^i_{coupling} \right\} ds} \quad (19)$$

Convolution representation (19) is the most compact form, hence it is of easy to perform numerical implementation.

Mathematical Model (4/5)

$$\boxed{\Psi = \underbrace{\Psi_{EQ}(\mathbf{C},\Theta)}_{\text{purely thermoelastic}} + \underbrace{\Psi_{NEQ}(\mathbf{C},\Theta,\boldsymbol{\alpha}_i)}_{\text{non-equilibrium part}}} \tag{20}$$

$$\Psi_{NEQ}(\mathbf{C},\Theta,\boldsymbol{\alpha}_i) = \sum_{i=1}^{n}\left[E_i\,|\boldsymbol{\alpha}_i|^2 - 2\partial_{\mathbf{C}}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta) : \boldsymbol{\alpha}_i + \Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta)\right] \tag{21}$$

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How to get expression (21) ?

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How to get expression (21) ?

Brief answer:

Because, we have proposed (22), motivated by (23) and (24):

$$\dot{\boldsymbol{\alpha}}_i = \frac{1}{\eta^*}\mathbf{Q}^i \tag{22}$$

$$\mathbf{Q}^i = -\partial_{\boldsymbol{\alpha}_i}\Psi = -\partial_{\boldsymbol{\alpha}_i}\Psi_{NEQ}(\mathbf{C},\Theta,\boldsymbol{\alpha}_i) \tag{23}$$

$$\partial^2_{\alpha_i\alpha_i}\Psi_{NEQ} = E_i(\Theta) \tag{24}$$

Therefore, by integration (24) twice, time derivative (23) with substitution of (22), expression (21) is achieved.

Mathematical Model (5/5) IN SHORT

$$\Psi = \underbrace{\Psi_{EQ}(\mathbf{C}, \Theta)}_{\text{purely thermoelastic}} + \underbrace{\Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i)}_{\text{non-equilibrium part}} \tag{25}$$

$$\Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i) = \sum_{i=1}^{n} \left[E_i \left|\boldsymbol{\alpha}_i\right|^2 - 2\partial_{\mathbf{C}} \Psi^{\Gamma}_{EQ}(\mathbf{C}, \Theta) : \boldsymbol{\alpha}_i + \Psi^{\Gamma}_{EQ}(\mathbf{C}, \Theta) \right]$$

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$$\mathbf{S} = 2\partial_{\mathbf{C}}\Psi_{EQ}(\mathbf{C},\Theta) + \sum_{i=1}^{n}\left[2\partial_{\mathbf{C}}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta) - 4\partial^2_{\mathbf{CC}}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta)\boldsymbol{\alpha}_i\right]$$

$$\eta = -\partial_{\Theta}\Psi_{EQ}(\mathbf{C},\Theta)$$

$$+\sum_{i=1}^{n}\left[-\partial_{\Theta}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta) + 2\partial_{\mathbf{C}\Theta}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta):\boldsymbol{\alpha}_i - \partial_{\Theta}E_i(\Theta)\left|\boldsymbol{\alpha}_i\right|^2\right]$$

$$\mathbf{Q}^i = e^{\frac{-t}{\tau}}\mathbf{Q}^i_0 + \int_{0^+}^{t} e^{\frac{-(t-s)}{\tau}}\left\{\frac{d}{ds}\left[2\partial_{\mathbf{C}}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta)\right] - 2\partial_{\Theta}E_i(\Theta)\boldsymbol{\alpha}_i\frac{d\Theta}{ds}\right\}ds$$

$$\boldsymbol{\alpha}_i = \frac{1}{E_i}2\partial_{\mathbf{C}}\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta) - \frac{\mathbf{Q}^i}{2E_i};\ \boxed{\Psi^{\Gamma}_{EQ}(\mathbf{C},\Theta) = \gamma_i\Psi_{EQ}(\mathbf{C},\Theta)} \tag{26}$$

Dilatational-Deviatoric Multiplicative Split

Dilatational-Deviatoric Multiplicative Split (1/2)

- Motivation: Polymeric materials behave strongly in shear, weakly in dilatation (Malvern [1969]).
- Proposed by Flory [1966]:

$$\boxed{\begin{aligned} \mathbf{F} &= (J^{1/3}\mathbf{I})\tilde{\mathbf{F}} \qquad (27) \\ \text{with} \qquad det(\tilde{\mathbf{F}}) &\equiv 1 \qquad (28) \end{aligned}}$$

where $\mathbf{F}$ is deformation gradient, J Jacobian, $\tilde{\mathbf{F}}$ deviatoric part and $(J^{1/3}\mathbf{I})$ dilatational part.

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where $\mathbf{F}$ is deformation gradient, J Jacobian, $\tilde{\mathbf{F}}$ deviatoric part and $(J^{1/3}\mathbf{I})$ dilatational part.
It follows:

$$\boxed{\begin{aligned} \tilde{\mathbf{C}} &= \tilde{\mathbf{F}}^{\mathsf{T}}\tilde{\mathbf{F}} = (J^{-2/3}\mathbf{I})\mathbf{C} && (29)\\ \partial_{\mathbf{C}}\tilde{\mathbf{C}} &= J^{-2/3}(\mathbb{I} - \frac{1}{3}\mathbf{C} \otimes \mathbf{C}^{-1}) && (30)\\ 2\partial_{\mathbf{C}}J &= J\mathbf{C}^{-1} && (31)\end{aligned}}$$

Dilatational-Deviatoric Multiplicative Split (2/2)

$$\boxed{\Psi = \Psi_{EQ}(\mathbf{C}, \Theta) + \Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i)} \tag{32}$$

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Consideration of specific Helmholtz free energy function $\Psi_{EQ}(\mathbf{C}, \Theta)$:

$$\Psi_{EQ}(\mathbf{C}, \Theta) = \Psi_{EQ}^{vol}(J, \Theta) + \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) + T(\Theta) \tag{33}$$

where

$$\begin{aligned} \Psi_{EQ}^{vol}(J, \Theta) &: \text{ convex volumetric response,} \\ \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) &: \text{ poly-convex deviatoric response,} \\ T(\Theta) &: \text{ purely thermal contribution.} \end{aligned}$$

which can be recast into:

$$\boxed{\Psi_{EQ}(\mathbf{C}, \Theta) = \zeta \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta)} \tag{34}$$

where $\zeta \in (0, \infty)$.

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where $\zeta \in (0, \infty)$.
Next step: Rewrite $\Psi_{EQ}(\mathbf{C}, \Theta) \rightarrow \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta)$ for $\Psi, \mathbf{S}, \eta, \mathbf{Q}$ and $\boldsymbol{\alpha}_i$.

Summary and conclusion

Summary 1/3

(i) Helmholtz free energy function:

$$\Psi = \Psi_{EQ}(\mathbf{C}, \Theta) + \Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i) \tag{35}$$

$$\Psi_{NEQ}(\mathbf{C}, \Theta, \boldsymbol{\alpha}_i) = \sum_{i=1}^{n} \left[E_i \left|\boldsymbol{\alpha}_i\right|^2 - 2\partial_{\mathbf{C}} \Psi^{\Gamma}_{EQ}(\mathbf{C}, \Theta) : \boldsymbol{\alpha}_i + \Psi^{\Gamma}_{EQ}(\mathbf{C}, \Theta) \right] \tag{36}$$

$$\Psi_{EQ}(\mathbf{C}, \Theta) = \Psi^{vol}_{EQ}(J, \Theta) + \Psi^{iso}_{EQ}(\tilde{\mathbf{C}}, \Theta) + T(\Theta) \tag{37}$$

and

$$\Psi^{\Gamma}_{EQ}(\mathbf{C}, \Theta) = \gamma_i \Psi_{EQ}(\mathbf{C}, \Theta) \equiv \gamma_i \zeta \Psi^{iso}_{EQ}(\tilde{\mathbf{C}}, \Theta) \equiv \beta \Psi^{iso}_{EQ}(\tilde{\mathbf{C}}, \Theta) \tag{38}$$

where free energy factor $\beta = \gamma_i \zeta$ and $\beta \in (0, \infty)$.

Summary 2/3

(ii) Second Piola-Kirchhoff stress:

$$
\begin{aligned}
S = {} & 2\partial_{\mathbf{C}} \Psi_{EQ}(\mathbf{C}, \Theta) \\
& + \sum_{i=1}^{m} \beta \left\{ J^{\frac{-2}{3}} \, Dev \left[2\partial_{\tilde{\mathbf{C}}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) \right] - 2\partial_{\mathbf{C}} \left\{ J^{\frac{-2}{3}} \, Dev \left[2\partial_{\tilde{\mathbf{C}}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) \right] \right\} \right\}
\end{aligned}
\tag{39}
$$

Summary 2/3

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$$S = 2\partial_{\mathbf{C}} \Psi_{EQ}(\mathbf{C}, \Theta) + \sum_{i=1}^{m} \beta \left\{ J^{\frac{-2}{3}} Dev \left[2\partial_{\tilde{\mathbf{C}}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) \right] - 2\partial_{\mathbf{C}} \left\{ J^{\frac{-2}{3}} Dev \left[2\partial_{\tilde{\mathbf{C}}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) \right] \right\} \right\} \tag{39}$$

(iii) Entropy:

$$\eta = -\partial_{\Theta} \Psi_{EQ}(\mathbf{C}, \Theta) + \sum_{i=1}^{n} \left[\beta \left(-\partial_{\Theta} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) + 2\partial_{\mathbf{C}\Theta} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) : \boldsymbol{\alpha}_i \right) - \partial_{\Theta} E_i \left| \boldsymbol{\alpha}_i \right|^2 \right] \tag{40}$$

Summary 3/3

$\boxed{\text{(iv) Dissipation:}}$

$$\mathcal{D}_{loc}^{red} = \sum_{i=1}^{n} \frac{1}{\tilde{\eta}_i^*} \left| \tilde{\mathbf{Q}}^i \right|^2 \geq 0 \tag{41}$$

with
$$\begin{aligned} \tilde{\mathbf{Q}}^i &= e^{\frac{-t}{\tilde{\tau}_i}} \tilde{\mathbf{Q}}_0^i \\ &+ \int_{0^+}^{t} e^{\frac{-(t-s)}{\tilde{\tau}_i}} \left\{ \frac{d}{ds} [2\partial_{\mathbf{C}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta)] - 2\partial_{\Theta} E_i \boldsymbol{\alpha}_i \frac{d\Theta}{ds} \right\} ds \end{aligned} \tag{42}$$

and
$$\boldsymbol{\alpha}_i = \frac{1}{E_i} \beta_i J^{\frac{-2}{3}} Dev[\partial_{\tilde{\mathbf{C}}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta)] - \frac{1}{2E_i} \tilde{\mathbf{Q}}^i \tag{43}$$

(iv) Dissipation:

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with
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$$\boldsymbol{\alpha}_i = \frac{1}{E_i} \beta_i J^{\frac{-2}{3}} Dev[\partial_{\tilde{\mathbf{C}}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta)] - \frac{1}{2E_i} \tilde{\mathbf{Q}}^i \tag{43}$$

(v) Evolution equation:

$$\dot{\tilde{\mathbf{Q}}}^i + \frac{\tilde{\mathbf{Q}}^i}{\tilde{\tau}_i} = \beta_i \frac{d}{dt} \left\{ J^{\frac{-2}{3}} Dev \left[2\partial_{\tilde{\mathbf{C}}} \Psi_{EQ}^{iso}(\tilde{\mathbf{C}}, \Theta) \right] \right\} - 2\partial_{\Theta} E_i \boldsymbol{\alpha}_i \frac{d\Theta}{dt} \tag{44}$$

$$\tilde{\mathbf{Q}}^i|_{t=0} = \tilde{\mathbf{Q}}_0^i \tag{45}$$

Conclusion

Goals:

- To develop a constitutive model compatible with 2nd law of thermodynamics and to suit FEM numerical treatment. $\rightarrow$ Satisfied
- 3D frame for viscoelastic constitutive model at finite thermomechanical changes. $\rightarrow$ Satisfied

Remarks:

- Free energy function is decomposed into purely thermoelastic part and non-equilibrium part. $\rightarrow$ Applied
- Volumetric/Deviatoric multiplicative decomposition on deformation gradient. $\rightarrow$ Applied

Suggested future work:

- A numerical realization/implementation.

Questions

Backup slide 1: Dilatational part $J^{1/3}$.I

Stretches λ_i $(i = 1, 2, 3)$ of a 3D-body in 3 directions x, y, z are:

$$\lambda_1 = \frac{|d\mathbf{x}|}{|d\mathbf{X}|} \tag{46}$$

$$\lambda_2 = \frac{|d\mathbf{y}|}{|d\mathbf{Y}|} \tag{47}$$

$$\lambda_3 = \frac{|d\mathbf{z}|}{|d\mathbf{Z}|} \tag{48}$$

$$\Rightarrow \lambda_1\lambda_2\lambda_3 = \frac{|d\mathbf{x}|}{|d\mathbf{X}|}\frac{|d\mathbf{y}|}{|d\mathbf{Y}|}\frac{|d\mathbf{z}|}{|d\mathbf{Z}|} \equiv \frac{dv}{dV} \equiv det(\mathbf{F}) \equiv J \tag{49}$$

Consideration of **isotropy**, we have $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$, hence:

$$\lambda^3 = J \tag{50}$$

$$\therefore \boxed{\lambda = J^{1/3}} \tag{51}$$

Backup slide 2: Why $\partial_\varepsilon \Psi(\varepsilon, \Theta) = 2\partial_{\mathbf{C}} \Psi(\mathbf{C}, \Theta)$?

$$\boxed{\partial_{\boldsymbol{\varepsilon}} \Psi(\boldsymbol{\varepsilon}, \Theta) = 2\partial_{\mathbf{C}} \Psi(\mathbf{C}, \Theta)} \tag{52}$$

Example:

$$\Psi(e) = 5e + 1 \tag{53}$$

$$\therefore \partial_e \Psi(e) = 5 \tag{54}$$

$$\because e = \frac{1}{2}(c - 1) \tag{55}$$

$$\Rightarrow \Psi(c) = \frac{5}{2}c - \frac{3}{2} \tag{56}$$

$$\therefore \partial_c \Psi(c) = \frac{5}{2} \tag{57}$$

Comparison equations (54) and (57) yields:

$$\partial_e \Psi = 2\partial_c \Psi \tag{58}$$

$$\therefore \boxed{\partial_{\boldsymbol{\varepsilon}} \Psi(\boldsymbol{\varepsilon}, \Theta) = 2\partial_{\mathbf{C}} \Psi(\mathbf{C}, \Theta)} \tag{59}$$

Backup slide 3: Driving forces $\mathbf{Q}^i$ in viscoelastic theory

Constitutive expression reads:

$$\boxed{\begin{aligned} \sigma &= \partial_\varepsilon \hat{\Psi} = E_0\varepsilon + \sum_{i=1}^{n} E_i(\varepsilon - \alpha_i) && (60) \\ Q_i &= -\partial_{\alpha_i}\hat{\Psi} = E_i(\varepsilon - \alpha_i) && (61) \\ \dot{\alpha}_i &= \frac{1}{\eta_i} Q_i && (62) \end{aligned}}$$

Consider the evolution equation, i.e. time derivative of (61):

$$\dot{Q}_i = E_i(\dot{\varepsilon} - \dot{\alpha}_i) \tag{63}$$

Insertion (62) to (63) yield:

$$\therefore \boxed{\dot{Q}_i + \frac{1}{\tau_i} Q_i = E_i \dot{\varepsilon}} \tag{64}$$