

Self-exercise - SRU00

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1 Step-by-step with A1-SRU00

A function which takes its argument as a function is called functional.

$$\boxed{\text{functional} = \text{function of function}} \quad (1)$$

Example 1. *Examine the first variation of the functional F in v direction*

$$F : \begin{cases} C^2([-1, 1]) \rightarrow \mathbb{R}, \\ y \mapsto F(y) := \int_{-1}^1 \left(y(x)y'(x) + e^{y'(x)^2} \right) dx, \end{cases} \quad \text{with } y : \begin{cases} [-1, 1] \rightarrow \mathbb{R}, \\ x \mapsto y(x), \end{cases} \quad (2)$$

where C^2 is called space of continuous functions, i.e. function space. The number 2 in C^2 indicates that those functions living in C^2 are those continuously twice differentiable. Herein, y is such function living in C^2 , as shown in (2)₁. Besides, the function y takes x as its only variable. In this case, the only variable x is defined on the closed interval $[-1, 1]$, as shown in (2)₂.

Approach: By using the definition, as follows

$$\boxed{\delta F(y; v) := \lim_{s \rightarrow 0} \frac{F(y + sv) - F(y)}{s}}, \quad (3)$$

where δF is called variation of the functional F , and the symbol δ stands for variation.

Interesting fact 1. *The symbol δ seen in variation of a functional is not the same one often seen in partial derivative, which is ∂ . In \LaTeX their commands are called, either from *maths'* package such as **amsmath** or **mathtools**, as follows*

$$\begin{array}{lll} \delta & \rightarrow \backslash delta & \rightarrow \text{variation of a functional} \\ \partial & \rightarrow \backslash partial & \rightarrow \text{partial derivative of a multi-variable function} \end{array} \quad (4)$$

In order to find the variation of the functional F in v direction, we need to compute the limit, as shown in (3).

$$\begin{aligned}\delta F(y; v) &:= \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left((y + s v)(x) (y + s v)'(x) + \exp \left((y + s v)'^2(x) \right) \right) dx \quad (5)\end{aligned}$$

$$- \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y(x) y'(x) + \exp \left(y'^2(x) \right) \right) dx \quad (6)$$

Let us now forget the argument (x) in (5) and (6) for a while, i.e. by dropping it temporarily, just for a sake of a better overview. Then, the new expression without (x) will take the form

$$\begin{aligned}\delta F(y; v) &:= \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left((y + s v) (y + s v)' + \exp \left((y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + \exp \left(y'^2 \right) \right) dx \quad (7)\end{aligned}$$

which leads to further analysis, as follows

$$\begin{aligned}\delta F(y; v) &\stackrel{(7)}{=} \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left((y + s v)(y' + s v') + \exp \left((y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + s y v' + s v y' + s^2 v v' + \exp \left((y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + s y v' + s v y' + s^2 v v' - y y' \right) dx \\ &\quad + \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left((y + s v)'^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \underbrace{\left(s y v' + s v y' + s^2 v v' \right) dx}_{:=\mathcal{A}} \\ &\quad + \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \underbrace{\left(\exp \left((y + s v)'^2 \right) - \exp \left(y'^2 \right) \right) dx}_{:=\mathcal{B}} \quad (8)\end{aligned}$$

Hence, it leads to

$$\therefore \boxed{\delta F(y; v) = \mathcal{A} + \mathcal{B}.} \quad (9)$$

Next, we consider part \mathcal{A} , as follows

$$\begin{aligned} \mathcal{A} &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(s y v' + s v y' + s^2 v v' \right) dx = \lim_{s \rightarrow 0} \int_{-1}^1 \left(y v' + v y' + s v v' \right) dx \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y v' + v y' \right) dx + \lim_{s \rightarrow 0} \int_{-1}^1 (s v v') dx \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y v' + v y' \right) dx + \underbrace{\lim_{s \rightarrow 0} \left(s \int_{-1}^1 (v v') dx \right)}_{=0} \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y(x) v'(x) + v(x) y'(x) \right) dx \quad (10) \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y(x) v(x) \right)' dx = \int_{-1}^1 \frac{d}{dx} \left(y(x) v(x) \right) dx \\ &= \int_{-1}^1 \frac{d}{dx} \left(y(x) v(x) \right) dx = \left(y(x) v(x) \right) \Big|_{x=-1}^{x=1} = y(1) v(1) - y(-1) v(-1), \quad (11) \end{aligned}$$

where (x) has returned back in (10) again in order to get ready to compute the follow-up integral. Therefore, solution to \mathcal{A} yields

$$\therefore \boxed{\mathcal{A} = y(1) v(1) - y(-1) v(-1).} \quad (12)$$

Besides, the computation of part \mathcal{B} goes as follows

$$\begin{aligned} \mathcal{B} &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left((y + s v)^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left(y'^2 + 2 s y' v' + s^2 v'^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left(y'^2 \right) \exp \left(2 s y' v' + s^2 v'^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left(y'^2 \right) \left(\exp \left(2 s y' v' + s^2 v'^2 \right) - 1 \right) \right) dx \\ &= \int_{-1}^1 \left(\exp \left(y'^2 \right) \underbrace{\lim_{s \rightarrow 0} \frac{\exp \left(2 s y' v' + s^2 v'^2 \right) - 1}{s}}_{:=\mathcal{C}} \right) dx, \quad (13) \end{aligned}$$

where a further result of \mathcal{C} in (13) is computed by using l'Hôpital's rule, as follows

$$\begin{aligned}
\mathcal{C} &= \lim_{s \rightarrow 0} \frac{\exp(2 s y' v' + s^2 v'^2) - 1}{s} \\
&= \lim_{s \rightarrow 0} \left((2 y' v' + 2 s v'^2) \exp(2 s y' v' + s^2 v'^2) \right) \\
&= 2 y' v' \\
&= 2 y'(x) v'(x).
\end{aligned} \tag{14}$$

Substitution of (14) into (13) leads to

$$\therefore \quad \boxed{\mathcal{B} = 2 \int_{-1}^1 \exp(y'^2(x)) y'(x) v'(x) dx.} \tag{15}$$

Finally, by inserting (12) and (15) into (9) the final result yields

$$\therefore \quad \boxed{\delta F(y; v) = y(1) v(1) - y(-1) v(-1) + 2 \int_{-1}^1 y'(x) v'(x) \exp(y'^2(x)) dx.} \tag{16}$$

2 Step-by-step with A4-SRU00

Example 2. Examine the following initial value problem (IVP)

$$y'''(t) + y'(t) = t y(t), \quad (17)$$

with initial conditions (IC) given as follows

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2. \quad (18)$$

- i) Transform the above third order ordinary differential equation (ODE) to a system of first order ODEs.
- ii) Solve the system i) numerically by using explicit Euler method with 2 steps.

Approach:

- i) We need to transform the third order ODE given in (17) into a system of first order ODEs.

Question 1. But.. how many ODEs shall we need in total for this system of first order ODEs?

Ans: 3.

Observation 1. Since the highest order of derivative in (17) is 3, i.e. as we can see there are three ' from $y'''(t)$, the given ODE is called third order ODE. Besides, the lowest order in (17) is 0, i.e. as we can see from $y(t)$. Therefore, we shall need a system of **three** first order ODEs for the given third order ODE in (17).

Question 2. Which variable from (17) shall we need to take and define, so that we are able to transform (17) into a system of first order ODEs corresponding to the given initial conditions given in (18), e.g. is it ideal to define $y'''(t) := u_3(t)$?

Ans: $y(t)$, $y'(t)$ and $y''(t)$; and it is not ideal to go with $y'''(t) := u_3(t)$.

Observation 2. The three initial conditions given $y(0)$, $y'(0)$ and $y''(0)$, as shown in (18), give rise to the fact that we shall need to define three new variables for $y(t)$, $y'(t)$ and $y''(t)$. This choice is ideal for the ODE with the given ICs because we will be able to apply these ICs directly for the new system of three ODEs.

Therefore, we shall need three first order ODEs for the new system. Besides, the transformation is done by setting new variables for $y(t)$, $y'(t)$ and $y''(t)$.

Transformation: We now need to set $y(t)$, $y'(t)$ and $y''(t)$ to new variables, let's say $u_0(t)$, $u_1(t)$, and $u_2(t)$, respectively, as follows

$$y(t) := u_0(t), \quad (19)$$

$$y'(t) := u_1(t), \quad (20)$$

$$y''(t) := u_2(t). \quad (21)$$

Then, by taking the derivative of the new variable $u_0(t)$, $u_1(t)$ and $u_2(t)$, as newly defined in (19), (20) and (21), we obtain the following relations

$$\boxed{\begin{aligned} u'_0(t) &\stackrel{(19)}{=} y'(t) \stackrel{(20)}{=} u_1(t), \\ u'_1(t) &\stackrel{(20)}{=} y''(t) \stackrel{(21)}{=} u_2(t), \\ u'_2(t) &\stackrel{(21)}{=} y'''(t) \stackrel{(17)}{=} t y(t) - y'(t) \stackrel{(17)}{=} t u_0(t) - u_1(t), \end{aligned}} \quad (22)$$

which is summarized, together with keeping in mind about the order of the index i from $u_i(t)|_{i=0,1,2}$, as follows

$$\begin{aligned} u'_0(t) &= & u_1(t) \\ u'_1(t) &= & u_2(t) \\ u'_2(t) &= t u_0(t) - u_1(t) \end{aligned} \quad (23)$$

which gives rise to the following system

$$\begin{aligned} u'_0(t) &= 0 u_0(t) & 1 u_1(t) & 0 u_2(t) \\ u'_1(t) &= 0 u_0(t) & 0 u_1(t) & 1 u_2(t) \\ u'_2(t) &= t u_0(t) & -1 u_1(t) & 0 u_2(t) \end{aligned} \quad (24)$$

from where we can write the system in form of

$$\mathbf{u}' = A\mathbf{u} \quad (25)$$

as follows

$$\underbrace{\begin{pmatrix} u_0(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}'}_{:=\mathbf{u}'} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{:=A} \underbrace{\begin{pmatrix} u_0(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}}_{:=\mathbf{u}} \quad (26)$$

where the relation between A and \mathbf{u} in $A\mathbf{u}$ is matrix-vector multiplication. Besides, by substituting (19) (20) and (21) into (18), respectively, we obtain the initial conditions for the new system in (26)

$$\boxed{\begin{aligned} u_0(0) &= 0, \\ u_1(0) &= 1, \\ u_2(0) &= 2, \end{aligned}} \quad (27)$$

which form an equivalent system of three first-order ODEs.

- ii) We need to solve the system of first order ODEs obtained in i) numerically by using explicit Euler method.

Summary 1. *Given the system of first order ODEs as follows*

$$\mathbf{u}'(t) = A\mathbf{u}(t) \quad (28)$$

where the vector \mathbf{u} is defined as $\mathbf{u} := (u_0, u_1, u_2)^\top$, and the matrix A is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix} \quad (29)$$

with initial conditions

$$u_0(0) = 0, \quad u_1(0) = 1, \quad u_2(0) = 2. \quad (30)$$

Solve this system numerically by using the explicit Euler method.

Approach: We need to approximate $\mathbf{u}'(t)$ arising in (28) by some numerical schemes, e.g. explicit Euler method.

Observation 3. *We consider Taylor expansion for $\mathbf{u}(t)$ around point $t = t_0$*

$$\mathbf{u}(t) = \mathbf{u}(t_0) + \mathbf{u}'(t_0)(t - t_0) + \mathcal{O}((t - t_0)^2), \quad (31)$$

which, for the sake of convergence, requires $h := t - t_0$ to be small. Since $h := t - t_0$, we equally have $t = h + t_0$. Substitution of $h := t - t_0$ and $t = t_0 + h$ to (31) leads to the following expression

$$\mathbf{u}(t_0 + h) = \mathbf{u}(t_0) + h\mathbf{u}'(t_0) + \mathcal{O}(h^2), \quad (32)$$

which is the new expression of Taylor expansion in terms of point t_0 , around which the function $\mathbf{u}(t)$ is approximated by the Taylor expansion, instead of the classical expression in terms of the variable t as seen in (31). Since point t_0 , although it implied a fixed point, arising in (32) can be chosen arbitrarily, we can, therefore, substitute it for t , as follows

$$\mathbf{u}(t + h) = \mathbf{u}(t) + h\mathbf{u}'(t) + \mathcal{O}(h^2), \quad (33)$$

Now, the expression (32) gives rise to a useful expression to approximate $\mathbf{u}'(t)$ numerically, as follows

$$\therefore \boxed{\mathbf{u}'(t) \approx \frac{\mathbf{u}(t + h) - \mathbf{u}(t)}{h}}. \quad (34)$$

Substitution of (34) into (28) yields

$$\frac{\mathbf{u}(t + h) - \mathbf{u}(t)}{h} = A\mathbf{u}(t) \quad (35)$$

which leads to the formula of explicit Euler method

$$\boxed{\mathbf{u}^{(j+1)} = \mathbf{u}^{(j)} + h A \mathbf{u}^{(j)},} \quad (36)$$

where $\mathbf{u}^{(j)}$ is the numerical solution by Euler method after j steps, and components of $\mathbf{u}^{(j)}$ is known as

$$\mathbf{u}^{(j)} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}^{(j)}, \quad (37)$$

which, together with (36), leads to the following expressions

$$\begin{aligned} \mathbf{u}^{(0)} &= \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \\ \mathbf{u}^{(1)} &= \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} + h \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + h \begin{pmatrix} 1 \\ 2 \\ 0 - 1 \end{pmatrix} \\ \therefore &= \begin{pmatrix} h \\ 1 + 2h \\ 2 - h \end{pmatrix}, \\ \mathbf{u}^{(2)} &= \begin{pmatrix} u_0^{(1)} \\ u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} + h \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} u_0^{(1)} \\ u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} h \\ 1 + 2h \\ 2 - h \end{pmatrix} + h \begin{pmatrix} 1 + 2h \\ 2 - h \\ h \cdot h - (1 + 2h) \end{pmatrix} \\ &= \begin{pmatrix} 2h(1 + h) \\ 1 + 4h - h^2 \\ 2 - 2h - 2h^2 + h^3 \end{pmatrix}. \end{aligned} \quad (38)$$