Self-exercise - SRU05
Response to questions

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1 Hint for A1-HA05

Example 1. *Given the set $\mathcal{M} \in \mathbb{R}^2$ bounded by three points A, B, and C, as shown in Figure 1.*

1. *Examine the integral of a function $f(x,y)$ over set $\mathcal{M}$.*
2. *Compute the area of $\mathcal{M}$ by setting $f(x,y) = 1$.*

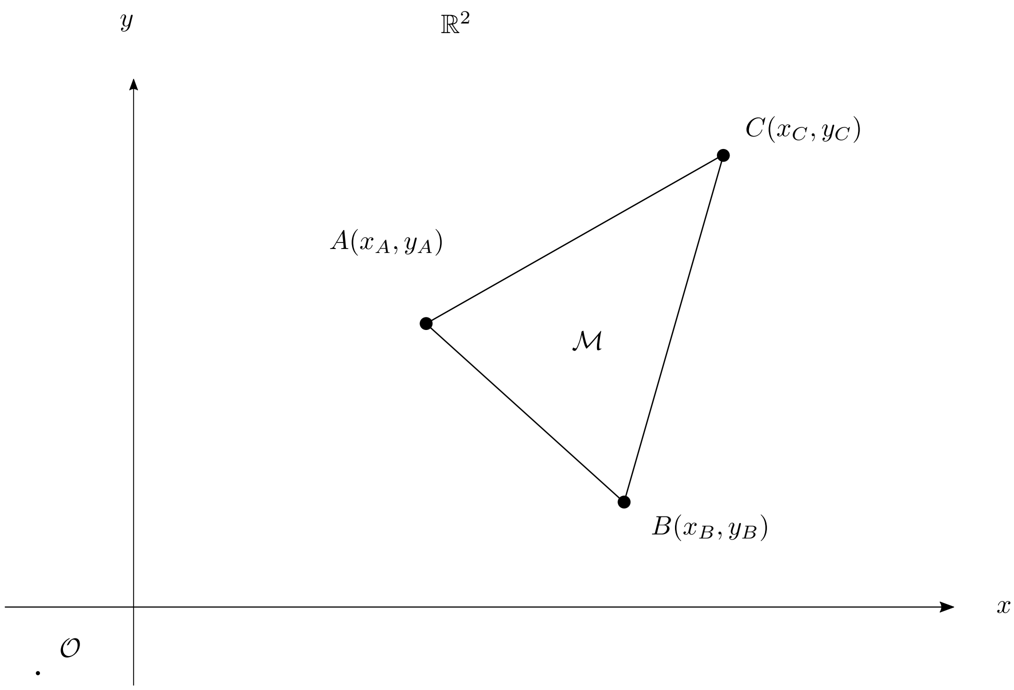

Figure 1: Set $M \in \mathbb{R}^2$.

Approach:
We will look at the Figure 1 and consider it in two scenarios as follows

1. Scenario 1: Figure 2,
2. Scenario 2: Figure 3,

which we will see later on that the two approaches will actually lead to the same solution, thanks to Fubini's theorem [1].

Recall 1. *Equation of a line* (l) *in general reads*

$$(l): \quad y = a\,x + b, \tag{1}$$

where a *is the slope of the line, and* b *is the point where the line* (l) *crosses over the vertical line of the axis. Hence, when this line* (l) *going through point* $P(x_P, y_P)$ *and point* $Q(x_Q, y_Q)$*, it leads to the following two relations*

$$y_P = a\,x_P + b, \tag{2}$$
$$y_Q = a\,x_Q + b, \tag{3}$$

which leads to the slope a *and vertical point* b *as follows*

$$a = \frac{y_P - y_Q}{x_P - x_Q}, \tag{4}$$
$$b = \frac{-y_P\,x_Q + y_Q\,x_P}{x_P - x_Q}. \tag{5}$$

Finally, equation of a line (l) *passing through point* $P(x_P, y_P)$ *and point* $Q(x_Q, y_Q)$ *in general takes the form*

$$\therefore \quad \boxed{y = \frac{y_P - y_Q}{x_P - x_Q}\,x + \frac{-y_P\,x_Q + y_Q\,x_P}{x_P - x_Q}.} \tag{6}$$

By taking advantage of the formula (6) we are ready to obtain the following three line functions for (AB), (AC), and (BC) as follows

$$\boxed{\begin{aligned} (AB): &\quad y = -x + 3, \\ (AC): &\quad y = \frac{1}{2}x + \frac{3}{2}, \\ (BC): &\quad y = 2\,x - 3, \end{aligned}} \tag{7}$$

which we, alternatively, we can also write those expressions in terms of x, as follows

$$\boxed{\begin{aligned} (AB): &\quad x = -y + 3, \\ (AC): &\quad x = 2\,y - 3, \\ (BC): &\quad x = \frac{1}{2}y + \frac{3}{2}. \end{aligned}} \tag{8}$$

Note in passing that we have used point $A(1, 2)$, $B(2, 1)$, and $C(3, 3)$.

1. Scenario 1: Figure 2

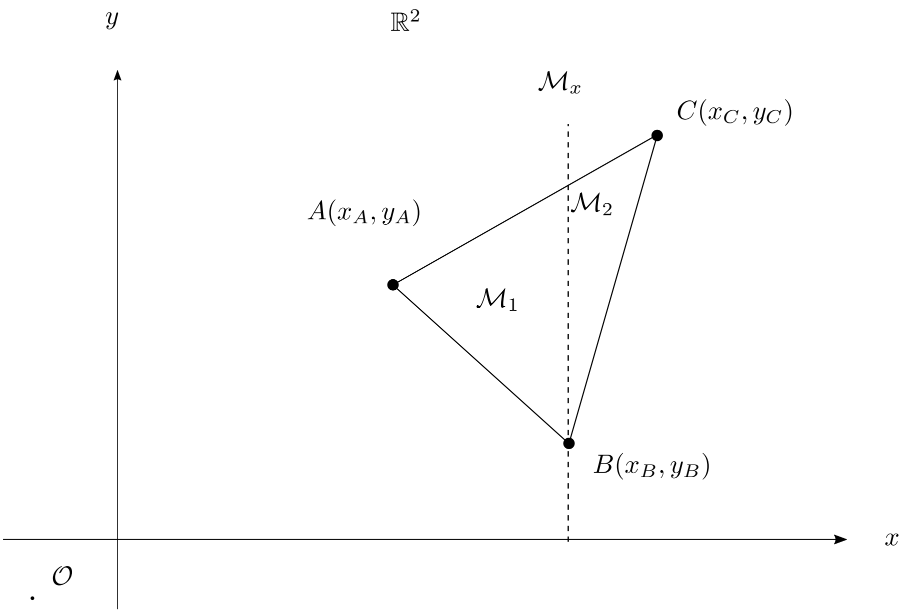

Figure 2: Set $\mathcal{M} \in \mathbb{R}^2$ where $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ and split by $\mathcal{M}_x$.

By following the $\mathcal{M}_x$ split and using the box (7) we obtain the set $\mathcal{M}_1$

$$\mathcal{M}_1 = \left\{ y \in \mathbb{R} \,\middle|\, -x + 3 \leq y \leq \frac{1}{2}x + \frac{3}{2} \right\}, \tag{9}$$

which is due to the fact that while following the y-axis we first come across the line (AB) and then the line (AC). Similarly, the set $\mathcal{M}_2$ is obtained as follows

$$\mathcal{M}_2 = \left\{ y \in \mathbb{R} \,\middle|\, 2x - 3 \leq y \leq \frac{1}{2}x + \frac{3}{2} \right\}, \tag{10}$$

which, again, is due the fact that we first face to the line (BC) and then (AC) while following the y-axis. Therefore, we obtain

$$\begin{aligned} \int_{\mathcal{M}} f(x,y)\, d\lambda_2 &= \int_{\mathcal{M}_1} f(x,y)\, d\lambda_2 + \int_{\mathcal{M}_2} f(x,y)\, d\lambda_2 \\ &= \int_{\mathbb{R}} (\int_{(\mathcal{M}_1)_x} f(x,y)\, dy)\, dx + \int_{\mathbb{R}} (\int_{(\mathcal{M}_2)_x} f(x,y)\, dy)\, dx \\ &= \int_{1=x_A}^{2=x_B} (\int_{-x+3}^{x/2+3/2} f(x,y)\, dy)\, dx \\ &\qquad + \int_{2=x_B}^{3=x_C} (\int_{2x-3}^{x/2+3/2} f(x,y)\, dy)\, dx \end{aligned} \tag{11}$$

which will results in $3/2$ when $f(x,y) = 1$.

2. Scenario 2: Figure 3

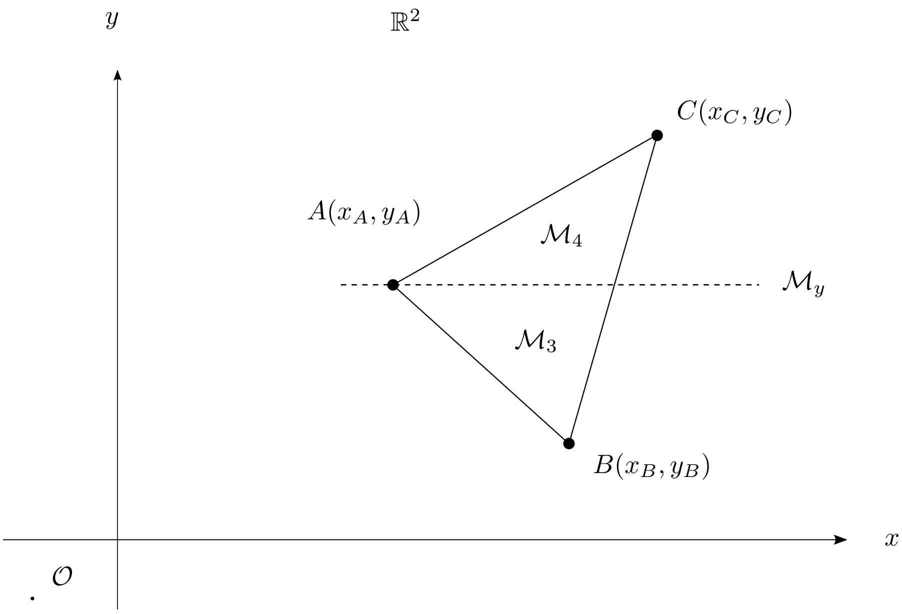

Figure 3: Set $\mathcal{M} \in \mathbb{R}^2$ where $\mathcal{M} = \mathcal{M}_3 \cup \mathcal{M}_4$ and split by $\mathcal{M}_y$.

By following the $\mathcal{M}_y$ split and using the box (8) we obtain the set $\mathcal{M}_3$

$$\mathcal{M}_3 = \Big\{ x \in \mathbb{R} \,\Big|\, -y + 3 \leq x \leq \frac{1}{2}y + \frac{3}{2} \Big\}, \tag{12}$$

which is due to the fact that while following the x-axis we first come across the line (AB) and then the line (BC). Similarly, the set $\mathcal{M}_4$ is obtained as follows

$$\mathcal{M}_4 = \Big\{ x \in \mathbb{R} \,\Big|\, 2y - 3 \leq x \leq \frac{1}{2}y + \frac{3}{2} \Big\}, \tag{13}$$

which, again, is due the fact that we first face to the line (AC) and then (BC) while following the x-axis. Therefore, we obtain

$$\begin{aligned}
\int_{\mathcal{M}} f(x,y)\, d\lambda_2 &= \int_{\mathcal{M}_3} f(x,y)\, d\lambda_2 + \int_{\mathcal{M}_4} f(x,y)\, d\lambda_2 \\
&= \int_{\mathbb{R}} \Big(\int_{(\mathcal{M}_3)_y} f(x,y)\, dx \Big)\, dy + \int_{\mathbb{R}} \Big(\int_{(\mathcal{M}_4)_y} f(x,y)\, dx \Big)\, dy \\
&= \int_{1=y_B}^{2=y_A} \Big(\int_{-y+3}^{y/2+3/2} f(x,y)\, dx \Big)\, dy \\
&\qquad + \int_{2=y_A}^{3=y_C} \Big(\int_{2y-3}^{y/2+3/2} f(x,y)\, dx \Big)\, dy
\end{aligned} \tag{14}$$

which will results in $3/2$ when $f(x,y) = 1$.

References

[1] Fubini's theorem. https://en.wikipedia.org/wiki/Fubini%27s_theorems. Accessed: 2022-11-19.