Global exercise - GUE09

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Content covered:

- ✓ Analysis:
  - (i) Step-by-step with Line integral of scalar field
  - (ii) Step-by-step with Line integral of vector field
- ✓ Numerics: Demo programming exercise 03 (PRU02)

Recall 1. *Line integral of a scalar field $\phi : \Omega \to \mathbb{R}$ is defined as follows*

$$\int_\Gamma \phi \, ds := \int_a^b \phi(\gamma(t)) \big\| \gamma'(t) \big\| \, dt \tag{1}$$

Recall 2. *Line integral of a vector field $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is defined as follows*

$$\int_\Gamma \boldsymbol{f} \cdot d\boldsymbol{x} := \int_a^b \big\langle \boldsymbol{f}(\gamma(t)), \gamma'(t) \big\rangle \, dt \tag{2}$$

1 Step-by-step with Line integral of scalar field

Example 1. *Examine the following trajectory* $\Gamma = \gamma([0, 2\pi])$

$$\gamma : \begin{cases} [0, 2\pi] \to \mathbb{R}^3, \\ (t) \mapsto \gamma(t) := \begin{pmatrix} \cos(t) \\ \sin(t) \\ ht \end{pmatrix}. \end{cases} \tag{3}$$

The curve γ *is known as the helix. We now would like to compute the total mass* $\mathcal{M}$ *of the helix whose density function* ρ *defined as follows*

$$\rho : \begin{cases} \Gamma \to \mathbb{R}, \\ (x, y, z) \mapsto \rho(x, y, z) := z. \end{cases} \tag{4}$$

Approach: By using recall 1.

Observation 1. *Since the parametrized curve* γ *from the given trajectory or path* Γ *is already known, we, therefore, do not have to seek any further parametrized curve. However, in case this information is not yet already given, it is necessary to find such parametrized curve, i.e. the section 2 is such an example.*

The mass of the helix is computed as follows

$$\therefore \quad \boxed{\begin{aligned} \mathcal{M}^{\text{helix}} &= \int_0^{2\pi} \rho \left\|\gamma'(t)\right\|_2 dt \\ &= \int_0^{2\pi} ht \sqrt{\sin^2 t + \cos^2 t + h^2}\, dt \\ &= \int_0^{2\pi} h\sqrt{1 + h^2}\, t\, dt \\ &= h\sqrt{1 + h^2}\, \frac{t^2}{2}\Big|_0^{2\pi} \\ &= 2\pi^2 h\sqrt{1 + h^2}. \end{aligned}}$$

2 Step-by-step with Line integral of vector field

Example 2. *Examine the following trajectory* Γ_1 *as shown in Figure 1 from the origin point* $(0,0)$ *to point* $(1,1)$*. Besides, the vector field* f *is given as follows*

$$\boldsymbol{f}: \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2, \\ (x,y) \mapsto \boldsymbol{f}(x,y) = \begin{pmatrix} xe^y \\ \sin(x) + y \end{pmatrix}, \end{cases}$$

We would like now to compute the line integral of the vector field $\boldsymbol{f}$*, i.e. Work integral/Arbeitsintegral*

$$\int_{\Gamma_1} \boldsymbol{f} \cdot d\boldsymbol{x}. \tag{5}$$

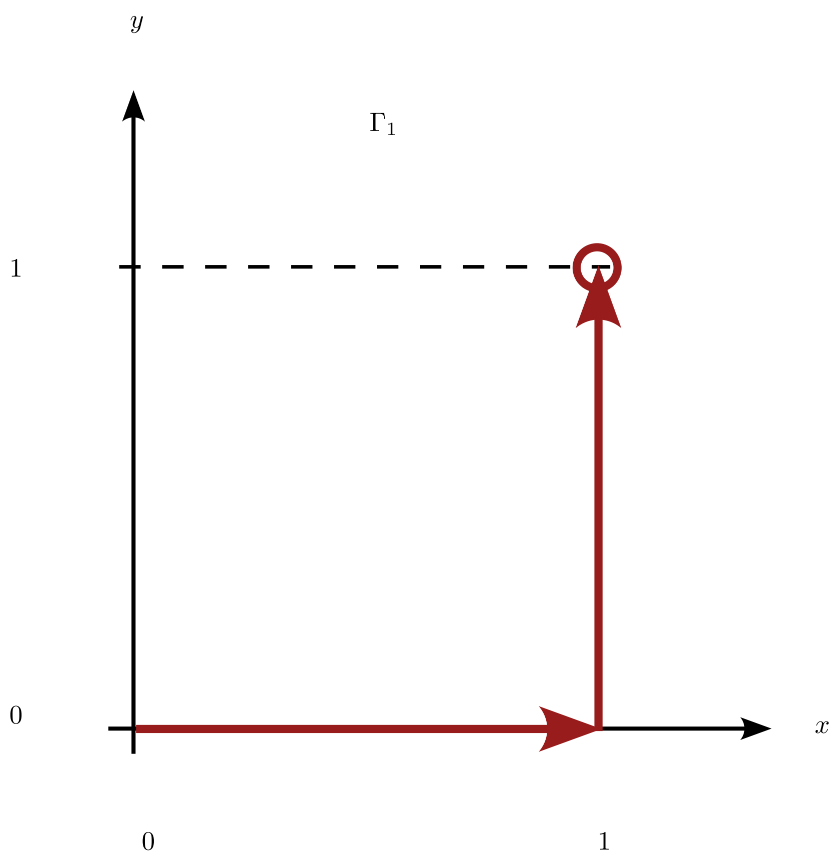

Figure 1: Discretization of domain Ω with n points.

Approach: By using recall 2.

Observation 2. *There is no parametrized curve for the trajectory Γ_1. Therefore, we shall need to find a parametrized curve for the path Γ_1. Since the path Γ_1 is a union of two smaller paths l_1 and l_2, we obtain the following relation*

$$\boxed{\Gamma_1 := \mathtt{Image}(l_1) \cup \mathtt{Image}(l_2)}$$

where the parametrized l_1 and l_2, and their gradients take the forms

$$l_1(t) := \begin{pmatrix} t \\ 0 \end{pmatrix} \text{ for } t \in [0,1] \Rightarrow l_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$l_2(t) := \begin{pmatrix} 1 \\ t \end{pmatrix} \text{ for } t \in [0,1] \Rightarrow l_2' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Substitution of $l_1(t)$ and $l_2(t)$ into $\boldsymbol{f}$ leads to the following expressions

$$\boldsymbol{f}(l_1(t)) = \begin{pmatrix} te^0 \\ \sin(t) + 0 \end{pmatrix} = \begin{pmatrix} t \\ \sin(t) \end{pmatrix},$$
$$\boldsymbol{f}(l_2(t)) = \begin{pmatrix} 1e^t \\ \sin(1) + t \end{pmatrix} = \begin{pmatrix} e^t \\ \sin(1) + t \end{pmatrix}.$$

The line integral of the vector field is computed as follows

$$\boxed{\begin{aligned} \int_{\Gamma_1} \boldsymbol{f} \cdot d\boldsymbol{x} &= \int_{l_1} \boldsymbol{f} \cdot d\boldsymbol{x} + \int_{l_2} \boldsymbol{f} \cdot d\boldsymbol{x} \\ &= \int_0^1 \Big\langle f(l_1(t)), l_1'(t) \Big\rangle \, dt + \int_0^1 \Big\langle f(l_2(t)), l_2'(t) \Big\rangle \, dt \\ &= \int_0^1 t \, dt + \int_0^1 \big(\sin(1) + t\big) \, dt \\ &= \sin(1) + \int_0^1 2t \, dt \\ &= \sin(1) + \Big[t^2\Big]_0^1 \end{aligned}}$$

Therefore, we obtain

$$\therefore \quad \boxed{\int_{\Gamma_1} \boldsymbol{f} \cdot d\boldsymbol{x} = \sin(1) + 1.}$$