Self-exercise - SRU01
Response to questions

Tuan Vo

23th October, 2022

1 Step-by-step with A1a-SRU01

Example 1. *Examine the first variation of the functional F in v direction*

$$F : \begin{cases} D \to \mathbb{R}, \\ y \mapsto F(y) := \dfrac{1}{2} \displaystyle\int_{-1}^{1} \left(x\, y'(x)\right)^2 dx, \end{cases} \tag{1}$$

where

$$y \in D = \left\{ \omega \in C^2([-1,1]) : \omega(-1) = \omega(1) = 1 \right\}, \tag{2}$$
$$v \in D_0 = \left\{ \omega \in C^2([-1,1]) : \omega(-1) = \omega(1) = 0 \right\}. \tag{3}$$

Approach: By using the definition, as follows

$$\boxed{\delta F(y; v) := \lim_{t \to 0} \frac{F(y + tv) - F(y)}{t}.} \tag{4}$$

In order to find the variation of the functional F, which fulfills (1) and (2), in v direction satisfying (2), we need to compute the limit, by definiton, as shown in (4). First, we shall write down the corresponding term for $F(y + t\,v)$ based on the functional $F(y)$ given in (1), which means

$$F(y + t\,v) = \int_{-1}^{1} \Big(x\,(y + t\,v)'(x)\Big)^2 dx, \tag{5}$$

then by substituting both (5) and (1) into (4) we obtain

$$\begin{aligned} \delta F(y; v) &:= \lim_{t \to 0} \frac{F(y + t\,v) - F(y)}{t} \\ &= \lim_{t \to 0} \frac{1}{t} \left(\frac{1}{2} \int_{-1}^{1} \Big(x\,(y + t\,v)'(x)\Big)^2 dx - \frac{1}{2} \int_{-1}^{1} \Big(x\,y'(x)\Big)^2 dx \right) \\ &= \lim_{t \to 0} \frac{1}{2t} \left(\int_{-1}^{1} \Big(x\,(y(x) + t\,v(x))'\Big)^2 dx - \int_{-1}^{1} \Big(x\,y'(x)\Big)^2 dx \right), \end{aligned} \tag{6}$$

where we will now compute those terms within the parenthesis, as follows

$$\begin{aligned}
\delta F(y;v) &= \lim_{t\to 0}\frac{1}{2t}\left(\int_{-1}^{1}\Big(x\,(y'(x)+t\,v'(x))\Big)^2\,dx-\int_{-1}^{1}\Big(x\,y'(x)\Big)^2\,dx\right)\\
&= \lim_{t\to 0}\frac{1}{2t}\left(\int_{-1}^{1}x^2\,\Big(y'(x)+t\,v'(x)\Big)^2\,dx-\int_{-1}^{1}x^2\,(y'(x))^2\,dx\right)\\
&= \lim_{t\to 0}\frac{1}{2t}\int_{-1}^{1}x^2\,\Big((y'(x))^2+2\,y'(x)\,t\,v'(x)+t^2\,(v'(x))^2\Big)\,dx-\int_{-1}^{1}x^2\,(y'(x))^2\,dx\\
&= \lim_{t\to 0}\frac{1}{2t}\int_{-1}^{1}\Big(\cancel{x^2\,(y'(x))^2}+2\,x^2\,y'(x)\,t\,v'(x)+x^2\,t^2\,(v'(x))^2-\cancel{x^2\,(y'(x))^2}\Big)\,dx\\
&= \lim_{t\to 0}\frac{1}{2t}\int_{-1}^{1}\Big(2\,x^2\,y'(x)\,t\,v'(x)+x^2\,t^2\,(v'(x))^2\Big)\,dx. \qquad (7)
\end{aligned}$$

Next, the expression in (7) is computed further, by first computing terms within the integrand going with t and then splitting into two terms going with the limit, as follows

$$\begin{aligned}
\delta F(y;v) &\overset{(7)}{=} \lim_{t\to 0}\frac{1}{2t}\int_{-1}^{1}\Big(2\,x^2\,y'(x)\,t\,v'(x)+x^2\,t^2\,(v'(x))^2\Big)\,dx\\
&= \lim_{t\to 0}\int_{-1}^{1}\frac{2\,x^2\,y'(x)\,t\,v'(x)+x^2\,t^2\,(v'(x))^2}{2t}\,dx\\
&= \lim_{t\to 0}\int_{-1}^{1}\Big(x^2\,y'(x)\,v'(x)+\frac{1}{2}x^2\,t\,(v'(x))^2\Big)\,dx\\
&= \lim_{t\to 0}\int_{-1}^{1}x^2\,y'(x)\,v'(x)\,dx+\lim_{t\to 0}\frac{1}{2}\int_{-1}^{1}x^2\,t\,(v'(x))^2\,dx\\
&= \int_{-1}^{1}x^2\,y'(x)\,v'(x)\,dx+\underbrace{\lim_{t\to 0}\frac{t}{2}\int_{-1}^{1}x^2\,(v'(x))^2\,dx}_{=0}\,. \qquad (8)
\end{aligned}$$

Note in passing that the term t can be brought out of the integral arising in (8) because t is just a constant for this integral. It is also worthy to note that bringing a limit from outside to inside an integral is critical as the final result is not always the same. Herein, for the limit showing in (8) we have been not doing so, but rather bringing the t out of the integral. Finally, the first variation of functional F is processed further as follows

$$\delta F(y;v) \overset{(8)}{=} \int_{-1}^{1}x^2\,y'(x)\,v'(x)\,dx. \qquad (9)$$

We will compute (9) by taking into consideration the **integration by parts**, as follows

$$\boxed{\int a'b = ab - \int ab'} \tag{10}$$

where we will assign

$$\begin{cases} a := v(x), \\ b := x^2\, y'(x), \end{cases} \tag{11}$$

which leads to

$$\begin{aligned}
\delta F(y; v) &\overset{(9)}{=} \int_{-1}^{1} \underbrace{v'(x)}_{a'}\, \underbrace{x^2\, y'(x)}_{b}\, dx \\
&\overset{(11)}{\underset{(10)}{=}} \left(\underbrace{v(x)}_{a}\, \underbrace{x^2\, y'(x)}_{b} \right)\Bigg|_{x=-1}^{x=1} - \int_{-1}^{1} \underbrace{v(x)}_{a}\, \underbrace{\frac{d(x^2\, y'(x))}{dx}}_{b'}\, dx \\
&= \underbrace{v(1)}_{=0}\,(1)^2\, \underbrace{y'(1)}_{=1} - \underbrace{v(-1)}_{=0}\,(-1)^2\, \underbrace{y'(-1)}_{=1} - \int_{-1}^{1} v(x) \frac{d(x^2\, y'(x))}{dx}\, dx \qquad (12) \\
&= -\int_{-1}^{1} v(x) \frac{d(x^2\, y'(x))}{dx}\, dx. \qquad (13)
\end{aligned}$$

Note in passing that the definition in (2) and (3) yield the following equalities

$$y(1) = y(-1) = 1, \tag{14}$$
$$v(1) = v(-1) = 0, \tag{15}$$

which we have used in (12). Therefore, we obtain the final result

$$\therefore \quad \boxed{\delta F(y; v) = -\int_{-1}^{1} v(x) \frac{d(x^2\, y'(x))}{dx}\, dx, \quad \forall\, v(x) \in D_0,\ y(x) \in D.} \tag{16}$$