Global exercise - GUE08

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Content covered:

✓ Numerics:

1. Review: QR-decomposition by using
   (a) Givens-Rotation
   (b) Householder-Reflection
2. QR algorithm to find all eigenvalues of a matrix A.

✓ Analysis: Application of Hölder's inequality to approximate integral

1 Numerics: Review of QR-decomposition

There are two main methods used to decompose a matrix into an orthogonal matrix Q and a right upper triangular matrix R

1. Givens-Rotation: ideally for **sparse** matrices.
   → Detect non-zero entries standing below the diagonal,
   → Clean them up by applying the corresponding Givens-Rotation matrix.

2. Householder-Reflection: ideally for **dense** matrices

1.1 Step-by-step with Givens-Rotation

Example 1. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{pmatrix}_{3\times 3}. \tag{1}$$

The only entry to be cleaned up is A_{32}. Therefore, the Givens-Rotation matrix is

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2 + b^2} = \sqrt{A_{22}^2 + A_{32}^2} = \sqrt{2}, \\ c = a/r = -1/\sqrt{2}, \\ s = -b/r = -1/\sqrt{2}, \end{cases} \tag{2}$$

$$\begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{pmatrix}$$

Figure 1: The entry A_{32} of matrix A is the only one that we may clean up so as to obtain a right upper triangular matrix. Givens-Rotation matrix is G_{32}.

Then, applying G_{32} onto A from the left leads to

$$G_{32}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 2 \\ 0 & \sqrt{2} & 6/\sqrt{2} \\ 0 & 0 & -6/\sqrt{2} \end{pmatrix} =: B, \tag{3}$$

which has already a form of a right upper triangular matrix R. Therefore, we obtain

$$G_{32}A = R, \tag{4}$$

which, equally, leads to

$$G_{32}A = R \Leftrightarrow G_{32}^{-1}G_{32}A = G_{32}^{-1}R \Leftrightarrow A = G_{32}^{-1}R \Leftrightarrow A = G_{32}^{\top}R. \tag{5}$$

Note in passing that $G_{32}^{-1} = G_{32}^{\top}$ in the previous step is due to the fact that the matrix G_{32} itself is orthogonal. By assigning $Q := G_{32}^{\top}$ we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{32}^{\top}R = QR.}$$

Example 2. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix}_{3\times 3}. \tag{6}$$

The only entry to be cleaned up is A_{21}. Therefore, the Givens-Rotation matrix is

$$G_{21} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2+b^2} = \sqrt{A_{11}^2 + A_{21}^2} = \sqrt{26}, \\ c = a/r = 5/\sqrt{26}, \\ s = -b/r = -1/\sqrt{26}, \end{cases} \tag{7}$$

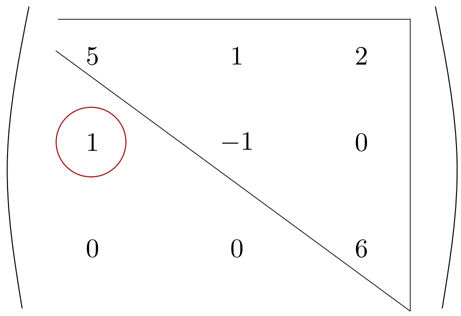

Figure 2: The entry A_{21} of matrix A is the only one that we may clean up so as to obtain a right upper triangular matrix. Givens-Rotation matrix is G_{21}.

Then, applying G_{21} onto A from the left leads to

$$G_{21}A = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{26} & 4/\sqrt{26} & 10/\sqrt{26} \\ 0 & -6/\sqrt{26} & -2/\sqrt{26} \\ 0 & 0 & 6 \end{pmatrix} \tag{8}$$

which has already a form of a right upper triangular matrix R. Therefore, we obtain

$$G_{21}A = R, \tag{9}$$

which, equally, leads to

$$G_{21}A = R \Leftrightarrow G_{21}^{-1}G_{21}A = G_{21}^{-1}R \Leftrightarrow A = G_{21}^{-1}R \Leftrightarrow A = G_{21}^{\top}R. \tag{10}$$

Note in passing that $G_{21}^{-1} = G_{21}^{\top}$ in the previous step is due to the fact that the matrix G_{21} itself is orthogonal. By assigning $Q := G_{21}^{\top}$ we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{21}^{\top}R = QR.}$$

Example 3. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 6 \end{pmatrix}_{3\times 3}. \tag{11}$$

The only entry to be cleaned up is A_{31}. Therefore, the Givens-Rotation matrix is

$$G_{31} = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2+b^2} = \sqrt{A_{11}^2 + A_{31}^2} = \sqrt{26}, \\ c = a/r = 5/\sqrt{26}, \\ s = -b/r = -1/\sqrt{26}, \end{cases} \tag{12}$$

Then, applying G_{31} onto A from the left leads to

$$G_{31}A = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & -1 & 0 \\ 0 & -1/\sqrt{26} & 28/\sqrt{26} \end{pmatrix} =: B, \tag{13}$$

which is not yet in forms of a right upper triangular matrix R. Herein, we still need to perform one more Givens-Rotation on the later matrix to clean up the entry B_{32}. The second Givens-Rotation takes the form

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r_2 = \sqrt{a_2^2+b_2^2} = \sqrt{B_{11}^2 + B_{31}^2} = 9/\sqrt{78}, \\ c_2 = a_2/r_2 = -\sqrt{78}/9, \\ s_2 = -b_2/r_2 = \sqrt{3}/9, \end{cases} \tag{14}$$

Next, applying G_{32} onto $B = G_{31}A$ leads to

$$G_{32}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & -1 & 0 \\ 0 & -1/\sqrt{26} & 28/\sqrt{26} \end{pmatrix} \tag{15}$$

which yields

$$G_{32}B = \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & 3\sqrt{3}/\sqrt{26} & -28\sqrt{3}/(9\sqrt{26}) \\ 0 & 0 & -28\sqrt{3}/9 \end{pmatrix} \tag{16}$$

which now has a form of a right upper triangular matrix R. Therefore, we obtain

$$G_{32}B = R \Leftrightarrow G_{32}G_{31}A = R \Leftrightarrow A = G_{31}^{-1}G_{32}^{-1}R \Leftrightarrow A = G_{31}^{\top}G_{32}^{\top}R. \tag{17}$$

By assigning $Q := G_{31}^{\top}G_{32}^{\top}$ we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{31}^{\top}G_{32}^{\top}R = QR.}$$

Example 4. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 6 \end{pmatrix}_{3\times 3}. \tag{18}$$

$$\begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 6 \end{pmatrix}$$

Figure 3: The entry A_{21}, A_{31}, and A_{32} of matrix A are the three entries that we would like clean up so that we may obtain a right upper triangular matrix. Givens-Rotation matrix for A_{21}, A_{31}, and A_{32} is G_{21}, G_{31}, and G_{32}, respectively. The order of applying G_{21} first or G_{31} or G_{32} is no matter, but the coefficients used to build the Givens-Rotation matrix r, a, and b have to be carefully computed, since they become various after every time applying the rotation.

The Givens-Rotation matrix G_{21} takes the form

$$G_{21} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19}$$

The Givens-Rotation matrix G_{31} takes the form

$$G_{31} = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \tag{20}$$

The Givens-Rotation matrix G_{32} takes the form

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \tag{21}$$

2 Numerics: QR-algorithm

Observation 1 (**QR-algorithm** vs. **QR-decomposition**). *QR-algorithm (as seen in Maths III) is not QR-decomposition (as seen in Maths II). QR-algorithm (Maths III) is an algorithm used to find* ***all*** *eigenvalues of a matrix A* ***numerically****. Meanwhile, QR-decomposition is a technique in linear algebra used to decompose a matrix A into an orthogonal matrix Q and a right upper triangular matrix R. Nevertheless, we will still need QR-decomposition for QR-algorithm.*

3 Analysis: Application of Hölder's inequality

Observation 2. *Sometimes we would like to estimate whether an integral, from complicated to very complicated, is bounded at a certain value or not, without actually compute it. The Hölder's inequality is a useful mathematical tool for such situation.*

Example 5. *Estimate the value of the following integral*

$$\int_{\Omega} (x+2)^{-3/5} \exp\left(-2x/3\right) dx$$

Observation 3. *The given integral is complicated to compute, but it would be easier if we may split the exponential term away from the polynomial term.*

Approach: Setting

$$f(x) := (x+2)^{-3/5} \tag{22}$$

$$g(x) := \exp\left(-2x/3\right) \tag{23}$$

Given $f(x) \in L^6(\Omega)$, then we can compute

$$\begin{aligned} \left\| f(x) \right\|_{L^6(\Omega)}^6 &= \int_0^\infty \quad |f(x)|^6 \, dx \\ &= \int_0^\infty \quad (x+2)^{-3 \cdot 6/5} \, dx \\ &= -\frac{5}{13}(x+2)^{-13/5} \bigg|_{x=0}^{x=\infty} = \frac{5}{13} \cdot 2^{-13/5} < \infty. \end{aligned} \tag{24}$$

Therefore, we obtain

$$\therefore \quad \boxed{\left\| f(x) \right\|_{L^6(\Omega)}^6 = \frac{5}{13} \cdot 2^{-13/5} < \infty.} \tag{25}$$

Besides, the following relation hold for Hölder's inequality

$$\frac{1}{6} + \frac{1}{q} = 1 \Leftrightarrow q = \frac{6}{5} \tag{26}$$

which leads to $g(x) \in L^q(\Omega)$:

$$
\begin{aligned}
\left\| g(x) \right\|_{L^{6/5}(\Omega)}^{6/5} &= \int_0^\infty |g(x)|^{6/5} \, dx \\
&= \int_0^\infty \exp\left(-\frac{2}{3} \cdot \frac{6}{5} \cdot x \right) dx \\
&= -\frac{5}{4} \exp\left(-\frac{4}{5} x \right) \Bigg|_{x=0}^{x=\infty} = \frac{5}{4} < \infty.
\end{aligned} \tag{27}
$$

Therefore, we obtain

$$
\therefore \quad \boxed{\left\| g(x) \right\|_{L^{6/5}(\Omega)}^{6/5} = \frac{5}{4} < \infty.} \tag{28}
$$

Finally, we can estimate the integral by using the Hölder's inequality

$$
\therefore \quad \boxed{
\begin{aligned}
\int_\Omega \frac{\exp(-2x/3)}{\sqrt[5]{(x+2)^3}} \, dx = \left\| f(x) \, g(x) \right\|_{L^1(\Omega)} &\leq \left\| f(x) \right\|_{L^p(\Omega)} \left\| g(x) \right\|_{L^q(\Omega)} \\
&= \left\| f(x) \right\|_{L^6(\Omega)} \left\| g(x) \right\|_{L^{6/5}(\Omega)} \\
&= \left(\frac{5 \cdot 2^{-13/5}}{13} \right)^{1/6} \left(\frac{5}{4} \right)^{5/6} \\
&< \infty.
\end{aligned}
} \tag{29}
$$

Example 6. *Estimate the value of the following integral*

$$
\int_\Omega \underbrace{\exp(-2x/3)}_{f(x)} \underbrace{(x+2)^{-4/3}}_{g(x)} \, dx
$$

Similarly, we estimate the value of the integral for $p = 3/2$ and $q = 3$ as follows

$$
\begin{aligned}
\int_\Omega \frac{\exp(-2x/3)}{\sqrt[3]{(x+2)^4}} \, dx &\leq \underbrace{\left(\int_0^\infty \left(\exp(-2x/3) \right)^{3/2} dx \right)^{2/3}}_{\|f(x)\|_{L^{3/2}(\Omega)}} \underbrace{\left(\int_0^\infty \left((x+2)^{-4/3} \right)^3 dx \right)^{1/3}}_{\|g(x)\|_{L^3(\Omega)}} \\
&= 1^{2/3} \cdot \left(\frac{1}{24} \right)^{1/3} = \frac{1}{\sqrt[3]{24}}.
\end{aligned}
$$

Therefore, the integral is easily estimated

$$
\therefore \quad \boxed{\int_\Omega \frac{\exp(-2x/3)}{\sqrt[3]{(x+2)^4}} \, dx \leq \frac{1}{\sqrt[3]{24}}.}
$$