

# Self-exercise - SRU00

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## 1 Step-by-step with A1-SRU00

A function which takes its argument as a function is called functional.

$$\boxed{\text{functional} = \text{function of function}} \quad (1)$$

**Example 1.** *Examine the first variation of the functional  $F$  in  $v$  direction*

$$F : \begin{cases} C^2([-1, 1]) \rightarrow \mathbb{R}, \\ y \mapsto F(y) := \int_{-1}^1 \left( y(x)y'(x) + e^{y'(x)^2} \right) dx, \end{cases} \quad \text{with } y : \begin{cases} [-1, 1] \rightarrow \mathbb{R}, \\ x \mapsto y(x), \end{cases} \quad (2)$$

where  $C^2$  is called space of continuous functions, i.e. function space. The number 2 in  $C^2$  indicates that those functions living in  $C^2$  are those continuously twice differentiable. Herein,  $y$  is such function living in  $C^2$ , as shown in (2)<sub>1</sub>. Besides, the function  $y$  takes  $x$  as its only variable. In this case, the only variable  $x$  is defined on the closed interval  $[-1, 1]$ , as shown in (2)<sub>2</sub>.

Approach: By using the definition, as follows

$$\boxed{\delta F(y; v) := \lim_{s \rightarrow 0} \frac{F(y + sv) - F(y)}{s}}, \quad (3)$$

where  $\delta F$  is called variation of the functional  $F$ , and the symbol  $\delta$  stands for variation.

**Interesting fact 1.** *The symbol  $\delta$  seen in variation of a functional is not the same one often seen in partial derivative, which is  $\partial$ . In  $\text{\LaTeX}$  their commands are called, either from *maths'* package such as **amsmath** or **mathtools**, as follows*

$$\begin{array}{lll} \delta & \rightarrow \backslash delta & \rightarrow \text{variation of a functional} \\ \partial & \rightarrow \backslash partial & \rightarrow \text{partial derivative of a multi-variable function} \end{array} \quad (4)$$

In order to find the variation of the functional  $F$  in  $v$  direction, we need to compute the limit, as shown in (3).

$$\begin{aligned}\delta F(y; v) &:= \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( (y + s v)(x) (y + s v)'(x) + \exp \left( (y + s v)'^2(x) \right) \right) dx \quad (5)\end{aligned}$$

$$- \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( y(x) y'(x) + \exp \left( y'^2(x) \right) \right) dx \quad (6)$$

Let us now forget the argument  $(x)$  in (5) and (6) for a while, i.e. by dropping it temporarily, just for a sake of a better overview. Then, the new expression without  $(x)$  will take the form

$$\begin{aligned}\delta F(y; v) &:= \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( (y + s v) (y + s v)' + \exp \left( (y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( y y' + \exp \left( y'^2 \right) \right) dx \quad (7)\end{aligned}$$

which leads to further analysis, as follows

$$\begin{aligned}\delta F(y; v) &\stackrel{(7)}{=} \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( (y + s v)(y' + s v') + \exp \left( (y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( y y' + \exp \left( y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( y y' + s y v' + s v y' + s^2 v v' + \exp \left( (y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( y y' + \exp \left( y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( y y' + s y v' + s v y' + s^2 v v' - y y' \right) dx \\ &\quad + \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( \exp \left( (y + s v)'^2 \right) - \exp \left( y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \underbrace{\left( s y v' + s v y' + s^2 v v' \right) dx}_{:= \mathcal{A}} \\ &\quad + \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \underbrace{\left( \exp \left( (y + s v)'^2 \right) - \exp \left( y'^2 \right) \right) dx}_{:= \mathcal{B}} \quad (8)\end{aligned}$$

Hence, it leads to

$$\therefore \boxed{\delta F(y; v) = \mathcal{A} + \mathcal{B}.} \quad (9)$$

Next, we consider part  $\mathcal{A}$ , as follows

$$\begin{aligned} \mathcal{A} &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( s y v' + s v y' + s^2 v v' \right) dx = \lim_{s \rightarrow 0} \int_{-1}^1 \left( y v' + v y' + s v v' \right) dx \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left( y v' + v y' \right) dx + \lim_{s \rightarrow 0} \int_{-1}^1 (s v v') dx \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left( y v' + v y' \right) dx + \underbrace{\lim_{s \rightarrow 0} \left( s \int_{-1}^1 (v v') dx \right)}_{=0} \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left( y(x) v'(x) + v(x) y'(x) \right) dx \quad (10) \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left( y(x) v(x) \right)' dx = \int_{-1}^1 \frac{d}{dx} \left( y(x) v(x) \right) dx \\ &= \int_{-1}^1 \frac{d}{dx} \left( y(x) v(x) \right) dx = \left( y(x) v(x) \right) \Big|_{x=-1}^{x=1} = y(1) v(1) - y(-1) v(-1), \quad (11) \end{aligned}$$

where  $(x)$  has returned back in (10) again in order to get ready to compute the follow-up integral. Therefore, solution to  $\mathcal{A}$  yields

$$\therefore \boxed{\mathcal{A} = y(1) v(1) - y(-1) v(-1).} \quad (12)$$

Besides, the computation of part  $\mathcal{B}$  goes as follows

$$\begin{aligned} \mathcal{B} &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( \exp \left( (y + s v)^2 \right) - \exp \left( y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( \exp \left( y'^2 + 2 s y' v' + s^2 v'^2 \right) - \exp \left( y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( \exp \left( y'^2 \right) \exp \left( 2 s y' v' + s^2 v'^2 \right) - \exp \left( y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left( \exp \left( y'^2 \right) \left( \exp \left( 2 s y' v' + s^2 v'^2 \right) - 1 \right) \right) dx \\ &= \int_{-1}^1 \left( \exp \left( y'^2 \right) \underbrace{\lim_{s \rightarrow 0} \frac{\exp \left( 2 s y' v' + s^2 v'^2 \right) - 1}{s}}_{:=\mathcal{C}} \right) dx, \quad (13) \end{aligned}$$

where a further result of  $\mathcal{C}$  in (13) is computed by using l'Hôpital's rule, as follows

$$\begin{aligned}
\mathcal{C} &= \lim_{s \rightarrow 0} \frac{\exp(2 s y' v' + s^2 v'^2) - 1}{s} \\
&= \lim_{s \rightarrow 0} \left( (2 y' v' + 2 s v'^2) \exp(2 s y' v' + s^2 v'^2) \right) \\
&= 2 y' v' \\
&= 2 y'(x) v'(x).
\end{aligned} \tag{14}$$

Substitution of (14) into (13) leads to

$$\therefore \quad \boxed{\mathcal{B} = 2 \int_{-1}^1 \exp(y'^2(x)) y'(x) v'(x) dx.} \tag{15}$$

Finally, by inserting (12) and (15) into (9) the final result yields

$$\therefore \quad \boxed{\delta F(y; v) = y(1) v(1) - y(-1) v(-1) + 2 \int_{-1}^1 y'(x) v'(x) \exp(y'^2(x)) dx.} \tag{16}$$

## 2 Step-by-step with A4-SRU00

**Example 2.** *Examine the following initial value problem (IVP)*

$$y'''(t) + y'(t) = t y(t), \quad (17)$$

*with initial conditions (IC) given as follows*

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2. \quad (18)$$

- i) *Transform the above third order ordinary differential equation (ODE) to a system of first order ODEs.*
- ii) *Solve the system i) numerically by using explicit Euler method with 2 steps.*

Approach:

- i) We need to transform the third order ODE given in (17) into a system of first order ODEs.

**Question 1.** *But.. how many ODEs shall we need in total for this system of first order ODEs?*

**Ans:** 3.

**Observation 1.** *Since the highest order of derivative in (17) is 3, i.e. as we can see there are three ' from  $y'''(t)$ , the given ODE is called third order ODE. Besides, the lowest order in (17) is 0, i.e. as we can see from  $y(t)$ . Therefore, we shall need a system of **three** first order ODEs for the given third order ODE in (17).*

**Question 2.** *Which variable from (17) shall we need to take and define, so that we are able to transform (17) into a system of first order ODEs corresponding to the given initial conditions given in (18), e.g. is it ideal to define  $y'''(t) := u_3(t)$ ?*

**Ans:**  $y(t)$ ,  $y'(t)$  and  $y''(t)$ ; and it is not ideal to go with  $y'''(t) := u_3(t)$ .

**Observation 2.** *The three initial conditions given  $y(0)$ ,  $y'(0)$  and  $y''(0)$ , as shown in (18), give rise to the fact that we shall need to define three new variables for  $y(t)$ ,  $y'(t)$  and  $y''(t)$ . This choice is ideal for the ODE with the given ICs because we will be able to apply these ICs directly for the new system of three ODEs.*

Therefore, we shall need three first order ODEs for the new system. Besides, the transformation is done by setting new variables for  $y(t)$ ,  $y'(t)$  and  $y''(t)$ .

Transformation: We now need to set  $y(t)$ ,  $y'(t)$  and  $y''(t)$  to new variables, let's say  $u_0(t)$ ,  $u_1(t)$ , and  $u_2(t)$ , respectively, as follows

$$y(t) := u_0(t), \quad (19)$$

$$y'(t) := u_1(t), \quad (20)$$

$$y''(t) := u_2(t). \quad (21)$$

Then, by taking the derivative of the new variable  $u_0(t)$ ,  $u_1(t)$  and  $u_2(t)$ , as newly defined in (19), (20) and (21), we obtain the following relations

$$\boxed{\begin{aligned} u'_0(t) &\stackrel{(19)}{=} y'(t) \stackrel{(20)}{=} u_1(t), \\ u'_1(t) &\stackrel{(20)}{=} y''(t) \stackrel{(21)}{=} u_2(t), \\ u'_2(t) &\stackrel{(21)}{=} y'''(t) \stackrel{(17)}{=} t y(t) - y'(t) \stackrel{(17)}{=} t u_0(t) - u_1(t), \end{aligned}} \quad (22)$$

which is summarized, together with keeping in mind about the order of the index  $i$  from  $u_i(t)|_{i=0,1,2}$ , as follows

$$\begin{aligned} u'_0(t) &= & u_1(t) \\ u'_1(t) &= & u_2(t) \\ u'_2(t) &= t u_0(t) - u_1(t) \end{aligned} \quad (23)$$

which gives rise to the following system

$$\begin{aligned} u'_0(t) &= 0 u_0(t) & 1 u_1(t) & 0 u_2(t) \\ u'_1(t) &= 0 u_0(t) & 0 u_1(t) & 1 u_2(t) \\ u'_2(t) &= t u_0(t) & -1 u_1(t) & 0 u_2(t) \end{aligned} \quad (24)$$

from where we can write the system in form of

$$\mathbf{u}' = A\mathbf{u} \quad (25)$$

as follows

$$\underbrace{\begin{pmatrix} u_0(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}'}_{:=\mathbf{u}'} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{:=A} \underbrace{\begin{pmatrix} u_0(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}}_{:=\mathbf{u}} \quad (26)$$

where the relation between  $A$  and  $\mathbf{u}$  in  $A\mathbf{u}$  is matrix-vector multiplication. Besides, by substituting (19) (20) and (21) into (18), respectively, we obtain the initial conditions for the new system in (26)

$$\boxed{\begin{aligned} u_0(0) &= 0, \\ u_1(0) &= 1, \\ u_2(0) &= 2, \end{aligned}} \quad (27)$$

which form an equivalent system of three first-order ODEs.

- ii) We need to solve the system of first order ODEs obtained in i) numerically by using explicit Euler method.

**Summary 1.** *Given the system of first order ODEs as follows*

$$\mathbf{u}'(t) = A\mathbf{u}(t) \quad (28)$$

where the vector  $\mathbf{u}$  is defined as  $\mathbf{u} := (u_0, u_1, u_2)^\top$ , and the matrix  $A$  is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix} \quad (29)$$

with initial conditions

$$u_0(0) = 0, \quad u_1(0) = 1, \quad u_2(0) = 2. \quad (30)$$

*Solve this system numerically by using the explicit Euler method.*

Approach: We need to approximate  $\mathbf{u}'(t)$  arising in (28) by some numerical schemes, e.g. explicit Euler method.

**Observation 3.** *We consider Taylor expansion for  $\mathbf{u}(t)$  around point  $t = t_0$*

$$\mathbf{u}(t) = \mathbf{u}(t_0) + \mathbf{u}'(t_0)(t - t_0) + \mathcal{O}((t - t_0)^2), \quad (31)$$

which, for the sake of convergence, requires  $h := t - t_0$  to be small. Since  $h := t - t_0$ , we equally have  $t = h + t_0$ . Substitution of  $h := t - t_0$  and  $t = t_0 + h$  to (31) leads to the following expression

$$\mathbf{u}(t_0 + h) = \mathbf{u}(t_0) + h\mathbf{u}'(t_0) + \mathcal{O}(h^2), \quad (32)$$

which is the new expression of Taylor expansion in terms of point  $t_0$ , around which the function  $\mathbf{u}(t)$  is approximated by the Taylor expansion, instead of the classical expression in terms of the variable  $t$  as seen in (31). Since point  $t_0$ , although it implied a fixed point, arising in (32) can be chosen arbitrarily, we can, therefore, substitute it for  $t$ , as follows

$$\mathbf{u}(t + h) = \mathbf{u}(t) + h\mathbf{u}'(t) + \mathcal{O}(h^2), \quad (33)$$

Now, the expression (32) gives rise to a useful expression to approximate  $\mathbf{u}'(t)$  numerically, as follows

$$\therefore \boxed{\mathbf{u}'(t) \approx \frac{\mathbf{u}(t + h) - \mathbf{u}(t)}{h}}. \quad (34)$$

Substitution of (34) into (28) yields

$$\frac{\mathbf{u}(t + h) - \mathbf{u}(t)}{h} = A\mathbf{u}(t) \quad (35)$$

which leads to the formula of explicit Euler method

$$\boxed{\mathbf{u}^{(j+1)} = \mathbf{u}^{(j)} + h A \mathbf{u}^{(j)},} \quad (36)$$

where  $\mathbf{u}^{(j)}$  is the numerical solution by Euler method after  $j$  steps, and components of  $\mathbf{u}^{(j)}$  is known as

$$\mathbf{u}^{(j)} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}^{(j)}, \quad (37)$$

which, together with (36), leads to the following expressions

$$\begin{aligned} \mathbf{u}^{(0)} &= \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \\ \mathbf{u}^{(1)} &= \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} + h \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + h \begin{pmatrix} 1 \\ 2 \\ 0 - 1 \end{pmatrix} \\ \therefore &= \begin{pmatrix} h \\ 1 + 2h \\ 2 - h \end{pmatrix}, \\ \mathbf{u}^{(2)} &= \begin{pmatrix} u_0^{(1)} \\ u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} + h \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} u_0^{(1)} \\ u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} h \\ 1 + 2h \\ 2 - h \end{pmatrix} + h \begin{pmatrix} 1 + 2h \\ 2 - h \\ h \cdot h - (1 + 2h) \end{pmatrix} \\ &= \begin{pmatrix} 2h(1 + h) \\ 1 + 4h - h^2 \\ 2 - 2h - 2h^2 + h^3 \end{pmatrix}. \end{aligned} \quad (38)$$