Self-exercise - SRU03
Response to questions

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1 Step-by-step with A3-SRU03

Example 1. *Let the linear m-step method be given by coefficients $\alpha_l \in \mathbb{R}$ and $\beta_l \in \mathbb{R}$ where $l = 0, 1, \ldots, m$, satisfying the following system of equations*

$$\sum_{l=0}^{m} a_l = 0, \tag{1}$$

$$\sum_{l=0}^{m} (l^q \alpha_l - q\, l^{q-1} \beta_l) = 0, \quad \textit{for} \quad q = 1, 2, 3, \ldots, p. \tag{2}$$

Then the linear m-step method has the consistency order of p.

Consider the following ordinary differential equation (ODE) taking the form

$$x'(t) = f(t, x(t)), \quad \text{with} \quad x(t_0) = x_0, \tag{3}$$

which may be approximated by a linear multistep method, taking the general form

$$\boxed{\sum_{l=0}^{m} \alpha_l\, x(t_j + lh) = h \sum_{l=0}^{m} \beta_l\, f(t_j + lh, x(t_j + lh)).} \tag{4}$$

Observation 1. *There are 6 symbols arising in (1), (2), (4) to be aware of*

1. *m is the total number of steps used in the multistep method.*
2. *$\alpha \in \mathbb{R}$ is the coefficient for the linear combination of $x(t_j + lh)$*
3. *$\beta \in \mathbb{R}$ is the coefficient for the linear combination of $f(t_j + lh, x(t_j + lh))$.*
4. *p is the consistency order.*
5. *l is the dummy index, running from 0 to m.*
6. *q is the dummy index, running from 1 to p.*

Observation 2. *The multistep method arising in (4) is a **linear** multistep method, or specifically **linear** m-step method. The term **linear** coming along with m-step method, i.e. linear m-step method, emphasizes the fact that there is actually a **linear combination** of the terms $x(t_j + lh)$ with coefficients α_l on the LHS of (4), and a **linear combination** of the terms $f(t_j + lh, x(t_j + lh))$ with coefficients β_l on the RHS of (4). This linear m-step method uses the information from the previous m steps to compute the value for the next step. In details, let us examine the formula (4) which is written again as follows*

$$\sum_{l=0}^{m} \alpha_l \, x(t_j + lh) = h \sum_{l=0}^{m} \beta_l \, f(t_j + lh, x(t_j + lh)), \tag{5}$$

whose LHS is written in its entirety as follows

$$\begin{aligned}\sum_{l=0}^{m} \alpha_l \, x(t_j + lh) &= \alpha_0 \, x(t_j) + \alpha_1 \, x(t_j + h) + \alpha_2 \, x(t_j + 2h) + \dots \\ &\quad + \dots + \alpha_{m-1} \, x(t_j + (m-1)h) + \alpha_m \, x(t_j + mh),\end{aligned} \tag{6}$$

whereas the RHS of (5) has its expansion as follows

$$\begin{aligned}\sum_{l=0}^{m} \beta_l \, f(t_j + lh, x(t_j + lh)) = \beta_0 \, f(t_j, x(t_j)) &+ \beta_1 \, f(t_j + h, x(t_j + h)) \\ &+ \beta_2 \, f(t_j + 2h, x(t_j + 2h)) \\ &+ \dots + \\ &+ \beta_{m-1} \, f(t_j + (m-1)h, x(t_j + (m-1)h)) \\ &+ \beta_m \, f(t_j + mh, x(t_j + mh)).\end{aligned} \tag{7}$$

The insertion of (6) and (7) into (5) leads to

$$\begin{aligned}&\alpha_0 \, x(t_j) + \alpha_1 \, x(t_j + h) + \alpha_2 \, x(t_j + 2h) \\ &+ \dots \\ &+ \alpha_{m-1} \, x(t_j + (m-1)h) + \alpha_m \, x(t_j + mh) \\ &= \\ &\beta_0 \, f(t_j, x(t_j)) + \beta_1 \, f(t_j + h, x(t_j + h)) + \beta_2 \, f(t_j + 2h, x(t_j + 2h)) \\ &+ \dots + \\ &+ \beta_{m-1} \, f(t_j + (m-1)h, x(t_j + (m-1)h)) + \beta_m \, f(t_j + mh, x(t_j + mh)),\end{aligned} \tag{8}$$

which tells us the fact that the value $x(t_j + mh)$ is approximated by using all information from previous steps, based on

1. $x(t_j)$, $x(t_j + h)$, $\dots$, $x(t_j + (m-1)h)$.
2. $f(t_j, x(t_j))$, $f(t_j + h, x(t_j + h))$, $\dots$, $f(t_j + mh, x(t_j + mh))$.
3. *Coefficients α_l and β_l.*

Observation 3. *The appearance of the green term $f(t_j + mh, x(t_j + mh))$ arising in (8) plays a significant role, meaning that*

1. $\beta_m = 0$: *the green term is switched off, and the scheme becomes **explicit**.*
2. $\beta_m \neq 0$: *the green term is switched on, and the scheme becomes **implicit**.*

Observation 4. *Specific choices of α_l and β_l leads to some familiar methods. Especially, when $m = 1$ the linear m-step method becomes single-step method, or one-step method. If so, according to (4) there will be 4 coefficients $(\alpha_0, \alpha_1, \beta_0, \beta_1)$ to be defined. For example:*

1. $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 1, 0)$ *we obtain explicit Euler.*
2. $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 0, 1)$ *we obtain implicit Euler.*

When the time t is considered at a specific discretized time point $t = t_j + lh$, the ODE in (3) yields

$$x'(t_j + lh) = f(t_j + lh, x(t_j + lh)), \tag{9}$$

Substitution of (9) into (4) leads to

$$\sum_{l=0}^{m} \alpha_l \, x(t_j + lh) = h \sum_{l=0}^{m} \beta_l \, x'(t_j + lh), \tag{10}$$

which, by group the summation sign together, leads equally to

$$\therefore \quad \boxed{\sum_{l=0}^{m} \left(\alpha_l \, x(t_j + lh) - h \, \beta_l \, x'(t_j + lh)\right) = 0.} \tag{11}$$

Then, the order of consistency is figured out by taking advantage of using Taylor's expansion for $x(t_j + lh)$ and $x'(t_j + lh)$. By using (24) and (27) in Observation 5 we obtain

$$x(t_j + lh) = \sum_{n=0}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} + \mathcal{O}((lh)^{q+1}), \tag{12}$$

$$x'(t_j + lh) = \sum_{n=0}^{r-1} \frac{x^{(n)}(t_j)\,(lh)^{n-1}}{(n-1)!} + \mathcal{O}((lh)^{r}), \tag{13}$$

where, according to (24) and (27) in Observation 5, the following two substitutions have been performed for (12) and (13)

$$a \to t_j, \tag{14}$$

$$h \to lh, \tag{15}$$

which means that a is replaced by t_j while h is by lh. Furthermore, expression (12) can be split into two main parts, where the first part is going with index $n = 0$, and the second part with $n \in [1, q]$, as follows

$$\begin{aligned} x(t_j + lh) &= \sum_{n=0}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} + \mathcal{O}((lh)^{q+1}) \\ &= \frac{x^{(0)}(t_j)\,(lh)^0}{0!} + \sum_{n=1}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} + \mathcal{O}((lh)^{q+1}) \\ &= x(t_j) + \sum_{n=1}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} + \mathcal{O}((lh)^{q+1}), \end{aligned} \tag{16}$$

which reads

$$\therefore \quad \boxed{x(t_j + lh) = x(t_j) + \sum_{n=1}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} + \mathcal{O}((lh)^{q+1}).} \tag{17}$$

Besides, let the index in expression (13) be starting from $n = 1$, instead of $n = 0$, we then obtain

$$\begin{aligned} x'(t_j + lh) &= \sum_{n=0}^{r-1} \frac{x^{(n)}(t_j)\,(lh)^{n-1}}{(n-1)!} + \mathcal{O}((lh)^r) \\ &= \sum_{n=1}^{r} \frac{x^{(n)}(t_j)\,(lh)^{n-1}}{(n-1)!} + \mathcal{O}((lh)^r), \end{aligned} \tag{18}$$

which means that every element within the summation sign going with $(n + 1)$ becomes n, and element going with n will become $(n - 1)$. Further expressions of (18) lead to

$$\begin{aligned} x'(t_j + lh) &= \sum_{n=1}^{r+1} \frac{n\,x^{(n)}(t_j)\,(lh)^{n-1}}{n!} + \mathcal{O}((lh)^r) \\ &= \sum_{n=1}^{r+1} \frac{n\,x^{(n)}(t_j)\,l^{n-1}h^n}{n!\,h} + \mathcal{O}((lh)^r), \end{aligned} \tag{19}$$

which reads

$$\therefore \quad \boxed{x'(t_j + lh) = \sum_{n=1}^{r+1} \frac{n\,x^{(n)}(t_j)\,l^{n-1}h^n}{n!\,h} + \mathcal{O}((lh)^r).} \tag{20}$$

Next, by substituting (17) and (20) into (11) we obtain the following relation

$$\begin{aligned} \sum_{l=0}^{m} \Bigg(\alpha_l\, x(t_j) + \alpha_l \sum_{n=1}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} + \alpha_l\, \mathcal{O}((lh)^{q+1}) \\ - h\,\beta_l \sum_{n=1}^{r+1} \frac{n\,x^{(n)}(t_j)\,l^{n-1}h^n}{n!\,h} - h\,\beta_l\, \mathcal{O}((lh)^r) \Bigg) = 0, \end{aligned} \tag{21}$$

which leads to

$$
\begin{aligned}
&\sum_{l=0}^{m} \alpha_l\, x(t_j) + \sum_{l=0}^{m} \alpha_l \sum_{n=1}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} + \sum_{l=0}^{m} \alpha_l \underbrace{\mathcal{O}((lh)^{q+1})}_{\mathcal{O}(h^{q+1})} \\
&\qquad\qquad - \sum_{l=0}^{m} h\,\beta_l \sum_{n=1}^{r+1} \frac{n\,x^{(n)}(t_j)\,l^{n-1}h^n}{n!\,h} - \sum_{l=0}^{m} \beta_l \underbrace{h\,\mathcal{O}((lh)^r)}_{\mathcal{O}(h^{r+1})} = 0, \\
\Leftrightarrow &\sum_{l=0}^{m} \alpha_l\, x(t_j) + \sum_{l=0}^{m} \left(\alpha_l \sum_{n=1}^{q} \frac{x^{(n)}(t_j)\,(lh)^n}{n!} - h\,\beta_l \sum_{n=1}^{r+1} \frac{n\,x^{(n)}(t_j)\,l^{n-1}h^n}{n!\,h} \right) \\
&\qquad\qquad + \sum_{l=0}^{m} \left(\underbrace{\alpha_l\,\mathcal{O}(h^{q+1}) - \beta_l\,\mathcal{O}(h^{r+1})}_{\gamma_l \mathcal{O}(h^{q+1})} \right) = 0 \\
\Leftrightarrow &\sum_{l=0}^{m} \alpha_l\, x(t_j) + \sum_{l=0}^{m} \left(\alpha_l \sum_{n=1}^{q} \frac{x^{(n)}(t_j)\,l^n h^n}{n!} - \beta_l \sum_{n=1}^{r+1} \frac{n\,x^{(n)}(t_j)\,l^{n-1}h^n}{n!} \right) \\
&\qquad\qquad + \sum_{l=0}^{m} \gamma_l\,\mathcal{O}(h^{q+1}) = 0
\end{aligned}
\tag{22}
$$

Observation 5. *Taylor's expansion of function $x(t)$ around point a reads*

$$\begin{aligned} x(t) &= x(a) + \frac{x'(a)\,(t-a)}{1!} + \frac{x''(a)\,(t-a)^2}{2!} + \cdots + \frac{x^{(p)}(a)\,(t-a)^p}{p!} \\ &\qquad + \mathcal{O}((t-a)^{p+1}) \\ &= \sum_{n=0}^{p} \frac{x^{(n)}(a)\,(t-a)^n}{n!} + \mathcal{O}((t-a)^{p+1}). \end{aligned} \tag{23}$$

Then, by setting $h := t - a$ we obtain $t = a + h$; hence, the (23) becomes

$$\begin{aligned} x(a+h) &= x(a) + \frac{x'(a)\,h}{1!} + \frac{x''(a)\,h^2}{2!} + \cdots + \frac{x^{(p)}(a)\,h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^{p} \frac{x^{(n)}(a)\,h^n}{n!} + \mathcal{O}(h^{p+1}). \end{aligned} \tag{24}$$

Since the point a can be chosen arbitrarily, we can set it as a variable t, yielding

$$\begin{aligned} x(t+h) &= x(t) + \frac{x'(t)\,h}{1!} + \frac{x''(t)\,h^2}{2!} + \cdots + \frac{x^{(p)}(t)\,h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^{p} \frac{x^{(n)}(t)\,h^n}{n!} + \mathcal{O}(h^{p+1}), \end{aligned} \tag{25}$$

which, by taking derivative w.r.t. variable t on both sides of (25), gives rise to

$$\begin{aligned} x'(t+h) &= x'(t) + \frac{x''(t)\,h}{1!} + \frac{x'''(t)\,h^2}{2!} + \cdots + \frac{x^{(p+1)}(t)\,h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^{p} \frac{x^{(n+1)}(t)\,h^n}{n!} + \mathcal{O}(h^{p+1}), \end{aligned} \tag{26}$$

which can also be written as follows

$$\begin{aligned} x'(t+h) &= x'(t) + \frac{x''(t)\,h}{1!} + \frac{x'''(t)\,h^2}{2!} + \cdots + \frac{x^{(p)}(t)\,h^{p-1}}{(p-1)!} + \mathcal{O}(h^p) \\ &= \sum_{n=0}^{p-1} \frac{x^{(n+1)}(t)\,h^n}{n!} + \mathcal{O}(h^p). \end{aligned} \tag{27}$$

Note in passing that the only difference between (26) and (27) is the order p, i.e. we expand till p in (26), but only till $(p-1)$ in (27). It turns out that the expression (27) helps our analysis in this exercise become more succinct.

$$x(t_j + lh) = x(t_j) \tag{28}$$