

# Self-exercise - SRU05

## Response to questions

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### 1 Hint for A1-HA05

**Example 1.** Let the linear  $m$ -step method be given by coefficients  $\alpha_l \in \mathbb{R}$  and  $\beta_l \in \mathbb{R}$  where  $l = 0, 1, \dots, m$ , satisfying the following system of equations

$$\sum_{l=0}^m a_l = 0, \quad (1)$$

$$\sum_{l=0}^m (l^q \alpha_l - q l^{q-1} \beta_l) = 0, \quad \text{for } q = 1, 2, 3, \dots, p. \quad (2)$$

Then the linear  $m$ -step method has the consistency order of  $p$ .

Consider the following ordinary differential equation (ODE) taking the form

$$x'(t) = f(t, x(t)), \quad \text{with } x(t_0) = x_0, \quad (3)$$

which may be approximated by a linear multistep method, taking the general form

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)). \quad (4)$$

**Observation 1.** There are 6 symbols arising in (1), (2), (4) to be aware of

1.  $m$  is the total number of steps used in the multistep method.
2.  $\alpha \in \mathbb{R}$  is the coefficient for the linear combination of  $x(t_j + lh)$
3.  $\beta \in \mathbb{R}$  is the coefficient for the linear combination of  $f(t_j + lh, x(t_j + lh))$ .
4.  $p$  is the consistency order.
5.  $l$  is the dummy index, running from 0 to  $m$ .
6.  $q$  is the dummy index, running from 1 to  $p$ .

**Observation 2.** The multistep method arising in (4) is a **linear** multistep method, or specifically **linear**  $m$ -step method. The term **linear** coming along with  $m$ -step method, i.e. linear  $m$ -step method, emphasizes the fact that there is actually a **linear combination** of the terms  $x(t_j + lh)$  with coefficients  $\alpha_l$  on the LHS of (4), and a **linear combination** of the terms  $f(t_j + lh, x(t_j + lh))$  with coefficients  $\beta_l$  on the RHS of (4). This linear  $m$ -step method uses the information from the previous  $m$  steps to compute the value for the next step. In details, let us examine the formula (4) which is written again as follows

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)), \quad (5)$$

whose LHS is written in its entirety as follows

$$\begin{aligned} \sum_{l=0}^m \alpha_l x(t_j + lh) &= \alpha_0 x(t_j) + \alpha_1 x(t_j + h) + \alpha_2 x(t_j + 2h) + \dots \\ &\quad + \dots + \alpha_{m-1} x(t_j + (m-1)h) + \alpha_m x(t_j + mh), \end{aligned} \quad (6)$$

whereas the RHS of (5) has its expansion as follows

$$\begin{aligned} \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)) &= \beta_0 f(t_j, x(t_j)) + \beta_1 f(t_j + h, x(t_j + h)) \\ &\quad + \beta_2 f(t_j + 2h, x(t_j + 2h)) \\ &\quad + \dots + \\ &\quad + \beta_{m-1} f(t_j + (m-1)h, x(t_j + (m-1)h)) \\ &\quad + \beta_m f(t_j + mh, x(t_j + mh)). \end{aligned} \quad (7)$$

The insertion of (6) and (7) into (5) leads to

$$\begin{aligned} &\alpha_0 x(t_j) + \alpha_1 x(t_j + h) + \alpha_2 x(t_j + 2h) \\ &\quad + \dots \\ &\quad + \alpha_{m-1} x(t_j + (m-1)h) + \alpha_m x(t_j + mh) \\ &= \\ &\beta_0 f(t_j, x(t_j)) + \beta_1 f(t_j + h, x(t_j + h)) + \beta_2 f(t_j + 2h, x(t_j + 2h)) \\ &\quad + \dots + \\ &\quad + \beta_{m-1} f(t_j + (m-1)h, x(t_j + (m-1)h)) + \beta_m f(t_j + mh, x(t_j + mh)), \end{aligned} \quad (8)$$

which tells us the fact that the value  $x(t_j + mh)$  is approximated by using all information from previous steps, based on

1.  $x(t_j), x(t_j + h), \dots, x(t_j + (m-1)h)$ .
2.  $f(t_j, x(t_j)), f(t_j + h, x(t_j + h)), \dots, f(t_j + mh, x(t_j + mh))$ .
3. Coefficients  $\alpha_l$  and  $\beta_l$ .

**Observation 3.** The appearance of the green term  $f(t_j + mh, x(t_j + mh))$  arising in (8) plays a significant role, meaning that

1.  $\beta_m = 0$ : the green term is switched off, and the scheme becomes **explicit**.
2.  $\beta_m \neq 0$ : the green term is switched on, and the scheme becomes **implicit**.

**Observation 4.** Specific choices of  $\alpha_l$  and  $\beta_l$  leads to some familiar methods. Especially, when  $m = 1$  the linear  $m$ -step method becomes single-step method, or one-step method. If so, according to (4) there will be 4 coefficients ( $\alpha_0, \alpha_1, \beta_0, \beta_1$ ) to be defined. For example:

1.  $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 1, 0)$  we obtain explicit Euler.
2.  $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 0, 1)$  we obtain implicit Euler.

When the time  $t$  is considered at a specific discretized time point  $t = t_j + lh$ , the ODE in (3) yields

$$x'(t_j + lh) = f(t_j + lh, x(t_j + lh)), \quad (9)$$

Substitution of (9) into (4) leads to

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l x'(t_j + lh), \quad (10)$$

which, by group the summation sign together, leads equally to

$$\therefore \sum_{l=0}^m (\alpha_l x(t_j + lh) - h \beta_l x'(t_j + lh)) = 0. \quad (11)$$

Then, the order of consistency is figured out by taking advantage of using Taylor's expansion for  $x(t_j + lh)$  and  $x'(t_j + lh)$ . By using (25) and (28) in Observation 5 we obtain

$$x(t_j + lh) = \sum_{n=0}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}), \quad (12)$$

$$x'(t_j + lh) = \sum_{n=0}^{r-1} \frac{x^{(n+1)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^r), \quad (13)$$

where, according to (25) and (28) in Observation 5, the following two substitutions have been performed for (12) and (13)

$$a \rightarrow t_j, \quad (14)$$

$$h \rightarrow lh, \quad (15)$$

which means that  $a$  is replaced by  $t_j$  while  $h$  is by  $lh$ . Furthermore, expression (12) can be split into two main parts, where the first part is going with index  $n = 0$ , and the second part with  $n \in [1, q]$ , as follows

$$\begin{aligned} x(t_j + lh) &= \sum_{n=0}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}) \\ &= \frac{x^{(0)}(t_j) (lh)^0}{0!} + \sum_{n=1}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}) \\ &= x(t_j) + \sum_{n=1}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}), \end{aligned} \quad (16)$$

which reads

$$\therefore \boxed{x(t_j + lh) = x(t_j) + \sum_{n=1}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}).} \quad (17)$$

Besides, let the index in expression (13) be starting from  $n = 1$ , instead of  $n = 0$ , we then obtain

$$\begin{aligned} x'(t_j + lh) &= \sum_{n=0}^{r-1} \frac{x^{(n+1)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^r) \\ &= \sum_{n=1}^r \frac{x^{(n)}(t_j) (lh)^{n-1}}{(n-1)!} + \mathcal{O}((lh)^r), \end{aligned} \quad (18)$$

which means that every element within the summation sign going with  $(n + 1)$  becomes  $n$ , and element going with  $n$  will become  $(n - 1)$ . Further expressions of (18) lead to

$$\begin{aligned} x'(t_j + lh) &= \sum_{n=1}^r \frac{n x^{(n)}(t_j) (lh)^{n-1}}{n!} + \mathcal{O}((lh)^r) \\ &= \sum_{n=1}^r \frac{n x^{(n)}(t_j) l^{n-1} h^n}{n! h} + \mathcal{O}((lh)^r), \end{aligned} \quad (19)$$

which reads

$$\therefore \boxed{x'(t_j + lh) = \sum_{n=1}^r \frac{n x^{(n)}(t_j) l^{n-1} h^n}{n! h} + \mathcal{O}((lh)^r).} \quad (20)$$

Next, by substituting (17) and (20) into (11) we obtain the following relation

$$\begin{aligned} \sum_{l=0}^m \left( \alpha_l x(t_j) + \alpha_l \sum_{n=1}^p \frac{x^{(n)}(t_j) (lh)^n}{n!} + \alpha_l \mathcal{O}((lh)^{p+1}) \right. \\ \left. - h \beta_l \sum_{n=1}^p \frac{n x^{(n)}(t_j) l^{n-1} h^n}{n! h} - h \beta_l \mathcal{O}((lh)^p) \right) = 0, \end{aligned} \quad (21)$$

where the dummy indices  $q$  and  $r$  have been switched to  $p$  in order to match our analysis for the given problem, leading to

$$\begin{aligned}
& \sum_{l=0}^m \alpha_l x(t_j) + \sum_{l=0}^m \left( \alpha_l \sum_{n=1}^p \frac{x^{(n)}(t_j) (lh)^n}{n!} \right) + \sum_{l=0}^m \alpha_l \underbrace{\mathcal{O}((lh)^{p+1})}_{\mathcal{O}(h^{p+1})} \\
& - \sum_{l=0}^m \left( h \beta_l \sum_{n=1}^p \frac{n x^{(n)}(t_j) l^{n-1} h^n}{n! h} \right) - \sum_{l=0}^m \beta_l \underbrace{h \mathcal{O}((lh)^p)}_{\mathcal{O}(h^{p+1})} = 0, \\
& \Leftrightarrow \sum_{l=0}^m \alpha_l x(t_j) + \sum_{l=0}^m \left( \alpha_l \sum_{n=1}^p \frac{x^{(n)}(t_j) (lh)^n}{n!} - h \beta_l \sum_{n=1}^p \frac{n x^{(n)}(t_j) l^{n-1} h^n}{n! h} \right) \\
& \quad + \sum_{l=0}^m \left( \underbrace{\alpha_l \mathcal{O}(h^{p+1}) - \beta_l \mathcal{O}(h^{p+1})}_{\gamma_l \mathcal{O}(h^{p+1})} \right) = 0 \\
& \Leftrightarrow \sum_{l=0}^m \alpha_l x(t_j) + \sum_{l=0}^m \left( \alpha_l \sum_{n=1}^p \frac{x^{(n)}(t_j) l^n h^n}{n!} - \beta_l \sum_{n=1}^p \frac{n x^{(n)}(t_j) l^{n-1} h^n}{n!} \right) \\
& \quad + \sum_{l=0}^m \gamma_l \mathcal{O}(h^{p+1}) = 0 \\
& \Leftrightarrow \sum_{l=0}^m \alpha_l x(t_j) + \sum_{n=1}^p \left( \frac{x^{(n)}(t_j) h^n}{n!} \sum_{l=0}^m \left( \alpha_l l^n - \beta_l n l^{n-1} \right) \right) + \sum_{l=0}^m \gamma_l \mathcal{O}(h^{p+1}) = 0
\end{aligned} \tag{22}$$

Finally, the expression in (22) leads us to the fact that the linear m-step method has the consistency order of  $p$ , according to the red term in (22), when the blue terms in (22) vanish, meaning that

$$\therefore \quad \boxed{
\begin{aligned}
& \sum_{l=0}^m \alpha_l = 0, \\
& \sum_{l=0}^m \left( \alpha_l l^n - \beta_l n l^{n-1} \right) = 0, \quad \text{for } n = 1, 2, 3, \dots, p.
\end{aligned}
} \tag{23}$$

where  $p$  is the order of consistency.

**Observation 5.** *Taylor's expansion of function  $x(t)$  around point  $a$  reads*

$$\begin{aligned} x(t) &= x(a) + \frac{x'(a)(t-a)}{1!} + \frac{x''(a)(t-a)^2}{2!} + \dots + \frac{x^{(p)}(a)(t-a)^p}{p!} \\ &\quad + \mathcal{O}((t-a)^{p+1}) \\ &= \sum_{n=0}^p \frac{x^{(n)}(a)(t-a)^n}{n!} + \mathcal{O}((t-a)^{p+1}). \end{aligned} \quad (24)$$

Then, by setting  $h := t - a$  we obtain  $t = a + h$ ; hence, the (24) becomes

$$\begin{aligned} x(a+h) &= x(a) + \frac{x'(a)h}{1!} + \frac{x''(a)h^2}{2!} + \dots + \frac{x^{(p)}(a)h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^p \frac{x^{(n)}(a)h^n}{n!} + \mathcal{O}(h^{p+1}). \end{aligned} \quad (25)$$

Since the point  $a$  can be chosen arbitrarily, we can set it as a variable  $t$ , yielding

$$\begin{aligned} x(t+h) &= x(t) + \frac{x'(t)h}{1!} + \frac{x''(t)h^2}{2!} + \dots + \frac{x^{(p)}(t)h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^p \frac{x^{(n)}(t)h^n}{n!} + \mathcal{O}(h^{p+1}), \end{aligned} \quad (26)$$

which, by taking derivative w.r.t. variable  $t$  on both sides of (26), gives rise to

$$\begin{aligned} x'(t+h) &= x'(t) + \frac{x''(t)h}{1!} + \frac{x'''(t)h^2}{2!} + \dots + \frac{x^{(p+1)}(t)h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^p \frac{x^{(n+1)}(t)h^n}{n!} + \mathcal{O}(h^{p+1}), \end{aligned} \quad (27)$$

which can also be written as follows

$$\begin{aligned} x'(t+h) &= x'(t) + \frac{x''(t)h}{1!} + \frac{x'''(t)h^2}{2!} + \dots + \frac{x^{(p)}(t)h^{p-1}}{(p-1)!} + \mathcal{O}(h^p) \\ &= \sum_{n=0}^{p-1} \frac{x^{(n+1)}(t)h^n}{n!} + \mathcal{O}(h^p). \end{aligned} \quad (28)$$

Note in passing that the only difference between (27) and (28) is the order  $p$ , i.e. we expand till  $p$  in (27), but only till  $(p-1)$  in (28). It turns out that the expression (28) helps our analysis in this exercise become more succinct.