Global exercise - GUE09

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Content covered:

✓ Analysis:

(i) Step-by-step with Line integral of scalar field
(ii) Step-by-step with Line integral of vector field

✓ Numerics:

1. Singular value decomposition (SVD)
2. Demo mini programming exercise (A1-HA09)

Recall 1. *Line integral of a scalar field $\phi : \Omega \to \mathbb{R}$ is defined as follows*

$$\int_\Gamma \phi \, ds := \int_a^b \phi(\gamma(t)) \big\| \gamma'(t) \big\| \, dt \tag{1}$$

Recall 2. *Line integral of a vector field $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is defined as follows*

$$\int_\Gamma \boldsymbol{f} \cdot d\boldsymbol{x} := \int_a^b \big\langle \boldsymbol{f}(\gamma(t)), \gamma'(t) \big\rangle \, dt \tag{2}$$

1 Step-by-step with Line integral of scalar field

Example 1. *Examine the following trajectory* $\Gamma = \gamma([0, 2\pi])$

$$\gamma : \begin{cases} [0, 2\pi] \to \mathbb{R}^3, \\ (t) \mapsto \gamma(t) := \begin{pmatrix} \cos(t) \\ \sin(t) \\ ht \end{pmatrix}. \end{cases} \tag{3}$$

The curve γ *is known as the helix. We now would like to compute the total mass* $\mathcal{M}$ *of the helix whose density function* ρ *defined as follows*

$$\rho : \begin{cases} \Gamma \to \mathbb{R}, \\ (x, y, z) \mapsto \rho(x, y, z) := z. \end{cases} \tag{4}$$

Approach: By using recall 1.

Observation 1. *Since the parametrized curve* γ *from the given trajectory or path* Γ *is already known, we, therefore, do not have to seek any further parametrized curve. However, in case this information is not yet already given, it is necessary to find such parametrized curve, i.e. the section 2 is such an example.*

The mass of the helix is computed as follows

$$\therefore \quad \boxed{\begin{aligned} \mathcal{M}^{\text{helix}} &= \int_0^{2\pi} \rho \left\|\gamma'(t)\right\|_2 dt \\ &= \int_0^{2\pi} ht \sqrt{\sin^2 t + \cos^2 t + h^2}\, dt \\ &= \int_0^{2\pi} h\sqrt{1 + h^2}\, t\, dt \\ &= h\sqrt{1 + h^2}\, \frac{t^2}{2}\bigg|_0^{2\pi} \\ &= 2\pi^2 h\sqrt{1 + h^2}. \end{aligned}}$$

2 Step-by-step with Line integral of vector field

Example 2. *Examine the following trajectory* Γ_1 *as shown in Figure 1 from the origin point* $(0,0)$ *to point* $(1,1)$*. Besides, the vector field* f *is given as follows*

$$\boldsymbol{f}: \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2, \\ (x,y) \mapsto \boldsymbol{f}(x,y) = \begin{pmatrix} xe^y \\ \sin(x) + y \end{pmatrix}, \end{cases}$$

We would like now to compute the line integral of the vector field $\boldsymbol{f}$*, i.e. Work integral/Arbeitsintegral*

$$\int_{\Gamma_1} \boldsymbol{f} \cdot d\boldsymbol{x}. \tag{5}$$

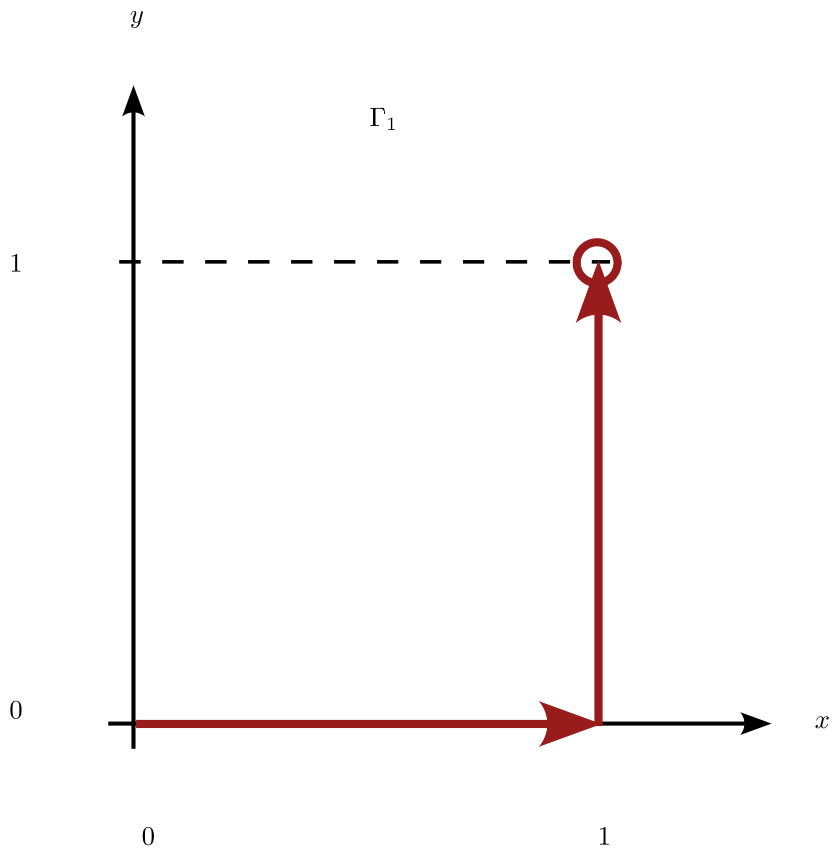

Figure 1: Discretization of domain Ω with n points.

Approach: By using recall 2.

Observation 2. *There is no parametrized curve for the trajectory* Γ_1*. Therefore, we shall need to find a parametrized curve for the path* Γ_1*. Since the path* Γ_1 *is a union of two smaller paths* l_1 *and* l_2*, we obtain the following relation*

$$\boxed{\Gamma_1 := \texttt{Image}(l_1) \cup \texttt{Image}(l_2)}$$

where the parametrized l_1 *and* l_2*, and their gradients take the forms*

$$l_1(t) := \begin{pmatrix} t \\ 0 \end{pmatrix} \text{ for } t \in [0,1] \Rightarrow l_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$l_2(t) := \begin{pmatrix} 1 \\ t \end{pmatrix} \text{ for } t \in [0,1] \Rightarrow l_2' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Substitution of $l_1(t)$ *and* $l_2(t)$ *into* $\boldsymbol{f}$ *leads to the following expressions*

$$\boldsymbol{f}(l_1(t)) = \begin{pmatrix} te^0 \\ \sin(t) + 0 \end{pmatrix} = \begin{pmatrix} t \\ \sin(t) \end{pmatrix},$$
$$\boldsymbol{f}(l_2(t)) = \begin{pmatrix} 1e^t \\ \sin(1) + t \end{pmatrix} = \begin{pmatrix} e^t \\ \sin(1) + t \end{pmatrix}.$$

The line integral of the vector field is computed as follows

$$\boxed{\begin{aligned} \int_{\Gamma_1} \boldsymbol{f} \cdot d\boldsymbol{x} &= \int_{l_1} \boldsymbol{f} \cdot d\boldsymbol{x} + \int_{l_2} \boldsymbol{f} \cdot d\boldsymbol{x} \\ &= \int_0^1 \Big\langle f(l_1(t)), l_1'(t) \Big\rangle \, dt + \int_0^1 \Big\langle f(l_2(t)), l_2'(t) \Big\rangle \, dt \\ &= \int_0^1 t \, dt + \int_0^1 \big(\sin(1) + t\big) \, dt \\ &= \sin(1) + \int_0^1 2t \, dt \\ &= \sin(1) + \Big[t^2\Big]_0^1 \end{aligned}}$$

Therefore, we obtain

$$\therefore \quad \boxed{\int_{\Gamma_1} \boldsymbol{f} \cdot d\boldsymbol{x} = \sin(1) + 1.}$$

3 Demo mini programming exercise (A1-HA09)

```
clear all;
close all;
clc;
img = flipud(imread('rogowski_in_color.jpeg'));
size(img)
image(img)
A = double(rgb2gray(img));
size(A)
image(A)
pcolor(A);
shading interp; colormap('gray');
axis equal
[U,S,V] = svd(A);
figure(1)
set(gcf,'Position',[10 200 700 700])
semilogy(diag(S), '--', 'LineWidth', 2, 'Color', '[0.8500 0.3250 0.0980]');
title('Singular values versus indices in semi-log scale',...
    'fontsize', 20,...
    'interpreter','latex');
xlabel('Index of singular value [-]', 'fontsize', 15, 'interpreter','latex');
ylabel('Singular values [-]', 'fontsize', 15, 'interpreter', 'latex');
xticks([0:250:2500, 2731])
set(gca,'TickLabelInterpreter','latex');
grid on
% print -depsc pics_rogowski_svd.eps
print -dpng pics_rogowski_svd.png
%------------------------------------------------
n = 100;
S_re = S * diag([ones(1,n), zeros(1,size(S,2)-n)]);
figure(2)
set(gcf,'Position',[900 200 600 400])
pcolor(U*S_re*V'); shading interp; colormap('gray'); axis equal
title('Number of singular values used: $n=100/2731$',...
    'fontsize', 20,...
    'interpreter','latex');
xlabel('Width of image matrix', 'fontsize', 15, 'interpreter','latex');
ylabel('Height of image matrix', 'fontsize', 15, 'interpreter','latex');
xticks([0:500:3500,4096])
yticks([0:250:2500,2731])
```

```
40 % ylim([-30 950])
41 set(gca,'TickLabelInterpreter','latex');
42 grid on
43 % print -depsc -tiff -r300 -painters plots.eps
44 print -dpng plots.png
```

Remark 1. *Depending on the size of photos to be examined, number in lines 22, 38, and 39 should be adjusted accordingly.*

Remark 2. *The lines of* `matlab` *code above is uploaded in Moodle and ready for running tests.*