Self-exercise - SRU03
Response to questions

Tuan Vo

09$^{\text{th}}$ November, 2022

1 Step-by-step with A3-SRU03

2 Step-by-step with A4-SRU03

Example 1. *Examine the consistency order of the following explicit numerical scheme*

$$y^{j+1} = y^j + \frac{h}{2}\Big(f(t^j, y^j) + f(t^{j+1}, y^j + hf(t^j, y^j))\Big), \tag{1}$$

which is used to approximate the ordinary differential equation (ODE):

$$y'(t) = f(t, y(t)). \tag{2}$$

Approach: The exact solution at time $t+h$ is obtained by Taylor series expansion as

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2}y''(t) + \mathcal{O}(h^3)$$

with

$$\begin{aligned} y' &= f\big(t, y\big) \\ y'' &= \frac{\mathrm{d}}{\mathrm{d}t}f\big(t, y\big) = f_t\big(t, y(t)\big) + f_y\big(t, y(t)\big)y'(t) = f_t\big(t, y(t)\big) + f_y\big(t, y(t)\big)f\big(t, y(t)\big). \end{aligned}$$

Thus

$$y(t+h) = y^j + h\,f(t, y^j) + \frac{h^2}{2}\Big(f_t(t, y^j) + f_y(t, y^j)\,f(t, y^j)\Big) + \mathcal{O}(h^3)$$

The approximation can be written as

$$y^{j+1} = y^j + h\,\Phi(h), \quad \text{with} \quad \Phi(h) = \frac{1}{2}\Big[f(t, y^j) + f(t+h, y^j + hf(t, y^j))\Big].$$

We now expand $\Phi(h)$ around 0, i.e., $\Phi(h) = \Phi(0) + h\,\Phi'(0) + \mathcal{O}(h^2)$. Here

$$\Phi'(h) = \frac{1}{2}f_t(t+h, y^j + hf(t, y^j)) + \frac{1}{2}f_y(t+h, y^j + hf(t, y^j))f(t, y^j).$$

Thus

$$\Phi(0) = f(t, y^j), \quad \Phi'(0) = \frac{1}{2}f_t(t, y^j) + \frac{1}{2}f_y(t, y^j)f(t, y^j)$$

and

$$\Phi(h) = f(t, y^j) + \frac{h}{2}\Big(f_t(t, y^j) + f_y(t, y^j)\, f(t, y^j)\Big) + \mathcal{O}(h^2).$$

Therefore

$$y^{j+1} = y^j + h\, f(t, y^j) + \frac{h^2}{2}\Big(f_t(t, y^j) + f_y(t, y^j)\, f(t, y^j)\Big) + \mathcal{O}(h^3).$$

The error $\left|y(t+h) - y^{j+1}\right| = \mathcal{O}(h^3)$ and thus the method has consistency order 2.