Global exercise - GUE08

Tuan Vo

08th December, 2022

Content covered:

✓ Numerics:

1. Review: QR-decomposition by using
   (a) Givens-Rotation
   (b) Householder-Reflection
2. QR algorithm to find all eigenvalues of a matrix A.

✓ Analysis: Application of Hölder's inequality to approximate integral

1 Numerics: Review of QR-decomposition

There are two main methods used to decompose a matrix into an orthogonal matrix Q and a right upper triangular matrix R

1. Givens-Rotation: ideally for **sparse** matrices.
   → Detect non-zero entries standing below the diagonal,
   → Clean them up by applying the corresponding Givens-Rotation matrix.

2. Householder-Reflection: ideally for **dense** matrices

1.1 Step-by-step with Givens-Rotation

Example 1. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{pmatrix}_{3\times 3}. \tag{1}$$

The only entry to be cleaned up is A_{32}. Therefore, the Givens-Rotation matrix is

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2+b^2} = \sqrt{A_{22}^2 + A_{32}^2} = \sqrt{2}, \\ c = a/r = -1/\sqrt{2}, \\ s = -b/r = -1/\sqrt{2}, \end{cases} \tag{2}$$

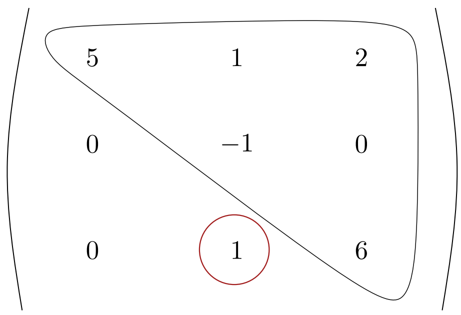

Figure 1: The entry A_{32} of matrix A is the only one that we may clean up so as to obtain a right upper triangular matrix. Givens-Rotation matrix is G_{32}.

Then, applying G_{32} onto A from the left leads to

$$G_{32}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 2 \\ 0 & \sqrt{2} & 6/\sqrt{2} \\ 0 & 0 & -6/\sqrt{2} \end{pmatrix} =: B, \tag{3}$$

which has already a form of a right upper triangular matrix R. Therefore, we obtain

$$G_{32}A = R, \tag{4}$$

which, equally, leads to

$$G_{32}A = R \Leftrightarrow G_{32}^{-1}G_{32}A = G_{32}^{-1}R \Leftrightarrow A = G_{32}^{-1}R \Leftrightarrow A = G_{32}^{\top}R. \tag{5}$$

Note in passing that $G_{32}^{-1} = G_{32}^{\top}$ in the previous step is due to the fact that the matrix G_{32} itself is orthogonal. By assigning $Q := G_{32}^{\top}$ we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{32}^{\top}R = QR.}$$

Example 2. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix}_{3\times 3}. \tag{6}$$

The only entry to be cleaned up is A_{21}. Therefore, the Givens-Rotation matrix is

$$G_{21} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2+b^2} = \sqrt{A_{11}^2 + A_{21}^2} = \sqrt{26}, \\ c = a/r = 5/\sqrt{26}, \\ s = -b/r = -1/\sqrt{26}, \end{cases} \tag{7}$$

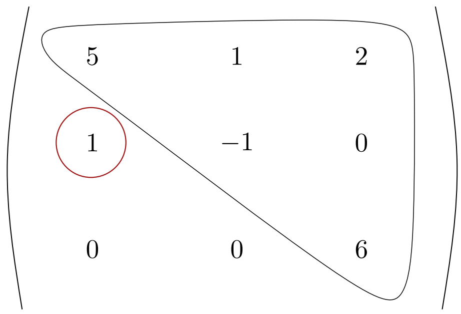

Figure 2: The entry A_{21} of matrix A is the only one that we may clean up so as to obtain a right upper triangular matrix. Givens-Rotation matrix is G_{21}.

Then, applying G_{21} onto A from the left leads to

$$G_{21}A = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{26} & 4/\sqrt{26} & 10/\sqrt{26} \\ 0 & -6/\sqrt{26} & -2/\sqrt{26} \\ 0 & 0 & 6 \end{pmatrix} \tag{8}$$

which has already a form of a right upper triangular matrix R. Therefore, we obtain

$$G_{21}A = R, \tag{9}$$

which, equally, leads to

$$G_{21}A = R \Leftrightarrow G_{21}^{-1}G_{21}A = G_{21}^{-1}R \Leftrightarrow A = G_{21}^{-1}R \Leftrightarrow A = G_{21}^{\top}R. \tag{10}$$

Note in passing that $G_{21}^{-1} = G_{21}^{\top}$ in the previous step is due to the fact that the matrix G_{21} itself is orthogonal. By assigning $Q := G_{21}^{\top}$ we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{21}^{\top}R = QR.}$$

Example 3. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 6 \end{pmatrix}_{3\times 3} . \tag{11}$$

The only entry to be cleaned up is A_{31}. Therefore, the Givens-Rotation matrix is

$$G_{31} = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2+b^2} = \sqrt{A_{11}^2 + A_{31}^2} = \sqrt{26}, \\ c = a/r = 5/\sqrt{26}, \\ s = -b/r = -1/\sqrt{26}, \end{cases} \tag{12}$$

Then, applying G_{31} onto A from the left leads to

$$G_{31}A = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & -1 & 0 \\ 0 & -1/\sqrt{26} & 28/\sqrt{26} \end{pmatrix} =: B, \tag{13}$$

which is not yet in forms of a right upper triangular matrix R. Herein, we still need to perform one more Givens-Rotation on the later matrix to clean up the entry B_{32}. The second Givens-Rotation takes the form

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix}_{3\times 3} \quad \text{where} \quad \begin{cases} r_2 = \sqrt{a_2^2+b_2^2} = \sqrt{B_{11}^2 + B_{31}^2} = 9/\sqrt{78}, \\ c_2 = a_2/r_2 = -\sqrt{78}/9, \\ s_2 = -b_2/r_2 = \sqrt{3}/9, \end{cases} \tag{14}$$

Next, applying G_{32} onto $B = G_{31}A$ leads to

$$G_{32}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & -1 & 0 \\ 0 & -1/\sqrt{26} & 28/\sqrt{26} \end{pmatrix} \tag{15}$$

which yields

$$G_{32}B = \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & 3\sqrt{3}/\sqrt{26} & -28\sqrt{3}/(9\sqrt{26}) \\ 0 & 0 & -28\sqrt{3}/9 \end{pmatrix} \tag{16}$$

which now has a form of a right upper triangular matrix R. Therefore, we obtain

$$G_{32}B = R \Leftrightarrow G_{32}G_{31}A = R \Leftrightarrow A = G_{31}^{-1}G_{32}^{-1}R \Leftrightarrow A = G_{31}^{\top}G_{32}^{\top}R. \tag{17}$$

By assigning $Q := G_{31}^{\top}G_{32}^{\top}$ we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{31}^{\top}G_{32}^{\top}R = QR.}$$

Example 4. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 6 \end{pmatrix}_{3\times 3} . \tag{18}$$

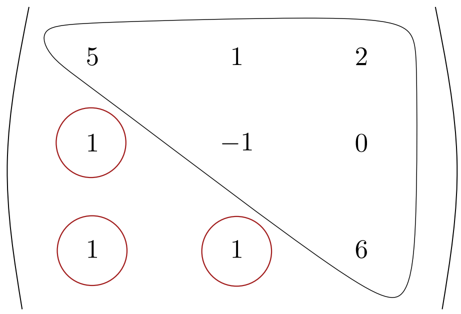

Figure 3: The entry A_{21}, A_{31}, and A_{32} of matrix A are the three entries that we would like clean up so that we may obtain a right upper triangular matrix. Givens-Rotation matrix for A_{21}, A_{31}, and A_{32} is G_{21}, G_{31}, and G_{32}, respectively. The order of applying G_{21} first or G_{31} or G_{32} is no matter, but the coefficients used to build the Givens-Rotation matrix r, a, and b have to be carefully computed, since they become various after every time applying the rotation.

The Givens-Rotation matrix G_{21} takes the form

$$G_{21} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19}$$

The Givens-Rotation matrix G_{31} takes the form

$$G_{31} = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \tag{20}$$

The Givens-Rotation matrix G_{32} takes the form

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \tag{21}$$

1.2 Step-by-step with Householder-Reflection

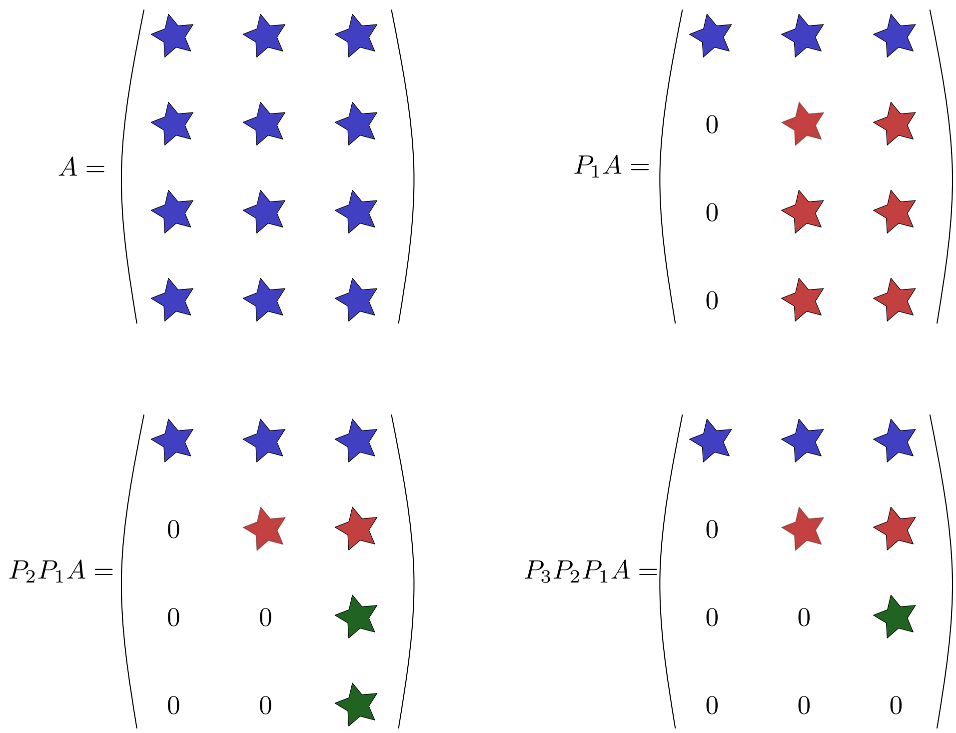

Figure 4: How Householder-Reflection actually works: Given matrix A. We would like to apply a matrix P_1 on the left side of matrix A in such a way that the resulting matrix has the first row unchanged, and any first entry from the second row onwards turn to zero. This procedure is repeated until the final resulting matrix becomes a right upper triangular matrix.

Let us consider the first column of matrix A

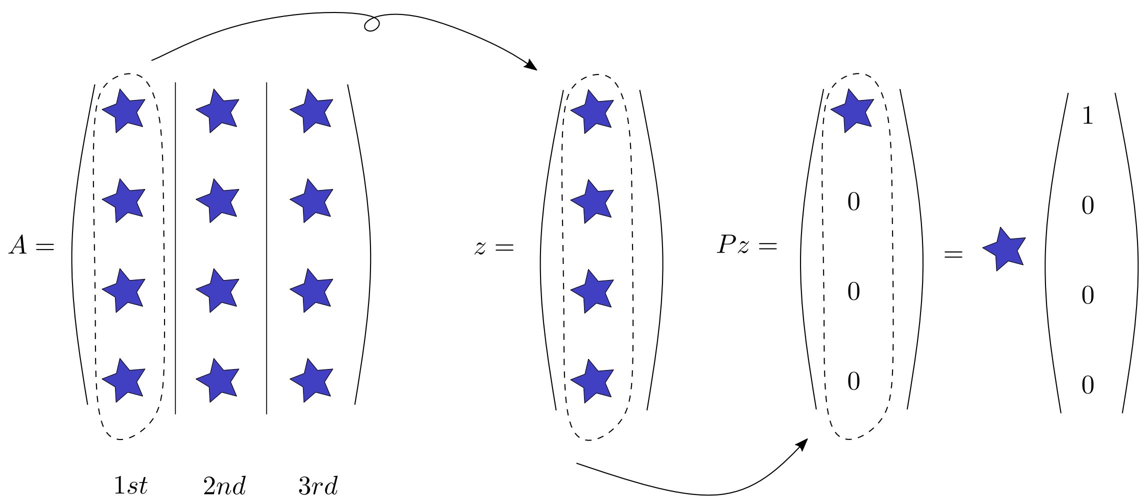

Figure 5: Consider the first column of matrix A.

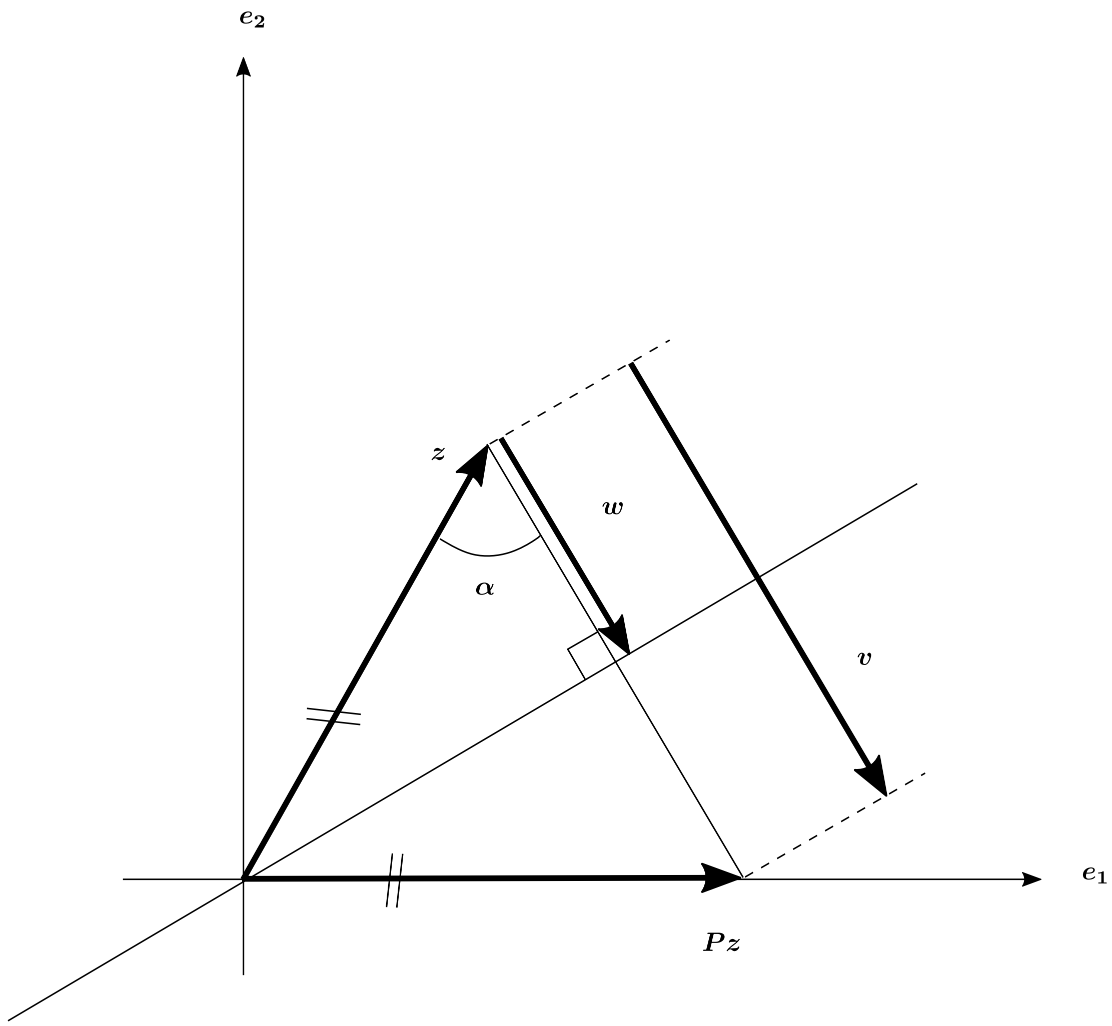

Figure 6: Consider the first column of matrix A.

Vector $\boldsymbol{z}$ is known. Then, we would like to apply matrix P on $\boldsymbol{z}$ such a way that the resulting vector $\boldsymbol{Pz}$ is not streching and laying along $\boldsymbol{e_1}$ axis. The following relation holds

$$\boxed{Pz = z + v} \tag{22}$$

but we also have

$$\|Pz\| = \|z\| \tag{23}$$

hence, the vector Pz should take the form

$$Pz = \|z\|\, e_1 \tag{24}$$

Substitution

$$\|z\|\, e_1 = z + v \tag{25}$$

which leads to

$$\boxed{v = \|z\|\, e_1 - z} \tag{26}$$

Next, we shall need to find a multiplicative expression of v w.r.t. z. Hence, we will consider the angle between v and z as follows

$$-z \cdot v = \|z\|\|v\| \cos(\alpha) \Rightarrow \cos(\alpha) = \frac{-z \cdot v}{\|z\|\|v\|}. \tag{27}$$

Besides, the following relation also holds

$$\cos(\alpha) = \frac{\|w\|}{\|z\|} \Rightarrow \|w\| = \|z\| \cos(\alpha). \tag{28}$$

Substitution of (27) into (28) yields

$$\|w\| = \|z\| \frac{-z \cdot v}{\|z\|\|v\|} = \frac{-z \cdot v}{\|v\|}, \tag{29}$$

which further leads to

$$\begin{aligned}
v = 2w = 2\|w\| \frac{w}{\|w\|} &= 2\|w\| \frac{v}{\|v\|} && (30)\\
&= 2\frac{-z \cdot v}{\|v\|} \frac{v}{\|v\|} && (31)\\
&= 2\frac{-v \cdot z}{\|v\|} \frac{v}{\|v\|} && (32)\\
&= 2\frac{v}{\|v\|} \frac{-v \cdot z}{\|v\|} && (33)\\
&= -2\frac{v}{\|v\|} \frac{v^\top z}{\|v\|} && (34)\\
&= -2\frac{vv^\top z}{\|v\|^2}. && (35)
\end{aligned}$$

Finally, substitution (30) into (22) results in

$$\begin{aligned}
Pz = z - 2\frac{vv^\top z}{\|v\|^2} &= Iz - 2\frac{vv^\top z}{\|v\|^2} && (36)\\
&= Iz - 2\frac{vv^\top}{\|v\|^2} z && (37)\\
&= \left(I - 2\frac{vv^\top}{v^\top v}\right) z && (38)
\end{aligned}$$

Therefore, the Householder-Reflection matrix takes the form

$$\therefore \quad \boxed{P := I - 2\frac{vv^\top}{v^\top v},} \tag{39}$$

which can be narrated that applying the Householder-Reflection matrix P onto any vector z from the left, one obtain reflected image of vector z over a hyperplane.

Example 5. *Examine the following matrix A given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 6 \end{pmatrix}_{3\times 3}. \tag{40}$$

Based on (26) we obtain

$$v = \|z\|\, e_1 - z = \sqrt{5^2 + 1^2 + 1^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{27} - 5 \\ -1 \\ -1 \end{pmatrix} \tag{41}$$

Hence, the components $vv^\top$ and $v^\top v$ are computed as follows

$$vv^\top = \begin{pmatrix} \sqrt{27} - 5 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} \sqrt{27} - 5 & -1 & -1 \end{pmatrix} = \cdots \tag{42}$$

Besides,

$$v^\top v = \begin{pmatrix} \sqrt{27} - 5 & -1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{27} - 5 \\ -1 \\ -1 \end{pmatrix} = 54 - 10\sqrt{27} \tag{43}$$

which leads to

$$P_1 = I - 2\frac{vv^\top}{v^\top v} = \cdots = \tag{44}$$

Therefore, we obtain

$$\boxed{P_1 A = \cdots = \begin{pmatrix} 5.1962 & 0.9623 & 3.0792 \\ 0 & -0.8075 & -5.5019 \\ 0 & 1.1925 & 0.4981 \end{pmatrix}} \tag{45}$$

Since the matrix $P_1 A$ is not yet in forms of a right upper triangular matrix, we will need to perform another Householder-Reflection, as follows

$$v_2 = \|z_2\|\, e_1 - z_2 = \sqrt{0.8075^2 + 1.1925^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{27} - 5 \\ -1 \\ -1 \end{pmatrix} \tag{46}$$

2 Numerics: QR-algorithm

Observation 1 (**QR-algorithm** vs. **QR-decomposition**). *QR-algorithm (as seen in Maths III) is not QR-decomposition (as seen in Maths II). QR-algorithm (Maths III) is an algorithm used to find* ***all*** *eigenvalues of a matrix A* ***numerically****. Meanwhile, QR-decomposition is a technique in linear algebra used to decompose a matrix A into an orthogonal matrix Q and a right upper triangular matrix R. Nevertheless, we will still need QR-decomposition for QR-algorithm.*

3 Analysis: Application of Hölder's inequality

Observation 2. *Sometimes we would like to estimate whether an integral, from complicated to very complicated, is bounded at a certain value or not, without actually compute it. The Hölder's inequality is a useful mathematical tool for such situation.*

Example 6. *Estimate the value of the following integral*

$$\int_{\Omega} (x+2)^{-3/5} \exp(-2x/3)\, dx$$

Observation 3. *The given integral is complicated to compute, but it would be easier if we may split the exponential term away from the polynomial term.*

Approach: Setting

$$f(x) := (x+2)^{-3/5} \tag{47}$$
$$g(x) := \exp(-2x/3) \tag{48}$$

Given $f(x) \in L^6(\Omega)$, then we can compute

$$\begin{aligned} \left\| f(x) \right\|_{L^6(\Omega)}^6 &= \int_0^\infty |f(x)|^6 \, dx \\ &= \int_0^\infty (x+2)^{-3 \cdot 6/5} \, dx \\ &= -\frac{5}{13}(x+2)^{-13/5} \Big|_{x=0}^{x=\infty} = \frac{5}{13} \cdot 2^{-13/5} < \infty. \end{aligned} \tag{49}$$

Therefore, we obtain

$$\therefore \quad \boxed{\left\| f(x) \right\|_{L^6(\Omega)}^6 = \frac{5}{13} \cdot 2^{-13/5} < \infty.} \tag{50}$$

Besides, the following relation hold for Hölder's inequality

$$\frac{1}{6} + \frac{1}{q} = 1 \Leftrightarrow q = \frac{6}{5} \tag{51}$$

which leads to $g(x) \in L^q(\Omega)$:

$$\begin{aligned}
\left\| g(x) \right\|_{L^{6/5}(\Omega)}^{6/5} &= \int_0^\infty |g(x)|^{6/5}\, dx \\
&= \int_0^\infty \exp\left(-\frac{2}{3}\cdot\frac{6}{5}\cdot x\right) dx \\
&= -\frac{5}{4}\exp\left(-\frac{4}{5}x\right)\Bigg|_{x=0}^{x=\infty} = \frac{5}{4} < \infty.
\end{aligned} \tag{52}$$

Therefore, we obtain

$$\therefore \quad \boxed{\left\| g(x) \right\|_{L^{6/5}(\Omega)}^{6/5} = \frac{5}{4} < \infty.} \tag{53}$$

Finally, we can estimate the integral by using the Hölder's inequality

$$\therefore \quad \boxed{\begin{aligned}
\int_\Omega \frac{\exp\left(-2x/3\right)}{\sqrt[5]{(x+2)^3}}\, dx = \left\| f(x)\, g(x) \right\|_{L^1(\Omega)} &\leq \left\| f(x) \right\|_{L^p(\Omega)} \left\| g(x) \right\|_{L^q(\Omega)} \\
&= \left\| f(x) \right\|_{L^6(\Omega)} \left\| g(x) \right\|_{L^{6/5}(\Omega)} \\
&= \left(\frac{5\cdot 2^{-13/5}}{13}\right)^{1/6} \left(\frac{5}{4}\right)^{5/6} \\
&< \infty.
\end{aligned}} \tag{54}$$

Example 7. *Estimate the value of the following integral*

$$\int_\Omega \underbrace{\exp\left(-2x/3\right)}_{f(x)} \underbrace{(x+2)^{-4/3}}_{g(x)}\, dx \tag{55}$$

Similarly, we estimate the value of the integral for $p = 3/2$ and $q = 3$ as follows

$$\begin{aligned}
\int_\Omega \frac{\exp\left(-2x/3\right)}{\sqrt[3]{(x+2)^4}}\, dx &\leq \underbrace{\left(\int_0^\infty \left(\exp\left(-2x/3\right)\right)^{3/2} dx\right)^{2/3}}_{\|f(x)\|_{L^{3/2}(\Omega)}} \underbrace{\left(\int_0^\infty \left((x+2)^{-4/3}\right)^3 dx\right)^{1/3}}_{\|g(x)\|_{L^3(\Omega)}} \\
&= 1^{2/3}\cdot\left(\frac{1}{24}\right)^{1/3} = \frac{1}{\sqrt[3]{24}}.
\end{aligned} \tag{56}$$

Therefore, the integral is easily estimated

$$\therefore \quad \boxed{\int_\Omega \frac{\exp\left(-2x/3\right)}{\sqrt[3]{(x+2)^4}}\, dx \leq \frac{1}{\sqrt[3]{24}}.} \tag{57}$$