Global exercise - GUE13-extra + SRU13

Tuan Vo + Jan Habscheid

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1 Step-by-step with A2-SRU12

Examine the surface integral in the Stokes' theorem.

(i) Parametrized surface S given in the cylindrical coordinate

$$\Phi(s,t) = \begin{pmatrix} s\cos(t) \\ s\sin(t) \\ t \end{pmatrix} \quad \forall\, s \in [0,1],\ t \in [0,\pi/2]. \tag{1}$$

$$\text{Surface } S : \Phi(s,t) = \Big(s\,\cos(t), s\,\sin(t),\, t\Big)^T,\ \ s \in [0,1],\ t \in [0,\pi/2]$$

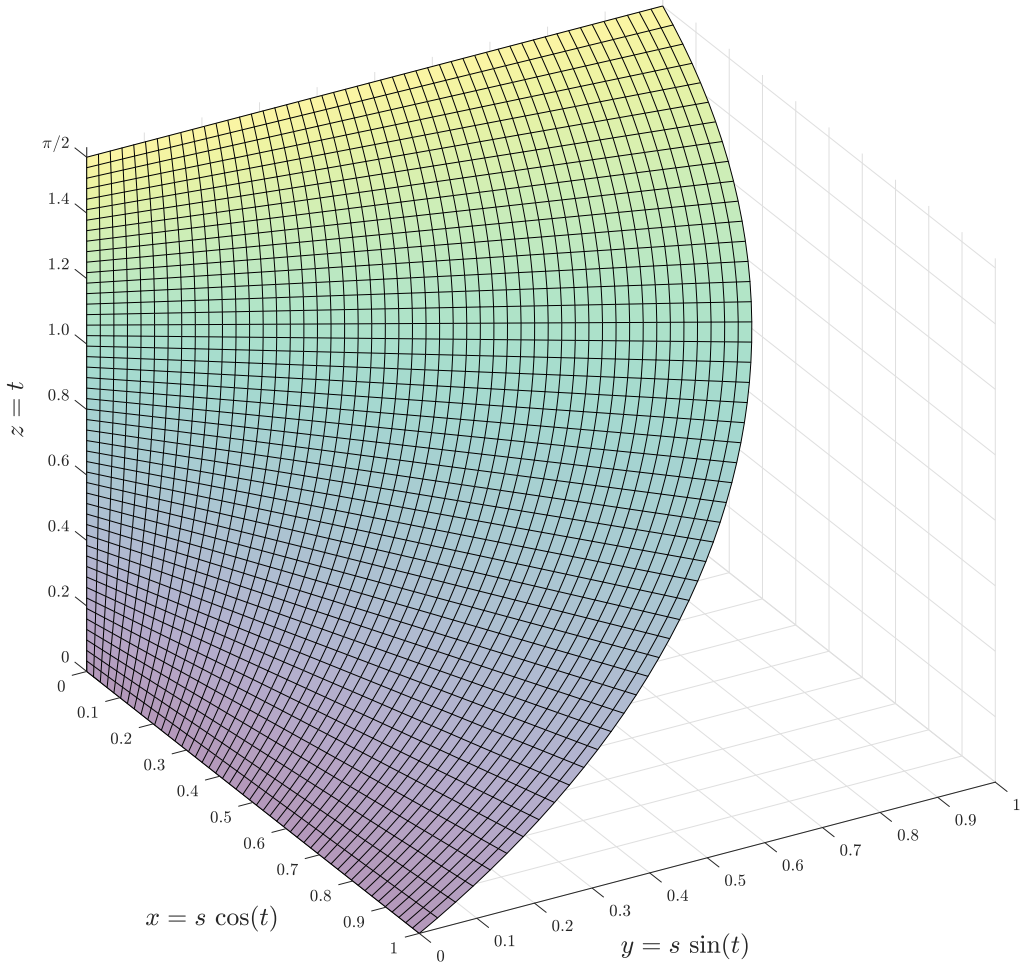

Figure 1: Surface S.

(ii) **Outward normal vector** of the surface S

$$\boldsymbol{n} := \partial_s\Phi(s,t) \times \partial_t\Phi(s,t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} \times \begin{pmatrix} -s\,\sin(t) \\ s\,\cos(t) \\ 1 \end{pmatrix}$$
$$= \begin{vmatrix} i & j & k \\ \cos(t) & \sin(t) & 0 \\ -s\,\sin(t) & s\,\cos(t) & 1 \end{vmatrix} = \begin{pmatrix} \sin(t) \\ -\cos(t) \\ s \end{pmatrix} \tag{2}$$

> **Observation 1.** *Note in passing that the z-component of the above normal vector for the surface S is already positive, which satisfies the convention mentioned in this exercise for an* ***outward*** *normal vector. It is due to the fact that the cross product $\partial_s\Phi(s,t) \times \partial_t\Phi(s,t)$ is different from $\partial_t\Phi(s,t) \times \partial_s\Phi(s,t)$ in terms of sign. We can have a quick check as follows*
>
> $$\partial_t\Phi(s,t) \times \partial_s\Phi(s,t) = \begin{pmatrix} -s\,\sin(t) \\ s\,\cos(t) \\ 1 \end{pmatrix} \times \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} \tag{3}$$
> $$= \begin{vmatrix} i & j & k \\ -s\,\sin(t) & s\,\cos(t) & 1 \\ \cos(t) & \sin(t) & 0 \end{vmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ -s \end{pmatrix} = -\boldsymbol{n} \tag{4}$$
>
> *whose z-component is obviously negative, and according to the convention mentioned in this exercise this normal vector is* ***not*** *an outward normal vector. Nevertheless, either option is still correct, and we just have to be consistent with the option that we have chosen. In the context of this exercise, we will go with the positive z-component.*

(iii) **Norm** of the outward normal vector

$$\|\boldsymbol{n}\|_2 := \|\partial_s\Phi(s,t) \times \partial_t\Phi(s,t)\|_2 \overset{(2)}{=} \sqrt{\sin^2(t) + \big(-\cos(t)\big)^2 + s} = \sqrt{1+s} \tag{5}$$

(iv) **Outward unit normal vector**: Normalization of the normal vector

$$\boldsymbol{\nu} := \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|_2} \overset{(2),(5)}{=} \frac{1}{\sqrt{1+s}} \begin{pmatrix} \sin(t) \\ -\cos(t) \\ s \end{pmatrix} \tag{6}$$

(v) Now, consider a vector field $\boldsymbol{f}$ defined as follows

$$\boldsymbol{f} : \begin{cases} \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3 \\ (x,y,z) \mapsto \boldsymbol{f}(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix} := \begin{pmatrix} z \\ x \\ y \end{pmatrix}. \end{cases} \tag{7}$$

(vi) The Stokes' theorem requires the equality of a surface integral and a line integral, meaning that

$$\int_S (\nabla \times \boldsymbol{f}) \cdot d\boldsymbol{\sigma} = \oint_{\partial S} \boldsymbol{f} \cdot d\boldsymbol{s} \tag{8}$$

(vii) Examine the surface integral:

$$\int_S (\nabla \times \boldsymbol{f}) \cdot d\boldsymbol{\sigma} \tag{9}$$

which is also written as follows

$$\int_S (\nabla \times \boldsymbol{f}) \cdot d\boldsymbol{\sigma} = \int_S (\nabla \times \boldsymbol{f}) \cdot \boldsymbol{\nu}\, d\sigma = \int_S \Big\langle (\nabla \times \boldsymbol{f}), \boldsymbol{\nu} \Big\rangle\, d\sigma \tag{10}$$

The curl of $\boldsymbol{f}$ is computed straightforwardly as follows

$$\begin{aligned}
\nabla \times \boldsymbol{f} &= \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix} \\
&= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_1(x,y,z) & f_2(x,y,z) & f_3(x,y,z) \end{vmatrix} = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ -\partial_x f_3 + \partial_z f_1 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix} \overset{(7)}{=} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{aligned} \tag{11}$$

Observation 2. *The integral* $(10)_3$ *is a surface integral of the first kind, i.e. surface integral of a* ***scalar*** *field. It is due to the fact that the inner product between the curl of vector field* $\boldsymbol{f}$*, i.e.* $(\nabla \times \boldsymbol{f})$*, and the outward unit normal vector* $\boldsymbol{\nu}$ *results in a* ***scalar****, as seen at* (11).

Recall 1. *Surface integral of a* ***scalar field*** $\phi : \Omega \to \mathbb{R}$ *is defined as*

$$\int_\Gamma \phi\, dA := \int_B \phi(\Phi(p,q)) \left\| \Phi_p \times \Phi_q \right\| d(p,q) \tag{12}$$

Note in passing that we can also write $\Phi_p \times \Phi_q = \partial_p \Phi \times \partial_q \Phi$.

Therefore, the integral $(10)_3$ is computed as follows

$$\int_S \Big\langle (\nabla \times \boldsymbol{f}), \boldsymbol{\nu} \Big\rangle \, d\sigma \overset{(11)}{=} \int_S \Big\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{\nu} \Big\rangle \, d\sigma \tag{13}$$

$$\overset{(12)}{=} \int_0^1 \int_0^{\pi/2} \Big\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{\nu} \Big\rangle \big\| \Phi_s \times \Phi_t \big\|_2 \, dt \, ds \tag{14}$$

$$\overset{(2)}{=} \int_0^1 \int_0^{\pi/2} \Big\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{\nu} \Big\rangle \big\| \boldsymbol{n} \big\|_2 \, dt \, ds \tag{15}$$

$$\overset{(6)}{=} \int_0^1 \int_0^{\pi/2} \Big\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|_2} \Big\rangle \big\| \boldsymbol{n} \big\|_2 \, dt \, ds \tag{16}$$

$$= \int_0^1 \int_0^{\pi/2} \Big\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{n} \Big\rangle \, dt \, ds \tag{17}$$

$$\overset{(2)}{=} \int_0^1 \int_0^{\pi/2} \Big\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \sin(t) \\ -\cos(t) \\ s \end{pmatrix} \Big\rangle \, dt \, ds \tag{18}$$

$$\overset{(2)}{=} \int_0^1 \int_0^{\pi/2} \Big(\sin(t) - \cos(t) + s \Big) \, dt \, ds \tag{19}$$

$$= \dots \tag{20}$$

$$= \frac{\pi}{4} \tag{21}$$

Consequentially, we need the unit normal vector $\boldsymbol{\nu}$. However, we may not need to compute its result explicitly, i.e. no need (5). This evidence points to the fact that the norm of the normal vector is canceled within integral, which can be seen at (16), due to the sense of the surface integral of the first kind. Therefore, it was actually not necessary to compute the norm, and we may compute the integral directly with the non-normalized normal vector $\boldsymbol{n}$.

2 Step-by-step with A2d-SRU13

Given

$$\nabla \times \boldsymbol{v} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \boldsymbol{e}_i\, \partial_{x_j} v_k \tag{22}$$

Evaluate

$$\begin{aligned}
\nabla \times (\nabla \times \boldsymbol{v}) &= \sum_{ijk} \epsilon_{ijk} \boldsymbol{e}_i \partial_{x_j} (\nabla \times \boldsymbol{v})_k && (23)\\
&= \sum_{ijk} \epsilon_{ijk} \boldsymbol{e}_i \partial_{x_j} \left(\sum_{lm} \epsilon_{klm} \partial_{x_l} v_m \right) && (24)\\
&= \sum_{ijk} \sum_{lm} \epsilon_{ijk} \boldsymbol{e}_i \partial_{x_j} \epsilon_{klm} \partial_{x_l} v_m && (25)\\
&= \sum_{ijklm} \epsilon_{kij} \epsilon_{klm} \boldsymbol{e}_i \partial_{x_j} \partial_{x_l} v_m && (26)\\
&= \sum_{ijklm} \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \right) \boldsymbol{e}_i \partial_{x_j} \partial_{x_l} v_m && (27)\\
&= \sum_{ijklm} \left(\delta_{il}\delta_{jm} \boldsymbol{e}_i \partial_{x_j} \partial_{x_l} v_m - \delta_{im}\delta_{jl} \boldsymbol{e}_i \partial_{x_j} \partial_{x_l} v_m \right) && (28)\\
&= \sum_{ijklm} \left(\boldsymbol{e}_i \partial_{x_j} \partial_{x_i} v_j - \boldsymbol{e}_i \partial_{x_j} \partial_{x_j} v_i \right) && (29)\\
&= \sum_{ij} \left(\boldsymbol{e}_i \partial_{x_j} \partial_{x_i} v_j - \boldsymbol{e}_i \partial_{x_j} \partial_{x_j} v_i \right) = \nabla(\nabla \cdot \boldsymbol{v}) - \Delta \boldsymbol{v} && (30)
\end{aligned}$$

Note in passing that

(i) From step (26) to step (27) we have used the equality

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \tag{31}$$

which also implies that there is no more index k from step (27) onwards.

(ii) From step (28) to step (29) we have used the equality

$$\begin{aligned}
\delta_{il}\, \partial_{x_l} = \partial_{x_i} \quad &\text{and} \quad \delta_{jm}\, v_m = v_j && (32)\\
\delta_{im}\, v_m = v_i \quad &\text{and} \quad \delta_{jl}\, \partial_{x_l} = \partial_{x_j} && (33)
\end{aligned}$$

which implies that the indices l and m are reduced.

Therefore, we have only two indices i and j.