Global exercise - GUE10

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22nd December, 2022

Content covered:

- ✓ Analysis: Review H9A3 and H9A4
- ✓ Analysis: Orientable surface and surface integral
- ✓ Numerics: Demo programming exercise 3 (PRU02)

1 Review H9A3

(white board)

2 Review H9A4

(white board)

3 Analysis: Orientable integral

Example 1. *Examine the* ***Möbius band*** *given as follows*

$$\vec{\gamma}:\begin{cases}(-1,1)\times(0,2\pi)\to\mathbb{R}^3,\\ (t,\phi)\mapsto\vec{\gamma}(t,\phi):=\begin{pmatrix}\cos(\phi)\\ \sin(\phi)\\ 0\end{pmatrix}+\frac{t}{2}\begin{pmatrix}\cos(\phi)\cos(\phi/2)\\ \sin(\phi)\cos(\phi/2)\\ \sin(\phi/2)\end{pmatrix}.\end{cases} \tag{1}$$

Show that this surface is ***not orientable****.*

Approach:

Proof. The normal field is computed as follows

$$\begin{aligned} n(t,\phi) &= \pm\frac{\partial_t\vec{\gamma}\times\partial_\phi\vec{\gamma}}{\|\partial_t\vec{\gamma}\times\partial_\phi\vec{\gamma}\|}\\ &= \frac{\mp 1}{a}\begin{pmatrix}\cos(\phi)\sin(\frac{\phi}{2})-\frac{t}{4}\sin(\phi)(1-\cos(\phi))\\ \sin(\phi)\sin(\phi/2)+\frac{t}{4}(\sin^2(\phi)+\cos(\phi))\\ -\cos(\phi/2)-\frac{t}{4}(1+\cos(\phi))\end{pmatrix}\end{aligned}$$

with

$$a=\sqrt{1+t\cos(\phi/2)+\frac{t^2}{16}(3+2\cos(\phi))}.$$

We choose the positive sign and consider the position $(1,0,0)^T=\vec{\gamma}(0,0)=\vec{\gamma}(0,2\pi)$:

$$\lim_{\phi\to 0} n(0,\phi)=\begin{pmatrix}0\\ 0\\ -1\end{pmatrix}\neq\lim_{\phi\to 2\pi} n(0,\phi)=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Since the normal field is not continuous on

$$\gamma([-1,1]\times[0,2\pi])$$

the surface is **not orientable**. □

4 Analysis: Surface integral

Example 2. *Examine the lateral surface of the* ***Frustum*** *given as follows*

$$K = \left\{ \boldsymbol{x} \in \mathbb{R}^3,\, 0 < r \leq R \,\middle|\, 0 \leq x_3 < H,\, 0 \leq x_1^2 + x_2^2 < \left(R - \frac{R-r}{H} x_3 \right)^2 \right\}.$$

where H is the height, and r is the radius.

Approach: By using the cylinder coordination $\boldsymbol{x} = (r\cos(\phi), r\sin(\phi), z)^T$ we obtain the parametrization of the lateral surface of the frustum, i.e. truncated cone

$$\boldsymbol{\gamma} : \begin{cases} (0, 2\pi) \times (0, H) \to \mathbb{R}^3, \\ (\phi, z) \mapsto \boldsymbol{\gamma}(\phi, z) := \begin{pmatrix} \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ \left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ z \end{pmatrix} \end{cases}.$$

Wir erhalten das Normalenfeld

$$\boxed{\begin{aligned} \partial_\phi \boldsymbol{\gamma}(\phi, z) \times \partial_z \boldsymbol{\gamma}(\phi, z) &= \begin{pmatrix} -\left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ 0 \end{pmatrix} \times \begin{pmatrix} -\frac{R-r}{H} \cos(\phi) \\ -\frac{R-r}{H} \sin(\phi) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ \left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ \left(R - \frac{R-r}{H} z\right) \frac{R-r}{H} \end{pmatrix} \end{aligned}}$$

und die Norm des Normalenfelds wird wie folgt berechnet

$$\boxed{||\partial_\phi \boldsymbol{\gamma}(\phi, z) \times \partial_z \boldsymbol{\gamma}(\phi, z)|| = \left(R - \frac{R-r}{H} z\right) \sqrt{1 + \frac{(R-r)^2}{H^2}}.}$$

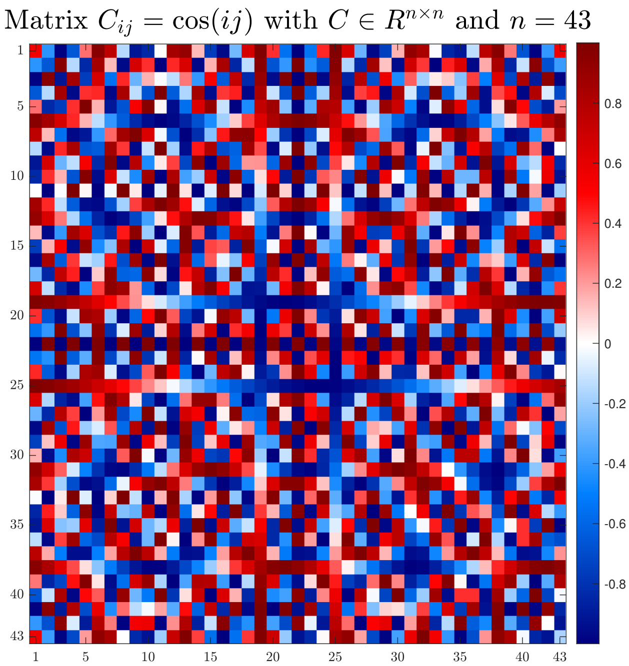
Somit ergibt sich für den Flächeninhalt der Mantelfläche

$$\begin{aligned} \mathcal{A} &= \int_0^H \int_0^{2\pi} ||\partial_\phi \boldsymbol{\gamma} \times \partial_z \boldsymbol{\gamma}||\, d\phi\, dz \\ &= 2\pi \sqrt{1 + \frac{(R-r)^2}{H^2}} \int_0^H \left(R - \frac{R-r}{H} z\right) dt = \pi \sqrt{1 + \frac{(R-r)^2}{H^2}}\, \frac{H}{R-r} \left(R^2 - r^2\right). \end{aligned}$$

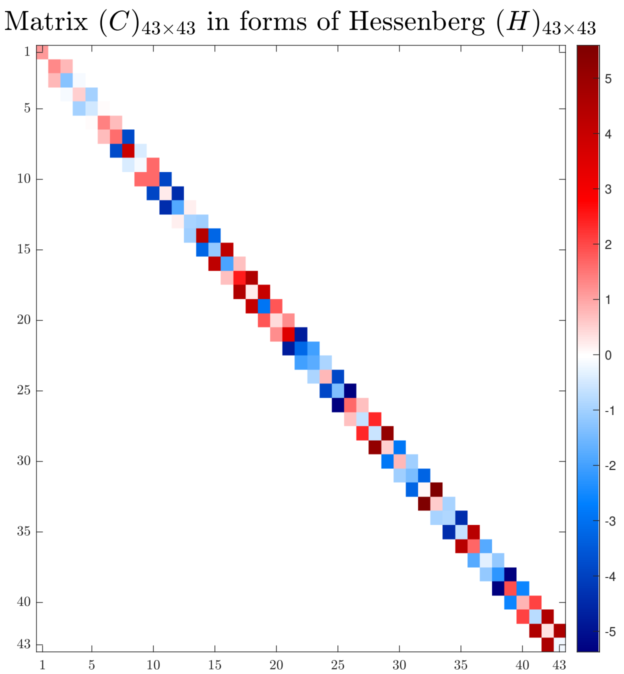
Daher ist der Flächeninhalt der Mantelfläche

$$\therefore \quad \boxed{\mathcal{A} = \int_0^H \int_0^{2\pi} ||\partial_\phi \boldsymbol{\gamma} \times \partial_z \boldsymbol{\gamma}||\, d\phi\, dz = \pi \sqrt{H^2 + (R-r)^2}(R+r).}$$

5 Numerics: Demo programming exercise 3 PRU(02)



(a)



(b)

Figure 1: Matrix size 43×43 and its Hessenberg form.

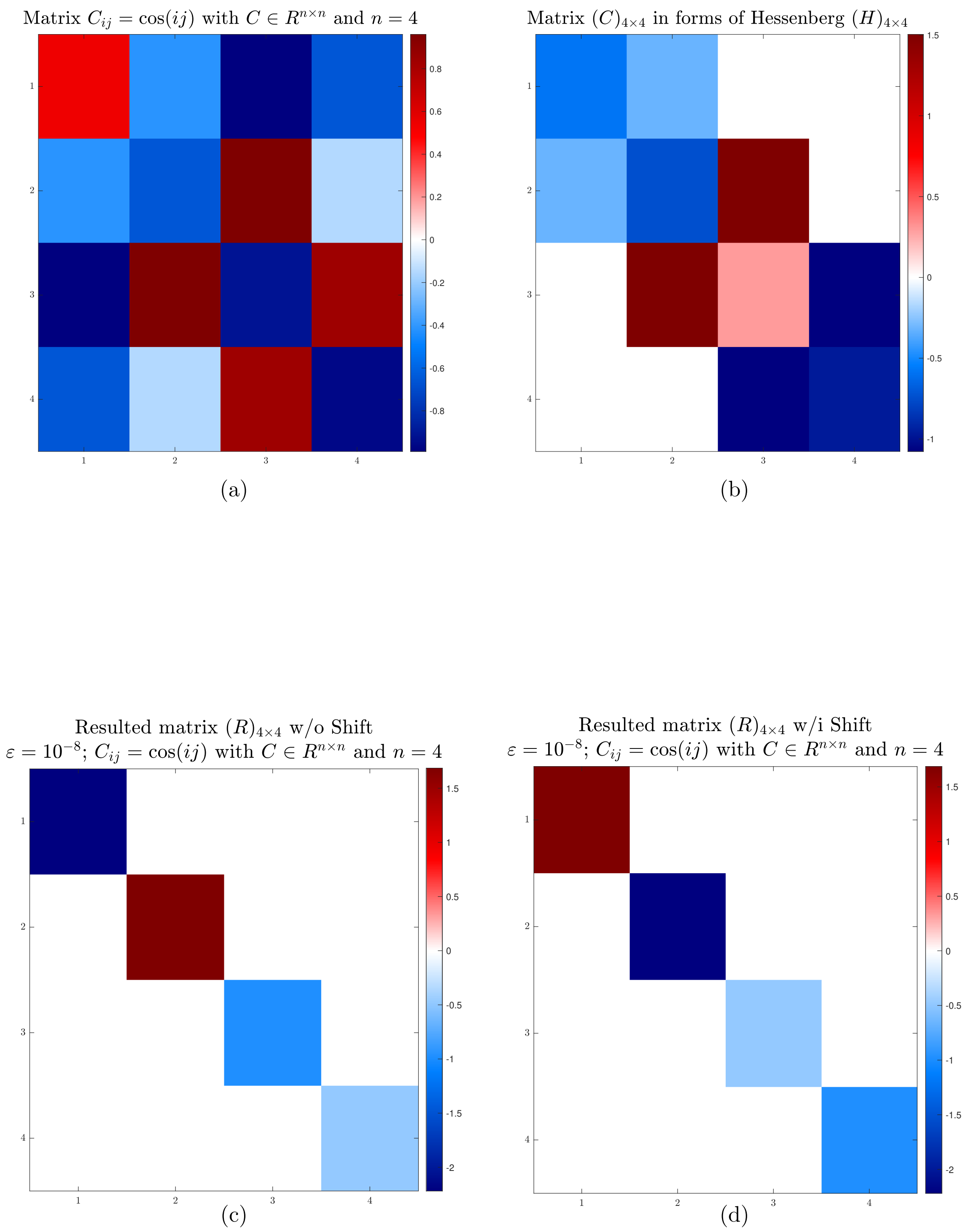
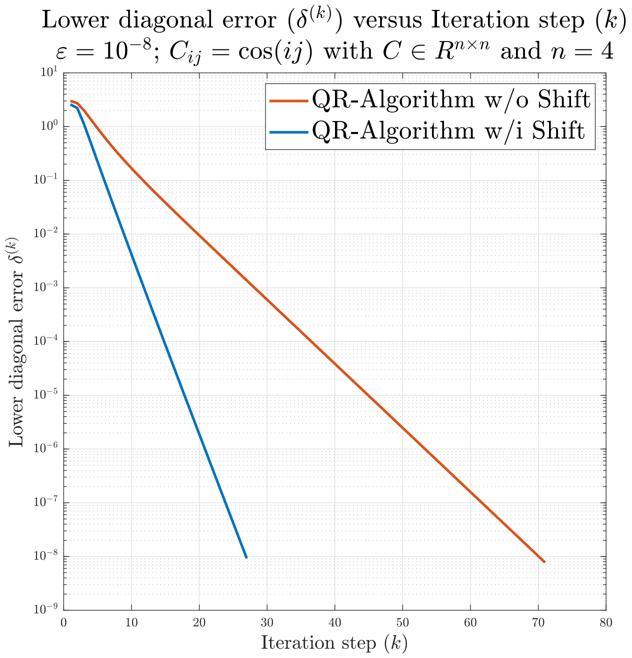
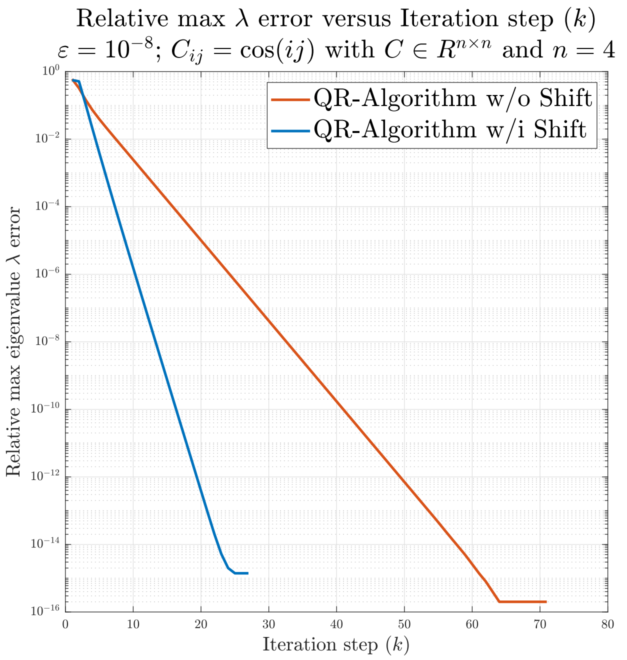


Figure 2: Matrix size 4×4 and its Hessenberg form.



(a)



(b)

Figure 3: Error plots for matrix size 4×4.