

# Self-exercise - SRU03

## Response to questions

Tuan Vo

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### 1 Step-by-step with A3-SRU03

**Example 1.** Let the linear  $m$ -step method be given by coefficients  $\alpha_l \in \mathbb{R}$  and  $\beta_l \in \mathbb{R}$  where  $l = 0, 1, \dots, m$ , satisfying the following system of equations

$$\sum_{l=0}^m a_l = 0, \quad (1)$$

$$\sum_{l=0}^m (l^q \alpha_l - q l^{q-1} \beta_l) = 0, \quad \text{for } q = 1, 2, 3, \dots, p. \quad (2)$$

Then the linear  $m$ -step method has the consistency order of  $p$ .

Consider the following ordinary differential equation (ODE) taking the form

$$x'(t) = f(t, x(t)), \quad \text{with } x(t_0) = x_0, \quad (3)$$

which may be approximated by a linear multistep method, taking the general form

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)). \quad (4)$$

**Observation 1.** There are 6 symbols arising in (1), (2), (4) to be aware of

1.  $m$  is the total number of steps used in the multistep method.
2.  $\alpha \in \mathbb{R}$  is the coefficient for the linear combination of  $x(t_j + lh)$
3.  $\beta \in \mathbb{R}$  is the coefficient for the linear combination of  $f(t_j + lh, x(t_j + lh))$ .
4.  $p$  is the consistency order.
5.  $l$  is the dummy index, running from 0 to  $m$ .
6.  $q$  is the dummy index, running from 1 to  $p$ .

**Observation 2.** The multistep method arising in (4) is a **linear** multistep method, or specifically **linear**  $m$ -step method. The term **linear** coming along with  $m$ -step method, i.e. linear  $m$ -step method, emphasizes the fact that there is actually a **linear combination** of the term  $x(t_j + lh)$  with coefficients  $\alpha_l$  on the LHS of (4), and a **linear combination** of the term  $f(t_j + lh, x(t_j + lh))$  with coefficients  $\beta_l$  on the RHS of (4). This linear  $m$ -step method uses the information from the previous  $m$  steps to compute the value for the next step. In details, let us examine the formula (4) which is written again as follows

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)), \quad (5)$$

whose LHS is written in its entirety as follows

$$\begin{aligned} \sum_{l=0}^m \alpha_l x(t_j + lh) &= \alpha_0 x(t_j) + \alpha_1 x(t_j + h) + \alpha_2 x(t_j + 2h) + \dots \\ &\quad + \dots + \alpha_{m-1} x(t_j + (m-1)h) + \alpha_m x(t_j + mh), \end{aligned} \quad (6)$$

whereas the RHS of (5) has its expansion as follows

$$\begin{aligned} \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)) &= \beta_0 f(t_j, x(t_j)) + \beta_1 f(t_j + h, x(t_j + h)) \\ &\quad + \beta_2 f(t_j + 2h, x(t_j + 2h)) \\ &\quad + \dots + \\ &\quad + \beta_{m-1} f(t_j + (m-1)h, x(t_j + (m-1)h)) \\ &\quad + \beta_m f(t_j + mh, x(t_j + mh)). \end{aligned} \quad (7)$$

The insertion of (6) and (7) into (5) leads to

$$\begin{aligned} &\alpha_0 x(t_j) + \alpha_1 x(t_j + h) + \alpha_2 x(t_j + 2h) \\ &\quad + \dots \\ &\quad + \alpha_{m-1} x(t_j + (m-1)h) + \alpha_m x(t_j + mh) \\ &= \\ &\beta_0 f(t_j, x(t_j)) + \beta_1 f(t_j + h, x(t_j + h)) + \beta_2 f(t_j + 2h, x(t_j + 2h)) \\ &\quad + \dots + \\ &\quad + \beta_{m-1} f(t_j + (m-1)h, x(t_j + (m-1)h)) + \beta_m f(t_j + mh, x(t_j + mh)), \end{aligned} \quad (8)$$

which tells us the fact that the value  $x(t_j + mh)$  is approximated by using all information from previous steps, based on

1.  $x(t_j), x(t_j + h), \dots, x(t_j + (m-1)h)$ .
2.  $f(t_j, x(t_j)), f(t_j + h, x(t_j + h)), \dots, f(t_j + mh, x(t_j + mh))$ .
3. Coefficients  $\alpha_l$  and  $\beta_l$ .

**Observation 3.** *The appearance of the green term  $f(t_j + mh, x(t_j + mh))$  arising in (8) plays a significant role, meaning that*

1.  $\beta_m = 0$ : *the green term is switched off, and the scheme becomes **explicit**.*
2.  $\beta_m \neq 0$ : *the green term is switched on, and the scheme becomes **implicit**.*

## 2 Step-by-step with A4-SRU03

**Example 2.** *Examine the consistency order of the following explicit numerical scheme*

$$y^{j+1} = y^j + \frac{h}{2} \left( f(t^j, y^j) + f(t^{j+1}, y^j + hf(t^j, y^j)) \right), \quad (9)$$

*which is used to approximate the ordinary differential equation (ODE):*

$$y'(t) = f(t, y(t)). \quad (10)$$

Approach: The exact solution at time  $t+h$  is obtained by Taylor series expansion as

$$y(t+h) = y(t) + h y'(t) + \frac{h^2}{2} y''(t) + \mathcal{O}(h^3)$$

with

$$\begin{aligned} y' &= f(t, y) \\ y'' &= \frac{d}{dt} f(t, y) = f_t(t, y(t)) + f_y(t, y(t)) y'(t) = f_t(t, y(t)) + f_y(t, y(t)) f(t, y(t)). \end{aligned}$$

Thus

$$y(t+h) = y^j + h f(t, y^j) + \frac{h^2}{2} \left( f_t(t, y^j) + f_y(t, y^j) f(t, y^j) \right) + \mathcal{O}(h^3)$$

The approximation can be written as

$$y^{j+1} = y^j + h \Phi(h), \quad \text{with} \quad \Phi(h) = \frac{1}{2} \left[ f(t, y^j) + f(t+h, y^j + hf(t, y^j)) \right].$$

We now expand  $\Phi(h)$  around 0, i.e.,  $\Phi(h) = \Phi(0) + h \Phi'(0) + \mathcal{O}(h^2)$ . Here

$$\Phi'(h) = \frac{1}{2} f_t(t+h, y^j + hf(t, y^j)) + \frac{1}{2} f_y(t+h, y^j + hf(t, y^j)) f(t, y^j).$$

Thus

$$\Phi(0) = f(t, y^j), \quad \Phi'(0) = \frac{1}{2} f_t(t, y^j) + \frac{1}{2} f_y(t, y^j) f(t, y^j)$$

and

$$\Phi(h) = f(t, y^j) + \frac{h}{2} \left( f_t(t, y^j) + f_y(t, y^j) f(t, y^j) \right) + \mathcal{O}(h^2).$$

Therefore

$$y^{j+1} = y^j + h f(t, y^j) + \frac{h^2}{2} \left( f_t(t, y^j) + f_y(t, y^j) f(t, y^j) \right) + \mathcal{O}(h^3).$$

The error  $|y(t+h) - y^{j+1}| = \mathcal{O}(h^3)$  and thus the method has consistency order 2.