Self-exercise - SRU03
Response to questions

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09th November, 2022

1 Step-by-step with A3-SRU03

Example 1. *Let the linear m-step method be given by coefficients $\alpha_l \in \mathbb{R}$ and $\beta_l \in \mathbb{R}$ where $l = 0, 1, \dots, m$, satisfying the following system of equations*

$$\sum_{l=0}^{m} a_l = 0, \tag{1}$$

$$\sum_{l=0}^{m} (l^q \alpha_l - q\, l^{q-1} \beta_l) = 0, \quad \text{for} \quad q = 1, 2, 3, \dots, p. \tag{2}$$

Then the linear m-step method has the consistency order of p.

Consider the following ordinary differential equation (ODE) taking the form

$$x'(t) = f(t, x(t)), \quad \text{with} \quad x(t_0) = x_0, \tag{3}$$

which may be approximated by a linear multistep method, taking the general form

$$\boxed{\sum_{l=0}^{m} \alpha_l\, x(t_j + lh) = h \sum_{l=0}^{m} \beta_l\, f(t_j + lh, x(t_j + lh)).} \tag{4}$$

Observation 1. *There are 6 symbols arising in (1), (2), (4) to be aware of*

1. *m is the total number of steps used in the multistep method.*
2. *$\alpha \in \mathbb{R}$ is the coefficient for the linear combination of $x(t_j + lh)$*
3. *$\beta \in \mathbb{R}$ is the coefficient for the linear combination of $f(t_j + lh, x(t_j + lh))$.*
4. *p is the consistency order.*
5. *l is the dummy index, running from 0 to m.*
6. *q is the dummy index, running from 1 to p.*

Observation 2. *The multistep method arising in (4) is a* ***linear*** *multistep method, or specifically* ***linear*** *m-step method. The term* ***linear*** *coming along with m-step method, i.e. linear m-step method, emphasizes the fact that there is actually a* ***linear combination*** *of the terms* $x(t_j + lh)$ *with coefficients* α_l *on the LHS of (4), and a* ***linear combination*** *of the terms* $f(t_j + lh, x(t_j + lh))$ *with coefficients* β_l *on the RHS of (4). This linear m-step method uses the information from the previous m steps to compute the value for the next step. In details, let us examine the formula (4) which is written again as follows*

$$\sum_{l=0}^{m} \alpha_l\, x(t_j + lh) = h \sum_{l=0}^{m} \beta_l\, f(t_j + lh, x(t_j + lh)), \tag{5}$$

whose LHS is written in its entirety as follows

$$\begin{aligned}\sum_{l=0}^{m} \alpha_l\, x(t_j + lh) &= \alpha_0\, x(t_j) + \alpha_1\, x(t_j + h) + \alpha_2\, x(t_j + 2h) + \dots \\ &\quad + \dots + \alpha_{m-1}\, x(t_j + (m-1)h) + \alpha_m\, x(t_j + mh),\end{aligned} \tag{6}$$

whereas the RHS of (5) has its expansion as follows

$$\begin{aligned}\sum_{l=0}^{m} \beta_l\, f(t_j + lh, x(t_j + lh)) = \beta_0\, f(t_j, x(t_j)) &+ \beta_1\, f(t_j + h, x(t_j + h)) \\ &+ \beta_2\, f(t_j + 2h, x(t_j + 2h)) \\ &+ \dots + \\ &+ \beta_{m-1}\, f(t_j + (m-1)h, x(t_j + (m-1)h)) \\ &+ \beta_m\, f(t_j + mh, x(t_j + mh)).\end{aligned} \tag{7}$$

The insertion of (6) and (7) into (5) leads to

$$\begin{aligned}&\alpha_0\, x(t_j) + \alpha_1\, x(t_j + h) + \alpha_2\, x(t_j + 2h) \\ &+ \dots \\ &+ \alpha_{m-1}\, x(t_j + (m-1)h) + \alpha_m\, x(t_j + mh) \\ &= \\ &\beta_0\, f(t_j, x(t_j)) + \beta_1\, f(t_j + h, x(t_j + h)) + \beta_2\, f(t_j + 2h, x(t_j + 2h)) \\ &+ \dots + \\ &+ \beta_{m-1}\, f(t_j + (m-1)h, x(t_j + (m-1)h)) + \beta_m\, f(t_j + mh, x(t_j + mh)),\end{aligned} \tag{8}$$

which tells us the fact that the value $x(t_j + mh)$ *is approximated by using all information from previous steps, based on*

1. $x(t_j)$, $x(t_j + h)$, $\dots$, $x(t_j + (m-1)h)$.
2. $f(t_j, x(t_j))$, $f(t_j + h, x(t_j + h))$, $\dots$, $f(t_j + mh, x(t_j + mh))$.
3. *Coefficients* α_l *and* β_l.

Observation 3. *The apprearance of the green term $f(t_j + mh, x(t_j + mh))$ arising in (8) plays a significant role, meaning that*

1. $\beta_m = 0$*: the green term is switched off, and the scheme becomes* ***explicit****.*

2. $\beta_m \neq 0$*: the green term is switched on, and the scheme becomes* ***implicit****.*

Observation 4. *Specific choices of α_l and β_l leads to some familiar methods. Especially, when $m = 1$ the linear m-step method becomes single-step method, or one-step method. If so, according to (4) there will be 4 coefficients $(\alpha_0, \alpha_1, \beta_0, \beta_1)$ to be defined. For example:*

1. $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 1, 0)$ *we obtain explicit Euler.*

2. $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 0, 1)$ *we obtain implicit Euler.*

When the time t is considered at a specific discretized time point $t = t_j + lh$, the ODE in (3) yields

$$x'(t_j + lh) = f(t_j + lh, x(t_j + lh)), \tag{9}$$

Substitution of (9) into (4) leads to

$$\sum_{l=0}^{m} \alpha_l \, x(t_j + lh) = h \sum_{l=0}^{m} \beta_l \, x'(t_j + lh), \tag{10}$$

which, by group the summation sign together, leads equally to

$$\sum_{l=0}^{m} (\alpha_l \, x(t_j + lh) - h \, \beta_l \, x'(t_j + lh)) = 0. \tag{11}$$

Then, the order of consistency is figured out by taking advantage of using Taylor's expansion for $x(t_j + lh)$ and $x'(t_j + lh)$. By using (15) and (16) in Observation 5 we obtain

$$x(t_j + lh) = \tag{12}$$
$$x'(t_j + lh) = \tag{13}$$

Observation 5. *Taylor's expansion of function $x(t)$ around point a reads*

$$\begin{aligned} x(t) &= x(a) + \frac{x'(a)(t-a)}{1!} + \frac{x''(a)(t-a)^2}{2!} + \cdots + \frac{x^{(n)}(a)(t-a)^n}{n!} \\ &\qquad + \mathcal{O}((t-a)^{n+1}) \\ &= \sum_{j=0}^{n} \frac{x^{(n)}(a)(t-a)^n}{n!} + \mathcal{O}((t-a)^{n+1}). \end{aligned} \tag{14}$$

Then, by setting $h := t-a$ we obtain $t = a+h$; hence, the (14) becomes

$$\begin{aligned} x(a+h) &= x(a) + \frac{x'(a)h}{1!} + \frac{x''(a)h^2}{2!} + \cdots + \frac{x^{(n)}(a)h^n}{n!} + \mathcal{O}(h^{n+1}) \\ &= \sum_{j=0}^{n} \frac{x^{(n)}(a)h^n}{n!} + \mathcal{O}(h^{n+1}), \end{aligned} \tag{15}$$

which gives rise to

$$\begin{aligned} x'(a+h) &= x'(a) + \frac{x''(a)h}{1!} + \frac{x'''(a)h^2}{2!} + \cdots + \frac{x^{(n+1)}(a)h^n}{n!} + \mathcal{O}(h^{n+1}) \\ &= \sum_{j=0}^{n} \frac{x^{(n+1)}(a)h^n}{n!} + \mathcal{O}(h^{n+1}), \end{aligned} \tag{16}$$

$$x(t_j + lh) = x(t_j) \tag{17}$$

2 Step-by-step with A4-SRU03

Example 2. *Examine the consistency order of the following explicit numerical scheme*

$$y^{j+1} = y^j + \frac{h}{2}\Big(f(t^j, y^j) + f(t^{j+1}, y^j + hf(t^j, y^j))\Big), \tag{18}$$

which is used to approximate the ordinary differential equation (ODE):

$$y'(t) = f(t, y(t)). \tag{19}$$

Approach: The exact solution at time $t+h$ is obtained by Taylor series expansion as

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2}y''(t) + \mathcal{O}(h^3)$$

with

$$\begin{aligned} y' &= f\big(t, y\big) \\ y'' &= \frac{\mathrm{d}}{\mathrm{d}t}f\big(t, y\big) = f_t\big(t, y(t)\big) + f_y\big(t, y(t)\big)y'(t) = f_t\big(t, y(t)\big) + f_y\big(t, y(t)\big)f\big(t, y(t)\big). \end{aligned}$$

Thus

$$y(t+h) = y^j + h\,f(t, y^j) + \frac{h^2}{2}\Big(f_t(t, y^j) + f_y(t, y^j)\,f(t, y^j)\Big) + \mathcal{O}(h^3)$$

The approximation can be written as

$$y^{j+1} = y^j + h\,\Phi(h), \quad \text{with} \quad \Phi(h) = \frac{1}{2}\Big[f(t, y^j) + f(t+h, y^j + hf(t, y^j))\Big].$$

We now expand $\Phi(h)$ around 0, i.e., $\Phi(h) = \Phi(0) + h\,\Phi'(0) + \mathcal{O}(h^2)$. Here

$$\Phi'(h) = \frac{1}{2}f_t(t+h, y^j + hf(t, y^j)) + \frac{1}{2}f_y(t+h, y^j + hf(t, y^j))f(t, y^j).$$

Thus

$$\Phi(0) = f(t, y^j), \quad \Phi'(0) = \frac{1}{2}f_t(t, y^j) + \frac{1}{2}f_y(t, y^j)f(t, y^j)$$

and

$$\Phi(h) = f(t, y^j) + \frac{h}{2}\Big(f_t(t, y^j) + f_y(t, y^j)\,f(t, y^j)\Big) + \mathcal{O}(h^2).$$

Therefore

$$y^{j+1} = y^j + h\,f(t, y^j) + \frac{h^2}{2}\Big(f_t(t, y^j) + f_y(t, y^j)\,f(t, y^j)\Big) + \mathcal{O}(h^3).$$

The error $\left|y(t+h) - y^{j+1}\right| = \mathcal{O}(h^3)$ and thus the method has consistency order 2.