Global exercise - GUE11

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12th January, 2023

Summary of content covered: Analysis

- Review line integral of the 1st kind (scalar field) and 2nd kind (vector field)
- Surface integral of the 1st kind (scalar field) and 2nd kind (vector field)
- Gauss's theorem and its useful applications

1 Course evaluation

The evaluation for the course

22W-11.03530 (L) Mathematische Grundlagen III (CES) (Übung)

has been opened from now on until **Friday 20.01.2023 23:59:00** via QR code

Figure 1: Evaluation QR-code.

or via the following link
https://www.campus.rwth-aachen.de/evasys/online.php?pswd=8J5FHC8JH9

2 Summary: Line integral and Surface integral

2.1 First kind: Scalar field

Recall 1. *Line integral of a* ***scalar field*** $\phi : \Omega \to \mathbb{R}$ *is defined as follows*

$$\int_\Gamma \phi \, ds := \int_a^b \phi(\gamma(t)) \left\|\gamma'(t)\right\| dt \tag{1}$$

$\to$ Intuitively: Mass

Recall 2. *Surface integral of a* ***scalar field*** $\phi : \Omega \to \mathbb{R}$ *is defined as follows*

$$\int_\Gamma \phi \, dA := \int_B \phi(\Phi(p,q)) \left\|\Phi_p \times \Phi_q\right\| d(p,q) \tag{2}$$

$\to$ Intuitively: Mass

2.2 Second kind: Vector field

Recall 3. *Line integral of a* ***vector field*** $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ *is defined as follows*

$$\int_\Gamma \boldsymbol{f} \cdot d\boldsymbol{x} := \int_a^b \Big\langle \boldsymbol{f}(\gamma(t)), \gamma'(t) \Big\rangle \, dt \tag{3}$$

$\to$ Intuitively: Work done

Recall 4. *Surface integral of a* ***vector field*** $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ *is defined as follows*

$$\int_\Gamma \langle \boldsymbol{f}, \boldsymbol{v} \rangle \, dA = \int_\Gamma \boldsymbol{f} \cdot d\boldsymbol{A} := \int_B \Big\langle \phi(\Phi(p,q)), \Phi_p \times \Phi_q \Big\rangle \, d(p,q) \tag{4}$$

$\to$ Intuitively: Flow

3 Step-by-step: Surface integral of the 1st kind

Example 1. *Examine Given surface integral*

$$\int_{\Omega} 2\sqrt{4 - x^2 - y^2}\, d\sigma \tag{5}$$

where $\Omega = S \cap Z$ *is the portion of the sphere*

$$S = \Big\{(x, y, z) \in \mathbb{R}^3 : z \geq 0,\ x^2 + y^2 + z^2 = 4\Big\} \tag{6}$$

inside the cylinder

$$Z = \Big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\Big\} \tag{7}$$

1. *Define spherical coordinates in* Ω *(parameterize the surface of interest* Ω*).*
2. *Calculate the surface integral (5) using the defined spherical coordinates.*

Approach:
On the surface Ω, we have $r = 2$, so we choose the sphere coordinate with:

$$\Phi(\phi, \theta) = (2 \cdot \cos\phi \cdot \sin\theta, 2 \cdot \sin\phi \cdot \sin\theta, 2 \cdot \cos\theta)^T, \quad \phi \in [0, 2\pi], \theta \in [0, \frac{\pi}{6}] \tag{8}$$

Hence, we obtain

$$\left\| \frac{\partial \Phi}{\partial \phi} \times \frac{\partial \Phi}{\partial \theta} \right\| = 2^2 \cdot |\sin\theta| \tag{9}$$

Furthermore, on the surface we have $x^2 + y^2 = 4$ so that

$$2\sqrt{4 - x^2 - y^2} = 2\sqrt{z^2} = 2 \cdot z = 4 \cdot \cos\theta \tag{10}$$

Therefore, we obtain

$$\therefore \quad \boxed{\begin{aligned} \int_{\Omega} 2\sqrt{4 - x^2 - y^2}\, d\sigma &= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} 4\cos(\theta) \cdot 2^2 \sin\theta \, d\theta d\phi \\ &= 32\pi \int_0^{\frac{\pi}{6}} \cos\theta \cdot \sin(\theta)\, d\theta \\ &= 4\pi \end{aligned}} \tag{11}$$

4 Step-by-step: Surface integral of the 2nd kind

Example 2. *Examine (i) the normalized normal vector of the surface F and (ii) the surface integral of a vector field based on the surface F given as follows*

$$F = \left\{ \alpha \in [0,1],\ \beta \in \Big[0, 2\pi\Big) \,\middle|\, \Big(e^{\alpha}, \alpha\cos(\beta), \alpha\sin(\beta)\Big) \right\}, \tag{12}$$

and the vector field $\boldsymbol{f}$ given by $\boldsymbol{f}(x,y,z) = \Big(x\sqrt{y^2+z^2}, y, z\Big)$.

Approach:

(i) The normal vector of the surface is computed as follows

$$\boldsymbol{n} = \partial_\alpha F \times \partial_\beta F = \begin{pmatrix} e^{\alpha} \\ \cos(\beta) \\ \sin(\beta) \end{pmatrix} \times \begin{pmatrix} 0 \\ -\alpha\sin(\beta) \\ \alpha\cos(\beta) \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} \alpha\cos^2(\beta) + \alpha\sin^2(\beta) \\ -\alpha\cos(\beta)e^{\alpha} \\ -\alpha\sin(\beta)e^{\alpha} \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} \alpha \\ -\alpha\cos(\beta)e^{\alpha} \\ -\alpha\sin(\beta)e^{\alpha} \end{pmatrix}, \tag{15}$$

Observation 1. *Be aware of the sign switching $(+,-,+)$ when using* ***co-factor method*** *to compute the determinant*

$$\partial_\alpha F \times \partial_\beta F = \begin{vmatrix} i & j & k \\ e^{\alpha} & \cos(\beta) & \sin(\beta) \\ 0 & -\alpha\sin(\beta) & \alpha\cos(\beta) \end{vmatrix} \tag{16}$$

$$= i\begin{vmatrix} \cos(\beta) & \sin(\beta) \\ -\alpha\sin(\beta) & \alpha\cos(\beta) \end{vmatrix} - j\begin{vmatrix} e^{\alpha} & \sin(\beta) \\ 0 & \alpha\cos(\beta) \end{vmatrix} + k\begin{vmatrix} e^{\alpha} & \cos(\beta) \\ 0 & -\alpha\sin(\beta) \end{vmatrix} \tag{17}$$

and the norm of the normal vector is computed as follows

$$\|\boldsymbol{n}\|_2 = \sqrt{\alpha^2 + \alpha^2\cos^2(\beta)e^{2\alpha} + \alpha^2\sin^2(\beta)e^{2\alpha}} = \alpha\sqrt{1+e^{2\alpha}}. \tag{18}$$

which leads to the normalized normal vector of the surface F as follows

$$\boxed{\boldsymbol{\nu} = \left(1+e^{2\alpha}\right)^{-1/2} \begin{pmatrix} 1 \\ -\cos(\beta)e^{\alpha} \\ -\sin(\beta)e^{\alpha} \end{pmatrix}.} \tag{19}$$

(ii) We now compute the surface integral of the second kind as follows

- By using $\boldsymbol{n}$

$$\boxed{\begin{aligned}\int_F \Big\langle \boldsymbol{f}, \boldsymbol{n} \Big\rangle \, d\sigma &= \int_0^1 \int_0^{2\pi} \Big\langle \boldsymbol{f}\big(F(\alpha,\beta)\big), \boldsymbol{n} \Big\rangle \, d\beta \, d\alpha \\ &= \dots\end{aligned}} \tag{20}$$

- By using $\boldsymbol{\nu}$

$$\boxed{\begin{aligned}\int_F \Big\langle \boldsymbol{f}, \boldsymbol{\nu} \Big\rangle \, d\sigma &= \int_0^1 \int_0^{2\pi} \Big\langle \boldsymbol{f}\big(F(\alpha,\beta)\big), \boldsymbol{\nu} \Big\rangle \, d\beta \, d\alpha \\ &= \dots\end{aligned}} \tag{21}$$

5 Step-by-step with Gauss's theorem example

Example 3. *Examine the vector field* $\boldsymbol{f}$ *with*

$$\boldsymbol{f} : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2, \\ (x, y) \mapsto \boldsymbol{f}(x, y) = \left(x + y, y^2\right), \end{cases} \tag{22}$$

and let $B \subset \mathbb{R}^2$ *be a triangle with vertices at* $(0, 0), (1, 0)$ *and* $(1, 2)$*. Verify the Gauss's theorem, by calculating the two integrals* $\int_{\partial B} \boldsymbol{f} \cdot \boldsymbol{\nu} \, ds$ *and* $\int_B \operatorname{div} \boldsymbol{f} \, dA$*, and compare them.*

Approach:

Observation 2 (2nd kind line integral)**.** *We note that the integral*

$$\int_{\partial B} \boldsymbol{f} \cdot \boldsymbol{\nu} \, ds \tag{23}$$

is the line integral of the second kind, i.e. line integral of a vector field.

Observation 3 (1st kind surface integral)**.** *We note that the integral*

$$\int_B \operatorname{div} \boldsymbol{f} \, dA \tag{24}$$

is the surface integral of the first kind, i.e. surface integral of a scalar field. Note in passing that divergence of vector field $\boldsymbol{f} \in \mathbb{R}^3$ *yields a scalar*

$$\operatorname{div} \boldsymbol{f} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3. \tag{25}$$

Likewise, divergence of vector field $\boldsymbol{f} \in \mathbb{R}^2$ *yields a scalar*

$$\operatorname{div} \boldsymbol{f} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \partial_x f_1 + \partial_y f_2. \tag{26}$$

(i) First, we compute the surface integral of the scalar field

$$\therefore \quad \boxed{\begin{aligned} \int_B \operatorname{div} \boldsymbol{f}\, dA &= \int_0^1 \int_0^{2x} \operatorname{div} \begin{pmatrix} x+y \\ y^2 \end{pmatrix} dy\, dx \\ &= \int_0^1 \int_0^{2x} \Big(1+2y\Big)\, dy\, dx \\ &= \int_0^1 \Big(4x^2+2x\Big)\, dx \\ &= \dots \end{aligned}} \tag{27}$$

where we have computed the divergence of vector field $\boldsymbol{f}$ as follows

$$\operatorname{div} \boldsymbol{f} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} x+y \\ y^2 \end{pmatrix} \tag{28}$$

$$= \partial_x f_1 + \partial_y f_2 \tag{29}$$

$$= \partial_x (x+y) + \partial_y (y^2) \tag{30}$$

$$= \dots \tag{31}$$

(ii) Since the area B is bounded by the three curves, we obtain the following relations

$$\boxed{\begin{array}{lllll} \gamma_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix},\ t \in [0,1] & \rightarrow & \gamma_1'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \rightarrow & \|\gamma_1'\|_2 = 1 \\ \gamma_2(t) = \begin{pmatrix} 1 \\ t \end{pmatrix},\ t \in [0,2] & \rightarrow & \gamma_2'(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \rightarrow & \|\gamma_2'\|_2 = 1 \\ \gamma_3(t) = \begin{pmatrix} 1-t \\ 2-2t \end{pmatrix},\ t \in [0,1] & \rightarrow & \gamma_3'(t) = \begin{pmatrix} -1 \\ -2 \end{pmatrix} & \rightarrow & \|\gamma_3'\|_2 = \sqrt{5} \end{array}} \tag{32}$$

Observation 4. *Parametrization of a line function may lead to various options. However, we should be aware of the limit of the parameter variable. For example*

$$\gamma_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \tag{33}$$

which leads to new limit applied on t taking $[0,1]$ instead of $[0,2]$

$$\gamma_2(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix},\ t \in [0,1] \quad \rightarrow \quad \gamma_2'(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \rightarrow \quad \|\gamma_2'\|_2 = 2 \tag{34}$$

Likewise, normal vectors take the form

$$\boxed{\boldsymbol{\nu}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ on } \gamma_1(t), \qquad \boldsymbol{\nu}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ on } \gamma_2(t), \qquad \boldsymbol{\nu}_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{5}} \text{ on } \gamma_3(t),} \tag{35}$$

which have been computed based on the following relations

$$\boldsymbol{\nu}_1 \cdot \gamma_1'(t) \stackrel{!}{=} 0 \tag{36}$$
$$\boldsymbol{\nu}_2 \cdot \gamma_2'(t) \stackrel{!}{=} 0 \tag{37}$$
$$\boldsymbol{\nu}_3 \cdot \gamma_3'(t) \stackrel{!}{=} 0 \tag{38}$$

Now, we compute the line integral of the second kine $\int_{\partial B} \boldsymbol{f} \cdot \boldsymbol{\nu}\, ds$

$$\int_{\partial B} \boldsymbol{f} \cdot \boldsymbol{\nu}\, ds = \int_{\partial B = \gamma_1(t)+\gamma_2(t)+\gamma_3(t)} \boldsymbol{f} \cdot \boldsymbol{\nu}\, ds \tag{39}$$
$$= \int_{\gamma_1(t)} \boldsymbol{f} \cdot \boldsymbol{\nu}_1\, ds + \int_{\gamma_2(t)} \boldsymbol{f} \cdot \boldsymbol{\nu}_2\, ds + \int_{\gamma_3(t)} \boldsymbol{f} \cdot \boldsymbol{\nu}_3\, ds \tag{40}$$
$$= \int_{\gamma_1(t)} \boldsymbol{f}(\gamma_1'(t)) \cdot \boldsymbol{\nu}_1\, ds + \int_{\gamma_2(t)} \boldsymbol{f} \cdot \boldsymbol{\nu}_2\, ds + \int_{\gamma_3(t)} \boldsymbol{f} \cdot \boldsymbol{\nu}_3\, ds \tag{41}$$
$$= \int_0^1 \begin{pmatrix} t+0 \\ 0^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt + \int_0^2 \begin{pmatrix} 1+t \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt \tag{42}$$
$$+ \int_0^1 \begin{pmatrix} 1-t+2-2t \\ 4-8t+4t^2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \frac{\sqrt{5}}{\sqrt{5}}\, dt \tag{43}$$
$$= \int_0^2 (1+t)\, dt + \int_0^1 (4t^2 - 2t - 2)\, dt = \frac{7}{3}. \tag{44}$$

Therefore, we have verified the Gauss's theorem

$$\therefore \quad \boxed{\int_{\partial B} \boldsymbol{f} \cdot \boldsymbol{\nu} = \int_B \operatorname{div} \boldsymbol{f} \, d\lambda_2.} \tag{45}$$