Global exercise - GUE11

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Content covered:

- ✓ Analysis: Surface integral of the first kind (scalar field)
- ✓ Analysis: Surface integral of the second kind (vector field)
- ✓ Analysis: Gauss's theorem

1 Analysis: Orientable integral

Example 1. *Examine the* ***Möbius band*** *given as follows*

$$\vec{\gamma} : \begin{cases} (-1,1) \times (0, 2\pi) \to \mathbb{R}^3, \\ (t,\phi) \mapsto \vec{\gamma}(t,\phi) := \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix} + \frac{t}{2} \begin{pmatrix} \cos(\phi)\cos(\phi/2) \\ \sin(\phi)\cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}. \end{cases} \tag{1}$$

Show that this surface is ***not orientable****.*

Approach:

Proof. The normal field is computed as follows

$$\begin{aligned} n(t,\phi) &= \pm \frac{\partial_t \vec{\gamma} \times \partial_\phi \vec{\gamma}}{\|\partial_t \vec{\gamma} \times \partial_\phi \vec{\gamma}\|} \\ &= \frac{\mp 1}{a} \begin{pmatrix} \cos(\phi)\sin(\frac{\phi}{2}) - \frac{t}{4}\sin(\phi)(1-\cos(\phi)) \\ \sin(\phi)\sin(\phi/2) + \frac{t}{4}(\sin^2(\phi) + \cos(\phi)) \\ -\cos(\phi/2) - \frac{t}{4}(1+\cos(\phi)) \end{pmatrix} \end{aligned}$$

with

$$a = \sqrt{1 + t\cos(\phi/2) + \frac{t^2}{16}(3 + 2\cos(\phi))}.$$

We choose the positive sign and consider the position $(1,0,0)^T = \vec{\gamma}(0,0) = \vec{\gamma}(0,2\pi)$:

$$\lim_{\phi \to 0} n(0,\phi) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \neq \lim_{\phi \to 2\pi} n(0,\phi) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the normal field is not continuous on

$$\gamma([-1,1] \times [0,2\pi])$$

the surface is **not orientable**. □

2 Analysis: Surface integral

Example 2. *Examine the lateral surface of the* ***Frustum*** *given as follows*

$$K = \left\{ \boldsymbol{x} \in \mathbb{R}^3,\, 0 < r \le R \,\middle|\, 0 \le x_3 < H,\, 0 \le x_1^2 + x_2^2 < \left(R - \frac{R-r}{H} x_3 \right)^2 \right\}.$$

where H is the height, and r is the radius.

Approach: By using the cylinder coordination $\boldsymbol{x} = (r\cos(\phi), r\sin(\phi), z)^T$ we obtain the parametrization of the lateral surface of the frustum, i.e. truncated cone

$$\boldsymbol{\gamma} : \begin{cases} (0, 2\pi) \times (0, H) \to \mathbb{R}^3, \\ (\phi, z) \mapsto \boldsymbol{\gamma}(\phi, z) := \begin{pmatrix} \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ \left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ z \end{pmatrix} \end{cases}$$

which leads to the following expression

$$\boxed{\begin{aligned} \partial_\phi \boldsymbol{\gamma}(\phi, z) \times \partial_z \boldsymbol{\gamma}(\phi, z) &= \begin{pmatrix} -\left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ 0 \end{pmatrix} \times \begin{pmatrix} -\frac{R-r}{H} \cos(\phi) \\ -\frac{R-r}{H} \sin(\phi) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ \left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ \left(R - \frac{R-r}{H} z\right) \frac{R-r}{H} \end{pmatrix} \end{aligned}}$$

where the normalization yields

$$\boxed{\left\| \partial_\phi \boldsymbol{\gamma}(\phi, z) \times \partial_z \boldsymbol{\gamma}(\phi, z) \right\| = \left(R - \frac{R-r}{H} z \right) \sqrt{1 + \frac{(R-r)^2}{H^2}}.}$$

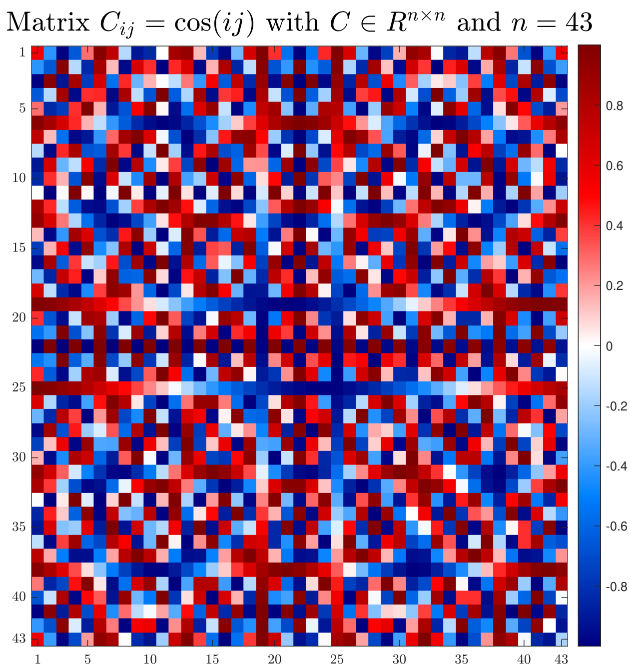
After that, the lateral surface of the truncated cone is computed as follows

$$\mathcal{A} = \int_0^H \int_0^{2\pi} \left\| \partial_\phi \boldsymbol{\gamma} \times \partial_z \boldsymbol{\gamma} \right\| d\phi\, dz = 2\pi \sqrt{1 + \frac{(R-r)^2}{H^2}} \int_0^H \left(R - \frac{R-r}{H} z \right) dt = \ldots$$

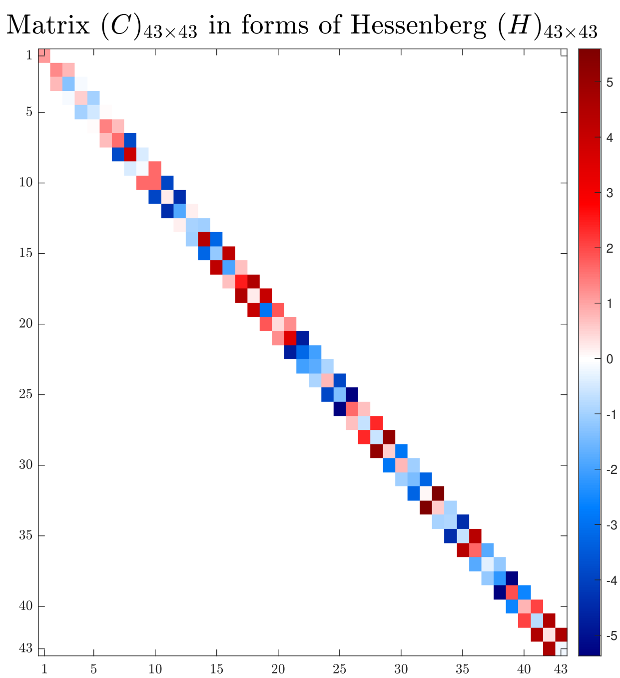
Therefore, we obtain

$$\therefore \quad \boxed{\mathcal{A} = \int_0^H \int_0^{2\pi} ||\partial_\phi \boldsymbol{\gamma} \times \partial_z \boldsymbol{\gamma}||\, d\phi\, dz = \ldots.}$$

3 Numerics: Demo programming exercise 3 PRU(02)



(a)



(b)

Figure 1: Matrix size 43×43 and its Hessenberg form.

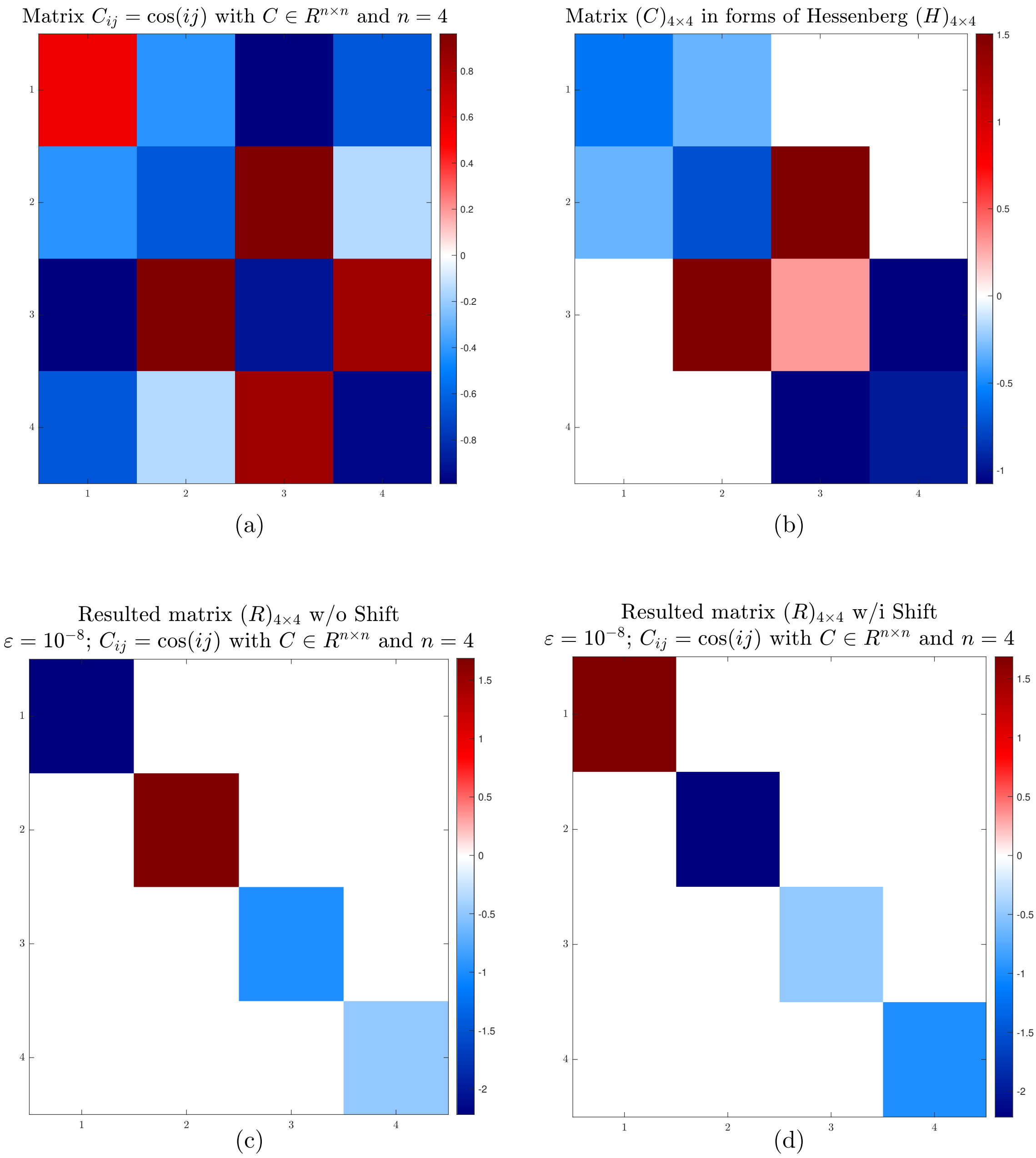
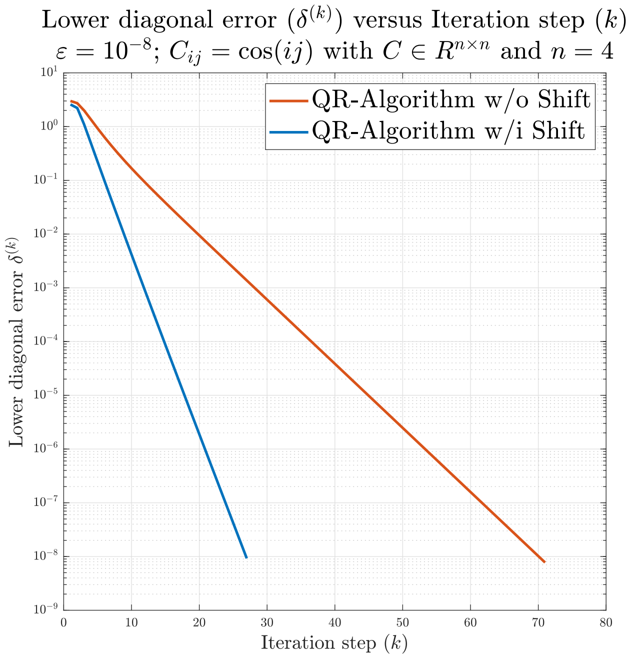
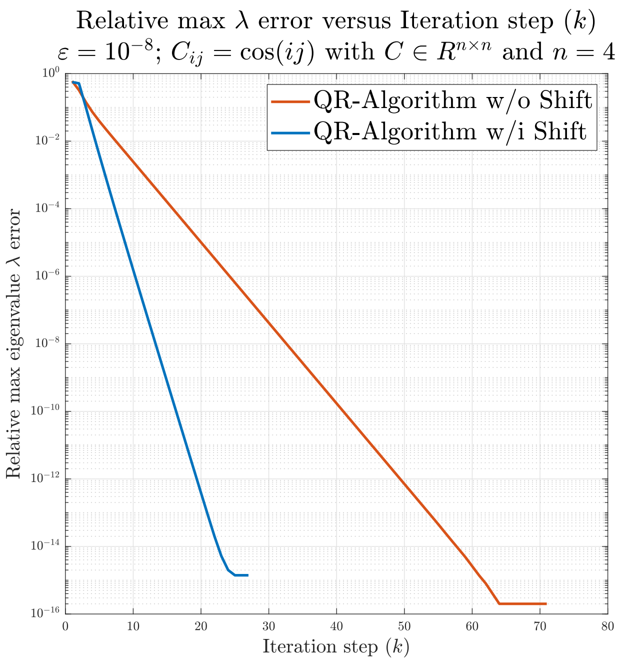


Figure 2: Matrix size 4×4 and its Hessenberg form.

Observation 1. *For the sake of rapid checks and coding prototype we may first start with a matrix sized* 4×4 *for the case* $C_{ij} = \sin(ij)$ *as given in the programming exercise.*



(a)



(b)

Figure 3: Error plots for matrix size 4×4.