Self-exercise - SRU00

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1 Step-by-step with A1-SRU00

A function which take its argument as a function is called functional.

$$\boxed{\text{functional} = \text{function of function}} \tag{1}$$

Example 1. *Examine the first variation of the functional F in v direction*

$$F : \begin{cases} C^2([-1,1]) \to \mathbb{R}, \\ y \mapsto F(y) := \displaystyle\int_{-1}^{1} \left(y(x)y'(x) + e^{y'(x)^2} \right) dx, \end{cases} \quad \text{with } y : \begin{cases} [-1,1] \to \mathbb{R}, \\ x \mapsto y(x), \end{cases} \tag{2}$$

where C^2 is called space of continuous functions, i.e. function space. The number 2 in C^2 indicates that those functions living in C^2 are those continuously twice differentiable. Herein, y is such function living in C^2, as shown in $(2)_1$. Besides, the function y takes x as its only variable. In this case, the only variable x is defined on the closed interval $[-1,1]$, as shown in $(2)_2$.

Approach: By using the definition, as follows

$$\boxed{\delta F(y; v) := \lim_{s \to 0} \frac{F(y + sv) - F(y)}{s},} \tag{3}$$

where δF is called variation of the functional F, and the symbol δ stands for variation.

Interesting fact 1. *The symbol δ seen in variation of a functional is not the same one often seen in partial derivative, which is ∂. In LaTeX their commands are called, either from maths' package such as* ***amsmath*** *or* ***mathtools****, as follows*

$$\begin{aligned} \delta \quad &\to \backslash delta &&\to \quad \textit{variation of a functional} \\ \partial \quad &\to \backslash partial &&\to \quad \textit{partial derivative of a multi-variable function} \end{aligned} \tag{4}$$

In order to find the variation of the functional F in v direction, we need to compute the limit, as shown in (3).

$$\begin{aligned}
\delta F(y;v) &:= \lim_{s\to 0} \frac{F(y+s\,v)-F(y)}{s} \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big((y+s\,v)(x)\,(y+s\,v)'(x) + \exp\Big((y+s\,v)'^2(x) \Big) \Big)\, dx \qquad (5) \\
&\qquad - \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(y(x)y'(x) + \exp\Big(y'^2(x) \Big) \Big)\, dx \qquad (6)
\end{aligned}$$

Let us now forget the argument (x) in (5) and (6) for a while, i.e. by dropping it temporarily, just for a sake of a better overview. Then, the new expression without (x) will take the form

$$\begin{aligned}
\delta F(y;v) &:= \lim_{s\to 0} \frac{F(y+s\,v)-F(y)}{s} \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big((y+s\,v)\,(y+s\,v)' + \exp\Big((y+s\,v)'^2 \Big) \Big)\, dx \\
&\qquad - \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(y\,y' + \exp\Big(y'^2 \Big) \Big)\, dx \qquad (7)
\end{aligned}$$

which leads to further analysis, as follows

$$\begin{aligned}
\delta F(y;v) &\overset{(7)}{:=} \lim_{s\to 0} \frac{F(y+s\,v)-F(y)}{s} \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big((y+s\,v)(y'+s\,v') + \exp\Big((y+s\,v)'^2 \Big) \Big)\, dx \\
&\qquad - \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(y\,y' + \exp\Big(y'^2 \Big) \Big)\, dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(y\,y' + s\,y\,v' + s\,v\,y' + s^2\,v\,v' + \exp\Big((y+s\,v)'^2 \Big) \Big)\, dx \\
&\qquad - \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(y\,y' + \exp\Big(y'^2 \Big) \Big)\, dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(y\,y' + s\,y\,v' + s\,v\,y' + s^2\,v\,v' - y\,y' \Big)\, dx \\
&\qquad + \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\Big((y+s\,v)'^2 \Big) - \exp\Big(y'^2 \Big) \Big)\, dx \\
&= \underbrace{\lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(s\,y\,v' + s\,v\,y' + s^2\,v\,v' \Big)\, dx}_{:=\mathcal{A}} \\
&\qquad + \underbrace{\lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\Big((y+s\,v)'^2 \Big) - \exp\Big(y'^2 \Big) \Big)\, dx}_{:=\mathcal{B}} \qquad (8)
\end{aligned}$$

Hence, it leads to

$$\therefore \quad \boxed{\delta F(y; v) = \mathcal{A} + \mathcal{B}.} \tag{9}$$

Next, we consider part $\mathcal{A}$, as follows

$$\begin{aligned}
\mathcal{A} &= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(s\, y\, v' + s\, v\, y' + s^2\, v\, v'\Big)\, dx = \lim_{s\to 0} \int_{-1}^{1} \Big(y\, v' + v\, y' + s\, v\, v'\Big)\, dx \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y\, v' + v\, y'\Big)\, dx + \lim_{s\to 0} \int_{-1}^{1} (s\, v\, v')\, dx \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y\, v' + v\, y'\Big)\, dx + \underbrace{\lim_{s\to 0} \left(s \int_{-1}^{1} (v\, v')\, dx\right)}_{=0} \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y(x)\, v'(x) + v(x)\, y'(x)\Big)\, dx \qquad (10) \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y(x)\, v(x)\Big)'\, dx = \int_{-1}^{1} \frac{d}{dx}\Big(y(x)\, v(x)\Big)\, dx \\
&= \int_{-1}^{1} \frac{d}{dx}\Big(y(x)\, v(x)\Big)\, dx = \Big(y(x)\, v(x)\Big)\Big|_{x=-1}^{x=1} = y(1)\, v(1) - y(-1)\, v(-1), \qquad (11)
\end{aligned}$$

where (x) has returned back in (10) again in order to get ready to compute the follow-up integral. Therefore, solution to $\mathcal{A}$ yields

$$\therefore \quad \boxed{\mathcal{A} = y(1)\, v(1) - y(-1)\, v(-1).} \tag{12}$$

Besides, the computation of part $\mathcal{B}$ goes as follows

$$\begin{aligned}
\mathcal{B} &= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\Big((y + s\, v)'^2\Big) - \exp\left(y'^2\right)\Big)\, dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\left(y'^2 + 2\, s\, y'\, v' + s^2\, v'^2\right) - \exp\left(y'^2\right)\Big)\, dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\left(y'^2\right) \exp\left(2\, s\, y'\, v' + s^2\, v'^2\right) - \exp\left(y'^2\right)\Big)\, dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \bigg(\exp\left(y'^2\right)\Big(\exp\left(2\, s\, y'\, v' + s^2\, v'^2\right) - 1\Big)\bigg)\, dx \\
&= \int_{-1}^{1} \left(\exp\left(y'^2\right) \underbrace{\lim_{s\to 0} \frac{\exp\left(2\, s\, y'\, v' + s^2\, v'^2\right) - 1}{s}}_{:=\mathcal{C}}\right) dx, \qquad (13)
\end{aligned}$$

where a further result of $\mathcal{C}$ in (13) is computed by using l'Hôspital's rule, as follows

$$\begin{aligned}
\mathcal{C} &= \lim_{s\to 0} \frac{\exp\left(2\,s\,y'\,v' + s^2\,v'^2\right) - 1}{s} \\
&= \lim_{s\to 0} \Big(\left(2\,y'\,v' + 2\,s\,v'^2\right) \exp\left(2\,s\,y'\,v' + s^2\,v'^2\right) \Big) \\
&= 2\,y'\,v' \\
&= 2\,y'(x)\,v'(x). \qquad (14)
\end{aligned}$$

Substitution of (14) into (13) leads to

$$\therefore \quad \boxed{\mathcal{B} = 2 \int_{-1}^{1} \exp\left(y'^2(x)\right) y'(x)\,v'(x)\,dx.} \qquad (15)$$

Finally, by inserting (12) and (15) into (9) the final result yields

$$\therefore \quad \boxed{\delta F(y;v) = y(1)\,v(1) - y(-1)\,v(-1) + 2 \int_{-1}^{1} y'(x)\,v'(x) \exp\left(y'^2(x)\right) dx.} \qquad (16)$$