Global exercise - GUE11

Tuan Vo

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Content covered:

- ✓ Analysis: Review line integral of the first kind and second kind
- ✓ Analysis: Surface integral of the first kind (scalar field)
- ✓ Analysis: Surface integral of the second kind (vector field)
- ✓ Analysis: Gauss's theorem

1 Analysis: Surface integral of the first kind

Example 1. *Examine the lateral surface of the* ***Frustum*** *given as follows*

$$K = \left\{ \boldsymbol{x} \in \mathbb{R}^3,\, 0 < r \leq R \,\middle|\, 0 \leq x_3 < H,\, 0 \leq x_1^2 + x_2^2 < \left(R - \frac{R-r}{H} x_3 \right)^2 \right\}.$$

where H is the height, and r is the radius.

Approach: By using the cylinder coordination $\boldsymbol{x} = (r\cos(\phi), r\sin(\phi), z)^T$ we obtain the parametrization of the lateral surface of the frustum, i.e. truncated cone

$$\boldsymbol{\gamma} : \begin{cases} (0, 2\pi) \times (0, H) \to \mathbb{R}^3, \\ (\phi, z) \mapsto \boldsymbol{\gamma}(\phi, z) := \begin{pmatrix} \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ \left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ z \end{pmatrix} \end{cases}$$

which leads to the following expression

$$\boxed{\begin{aligned} \partial_\phi \boldsymbol{\gamma}(\phi, z) \times \partial_z \boldsymbol{\gamma}(\phi, z) &= \begin{pmatrix} -\left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ 0 \end{pmatrix} \times \begin{pmatrix} -\frac{R-r}{H} \cos(\phi) \\ -\frac{R-r}{H} \sin(\phi) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \left(R - \frac{R-r}{H} z\right) \cos(\phi) \\ \left(R - \frac{R-r}{H} z\right) \sin(\phi) \\ \left(R - \frac{R-r}{H} z\right) \frac{R-r}{H} \end{pmatrix} \end{aligned}}$$

where the normalization yields

$$\boxed{\left\| \partial_\phi \boldsymbol{\gamma}(\phi, z) \times \partial_z \boldsymbol{\gamma}(\phi, z) \right\| = \left(R - \frac{R-r}{H} z \right) \sqrt{1 + \frac{(R-r)^2}{H^2}}.}$$

After that, the lateral surface of the truncated cone is computed as follows

$$\mathcal{A} = \int_0^H \int_0^{2\pi} \left\| \partial_\phi \boldsymbol{\gamma} \times \partial_z \boldsymbol{\gamma} \right\| d\phi \, dz = 2\pi \sqrt{1 + \frac{(R-r)^2}{H^2}} \int_0^H \left(R - \frac{R-r}{H} z \right) dt = \dots$$

Therefore, we obtain

$$\therefore \quad \boxed{\mathcal{A} = \int_0^H \int_0^{2\pi} ||\partial_\phi \boldsymbol{\gamma} \times \partial_z \boldsymbol{\gamma}|| \, d\phi \, dz = \dots .}$$

2 Evaluation