

# Global exercise - GUE08

Tuan Vo

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Content covered:

✓ Numerics:

1. Review: QR-decomposition by using
  - (a) Givens-Rotation
  - (b) Householder-Reflection
2. QR algorithm to find all eigenvalues of a matrix  $A$ .

✓ Analysis: Application of Hölder's inequality to approximate integral

## 1 Numerics: Review of QR-decomposition

There are two main methods used to decompose a matrix into an orthogonal matrix  $Q$  and a right upper triangular matrix  $R$

1. Givens-Rotation: ideally for **sparse** matrices.
  - Detect non-zero entries standing below the diagonal,
  - Clean them up by applying the corresponding Givens-Rotation matrix.
2. Householder-Reflection: ideally for **dense** matrices

## 1.1 Step-by-step with Givens-Rotation

**Example 1.** *Examine the following matrix  $A$  given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{pmatrix}_{3 \times 3}. \quad (1)$$

The only entry to be cleaned up is  $A_{32}$ . Therefore, the Givens-Rotation matrix is

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}_{3 \times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2 + b^2} = \sqrt{A_{22}^2 + A_{32}^2} = \sqrt{2}, \\ c = a/r = -1/\sqrt{2}, \\ s = -b/r = -1/\sqrt{2}, \end{cases} \quad (2)$$

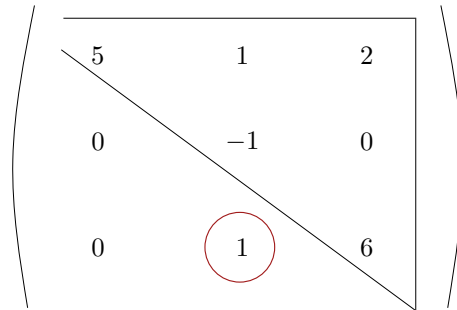


Figure 1: The entry  $A_{32}$  of matrix  $A$  is the only one that we may clean up so as to obtain a right upper triangular matrix. Givens-Rotation matrix is  $G_{32}$ .

Then, applying  $G_{32}$  onto  $A$  from the left leads to

$$G_{32}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 2 \\ 0 & \sqrt{2} & 6/\sqrt{2} \\ 0 & 0 & -6/\sqrt{2} \end{pmatrix} =: B, \quad (3)$$

which has already a form of a right upper triangular matrix  $R$ . Therefore, we obtain

$$G_{32}A = R, \quad (4)$$

which, equally, leads to

$$G_{32}A = R \Leftrightarrow G_{32}^{-1}G_{32}A = G_{32}^{-1}R \Leftrightarrow A = G_{32}^{-1}R \Leftrightarrow A = G_{32}^{\top}R. \quad (5)$$

Note in passing that  $G_{32}^{-1} = G_{32}^{\top}$  in the previous step is due to the fact that the matrix  $G_{32}$  itself is orthogonal. By assigning  $Q := G_{32}^{\top}$  we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{32}^{\top}R = QR.}$$

**Example 2.** *Examine the following matrix  $A$  given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix}_{3 \times 3}. \quad (6)$$

The only entry to be cleaned up is  $A_{21}$ . Therefore, the Givens-Rotation matrix is

$$G_{21} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3 \times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2 + b^2} = \sqrt{A_{11}^2 + A_{21}^2} = \sqrt{26}, \\ c = a/r = 5/\sqrt{26}, \\ s = -b/r = -1/\sqrt{26}, \end{cases} \quad (7)$$

Figure 2: The entry  $A_{21}$  of matrix  $A$  is the only one that we may clean up so as to obtain a right upper triangular matrix. Givens-Rotation matrix is  $G_{21}$ .

Then, applying  $G_{21}$  onto  $A$  from the left leads to

$$G_{21}A = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{26} & 4/\sqrt{26} & 10/\sqrt{26} \\ 0 & -6/\sqrt{26} & -2/\sqrt{26} \\ 0 & 0 & 6 \end{pmatrix} \quad (8)$$

which has already a form of a right upper triangular matrix  $R$ . Therefore, we obtain

$$G_{21}A = R, \quad (9)$$

which, equally, leads to

$$G_{21}A = R \Leftrightarrow G_{21}^{-1}G_{21}A = G_{21}^{-1}R \Leftrightarrow A = G_{21}^{-1}R \Leftrightarrow A = G_{21}^{\top}R. \quad (10)$$

Note in passing that  $G_{21}^{-1} = G_{21}^{\top}$  in the previous step is due to the fact that the matrix  $G_{21}$  itself is orthogonal. By assigning  $Q := G_{21}^{\top}$  we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{21}^{\top}R = QR.}$$

**Example 3.** Examine the following matrix  $A$  given as follows

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 6 \end{pmatrix}_{3 \times 3}. \quad (11)$$

The only entry to be cleaned up is  $A_{31}$ . Therefore, the Givens-Rotation matrix is

$$G_{31} = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix}_{3 \times 3} \quad \text{where} \quad \begin{cases} r = \sqrt{a^2 + b^2} = \sqrt{A_{11}^2 + A_{31}^2} = \sqrt{26}, \\ c = a/r = 5/\sqrt{26}, \\ s = -b/r = -1/\sqrt{26}, \end{cases} \quad (12)$$

Then, applying  $G_{31}$  onto  $A$  from the left leads to

$$G_{31}A = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & -1 & 0 \\ 0 & -1/\sqrt{26} & 28/\sqrt{26} \end{pmatrix} =: B, \quad (13)$$

which is not yet in forms of a right upper triangular matrix  $R$ . Herein, we still need to perform one more Givens-Rotation on the later matrix to clean up the entry  $B_{32}$ . The second Givens-Rotation takes the form

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix}_{3 \times 3} \quad \text{where} \quad \begin{cases} r_2 = \sqrt{a_2^2 + b_2^2} = \sqrt{B_{11}^2 + B_{31}^2} = 9/\sqrt{78}, \\ c_2 = a_2/r_2 = -\sqrt{78}/9, \\ s_2 = -b_2/r_2 = \sqrt{3}/9, \end{cases} \quad (14)$$

Next, applying  $G_{32}$  onto  $B = G_{31}A$  leads to

$$G_{32}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & -1 & 0 \\ 0 & -1/\sqrt{26} & 28/\sqrt{26} \end{pmatrix} \quad (15)$$

which yields

$$G_{32}B = \begin{pmatrix} \sqrt{26} & 5/\sqrt{26} & 16/\sqrt{26} \\ 0 & 3\sqrt{3}/\sqrt{26} & -28\sqrt{3}/(9\sqrt{26}) \\ 0 & 0 & -28\sqrt{3}/9 \end{pmatrix} \quad (16)$$

which now has a form of a right upper triangular matrix  $R$ . Therefore, we obtain

$$G_{32}B = R \Leftrightarrow G_{32}G_{31}A = R \Leftrightarrow A = G_{31}^{-1}G_{32}^{-1}R \Leftrightarrow A = G_{31}^{\top}G_{32}^{\top}R. \quad (17)$$

By assigning  $Q := G_{31}^{\top}G_{32}^{\top}$  we arrive at QR-decomposition

$$\therefore \quad \boxed{A = G_{31}^{\top}G_{32}^{\top}R = QR.}$$

**Example 4.** *Examine the following matrix  $A$  given as follows*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 6 \end{pmatrix}_{3 \times 3}. \quad (18)$$

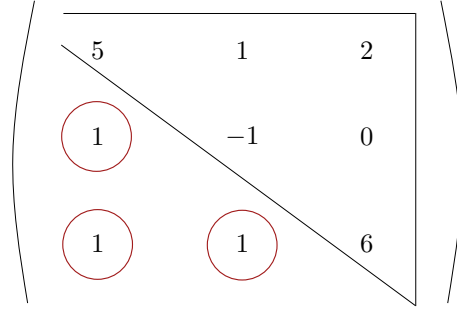


Figure 3: The entry  $A_{21}$ ,  $A_{31}$ , and  $A_{32}$  of matrix  $A$  are the three entries that we would like clean up so that we may obtain a right upper triangular matrix. Givens-Rotation matrix for  $A_{21}$ ,  $A_{31}$ , and  $A_{32}$  is  $G_{21}$ ,  $G_{31}$ , and  $G_{32}$ , respectively. The order of applying  $G_{21}$  first or  $G_{31}$  or  $G_{32}$  is no matter, but the coefficients used to build the Givens-Rotation matrix  $r$ ,  $a$ , and  $b$  have to be carefully computed, since they become various after every time applying the rotation.

The Givens-Rotation matrix  $G_{21}$  takes the form

$$G_{21} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (19)$$

The Givens-Rotation matrix  $G_{31}$  takes the form

$$G_{31} = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \quad (20)$$

The Givens-Rotation matrix  $G_{32}$  takes the form

$$G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \quad (21)$$

## 2 Numerics: QR-algorithm

**Observation 1 (QR-algorithm vs. QR-decomposition).** *QR-algorithm (as seen in Maths III) is not QR-decomposition (as seen in Maths II). QR-algorithm (Maths III) is an algorithm used to find **all** eigenvalues of a matrix  $A$  **numerically**. Meanwhile, QR-decomposition is a technique in linear algebra used to decompose a matrix  $A$  into an orthogonal matrix  $Q$  and a right upper triangular matrix  $R$ . Nevertheless, we will still need QR-decomposition for QR-algorithm.*

### 3 Analysis: Application of Hölder's inequality

**Observation 2.** *Sometimes we would like to estimate whether an integral, from complicated to very complicated, is bounded or not, without actually compute it. The Hölder's inequality is an ideally mathematical tool for such situation.*

**Example 5.** *Examine the following integral*

$$\int_{\Omega} (x+2)^{-3/5} \exp(-2x/3) dx$$

Approach: