

Self-exercise - SRU00

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1 Step-by-step with A1-SRU00

A function which takes its argument as a function is called functional.

$$\boxed{\text{functional} = \text{function of function}} \quad (1)$$

Example 1. Examine the first variation of the functional F in v direction

$$F : \begin{cases} C^2([-1, 1]) \rightarrow \mathbb{R}, \\ y \mapsto F(y) := \int_{-1}^1 \left(y(x)y'(x) + e^{y'(x)^2} \right) dx, \end{cases} \quad \text{with } y : \begin{cases} [-1, 1] \rightarrow \mathbb{R}, \\ x \mapsto y(x), \end{cases} \quad (2)$$

where C^2 is called space of continuous functions, i.e. function space. The number 2 in C^2 indicates that those functions living in C^2 are those continuously twice differentiable. Herein, y is such function living in C^2 , as shown in (2)₁. Besides, the function y takes x as its only variable. In this case, the only variable x is defined on the closed interval $[-1, 1]$, as shown in (2)₂.

Approach: By using the definition, as follows

$$\boxed{\delta F(y; v) := \lim_{s \rightarrow 0} \frac{F(y + sv) - F(y)}{s}}, \quad (3)$$

where δF is called variation of the functional F , and the symbol δ stands for variation.

Interesting fact 1. The symbol δ seen in variation of a functional is not the same one often seen in partial derivative, which is ∂ . In \LaTeX their commands are called, either from *maths'* package such as **amsmath** or **mathtools**, as follows

$$\begin{array}{lll} \delta & \rightarrow \backslash delta & \rightarrow \text{variation of a functional} \\ \partial & \rightarrow \backslash partial & \rightarrow \text{partial derivative of a multi-variable function} \end{array} \quad (4)$$

In order to find the variation of the functional F in v direction, we need to compute the limit, as shown in (3).

$$\begin{aligned}\delta F(y; v) &:= \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left((y + s v)(x) (y + s v)'(x) + \exp \left((y + s v)'^2(x) \right) \right) dx \quad (5)\end{aligned}$$

$$- \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y(x) y'(x) + \exp \left(y'^2(x) \right) \right) dx \quad (6)$$

Let us now forget the argument (x) in (5) and (6) for a while, i.e. by dropping it temporarily, just for a sake of a better overview. Then, the new expression without (x) will take the form

$$\begin{aligned}\delta F(y; v) &:= \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left((y + s v) (y + s v)' + \exp \left((y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + \exp \left(y'^2 \right) \right) dx \quad (7)\end{aligned}$$

which leads to further analysis, as follows

$$\begin{aligned}\delta F(y; v) &\stackrel{(7)}{=} \lim_{s \rightarrow 0} \frac{F(y + s v) - F(y)}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left((y + s v)(y' + s v') + \exp \left((y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + s y v' + s v y' + s^2 v v' + \exp \left((y + s v)'^2 \right) \right) dx \\ &\quad - \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(y y' + s y v' + s v y' + s^2 v v' - y y' \right) dx \\ &\quad + \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left((y + s v)'^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \underbrace{\left(s y v' + s v y' + s^2 v v' \right) dx}_{:=\mathcal{A}} \\ &\quad + \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \underbrace{\left(\exp \left((y + s v)'^2 \right) - \exp \left(y'^2 \right) \right) dx}_{:=\mathcal{B}} \quad (8)\end{aligned}$$

Hence, it leads to

$$\therefore \boxed{\delta F(y; v) = \mathcal{A} + \mathcal{B}.} \quad (9)$$

Next, we consider part \mathcal{A} , as follows

$$\begin{aligned} \mathcal{A} &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(s y v' + s v y' + s^2 v v' \right) dx = \lim_{s \rightarrow 0} \int_{-1}^1 \left(y v' + v y' + s v v' \right) dx \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y v' + v y' \right) dx + \lim_{s \rightarrow 0} \int_{-1}^1 (s v v') dx \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y v' + v y' \right) dx + \underbrace{\lim_{s \rightarrow 0} \left(s \int_{-1}^1 (v v') dx \right)}_{=0} \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y(x) v'(x) + v(x) y'(x) \right) dx \quad (10) \\ &= \lim_{s \rightarrow 0} \int_{-1}^1 \left(y(x) v(x) \right)' dx = \int_{-1}^1 \frac{d}{dx} \left(y(x) v(x) \right) dx \\ &= \int_{-1}^1 \frac{d}{dx} \left(y(x) v(x) \right) dx = \left(y(x) v(x) \right) \Big|_{x=-1}^{x=1} = y(1) v(1) - y(-1) v(-1), \quad (11) \end{aligned}$$

where (x) has returned back in (10) again in order to get ready to compute the follow-up integral. Therefore, solution to \mathcal{A} yields

$$\therefore \boxed{\mathcal{A} = y(1) v(1) - y(-1) v(-1).} \quad (12)$$

Besides, the computation of part \mathcal{B} goes as follows

$$\begin{aligned} \mathcal{B} &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left((y + s v)^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left(y'^2 + 2 s y' v' + s^2 v'^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left(y'^2 \right) \exp \left(2 s y' v' + s^2 v'^2 \right) - \exp \left(y'^2 \right) \right) dx \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \int_{-1}^1 \left(\exp \left(y'^2 \right) \left(\exp \left(2 s y' v' + s^2 v'^2 \right) - 1 \right) \right) dx \\ &= \int_{-1}^1 \left(\exp \left(y'^2 \right) \underbrace{\lim_{s \rightarrow 0} \frac{\exp \left(2 s y' v' + s^2 v'^2 \right) - 1}{s}}_{:=\mathcal{C}} \right) dx, \quad (13) \end{aligned}$$

where a further result of \mathcal{C} in (13) is computed by using l'Hôpital's rule, as follows

$$\begin{aligned}
\mathcal{C} &= \lim_{s \rightarrow 0} \frac{\exp(2 s y' v' + s^2 v'^2) - 1}{s} \\
&= \lim_{s \rightarrow 0} \left((2 y' v' + 2 s v'^2) \exp(2 s y' v' + s^2 v'^2) \right) \\
&= 2 y' v' \\
&= 2 y'(x) v'(x).
\end{aligned} \tag{14}$$

Substitution of (14) into (13) leads to

$$\therefore \quad \boxed{\mathcal{B} = 2 \int_{-1}^1 \exp(y'^2(x)) y'(x) v'(x) dx.} \tag{15}$$

Finally, by inserting (12) and (15) into (9) the final result yields

$$\therefore \quad \boxed{\delta F(y; v) = y(1) v(1) - y(-1) v(-1) + 2 \int_{-1}^1 y'(x) v'(x) \exp(y'^2(x)) dx.} \tag{16}$$