Self-exercise - SRU00

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1 Step-by-step with A1-SRU00

A function which takes its argument as a function is called functional.

$$\boxed{\text{functional} = \text{function of function}} \tag{1}$$

Example 1. *Examine the first variation of the functional F in v direction*

$$F : \begin{cases} C^2([-1,1]) \to \mathbb{R}, \\ y \mapsto F(y) := \displaystyle\int_{-1}^{1} \left(y(x)y'(x) + e^{y'(x)^2} \right) dx, \end{cases} \quad \text{with } y : \begin{cases} [-1,1] \to \mathbb{R}, \\ x \mapsto y(x), \end{cases} \tag{2}$$

where C^2 is called space of continuous functions, i.e. function space. The number 2 in C^2 indicates that those functions living in C^2 are those continuously twice differentiable. Herein, y is such function living in C^2, as shown in $(2)_1$. Besides, the function y takes x as its only variable. In this case, the only variable x is defined on the closed interval $[-1,1]$, as shown in $(2)_2$.

Approach: By using the definition, as follows

$$\boxed{\delta F(y;v) := \lim_{s\to 0} \frac{F(y+sv) - F(y)}{s},} \tag{3}$$

where δF is called variation of the functional F, and the symbol δ stands for variation.

Interesting fact 1. *The symbol δ seen in variation of a functional is not the same one often seen in partial derivative, which is ∂. In LaTeX their commands are called, either from maths' package such as* ***amsmath*** *or* ***mathtools****, as follows*

$$\begin{aligned} &\delta && \to \backslash delta && \to && \textit{variation of a functional} \\ &\partial && \to \backslash partial && \to && \textit{partial derivative of a multi-variable function} \end{aligned} \tag{4}$$

In order to find the variation of the functional F in v direction, we need to compute the limit, as shown in (3).

$$\delta F(y;v) := \lim_{s\to 0} \frac{F(y+s\,v)-F(y)}{s}$$

$$= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left((y+s\,v)(x)\,(y+s\,v)'(x) + \exp\left((y+s\,v)'^2(x) \right) \right)\,dx \tag{5}$$

$$- \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(y(x)y'(x) + \exp\left(y'^2(x) \right) \right)\,dx \tag{6}$$

Let us now forget the argument (x) in (5) and (6) for a while, i.e. by dropping it temporarily, just for a sake of a better overview. Then, the new expression without (x) will take the form

$$\begin{aligned}
\delta F(y;v) &:= \lim_{s\to 0} \frac{F(y+s\,v)-F(y)}{s} \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left((y+s\,v)\,(y+s\,v)' + \exp\left((y+s\,v)'^2 \right) \right)\,dx \\
&\qquad - \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(y\,y' + \exp\left(y'^2 \right) \right)\,dx
\end{aligned} \tag{7}$$

which leads to further analysis, as follows

$$\begin{aligned}
\delta F(y;v) &\overset{(7)}{:=} \lim_{s\to 0} \frac{F(y+s\,v)-F(y)}{s} \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left((y+s\,v)(y'+s\,v') + \exp\left((y+s\,v)'^2 \right) \right)\,dx \\
&\qquad - \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(y\,y' + \exp\left(y'^2 \right) \right)\,dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(y\,y' + s\,y\,v' + s\,v\,y' + s^2\,v\,v' + \exp\left((y+s\,v)'^2 \right) \right)\,dx \\
&\qquad - \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(y\,y' + \exp\left(y'^2 \right) \right)\,dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(y\,y' + s\,y\,v' + s\,v\,y' + s^2\,v\,v' - y\,y' \right)\,dx \\
&\qquad + \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(\exp\left((y+s\,v)'^2 \right) - \exp\left(y'^2 \right) \right)\,dx \\
&= \underbrace{\lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(s\,y\,v' + s\,v\,y' + s^2\,v\,v' \right)\,dx}_{:=\mathcal{A}} \\
&\qquad + \underbrace{\lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(\exp\left((y+s\,v)'^2 \right) - \exp\left(y'^2 \right) \right)\,dx}_{:=\mathcal{B}}
\end{aligned} \tag{8}$$

Hence, it leads to

$$\therefore \quad \boxed{\delta F(y;v) = \mathcal{A} + \mathcal{B}.} \tag{9}$$

Next, we consider part $\mathcal{A}$, as follows

$$\begin{aligned}
\mathcal{A} &= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(s\,y\,v' + s\,v\,y' + s^2\,v\,v' \Big)\,dx = \lim_{s\to 0} \int_{-1}^{1} \Big(y\,v' + v\,y' + s\,v\,v' \Big)\,dx \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y\,v' + v\,y' \Big)\,dx + \lim_{s\to 0} \int_{-1}^{1} (s\,v\,v')\,dx \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y\,v' + v\,y' \Big)\,dx + \underbrace{\lim_{s\to 0} \left(s \int_{-1}^{1} (v\,v')\,dx \right)}_{=0} \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y(x)\,v'(x) + v(x)\,y'(x) \Big)\,dx \qquad (10) \\
&= \lim_{s\to 0} \int_{-1}^{1} \Big(y(x)\,v(x) \Big)'\,dx = \int_{-1}^{1} \frac{d}{dx} \Big(y(x)\,v(x) \Big)\,dx \\
&= \int_{-1}^{1} \frac{d}{dx} \Big(y(x)\,v(x) \Big)\,dx = \Big(y(x)\,v(x) \Big)\Big|_{x=-1}^{x=1} = y(1)\,v(1) - y(-1)\,v(-1), \qquad (11)
\end{aligned}$$

where (x) has returned back in (10) again in order to get ready to compute the follow-up integral. Therefore, solution to $\mathcal{A}$ yields

$$\therefore \quad \boxed{\mathcal{A} = y(1)\,v(1) - y(-1)\,v(-1).} \tag{12}$$

Besides, the computation of part $\mathcal{B}$ goes as follows

$$\begin{aligned}
\mathcal{B} &= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\Big((y + s\,v)'^2 \Big) - \exp\left(y'^2 \right) \Big)\,dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\left(y'^2 + 2\,s\,y'\,v' + s^2\,v'^2 \right) - \exp\left(y'^2 \right) \Big)\,dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \Big(\exp\left(y'^2 \right) \exp\left(2\,s\,y'\,v' + s^2\,v'^2 \right) - \exp\left(y'^2 \right) \Big)\,dx \\
&= \lim_{s\to 0} \frac{1}{s} \int_{-1}^{1} \left(\exp\left(y'^2 \right) \Big(\exp\left(2\,s\,y'\,v' + s^2\,v'^2 \right) - 1 \Big) \right)\,dx \\
&= \int_{-1}^{1} \left(\exp\left(y'^2 \right) \underbrace{\lim_{s\to 0} \frac{\exp\left(2\,s\,y'\,v' + s^2\,v'^2 \right) - 1}{s}}_{:=\mathcal{C}} \right)\,dx, \qquad (13)
\end{aligned}$$

where a further result of $\mathcal{C}$ in (13) is computed by using l'Hôspital's rule, as follows

$$\begin{aligned}
\mathcal{C} &= \lim_{s\to 0} \frac{\exp\left(2\,s\,y'\,v' + s^2\,v'^2\right) - 1}{s} \\
&= \lim_{s\to 0} \Big(\left(2\,y'\,v' + 2\,s\,v'^2\right) \exp\left(2\,s\,y'\,v' + s^2\,v'^2\right) \Big) \\
&= 2\,y'\,v' \\
&= 2\,y'(x)\,v'(x).
\end{aligned} \tag{14}$$

Substitution of (14) into (13) leads to

$$\therefore \quad \boxed{\mathcal{B} = 2 \int_{-1}^{1} \exp\left(y'^2(x)\right) y'(x)\,v'(x)\,dx.} \tag{15}$$

Finally, by inserting (12) and (15) into (9) the final result yields

$$\therefore \quad \boxed{\delta F(y;v) = y(1)\,v(1) - y(-1)\,v(-1) + 2 \int_{-1}^{1} y'(x)\,v'(x) \exp\left(y'^2(x)\right) dx.} \tag{16}$$

2 Step-by-step with A4-SRU00

Example 2. *Examine the following initial value problem (IVP)*

$$y'''(t) + y'(t) = t\, y(t), \tag{17}$$

with initial conditions (IC) given as follows

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2. \tag{18}$$

i) Transform the above third order ordinary differential equation (ODE) to a system of first order ODEs.

ii) Solve the system i) numerically by using explicit Euler method with 2 steps.

Approach:

i) We need to transform the third order ODE given in (17) into a system of first order ODEs.

Question 1. *But.. how many ODEs shall we need in total for this system of first order ODEs?*
Ans: *3.*

Observation 1. *Since the highest order of derivative in (17) is 3, i.e. as we can see there are three $'$ from $y'''(t)$, the given ODE is called third order ODE. Besides, the lowest order in (17) is 0, i.e. as we can see from $y(t)$. Therefore, we shall need a system of* ***three*** *first order ODEs for the given third order ODE in (17).*

Question 2. *Which variable from (17) shall we need to take and define, so that we are able to transform (17) into a system of first order ODEs corresponding to the given initial conditions given in (18), e.g. is it ideal to define $y'''(t) := u_3(t)$?*
Ans: *$y(t)$, $y'(t)$ and $y''(t)$; and it is not ideal to go with $y'''(t) := u_3(t)$.*

Observation 2. *The three initial conditions given $y(0)$, $y'(0)$ and $y''(0)$, as shown in (18), give rise to the fact that we shall need to define three new variables for $y(t)$, $y'(t)$ and $y''(t)$. This choice is ideal for the ODE with the given ICs because we will be able to apply these ICs directly for the new system of three ODEs.*

Therefore, we shall need three first order ODEs for the new system. Besides, the transformation is done by setting new variables for $y(t)$, $y'(t)$ and $y''(t)$.

Transformation: We now need to set $y(t)$, $y'(t)$ and $y''(t)$ to new variables, let's say $u_0(t)$, $u_1(t)$, and $u_2(t)$, respectively, as follows

$$\begin{aligned} y(t) &:= u_0(t), && (19)\\ y'(t) &:= u_1(t), && (20)\\ y''(t) &:= u_2(t). && (21) \end{aligned}$$

Then, by taking the derivative of the new variable $u_0(t)$, $u_1(t)$ and $u_2(t)$, as newly defined in (19), (20) and (21), we obtain the following relations

$$\boxed{\begin{aligned} &u_0'(t) \overset{(19)}{=} y'(t) \overset{(20)}{=} u_1(t),\\ &u_1'(t) \overset{(20)}{=} y''(t) \overset{(21)}{=} u_2(t),\\ &u_2'(t) \overset{(21)}{=} y'''(t) \overset{(17)}{=} t\,y(t) - y'(t) \overset{(17)}{\overset{(17)}{=}} t\,u_0(t) - u_1(t), \end{aligned}} \tag{22}$$

which is summarized, together with keeping in mind about the order of the index i from $u_i(t)|_{i=0,1,2}$, as follows

$$\begin{array}{rcccc} u_0'(t) & = & & u_1(t) & \\ u_1'(t) & = & & & u_2(t) \\ u_2'(t) & = & t\,u_0(t) & -u_1(t) & \end{array} \tag{23}$$

which gives rise to the following system

$$\begin{array}{rcccc} u_0'(t) & = & 0\,u_0(t) & 1\,u_1(t) & 0\,u_2(t) \\ u_1'(t) & = & 0\,u_0(t) & 0\,u_1(t) & 1\,u_2(t) \\ u_2'(t) & = & t\,u_0(t) & -1\,u_1(t) & 0\,u_2(t) \end{array} \tag{24}$$

from where we can write the system in form of

$$\boldsymbol{u}' = A\boldsymbol{u} \tag{25}$$

as follows

$$\boxed{\underbrace{\begin{pmatrix} u_0(t)\\ u_1(t)\\ u_2(t) \end{pmatrix}'}_{:=\boldsymbol{u}'} = \underbrace{\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ t & -1 & 0 \end{pmatrix}}_{:=A} \underbrace{\begin{pmatrix} u_0(t)\\ u_1(t)\\ u_2(t) \end{pmatrix}}_{:=\boldsymbol{u}}} \tag{26}$$

where the relation between A and $\boldsymbol{u}$ in $A\boldsymbol{u}$ is matrix-vector multiplication. Besides, by substituting (19) (20) and (21) into (18), respectively, we obtain the initial conditions for the new system in (26)

$$\boxed{\begin{aligned} u_0(0) &= 0,\\ u_1(0) &= 1,\\ u_2(0) &= 2, \end{aligned}} \tag{27}$$

which form an equivalent system of three first-order ODEs.

ii) We need to solve the system of first order ODEs obtained in i) numerically by using explicit Euler method.

Summary 1. *Given the system of first order ODEs as follows*

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) \tag{28}$$

where the vector $\boldsymbol{u}$ *is defined as* $\boldsymbol{u} := (u_0, u_1, u_2)^\top$, *and the matrix* A *is*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix} \tag{29}$$

with initial conditions

$$u_0(0) = 0, \quad u_1(0) = 1, \quad u_2(0) = 2. \tag{30}$$

Solve this system numerically by using the explicit Euler method.

Approach: We need to approximate $\boldsymbol{u}'(t)$ arising in (28) by some numerical schemes, e.g. explicit Euler method.

Observation 3. *We consider Taylor expansion for* $\boldsymbol{u}(t)$ *around point* $t = t_0$

$$\boldsymbol{u}(t) = \boldsymbol{u}(t_0) + \boldsymbol{u}'(t_0)(t - t_0) + \mathcal{O}((t - t_0)^2), \tag{31}$$

which, for the sake of convergence, requires $h := t - t_0$ *to be small. Since* $h := t - t_0$, *we equally have* $t = h + t_0$. *Substitution of* $h := t - t_0$ *and* $t = t_0 + h$ *to (31) leads to the following expression*

$$\boldsymbol{u}(t_0 + h) = \boldsymbol{u}(t_0) + h\boldsymbol{u}'(t_0) + \mathcal{O}(h^2), \tag{32}$$

which is the new expression of Taylor expansion in terms of point t_0, *around which the function* $\boldsymbol{u}(t)$ *is approximated by the Taylor expansion, instead of the classical expression in terms of the variable* t *as seen in (31). Since point* t_0, *although it implied a fixed point, arising in (32) can be chosen arbitrarily, we can, therefore, substitute it for* t, *as follows*

$$\boldsymbol{u}(t + h) = \boldsymbol{u}(t) + h\boldsymbol{u}'(t) + \mathcal{O}(h^2), \tag{33}$$

Now, the expression (32) gives rise to a useful expression to approximate $\boldsymbol{u}'(t)$ *numerically, as follows*

$$\therefore \quad \boxed{\boldsymbol{u}'(t) \approx \frac{\boldsymbol{u}(t + h) - \boldsymbol{u}(t)}{h}.} \tag{34}$$

Substitution of (34) into (28) yields

$$\frac{\boldsymbol{u}(t + h) - \boldsymbol{u}(t)}{h} = A\boldsymbol{u}(t) \tag{35}$$

which leads to the formula of explicit Euler method

$$\boxed{\boldsymbol{u}^{(j+1)} = \boldsymbol{u}^{(j)} + h\, A\boldsymbol{u}^{(j)},} \tag{36}$$

where $\boldsymbol{u}^{(j)}$ is the numerical solution by Euler method after j steps, and components of $\boldsymbol{u}^{(j)}$ is known as

$$\boldsymbol{u}^{(j)} = \begin{pmatrix} u_0^{(k)} \\ u_1^{(k)} \\ u_2^{(k)} \end{pmatrix}^{(j)}, \tag{37}$$

which, together with (36), leads to the following expressions

$$\therefore \quad \boxed{\begin{aligned}
\boldsymbol{u}^{(0)} &= \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \\
\\
\boldsymbol{u}^{(1)} &= \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} + h \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + h \begin{pmatrix} 1 \\ 2 \\ 0-1 \end{pmatrix} \\
&= \begin{pmatrix} h \\ 1+2h \\ 2-h \end{pmatrix}, \\
\\
\boldsymbol{u}^{(2)} &= \begin{pmatrix} u_0^{(1)} \\ u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} + h \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} u_0^{(1)} \\ u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} \\
&= \begin{pmatrix} h \\ 1+2h \\ 2-h \end{pmatrix} + h \begin{pmatrix} 1+2h \\ 2-h \\ h \cdot h - (1+2h) \end{pmatrix} \\
&= \begin{pmatrix} 2h(1+h) \\ 1+4h-h^2 \\ 2-2h-2h^2+h^3 \end{pmatrix}.
\end{aligned}} \tag{38}$$