

# Self-exercise - SRU03

## Response to questions

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### 1 Step-by-step with A3-SRU03

**Example 1.** Let the linear  $m$ -step method be given by coefficients  $\alpha_l \in \mathbb{R}$  and  $\beta_l \in \mathbb{R}$  where  $l = 0, 1, \dots, m$ , satisfying the following system of equations

$$\sum_{l=0}^m a_l = 0, \quad (1)$$

$$\sum_{l=0}^m (l^q \alpha_l - q l^{q-1} \beta_l) = 0, \quad \text{for } q = 1, 2, 3, \dots, p. \quad (2)$$

Then the linear  $m$ -step method has the consistency order of  $p$ .

Consider the following ordinary differential equation (ODE) taking the form

$$x'(t) = f(t, x(t)), \quad \text{with } x(t_0) = x_0, \quad (3)$$

which may be approximated by a linear multistep method, taking the general form

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)). \quad (4)$$

**Observation 1.** There are 6 symbols arising in (1), (2), (4) to be aware of

1.  $m$  is the total number of steps used in the multistep method.
2.  $\alpha \in \mathbb{R}$  is the coefficient for the linear combination of  $x(t_j + lh)$
3.  $\beta \in \mathbb{R}$  is the coefficient for the linear combination of  $f(t_j + lh, x(t_j + lh))$ .
4.  $p$  is the consistency order.
5.  $l$  is the dummy index, running from 0 to  $m$ .
6.  $q$  is the dummy index, running from 1 to  $p$ .

**Observation 2.** The multistep method arising in (4) is a **linear** multistep method, or specifically **linear**  $m$ -step method. The term **linear** coming along with  $m$ -step method, i.e. linear  $m$ -step method, emphasizes the fact that there is actually a **linear combination** of the terms  $x(t_j + lh)$  with coefficients  $\alpha_l$  on the LHS of (4), and a **linear combination** of the terms  $f(t_j + lh, x(t_j + lh))$  with coefficients  $\beta_l$  on the RHS of (4). This linear  $m$ -step method uses the information from the previous  $m$  steps to compute the value for the next step. In details, let us examine the formula (4) which is written again as follows

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)), \quad (5)$$

whose LHS is written in its entirety as follows

$$\begin{aligned} \sum_{l=0}^m \alpha_l x(t_j + lh) &= \alpha_0 x(t_j) + \alpha_1 x(t_j + h) + \alpha_2 x(t_j + 2h) + \dots \\ &\quad + \dots + \alpha_{m-1} x(t_j + (m-1)h) + \alpha_m x(t_j + mh), \end{aligned} \quad (6)$$

whereas the RHS of (5) has its expansion as follows

$$\begin{aligned} \sum_{l=0}^m \beta_l f(t_j + lh, x(t_j + lh)) &= \beta_0 f(t_j, x(t_j)) + \beta_1 f(t_j + h, x(t_j + h)) \\ &\quad + \beta_2 f(t_j + 2h, x(t_j + 2h)) \\ &\quad + \dots + \\ &\quad + \beta_{m-1} f(t_j + (m-1)h, x(t_j + (m-1)h)) \\ &\quad + \beta_m f(t_j + mh, x(t_j + mh)). \end{aligned} \quad (7)$$

The insertion of (6) and (7) into (5) leads to

$$\begin{aligned} &\alpha_0 x(t_j) + \alpha_1 x(t_j + h) + \alpha_2 x(t_j + 2h) \\ &\quad + \dots \\ &\quad + \alpha_{m-1} x(t_j + (m-1)h) + \alpha_m x(t_j + mh) \\ &= \\ &\beta_0 f(t_j, x(t_j)) + \beta_1 f(t_j + h, x(t_j + h)) + \beta_2 f(t_j + 2h, x(t_j + 2h)) \\ &\quad + \dots + \\ &\quad + \beta_{m-1} f(t_j + (m-1)h, x(t_j + (m-1)h)) + \beta_m f(t_j + mh, x(t_j + mh)), \end{aligned} \quad (8)$$

which tells us the fact that the value  $x(t_j + mh)$  is approximated by using all information from previous steps, based on

1.  $x(t_j), x(t_j + h), \dots, x(t_j + (m-1)h)$ .
2.  $f(t_j, x(t_j)), f(t_j + h, x(t_j + h)), \dots, f(t_j + mh, x(t_j + mh))$ .
3. Coefficients  $\alpha_l$  and  $\beta_l$ .

**Observation 3.** The appearance of the green term  $f(t_j + mh, x(t_j + mh))$  arising in (8) plays a significant role, meaning that

1.  $\beta_m = 0$ : the green term is switched off, and the scheme becomes **explicit**.
2.  $\beta_m \neq 0$ : the green term is switched on, and the scheme becomes **implicit**.

**Observation 4.** Specific choices of  $\alpha_l$  and  $\beta_l$  leads to some familiar methods. Especially, when  $m = 1$  the linear  $m$ -step method becomes single-step method, or one-step method. If so, according to (4) there will be 4 coefficients ( $\alpha_0, \alpha_1, \beta_0, \beta_1$ ) to be defined. For example:

1.  $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 1, 0)$  we obtain explicit Euler.
2.  $(\alpha_0, \alpha_1, \beta_0, \beta_1) = (-1, 1, 0, 1)$  we obtain implicit Euler.

When the time  $t$  is considered at a specific discretized time point  $t = t_j + lh$ , the ODE in (3) yields

$$x'(t_j + lh) = f(t_j + lh, x(t_j + lh)), \quad (9)$$

Substitution of (9) into (4) leads to

$$\sum_{l=0}^m \alpha_l x(t_j + lh) = h \sum_{l=0}^m \beta_l x'(t_j + lh), \quad (10)$$

which, by group the summation sign together, leads equally to

$$\sum_{l=0}^m (\alpha_l x(t_j + lh) - h \beta_l x'(t_j + lh)) = 0. \quad (11)$$

Then, the order of consistency is figured out by taking advantage of using Taylor's expansion for  $x(t_j + lh)$  and  $x'(t_j + lh)$ . By using (19) and (21) in Observation 5 we obtain

$$x(t_j + lh) = \sum_{n=0}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}), \quad (12)$$

$$x'(t_j + lh) = \sum_{n=0}^r \frac{x^{(n+1)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{r+1}), \quad (13)$$

where, according to (19) and (21) in Observation 5, the following two substitutions have been performed for (12) and (13), meaning that

$$a \rightarrow t_j, \quad (14)$$

$$h \rightarrow lh. \quad (15)$$

Furthermore, expression (12) can be split into two main parts, where the first part is going with index  $n = 0$ , and the second part with  $n \geq 1$ , as follows

$$\begin{aligned}
x(t_j + lh) &= \sum_{n=0}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}) \\
&= \frac{x^{(0)}(t_j) (lh)^0}{0!} + \sum_{n=1}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}) \\
&= x(t_j) + \sum_{n=1}^q \frac{x^{(n)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{q+1}).
\end{aligned} \tag{16}$$

Besides, shifting index starting from  $n = 0$  to  $n = 1$  for expression (13) yields

$$\begin{aligned}
x'(t_j + lh) &= \sum_{n=0}^r \frac{x^{(n+1)}(t_j) (lh)^n}{n!} + \mathcal{O}((lh)^{r+1}) \\
&= \sum_{n=1}^{r+1} \frac{x^{(n)}(t_j) (lh)^{n-1}}{(n-1)!} + \mathcal{O}((lh)^{r+1}) \\
&= \sum_{n=1}^{r+1} \frac{n x^{(n)}(t_j) (lh)^{n-1}}{n!} + \mathcal{O}((lh)^{r+1}) \\
&= \sum_{n=1}^{r+1} \frac{n x^{(n)}(t_j) l^{n-1} h^n}{n! h} + \mathcal{O}((lh)^{r+1})
\end{aligned} \tag{17}$$

**Observation 5.** *Taylor's expansion of function  $x(t)$  around point  $a$  reads*

$$\begin{aligned} x(t) &= x(a) + \frac{x'(a)(t-a)}{1!} + \frac{x''(a)(t-a)^2}{2!} + \dots + \frac{x^{(p)}(a)(t-a)^p}{p!} \\ &\quad + \mathcal{O}((t-a)^{p+1}) \\ &= \sum_{n=0}^p \frac{x^{(n)}(a)(t-a)^n}{n!} + \mathcal{O}((t-a)^{p+1}). \end{aligned} \quad (18)$$

Then, by setting  $h := t - a$  we obtain  $t = a + h$ ; hence, the (18) becomes

$$\begin{aligned} x(a+h) &= x(a) + \frac{x'(a)h}{1!} + \frac{x''(a)h^2}{2!} + \dots + \frac{x^{(p)}(a)h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^p \frac{x^{(n)}(a)h^n}{n!} + \mathcal{O}(h^{p+1}). \end{aligned} \quad (19)$$

Since the point  $a$  can be chosen arbitrarily, we can set it as a variable  $t$ , yielding

$$\begin{aligned} x(t+h) &= x(t) + \frac{x'(t)h}{1!} + \frac{x''(t)h^2}{2!} + \dots + \frac{x^{(p)}(t)h^p}{p!} + \mathcal{O}(h^{p+1}) \\ &= \sum_{n=0}^p \frac{x^{(n)}(t)h^n}{n!} + \mathcal{O}(h^{p+1}), \end{aligned} \quad (20)$$

which, by taking derivative w.r.t.  $t$  on both sides of (20), gives rise to

$$\begin{aligned} x'(t+h) &= x'(t) + \frac{x''(t)h}{1!} + \frac{x'''(t)h^2}{2!} + \dots + \frac{x^{(p+1)}(t)h^p}{p!} + \mathcal{O}(h^p) \\ &= \sum_{n=0}^p \frac{x^{(n+1)}(t)h^n}{n!} + \mathcal{O}(h^p). \end{aligned} \quad (21)$$

$$x(t_j + lh) = x(t_j) \quad (22)$$