

Global exercise - GUE11

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SUMMARY OF CONTENT COVERED: Analysis

- Review line integral of the 1st kind (scalar field) and 2nd kind (vector field)
- Surface integral of the 1st kind (scalar field) and 2nd kind (vector field)
- Gauss's theorem and its useful applications

1 Course evaluation

The evaluation for the course

22W-11.03530 (L) Mathematische Grundlagen III (CES) (Übung)

has been opened from now on until **Friday 20.01.2023 23:59:00** via QR code



Figure 1: Evaluation QR-code.

or via the following link

<https://www.campus.rwth-aachen.de/evasys/online.php?pswd=8J5FHC8JH9>

2 Summary: Line integral and Surface integral

2.1 First kind: Scalar field

Recall 1. *Line integral of a **scalar field** $\phi : \Omega \rightarrow \mathbb{R}$ is defined as follows*

$$\int_{\Gamma} \phi \, ds := \int_a^b \phi(\gamma(t)) \|\gamma'(t)\| \, dt \quad (1)$$

→ Intuitively: Mass

Recall 2. *Surface integral of a **scalar field** $\phi : \Omega \rightarrow \mathbb{R}$ is defined as follows*

$$\int_{\Gamma} \phi \, dA := \int_B \phi(\Phi(p, q)) \|\Phi_p \times \Phi_q\| \, d(p, q) \quad (2)$$

→ Intuitively: Mass

2.2 Second kind: Vector field

Recall 3. *Line integral of a **vector field** $\mathbf{f} : \Omega \rightarrow \mathbb{R}^n$ is defined as follows*

$$\int_{\Gamma} \mathbf{f} \cdot d\mathbf{x} := \int_a^b \langle \mathbf{f}(\gamma(t)), \gamma'(t) \rangle \, dt \quad (3)$$

→ Intuitively: Work done

Recall 4. *Surface integral of a **vector field** $\mathbf{f} : \Omega \rightarrow \mathbb{R}^n$ is defined as follows*

$$\int_{\Gamma} \langle \mathbf{f}, \mathbf{v} \rangle \, dA = \int_{\Gamma} \mathbf{f} \cdot d\mathbf{A} := \int_B \langle \phi(\Phi(p, q)), \Phi_p \times \Phi_q \rangle \, d(p, q) \quad (4)$$

→ Intuitively: Flow

3 Step-by-step: Surface integral of the 1st kind

Example 1. *Examine Given surface integral*

$$\int_{\Omega} 2\sqrt{4-x^2-y^2} d\sigma \quad (5)$$

where $\Omega = S \cap Z$ is the portion of the sphere

$$S = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0, x^2 + y^2 + z^2 = 4\} \quad (6)$$

inside the cylinder

$$Z = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\} \quad (7)$$

1. Define spherical coordinates in Ω (parameterize the surface of interest Ω).
2. Calculate the surface integral (5) using the defined spherical coordinates.

Approach:

On the surface Ω , we have $r = 2$, so we choose the sphere coordinate with:

$$\Phi(\phi, \theta) = (2 \cdot \cos \phi \cdot \sin \theta, 2 \cdot \sin \phi \cdot \sin \theta, 2 \cdot \cos \theta)^T, \quad \phi \in [0, 2\pi], \theta \in [0, \frac{\pi}{6}] \quad (8)$$

Hence, we obtain

$$\left\| \frac{\partial \Phi}{\partial \phi} \times \frac{\partial \Phi}{\partial \theta} \right\| = 2^2 \cdot |\sin \theta| \quad (9)$$

Furthermore, on the surface we have $x^2 + y^2 = 4 - z^2$ so that

$$2\sqrt{4-x^2-y^2} = 2\sqrt{z^2} = 2 \cdot z = 4 \cdot \cos \theta \quad (10)$$

Therefore, we obtain

$$\begin{aligned} \int_{\Omega} 2\sqrt{4-x^2-y^2} d\sigma &= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} 4 \cos(\theta) \cdot 2^2 \sin \theta d\theta d\phi \\ \therefore &= 32\pi \int_0^{\frac{\pi}{6}} \cos \theta \cdot \sin(\theta) d\theta \\ &= 4\pi \end{aligned} \quad (11)$$

4 Step-by-step: Surface integral of the 2nd kind

Example 2. Examine (i) the normalized normal vector of the surface F and (ii) the surface integral of a vector field based on the surface F given as follows

$$F = \left\{ \alpha \in [0, 1], \beta \in [0, 2\pi) \mid \left(e^\alpha, \alpha \cos(\beta), \alpha \sin(\beta) \right) \right\}, \quad (12)$$

and the vector field \mathbf{f} given by $\mathbf{f}(x, y, z) = (x\sqrt{y^2 + z^2}, y, z)$.

Approach:

(i) The normal vector of the surface is computed as follows

$$\mathbf{n} = \partial_\alpha F \times \partial_\beta F = \begin{pmatrix} e^\alpha \\ \cos(\beta) \\ \sin(\beta) \end{pmatrix} \times \begin{pmatrix} 0 \\ -\alpha \sin(\beta) \\ \alpha \cos(\beta) \end{pmatrix} \quad (13)$$

$$= \begin{pmatrix} \alpha \cos^2(\beta) + \alpha \sin^2(\beta) \\ -\alpha \cos(\beta)e^\alpha \\ -\alpha \sin(\beta)e^\alpha \end{pmatrix} \quad (14)$$

$$= \begin{pmatrix} \alpha \\ -\alpha \cos(\beta)e^\alpha \\ -\alpha \sin(\beta)e^\alpha \end{pmatrix}, \quad (15)$$

Observation 1. Be aware of the sign switching $(+, -, +)$ when using **co-factor method** to compute the determinant

$$\partial_\alpha F \times \partial_\beta F = \begin{vmatrix} i & j & k \\ e^\alpha & \cos(\beta) & \sin(\beta) \\ 0 & -\alpha \sin(\beta) & \alpha \cos(\beta) \end{vmatrix} \quad (16)$$

$$= i \begin{vmatrix} \cos(\beta) & \sin(\beta) \\ -\alpha \sin(\beta) & \alpha \cos(\beta) \end{vmatrix} - j \begin{vmatrix} e^\alpha & \sin(\beta) \\ 0 & \alpha \cos(\beta) \end{vmatrix} + k \begin{vmatrix} e^\alpha & \cos(\beta) \\ 0 & -\alpha \sin(\beta) \end{vmatrix} \quad (17)$$

and the norm of the normal vector is computed as follows

$$\|\mathbf{n}\|_2 = \sqrt{\alpha^2 + \alpha^2 \cos^2(\beta)e^{2\alpha} + \alpha^2 \sin^2(\beta)e^{2\alpha}} = \alpha\sqrt{1 + e^{2\alpha}}. \quad (18)$$

which leads to the normalized normal vector of the surface F as follows

$$\boldsymbol{\nu} = (1 + e^{2\alpha})^{-1/2} \begin{pmatrix} 1 \\ -\cos(\beta)e^\alpha \\ -\sin(\beta)e^\alpha \end{pmatrix}. \quad (19)$$

(ii) We now compute the surface integral of the second kind as follows

- By using \mathbf{n}

$$\boxed{\int_F \langle \mathbf{f}, \mathbf{n} \rangle d\sigma = \int_0^1 \int_0^{2\pi} \langle \mathbf{f}(F(\alpha, \beta)), \mathbf{n} \rangle d\beta d\alpha} \quad (20)$$
$$= \dots$$

- By using $\boldsymbol{\nu}$

$$\boxed{\int_F \langle \mathbf{f}, \boldsymbol{\nu} \rangle d\sigma = \int_0^1 \int_0^{2\pi} \langle \mathbf{f}(F(\alpha, \beta)), \boldsymbol{\nu} \rangle d\beta d\alpha} \quad (21)$$
$$= \dots$$

5 Step-by-step with Gauss's theorem example

Example 3. *Examine the vector field \mathbf{f} with*

$$\mathbf{f} : \begin{cases} \mathbb{R}^2 \rightarrow \mathbb{R}^2, \\ (x, y) \mapsto \mathbf{f}(x, y) = (x + y, y^2), \end{cases} \quad (22)$$

and let $B \subset \mathbb{R}^2$ be a triangle with vertices at $(0, 0)$, $(1, 0)$ and $(1, 2)$. Verify the Gauss's theorem, by calculating the two integrals $\int_{\partial B} \mathbf{f} \cdot \boldsymbol{\nu} ds$ and $\int_B \operatorname{div} \mathbf{f} dA$, and compare them.

Approach:

Observation 2 (2nd kind line integral). *We note that the integral*

$$\int_{\partial B} \mathbf{f} \cdot \boldsymbol{\nu} ds \quad (23)$$

is the line integral of the second kind, i.e. line integral of a vector field.

Observation 3 (1st kind surface integral). *We note that the integral*

$$\int_B \operatorname{div} \mathbf{f} dA \quad (24)$$

is the surface integral of the first kind, i.e. surface integral of a scalar field. Note in passing that divergence of vector field $\mathbf{f} \in \mathbb{R}^3$ yields a scalar

$$\operatorname{div} \mathbf{f} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3. \quad (25)$$

Likewise, divergence of vector field $\mathbf{f} \in \mathbb{R}^2$ yields a scalar

$$\operatorname{div} \mathbf{f} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \partial_x f_1 + \partial_y f_2. \quad (26)$$

(i) First, we compute the surface integral of the scalar field

$$\begin{aligned} \int_B \operatorname{div} \mathbf{f} \, dA &= \int_0^1 \int_0^{2x} \operatorname{div} \begin{pmatrix} x+y \\ y^2 \end{pmatrix} dy \, dx \\ &= \int_0^1 \int_0^{2x} (1+2y) dy \, dx \\ &= \int_0^1 (4x^2 + 2x) dx \\ &= \dots \end{aligned} \quad (27)$$

where we have computed the divergence of vector field \mathbf{f} as follows

$$\operatorname{div} \mathbf{f} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} x+y \\ y^2 \end{pmatrix} \quad (28)$$

$$= \partial_x f_1 + \partial_y f_2 \quad (29)$$

$$= \partial_x(x+y) + \partial_y(y^2) \quad (30)$$

$$= \dots \quad (31)$$

(ii) Since the area B is bounded by the three curves, we obtain the following relations

$$\begin{aligned} \gamma_1(t) &= \begin{pmatrix} t \\ 0 \end{pmatrix}, t \in [0, 1] & \rightarrow \gamma'_1(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \rightarrow \|\gamma'_1\|_2 &= 1 \\ \gamma_2(t) &= \begin{pmatrix} 1 \\ t \end{pmatrix}, t \in [0, 2] & \rightarrow \gamma'_2(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \rightarrow \|\gamma'_2\|_2 &= 1 \\ \gamma_3(t) &= \begin{pmatrix} 1-t \\ 2-2t \end{pmatrix}, t \in [0, 1] & \rightarrow \gamma'_3(t) &= \begin{pmatrix} -1 \\ -2 \end{pmatrix} & \rightarrow \|\gamma'_3\|_2 &= \sqrt{5} \end{aligned} \quad (32)$$

Observation 4. *Parametrization of a line function may lead to various options. However, we should be aware of the limit of the parameter variable. For example*

$$\gamma_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \quad (33)$$

which leads to new limit applied on t taking $[0, 1]$ instead of $[0, 2]$

$$\gamma_2(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}, t \in [0, 1] \quad \rightarrow \quad \gamma_2'(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \rightarrow \quad \|\gamma_2'\|_2 = 2 \quad (34)$$

Likewise, normal vectors take the form

$$\boldsymbol{\nu}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ on } \gamma_1(t), \quad \boldsymbol{\nu}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ on } \gamma_2(t), \quad \boldsymbol{\nu}_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{5}} \text{ on } \gamma_3(t), \quad (35)$$

which have been computed based on the following relations

$$\boldsymbol{\nu}_1 \cdot \gamma_1'(t) \stackrel{!}{=} 0 \quad (36)$$

$$\boldsymbol{\nu}_2 \cdot \gamma_2'(t) \stackrel{!}{=} 0 \quad (37)$$

$$\boldsymbol{\nu}_3 \cdot \gamma_3'(t) \stackrel{!}{=} 0 \quad (38)$$

Now, we compute the line integral of the second kind $\int_{\partial B} \mathbf{f} \cdot \boldsymbol{\nu} ds$

$$\int_{\partial B} \mathbf{f} \cdot \boldsymbol{\nu} ds = \int_{\partial B = \gamma_1(t) + \gamma_2(t) + \gamma_3(t)} \mathbf{f} \cdot \boldsymbol{\nu} ds \quad (39)$$

$$= \int_{\gamma_1(t)} \mathbf{f} \cdot \boldsymbol{\nu}_1 ds + \int_{\gamma_2(t)} \mathbf{f} \cdot \boldsymbol{\nu}_2 ds + \int_{\gamma_3(t)} \mathbf{f} \cdot \boldsymbol{\nu}_3 ds \quad (40)$$

$$= \int_0^1 \begin{pmatrix} t+0 \\ 0^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt + \int_0^2 \begin{pmatrix} 1+t \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt \quad (41)$$

$$+ \int_0^1 \begin{pmatrix} 1-t+2-2t \\ 4-8t+4t^2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \frac{\sqrt{5}}{\sqrt{5}} dt \quad (42)$$

$$= \int_0^2 (1+t) dt + \int_0^1 (4t^2 - 2t - 2) dt = \dots \quad (43)$$

Therefore, by comparing (27) and (39) we have verified the Gauss's theorem

$$\therefore \boxed{\int_{\partial B} \mathbf{f} \cdot \boldsymbol{\nu} ds = \int_B \operatorname{div} \mathbf{f} dA.} \quad (44)$$