Global Exercise - Gue08

Tuan Vo

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Content covered:

✓ Analysis: Eigenvalues/Eigenfunction of Laplace operators. How to

 * treat the inhomogeneous problem $\partial_t u(x,t) = \partial_{xx} u(x,t)$ with $u(L,t) \neq 0$.
 * treat the $\partial_{tt} u(x,t)$ term in $\partial_{tt} u(x,t) = \partial_{xx} u(x,t) + a$.

✓ Numerics: FDM consistency order.

1 Analysis

Example 1. *Examine the following problem:*

$$\partial_t u = \partial_x^2 u, \quad \textit{for } x \in \Omega = (0, L), \textit{ and } t > 0,$$

where the boundary conditions (BCs) at the left-most and right-most points for all time t with a constant $\alpha \in \mathbb{R}$ are defined, respectively, as follows

$$\begin{aligned} u(0,t) &= 0, \\ u(L,t) &= \alpha t \exp(-\alpha t), \end{aligned}$$

and the initial condition (IC) with a constant $u_0 \in \mathbb{R}$ is given as

$$u(x,0) = u_0 \; x(L-x).$$

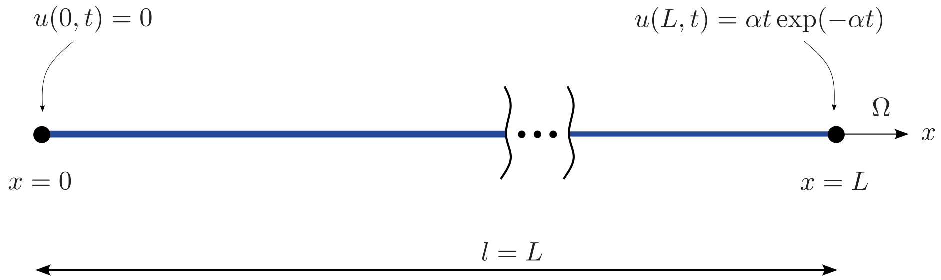

Figure 1: Domain Ω with given BCs.

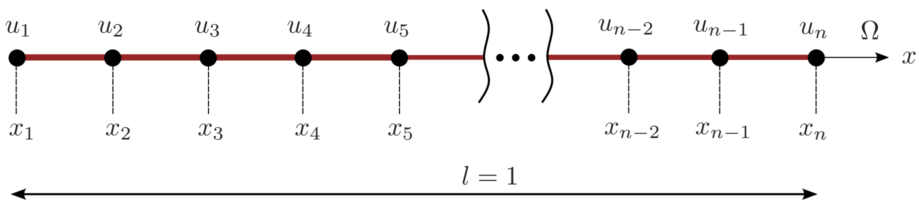

Figure 2: Discretization of domain Ω with n points.

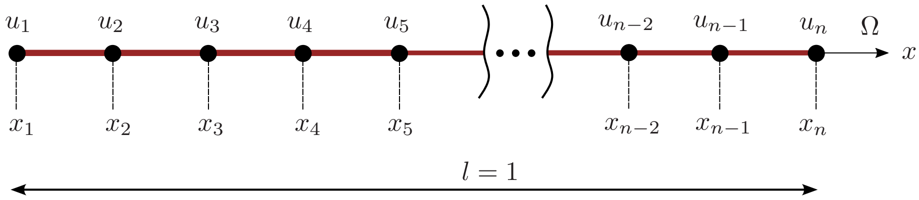

Figure 3: Discretization of domain Ω with n points.

2 Analysis: Spectral theory · Laplace operator application

Example 2. *Examine the following problem*

$$\begin{aligned}
\partial_t u(x,t) &= \partial_{xx} u(x,t) + a, && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= \sin(\pi x), && x \in (0,1),
\end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the source term* $a \in \mathbb{R}$ *in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 2 cont.)

(Example 2 cont.)

Example 3. *Examine the following problem*

$$
\begin{aligned}
\partial_t u(x,t) &= \partial_{xx} u(x,t) + \sin(\pi x), && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= a, && x \in (0,1),
\end{aligned}
$$

with constant IC $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

Example 4. *Examine the following problem*

$$
\begin{aligned}
\partial_{tt}u(x,t) &= \partial_{xx}u(x,t) + a, && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= x, && x \in (0,1),\\
u_t(x,0) &= 0, && x \in (0,1),
\end{aligned}
$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 4 cont.)