Global Exercise - Gue10

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Content covered:

- ✓ Programming exercise 02: Explanation.
- ✓ Analysis:
  - * Distributional derivative.
- ✓ Numerics:

1 Programming exercise 02: Explanation

Example 1. *Examine the following Convection-diffustion problem*

$$\begin{aligned}-\varepsilon\Delta u(x,y)+\cos\beta\,\frac{\partial u}{\partial x}(x,y)+\sin\beta\,\frac{\partial u}{\partial y}&=f(x,y),\ \ (x,y)\in\Omega=(0,1)^2,\\ u(x,y)&=g(x,y),\ \ (x,y)\ \text{on}\ \partial\Omega,\end{aligned}$$

2 Review: Example 3 · Gue08

Approach:

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}\left(h^3\right) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \mathcal{O}\left((2h)^3\right) \end{cases} \tag{1}$$

   which leads to

$$\alpha f(x) = \alpha f(x) \tag{2}$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2}f''(x) + \beta\mathcal{O}\left(h^3\right) \tag{3}$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma\mathcal{O}\left((2h)^3\right) \tag{4}$$

   Summation of (2), (3) and (4) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) = &(\alpha+\beta+\gamma)f(x) \\ &+ (\beta h + 2\gamma h)f'(x) \\ &+ \left(\frac{\beta h^2}{2} + 2\gamma h^2\right)f''(x) \\ &+ \beta\mathcal{O}\left(h^3\right) \\ &+ \gamma\mathcal{O}\left((2h)^3\right) \end{aligned} \tag{5}$$

   Since we would like to approximate $f'(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f'(x) \tag{6}$$

   Comparison (6) with (5) leads to the following 3 conditions

$$\begin{cases} \alpha+\beta+\gamma \overset{!}{=} 0 \\ \beta h + 2\gamma h \overset{!}{=} 1 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \overset{!}{=} 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \tag{7}$$

   which leads the expression (5) to the following equality

$$\begin{aligned} \alpha(x) + \beta f(x+h) + \gamma f(x+2h) &= f'(x) + \beta\mathcal{O}\left(h^3\right) + \gamma\mathcal{O}\left((2h)^3\right) \\ &= f'(x) + \frac{2}{h}\mathcal{O}\left(h^3\right) + \frac{-1}{2h}\mathcal{O}\left((2h)^3\right) \\ &= f'(x) + \mathcal{O}\left(h^2\right). \end{aligned} \tag{8}$$

   Therefore, $f'(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (7), with a consistency order of at least order 2, as shown in (8).

2. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach from $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns for 4 equations. Therefore, the highest consistency order we may obtain is of order 2.