Global Exercise - Gue08

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Content covered:

✓ Analysis: Eigenvalues/Eigenfunction of Laplace operators. How to

  * treat the inhomogeneous problem $\partial_t u(x,t) = \partial_{xx} u(x,t)$ with $u(L,t) \neq 0$.
  * treat the $\partial_{tt} u(x,t)$ term in $\partial_{tt} u(x,t) = \partial_{xx} u(x,t) + a$.

✓ Numerics: FDM consistency order.

1 Analysis

Example 1. *Examine the following problem:*

$$\partial_t u(x,t) = \partial_{xx} u(x,t), \quad \textit{for } x \in \Omega = (0,L), \textit{ and } t > 0, \tag{1}$$

where the boundary conditions (BCs) at the left-most and right-most points for all time t with a constant $\alpha \in \mathbb{R}$ are defined, respectively, as follows

$$\begin{cases} u(0,t) &= 0, \\ u(L,t) &= \alpha t \exp(-\alpha t), \end{cases} \tag{2}$$

and the initial condition (IC) with a constant $u_0 \in \mathbb{R}$ is given as

$$u(x,0) = u_0 \; x(L-x). \tag{3}$$

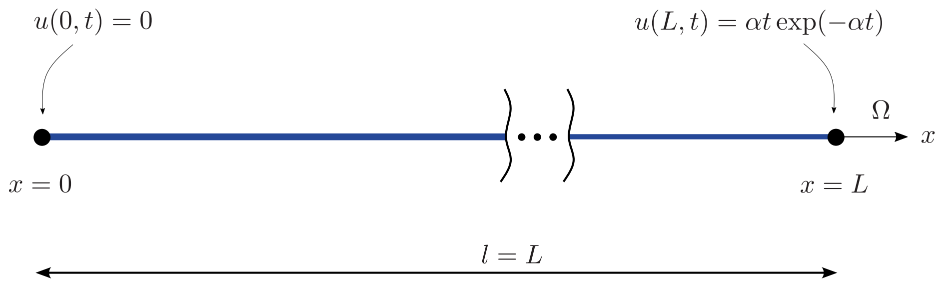

Figure 1: Domain Ω with given BCs.

Approach:

1. Homogenization of the BCs: We search for $u_{inh}(x,t)$ such that

$$\partial_{xx} u_{inh} = 0, \quad \text{with BCs: } \begin{cases} u_{inh}(0,t) = 0, \\ u_{inh}(L,t) = \alpha t \exp(-\alpha t), \end{cases} \tag{4}$$

which leads to

$$\boxed{u_{inh}(x,t) = \frac{x}{L}\alpha t \exp(-\alpha t).} \tag{5}$$

Likewise, we also obtain the following expression

$$\partial_t u_{inh} = \frac{x}{L}\alpha \exp(-\alpha t) - \frac{x}{L}\alpha^2 t \exp(-\alpha t) = \frac{x}{L}(1-\alpha t)\alpha \exp(-\alpha t) \tag{6}$$

Note in passing that we have used $u_{inh}(x,t) = ax + b, \forall a, b \in \mathbb{R}$ to probe the solution mentioned at (4), since it satisfies $\partial_{xx}(ax+b) = 0$. The solution (5) is obtained by considering BCs given in (4).

2. We define

$$\tilde{u}(x,t) := u(x,t) - u_{inh}(x,t) \Leftrightarrow u(x,t) = \tilde{u}(x,t) + u_{inh}(x,t) \tag{7}$$

By substituting (7) into (1) we obtain the following equality

$$\begin{aligned}
\partial_t(\tilde{u} + u_{inh}) = \partial_{xx}(\tilde{u} + u_{inh}) &\Leftrightarrow \partial_t \tilde{u} - \partial_{xx}\tilde{u} = \cancelto{0;\,\text{cf.}(4)}{\partial_{xx} u_{inh}} - \partial_t u_{inh} \\
&\Leftrightarrow \partial_t \tilde{u} - \partial_{xx}\tilde{u} = -\partial_t u_{inh} \\
&\Leftrightarrow \partial_t \tilde{u} - \partial_{xx}\tilde{u} \overset{(6)}{=} -\frac{x}{L}(1-\alpha t)\alpha \exp(-\alpha t)
\end{aligned} \tag{8}$$

Then, the homogenized problem reads

$$\boxed{\partial_t \tilde{u}(x,t) - \partial_{xx}\tilde{u}(x,t) = \tilde{f}(x,t), \quad \text{with } \begin{cases} \tilde{u}(0,t) = \tilde{u}(L,t) = 0, \\ \tilde{u}(x,0) = u_0\ x(L-x), \end{cases}} \tag{9}$$

where the source term $\tilde{f}(x,t)$ defined as $\tilde{f}(x,t) := -\frac{x}{L}(1-\alpha t)\alpha \exp(-\alpha)$.

3. We now need to find out the Eigenfunctions $\varphi(x)$ satisfying

$$-\partial_{xx}\varphi(x) = \lambda\varphi(x), \quad \text{with } \varphi(0) = \varphi(L) = 0. \tag{10}$$

By taking into consideration the Ansatz

$$\begin{cases} \varphi(x) = \exp(rx) \\ \varphi''(x) = r^2 \exp(rx) \end{cases} \tag{11}$$

We then obtain the following expressions

$$(10) \overset{(11)}{\Leftrightarrow} \exp(rx)(r^2 + \lambda) = 0 \Leftrightarrow r = \pm i\sqrt{\lambda} \tag{12}$$

The solution to (10) $\forall A, B, C_1, C_2 \in \mathbb{R}$ takes the following form

$$\begin{aligned}
\varphi(x) &= A \exp\left(i\sqrt{\lambda}x\right) + B \exp\left(-i\sqrt{\lambda}x\right) \\
&= A\left(\cos\left(\sqrt{\lambda}x\right) + i\sin\left(\sqrt{\lambda}x\right)\right) + B\left(\cos\left(\sqrt{\lambda}x\right) - i\sin\left(\sqrt{\lambda}x\right)\right) \\
&= (A+B)\cos\left(\sqrt{\lambda}x\right) + (A-B)i\sin\left(\sqrt{\lambda}x\right) \\
&= C_1 \cos\left(\sqrt{\lambda}x\right) + iC_2 \sin\left(\sqrt{\lambda}x\right)
\end{aligned} \tag{13}$$

Note in passing that Euler's formula $\exp(ix) = \cos(x) + i\sin(x)$ has been used in (13). Next, by applying BCs from (10) we obtain the explicit values for C_1 and eigenvalues λ_k for (13), as follows

$$\varphi(0) = 0 \overset{(13)}{\Leftrightarrow} C_1 = 0. \tag{14}$$

$$\varphi(L) = 0 \overset{(13)(14)}{\Leftrightarrow} iC_2 \sin\left(\sqrt{\lambda}L\right) = 0 \Leftrightarrow \lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad \forall\ k \in \mathbb{Z}. \tag{15}$$

Till this point we have obtained so far only **non-normalized** eigenfunctions $\varphi_k(x)$ and their eigenvalues λ_k as follows

$$\begin{cases} \varphi_k(x) = iC_2 \sin\left(\dfrac{k\pi}{L}x\right), \\ \lambda_k = \left(\dfrac{k\pi}{L}\right)^2, \end{cases} \quad \forall\ k \in \mathbb{Z}. \tag{16}$$

Next, the eigenfunction $\varphi_k(x)$ in (16) needs to be normalized. It goes as follows

$$\begin{aligned}
||\varphi_k(x)||_2 \overset{!}{=} 1 \Leftrightarrow & \sqrt{\int_0^L (\varphi_k(x))^2 \ dx} = 1 \\
& \overset{(16)}{\Leftrightarrow} \int_0^L \left(iC_2 \sin\left(\frac{k\pi}{L}x\right)\right)^2 \ dx = 1 \\
& \Leftrightarrow \int_0^L \sin^2\left(\frac{k\pi}{L}x\right) \ dx = \frac{-1}{(C_2)^2} \\
& \Leftrightarrow \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2k\pi}{L}x\right)\right) \ dx = \frac{-1}{(C_2)^2} \\
& \Leftrightarrow \frac{1}{2}x\Bigg|_{x=0}^{x=L} - \frac{1}{2}\frac{L}{2k\pi}\sin\left(\frac{2k\pi}{L}x\right)\Bigg|_{x=0}^{x=L} = \frac{-1}{(C_2)^2} \\
& \Leftrightarrow \frac{1}{2}L = \frac{-1}{(C_2)^2} \Leftrightarrow C_2 = \pm i\sqrt{\frac{2}{L}}
\end{aligned} \tag{17}$$

- $C_2 = i\sqrt{\frac{2}{L}} \overset{(16)}{\Rightarrow} \varphi_k(x) = -\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{h\pi}{L}x\right) \forall\ k, h \in \mathbb{Z}.$
- $C_2 = -i\sqrt{\frac{2}{L}} \overset{(16)}{\Rightarrow} \varphi_k(x) = +\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) \forall\ k \in \mathbb{Z}.$

Therefore, the **normalized** eigenfunctions $\varphi_k(x)$ and their eigenvalues λ_k take the following forms

$$\therefore \boxed{\begin{aligned} \varphi_k(x) &= \sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) \\ \lambda_k &= \left(\frac{k\pi}{L}\right)^2, \forall\ k \in \mathbb{Z} \end{aligned}}$$

4. Expand the source term $\tilde{f}(x,t)$ by using the normalized eigenfunction $\varphi_k(x)$. The source term $\tilde{f}(x,t)$ takes the following form

$$\tilde{f}(x,t) = -\frac{x}{L}(1-\alpha t)\alpha \exp(-\alpha t), \tag{18}$$

which can be written in an expansion of eigenfuctions

$$\boxed{\tilde{f}(x,t) = \sum_{k=1}^{\infty} \beta_k(t)\varphi_k(x) = \sum_{k=1}^{\infty} \beta_k(t)\sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right)} \tag{19}$$

where coefficient $\beta_k(t)$ is computed by projecting $\tilde{f}(x,t)$ on eigenfunctions $\varphi_k(x)$, as follows

$$\begin{aligned}
\beta_k(t) &= \langle \tilde{f}(x,t), \varphi_k(x)\rangle \\
&= \int_0^L \left(-\frac{x}{L}(1-\alpha t)\alpha \exp(-\alpha t)\right) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right)\, dx \\
&= \frac{-(1-\alpha t)\alpha \exp(-\alpha t)}{L} \sqrt{\frac{2}{L}} \int_0^L x \sin\left(\frac{k\pi}{L}x\right)\, dx \\
&= \frac{-(1-\alpha t)\alpha \exp(-\alpha t)}{L} \sqrt{\frac{2}{L}} \frac{(-1)^k}{k\pi} L^2 \\
&= -(-1)^k \frac{\sqrt{2L}}{k\pi}(1-\alpha t)\alpha \exp(-\alpha t).
\end{aligned}$$

5. Ansatz for solution $\tilde{u}(x,t)$ take the following form

$$\tilde{u}(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x). \tag{20}$$

By substituting (19) and (20) into (9) we obtain

$$\begin{aligned}
&\Leftrightarrow \sum_{k=1}^{\infty} \left(\partial_t \alpha_k(t)\varphi_k(x) - \alpha_k(t)\partial_{xx}\varphi_k(x) - \beta_k(t)\varphi_k(x)\right) = 0 \\
&\overset{(10)}{\Leftrightarrow} \sum_{k=1}^{\infty} \left(\alpha_k'(t)\varphi_k(x) + \alpha_k(t)\lambda_k\varphi_k(x) - \beta_k(t)\varphi_k(x)\right) = 0 \\
&\Leftrightarrow \sum_{k=1}^{\infty} \left(\alpha_k'(t) + \lambda_k\alpha_k(t) - \beta_k(t)\right)\varphi_k(x) = 0.
\end{aligned} \tag{21}$$

Since the eigenfunctions $\varphi_k(x)$ are orthonormal to each other, we arrive at the following equality

$$\boxed{\alpha_k'(t) + \lambda_k\alpha_k(t) - \beta_k(t) = 0.} \tag{22}$$

Solution to $\alpha_k(t)$ takes the form

$$\alpha_k(t) = \alpha_k(0)\exp(-\lambda_k t) + \exp(-\lambda_k t)\int_0^t \exp(\lambda_k \tau)\beta_k(\tau)\, d\tau,$$

whose derivation goes as follows

- Solving for homogeneous part:

$$\alpha_k'(t) + \lambda_k \alpha_k(t) = 0 \Leftrightarrow \alpha_k(t) = \alpha_k(0)\exp(-\lambda_k t) \tag{23}$$

- Inhomogeneous part: using the method *variation of constant*

Das Integral können wir weiter ausrechnen:

$$\begin{aligned}
\int_0^t e^{-\lambda_k(t-\tau)}\beta_k(\tau)\, d\tau &= e^{-\lambda_k t}\int_0^t e^{\lambda_k \tau}(-1)^k \frac{\sqrt{2L}}{k\pi}(1-a\tau)ae^{-a\tau}\, d\tau \\
&= e^{-\lambda_k t}(-1)^k \frac{\sqrt{2L}}{k\pi} a \int_0^t e^{\lambda_k \tau}(1-a\tau)e^{-a\tau}\, d\tau \\
&= e^{-\lambda_k t}(-1)^k \frac{\sqrt{2L}}{k\pi} a \left(\int_0^t e^{\lambda_k \tau}e^{-a\tau}\, d\tau - a\int_0^t \tau e^{\lambda_k \tau}e^{-a\tau}\, d\tau\right) \\
&= e^{-\lambda_k t}(-1)^k \frac{\sqrt{2L}}{k\pi} a \frac{(\lambda_k - (\lambda_k - a)at)e^{\lambda_k - at} - \lambda_k}{(\lambda_k - a)^2} \\
&= (-1)^k \frac{\sqrt{2L}}{k\pi} a \frac{(\lambda_k - (\lambda_k - a)at)e^{-at} - \lambda_k e^{-\lambda_k t}}{(\lambda_k - a)^2}
\end{aligned}$$

Es bleibt die Anfangsbedingung für $\alpha_k(0)$ zu bestimmen. Dazu muss die Anfangsbedingung für v entwickelt werden:

$$v(x,0) = \sum_{i=1}^{k} \alpha_k(0)\phi_k(x) \stackrel{!}{=} u_0\ x(L-x)$$

mit

$$\begin{aligned}
\alpha_k(0) &= \int_0^L u_0\ x(L-x)\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)\ dx \\
&= \begin{cases} u_0\sqrt{2L}\dfrac{4L^2}{k^3\pi^3}, & k \text{ ungerade}, \\ 0, & k \text{ gerade}. \end{cases}
\end{aligned}$$

Die Lösung v lautet insgesamt:

$$\begin{aligned}
v(x,t) &= \sum_{k=1}^{\infty} \alpha_k(t)\phi_k(x) \\
&= \sum_{k=1, k \text{ ungerade}}^{\infty} \frac{8u_0 L^2}{k^3\pi^3} e^{-\lambda_k t}\sin\left(\frac{k\pi}{L}x\right) \\
&+ \sum_{k=1}^{\infty} (-1)^k \frac{2a}{k\pi}\frac{(\lambda_k - (\lambda_k - a)at)e^{-at} - \lambda_k e^{-\lambda_k t}}{(\lambda_k - a)^2}\sin\left(\frac{k\pi}{L}x\right).
\end{aligned}$$

und die Lösung für u folgt aus:

$$\therefore \quad \boxed{u(x,t) = v(x,t) + \frac{x}{L}a\ t\ e^{-at}.}$$

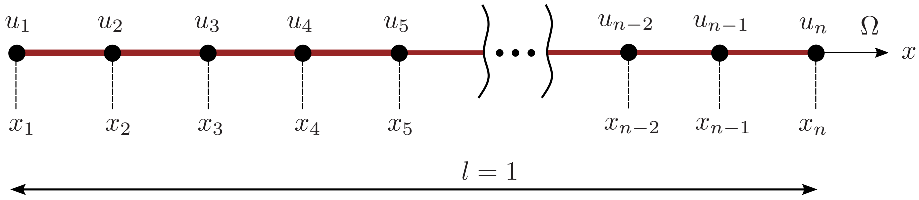

Figure 2: Discretization of domain Ω with n points.

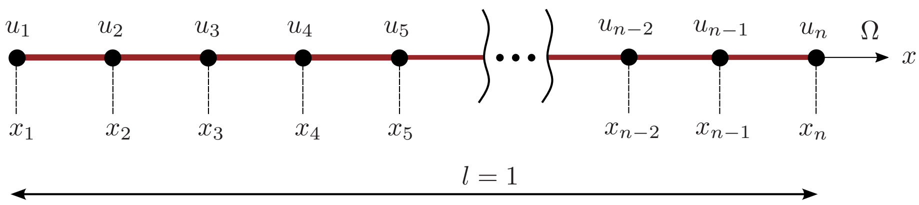

Figure 3: Discretization of domain Ω with n points.

2 Analysis: Spectral theory · Laplace operator application

Example 2. *Examine the following problem*

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx} u(x,t) + a, && x \in (0,1), t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(1,t) &= 0, && t > 0, \\ u(x,0) &= \sin(\pi x), && x \in (0,1), \end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the source term* $a \in \mathbb{R}$ *in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 2 cont.)

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Example 3. *Examine the following problem*

$$
\begin{aligned}
\partial_t u(x,t) &= \partial_{xx} u(x,t) + \sin(\pi x), && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= a, && x \in (0,1),
\end{aligned}
$$

with constant IC $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

Example 4. *Examine the following problem*

$$\begin{aligned}
\partial_{tt}u(x,t) &= \partial_{xx}u(x,t) + a, && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= x, && x \in (0,1),\\
u_t(x,0) &= 0, && x \in (0,1),
\end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from ∂_{xx}.*
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 4 cont.)