Global Exercise - Gue10

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Content to be covered:

- ✓ Programming exercise 02: Explanation.
- ✓ [Q&A] Insight review: Example 3 · Gue08
- ✓ Euler scheme $\begin{cases} \text{Explicit} \\ \text{Implicit} \end{cases} : \sum \longrightarrow$ Crank-Nicolson scheme
- ✓ Iterative solution method
- ✓ [Review HW10] Consistency error

1 Programming exercise 02: Explanation

Example 1. *Examine the following Convection-diffustion problem*

$$-\varepsilon\Delta u(x,y) + \cos\beta\,\frac{\partial u}{\partial x}(x,y) + \sin\beta\,\frac{\partial u}{\partial y} = f(x,y),\ \ (x,y)\in\Omega=(0,1)^2,$$
$$u(x,y) = g(x,y),\ \ (x,y)\ \text{on}\ \partial\Omega,$$

Main points of discretization:

$$-\varepsilon\,\partial_{xx}u(x,y) - \varepsilon\,\partial_{yy}u(x,y) + \cos\beta\,\partial_x u(x,y) + \sin\beta\,\partial_y u(x,y) = 1$$

Since $\beta = 5\pi/6$, it goes with $\cos\beta < 0$ and $\sin\beta > 0$. Therefore, we need Upwind for $\partial_x u(x,y)$ and Downwind for $\partial_y u(x,y)$, as follows

1. Central difference method applied for $\partial_{xx}u(x,y)$,
2. Central difference method applied for $\partial_{yy}u(x,y)$,
3. Upwind method applied for $\partial_x u(x,y)$,
4. Downwind method applied for $\partial_y u(x,y)$.

$$-\varepsilon\,\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} - \varepsilon\,\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2} + \cos\beta\,\frac{u_{i+1,j}-u_{i,j}}{h} + \sin\beta\,\frac{u_{i,j}-u_{i,j-1}}{h} = 1$$

...[further steps is followed by Problem 2.41 in the lecture note.]

2 [Q&A] Insight review: Example 3 · Gue08

Example 2. *Examine the consistency order of the following problem*

$$f'(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach:

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \mathcal{O}\left(h^3\right) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!} f'(x) + \frac{(2h)^2}{2!} f''(x) + \mathcal{O}\left((2h)^3\right) \end{cases} \tag{1}$$

which leads to

$$\alpha f(x) = \alpha f(x) \tag{2}$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}\left(h^3\right) \tag{3}$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}\left((2h)^3\right) \tag{4}$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) = &(\alpha + \beta + \gamma) f(x) \\ &+ (\beta h + 2\gamma h) f'(x) \\ &+ \left(\frac{\beta h^2}{2} + 2\gamma h^2\right) f''(x) \\ &+ \beta \mathcal{O}\left(h^3\right) \\ &+ \gamma \mathcal{O}\left((2h)^3\right) \end{aligned} \tag{5}$$

Since we would like to approximate $f'(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f'(x) \tag{6}$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \overset{!}{=} 0 \\ \beta h + 2\gamma h \overset{!}{=} 1 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \overset{!}{=} 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \tag{7}$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \tag{8}$$

which leads the expression (14) to the following equality

$$\begin{aligned}\alpha(x) + \beta f(x+h) + \gamma f(x+2h) &= f'(x) + \beta\mathcal{O}\left(h^3\right) + \gamma\mathcal{O}\left((2h)^3\right) \\ &= f'(x) + \frac{2}{h}\mathcal{O}\left(h^3\right) + \frac{-1}{2h}\mathcal{O}\left((2h)^3\right) \\ &= f'(x) + \mathcal{O}\left(h^2\right). \end{aligned} \tag{9}$$

Therefore, $f'(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (16), with a consistency order of at least order 2, as shown in (18).

2. Let's now expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach for $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns $\{\alpha, \beta, \gamma\}$ for 4 equations.

3. Therefore, the highest consistency order we may obtain is of order 2.

Example 3. *Examine the consistency order of the following problem*

$$f''(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach: The procedure goes as same as Example 2

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \dfrac{h^1}{1!} f'(x) + \dfrac{h^2}{2!} f''(x) + \mathcal{O}\left(h^3\right) \\ f(x+2h) = f(x) + \dfrac{(2h)^1}{1!} f'(x) + \dfrac{(2h)^2}{2!} f''(x) + \mathcal{O}\left((2h)^3\right) \end{cases} \tag{10}$$

which leads to

$$\alpha f(x) = \alpha f(x) \tag{11}$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}\left(h^3\right) \tag{12}$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}\left((2h)^3\right) \tag{13}$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) = & (\alpha + \beta + \gamma) f(x) \\ & + (\beta h + 2\gamma h) f'(x) \\ & + \left(\frac{\beta h^2}{2} + 2\gamma h^2\right) f''(x) \\ & + \beta \mathcal{O}\left(h^3\right) \\ & + \gamma \mathcal{O}\left((2h)^3\right) \end{aligned} \tag{14}$$

Since we would like to approximate $f''(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f''(x) \tag{15}$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \overset{!}{=} 0 \\ \beta h + 2\gamma h \overset{!}{=} 0 \\ \dfrac{\beta h^2}{2} + 2\gamma h^2 \overset{!}{=} 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/h^2 \end{pmatrix} \tag{16}$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} ?/h^2 \\ ??/h^2 \\ ???/h^2 \end{pmatrix} \tag{17}$$

which leads the expression (14) to the following equality

$$\begin{aligned} \alpha(x) + \beta f(x+h) + \gamma f(x+2h) &= f''(x) + \beta \mathcal{O}\left(h^3\right) + \gamma \mathcal{O}\left((2h)^3\right) \\ &= f''(x) + \frac{??}{h^2} \mathcal{O}\left(h^3\right) + \frac{???}{h^2} \mathcal{O}\left((2h)^3\right) \\ &= f''(x) + \mathcal{O}(h). \end{aligned} \tag{18}$$

Therefore, $f''(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (16), with a consistency order of at least order 1, as shown in (18).

2. Let's now expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach for $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns $\{\alpha, \beta, \gamma\}$ for 4 equations.

3. Therefore, the highest consistency order we may obtain is of order 1.

3 Euler scheme · Crank-Nicolson scheme

Example 4. *Examine the following problem for* $\alpha > 0$

$$\partial_t u(x,t) = \alpha\, \partial_{xx} u(x,t) + q(x,t), \quad t > 0,\ x \in [0,1], \tag{19}$$

with IC $u(x,0) = u_0(x)$*, and BC* $u(0,t) = u(1,t) = 0$.

Approach: Vertical line method, i.e. we first discretize space x, we then obtain a system of ODEs w.r.t. time t.

1. Second order discretization in space

$$\partial_{xx} u(x_j,t) \approx \frac{u(x_{j+1},t) - 2u(x_j,t) + u(x_{j-1},t)}{h^2}$$

   Assume $y_j(t) \approx u(x_j,t)$ yields

$$y_j'(t) = \frac{\alpha}{h^2}(y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)) + q(x_j,t).$$

   General form $y' = Ay + b(t)$ with

$$A = -\frac{\alpha}{h_x^2}\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

   and

$$b = \begin{pmatrix} q(x_1,t) \\ \vdots \\ q(x_{n-1},t) \end{pmatrix}.$$

   where the IC reads

$$y_i(0) = u_0(x_i), \quad i = 1,\dots,n-1$$

2. (a) Explicit Euler

$$\therefore \quad \boxed{y^{i+1} = y^i + h_t\left(Ay^i + b^i\right).}$$

   (b) Implicit Euler

$$\therefore \quad \boxed{y^{i+1} = y^i + h_t\left(Ay^{i+1} + b^{i+1}\right).}$$

   (c) Averaging explicit Euler and implicit Euler leads to Crank-Nicolson

$$\therefore \quad \boxed{\begin{aligned} y^{i+1} &= y^i + \frac{h_t}{2}\left(Ay^i + Ay^{i+1} + b^i + b^{i+1}\right) \\ &= y^i + \frac{h_t}{2}A(y^i + y^{i+1}) + \frac{h_t}{2}\left(b^i + b^{i+1}\right). \end{aligned}}$$

4 Iterative solution method

Example 5. *Examine the following problem: The iteration scheme*

$$\begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix} x^{k+1} = \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta b, \quad \delta \in \mathbb{R}.$$

is used to solve the following problem

$$\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} x = b, \quad x, b \ \in \mathbb{R}^2,$$

1. Determine the matrix C from the iteration matrix $I - CA$
2. Which value of $a \in \mathbb{R}$ can help this method to be converged in case of $\delta = 1$?

Approach: Followed by Bemerkung 3.3. from lecture note.

1. The iteration scheme takes the form

$$\boxed{x^{k+1} = (I - CA)x^k + Cb.}$$

which is derived directly from the following iteration setting function

$$\Phi : \begin{cases} \mathbb{R}^n \to \mathbb{R}^n, \\ x \mapsto \Phi(x) := x + C(b - Ax). \end{cases} \tag{20}$$

Moreover, when x^* is the solution of the system $Ax = b$, we should obtain the following expression

$$\begin{aligned} \Phi(x^*) &= x^* + C\underbrace{(b - Ax^*)}_{=0} \\ &= x^*. \end{aligned}$$

Back to the main problem, we now examine the following derivation

$$\begin{aligned} x^{k+1} &= \begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta \begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix}^{-1} b & (21) \\ &= \begin{pmatrix} 1 & 0 \\ \delta a & 1 \end{pmatrix} \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta \begin{pmatrix} 1 & 0 \\ \delta a & 1 \end{pmatrix} b & (22) \\ &= \begin{pmatrix} 1-\delta & \delta a \\ \delta a - \delta^2 a & \delta^2 a^2 + 1 - \delta \end{pmatrix} x^k + \begin{pmatrix} \delta & 0 \\ \delta^2 a & \delta \end{pmatrix} b & (23) \end{aligned}$$

Moreover, we also expect

$$x^{k+1} = (I - CA)x^k + Cb. \tag{24}$$

Comparison of (24) with (21) yields the matrix C as follows

$$\therefore \quad \boxed{C = \begin{pmatrix} \delta & 0 \\ \delta^2 a & \delta \end{pmatrix}} \tag{25}$$

2. For the case $\delta = 1$ we obtain the following iteration matrix $(I - CA)$

$$I - CA = \begin{pmatrix} 0 & a \\ 0 & a^2 \end{pmatrix}, \tag{26}$$

Observation: This matrix is of upper triangle. Hence, its eigenvalues are those on the main diagonal, i.e. we can recognize immediately the eigenvalues

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = a^2. \end{cases} \tag{27}$$

The iteration method is expected to be converged when the spectral radius of the iteration matrix $(I - CA)$ smaller than 1, i.e. we should have the following condition

$$\rho(I - CA) < 1 \Leftrightarrow a^2 < 1$$

which leads to

$$\therefore \quad \boxed{-1 < a < 1} \tag{28}$$

5 [Review HW10] Consistency error

Example 6. *Examine the consistency error of the following problem*

$$u''(x) - u'(x) + u(x) = 2x - 1 - x^2$$

with the exact solution is known, i.e. $u(x) = 1 - x^2$.

Approach: The consistency error reads

$$\left|\left| - \Delta_h u\big|_{\bar{\Omega}_h} - f\big|_{\Omega_h}\right|\right|_\infty = \left|\left|A_h u\big|_{\Omega_h} - f\big|_{\Omega_h}\right|\right|_\infty$$

Observation: The consistency error can be foreseen (and indeed) with the value 0, i.e. the numerical solution resembles the exact/analytical solution, since we have been using the following numerical scheme + the given information about the exact solution:

1. Second order discretization scheme is used to approximate $u''(x)$.
2. Second order discretization scheme is used to approximate $u'(x)$.
3. The exact solution, which is given, is of quadratics.