Global Exercise - Gue07

Tuan Vo

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1 Numerics: FDM · BC · Derivation of A_h & b_h

Example 1. *Discretization of the following convection-diffusion problem:*

$$-u''(x) - 5u'(x) = 3, \quad \textit{for } x \in (0,1) \tag{1}$$

with boundary conditions $u(0) = u(1) = 1$.

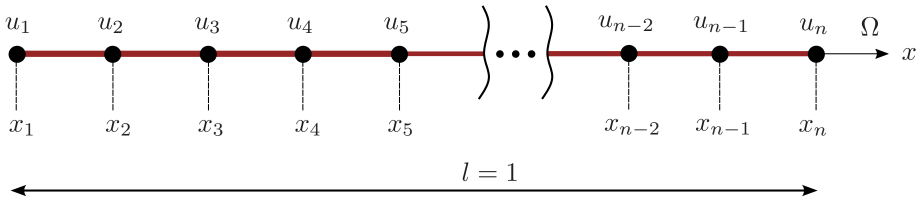

Figure 1: Discretization of domain Ω with n points.

Approach: Central method for $u''(x)$ is obtained as follows

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2). \tag{2}$$

Forward method for $u'(x)$ is obtained as follows

$$u'(x) = \frac{u(x+h) - u(x)}{h} + \mathcal{O}(h). \tag{3}$$

which leads to the system $A_h u_h = b_h$ where A_h and b_h are recognized as follows

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2+5h & -1-5h & & & & \\ -1 & 2+5h & -1-5h & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2+5h & -1-5h \\ & & & & -1 & 2+5h \end{pmatrix} \in \mathbb{R}^{(n-2)\times(n-2)}, \tag{4}$$

$$b_h = \begin{pmatrix} 3 + \dfrac{1}{h^2} \\ 3 \\ \vdots \\ 3 \\ 3 + \dfrac{1}{h^2} + \dfrac{5}{h} \end{pmatrix} \in \mathbb{R}^{(n-2)}. \tag{5}$$

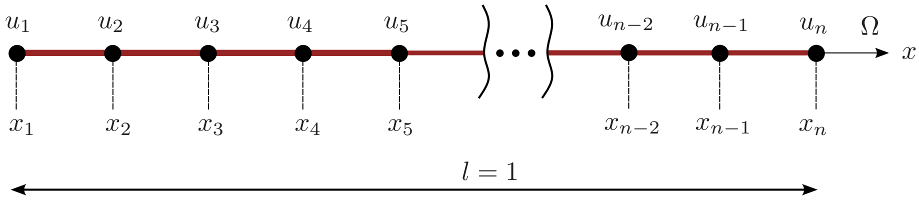

Figure 2: Discretization of domain Ω with n points.

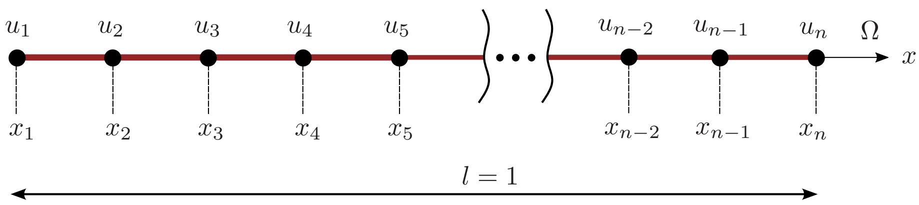

Figure 3: Discretization of domain Ω with n points.

2 Analysis: Spectral theory · Laplace operators

Example 2. *Examine the following problem*

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx} u(x,t) + a, && x \in (0,1), t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(1,t) &= 0, && t > 0, \\ u(x,0) &= \sin(\pi x), && x \in (0,1), \end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from ∂_{xx}..*
2. *Develop the source term $a \in \mathbb{R}$ in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

1. Die zum Problem geh"orenden Eigenfunktionen und Eigenwerte erf"ullen

$$\begin{aligned} \varphi''(x) + \lambda\varphi(x) &= 0, \quad x \in (0,1) \\ \varphi(0) = \varphi(1) &= 0 \end{aligned}$$

$$\Rightarrow \quad \varphi(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)$$

Mit den Randbedingungen f"ur φ erh"alt man

$$\begin{aligned} 0 &= \varphi(0) = C_2 \;\Rightarrow\; C_2 = 0 \\ 0 &= \varphi(1) = C_1 \sin(\sqrt{\lambda}x) \;\Rightarrow\; \lambda_k = (k\pi)^2,\; k \in \mathbb{N} \end{aligned}$$

$$\Rightarrow \varphi_k(x) = C_1 \sin(k\pi x),\; k \in \mathbb{N}$$

C_1 erh"alt man durch Normierung

$$\begin{aligned} 1 &\stackrel{!}{=} \int_0^1 \varphi_k(x)^2\,dx = C_1^2 \int_0^1 \sin^2(k\pi x)\,dx \\ &= C_1^2 \left[\frac{x}{2} - \frac{\sin(2k\pi x)}{4k\pi} \bigg|_0^1 \right] = C_1^2 \frac{1}{2} \\ \Rightarrow C_1 &= \sqrt{2} \end{aligned}$$

damit lauten die Eigenwerte und normierten Eigenfunktionen

$$\begin{aligned} \varphi_k(x) &= \sqrt{2}\sin(k\pi x) \\ \text{und} \quad \lambda_k &= (k\pi)^2 \text{ f"ur } k \in \mathbb{N} \end{aligned}$$

2. Die Entwicklung des konstanten Quellterms $a(x,t) = a$ in Eigenfunktionen lautet

$$\begin{aligned} a &= \sum_{k=1}^{\infty} \beta_k \varphi_k(x) \\ &= \sum_{k=1}^{\infty} \beta_k \sqrt{2} \sin(k\pi x) \end{aligned}$$

Die Koeffizienten berechnet man durch

$$\begin{aligned} \beta_k &= \int_0^1 a\sqrt{2}\sin(k\pi x)dx \\ &= a\sqrt{2}\left[-\frac{\cos(k\pi x)}{k\pi}\Big|_0^1 \right] \\ &= \frac{a\sqrt{2}}{k\pi}(1-\cos(k\pi)) = \begin{cases} \frac{2a\sqrt{2}}{k\pi} & ,k \text{ ungerade} \\ 0 & ,\text{sonst} \end{cases} \end{aligned}$$

3. F"ur die L"osung u wird der Eigenfunktionen-Ansatz

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x)$$

gew"ahlt, mit zeitabh"angigen Koeffizienten $\alpha_k(t)$, $t > 0, k \in \mathbb{N}$.
Einsetzen der Entwicklungen von $u(x,t)$ und a in die Differentialgleichung liefert

$$\begin{aligned} & (\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x))_t = (\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x))_{xx} + \sum_{k=1}^{\infty} \beta_k\varphi_k(x) \\ \Leftrightarrow \quad & \sum_{k=1}^{\infty} \alpha_k'(t)\varphi_k(x) = -\sum_{k=1}^{\infty} \alpha_k(t)\lambda_k\varphi_k(x) + \sum_{k=1}^{\infty} \beta_k\varphi_k(x) \\ \Leftrightarrow \quad & \sum_{k=1}^{\infty} (\alpha_k'(t) + \lambda_k\alpha_k(t) - \beta_k)\varphi_k(x) = 0 \end{aligned}$$

Wir erhalten eine gew"ohnliche Differentialgleichung f"ur die Koeffizienten $\alpha_k(t)$

$$\alpha_k'(t) = -\lambda_k\alpha_k(t) + \beta_k$$

mit der allgemeinen L"osung

$$\alpha_k(t) = \frac{\beta_k}{\lambda_k} + C_k e^{-\lambda_k t}.$$

Die Konstante C_k wird durch die Anfangsbedingung f"ur $\alpha_k(0)$ bestimmt. Dazu betrachten wir die Anfangsbedingung f"ur u:

$$\begin{aligned} u(x,0) = \sum_{k=1}^{\infty} \alpha_k(0)\sqrt{2}\sin(k\pi x) \overset{!}{=} \sin(\pi x) \\ \Rightarrow \alpha_1(0) \overset{!}{=} \frac{1}{\sqrt{2}} \text{ und } \alpha_k(0) = 0 \;\forall\; k > 1 \end{aligned}$$

Also gilt f"ur $k = 1$

$$\alpha_1(0) = \frac{\beta_1}{\lambda_1} + C_1 \stackrel{!}{=} \frac{1}{\sqrt{2}}$$
$$\Rightarrow C_1 = \frac{1}{\sqrt{2}} - \frac{\beta_1}{\lambda_1} = \frac{1}{\sqrt{2}} - \frac{2a\sqrt{2}}{\pi^3} = \frac{\pi^3 - 4a}{\sqrt{2}\pi^3}$$

und f"ur $k > 1$

$$\alpha_k(0) = \frac{\beta_k}{\lambda_k} + C_k \stackrel{!}{=} 0$$
$$\Rightarrow C_k = -\frac{\beta_k}{\lambda_k} = \begin{cases} -\frac{2a\sqrt{2}}{k^3\pi^3} & , k \text{ ungerade} \\ 0 & , \text{sonst} \end{cases}$$

Die L"osung u lautet daher insgesamt

$$\begin{aligned} u(x,t) &= \alpha_1(t)\sqrt{2}\sin(\pi x) + \sum_{k=2}^{\infty}(\frac{\beta_k}{\lambda_k} + C_k e^{-\lambda_k t})\sqrt{2}\sin(k\pi x) \\ &= \left(\frac{4a}{\pi^3} + (1 - \frac{4a}{\pi^3})e^{-\pi^2 t}\right)\sin(\pi x) + \sum_{\substack{k=2 \\ k \text{ ungerade}}}^{\infty} \left(\frac{4a}{k^3\pi^3} - \frac{4a}{k^3\pi^3}e^{-k^2\pi^2 t}\right)\sin(k\pi x) \end{aligned}$$

Example 3. *Examine the following problem*

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx} u(x,t) + \sin(\pi x), && x \in (0,1), t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(1,t) &= 0, && t > 0, \\ u(x,0) &= a, && x \in (0,1), \end{aligned}$$

with constant IC $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}..
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

1. Die zum Problem geh"orenden Eigenfunktionen und Eigenwerte erf"ullen

$$\begin{aligned} \varphi''(x) + \lambda \varphi(x) = 0, \quad x \in (0,1) \\ \varphi(0) = \varphi(1) = 0 \end{aligned}$$

$$\Rightarrow \quad \varphi(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$$

Mit den Randbedingungen f"ur φ erh"alt man

$$\begin{aligned} 0 = \varphi(0) = C_2 \;\Rightarrow\; C_2 = 0 \\ 0 = \varphi(1) = C_1 \sin(\sqrt{\lambda}) \;\Rightarrow\; \lambda_k = (k\pi)^2,\ k \in \mathbb{N} \end{aligned}$$

$$\Rightarrow \varphi_k(x) = C_1 \sin(k\pi x),\ k \in \mathbb{N}$$

C_1 erh"alt man durch Normierung

$$\begin{aligned} 1 \overset{!}{=} \int_0^1 \varphi_k(x)^2 \, dx &= C_1^2 \int_0^1 \sin^2(k\pi x) \, dx \\ &= C_1^2 \left[\frac{x}{2} - \frac{\sin(2k\pi x)}{4k\pi} \Big|_0^1 \right] = C_1^2 \frac{1}{2} \\ \Rightarrow C_1 = \sqrt{2} \end{aligned}$$

damit lauten die Eigenwerte und normierten Eigenfunktionen

$$\begin{aligned} \varphi_k(x) &= \sqrt{2} \sin(k\pi x) \\ \text{und} \quad \lambda_k &= (k\pi)^2 \text{ f"ur } k \in \mathbb{N} \end{aligned}$$

2. Die Entwicklung der konstanten Anfangsbedingung $a(x,t) = a$ in Eigenfunktionen lautet

$$\begin{aligned} a &= \sum_{k=1}^{\infty} \beta_k \varphi_k(x) \\ &= \sum_{k=1}^{\infty} \beta_k \sqrt{2} \sin(k\pi x) \end{aligned}$$

Die Koeffizienten berechnet man durch

$$\begin{aligned}
\beta_k &= \int_0^1 a\sqrt{2}\sin(k\pi x)dx \\
&= a\sqrt{2}\left[-\frac{\cos(k\pi x)}{k\pi}\bigg|_0^1\right] \\
&= \frac{a\sqrt{2}}{k\pi}(1-\cos(k\pi)) = \begin{cases} \frac{2a\sqrt{2}}{k\pi} & ,k \text{ ungerade} \\ 0 & ,\text{sonst.} \end{cases}
\end{aligned}$$

Der Quellterm hat offenbar die Entwicklung

$$\sin(\pi x) = \sum_{k=1}^{\infty} \gamma_k \sqrt{2}\sin(k\pi x)$$

$$\text{mit} \quad \gamma_k = \begin{cases} \frac{1}{\sqrt{2}} & ,k=1 \\ 0 & ,\text{sonst.} \end{cases}$$

3. F”ur die L”osung u wird der Eigenfunktionen-Ansatz

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x)$$

gew”ahlt, mit zeitabh”angigen Koeffizienten $\alpha_k(t)$, $t > 0, k \in \mathbb{N}$.
Einsetzen der Entwicklungen von $u(x,t)$ und $\sin(\pi x)$ in die Differentialgleichung liefert

$$\begin{aligned}
&(\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x))_t = (\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x))_{xx} + \sum_{k=1}^{\infty} \gamma_k\varphi_k(x) \\
\Leftrightarrow \quad &\sum_{k=1}^{\infty} \alpha_k'(t)\varphi_k(x) = -\sum_{k=1}^{\infty} \alpha_k(t)\lambda_k\varphi_k(x) + \sum_{k=1}^{\infty} \gamma_k\varphi_k(x) \\
\Leftrightarrow \quad &\sum_{k=1}^{\infty} (\alpha_k'(t) + \lambda_k\alpha_k(t) - \gamma_k)\varphi_k(x) = 0
\end{aligned}$$

Wir erhalten eine gew”ohnliche Differentialgleichung f”ur die Koeffizienten $\alpha_k(t)$

$$\alpha_k'(t) = -\lambda_k\alpha_k(t) + \gamma_k$$

mit der allgemeinen L”osung

$$\alpha_k(t) = \frac{\gamma_k}{\lambda_k} + C_k e^{-\lambda_k t}.$$

Die Konstante C_k wird durch die Anfangsbedingung f”ur $\alpha_k(0)$ bestimmt. Dazu betrachten wir die Anfangsbedingung f”ur u:

$$u(x,0) = \sum_{k=1}^{\infty} \alpha_k(0)\sqrt{2}\sin(k\pi x) \stackrel{!}{=} a = \sum_{k=0}^{\infty} \frac{2\sqrt{2}a}{(2k+1)\pi}\sqrt{2}\sin((2k+1)\pi x)$$

$$\Rightarrow \alpha_{2j}(0) = 0 \quad \text{und} \quad \alpha_{2j+1}(0) = \frac{2\sqrt{2}a}{(2j+1)\pi} \forall\ j$$

Also gilt f"ur $j = 0$

$$\alpha_1(0) = \frac{\gamma_1}{\lambda_1} + C_1 \overset{!}{=} \frac{2\sqrt{2}a}{\pi}$$
$$\Rightarrow C_1 = \frac{2\sqrt{2}a}{\pi} - \frac{\gamma_1}{\lambda_1} = \frac{2\sqrt{2}a}{\pi} - \frac{1}{\pi^2\sqrt{2}}$$

und f"ur $j > 0$

$$\alpha_{2j}(0) = \frac{\gamma_{2j}}{\lambda_{2j}} + C_{2j} = C_{2j} \overset{!}{=} 0$$
$$\Rightarrow C_{2j} = 0$$
$$\alpha_{2j+1}(0) = \frac{\gamma_{2j+1}}{\lambda_{2j+1}} + C_{2j+1} = C_{2j+1} \overset{!}{=} \frac{2\sqrt{2}a}{(2j+1)\pi}$$
$$\Rightarrow C_{2j+1} = \frac{2\sqrt{2}a}{(2j+1)\pi}$$

Die L"osung u lautet daher insgesamt

$$\begin{aligned} u(x,t) &= \left(\frac{\gamma_1}{\lambda_1} + \left(\frac{2\sqrt{2}a}{\pi} - \frac{1}{\pi^2\sqrt{2}}\right) e^{-\lambda_1 t}\right)\sqrt{2}\sin(\pi x) + \sum_{k=2}^{\infty} C_k e^{-\lambda_k t}\sqrt{2}\sin(k\pi x) \\ &= \left(\frac{4a}{\pi} - \frac{1}{\pi^2}\right) e^{-\pi^2 t}\sin(\pi x) + \frac{1}{\pi^2}\sin(\pi x) \\ &+ \sum_{j=1}^{\infty} \frac{4a}{(2j+1)\pi} e^{-(2j+1)^2\pi^2 t}\sin((2j+1)\pi x) \end{aligned}$$

Example 4. *Examine the following problem*

$$\begin{aligned}
\partial_{tt}u(x,t) &= \partial_{xx}u(x,t) + a, && x \in (0,1), t > 0\\
u(0,t) &= 0, && t > 0\\
u(1,t) &= 0, && t > 0\\
u(x,0) &= x, && x \in (0,1)\\
u_t(x,0) &= 0, && x \in (0,1)
\end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

Die Gleichung $y''(x) = -cy(x) + b$ besitzt die allg. Lösung

$$y(x) = \frac{b}{c} + k_1 \sin(\sqrt{c}x) + k_2 \cos(\sqrt{c}x), \quad k_1, k_2 \in \mathbb{R}$$

1. Die zum Problem geh"orenden Eigenfunktionen und Eigenwerte erf"ullen

$$\begin{aligned}
\varphi''(x) + \lambda\varphi(x) &= 0, \quad x \in (0,1)\\
\varphi(0) = \varphi(1) &= 0
\end{aligned}$$

$$\Rightarrow \quad \varphi(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)$$

Mit den Randbedingungen f"ur φ erh"alt man

$$\begin{aligned}
&0 = \varphi(0) = C_2 \Rightarrow C_2 = 0\\
&0 = \varphi(1) = C_1 \sin(\sqrt{\lambda}x) \Rightarrow \lambda_k = (k\pi)^2,\ k \in \mathbb{N}
\end{aligned}$$

$$\Rightarrow \varphi_k(x) = C_1 \sin(k\pi x),\ k \in \mathbb{N}$$

C_1 is obtained by taking into the consideration of normalization

$$\begin{aligned}
1 &\overset{!}{=} \int_0^1 \varphi_k(x)^2\,dx = C_1^2 \int_0^1 \sin^2(k\pi x)\,dx\\
&= C_1^2 \left[\frac{x}{2} - \frac{\sin(2k\pi x)}{4k\pi} \Bigg|_0^1 \right] = C_1^2 \frac{1}{2}
\end{aligned}$$

$$\Rightarrow C_1 = \sqrt{2}$$

damit lauten die Eigenwerte und normierten Eigenfunktionen

$$\begin{aligned}
\varphi_k(x) &= \sqrt{2}\sin(k\pi x)\\
\text{und} \quad \lambda_k &= (k\pi)^2 \text{ f"ur } k \in \mathbb{N}
\end{aligned}$$

2. Die Entwicklung des konstanten Quellterms $a(x,t) = a$ in Eigenfunktionen lautet

$$\begin{aligned} a &= \sum_{k=1}^{\infty} \beta_k \varphi_k(x) \\ &= \sum_{k=1}^{\infty} \beta_k \sqrt{2} \sin(k\pi x) \end{aligned}$$

Die Koeffizienten berechnet man durch

$$\begin{aligned} \beta_k &= \int_0^1 a\sqrt{2}\sin(k\pi x)dx \\ &= a\sqrt{2}\left[-\frac{\cos(k\pi x)}{k\pi}\bigg|_0^1 \right] \\ &= \frac{a\sqrt{2}}{k\pi}(1-\cos(k\pi)) = \begin{cases} \frac{2a\sqrt{2}}{k\pi} & , k \text{ ungerade} \\ 0 & , \text{sonst} \end{cases} \end{aligned}$$

Für die Entwicklung der Anfangsbedingung $u(x,0) = x$ gilt:

$$\begin{aligned} x &= \sum_{k=1}^{\infty} \gamma_k \varphi_k(x) \\ &= \sum_{k=1}^{\infty} \gamma_k \sqrt{2} \sin(k\pi x) \end{aligned}$$

mit

$$\begin{aligned} \gamma_k &= \int_0^1 x\sqrt{2}\sin(k\pi x)dx \\ &= \sqrt{2}\left[-x\cos(k\pi x)\frac{1}{k\pi} \right]_0^1 + \sqrt{2}\int_0^1 \frac{1}{k\pi}\cos(k\pi x)dx \\ &= \frac{\sqrt{2}}{k\pi}\cos(k\pi) + \frac{\sqrt{2}}{k\pi}\left[\frac{1}{k\pi}\sin(k\pi x) \right]_0^1 \\ &= (-1)^k \frac{\sqrt{2}}{k\pi} \end{aligned}$$

3. F"ur die L"osung u wird der Eigenfunktionen-Ansatz

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x)$$

gew"ahlt, mit zeitabh"angigen Koeffizienten $\alpha_k(t)$, $t > 0, k \in \mathbb{N}$.
Einsetzen der Entwicklungen von $u(x,t)$ und a in die Differentialgleichung

liefert

$$
\begin{aligned}
&(\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x))_{tt} = (\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x))_{xx} + \sum_{k=1}^{\infty} \beta_k\varphi_k(x) \\
\Leftrightarrow \quad & \sum_{k=1}^{\infty} \alpha_k''(t)\varphi_k(x) = -\sum_{k=1}^{\infty} \alpha_k(t)\lambda_k\varphi_k(x) + \sum_{k=1}^{\infty} \beta_k\varphi_k(x) \\
\Leftrightarrow \quad & \sum_{k=1}^{\infty} (\alpha_k''(t) + \lambda_k\alpha_k(t) - \beta_k)\varphi_k(x) = 0
\end{aligned}
$$

Wir erhalten eine gew"ohnliche Differentialgleichung f"ur die Koeffizienten $\alpha_k(t)$

$$\alpha_k''(t) = -\lambda_k\alpha_k(t) + \beta_k$$

mit der allgemeinen L"osung (Hinweis)

$$\alpha_k(t) = \frac{\beta_k}{\lambda_k} + C_{k,1}\sin(\sqrt{\lambda_k}t) + C_{k,2}\cos(\sqrt{\lambda_k}t).$$

Die Konstanten werden durch die Anfangsbedingungen bestimmt. Dazu gilt zunächst wegen $u_t(x,0) = 0$:

$$
\begin{aligned}
u_t(x,0) = \sum_{k=1}^{\infty} \alpha_k'(0)\sqrt{2}\sin(k\pi x) &\overset{!}{=} 0 \\
\Rightarrow \alpha_k'(0) = \sqrt{\lambda_k}C_{k,1}\cos(0) - \sqrt{\lambda_k}C_{k,2}\sin(0) &\overset{!}{=} 0 \\
\Rightarrow C_{k,1} &= 0
\end{aligned}
$$

Die Konstante $C_{k,2}$ wird durch die Anfangsbedingung $u(x,0) = x$ bestimmt. Dazu betrachte

$$
\begin{aligned}
u(x,0) = \sum_{k=1}^{\infty} \alpha_k(0)\sqrt{2}\sin(k\pi x) &\overset{!}{=} \sum_{k=1}^{\infty} \gamma_k\sqrt{2}\sin(k\pi x) \\
\Rightarrow \alpha_k(0) &\overset{!}{=} \gamma_k
\end{aligned}
$$

Also gilt

$$
\begin{aligned}
\alpha_k(0) = \frac{\beta_k}{\lambda_k} + C_{k,2} &\overset{!}{=} \gamma_k = (-1)^k\frac{\sqrt{2}}{k\pi} \\
\Rightarrow C_{k,2} &= \begin{cases} -\frac{2a\sqrt{2}}{(k\pi)^3} - \frac{\sqrt{2}}{k\pi}, k \text{ ungerade} \\ \frac{\sqrt{2}}{k\pi}, \text{ sonst} \end{cases}
\end{aligned}
$$

Die L"osung u lautet daher insgesamt

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty} \left(\frac{\beta_k}{\lambda_k} + C_{k,1}\sin(\sqrt{\lambda_k}t) + C_{k,2}\cos(\sqrt{\lambda_k}t)\right)\sqrt{2}\sin(k\pi x) \\
&= \sum_{\substack{k=1 \\ k \text{ gerade}}}^{\infty} \left[\frac{4a}{(k\pi)^3} + \left(\frac{2}{k\pi}\right)\cos(k\pi t)\right]\sin(k\pi x) \\
&+ \sum_{\substack{k=1 \\ k \text{ ungerade}}}^{\infty} \left[\frac{4a}{(k\pi)^3} - \left(\frac{2a}{(k\pi)^3} + \frac{2}{k\pi}\right)\cos(k\pi t)\right]\sin(k\pi x)
\end{aligned}
$$