Global Exercise - Gue10

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23rd June, 2022

Content covered:

- ✓ Programming exercise 02: Explanation.
- ✓ [Answer to question] Insight review: Example 3 · Gue08
- ✓ Euler scheme $\begin{cases}\text{Explicit}\\ \text{Implicit}\end{cases} : \sum \longrightarrow$ Crank-Nicolson scheme
- ✓ [Remark of Homework 10] Consistency error

1 Programming exercise 02: Explanation

Example 1. *Examine the following Convection-diffustion problem*

$$-\varepsilon\Delta u(x,y) + \cos\beta\,\frac{\partial u}{\partial x}(x,y) + \sin\beta\,\frac{\partial u}{\partial y} = f(x,y),\ \ (x,y)\in\Omega = (0,1)^2,$$
$$u(x,y) = g(x,y),\ \ (x,y)\ \textit{on}\ \partial\Omega,$$

Main points of discretization:

$$-\varepsilon\,\partial_{xx}u(x,y) - \varepsilon\,\partial_{yy}u(x,y) + \cos\beta\,\partial_x u(x,y) + \sin\beta\,\partial_y u(x,y) = 1$$

Since $\beta = 5\pi/6$, it goes with $\cos\beta < 0$ and $\sin\beta > 0$. Therefore, we need Upwind for $\partial_x u(x,y)$ and Downwind for $\partial_y u(x,y)$, as follows

1. Central difference method applied for $\partial_{xx}u(x,y)$,
2. Central difference method applied for $\partial_{yy}u(x,y)$,
3. Upwind method applied for $\partial_x u(x,y)$,
4. Downwind method applied for $\partial_y u(x,y)$.

$$-\varepsilon\,\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \varepsilon\,\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \cos\beta\,\frac{u_{i+1,j} - u_{i,j}}{h} + \sin\beta\,\frac{u_{i,j} - u_{i,j-1}}{h} = 1$$

$\dots$

2 Insight review: Example 3 · Gue08

Example 2. *Examine the consistency order of the following problem*

$$f'(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach:

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \mathcal{O}\left(h^3\right) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!} f'(x) + \frac{(2h)^2}{2!} f''(x) + \mathcal{O}\left((2h)^3\right) \end{cases} \tag{1}$$

which leads to

$$\alpha f(x) = \alpha f(x) \tag{2}$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}\left(h^3\right) \tag{3}$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}\left((2h)^3\right) \tag{4}$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) = & (\alpha + \beta + \gamma) f(x) \\ & + (\beta h + 2\gamma h) f'(x) \\ & + \left(\frac{\beta h^2}{2} + 2\gamma h^2\right) f''(x) \\ & + \beta \mathcal{O}\left(h^3\right) \\ & + \gamma \mathcal{O}\left((2h)^3\right) \end{aligned} \tag{5}$$

Since we would like to approximate $f'(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f'(x) \tag{6}$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \overset{!}{=} 0 \\ \beta h + 2\gamma h \overset{!}{=} 1 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \overset{!}{=} 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \tag{7}$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \tag{8}$$

which leads the expression (14) to the following equality

$$\begin{aligned}\alpha(x) + \beta f(x+h) + \gamma f(x+2h) &= f'(x) + \beta \mathcal{O}\left(h^3\right) + \gamma \mathcal{O}\left((2h)^3\right) \\ &= f'(x) + \frac{2}{h}\mathcal{O}\left(h^3\right) + \frac{-1}{2h}\mathcal{O}\left((2h)^3\right) \\ &= f'(x) + \mathcal{O}\left(h^2\right). \end{aligned} \tag{9}$$

Therefore, $f'(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (16), with a consistency order of at least order 2, as shown in (18).

2. Let's now expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach for $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns $\{\alpha, \beta, \gamma\}$ for 4 equations.

3. Therefore, the highest consistency order we may obtain is of order 2.

Example 3. *Examine the consistency order of the following problem*

$$f''(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach: The procedure goes as same as Example 2

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}\left(h^3\right) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \mathcal{O}\left((2h)^3\right) \end{cases} \tag{10}$$

which leads to

$$\alpha f(x) = \alpha f(x) \tag{11}$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}\left(h^3\right) \tag{12}$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}\left((2h)^3\right) \tag{13}$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) = {} & (\alpha+\beta+\gamma) f(x) \\ & + (\beta h + 2\gamma h) f'(x) \\ & + \left(\frac{\beta h^2}{2} + 2\gamma h^2\right) f''(x) \\ & + \beta \mathcal{O}\left(h^3\right) \\ & + \gamma \mathcal{O}\left((2h)^3\right) \end{aligned} \tag{14}$$

Since we would like to approximate $f''(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f''(x) \tag{15}$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \overset{!}{=} 0 \\ \beta h + 2\gamma h \overset{!}{=} 0 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \overset{!}{=} 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/h^2 \end{pmatrix} \tag{16}$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} ?/h^2 \\ ??/h^2 \\ ???/h^2 \end{pmatrix} \tag{17}$$

which leads the expression (14) to the following equality

$$\begin{aligned} \alpha(x) + \beta f(x+h) + \gamma f(x+2h) &= f''(x) + \beta \mathcal{O}\left(h^3\right) + \gamma \mathcal{O}\left((2h)^3\right) \\ &= f''(x) + \frac{??}{h^2}\mathcal{O}\left(h^3\right) + \frac{???}{h^2}\mathcal{O}\left((2h)^3\right) \\ &= f''(x) + \mathcal{O}\left(h\right). \end{aligned} \tag{18}$$

Therefore, $f''(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (16), with a consistency order of at least order 1, as shown in (18).

2. Let's now expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach for $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns $\{\alpha, \beta, \gamma\}$ for 4 equations.

3. Therefore, the highest consistency order we may obtain is of order 1.