Global Exercise - Gue09

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Content covered:

- ✓ Programming exercise 02: Introduction + Hints.
- ✓ Analysis:
  - * Expansion of $f(x, y)$ in eigenfunctions of Laplace operator (2D).
  - * Distributional derivative.
- ✓ Numerics: Ghost points in FDM + Neumann BCs.

1 Programming exercise 02: Introduction + Hints

Example 1. *Examine the following Convection-diffustion problem*

$$\begin{aligned} -\varepsilon \Delta u(x, y) + \cos\beta \, \frac{\partial u}{\partial x}(x, y) + \sin\beta \, \frac{\partial u}{\partial y} &= f(x, y), \ (x, y) \in \Omega = (0, 1)^2, \\ u(x, y) &= g(x, y), \ (x, y) \ \textit{on} \ \partial\Omega, \end{aligned}$$

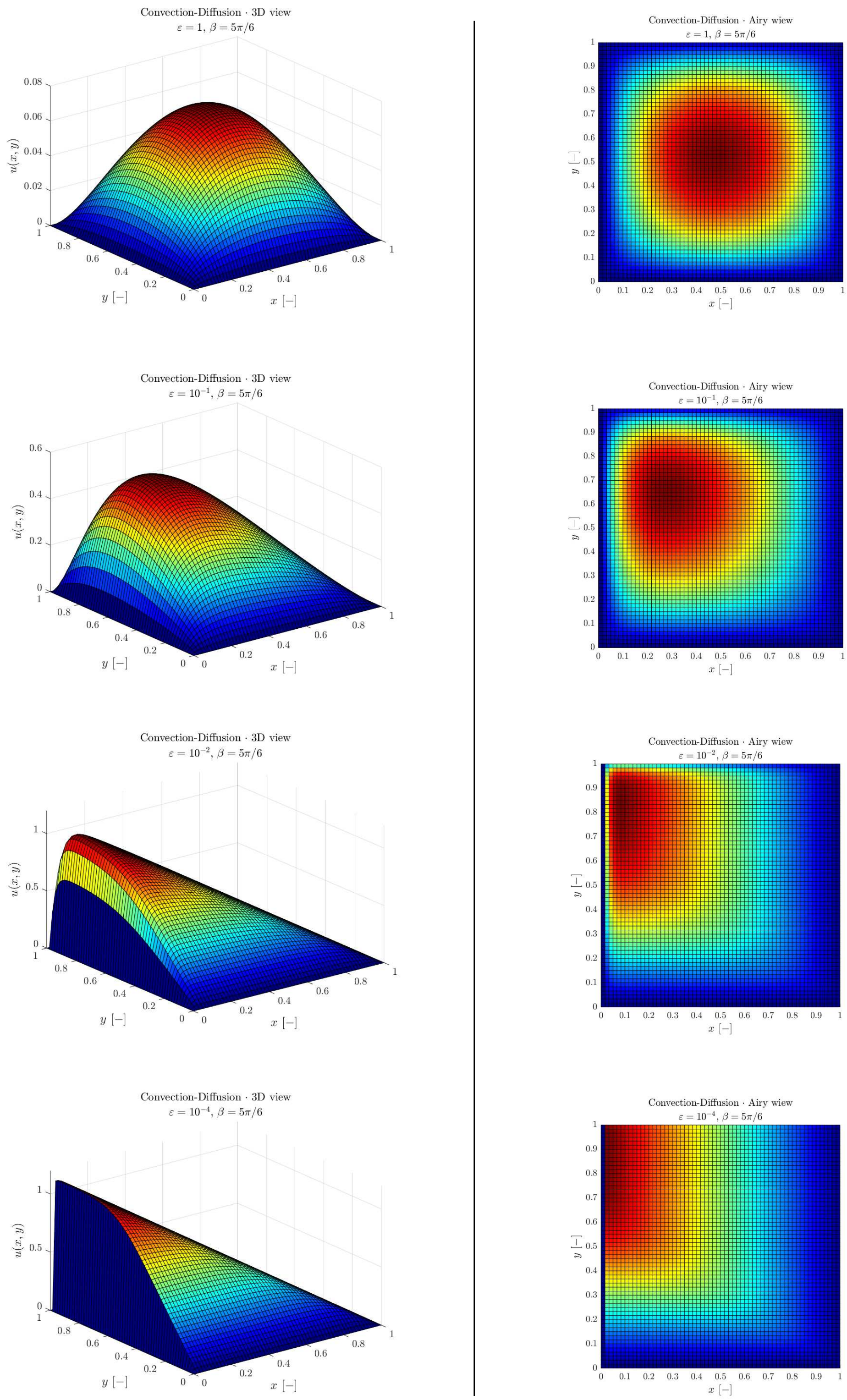


Figure 1: Comparison among different values of $\varepsilon = [1, 10^{-1}, 10^{-2}, 10^{-4}]$: Upwind differences used for discretizing convection terms $\partial_x u$ and $\partial_y u$; fixed $\beta = 5\pi/6$.

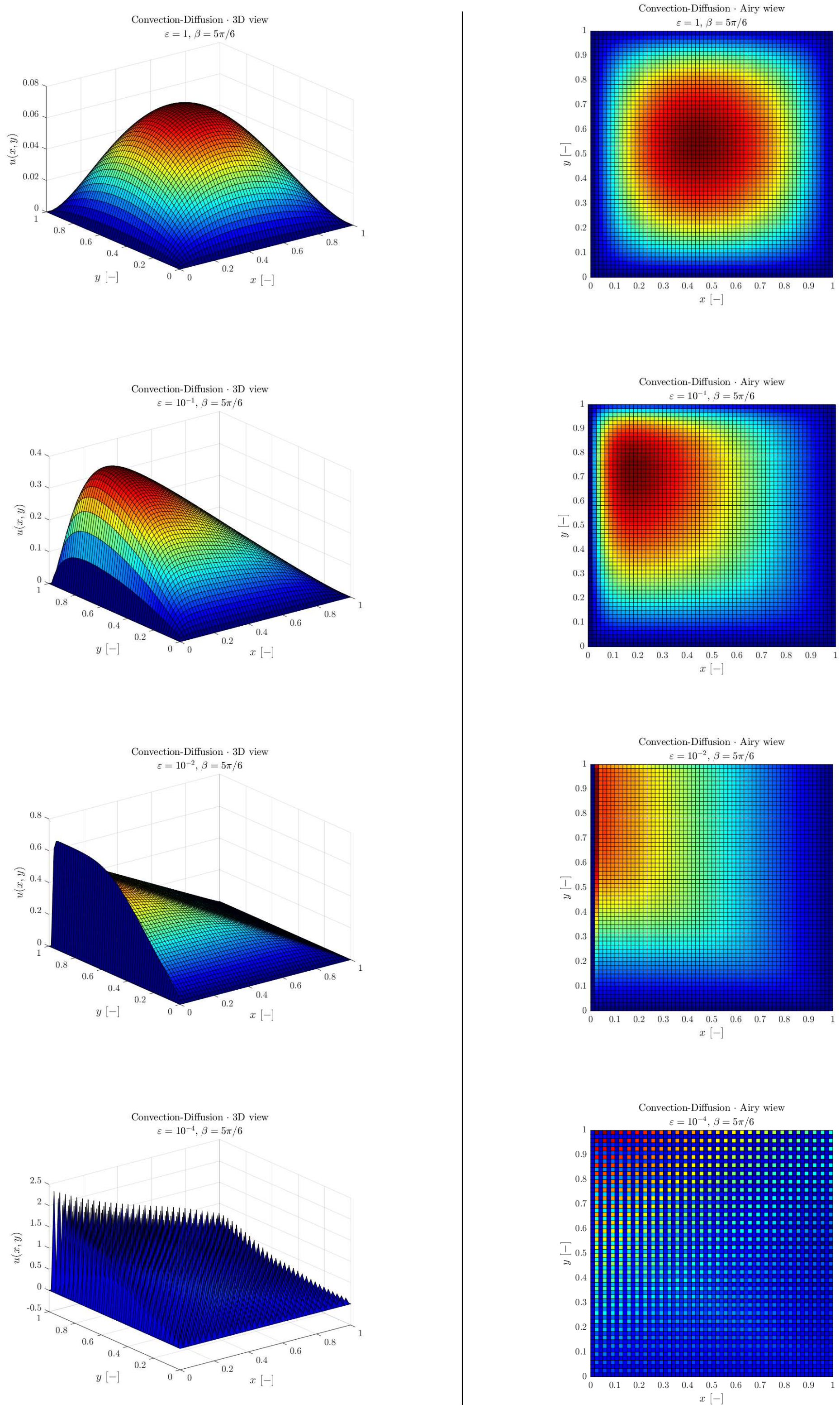
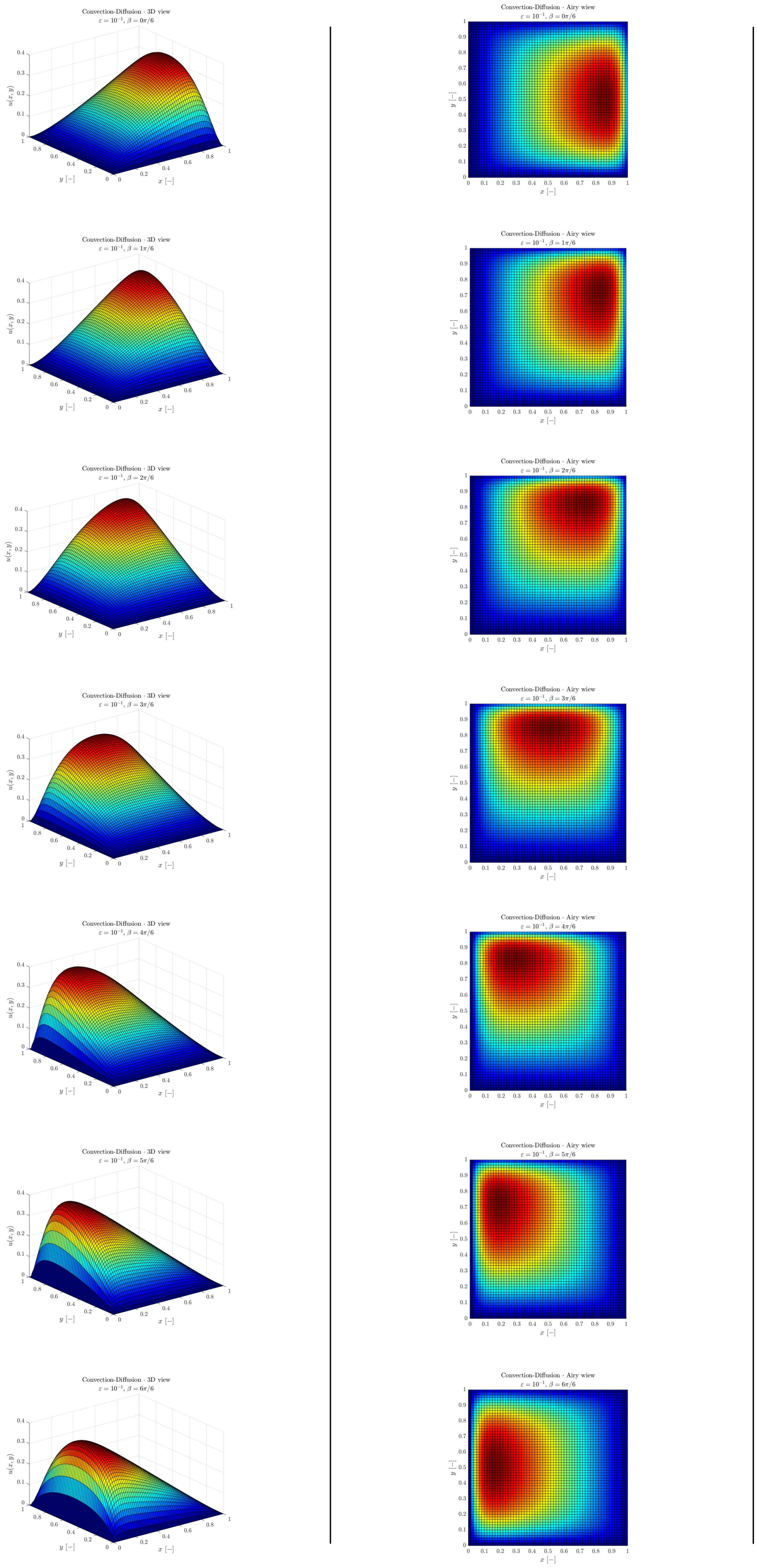


Figure 2: Comparison among different values of $\varepsilon = [1, 10^{-1}, 10^{-2}, 10^{-4}]$: Central difference used for discretizing convection terms $\partial_x u$ and $\partial_y u$; fixed $\beta = 5\pi/6$.



$$\beta = \frac{0\pi}{6} = 0$$

$$\beta = \frac{1\pi}{6}$$

$$\beta = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$\beta = \frac{3\pi}{6} = \frac{\pi}{2}$$

$$\beta = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$\beta = \frac{5\pi}{6}$$

$$\beta = \frac{6\pi}{6} = \pi$$

Figure 3: Comparison among different values of $\beta = \left[0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}, \pi\right]$: Upwind differences used for discretizing convection terms $\partial_x u$ and $\partial_y u$; fixed $\varepsilon = 10^{-1}$.

2 Analysis: 2D Eigenfuctions of Laplace operator

Recall 1.

Example 2. *Examine the following problem where the squared unit domain defined as $\Omega = [0,1]^2$ and the function $f(x,y)$ given as follows*

$$f(x,y) = xy(1-x)(1-y).$$

Expand the given function $f(x,y)$ to eigenfunctions derived from

$$\begin{aligned} -\Delta\phi &= \lambda\phi, \ \textit{in } \Omega, \\ \phi &= 0, \ \textit{on } \partial\Omega. \end{aligned}$$

Approach: The Eigenfunctions take the form

$$\boxed{\phi_{j,k}(x,y) = C_1 \sin(j\pi x)\sin(k\pi y).}$$

Normalization of eigenfunctions goes as follows

$$\boxed{\begin{aligned} ||\phi_{j,k}(x,y)||_2^2 \overset{!}{=} 1 &\Leftrightarrow \int_\Omega \phi_{j,k}^2(x,y)\ d\Omega \overset{!}{=} 1 \\ &\Leftrightarrow \int_0^1 \int_0^1 C_1^2 \sin^2(j\pi x)\sin^2(k\pi y)\ dxdy \overset{!}{=} 1 \Rightarrow C_1 = 2. \end{aligned}}$$

Derivation of eigenfunctions $\phi_{j,k}(x,y)$

Derivation of eigenfunctions $\phi_{j,k}(x, y)$ (cont.)

Expansion

$$f(x,y) = xy(1-x)(1-y) \stackrel{!}{=} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \alpha_{j,k}\phi_{j,k}(x,y)$$

where the coefficient $\alpha_{j,k}$ is computed as follows

$$\begin{aligned}
\alpha_{j,k} &= \int_{\Omega} f(x,y)\phi_{j,k}\ d\Omega \\
&= \int_0^1\int_0^1 xy(1-x)(1-y)\ 2\sin(j\pi x)\sin(k\pi y)\ dxdy \\
&= 2\int_0^1 x(1-x)\sin(j\pi x)\ dx \int_0^1 y(1-y)\sin(k\pi y)\ dy
\end{aligned}$$

which is computed firstly w.r.t. dx step-by-step as follows

$$\int_0^1 x(1-x)\sin(j\pi x)\ dx = \frac{2}{j^3\pi^3}(1-\cos(j\pi)) - \frac{1}{j^2\pi^2}\sin(j\pi)$$

The term goes with sinus function gets vanished, while the term goes with cosinus function takes the form $\cos(j\pi) = (-1)^j$. Similarly, it goes the same for dy. Finally, we obtain

$$\begin{aligned}
\alpha_{j,k} &= 2\frac{2}{j^3\pi^3}\left(1-(-1)^j\right)\frac{2}{k^3\pi^3}\left(1-(-1)^k\right) \\
&= \begin{cases} \dfrac{32}{j^3k^3\pi^6}, & \text{if } j,k \text{ odd,} \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

which leads to the final form of expansion of $f(x,y)$ as follows

$$\therefore \quad \boxed{f(x,y) = \sum_{j,k\ \text{odd}} \frac{32}{j^3k^3\pi^6} 2\sin(j\pi x)\sin(k\pi y).}$$

3 Analysis: Distributional derivative

Example 3. *Examine the following problem*

$$f : \begin{cases} \mathbb{R} \to \mathbb{R}, \\ x \mapsto f(x) = |x+1|. \end{cases}$$

4 Numerics: Ghost points in FDM

Example 4. *Examine the following FDM problem:*

$$u''(x) - u'(x) + u(x) = -x^2 + 2x - 1, \quad \text{for } \ x \in \Omega = (0,1),$$

where the boundary conditions are defined as follows

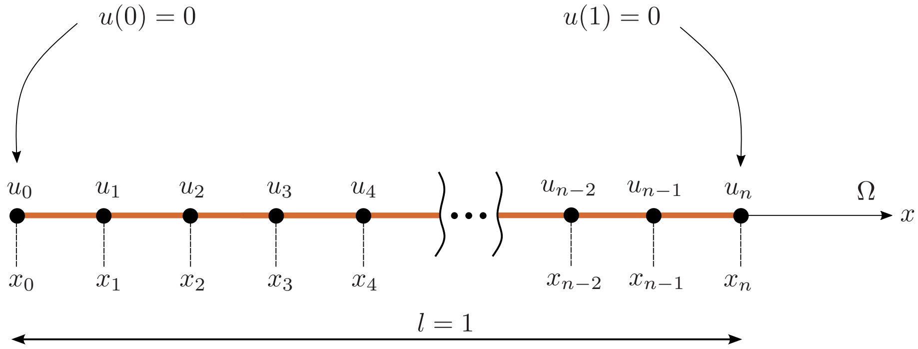


Figure 4: Discretization of domain Ω with $n+1$ points: From x_0 to x_n.

$$u(0) = 0, \quad u(1) = 0.$$

Determine the resulting system of linear equations $A_h u_h = b_h$ by using second order FDM scheme for both $u''(x)$ and $u'(x)$.

Approach: This problem exists no ghost point

The second order FDM scheme for both $u''(x)$ and $u'(x)$ take the following forms

$$\boxed{\begin{aligned} u''(x) &\approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \\ u'(x) &\approx \frac{u(x+h) - u(x-h)}{2h}, \end{aligned}}$$

Example 5. *Examine the following FDM problem:*

$$u''(x) - u'(x) + u(x) = -x^2 + 2x - 1, \quad \textit{for} \;\; x \in \Omega = (0,1),$$

where the boundary conditions are defined as follows

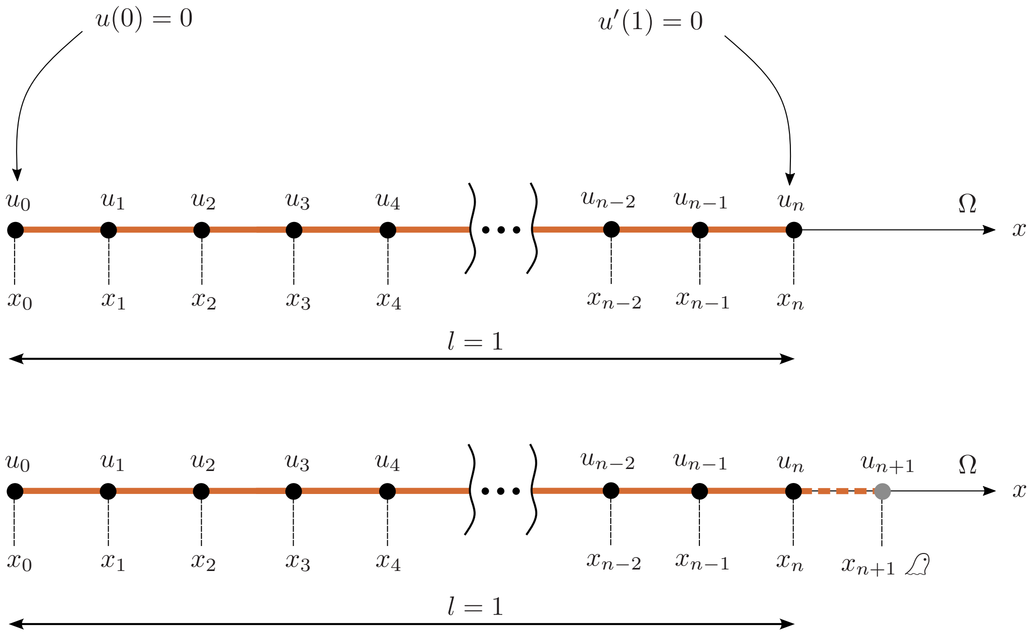


Figure 5: Discretization of domain Ω with $n+1$ points: From x_0 to x_n. Ghost point x_{n+1} is located on the right boundary $\partial\Omega_R$.

$$u(0) = 0, \quad u'(1) = 0.$$

Determine the resulting system of linear equations $A_h u_h = b_h$ by using second order FDM scheme for both $u''(x)$ and $u'(x)$.

Approach: This problem exists a ghost point on the RHS boundary $\partial\Omega_R$.

The second order FDM scheme for both $u''(x)$ and $u'(x)$ take the following forms

$$\boxed{\begin{aligned} u''(x) &\approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \\ u'(x) &\approx \frac{u(x+h) - u(x-h)}{2h}, \end{aligned}}$$

Example 6. *Examine the following FDM problem:*

$$u''(x) - u'(x) + u(x) = -x^2 + 2x - 1, \quad \textit{for} \;\; x \in \Omega = (0,1),$$

where the boundary conditions are defined as follows

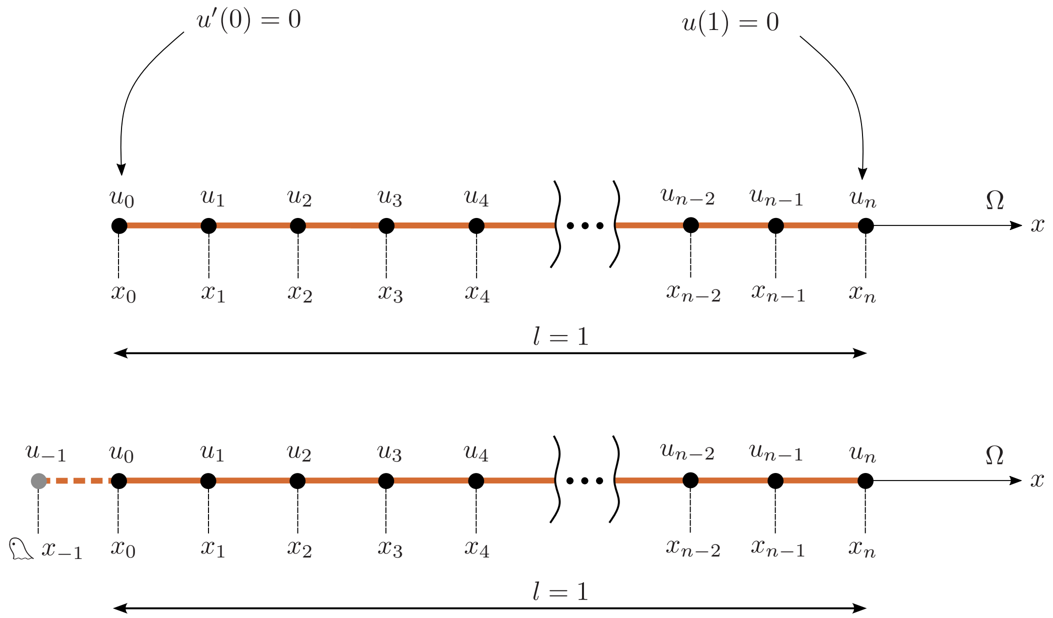


Figure 6: Discretization of domain Ω with $n+1$ points: From x_0 to x_n. Ghost point x_{n+1} is located on the left boundary $\partial\Omega_L$.

$$u'(0) = 0, \quad u(1) = 0.$$

Determine the resulting system of linear equations $A_h u_h = b_h$ by using second order FDM scheme for both $u''(x)$ and $u'(x)$.

Approach: This problem exists a ghost point on the LHS boundary $\partial\Omega_L$

The second order FDM scheme for both $u''(x)$ and $u'(x)$ take the following forms

$$\boxed{\begin{aligned} u''(x) &\approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \\ u'(x) &\approx \frac{u(x+h) - u(x-h)}{2h}, \end{aligned}}$$

1. Discretization for $n = 1, .., N-1$

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - \frac{u_{n+1} - u_{n-1}}{2h} + u_n = 2x_n - 1 - x_n^2,$$

which leads to the following grouped form

$$\boxed{(2+h)u_{n-1} + (2h^2 - 4)u_n + (2-h)u_{n+1} = 2h^2\left(2nh - 1 - (nh)^2\right).}$$

Left boundary exists **ghost** point u_{-1} taking the following expression

$$u'(0) = 0 \quad \Rightarrow \quad \frac{u_1 - u_{-1}}{2h} = 0 \quad \Rightarrow \quad u_1 = u_{-1}$$

which yields the first equation for the case $n = 0$ as follows

$$\boxed{(2h^2 - 4)u_0 + 4u_1 = -2h^2.}$$

The right boundary where $u(1) = 0$, meaning $u_N = 0$, is for the case of $n = N - 1$

$$\boxed{(2 + h)u_{N-2} + (2h^2 - 4)u_{N-1} = 2h^2(2(1 - h) - 1 - (1 - h)^2).}$$

The final system of equations takes the following form

$$\therefore \quad \boxed{A_h = \begin{pmatrix} 2h^2 - 4 & 4 & & & \\ 2 + h & 2h^2 - 4 & 2 - h & & \\ & 2 + h & 2h^2 - 4 & 2 - h & \\ & & \ddots & \ddots & \ddots \\ & & & 2 + h & 2h^2 - 4 \end{pmatrix}_{N \times N} ,}$$

$$\therefore \quad \boxed{u_h = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}_{N \times 1} , \qquad b_h = 2h^2 \begin{pmatrix} -1 \\ 2h - 1 - h^2 \\ 4h - 1 - (2h)^2 \\ \vdots \\ 2(N-1)h - 1 - ((N-1)h)^2 \end{pmatrix}_{N \times 1} .}$$

5 Evaluation

Figure 7: QR Code for evaluation of Global exercises.

Alternative link: here