Global Exercise - Gue11

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Content covered:

✓ [Review HW10] Numerics: Consistency error (cont.)

✓ [Review HW11 - A1] Analysis: Fundamental solution Laplace-operator in 2D.

✓ Numerics: $\begin{cases} \text{Jacobi iteration solution} \\ \text{Gauss-Seidel iteration solution} \end{cases}$

1 [Review HW10] Consistency error (cont.)

Example 1. *Examine the consistency error of the following problem*

$$u''(x) - u'(x) + u(x) = 2x - 1 - x^2$$

with the exact solution is known, i.e. $u(x) = 1 - x^2$.

2 [Review HW10 - A1] Fundamental solution

Example 2. *Examine the following problem: Show that the γ function defined as*

$$\gamma : \begin{cases} \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}, \\ \boldsymbol{x} \mapsto \gamma(\boldsymbol{x}) = -\dfrac{1}{2\pi} \ln |\boldsymbol{x}|, \end{cases} \tag{1}$$

is a fundamental solution of Laplace equation $-\Delta\gamma(\boldsymbol{x}) = \delta_0$ in $\mathbb{R}^2$.

Approach:

Proof. For all test functions φ in the compactly supported, infinitely differentiable function space, i.e. $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we need to show the following equality

$$\begin{aligned} -\Delta\gamma(\boldsymbol{x}) = \delta_0 &\Leftrightarrow \langle -\Delta\gamma(\boldsymbol{x}), \varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle -\partial_{xx}\gamma(\boldsymbol{x}) - \partial_{yy}\gamma(\boldsymbol{x}), \varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow -\langle \partial_{xx}\gamma(\boldsymbol{x}), \varphi\rangle - \langle \partial_{yy}\gamma(\boldsymbol{x}), \varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow -(-1)^2\langle \gamma(\boldsymbol{x}), \partial_{xx}\varphi\rangle - (-1)^2\langle \gamma(\boldsymbol{x}), \partial_{yy}\varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\partial_{xx}\varphi\rangle + \langle \gamma(\boldsymbol{x}), -\partial_{yy}\varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\partial_{xx}\varphi - \partial_{yy}\varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle = \langle \delta_0, \varphi\rangle \end{aligned} \tag{2}$$

yielding the following expression

$$\boxed{\langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle = \langle \delta_0, \varphi\rangle \Leftrightarrow \langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle \stackrel{!}{=} \varphi(0), \quad \forall\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).} \tag{3}$$

Setup: Let the test function $\varphi \in \mathcal{D}(\mathbb{R}^2) = \mathcal{C}_0^\infty(\mathbb{R}^2)$ be arbitrarily chosen with its support in a subset of a ball centered at O with radius R, i.e. $\operatorname{supp}(\varphi) \subset B_R(0)$. This setting leads to the fact

$$\begin{cases} \varphi(\boldsymbol{x}) \neq 0 \text{ in } B_R(0), \\ \varphi(\boldsymbol{x}) = 0 \text{ on } \partial B_R(0). \end{cases}$$

Besides, let another smaller ball centered at O with radius $\varepsilon \ll 1$ be defined $B_\varepsilon(0)$. Then, the domain to be considered is $\Omega_\varepsilon := B_R(0) \setminus B_\varepsilon(0)$, i.e. the *2D doughnut* as shown in Figure 1. By using Gauss divergence theorem twice we obtain the following equality:

$$\begin{aligned} \langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle &= -\int_{\mathbb{R}^2} \gamma(\boldsymbol{x})\Delta\varphi(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \lim_{\varepsilon\to 0^+} \left(-\int_{\Omega_\varepsilon} \gamma(\boldsymbol{x})\Delta\varphi(\boldsymbol{x})\, d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon\to 0^+} \left(-\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\, d\partial\Omega_\varepsilon + \int_{\Omega_\varepsilon} \nabla\gamma(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})\, d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon\to 0^+} \Bigg(-\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\, d\partial\Omega_\varepsilon \\ &\qquad + \left(\int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\, d\partial\Omega_\varepsilon - \int_{\Omega_\varepsilon} \cancelto{0}{\Delta\gamma(\boldsymbol{x})}\varphi(\boldsymbol{x})\, d\Omega_\varepsilon \right) \Bigg) \\ &= \lim_{\varepsilon\to 0^+} \left(-\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\, d\partial\Omega_\varepsilon + \int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\, d\partial\Omega_\varepsilon \right) \end{aligned} \tag{4}$$

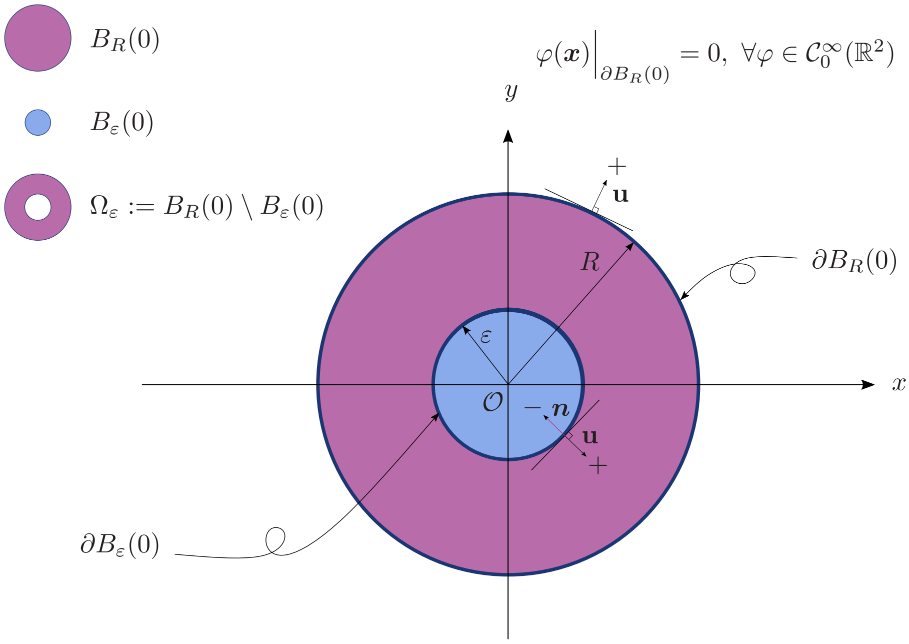

Figure 1: The *2D doughnut*: $\Omega_\varepsilon = B_R(0) \setminus B_\varepsilon(0)$.

Note in passing that $\Delta\gamma(\boldsymbol{x}) = 0$ from (4) due to the fact that

$$\boxed{\begin{aligned}
&\gamma(\boldsymbol{x}) = -\frac{1}{2\pi}\ln|\boldsymbol{x}| \Leftrightarrow \gamma(x,y) = -\frac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right) \\
&\quad \text{which leads to} \\
&\qquad \partial_x\gamma(x,y) = \frac{-x}{2\pi(x^2+y^2)} = -\frac{1}{2\pi}\frac{x}{|\boldsymbol{x}|^2} \\
&\qquad \partial_y\gamma(x,y) = \frac{-y}{2\pi(x^2+y^2)} = -\frac{1}{2\pi}\frac{y}{|\boldsymbol{x}|^2} \\
&\qquad \text{and} \\
&\qquad \partial_{xx}\gamma(x,y) = \frac{2x^2-2y^2}{4\pi^2(x^2+y^2)^2} \\
&\qquad \partial_{yy}\gamma(x,y) = \frac{2y^2-2x^2}{4\pi^2(x^2+y^2)^2} \\
&\qquad \text{hence} \\
&\qquad \nabla\gamma(\boldsymbol{x}) = (\partial_x\gamma(x,y), \partial_y\gamma(x,y)) = -\frac{1}{2\pi}\frac{(x,y)}{|\boldsymbol{x}|^2} = -\frac{1}{2\pi}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2} \\
&\qquad \Delta\gamma(\boldsymbol{x}) = \partial_{xx}\gamma(x,y) + \partial_{yy}\gamma(x,y) = 0
\end{aligned}} \tag{5}$$

Next, by decomposing the boundary $\partial\Omega_\varepsilon$ into $\partial B_R(0)$ and $\partial B_\varepsilon(0)$, i.e.

$$\partial\Omega_\varepsilon = \partial B_R(0) \cup \partial B_\varepsilon(0), \tag{6}$$

we consequently obtain from (4) the following expression

$$
\begin{aligned}
&\langle\gamma(\boldsymbol{x}), -\Delta\varphi\rangle \\
&= \lim_{\varepsilon\to 0^+}\left(-\int_{\partial\Omega_\varepsilon}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\,d\partial\Omega_\varepsilon + \int_{\partial\Omega_\varepsilon}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\,d\partial\Omega_\varepsilon\right) \\
&= \lim_{\varepsilon\to 0^+}\Bigg(-\left(\int_{\partial B_R(0)}\gamma(\boldsymbol{x})\cancelto{0}{\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}}\,d\partial B_R(0) + \int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\,d\partial B_\varepsilon(0)\right) \\
&\qquad + \left(\int_{\partial B_R(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\cancelto{0}{\varphi(\boldsymbol{x})}\,d\partial B_R(0) + \int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\,d\partial B_\varepsilon(0)\right)\Bigg) \\
&= \lim_{\varepsilon\to 0^+}\Bigg(-\left(\int_{\partial B_R(0)}\gamma(\boldsymbol{x})\cancelto{0}{\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}}\,d\partial B_R(0) - \int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right) \\
&\qquad + \left(\int_{\partial B_R(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\cancelto{0}{\varphi(\boldsymbol{x})}\,d\partial B_R(0) - \int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\varphi(\boldsymbol{x})\,d\partial B_\varepsilon(0)\right)\Bigg) \\
&= \lim_{\varepsilon\to 0^+}\left(\int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) - \int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\varphi(\boldsymbol{x})\,d\partial B_\varepsilon(0)\right) \\
&= \lim_{\varepsilon\to 0^+}\Bigg(-\int_{\partial B_\varepsilon(0)}(\varphi(\boldsymbol{x})-\varphi(0))\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) \\
&\qquad -\varphi(0)\int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) + \int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\Bigg) \\
&= \underbrace{\lim_{\varepsilon\to 0^+}\left(-\int_{\partial B_\varepsilon(0)}(\varphi(\boldsymbol{x})-\varphi(0))\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right)}_{:=A} \\
&\quad + \underbrace{\lim_{\varepsilon\to 0^+}\left(-\varphi(0)\int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right)}_{:=B} \\
&\quad + \underbrace{\lim_{\varepsilon\to 0^+}\left(\int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right)}_{:=C} = A+B+C.
\end{aligned} \tag{7}
$$

Note in passing that the test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ vanishes on boundary $\partial B_R(0)$. Next, computation of the directional derivative of $\gamma(\boldsymbol{x})$ goes as follows

$$\frac{\partial\gamma(\boldsymbol{x})}{\partial n} = \nabla\gamma(\boldsymbol{x})\cdot\boldsymbol{n} \overset{(5)}{=} -\frac{1}{2\pi}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}\cdot\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \frac{-1}{2\pi|\boldsymbol{x}|},$$

which leads to

$$
\begin{aligned}
\int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) &= \frac{-1}{2\pi}\int_{\partial B_\varepsilon(0)}\frac{1}{|\boldsymbol{x}|}\,d\partial B_\varepsilon(0) \\
&= \frac{-1}{2\pi}\int_{\partial B_\varepsilon(0)}\frac{1}{\varepsilon}\,d\partial B_\varepsilon(0) \\
&= \frac{-1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}d\partial B_\varepsilon(0) = \frac{-1}{2\pi\varepsilon}(2\pi\varepsilon) = -1
\end{aligned} \tag{8}
$$

Therefore, term B in (7) yields

$$\therefore \quad \boxed{B = \lim_{\varepsilon\to 0^+}\left(-\varphi(0)\int_{\partial B_\varepsilon(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial n}\, d\partial B_\varepsilon(0)\right) \overset{(8)}{=} \varphi(0)} \tag{9}$$

Besides, the analysis of term A goes as follows

$$\begin{aligned}
|A| &= \lim_{\varepsilon\to 0^+}\left|-\int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x})-\varphi(0))\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\, d\partial B_\varepsilon(0)\right| \\
&\le \lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(0)} \Big|\varphi(\boldsymbol{x})-\varphi(0)\Big|\left|\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\right| d\partial B_\varepsilon(0) \\
&\le \lim_{\varepsilon\to 0^+}\max_{|\boldsymbol{x}|=\varepsilon}|\varphi(\boldsymbol{x})-\varphi(0)|\int_{\partial B_\varepsilon(0)}\left|\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\right| d\partial B_\varepsilon(0) \\
&= \lim_{\varepsilon\to 0^+}\max_{|\boldsymbol{x}|=\varepsilon}|\nabla\varphi(\boldsymbol{x})(\boldsymbol{x}-\boldsymbol{0})|\int_{\partial B_\varepsilon(0)}\left|\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\right| d\partial B_\varepsilon(0) \\
&\le \lim_{\varepsilon\to 0^+}\underbrace{\max_{|\boldsymbol{x}|=\varepsilon}|\nabla\varphi(\boldsymbol{x})|}_{=M_1\in\mathbb{R},\ \forall\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2)}\underbrace{|\boldsymbol{x}|}_{\varepsilon}\int_{\partial B_\varepsilon(0)}\left|\underbrace{\frac{\partial\gamma(\boldsymbol{x})}{\partial n}}_{\overset{(8)}{=}-1}\right| d\partial B_\varepsilon(0) \\
&= \lim_{\varepsilon\to 0^+} M_1\varepsilon = 0
\end{aligned} \tag{10}$$

Note in passing that the Mean value theorem has been considered in the above analysis. Consequently, term A vanishes itself

$$\therefore \quad \boxed{A = \lim_{\varepsilon\to 0^+}\left(-\int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x})-\varphi(0))\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\, d\partial B_\varepsilon(0)\right) = 0} \tag{11}$$

Last but not least, consideration of term C leads to

$$\begin{aligned}
|C| &= \lim_{\varepsilon\to 0^+}\left|\int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\, d\partial B_\varepsilon(0)\right| \\
&= \lim_{\varepsilon\to 0^+}\left|\int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})\cdot\boldsymbol{n}\, d\partial B_\varepsilon(0)\right| \\
&\le \lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(0)} \Big|\gamma(\boldsymbol{x})\Big|\Big|\nabla\varphi(\boldsymbol{x})\Big|\Big|\boldsymbol{n}\Big|\, d\partial B_\varepsilon(0) \\
&\le \lim_{\varepsilon\to 0^+}\underbrace{\max_{\boldsymbol{x}\in B_R(0)}|\nabla\varphi|}_{=:M_2\in\mathbb{R},\ \forall\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2)}\underbrace{|\boldsymbol{n}|}_{=1}\int_{\partial B_\varepsilon(0)} |\gamma(\boldsymbol{x})|\, d\partial B_\varepsilon(0) \\
&= \lim_{\varepsilon\to 0^+} M_2\int_{\partial B_\varepsilon(0)} \frac{1}{2\pi}\ln|\boldsymbol{x}|\, d\partial B_\varepsilon(0) = \lim_{\varepsilon\to 0^+}\frac{M_2}{2\pi}\ln(\varepsilon)\underbrace{\int_{\partial B_\varepsilon(0)} d\partial B_\varepsilon(0)}_{=2\pi\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} M_2\varepsilon\ln(\varepsilon) = 0
\end{aligned} \tag{12}$$

As a consequence, term C also vanishes itself

$$\therefore \quad \boxed{C = \lim_{\varepsilon \to 0^+} \left(\int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial \varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) = 0} \tag{13}$$

Finally, substitution of (9), (11) and (13) into (7) we obtain

$$\therefore \quad \boxed{\langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).}$$

□