# Global Exercise - Gue11

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Content covered:

✓ [Review HW10] Numerics: Consistency error (cont.)

✓ [Review HW11 - A1] Analysis: Fundamental solution Laplace-operator in 2D.

✓ Numerics: $\begin{cases} \text{Jacobi iteration solution} \\ \text{Gauss-Seidel iteration solution} \end{cases}$

# 1 [Review HW10] Consistency error (cont.)

**Example 1.** *Examine the consistency error of the following problem*

$$u''(x) - u'(x) + u(x) = 2x - 1 - x^2$$

*with the exact solution is known, i.e.* $u(x) = 1 - x^2$.

# 2 [Review HW10 - A1] Fundamental solution

**Example 2.** *Examine the following problem: Show that the $\gamma$ function defined as*

$$\gamma : \begin{cases} \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}, \\ \boldsymbol{x} \mapsto \gamma(\boldsymbol{x}) = -\dfrac{1}{2\pi} \ln |\boldsymbol{x}|, \end{cases} \tag{1}$$

*is a fundamental solution of Laplace equation $-\Delta\gamma(\boldsymbol{x}) = \delta_0$ in $\mathbb{R}^2$.*

Approach:

*Proof.* For all test functions $\varphi$ in the compactly supported, infinitely differentiable function space, i.e. $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we need to show the following equality

$$\begin{aligned} -\Delta\gamma(\boldsymbol{x}) = \delta_0 &\Leftrightarrow \langle -\Delta\gamma(\boldsymbol{x}), \varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle -\partial_{xx}\gamma(\boldsymbol{x}) - \partial_{yy}\gamma(\boldsymbol{x}), \varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow -\langle \partial_{xx}\gamma(\boldsymbol{x}), \varphi \rangle - \langle \partial_{yy}\gamma(\boldsymbol{x}), \varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow -(-1)^2\langle \gamma(\boldsymbol{x}), \partial_{xx}\varphi \rangle - (-1)^2\langle \gamma(\boldsymbol{x}), \partial_{yy}\varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\partial_{xx}\varphi \rangle + \langle \gamma(\boldsymbol{x}), -\partial_{yy}\varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\partial_{xx}\varphi - \partial_{yy}\varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle = \langle \delta_0, \varphi \rangle \end{aligned} \tag{2}$$

yielding the following expression

$$\boxed{\langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle = \langle \delta_0, \varphi \rangle \Leftrightarrow \langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle \overset{!}{=} \varphi(0), \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).} \tag{3}$$

Setup: Let the test function $\varphi \in \mathcal{D}(\mathbb{R}^2) = \mathcal{C}_0^\infty(\mathbb{R}^2)$ be arbitrarily chosen with its support in a subset of a ball centered at $O$ with radius $R$, i.e. $\text{supp}(\varphi) \subset B_R(0)$. This setting leads to the fact

$$\begin{cases} \varphi(\boldsymbol{x}) \neq 0 \text{ in } B_R(0), \\ \varphi(\boldsymbol{x}) = 0 \text{ on } \partial B_R(0). \end{cases}$$

Besides, let another smaller ball centered at $O$ with radius $\varepsilon \ll 1$ be defined $B_\varepsilon(0)$. Then, the domain to be considered is $\Omega_\varepsilon := B_R(0) \setminus B_\varepsilon(0)$, i.e. the *2D doughnut* as shown in Figure 1. By using Gauss divergence theorem twice we obtain the following equality:

$$\begin{aligned} \langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle &= -\int_{\mathbb{R}^2} \gamma(\boldsymbol{x})\Delta\varphi(\boldsymbol{x}) \, d\boldsymbol{x} \\ &= \lim_{\varepsilon \to 0^+} \left( -\int_{\Omega_\varepsilon} \gamma(\boldsymbol{x})\Delta\varphi(\boldsymbol{x}) \, d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon \to 0^+} \left( -\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu} \, d\partial\Omega_\varepsilon + \int_{\Omega_\varepsilon} \nabla\gamma(\boldsymbol{x})\nabla\varphi(\boldsymbol{x}) \, d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon \to 0^+} \left( -\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu} \, d\partial\Omega_\varepsilon \right. \\ &\qquad\qquad \left. + \left( \int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x}) \, d\partial\Omega_\varepsilon - \int_{\Omega_\varepsilon} \underbrace{\Delta\gamma(\boldsymbol{x})}_{0}\varphi(\boldsymbol{x}) \, d\Omega_\varepsilon \right) \right) \\ &= \lim_{\varepsilon \to 0^+} \left( -\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu} \, d\partial\Omega_\varepsilon + \int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x}) \, d\partial\Omega_\varepsilon \right) \tag{4} \end{aligned}$$
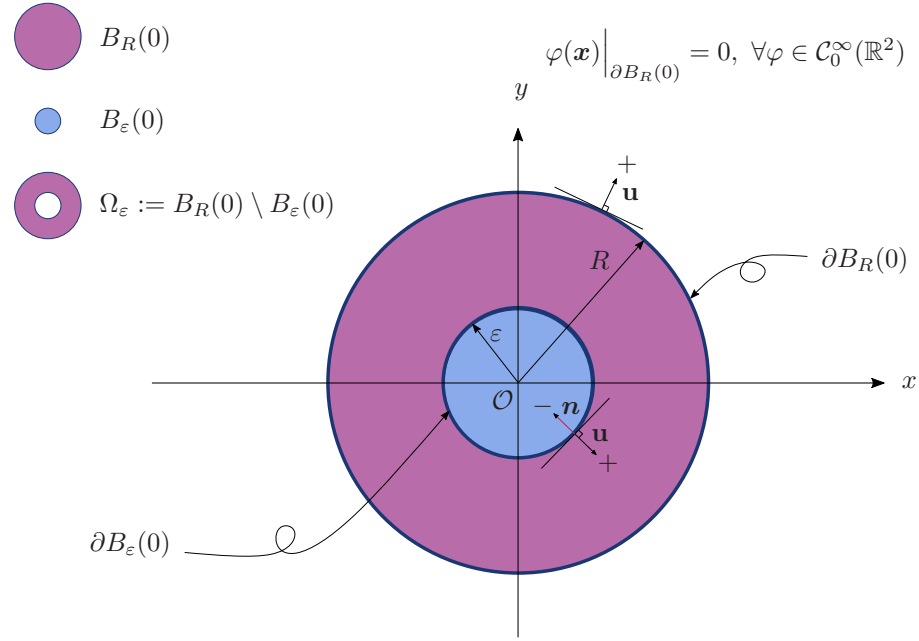
Figure 1: The *2D doughnut*: $\Omega_\varepsilon = B_R(0) \setminus B_\varepsilon(0)$.

Note in passing that $\Delta\gamma(\boldsymbol{x}) = 0$ from (4) due to the fact that

$$\gamma(\boldsymbol{x}) = -\frac{1}{2\pi}\ln|\boldsymbol{x}| \Leftrightarrow \gamma(x,y) = -\frac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)$$

which leads to

$$\partial_x\gamma(x,y) = \frac{-x}{2\pi(x^2+y^2)} = -\frac{1}{2\pi}\frac{x}{|\boldsymbol{x}|^2}$$

$$\partial_y\gamma(x,y) = \frac{-y}{2\pi(x^2+y^2)} = -\frac{1}{2\pi}\frac{y}{|\boldsymbol{x}|^2}$$

and

$$\partial_{xx}\gamma(x,y) = \frac{2x^2 - 2y^2}{4\pi^2(x^2+y^2)^2}$$

$$\partial_{yy}\gamma(x,y) = \frac{2y^2 - 2x^2}{4\pi^2(x^2+y^2)^2}$$

hence

$$\nabla\gamma(\boldsymbol{x}) = (\partial_x\gamma(x,y), \partial_y\gamma(x,y)) = -\frac{1}{2\pi}\frac{(x,y)}{|\boldsymbol{x}|^2} = -\frac{1}{2\pi}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}$$

$$\Delta\gamma(\boldsymbol{x}) = \partial_{xx}\gamma(x,y) + \partial_{yy}\gamma(x,y) = 0$$

(5)

Next, by decomposing the boundary $\partial\Omega_\varepsilon$ into $\partial B_R(0)$ and $\partial B_\varepsilon(0)$, i.e.

$$\partial\Omega_\varepsilon = \partial B_R(0) \cup \partial B_\varepsilon(0), \tag{6}$$

we consequently obtain from (4) the following expression

$$\langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle$$

$$= \lim_{\varepsilon \to 0^+} \left( -\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu} \, d\partial\Omega_\varepsilon + \int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu} \varphi(\boldsymbol{x}) \, d\partial\Omega_\varepsilon \right)$$

$$= \lim_{\varepsilon \to 0^+} \left( -\left( \int_{\partial B_R(0)} \gamma(\boldsymbol{x}) \cancelto{0}{\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}} \, d\partial B_R(0) + \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu} \, d\partial B_\varepsilon(0) \right) \right.$$

$$\left. + \left( \int_{\partial B_R(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu} \cancelto{0}{\varphi(\boldsymbol{x})} \, d\partial B_R(0) + \int_{\partial B_\varepsilon(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu} \varphi(\boldsymbol{x}) \, d\partial B_\varepsilon(0) \right) \right)$$

$$= \lim_{\varepsilon \to 0^+} \left( -\left( \int_{\partial B_R(0)} \gamma(\boldsymbol{x}) \cancelto{0}{\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}} \, d\partial B_R(0) - \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) \right.$$

$$\left. + \left( \int_{\partial B_R(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu} \cancelto{0}{\varphi(\boldsymbol{x})} \, d\partial B_R(0) - \int_{\partial B_\varepsilon(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial n} \varphi(\boldsymbol{x}) \, d\partial B_\varepsilon(0) \right) \right)$$

$$= \lim_{\varepsilon \to 0^+} \left( \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) - \int_{\partial B_\varepsilon(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial n} \varphi(\boldsymbol{x}) \, d\partial B_\varepsilon(0) \right)$$

$$= \lim_{\varepsilon \to 0^+} \left( -\int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x}) - \varphi(0)) \frac{\partial\gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right.$$

$$\left. -\varphi(0) \int_{\partial B_\varepsilon(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) + \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right)$$

$$= \underbrace{\lim_{\varepsilon \to 0^+} \left( -\int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x}) - \varphi(0)) \frac{\partial\gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right)}_{:=A}$$

$$+ \underbrace{\lim_{\varepsilon \to 0^+} \left( -\varphi(0) \int_{\partial B_\varepsilon(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right)}_{:=B}$$

$$+ \underbrace{\lim_{\varepsilon \to 0^+} \left( \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right)}_{:=C} = A + B + C. \tag{7}$$

Note in passing that the test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ vanishes on boundary $\partial B_R(0)$.
Next, computation of the directional derivative of $\gamma(\boldsymbol{x})$ goes as follows

$$\frac{\partial\gamma(\boldsymbol{x})}{\partial n} = \nabla\gamma(\boldsymbol{x}) \cdot \boldsymbol{n} \overset{(5)}{=} -\frac{1}{2\pi} \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \frac{-1}{2\pi|\boldsymbol{x}|}, \tag{8}$$

which leads to

$$\int_{\partial B_\varepsilon(0)} \frac{\partial\gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) = \frac{-1}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{1}{|\boldsymbol{x}|} \, d\partial B_\varepsilon(0)$$

$$= \frac{-1}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{1}{\varepsilon} \, d\partial B_\varepsilon(0)$$

$$= \frac{-1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(0)} d\partial B_\varepsilon(0) = \frac{-1}{2\pi\varepsilon} (2\pi\varepsilon) = -1 \tag{9}$$

Therefore, term $B$ in (7) yields

$$\therefore \quad \boxed{B = \lim_{\varepsilon \to 0^+} \left( -\varphi(0) \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) \overset{(9)}{=} \varphi(0)} \tag{10}$$

Besides, the analysis of term A goes as follows

$$|A| = \lim_{\varepsilon \to 0^+} \left| - \int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x}) - \varphi(0)) \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right|$$

$$\leq \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(0)} \left| \varphi(\boldsymbol{x}) - \varphi(0) \right| \left| \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \right| d\partial B_\varepsilon(0)$$

$$\leq \lim_{\varepsilon \to 0^+} \max_{|\boldsymbol{x}| = \varepsilon} |\varphi(\boldsymbol{x}) - \varphi(0)| \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \right| d\partial B_\varepsilon(0)$$

$$= \lim_{\varepsilon \to 0^+} \max_{|\boldsymbol{x}| = \varepsilon} |\nabla \varphi(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{0})| \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \right| d\partial B_\varepsilon(0)$$

$$\leq \lim_{\varepsilon \to 0^+} \underbrace{\max_{|\boldsymbol{x}| = \varepsilon} |\nabla \varphi(\boldsymbol{x})|}_{=M_1 \in \mathbb{R}, \ \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)} \underbrace{|\boldsymbol{x}|}_{\varepsilon} \underbrace{\int_{\partial B_\varepsilon(0)} \left| \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \right| d\partial B_\varepsilon(0)}_{\overset{(9)}{=} -1}$$

$$= \lim_{\varepsilon \to 0^+} M_1 \varepsilon = 0 \tag{11}$$

Note in passing that the Mean value theorem has been considered in the above analysis. Consequently, term A vanishes itself

$$\therefore \quad \boxed{A = \lim_{\varepsilon \to 0^+} \left( - \int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x}) - \varphi(0)) \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) = 0} \tag{12}$$

Last but not least, consideration of term $C$ leads to

$$|C| = \lim_{\varepsilon \to 0^+} \left| \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial \varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right|$$

$$= \lim_{\varepsilon \to 0^+} \left| \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \nabla \varphi(\boldsymbol{x}) \cdot \boldsymbol{n} \, d\partial B_\varepsilon(0) \right|$$

$$\leq \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(0)} \left| \gamma(\boldsymbol{x}) \right| \left| \nabla \varphi(\boldsymbol{x}) \right| |\boldsymbol{n}| \, d\partial B_\varepsilon(0)$$

$$\leq \lim_{\varepsilon \to 0^+} \underbrace{\max_{\boldsymbol{x} \in \overline{B_R(0)}} |\nabla \varphi|}_{=:M_2 \in \mathbb{R}, \ \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)} \underbrace{|\boldsymbol{n}|}_{=1} \int_{\partial B_\varepsilon(0)} |\gamma(\boldsymbol{x})| \, d\partial B_\varepsilon(0)$$

$$= \lim_{\varepsilon \to 0^+} M_2 \int_{\partial B_\varepsilon(0)} \frac{1}{2\pi} \ln |\boldsymbol{x}| \, d\partial B_\varepsilon(0) = \lim_{\varepsilon \to 0^+} \frac{M_2}{2\pi} \ln(\varepsilon) \underbrace{\int_{\partial B_\varepsilon(0)} d\partial B_\varepsilon(0)}_{=2\pi\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} M_2 \varepsilon \ln(\varepsilon) = 0 \tag{13}$$

As a consequence, term $C$ also vanishes itself

$$\therefore \quad \boxed{C = \lim_{\varepsilon \to 0^+} \left( \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial \varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) = 0} \tag{14}$$

Finally, substitution of (10), (12) and (14) into (7) we obtain

$$\therefore \quad \boxed{\langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).}$$

$\square$