

Global Exercise - Gue08

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Content covered:

- ✓ Analysis: Eigenvalues/Eigenfunction of Laplace operators. How to
 - * treat the inhomogeneous problem $\partial_t u(x, t) = \partial_{xx} u(x, t)$ with $u(L, t) \neq 0$.
 - * treat the $\partial_{tt} u(x, t)$ term in $\partial_{tt} u(x, t) = \partial_{xx} u(x, t) + a$.
- ✓ Numerics: FDM consistency order.

1 Analysis

Example 1. *Examine the following problem:*

$$\partial_t u(x, t) = \partial_{xx} u(x, t), \quad \text{for } x \in \Omega = (0, L), \text{ and } t > 0, \quad (1)$$

where the boundary conditions (BCs) at the left-most and right-most points for all time t with a constant $\mu \in \mathbb{R}$ are defined, respectively, as follows

$$\begin{cases} u(0, t) &= 0, \\ u(L, t) &= \mu t \exp(-\mu t), \end{cases} \quad (2)$$

and the initial condition (IC) with a constant $u_0 \in \mathbb{R}$ is given as

$$u(x, 0) = u_0 x(L - x). \quad (3)$$

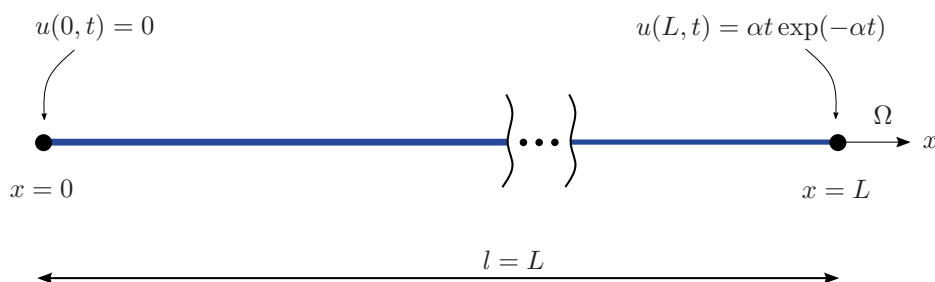


Figure 1: Domain Ω with given BCs.

Approach:

1. Homogenization of the BCs: We search for $u_{inh}(x, t)$ such that

$$\partial_{xx} u_{inh} = 0, \quad \text{with BCs: } \begin{cases} u_{inh}(0, t) = 0, \\ u_{inh}(L, t) = \mu t \exp(-\mu t), \end{cases} \quad (4)$$

which leads to

$$\boxed{u_{inh}(x, t) = \frac{x}{L} \mu t \exp(-\mu t).} \quad (5)$$

Likewise, we also obtain the following expression

$$\partial_t u_{inh} = \frac{x}{L} \mu \exp(-\mu t) - \frac{x}{L} \mu^2 t \exp(-\mu t) = \frac{x}{L} (1 - \mu t) \mu \exp(-\mu t) \quad (6)$$

Note in passing that we have used $u_{inh}(x, t) = ax + b, \forall a, b \in \mathbb{R}$ to probe the solution mentioned at (4), since it satisfies $\partial_{xx}(ax + b) = 0$. The solution (5) is obtained by considering BCs given in (4).

2. We define

$$\tilde{u}(x, t) := u(x, t) - u_{inh}(x, t) \Leftrightarrow u(x, t) = \tilde{u}(x, t) + u_{inh}(x, t) \quad (7)$$

By substituting (7) into (1) we obtain the following equality

$$\begin{aligned} \partial_t(\tilde{u} + u_{inh}) &= \partial_{xx}(\tilde{u} + u_{inh}) \Leftrightarrow \partial_t \tilde{u} - \partial_{xx} \tilde{u} = \cancel{\partial_{xx} u_{inh}}^{0; \text{cf. (4)}} - \partial_t u_{inh} \\ &\Leftrightarrow \partial_t \tilde{u} - \partial_{xx} \tilde{u} = -\partial_t u_{inh} \\ &\Leftrightarrow \partial_t \tilde{u} - \partial_{xx} \tilde{u} \stackrel{(6)}{=} -\frac{x}{L} (1 - \mu t) \mu \exp(-\mu t) \end{aligned} \quad (8)$$

Then, the homogenized problem reads

$$\boxed{\partial_t \tilde{u}(x, t) - \partial_{xx} \tilde{u}(x, t) = \tilde{f}(x, t), \quad \text{with} \quad \begin{cases} \tilde{u}(0, t) = \tilde{u}(L, t) = 0, \\ \tilde{u}(x, 0) = u_0 x(L - x), \end{cases}} \quad (9)$$

where the source term $\tilde{f}(x, t)$ defined as $\tilde{f}(x, t) := -\frac{x}{L} (1 - \mu t) \mu \exp(-\mu t)$.

3. We now need to find out the Eigenfunctions $\varphi(x)$ satisfying

$$-\partial_{xx} \varphi(x) = \lambda \varphi(x), \quad \text{with } \varphi(0) = \varphi(L) = 0. \quad (10)$$

By taking into consideration the Ansatz

$$\begin{cases} \varphi(x) = \exp(rx) \\ \varphi''(x) = r^2 \exp(rx) \end{cases} \quad (11)$$

We then obtain the following expressions

$$(10) \stackrel{(11)}{\Leftrightarrow} \exp(rx)(r^2 + \lambda) = 0 \Leftrightarrow r = \pm i\sqrt{\lambda} \quad (12)$$

The solution to (10) $\forall A, B, C_1, C_2 \in \mathbb{R}$ takes the following form

$$\begin{aligned} \varphi(x) &= A \exp(i\sqrt{\lambda}x) + B \exp(-i\sqrt{\lambda}x) \\ &= A \left(\cos(\sqrt{\lambda}x) + i \sin(\sqrt{\lambda}x) \right) + B \left(\cos(\sqrt{\lambda}x) - i \sin(\sqrt{\lambda}x) \right) \\ &= (A + B) \cos(\sqrt{\lambda}x) + (A - B)i \sin(\sqrt{\lambda}x) \\ &= C_1 \cos(\sqrt{\lambda}x) + iC_2 \sin(\sqrt{\lambda}x) \end{aligned} \quad (13)$$

Note in passing that Euler's formula $\exp(ix) = \cos(x) + i \sin(x)$ has been used in (13). Next, by applying BCs from (10) we obtain the explicit values for C_1 and eigenvalues λ_k for (13), as follows

$$\varphi(0) = 0 \stackrel{(13)}{\Leftrightarrow} C_1 = 0. \quad (14)$$

$$\varphi(L) = 0 \stackrel{(13)(14)}{\Leftrightarrow} iC_2 \sin(\sqrt{\lambda}L) = 0 \Leftrightarrow \lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad \forall k \in \mathbb{Z}. \quad (15)$$

Till this point we have obtained so far only **non-normalized** eigenfunctions $\varphi_k(x)$ and their eigenvalues λ_k as follows

$$\begin{cases} \varphi_k(x) = iC_2 \sin\left(\frac{k\pi}{L}x\right), \\ \lambda_k = \left(\frac{k\pi}{L}\right)^2, \end{cases} \quad \forall k \in \mathbb{Z}. \quad (16)$$

Next, the eigenfunction $\varphi_k(x)$ in (16) needs to be normalized. It goes as follows

$$\begin{aligned} \|\varphi_k(x)\|_2 &\stackrel{!}{=} 1 \Leftrightarrow \sqrt{\int_0^L (\varphi_k(x))^2 dx} = 1 \\ &\stackrel{(16)}{\Leftrightarrow} \int_0^L \left(iC_2 \sin\left(\frac{k\pi}{L}x\right)\right)^2 dx = 1 \\ &\Leftrightarrow \int_0^L \sin^2\left(\frac{k\pi}{L}x\right) dx = \frac{-1}{(C_2)^2} \\ &\Leftrightarrow \int_0^L \left(\frac{1}{2} - \frac{1}{2} \cos\left(\frac{2k\pi}{L}x\right)\right) dx = \frac{-1}{(C_2)^2} \\ &\Leftrightarrow \frac{1}{2}x \Big|_{x=0}^{x=L} - \frac{1}{2} \frac{L}{2k\pi} \sin\left(\frac{2k\pi}{L}x\right) \Big|_{x=0}^{x=L} = \frac{-1}{(C_2)^2} \\ &\Leftrightarrow \frac{1}{2}L = \frac{-1}{(C_2)^2} \Leftrightarrow C_2 = \pm i\sqrt{\frac{2}{L}} \end{aligned} \quad (17)$$

- $C_2 = i\sqrt{\frac{2}{L}} \stackrel{(16)}{\Rightarrow} \varphi_k(x) = -\sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{h\pi}{L}x\right) \forall k, h \in \mathbb{Z}.$
- $C_2 = -i\sqrt{\frac{2}{L}} \stackrel{(16)}{\Rightarrow} \varphi_k(x) = +\sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \forall k \in \mathbb{Z}.$

Therefore, the **normalized** eigenfunctions $\varphi_k(x)$ and their eigenvalues λ_k take the following forms

$$\therefore \boxed{\begin{aligned} \varphi_k(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \\ \lambda_k &= \left(\frac{k\pi}{L}\right)^2, \forall k \in \mathbb{Z} \end{aligned}}$$

4. Expand the source term $\tilde{f}(x, t)$ by using the normalized eigenfunction $\varphi_k(x)$. The source term $\tilde{f}(x, t)$ takes the following form

$$\tilde{f}(x, t) = -\frac{x}{L}(1 - \mu t)\mu \exp(-\mu t), \quad (18)$$

which can be written in an expansion of eigenfunctions

$$\tilde{f}(x, t) = \sum_{k=1}^{\infty} \beta_k(t) \varphi_k(x) = \sum_{k=1}^{\infty} \beta_k(t) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \quad (19)$$

where coefficient $\beta_k(t)$ is computed by projecting $\tilde{f}(x, t)$ on eigenfunctions $\varphi_k(x)$, as follows

$$\begin{aligned} \beta_k(t) &= \langle \tilde{f}(x, t), \varphi_k(x) \rangle \\ &= \int_0^L \left(-\frac{x}{L}(1 - \mu t)\mu \exp(-\mu t) \right) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) dx \\ &= \frac{-(1 - \mu t)\mu \exp(-\mu t)}{L} \sqrt{\frac{2}{L}} \int_0^L x \sin\left(\frac{k\pi}{L}x\right) dx \\ &= \frac{-(1 - \mu t)\mu \exp(-\mu t)}{L} \sqrt{\frac{2}{L}} \frac{(-1)^k}{k\pi} L^2 \\ &= -(-1)^k \frac{\sqrt{2L}}{k\pi} (1 - \mu t)\mu \exp(-\mu t). \end{aligned} \quad (20)$$

5. Ansatz for solution $\tilde{u}(x, t)$ take the following form

$$\tilde{u}(x, t) = \sum_{k=1}^{\infty} \alpha_k(t) \varphi_k(x). \quad (21)$$

By substituting (19) and (21) into (9) we obtain

$$\begin{aligned} &\Leftrightarrow \sum_{k=1}^{\infty} (\partial_t \alpha_k(t) \varphi_k(x) - \alpha_k(t) \partial_{xx} \varphi_k(x) - \beta_k(t) \varphi_k(x)) = 0 \\ &\stackrel{(10)}{\Leftrightarrow} \sum_{k=1}^{\infty} (\alpha'_k(t) \varphi_k(x) + \alpha_k(t) \lambda_k \varphi_k(x) - \beta_k(t) \varphi_k(x)) = 0 \\ &\Leftrightarrow \sum_{k=1}^{\infty} (\alpha'_k(t) + \lambda_k \alpha_k(t) - \beta_k(t)) \varphi_k(x) = 0. \end{aligned} \quad (22)$$

Since the eigenfunctions $\varphi_k(x)$ are orthonormal to each other, we induce from (22) and arrive at the following equality

$$\boxed{\alpha'_k(t) + \lambda_k \alpha_k(t) - \beta_k(t) = 0.} \quad (23)$$

This is the 1st order ODE with unknown $\alpha_k(t)$. The solution to $\alpha_k(t)$ takes the form

$$\alpha_k(t) = \alpha_k(0) \exp(-\lambda_k t) + \exp(-\lambda_k t) \int_0^t \exp(\lambda_k \tau) \beta_k(\tau) d\tau,$$

whose derivation goes as follows

- Solving for homogeneous part:

$$\alpha'_k(t) + \lambda_k \alpha_k(t) = 0 \Leftrightarrow \alpha_k^{hom}(t) = \alpha_k(0) \exp(-\lambda_k t). \quad (24)$$

- Inhomogeneous part: By using the method *variation of constant* we exploit the ansatz

$$\alpha_k^{inh}(t) = \alpha_k(t) \exp(-\lambda_k t). \quad (25)$$

Insertion of (25) into (23) leads to

$$\alpha'_k(t) \exp(-\lambda_k t) - \lambda_k \alpha_k(t) \exp(-\lambda_k t) + \lambda_k \alpha_k(t) \exp(-\lambda_k t) - \beta_k(t) = 0$$

where the two terms in the middle have been canceled out with each other. We then obtain the equality

$$\begin{aligned} \alpha'_k(t) \exp(-\lambda_k t) &= \beta_k(t) \\ \Leftrightarrow \alpha'_k(t) &= \exp(\lambda_k t) \beta_k(t) \\ \Leftrightarrow \alpha_k(t) &= \int_0^t \exp(\lambda_k \tau) \beta_k(\tau) d\tau \\ \Leftrightarrow \alpha_k(t) &\stackrel{(20)}{=} \int_0^t \exp(\lambda_k \tau) (-1)^k \frac{\sqrt{2L}}{k\pi} (1 - \mu\tau) \mu \exp(-\mu\tau) d\tau \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \int_0^t \exp(\lambda_k \tau) (1 - \mu\tau) \exp(-\mu\tau) d\tau \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \left(\int_0^t \exp(\lambda_k \tau - \mu\tau) d\tau - \mu \int_0^t \tau \exp(\lambda_k \tau - \mu\tau) d\tau \right) \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \frac{(\lambda_k - (\lambda_k - \mu)\mu t) \exp(\lambda_k - \mu t) - \lambda_k}{(\lambda_k - \mu)^2} \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \frac{(\lambda_k - (\lambda_k - \mu)\mu t) \exp(-\mu t) - \lambda_k \exp(-\lambda_k t)}{(\lambda_k - \mu)^2} \end{aligned} \quad (26)$$

Es bleibt die Anfangsbedingung für $\alpha_k(0)$ zu bestimmen. Dazu muss die Anfangsbedingung für v entwickelt werden:

$$v(x, 0) = \sum_{i=1}^k \alpha_k(0) \phi_k(x) \stackrel{!}{=} u_0 x(L - x)$$

mit

$$\begin{aligned} \alpha_k(0) &= \int_0^L u_0 x(L - x) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) dx \\ &= \begin{cases} u_0 \sqrt{2L} \frac{4L^2}{k^3 \pi^3}, & k \text{ ungerade,} \\ 0, & k \text{ gerade.} \end{cases} \end{aligned}$$

Die Lösung v lautet insgesamt:

$$\begin{aligned}
 v(x, t) &= \sum_{k=1}^{\infty} \alpha_k(t) \phi_k(x) \\
 &= \sum_{k=1, k \text{ ungerade}}^{\infty} \frac{8u_0 L^2}{k^3 \pi^3} e^{-\lambda_k t} \sin\left(\frac{k\pi}{L} x\right) \\
 &\quad + \sum_{k=1}^{\infty} (-1)^k \frac{2a}{k\pi} \frac{(\lambda_k - (\lambda_k - a)at) e^{-at} - \lambda_k e^{-\lambda_k t}}{(\lambda_k - a)^2} \sin\left(\frac{k\pi}{L} x\right).
 \end{aligned}$$

und die Lösung für u folgt aus:

$$\therefore \quad \boxed{u(x, t) = v(x, t) + \frac{x}{L} a t e^{-at}.}$$

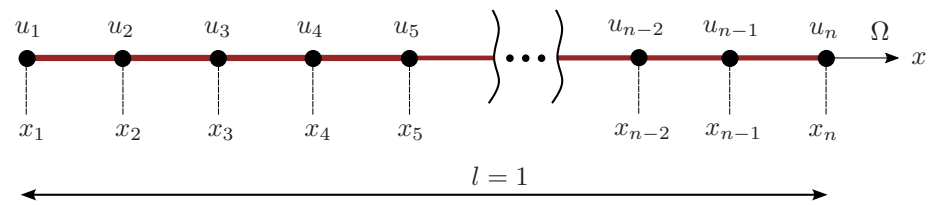


Figure 2: Discretization of domain Ω with n points.

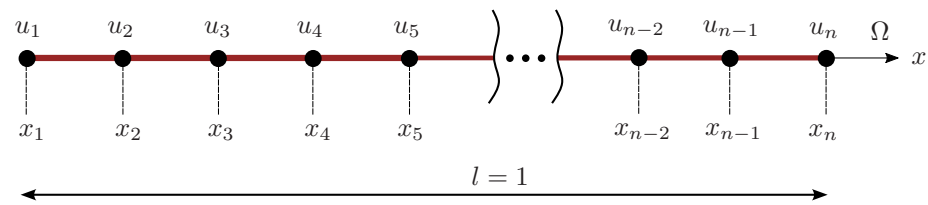


Figure 3: Discretization of domain Ω with n points.

2 Analysis: Spectral theory · Laplace operator application

Example 2. *Examine the following problem*

$$\begin{aligned}\partial_t u(x, t) &= \partial_{xx} u(x, t) + a, & x &\in (0, 1), t > 0, \\ u(0, t) &= 0, & t &> 0, \\ u(1, t) &= 0, & t &> 0, \\ u(x, 0) &= \sin(\pi x), & x &\in (0, 1),\end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from ∂_{xx} .*
2. *Develop the source term $a \in \mathbb{R}$ in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 2 cont.)

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Example 3. *Examine the following problem*

$$\begin{aligned}\partial_t u(x, t) &= \partial_{xx} u(x, t) + \sin(\pi x), & x &\in (0, 1), t > 0, \\ u(0, t) &= 0, & t &> 0, \\ u(1, t) &= 0, & t &> 0, \\ u(x, 0) &= a, & x &\in (0, 1),\end{aligned}$$

with constant IC $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from ∂_{xx} .*
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

Example 4. *Examine the following problem*

$$\begin{aligned}\partial_{tt}u(x,t) &= \partial_{xx}u(x,t) + a, & x &\in (0,1), t > 0, \\ u(0,t) &= 0, & t &> 0, \\ u(1,t) &= 0, & t &> 0, \\ u(x,0) &= x, & x &\in (0,1), \\ u_t(x,0) &= 0, & x &\in (0,1),\end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from ∂_{xx} .*
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 4 cont.)