Global Exercise - Gue12

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✓ Programming exercise 3: Introduction + hints

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1 Programming exercise 3: Introduction + Hints

1.1 Task a: 1D problem

1.1.1 Implicit Euler

Discretization of the following given PDE

$$\partial_t u = \partial_{xx} u \tag{1}$$

by using implicit Euler for time t and second order FDM for space x leads to

$$\frac{u_j^{n+1} - u_j^n}{dt} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{dx^2}, \tag{2}$$

which yields the form used for numerical updating as follows

$$\therefore \quad \boxed{u_j^{n+1} - \frac{dt}{dx^2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) = u_j^n.} \tag{3}$$

Note in passing that all entries on the LHS are unknown since they are u based on time $(n+1)$. These unknowns are computed based on the term on the RHS, which is already known, i.e. u at time n. Then, we obtain the compact form

$$M u_{(\cdot)}^{n+1} = u_j^n \Rightarrow u_{(\cdot)}^{n+1} = M \backslash u_j^n, \tag{4}$$

where matrix M is derived from the RHS of (3), as follows

$$\begin{pmatrix} 1+2dt/dx^2 & -dt/dx^2 & & & & \\ -dt/dx^2 & 1+2dt/dx^2 & -dt/dx^2 & & & \\ & -dt/dx^2 & 1+2dt/dx^2 & -dt/dx^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1+2dt/dx^2 & -dt/dx^2 \\ & & & & -dt/dx^2 & 1+2dt/dx^2 \end{pmatrix} \tag{5}$$

which can be recast in MATLAB code

Listing 1: Matrix M.

```
e = ones(N,1);
M = eye(N)-dt/dx^2*spdiags([e -2*e e],-1:1,N,N);
```

1.1.2 Crank-Nicolson

Discretization of the following given PDE

$$\partial_t u = \partial_{xx} u \tag{6}$$

by using Crank-Nicolson for time t and second order FDM for space x leads to

$$\frac{u_j^{n+1} - u_j^n}{dt} = \frac{1}{2}\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{dx^2} + \frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{dx^2}\right), \tag{7}$$

which yields the form used for numerical updating as follows

$$\therefore \quad \boxed{u_j^{n+1} - \frac{1}{2}\frac{dt}{dx^2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) = u_j^n + \frac{1}{2}\frac{dt}{dx^2}\left(u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}\right).} \tag{8}$$

Note in passing that all entries on the LHS are unknown since they are u based on time $(n+1)$. These unknowns are computed based on the term on the RHS, which is already known, i.e. u at time n. Then, we obtain the compact form

$$M u_{(\cdot)}^{n+1} = K u_{(\cdot)}^{n} \Rightarrow u_{(\cdot)}^{n+1} = M \backslash (K u_{(\cdot)}^{n}), \tag{9}$$

where matrix M is derived from the RHS of (8), as follows

$$\begin{pmatrix}
1 + dt/dx^2 & -dt/dx^2/2 & & & & \\
-dt/dx^2/2 & 1 + dt/dx^2 & -dt/dx^2/2 & & & \\
 & -dt/dx^2/2 & 1 + dt/dx^2 & -dt/dx^2/2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & & 1 + dt/dx^2 & -dt/dx^2/2 \\
 & & & & -dt/dx^2/2 & 1 + dt/dx^2
\end{pmatrix} \tag{10}$$

and matrix K, whose difference from M is only about the sign, takes the form

$$\begin{pmatrix}
1 - dt/dx^2 & -dt/dx^2/2 & & & & \\
-dt/dx^2/2 & 1 - dt/dx^2 & -dt/dx^2/2 & & & \\
 & -dt/dx^2/2 & 1 - dt/dx^2 & -dt/dx^2/2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & & 1 - dt/dx^2 & -dt/dx^2/2 \\
 & & & & -dt/dx^2/2 & 1 - dt/dx^2
\end{pmatrix} \tag{11}$$

which can be recast in MATLAB code

Listing 2: Matrix M and K.

```
e = ones(N,1);
M = eye(N)-dt/dx^2/2*spdiags([e -2*e e],-1:1,N,N);
K = eye(N)+dt/dx^2/2*spdiags([e -2*e e],-1:1,N,N);
```

1.2 Task b: 2D problem + Melting a penny

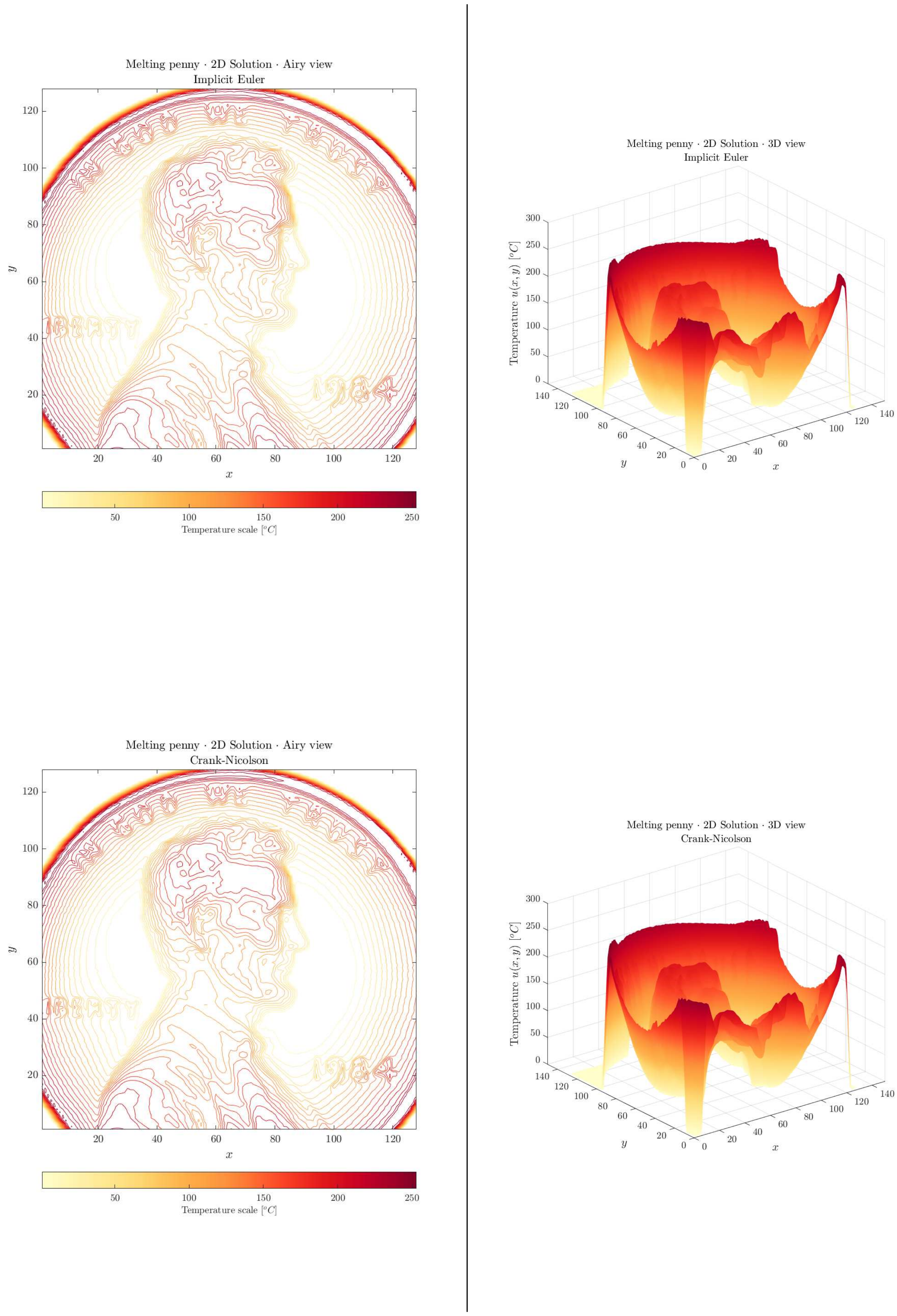

Figure 1: Melting penny: 2D solution; implicit Euler and Crank-Nicolson scheme.

2 Exam preparation

(Doing exercises together on the blackboard)

- Eigenvalue-Eigenfunction problem of Laplace operator in 2D
- Expand a function $f(x, y)$ in Eigenfunctions $\phi_{j,k}(x, y)$
- Solve PDE with Δ based on Eigenvalue-Eigenfunction problem of Laplace operator in 2D
- Distributional derivative
- Fundamental solution
- ...