

# Global Exercise - Gue08

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Content covered:

- ✓ Analysis: Eigenvalues/Eigenfunction of Laplace operators. How to
  - \* treat the inhomogeneous problem  $\partial_t u(x, t) = \partial_{xx} u(x, t)$  with  $u(L, t) \neq 0$ .
  - \* treat the  $\partial_{tt} u(x, t)$  term in  $\partial_{tt} u(x, t) = \partial_{xx} u(x, t) + a$ .
- ✓ Numerics: FDM consistency order.

## 1 Analysis

**Example 1.** *Examine the following problem:*

$$\partial_t u(x, t) = \partial_{xx} u(x, t), \quad \text{for } x \in \Omega = (0, L), \text{ and } t > 0, \quad (1)$$

where the boundary conditions (BCs) at the left-most and right-most points for all time  $t$  with a constant  $\mu \in \mathbb{R}$  are defined, respectively, as follows

$$\begin{cases} u(0, t) &= 0, \\ u(L, t) &= \mu t \exp(-\mu t), \end{cases} \quad (2)$$

and the initial condition (IC) with a constant  $u_0 \in \mathbb{R}$  is given as

$$u(x, 0) = u_0 x(L - x). \quad (3)$$

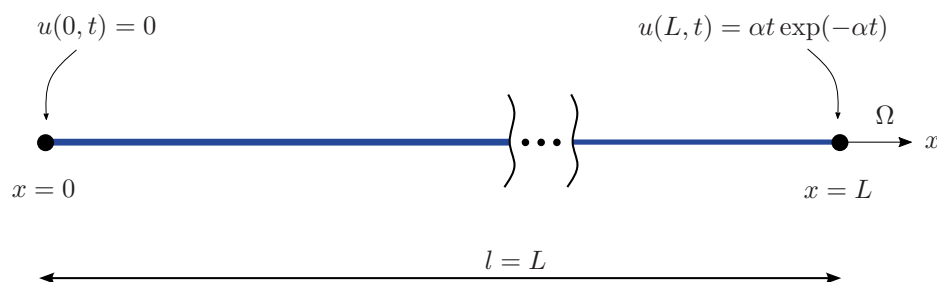


Figure 1: Domain  $\Omega$  with given BCs.

Approach:

1. Homogenization of the BCs: We search for  $u_{inh}(x, t)$  such that

$$\partial_{xx} u_{inh} = 0, \quad \text{with BCs: } \begin{cases} u_{inh}(0, t) = 0, \\ u_{inh}(L, t) = \mu t \exp(-\mu t), \end{cases} \quad (4)$$

which leads to

$$\boxed{u_{inh}(x, t) = \frac{x}{L} \mu t \exp(-\mu t).} \quad (5)$$

Likewise, we also obtain the following expression

$$\partial_t u_{inh} = \frac{x}{L} \mu \exp(-\mu t) - \frac{x}{L} \mu^2 t \exp(-\mu t) = \frac{x}{L} (1 - \mu t) \mu \exp(-\mu t) \quad (6)$$

Note in passing that we have used  $u_{inh}(x, t) = ax + b, \forall a, b \in \mathbb{R}$  to probe the solution mentioned at (4), since it satisfies  $\partial_{xx}(ax + b) = 0$ . The solution (5) is obtained by considering BCs given in (4).

2. We define

$$\tilde{u}(x, t) := u(x, t) - u_{inh}(x, t) \Leftrightarrow u(x, t) = \tilde{u}(x, t) + u_{inh}(x, t) \quad (7)$$

By substituting (7) into (1) we obtain the following equality

$$\begin{aligned} \partial_t(\tilde{u} + u_{inh}) &= \partial_{xx}(\tilde{u} + u_{inh}) \Leftrightarrow \partial_t \tilde{u} - \partial_{xx} \tilde{u} = \cancel{\partial_{xx} u_{inh}}^{0; \text{cf. (4)}} - \partial_t u_{inh} \\ &\Leftrightarrow \partial_t \tilde{u} - \partial_{xx} \tilde{u} = -\partial_t u_{inh} \\ &\Leftrightarrow \partial_t \tilde{u} - \partial_{xx} \tilde{u} \stackrel{(6)}{=} -\frac{x}{L} (1 - \mu t) \mu \exp(-\mu t) \end{aligned} \quad (8)$$

Then, the homogenized problem reads

$$\boxed{\partial_t \tilde{u}(x, t) - \partial_{xx} \tilde{u}(x, t) = \tilde{f}(x, t), \quad \text{with} \quad \begin{cases} \tilde{u}(0, t) = \tilde{u}(L, t) = 0, \\ \tilde{u}(x, 0) = u_0 x(L - x), \end{cases}} \quad (9)$$

where the source term  $\tilde{f}(x, t)$  defined as  $\tilde{f}(x, t) := -\frac{x}{L} (1 - \mu t) \mu \exp(-\mu t)$ .

3. We now need to find out the Eigenfunctions  $\varphi(x)$  satisfying

$$-\partial_{xx} \varphi(x) = \lambda \varphi(x), \quad \text{with } \varphi(0) = \varphi(L) = 0. \quad (10)$$

By taking into consideration the Ansatz

$$\begin{cases} \varphi(x) = \exp(rx) \\ \varphi''(x) = r^2 \exp(rx) \end{cases} \quad (11)$$

We then obtain the following expressions

$$(10) \stackrel{(11)}{\Leftrightarrow} \exp(rx)(r^2 + \lambda) = 0 \Leftrightarrow r = \pm i\sqrt{\lambda} \quad (12)$$

The solution to (10)  $\forall A, B, C_1, C_2 \in \mathbb{R}$  takes the following form

$$\begin{aligned} \varphi(x) &= A \exp(i\sqrt{\lambda}x) + B \exp(-i\sqrt{\lambda}x) \\ &= A \left( \cos(\sqrt{\lambda}x) + i \sin(\sqrt{\lambda}x) \right) + B \left( \cos(\sqrt{\lambda}x) - i \sin(\sqrt{\lambda}x) \right) \\ &= (A + B) \cos(\sqrt{\lambda}x) + (A - B)i \sin(\sqrt{\lambda}x) \\ &= C_1 \cos(\sqrt{\lambda}x) + iC_2 \sin(\sqrt{\lambda}x) \end{aligned} \quad (13)$$

Note in passing that Euler's formula  $\exp(ix) = \cos(x) + i \sin(x)$  has been used in (13). Next, by applying BCs from (10) we obtain the explicit values for  $C_1$  and eigenvalues  $\lambda_k$  for (13), as follows

$$\varphi(0) = 0 \stackrel{(13)}{\Leftrightarrow} C_1 = 0. \quad (14)$$

$$\varphi(L) = 0 \stackrel{(13)(14)}{\Leftrightarrow} iC_2 \sin(\sqrt{\lambda}L) = 0 \Leftrightarrow \lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad \forall k \in \mathbb{Z}. \quad (15)$$

Till this point we have obtained so far only **non-normalized** eigenfunctions  $\varphi_k(x)$  and their eigenvalues  $\lambda_k$  as follows

$$\begin{cases} \varphi_k(x) = iC_2 \sin\left(\frac{k\pi}{L}x\right), \\ \lambda_k = \left(\frac{k\pi}{L}\right)^2, \end{cases} \quad \forall k \in \mathbb{Z}. \quad (16)$$

Next, the eigenfunction  $\varphi_k(x)$  in (16) needs to be normalized. It goes as follows

$$\begin{aligned} \|\varphi_k(x)\|_2 &\stackrel{!}{=} 1 \Leftrightarrow \sqrt{\int_0^L (\varphi_k(x))^2 dx} = 1 \\ &\stackrel{(16)}{\Leftrightarrow} \int_0^L \left(iC_2 \sin\left(\frac{k\pi}{L}x\right)\right)^2 dx = 1 \\ &\Leftrightarrow \int_0^L \sin^2\left(\frac{k\pi}{L}x\right) dx = \frac{-1}{(C_2)^2} \\ &\Leftrightarrow \int_0^L \left(\frac{1}{2} - \frac{1}{2} \cos\left(\frac{2k\pi}{L}x\right)\right) dx = \frac{-1}{(C_2)^2} \\ &\Leftrightarrow \left.\frac{1}{2}x - \frac{1}{2} \frac{L}{2k\pi} \sin\left(\frac{2k\pi}{L}x\right)\right|_{x=0}^{x=L} = \frac{-1}{(C_2)^2} \\ &\Leftrightarrow \frac{1}{2}L = \frac{-1}{(C_2)^2} \Leftrightarrow C_2 = \pm i\sqrt{\frac{2}{L}} \end{aligned} \quad (17)$$

- $C_2 = i\sqrt{\frac{2}{L}} \stackrel{(16)}{\Rightarrow} \varphi_k(x) = -\sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{h\pi}{L}x\right) \forall k, h \in \mathbb{Z}.$
- $C_2 = -i\sqrt{\frac{2}{L}} \stackrel{(16)}{\Rightarrow} \varphi_k(x) = +\sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \forall k \in \mathbb{Z}.$

Therefore, the **normalized** eigenfunctions  $\varphi_k(x)$  and their eigenvalues  $\lambda_k$  take the following forms

$$\therefore \boxed{\begin{aligned} \varphi_k(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \\ \lambda_k &= \left(\frac{k\pi}{L}\right)^2, \forall k \in \mathbb{Z} \end{aligned}}$$

4. Expand the source term  $\tilde{f}(x, t)$  by using the normalized eigenfunction  $\varphi_k(x)$ . The source term  $\tilde{f}(x, t)$  takes the following form

$$\tilde{f}(x, t) = -\frac{x}{L}(1 - \mu t)\mu \exp(-\mu t), \quad (18)$$

which can be written in an expansion of eigenfunctions

$$\boxed{\tilde{f}(x, t) = \sum_{k=1}^{\infty} \beta_k(t) \varphi_k(x) = \sum_{k=1}^{\infty} \beta_k(t) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right)} \quad (19)$$

where coefficient  $\beta_k(t)$  is computed by projecting  $\tilde{f}(x, t)$  on eigenfunctions  $\varphi_k(x)$ , as follows

$$\begin{aligned} \beta_k(t) &= \langle \tilde{f}(x, t), \varphi_k(x) \rangle \\ &= \int_0^L \left( -\frac{x}{L}(1 - \mu t)\mu \exp(-\mu t) \right) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) dx \\ &= \frac{-(1 - \mu t)\mu \exp(-\mu t)}{L} \sqrt{\frac{2}{L}} \int_0^L x \sin\left(\frac{k\pi}{L}x\right) dx \\ &= \frac{-(1 - \mu t)\mu \exp(-\mu t)}{L} \sqrt{\frac{2}{L}} \frac{(-1)^k}{k\pi} L^2 \\ &= (-1)^k \frac{\sqrt{2L}}{k\pi} (1 - \mu t)\mu \exp(-\mu t). \end{aligned} \quad (20)$$

5. Ansatz for solution  $\tilde{u}(x, t)$  take the following form

$$\tilde{u}(x, t) = \sum_{k=1}^{\infty} \alpha_k(t) \varphi_k(x). \quad (21)$$

By substituting (19) and (21) into (9) we obtain

$$\begin{aligned} &\Leftrightarrow \sum_{k=1}^{\infty} (\partial_t \alpha_k(t) \varphi_k(x) - \alpha_k(t) \partial_{xx} \varphi_k(x) - \beta_k(t) \varphi_k(x)) = 0 \\ &\stackrel{(10)}{\Leftrightarrow} \sum_{k=1}^{\infty} (\alpha'_k(t) \varphi_k(x) + \alpha_k(t) \lambda_k \varphi_k(x) - \beta_k(t) \varphi_k(x)) = 0 \\ &\Leftrightarrow \sum_{k=1}^{\infty} (\alpha'_k(t) + \lambda_k \alpha_k(t) - \beta_k(t)) \varphi_k(x) = 0. \end{aligned} \quad (22)$$

Since the eigenfunctions  $\varphi_k(x)$  are orthonormal to each other, we induce from (22) and arrive at the following equality

$$\boxed{\alpha'_k(t) + \lambda_k \alpha_k(t) - \beta_k(t) = 0.} \quad (23)$$

This is the 1st order ODE with unknown  $\alpha_k(t)$ . The solution to  $\alpha_k(t)$  takes the form

$$\therefore \boxed{\alpha_k(t) = \alpha_k(0) \exp(-\lambda_k t) + \exp(-\lambda_k t) \int_0^t \exp(\lambda_k \tau) \beta_k(\tau) d\tau,}$$

whose derivation goes as follows

- Solving for homogeneous part:

$$\alpha'_k(t) + \lambda_k \alpha_k(t) = 0 \Leftrightarrow \alpha_k^{hom}(t) = \alpha_k(0) \exp(-\lambda_k t). \quad (24)$$

- Inhomogeneous part: By using the method *variation of constant* we exploit the ansatz

$$\alpha_k^{inh}(t) = \alpha_k(t) \exp(-\lambda_k t). \quad (25)$$

Insertion of (25) into (23) leads to

$$\alpha'_k(t) \exp(-\lambda_k t) - \lambda_k \alpha_k(t) \exp(-\lambda_k t) + \lambda_k \alpha_k(t) \exp(-\lambda_k t) - \beta_k(t) = 0$$

where the two terms in the middle have been canceled out with each other. We then obtain the equality

$$\begin{aligned} \alpha'_k(t) \exp(-\lambda_k t) &= \beta_k(t) \\ \Leftrightarrow \alpha'_k(t) &= \exp(\lambda_k t) \beta_k(t) \\ \Leftrightarrow \alpha_k(t) &= \int_0^t \exp(\lambda_k \tau) \beta_k(\tau) d\tau \\ \Leftrightarrow \alpha_k(t) &\stackrel{(20)}{=} \int_0^t \exp(\lambda_k \tau) (-1)^k \frac{\sqrt{2L}}{k\pi} (1 - \mu\tau) \mu \exp(-\mu\tau) d\tau \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \int_0^t \exp(\lambda_k \tau) (1 - \mu\tau) \exp(-\mu\tau) d\tau \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \left( \int_0^t \exp(\lambda_k \tau - \mu\tau) d\tau - \mu \int_0^t \tau \exp(\lambda_k \tau - \mu\tau) d\tau \right) \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \frac{(\lambda_k - (\lambda_k - \mu)\mu t) \exp(\lambda_k - \mu t) - \lambda_k}{(\lambda_k - \mu)^2} \\ \Leftrightarrow \alpha_k(t) &= (-1)^k \frac{\sqrt{2L}}{k\pi} \mu \frac{(\lambda_k - (\lambda_k - \mu)\mu t) \exp(-\mu t) - \lambda_k \exp(-\lambda_k t)}{(\lambda_k - \mu)^2} \end{aligned} \quad (26)$$

6. Find  $\alpha_k(0)$  by taking into consideration the IC of  $\tilde{u}(x, 0)$ , as follows

$$\tilde{u}(x, 0) = \sum_{k=1}^{\infty} \alpha_k(0) \varphi_k(x) \stackrel{(9)}{\Leftrightarrow} u_0 x(L - x) \stackrel{!}{=} \sum_{k=1}^{\infty} \alpha_k(0) \varphi_k(x) \quad (27)$$

where the coefficient  $\alpha_k(0)$  is computed as follows

$$\begin{aligned} \alpha_k(0) &= \langle \tilde{u}(x, 0), \varphi_k(x) \rangle \\ &= \int_0^L u_0 x(L - x) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L} x\right) dx = \begin{cases} u_0 \sqrt{2L} \frac{4L^2}{(k\pi)^3}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \end{aligned} \quad (28)$$

The solution to  $\tilde{u}(x, t)$  then reads

$$\begin{aligned}
 \tilde{u}(x, t) &= \sum_{k=1}^{\infty} \alpha_k(t) \varphi_k(x) \\
 &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{8u_0 L^2}{(k\pi)^3} \exp(-\lambda_k t) \sin\left(\frac{k\pi}{L}x\right) \\
 &\quad + \sum_{k=1}^{\infty} (-1)^k \frac{2\mu}{k\pi} \frac{(\lambda_k - (\lambda_k - \mu)\mu t) \exp(-\mu t) - \lambda_k \exp(-\lambda_k t)}{(\lambda_k - \mu)^2} \sin\left(\frac{k\pi}{L}x\right)
 \end{aligned}
 \tag{29}$$

7. Finally, by substituting (6) and (29) into (7) the solution to  $u(x, t)$  reads

$$\therefore \quad u(x, t) = \tilde{u}(x, t) + \frac{x}{L} \mu t \exp(-\mu t).$$

**Example 2.** *Examine the term  $\partial_{tt}u(x, t)$  in the following problem*

$$\partial_{tt}u(x, t) = \partial_{xx}u(x, t) + a, \quad \text{for } x \in \Omega = (0, 1), \quad t > 0. \quad (30)$$

Similarly to steps approaching to (23), which is shown in Example 1, we obtain a 2nd order ODE, instead of the 1st order ODE, as follows

$$\therefore \quad \boxed{\alpha_k''(t) + \lambda_k \alpha_k(t) - \beta_k = 0.} \quad (31)$$

Essentially, the equation (31) has the form

$$y''(x) = -cy(x) + b, \quad \forall c, b \in \mathbb{R}, \quad (32)$$

whose solution takes the following form

$$y(x) = \frac{b}{c} + C_1 \sin(\sqrt{c}x) + C_2 \cos(\sqrt{c}x), \quad \forall C_1, C_2 \in \mathbb{R}. \quad (33)$$

## 2 Numerics: FDM consistency order

**Example 3.** Examine the following approximation problem for  $f'(x)$  given as follows

$$f'(x) \approx \alpha f(x) + \beta f(x+h) + \gamma f(x+2h), \quad (34)$$

where constants  $\alpha, \beta, \gamma \in \mathbb{R}$ . Find  $\alpha, \beta, \gamma$  such that the FDM used to approximate  $f$  at point  $x$  gains consistency order as high as possible.

Approach: By using Taylor expansion for  $f(x+h)$  and  $f(x+2h)$  we obtain the following expressions

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \mathcal{O}(h^4) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \mathcal{O}(h^4) \end{cases} \quad (35)$$

which leads to

$$\alpha f(x) = \alpha f(x) \quad (36)$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \frac{\beta h^3}{6} f'''(x) + \mathcal{O}(h^4) \quad (37)$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \frac{4\gamma h^3}{3} f'''(x) + \mathcal{O}(h^4) \quad (38)$$

Summation of (36), (37) and (38) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= (\alpha + \beta + \gamma) f(x) \\ &\quad + (\beta h + 2\gamma h) f'(x) \\ &\quad + \left( \frac{\beta h^2}{2} + 2\gamma h^2 \right) f''(x) \\ &\quad + \left( \frac{\beta h^3}{6} + \frac{4\gamma h^3}{3} \right) f'''(x) + \mathcal{O}(h^4) \end{aligned} \quad (39)$$

Comparison of (39) with (34) yields the following equalities

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \beta h + 2\gamma h = 1 \\ \frac{\beta h^2}{2} + 2\gamma h^2 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \quad (40)$$

which leads to

$$af(x) + bf(x+h) + cf(x+2h) \approx f'(x) + f'''(x) \underbrace{\left( \frac{2}{h} \frac{h^3}{6} + \frac{-1}{2h} \frac{4h^3}{3} \right)}_{= -h^2/3} + \mathcal{O}(h^4). \quad (41)$$

Since the term  $-h^2/3$  from (41) is nonzero, we only need to expand those two Taylor expansions from (35) till  $\mathcal{O}(h^3)$ . Therefore, the consistency order, due to the factor  $h$  coming along with  $f'(x)$ , i.e.  $\mathcal{O}(h^3)/h$ , is guaranteed till at least order 2.