→ Small remark about HW08 (index $i = 1 \to 2$ only)

→ FDM: A_h & b_h @ Boundary condition $\begin{cases} u(0) \neq 0 \\ u(\ell) \neq 0 \end{cases}$

→ Spectral theory: Application of laplace operator

* $\partial_t u = \partial_{xx} u + a$
* $\partial_t u = \partial_{xx} u + f(x)$
* $\partial_{tt} u = \partial_{xx} u + a$

Global Exercise - Gue07

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1 Small remarks about HW08

Aufgabe 1. (Hyperbolische Erhaltungsgleichung)

Gegeben ist das PDE-System 1. Ordnung für $U : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^3$

$$\begin{cases} \partial_t u_1 + 3\partial_{x_1} u_2 + 2\partial_{x_2} u_1 + \partial_{x_2} u_2 & = 0, \\ \partial_t u_2 + \partial_{x_1} u_2 + 3\partial_{x_2} u_2 & = 0, \\ \partial_t u_3 + 2\partial_{x_1} u_2 + 3\partial_{x_1} u_3 + 2\partial_{x_2} u_3 & = 0. \end{cases}$$

|| No partial derivative w.r.t x_3

$\Rightarrow$ 2D

Zeigen Sie, dass das System mit den Flussfunktionen $F^{(i)} : \mathbb{R}^3 \to \mathbb{R}^3$ $(i = 1, 2)$ in der Form

$$\partial_t U + \mathrm{div} F(U) = \partial_t U + \sum_{i=1}^{2} \partial_{x_i} F^{(i)}(U) = 0$$

geschrieben werden kann, und überprüfen Sie, ob das System hyperbolisch ist. Dies ist der Fall, wenn die gerichtete Transportmatrix

$$A^{(\xi)}(U) = \left(\sum_{i=1}^{2} \xi_i \frac{\partial F^{(i)}}{\partial u_j} \right)_{i,j=1,2,3}$$

$1 \to 3$

$j = 1, 2, 3$

$i = 1, 2$

für alle Richtungen $\xi \in \mathbb{R}^2$ diagonalisierbar ist.

4 Punkte

$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$$\partial_t U + \mathrm{div} F(U) = \partial_t U + \underbrace{\partial_{x_1} F^{(1)}(U)}_{A^{(1)} \partial_{x_1} U} + \underbrace{\partial_{x_2} F^{(2)}(U)}_{A^{(2)} \partial_{x_2} U} = 0$$

Boundary condition non-zero

Matrix Vector

2 Numerics: FDM · BC · Derivation of A_h & b_h → RHS

$A_h u_n = b_h$

known · unknowns · known

Example 1. *Discretization of the following convection-diffusion problem:*

$$\boxed{-u''(x) - 5u'(x) = 3, \quad \textit{for } x \in (0,1)} \tag{1}$$

with boundary conditions $u(0) = u(1) = 1.$

Dirichlet BC

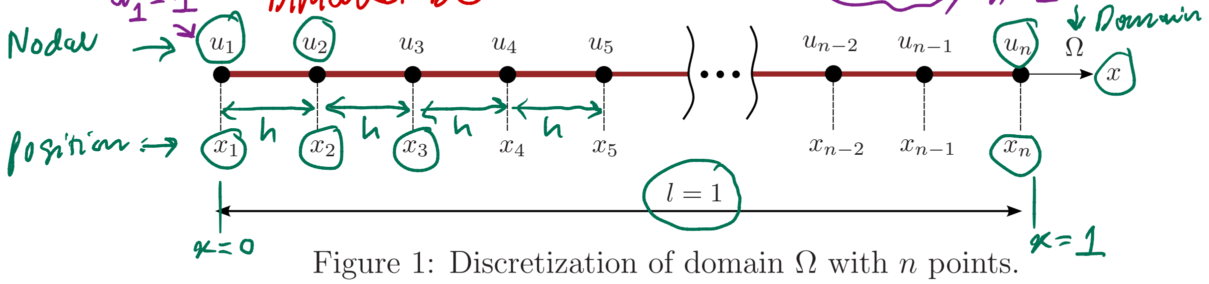


Figure 1: Discretization of domain Ω with n points.

Approach: Central method for $u''(x)$ is obtained as follows

$u(x+h) = \dots + O(h^4)$

$u(x-h) = \dots + O(h^4)$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2). \tag{2}$$

Forward method for $u'(x)$ is obtained as follows

$u(x+h) = \dots$

$u(x-h) = u(x_0) + u' + \dots$

$$u'(x) = \frac{u(x+h) - u(x)}{h} + \mathcal{O}(h). \tag{3}$$

which leads to the system $A_h u_h = b_h$ where A_h and b_h are recognized as follows

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2+5h & -1-5h & & & & \\ -1 & 2+5h & -1-5h & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2+5h & -1-5h \\ & & & & -1 & 2+5h \end{pmatrix} \in \mathbb{R}^{(n-2)\times(n-2)}, \tag{4}$$

$(n-2)\times(n-2)$

$$b_h = \begin{pmatrix} 3 + \frac{1}{h^2} \\ 3 \\ \vdots \\ 3 \\ 3 + \left(\frac{1}{h^2} + \frac{5}{h}\right) \end{pmatrix} \in \mathbb{R}^{(n-2)}. \tag{5}$$

$u_1 = 1$

$u_n = 1$

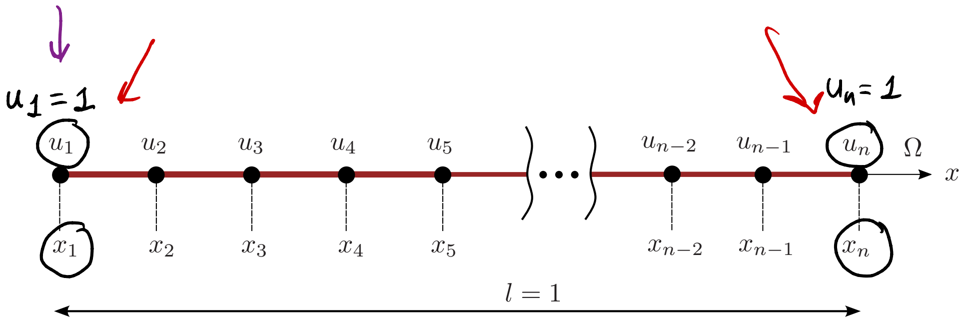


Figure 2: Discretization of domain Ω with n points.

$$u''(x) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \mathcal{O}(h^2) \qquad u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2).$$

$$u'(x) = \frac{u_{i+1} - u_i}{h} + \mathcal{O}(h) \qquad u'(x) = \frac{u(x+h) - u(x)}{h} + \mathcal{O}(h).$$

$$-u'' - 5u' = 3$$

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - 5\frac{u_{i+1} - u_i}{h} = 3$$

$$\frac{1}{h^2}\left(-u_{i-1} + 2u_i - u_{i+1} - 5h u_{i+1} + 5h u_i\right) = 3$$

$$\boxed{\frac{1}{h^2}\left(-u_{i-1} + (2+5h)u_i - (1+5h)u_{i+1}\right) = 3}$$

$i = 2, 3, \ldots, n-1$

u_1 u_n

$i=2$: $\frac{1}{h^2}\left(-u_1 + (2+5h)u_2 - (1+5h)u_3\right) = 3$

$i=3$: $\frac{1}{h^2}\left(-u_2 + (2+5h)u_3 - (1+5h)u_4\right) = 3$

...

$i = n-1$ $\frac{1}{h^2}\left(-u_{n-2} + (2+5h)u_{n-1} - (1+5h)u_n\right) = 3$

$\rightarrow$ only $(n-2)$ eqns

but n unknowns

$u_1\ u_2 \ldots u_n$

$u_1 = 1$ $\qquad\qquad$ $u_n = 1$

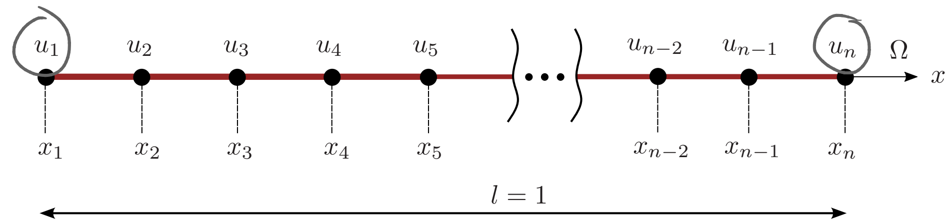


Figure 3: Discretization of domain Ω with n points.

known

$$\frac{1}{h^2}\underbrace{\begin{bmatrix} -1 & 2+5h & -1-5h & & & \\ & -1 & 2+5h & -1-5h & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2+5h & -1-5h \\ & & & & -1 \quad 2+5h & -1-5h \end{bmatrix}}_{(n-2)\times n}\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 3 \\ 3 \\ \vdots \\ 3 \\ 3 \\ 3 \end{bmatrix}}_{(n-2)\times 1}$$

known

u_h unknown only

$$\begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-1} \end{bmatrix}$$

$$\frac{1}{h^2}\left(-u_1 + (2+5h)u_2 - (1+5h)u_3\right) = 3$$

$$\frac{1}{h^2}\left((2+5h)u_2 - (1+5h)u_3\right) = 3 + u_1\frac{1}{h^2}$$

$u_1 = 0 \longrightarrow 3$

$u_1 = 1$

$u_n = 1$

$$\frac{1}{h^2}\left(-u_{n-2} + (2+5h)u_{n-1} - (1+5h)u_n\right) = 3$$

$$\frac{1}{h^2}\left(-u_{n-2} + (2+5h)u_{n-1}\right) = 3 + \frac{1}{h^2}(1+5h)u_n$$

$$= \boxed{3 + \frac{1}{h^2} + \frac{5}{h}}$$

□

3 Analysis: Spectral theory · Laplace operator application

Generally $\underline{\mathcal{L}(\partial_t)u} - \Delta u = f$ in $\Omega \times \mathbb{R}^+$

$\boxed{-\Delta u = f}$

$\boxed{\partial_t u = \Delta u + f}$ Ex2 & Ex3

$\boxed{\partial_{tt} u = \Delta u + f}$ Ex4

HW8 A2 ← (similarly)

$\mathcal{L}(\partial_t) = 0$: Poisson eqn

$\mathcal{L}(\partial_t) = \partial_t$: Diffusion / Heat conduction

$\mathcal{L}(\partial_t) = \partial_{tt}$: Wave eqn

$\mathcal{L}(\partial_t) = \partial_t - 1$: Reaction-Diffusion eqn

$(\partial_t - 1)u - \underbrace{\Delta u}_{\partial_{xx} u} = f \quad \Big\| \quad \boxed{\partial_t u = \partial_{xx} + u + f}$

polynomial

$\partial_t u = \partial_{xx} u + a$

$\begin{cases} a = 2 \\ a = 4 \end{cases}$

Example 2. *Examine the following problem*

$$\boxed{\partial_t u(x,t) = \partial_{xx} u(x,t) + a,} \qquad x \in (0,1), t > 0,$$

(BC) Boundary C $\begin{cases} u(0,t) = 0, & t > 0, \\ u(1,t) = 0, & t > 0, \end{cases}$ Dirichlet BC, time t

(IC) Initial $\quad u(x,0) = \sin(\pi x), \qquad x \in (0,1), \ \Omega$

with constant source term $a \in \mathbb{R}$.

Δu @ 1D

1. *Compute the Eigenvalues and the* $\boxed{\textit{normalized}}$ *Eigenfunctions from $\partial_{xx}\cdot$* Laplace operator
2. *Develop the source term $a \in \mathbb{R}$ in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

*Starting point

$$\boxed{-\varphi''(x) = \lambda \varphi(x)}$$

$\begin{cases} \lambda & \text{Eigenvalue} \\ \varphi(x) & \text{Eigenfunction} \end{cases}$

with $\begin{cases} \varphi(0) = 0 \\ \varphi(1) = 0 \end{cases}$

$$\varphi''(x) + \lambda \varphi(x) = 0$$

2nd order ODE

Ansatz: $\varphi(x) = e^{rx}$, $\varphi''(x) = r^2 e^{rx}$

$$r^2 e^{rx} + \lambda e^{rx} = 0$$

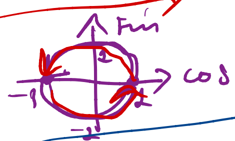
$$e^{rx}(r^2 + \lambda) = 0 \iff r^2 + \lambda = 0 \iff \boxed{r = \pm i\sqrt{\lambda}}$$

Euler formula $\to$ ($\sin(x) + \cos(\cdot)$)

$$\varphi(x) = Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} = \dots = C_1 \sin(\sqrt{\lambda}x) + \cancel{C_2 \cos(\sqrt{\lambda}x)}$$

Eigen function

$\varphi_x(0) = 0 = \cos(\cdot) = 0$



$k\pi, k \in \mathbb{N}$

$$\begin{cases} 0 = \varphi(0) = C_2 \implies C_2 = 0 \\ 0 = \varphi(1) = C_1 \sin(\sqrt{\lambda}x) \implies \boxed{\lambda_k = (k\pi)^2} \quad k \in \mathbb{N} \end{cases}$$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$$\therefore \quad \boxed{\varphi_k(x) = C_1 \sin(k\pi x), \quad k \in \mathbb{N}}$$

Eigenfunction

$$\|\varphi_k(x)\|_2^2 \overset{!}{=} 1 = \int_0^1 (\varphi_k(x))^2 = (C_1)^2 \int_0^1 \sin^2(k\pi x)\,dx = \dots = (C_1)^2 \frac{1}{2}$$

$\implies C_1 = \sqrt{2}$

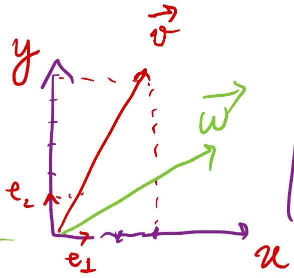
(Example 2 cont.)

$$\begin{cases} \text{Eigenfunction} & \boxed{\varphi_k(x) = \sqrt{2}\sin(k\pi x)} \\ \text{Eigenvalue} & \lambda_k = (k\pi)^2 \quad k \in \mathbb{N} \end{cases}$$

2) Coefficient, eigenfunction

$$a = \sum_{k=1}^{\infty} \beta_k \varphi_k(x)$$

How to compute β_k ?



$$\vec{v} = 3e_1 + 6e_2 = \beta_i e_i$$

$$\vec{v} \cdot e_i = \beta_i$$

$$\beta_k = \langle a, \varphi_k(x) \rangle$$

$$= \int_0^1 a \sqrt{2}\sin(k\pi x)\, dx$$

$\left(\sin(\pi x) \rightarrow \text{Ex3.2} \right)$

$$= a\sqrt{2}\int_0^1 \sin(k\pi x)\, dx$$

$$= \dots$$

$$= \frac{a\sqrt{2}}{k\pi}\left(1 - \cos(k\pi)\right)$$

$$= \begin{cases} \dfrac{2a\sqrt{2}}{k\pi} & k \text{: odd} \\ 0, & \text{otherwise} \end{cases}$$

3. new coefficient

$$w(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\, \varphi_k(x)$$

$$\partial_t u = \partial_{xx} u + a$$

$$\boxed{\partial_{xx}\varphi = \lambda\varphi}$$

$$\left(\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x)\right)_t = \left(\sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x)\right)_{xx} + \sum_{k=1}^{\infty} \beta_k \varphi_k(x)$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \alpha_k'(t)\varphi_k(x) = -\sum_{k=1}^{\infty} \alpha_k(t)\, \lambda_k \varphi_k(x) + \sum_{k=1}^{\infty} \beta_k \varphi_k(x)$$

(Example 2 cont.)

normalized ✓ | orthogonal

orthonormal

$$\Leftrightarrow \sum_{k=1}^{\infty} \underbrace{\left(\alpha_k'(t) + \lambda_k \alpha_k(t) - \beta_k\right)}_{\text{linear combination}} \varphi_k(x) = 0$$

$\therefore$

$$\alpha_k'(t) + \lambda_k \alpha_k(t) - \beta_k \overset{!}{=} 0$$

$$\int \frac{d\alpha_k(t)}{-\lambda_k \alpha_k(t) + \beta_k} = \int dt \quad \Leftrightarrow \cdots = \boxed{\alpha_k(t) = \frac{\beta_k}{\lambda_k} + C_k e^{-\lambda_k t}}$$

C_k ?

Initial condition IC ? $\Leftarrow$ Let's use it

$$u(x,0) = \sin(\pi x) = \sum_{k=1}^{\infty} \alpha_k(0) \sqrt{2} \sin(k\pi x)$$

$$\Leftrightarrow \begin{cases} \alpha_1(0) \overset{!}{=} \frac{1}{\sqrt{2}} & \Leftarrow \\ \alpha_k(0) = 0 \quad \forall k > 1 & \Leftarrow \end{cases}$$

OK for C_k ✓

$$u(x,t) = \alpha_1(t)\sqrt{2}\sin(\pi x) + \sum_{k=2}^{\infty}\left(\frac{\beta_k}{\lambda_k} + C_k e^{-\lambda_k t}\right)\sqrt{2}\sin(k\pi x)$$

$$= \left(\frac{4a}{\pi^3} + \left(1 - \frac{4a}{\pi^3}\right)e^{-\pi^2 t}\right)\sin(\pi x) + \sum_{\substack{k=2 \\ k \text{ odd}}}^{\infty}\left(\frac{4a}{k^3\pi^3} - \frac{4a}{k^3\pi^3}e^{-k^2\pi^2 t}\right)\sin(k\pi x)$$

$\square$

Example 3. *Examine the following problem*

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx} u(x,t) + \sin(\pi x), && x \in (0,1), t>0,\\ u(0,t) &= 0, && t>0,\\ u(1,t) &= 0, && t>0,\\ u(x,0) &= a, && x \in (0,1), \end{aligned}$$

BC

IC

with constant IC $a \in \mathbb{R}$.

✓

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.

   The same as Ex 2.1

2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

Ex 2.

As Ex 2.

Example 4. *Examine the following problem*

$$
\begin{aligned}
&\partial_{tt}u(x,t) = \partial_{xx}u(x,t) + a, && x \in (0,1), t>0,\\
\text{BC}\quad &u(0,t) = 0, && t>0,\\
&u(1,t) = 0, && t>0,\\
\text{IC}\quad &\begin{cases} u(x,0) = x, & x \in (0,1),\\ u_t(x,0) = 0, & x \in (0,1),\end{cases}
\end{aligned}
$$

with constant source term $a \in \mathbb{R}$.

✓ 1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.

The Same as Ex2.1

2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

As Ex 2.

→ Meet on Thursday

(Not on Zoom anymore)

→ Email agueri @ RWTH Moodle about Room next week.

(Example 4 cont.)