

# Global Exercise - Gue11

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Content covered:

- ✓ [Review HW10] Numerics: Consistency error (cont.)
- ✓ [Review HW11 - A1] Analysis: Fundamental solution Laplace-operator in 2D.
- ✓ Numerics:  $\begin{cases} \text{Jacobi iteration solution} \\ \text{Gauss-Seidel iteration solution} \end{cases}$

## 1 [Review HW10] Consistency error (cont.)

**Example 1.** *Examine the consistency error of the following problem*

$$u''(x) - u'(x) + u(x) = 2x - 1 - x^2$$

*with the Neumann and Dirichlet BC are given, respectively, as follows*

$$u'(0) = 0, \quad u(1) = 0.$$

*The exact solution is known, i.e.  $u(x) = 1 - x^2$ .*

Approach: **Definition 2.19** in the lecture note.

The discretization error of FDM takes the form

$$e_h := u|_{\bar{\Omega}_h} - u_h \tag{1}$$

Furthermore, the discretization error is estimated as follows

$$\begin{aligned} \|e_h\| &= \|u|_{\bar{\Omega}_h} - u_h\| \\ &= \|\Delta_h^{-1} \Delta_h (u|_{\bar{\Omega}_h} - u_h)\| \\ &= \|\Delta_h^{-1} (\Delta_h u|_{\bar{\Omega}_h} - \Delta_h u_h)\| \\ &= \|\Delta_h^{-1} (-\Delta_h u|_{\bar{\Omega}_h} + \Delta_h u_h)\| \\ &= \|\Delta_h^{-1} (-\Delta_h u|_{\bar{\Omega}_h} - f|_{\Omega_h})\| \\ &\leq \underbrace{\|\Delta_h^{-1}\|}_{\text{Stability}} \underbrace{\|-\Delta_h u|_{\bar{\Omega}_h} - f|_{\Omega_h}\|}_{\text{Consistency}} \end{aligned} \tag{2}$$

We need to estimate the consistency error

$$\boxed{\|-\Delta_h u|_{\bar{\Omega}_h} - f|_{\Omega_h}\|_{\infty} \rightarrow \|A_h u|_{\Omega_h} - f|_{\Omega_h}\|_{\infty}} \tag{3}$$

for the given problem.

By taking the discretization

$$u_n = u(x_n) \quad (4)$$

$$x_n = nh \quad (5)$$

where  $n = 0, \dots, N-1$ . The matrix  $A_h$  takes the form

$$\therefore A_h = \frac{1}{2h^2} \begin{pmatrix} 2h^2 - 4 & 4 & & & \\ 2 + h & 2h^2 - 4 & 2 - h & & \\ & 2 + h & 2h^2 - 4 & 2 - h & \\ & & \ddots & \ddots & \ddots \\ & & & 2 + h & 2h^2 - 4 \end{pmatrix}_{N \times N}, \quad (6)$$

The exact solution  $u(x) = 1 - x^2$  defined on grid  $\Omega_h$ , i.e.  $u|_{\Omega_h}$ , takes the form

$$\therefore u|_{\Omega_h} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix}_{N \times 1} = \begin{pmatrix} 1 - (0h)^2 \\ 1 - (1h)^2 \\ 1 - (2h)^2 \\ 1 - (3h)^2 \\ \vdots \\ 1 - ((N-2)h)^2 \\ 1 - ((N-1)h)^2 \end{pmatrix}_{N \times 1} \quad (7)$$

The function  $f(x) = 2x - 1 - x^2$  defined on grid  $\Omega_h$ , i.e.  $f|_{\Omega_h}$ , takes the form

$$\therefore f|_{\Omega_h} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{pmatrix}_{N \times 1} = \begin{pmatrix} 2(0h) - 1 - (0h)^2 \\ 2(1h) - 1 - (1h)^2 \\ 2(2h) - 1 - (2h)^2 \\ 2(3h) - 1 - (3h)^2 \\ \vdots \\ 2((N-2)h) - 1 - ((N-2)h)^2 \\ 2((N-1)h) - 1 - ((N-1)h)^2 \end{pmatrix}_{N \times 1} \quad (8)$$

Substitution of (6), (7) and (8) into (3) leads to

$$\begin{aligned}
& \left\| A_h u|_{\Omega_h} - f|_{\Omega_h} \right\|_{\infty} \\
&= \left\| \frac{1}{2h^2} \begin{pmatrix} 2h^2 - 4 & 4 & & & \\ 2 + h & 2h^2 - 4 & 2 - h & & \\ & 2 + h & 2h^2 - 4 & 2 - h & \\ & & \ddots & \ddots & \ddots \\ & & & 2 + h & 2h^2 - 4 \end{pmatrix} \begin{pmatrix} 1 - (0h)^2 \\ 1 - (1h)^2 \\ 1 - (2h)^2 \\ 1 - (3h)^2 \\ \vdots \\ 1 - ((N-2)h)^2 \\ 1 - ((N-1)h)^2 \end{pmatrix} - f|_{\Omega_h} \right\|_{\infty} \\
&= \left\| \begin{pmatrix} -1 \\ -1 + 1(2h) - (1h)^2 \\ -1 + 2(2h) - (2h)^2 \\ -1 + 3(2h) - (3h)^2 \\ \vdots \\ -1 + (N-2)(2h) - ((N-2)h)^2 \\ -1 + (N-1)(2h) - ((N-1)h)^2 \end{pmatrix} - f|_{\Omega_h} \right\|_{\infty} \\
&= \left\| \begin{pmatrix} -1 \\ -1 + 1(2h) - (1h)^2 \\ -1 + 2(2h) - (2h)^2 \\ -1 + 3(2h) - (3h)^2 \\ \vdots \\ -1 + (N-2)(2h) - ((N-2)h)^2 \\ -1 + (N-1)(2h) - ((N-1)h)^2 \end{pmatrix} - \begin{pmatrix} 2(0h) - 1 - (0h)^2 \\ 2(1h) - 1 - (1h)^2 \\ 2(2h) - 1 - (2h)^2 \\ 2(3h) - 1 - (3h)^2 \\ \vdots \\ 2((N-2)h) - 1 - ((N-2)h)^2 \\ 2((N-1)h) - 1 - ((N-1)h)^2 \end{pmatrix} \right\|_{\infty} \\
&= 0. \tag{9}
\end{aligned}$$

Finally, we obtain the consistency error

$$\therefore \quad \boxed{\left\| A_h u|_{\Omega_h} - f|_{\Omega_h} \right\|_{\infty} = 0.} \tag{10}$$

## 2 [Review HW10 - A1] Fundamental solution

**Example 2.** Examine the following problem: Show that the  $\gamma$  function defined as

$$\gamma : \begin{cases} \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}, \\ \mathbf{x} \mapsto \gamma(\mathbf{x}) = -\frac{1}{2\pi} \ln |\mathbf{x}|, \end{cases} \quad (11)$$

is a fundamental solution of Laplace equation  $-\Delta\gamma(\mathbf{x}) = \delta_0$  in  $\mathbb{R}^2$ .

Approach:

*Proof.* For all test functions  $\varphi$  in the compactly supported, infinitely differentiable function space, i.e.  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ , we need to show the following equality

$$\begin{aligned} -\Delta\gamma(\mathbf{x}) = \delta_0 &\Leftrightarrow \langle -\Delta\gamma(\mathbf{x}), \varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle -\partial_{xx}\gamma(\mathbf{x}) - \partial_{yy}\gamma(\mathbf{x}), \varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow -\langle \partial_{xx}\gamma(\mathbf{x}), \varphi \rangle - \langle \partial_{yy}\gamma(\mathbf{x}), \varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow -(-1)^2 \langle \gamma(\mathbf{x}), \partial_{xx}\varphi \rangle - (-1)^2 \langle \gamma(\mathbf{x}), \partial_{yy}\varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle \gamma(\mathbf{x}), -\partial_{xx}\varphi \rangle + \langle \gamma(\mathbf{x}), -\partial_{yy}\varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle \gamma(\mathbf{x}), -\partial_{xx}\varphi - \partial_{yy}\varphi \rangle = \langle \delta_0, \varphi \rangle \\ &\Leftrightarrow \langle \gamma(\mathbf{x}), -\Delta\varphi \rangle = \langle \delta_0, \varphi \rangle \end{aligned} \quad (12)$$

yielding the following expression

$$\boxed{\langle \gamma(\mathbf{x}), -\Delta\varphi \rangle = \langle \delta_0, \varphi \rangle \Leftrightarrow \langle \gamma(\mathbf{x}), -\Delta\varphi \rangle \stackrel{!}{=} \varphi(0), \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).} \quad (13)$$

Setup: Let the test function  $\varphi \in \mathcal{D}(\mathbb{R}^2) = \mathcal{C}_0^\infty(\mathbb{R}^2)$  be arbitrarily chosen with its support in a subset of a ball centered at  $O$  with radius  $R$ , i.e.  $\text{supp}(\varphi) \subset B_R(0)$ . This setting leads to the fact

$$\begin{cases} \varphi(\mathbf{x}) \neq 0 \text{ in } B_R(0), \\ \varphi(\mathbf{x}) = 0 \text{ on } \partial B_R(0). \end{cases} \quad (14)$$

Besides, let another smaller ball centered at  $O$  with radius  $\varepsilon \ll 1$  be defined  $B_\varepsilon(0)$ . Then, the domain to be considered is  $\Omega_\varepsilon := B_R(0) \setminus B_\varepsilon(0)$ , i.e. the *2D doughnut* as shown in Figure 1. By using Gauss divergence theorem twice we obtain the following equality:

$$\begin{aligned} \langle \gamma(\mathbf{x}), -\Delta\varphi \rangle &= - \int_{\mathbb{R}^2} \gamma(\mathbf{x}) \Delta\varphi(\mathbf{x}) d\mathbf{x} \\ &= \lim_{\varepsilon \rightarrow 0^+} \left( - \int_{\Omega_\varepsilon} \gamma(\mathbf{x}) \Delta\varphi(\mathbf{x}) d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left( - \int_{\partial\Omega_\varepsilon} \gamma(\mathbf{x}) \frac{\partial\varphi(\mathbf{x})}{\partial\nu} d\partial\Omega_\varepsilon + \int_{\Omega_\varepsilon} \nabla\gamma(\mathbf{x}) \nabla\varphi(\mathbf{x}) d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left( - \int_{\partial\Omega_\varepsilon} \gamma(\mathbf{x}) \frac{\partial\varphi(\mathbf{x})}{\partial\nu} d\partial\Omega_\varepsilon \right. \\ &\quad \left. + \left( \int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\mathbf{x})}{\partial\nu} \varphi(\mathbf{x}) d\partial\Omega_\varepsilon - \int_{\Omega_\varepsilon} \Delta\gamma(\mathbf{x}) \varphi(\mathbf{x}) d\Omega_\varepsilon \right) \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left( - \int_{\partial\Omega_\varepsilon} \gamma(\mathbf{x}) \frac{\partial\varphi(\mathbf{x})}{\partial\nu} d\partial\Omega_\varepsilon + \int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\mathbf{x})}{\partial\nu} \varphi(\mathbf{x}) d\partial\Omega_\varepsilon \right) \end{aligned} \quad (15)$$

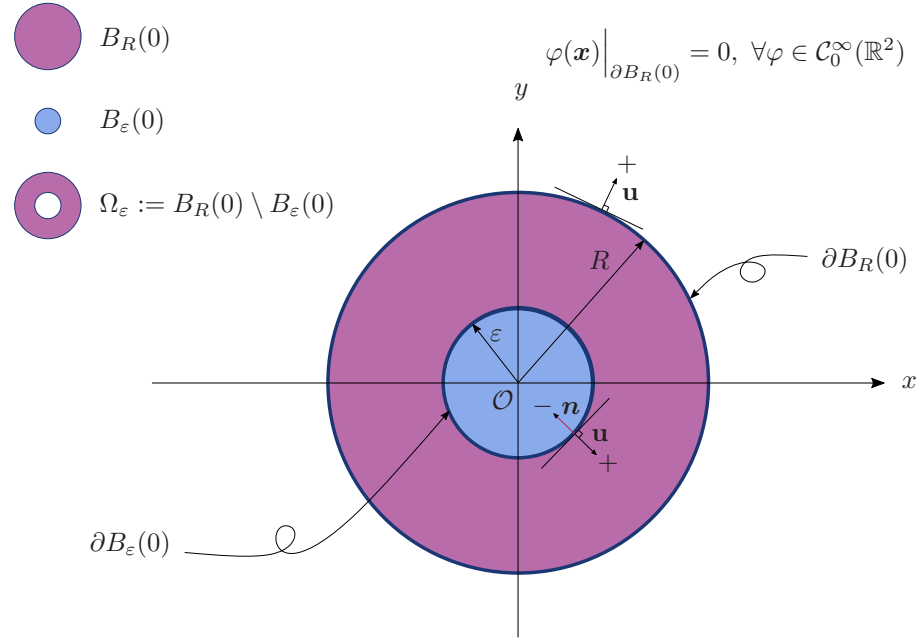


Figure 1: The 2D doughnut:  $\Omega_\epsilon = B_R(0) \setminus B_\epsilon(0)$ .

Note in passing that  $\Delta\gamma(\mathbf{x}) = 0$  from (15) due to the fact that

$$\begin{aligned}
 \gamma(\mathbf{x}) &= -\frac{1}{2\pi} \ln |\mathbf{x}| \Leftrightarrow \gamma(x, y) = -\frac{1}{2\pi} \ln \left( \sqrt{x^2 + y^2} \right) \\
 &\text{which leads to} \\
 \partial_x \gamma(x, y) &= \frac{-x}{2\pi(x^2 + y^2)} = -\frac{1}{2\pi} \frac{x}{|\mathbf{x}|^2} \\
 \partial_y \gamma(x, y) &= \frac{-y}{2\pi(x^2 + y^2)} = -\frac{1}{2\pi} \frac{y}{|\mathbf{x}|^2} \\
 &\text{and} \\
 \partial_{xx} \gamma(x, y) &= \frac{2x^2 - 2y^2}{4\pi^2(x^2 + y^2)^2} \\
 \partial_{yy} \gamma(x, y) &= \frac{2y^2 - 2x^2}{4\pi^2(x^2 + y^2)^2} \\
 &\text{hence} \\
 \nabla \gamma(\mathbf{x}) &= (\partial_x \gamma(x, y), \partial_y \gamma(x, y)) = -\frac{1}{2\pi} \frac{(x, y)}{|\mathbf{x}|^2} = -\frac{1}{2\pi} \frac{\mathbf{x}}{|\mathbf{x}|^2} \\
 \Delta \gamma(\mathbf{x}) &= \partial_{xx} \gamma(x, y) + \partial_{yy} \gamma(x, y) = 0
 \end{aligned} \tag{16}$$

Next, by decomposing the boundary  $\partial\Omega_\epsilon$  into  $\partial B_R(0)$  and  $\partial B_\epsilon(0)$ , i.e.

$$\partial\Omega_\epsilon = \partial B_R(0) \cup \partial B_\epsilon(0), \tag{17}$$

we consequently obtain from (15) the following expression

$$\begin{aligned}
& \langle \gamma(\mathbf{x}), -\Delta \varphi \rangle \\
&= \lim_{\varepsilon \rightarrow 0^+} \left( - \int_{\partial \Omega_\varepsilon} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial \nu} d\partial \Omega_\varepsilon + \int_{\partial \Omega_\varepsilon} \frac{\partial \gamma(\mathbf{x})}{\partial \nu} \varphi(\mathbf{x}) d\partial \Omega_\varepsilon \right) \\
&= \lim_{\varepsilon \rightarrow 0^+} \left( - \left( \int_{\partial B_R(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial \nu} d\partial B_R(0) + \int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial \nu} d\partial B_\varepsilon(0) \right) \right. \\
&\quad \left. + \left( \int_{\partial B_R(0)} \frac{\partial \gamma(\mathbf{x})}{\partial \nu} \varphi(\mathbf{x}) d\partial B_R(0) + \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\mathbf{x})}{\partial \nu} \varphi(\mathbf{x}) d\partial B_\varepsilon(0) \right) \right) \\
&= \lim_{\varepsilon \rightarrow 0^+} \left( - \left( \int_{\partial B_R(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial \nu} d\partial B_R(0) - \int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right) \right. \\
&\quad \left. + \left( \int_{\partial B_R(0)} \frac{\partial \gamma(\mathbf{x})}{\partial \nu} \varphi(\mathbf{x}) d\partial B_R(0) - \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\mathbf{x})}{\partial n} \varphi(\mathbf{x}) d\partial B_\varepsilon(0) \right) \right) \\
&= \lim_{\varepsilon \rightarrow 0^+} \left( \int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) - \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\mathbf{x})}{\partial n} \varphi(\mathbf{x}) d\partial B_\varepsilon(0) \right) \\
&= \lim_{\varepsilon \rightarrow 0^+} \left( - \int_{\partial B_\varepsilon(0)} (\varphi(\mathbf{x}) - \varphi(0)) \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right. \\
&\quad \left. - \varphi(0) \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) + \int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right) \\
&= \lim_{\varepsilon \rightarrow 0^+} \left( \underbrace{- \int_{\partial B_\varepsilon(0)} (\varphi(\mathbf{x}) - \varphi(0)) \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0)}_{:=A} \right) \\
&\quad + \lim_{\varepsilon \rightarrow 0^+} \left( \underbrace{- \varphi(0) \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0)}_{:=B} \right) \\
&\quad + \lim_{\varepsilon \rightarrow 0^+} \left( \underbrace{\int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0)}_{:=C} \right) = A + B + C. \tag{18}
\end{aligned}$$

Note in passing that the test function  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$  vanishes on boundary  $\partial B_R(0)$ . Next, computation of the directional derivative of  $\gamma(\mathbf{x})$  goes as follows

$$\frac{\partial \gamma(\mathbf{x})}{\partial n} = \nabla \gamma(\mathbf{x}) \cdot \mathbf{n} \stackrel{(16)}{=} -\frac{1}{2\pi} \frac{\mathbf{x}}{|\mathbf{x}|^2} \cdot \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{-1}{2\pi|\mathbf{x}|}, \tag{19}$$

which leads to

$$\begin{aligned}
\int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) &= \frac{-1}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{1}{|\mathbf{x}|} d\partial B_\varepsilon(0) \\
&= \frac{-1}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{1}{\varepsilon} d\partial B_\varepsilon(0) \\
&= \frac{-1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(0)} d\partial B_\varepsilon(0) = \frac{-1}{2\pi\varepsilon} (2\pi\varepsilon) = -1 \tag{20}
\end{aligned}$$

Therefore, term  $B$  in (18) yields

$$\therefore \boxed{B = \lim_{\varepsilon \rightarrow 0^+} \left( -\varphi(0) \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right) \stackrel{(20)}{=} \varphi(0)} \quad (21)$$

Besides, the analysis of term  $A$  goes as follows

$$\begin{aligned} |A| &= \lim_{\varepsilon \rightarrow 0^+} \left| - \int_{\partial B_\varepsilon(0)} (\varphi(\mathbf{x}) - \varphi(0)) \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right| \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \int_{\partial B_\varepsilon(0)} \left| \varphi(\mathbf{x}) - \varphi(0) \right| \left| \frac{\partial \gamma(\mathbf{x})}{\partial n} \right| d\partial B_\varepsilon(0) \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \max_{|\mathbf{x}|=\varepsilon} |\varphi(\mathbf{x}) - \varphi(0)| \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \gamma(\mathbf{x})}{\partial n} \right| d\partial B_\varepsilon(0) \\ &= \lim_{\varepsilon \rightarrow 0^+} \max_{|\mathbf{x}|=\varepsilon} |\nabla \varphi(\mathbf{x})(\mathbf{x} - \mathbf{0})| \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \gamma(\mathbf{x})}{\partial n} \right| d\partial B_\varepsilon(0) \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \underbrace{\max_{|\mathbf{x}|=\varepsilon} |\nabla \varphi(\mathbf{x})|}_{=M_1 \in \mathbb{R}, \forall \varphi \in C_0^\infty(\mathbb{R}^2)} \underbrace{|\mathbf{x}|}_{\varepsilon} \int_{\partial B_\varepsilon(0)} \underbrace{\left| \frac{\partial \gamma(\mathbf{x})}{\partial n} \right|}_{\stackrel{(20)}{=} -1} d\partial B_\varepsilon(0) \\ &= \lim_{\varepsilon \rightarrow 0^+} M_1 \varepsilon = 0 \end{aligned} \quad (22)$$

Note in passing that the Mean value theorem has been considered in the above analysis. Consequently, term  $A$  vanishes itself

$$\therefore \boxed{A = \lim_{\varepsilon \rightarrow 0^+} \left( - \int_{\partial B_\varepsilon(0)} (\varphi(\mathbf{x}) - \varphi(0)) \frac{\partial \gamma(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right) = 0} \quad (23)$$

Last but not least, consideration of term  $C$  leads to

$$\begin{aligned} |C| &= \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right| \\ &= \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \nabla \varphi(\mathbf{x}) \cdot \mathbf{n} d\partial B_\varepsilon(0) \right| \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \int_{\partial B_\varepsilon(0)} |\gamma(\mathbf{x})| |\nabla \varphi(\mathbf{x})| |\mathbf{n}| d\partial B_\varepsilon(0) \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \underbrace{\max_{\mathbf{x} \in \overline{B_R(0)}} |\nabla \varphi|}_{=: M_2 \in \mathbb{R}, \forall \varphi \in C_0^\infty(\mathbb{R}^2)} \underbrace{|\mathbf{n}|}_{=1} \int_{\partial B_\varepsilon(0)} |\gamma(\mathbf{x})| d\partial B_\varepsilon(0) \\ &= \lim_{\varepsilon \rightarrow 0^+} M_2 \int_{\partial B_\varepsilon(0)} \frac{1}{2\pi} \ln |\mathbf{x}| d\partial B_\varepsilon(0) = \lim_{\varepsilon \rightarrow 0^+} \frac{M_2}{2\pi} \ln(\varepsilon) \underbrace{\int_{\partial B_\varepsilon(0)} d\partial B_\varepsilon(0)}_{=2\pi\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} M_2 \varepsilon \ln(\varepsilon) = 0 \end{aligned} \quad (24)$$

As a consequence, term  $C$  also vanishes itself

$$\therefore \quad \boxed{C = \lim_{\varepsilon \rightarrow 0^+} \left( \int_{\partial B_\varepsilon(0)} \gamma(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial n} d\partial B_\varepsilon(0) \right) = 0} \quad (25)$$

Finally, substitution of (21), (23) and (25) into (18) we obtain

$$\therefore \quad \boxed{\langle \gamma(\mathbf{x}), -\Delta \varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).} \quad (26)$$

□



### 3 Numerics: Jacobi iteration solution

**Example 3.** *Examine the following problem:  $Ax = b$  with matrix  $A$*

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \quad (27)$$

*Perform the first step of Jacobi iteration given the initial value for the vector  $x^{(0)} = (1, 0, 0)^T$  and the RHS vector  $b = (3, 5, -1)^T$ .*

Approach: **Bemerkung 3.12** in the lecture note:

$$\boxed{Dx^{(k+1)} = (L + U)x^{(k)} + b.} \quad (28)$$

Note in passing that

$$A = D - L - U \quad (29)$$

where the matrix  $A$  is decomposed into

1. the diagonal matrix  $D = \begin{pmatrix} 2 & & \\ & -4 & \\ & & 3 \end{pmatrix}$
2. the lower triangular matrix  $L = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ -1 & -2 & 0 \end{pmatrix}$  with **sign changed**.
3. the upper triangular matrix  $U = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 2 \\ & & 0 \end{pmatrix}$  with **sign changed**.

By using (28) the computation of the **first step** of iteration based on the **Jacobi** scheme goes as follows

$$Dx^{(1)} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}, \quad (30)$$

which leads to the solution of the first step of Jacobi iteration

$$\therefore \boxed{x^{(1)} = \begin{pmatrix} 3/2 \\ -1 \\ -2/3 \end{pmatrix}.} \quad (31)$$

**Remark 1.** *The computation of **Jacobi** iteration requires the inverse of matrix  $D$ . Since matrix  $D$  is a diagonal matrix, its inverse is obtained by taking the inverse of every single entry on its diagonal, which is*

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad (32)$$

*and the inverse of  $D$  takes the form*

$$D^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}, \quad (33)$$

*where  $a, b, c \neq 0$ .*

## 4 Numerics: Gauss-Seidel iteration solution

**Example 4.** Examine the following problem:  $Ax = b$  with matrix  $A$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -4 & -2 \\ 1 & 2 & 3 \end{pmatrix}. \quad (34)$$

Perform the first step of Gauss-Seidel iteration given the initial value for the vector  $x^{(0)} = (1, 0, 0)^T$  and the RHS vector  $b = (3, 5, -1)^T$ .

Approach: **Bemerkung 3.17** in the lecture note:  
Either using

$$x^{(k+1)} = (I - (D - L)^{-1}A)x^{(k)} + (D - L)^{-1}b \quad (35)$$

or, equally, using

$$(D - L)x^{(k+1)} = Ux^{(k)} + b \quad (36)$$

Note in passing that

$$A = D - L - U \quad (37)$$

where the matrix  $A$  is decomposed into

1. the diagonal matrix  $D = \begin{pmatrix} 2 & & \\ & -4 & \\ & & 3 \end{pmatrix}$
2. the lower triangular matrix  $L = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ -1 & -2 & 0 \end{pmatrix}$  with **sign changed**.
3. the upper triangular matrix  $U = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 2 \\ & & 0 \end{pmatrix}$  with **sign changed**.

By using (36) the computation of the **first step** of iteration based on the **Gauss-Seidel** scheme goes as follows

$$(D - L)x^{(k+1)} = Ux^{(k)} + b, \quad (38)$$

which leads to

$$(D - L)x^{(1)} = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 2 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}. \quad (39)$$

Therefore, we obtain the solution after the first step

$$\therefore \boxed{x^{(1)} = (D - L)^{-1} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 & & \\ 1/8 & -1/4 & \\ -1/4 & 1/6 & 1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -7/8 \\ -1/4 \end{pmatrix}}. \quad (40)$$

**Remark 2.** The computation of **Gauss-Seidel** iteration requires the inverse of matrix  $D$ . Since the matrix  $(D - L)$  is a lower triangular matrix with non-zero entries on its diagonal, the matrix  $(D - L)$  has the following form

$$D - L = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}, \quad (41)$$

and the inverse  $(D - L)^{-1}$  takes the form

$$(D - L)^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ -b/(ad) & 1/d & 0 \\ (be - cd)/(adf) & -e/(df) & 1/f \end{pmatrix}, \quad (42)$$

where  $a, d, f \neq 0$ .