Global Exercise - Gue09

Tuan Vo

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Content covered:

- ✓ Programming exercise 02: Introduction + Hints.
- ✓ Analysis:
  - * Expansion of $f(x, y)$ in eigenfunctions of Laplace operator (2D).
  - * Distributional derivative. in the sense of distribution.
- ✓ Numerics: Ghost points 👻 in FDM + Neumann BCs.

We need this ghost point

1 Programming exercise 02: Introduction + Hints

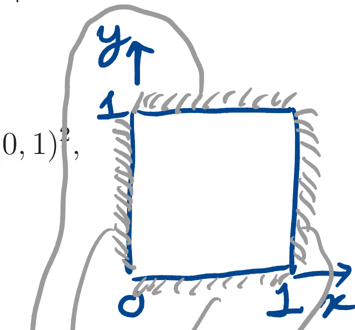
$\varepsilon = 1$ we would like take into account of diffusion

$\varepsilon = 10^{-4}$

Example 1. *Examine the following Convection-diffusion problem* (×sion)

$$-\varepsilon \underbrace{\Delta u(x, y)}_{\text{diffusion}} + \underbrace{\cos\beta \frac{\partial u}{\partial x}(x, y) + \sin\beta \frac{\partial u}{\partial y}}_{\text{convection}} = f(x, y), \ (x, y) \in \Omega = (0, 1)^2,$$

$$u(x, y) = g(x, y), \ (x, y) \ on \ \partial\Omega,$$



$u(x,y) = 0$ on $\partial\Omega$

Konvektions–Diffusionsgleichung

Die Lösung des Konvektions–Diffusionsproblems

$$-\varepsilon \Delta u(x, y) + \cos\beta \frac{\partial u}{\partial x}(x, y) + \sin\beta \frac{\partial u}{\partial y} = \underbrace{f(x, y)}_{=1}, \ (x, y) \in \Omega = (0, 1)^2,$$

$$u(x, y) = \underbrace{g(x, y)}_{0}, \ (x, y) \text{ auf } \partial\Omega,$$

mit $f(x, y) = 1$ und $g(x, y) = 0$ soll mittels eines Finiten-Differenzen Verfahrens bestimmt werden. Verwenden Sie dazu das regelmäßige Gitter (Meshing)

$$\Omega_h := \{(ih, jh) \mid 1 \leq i, j \leq n - 1)\},$$

closure $\quad \overline{\Omega}_h := \{(ih, jh) \mid 0 \leq i, j \leq n)\},$

mit der Gitterweite $h = 1/n, \ n \in \mathbb{N}$. Zur Diskretisierung des Laplace-Operators verwenden Sie die in der Vorlesung vorgestellte Differenzenformel mit zentralen Differenzen. Für die Diskretisierung der x-Ableitung des Konvektionsterms kann ein zentraler Differenzenquotient,

b) $$\cos\beta \frac{\partial u}{\partial x}(x, y) \approx \frac{\cos\beta}{2h} \left(u(x + h, y) - u(x - h, y)\right),$$

oder ein Upwind-Differenzenquotient

a) $$\cos\beta \frac{\partial u}{\partial x}(x, y) \approx \begin{cases} \frac{\cos\beta}{h} \left(u(x, y) - u(x - h, y)\right), & \cos\beta \geq 0, \\ \frac{\cos\beta}{h} \left(u(x + h, y) - u(x, y)\right), & \cos\beta < 0, \end{cases}$$

verwendet werden. Analog geht man bei der Diskretisierung der y-Ableitung vor.

a) Lösen Sie das lineare Gleichungssystem mit Hilfe der `matlab` Routine "\" für Ihre dünnbesetzte Matrix. Bestimmen Sie die Approximationen $u_{h,\varepsilon}$ von u für $n = 60$ und $\varepsilon = 1, 10^{-2}, 10^{-4}$, indem Sie einen *Upwind-Differenzenquotienten* zur Diskretisierung des Konvektionsterms verwenden. Dabei sei $\beta = 5\pi/6$. Plotten Sie die Lösung $u_{h,\varepsilon}$ mittels der `matlab` Funktion `surf`.

b) Verwenden Sie anstatt des *Upwind-Differenzenquotienten* nun den *zentralen Differenzenquotienten*. Vergleichen Sie die Lösung mit den Ergebnissen aus a).

$A_h u_h = f_h$

unknown

$u_h = A_h^{-1} f_h$

Ω_h

$u_h = A_h \backslash f_h$

b) is exactly like a) but using the central method

$\cos \frac{5\pi}{6} < 0$

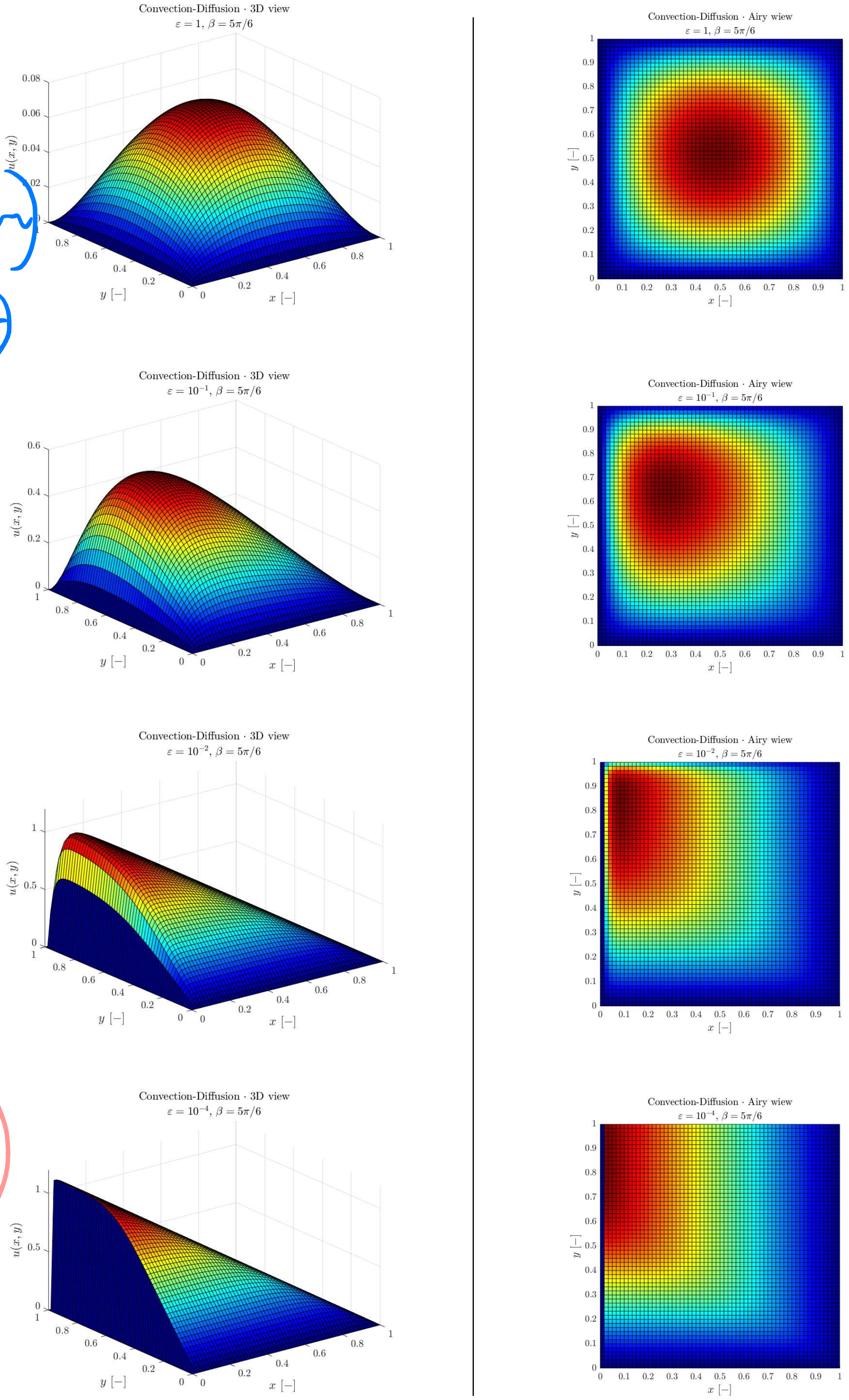
$\varepsilon = 1$ (strong diffusion) (dominantly)

$\varepsilon = 10^{-1}$

$\varepsilon = 10^{-2}$

(convection is dominant) $\varepsilon = 10^{-4}$

Case a)



✓ Still good

Figure 1: Comparison among different values of $\varepsilon = [1, 10^{-1}, 10^{-2}, 10^{-4}]$: Upwind differences used for discretizing convection terms $\partial_x u$ and $\partial_y u$; $\boxed{\text{fixed } \beta = 5\pi/6.}$

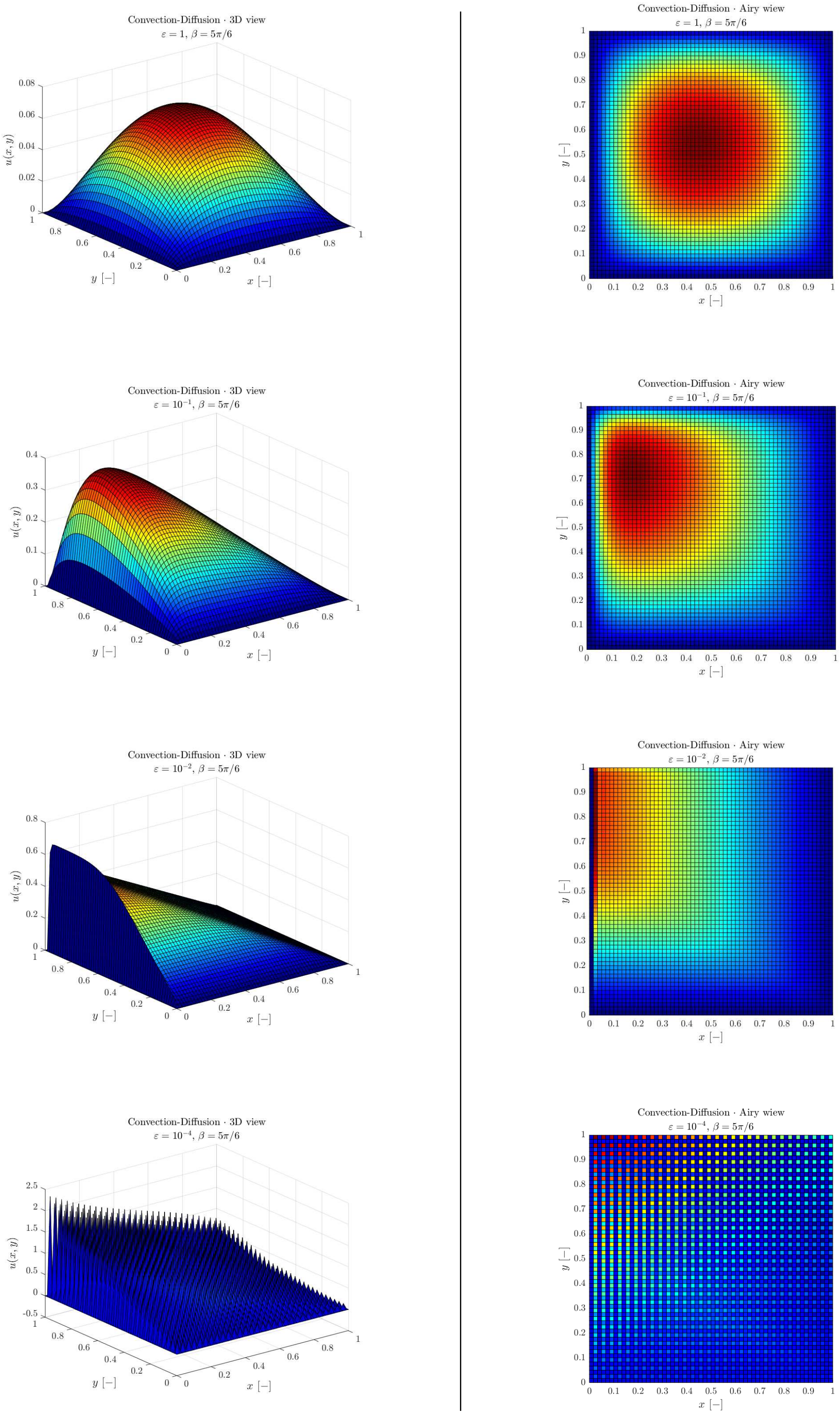
$\varepsilon = 1$

$\varepsilon = 10^{-1}$

$\varepsilon = 10^{-2}$

$\varepsilon = 10^{-4}$

Case b)



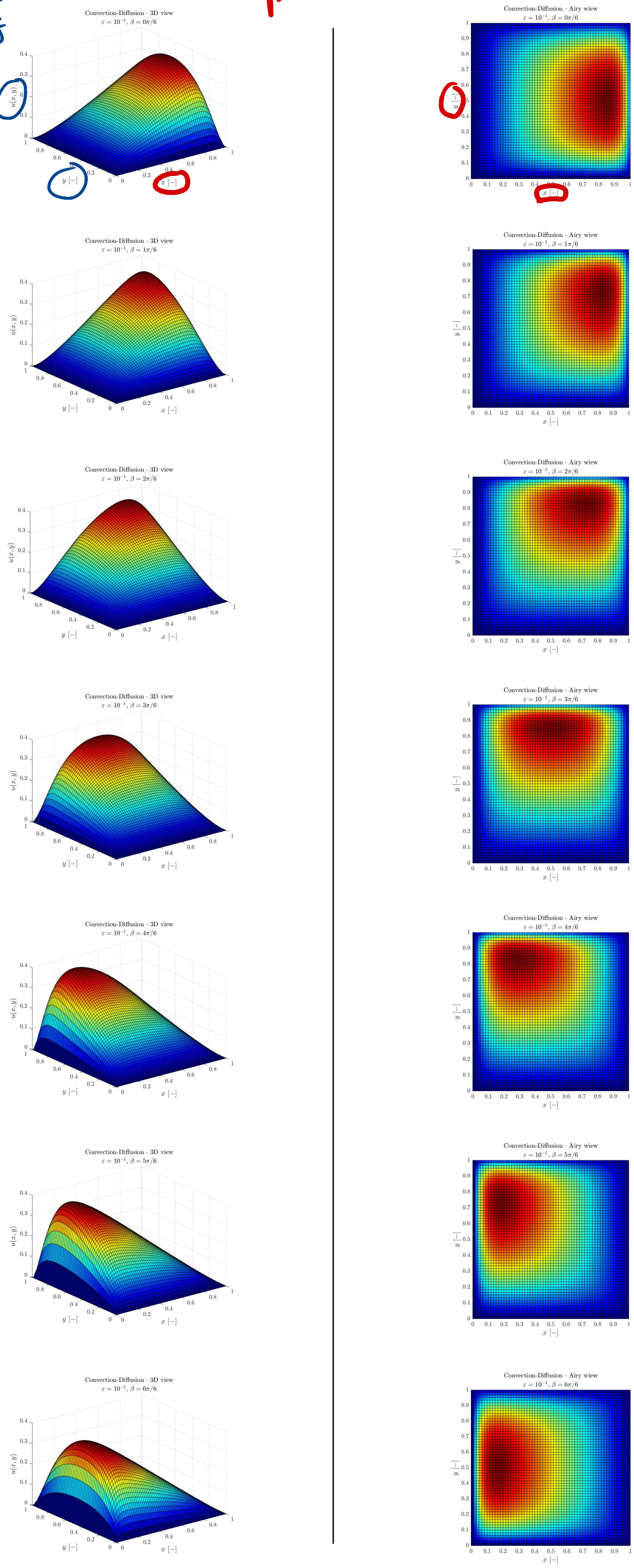
Not good any more

Figure 2: Comparison among different values of $\varepsilon = [1, 10^{-1}, 10^{-2}, 10^{-4}]$: Central difference used for discretizing convection terms $\partial_x u$ and $\partial_y u$; fixed $\boxed{\beta = 5\pi/6}$.

Convection: $\cos\beta \frac{\partial u}{\partial x} + \cancel{\sin\beta \frac{\partial u}{\partial y}}$

$\beta = 0$

@MATLAB surf



$\beta = \frac{0\pi}{6} = 0$

Strong in x-direction

$\beta = \frac{1\pi}{6}$

$\beta = \frac{2\pi}{6} = \frac{\pi}{3}$

$\beta = \frac{3\pi}{6} = \frac{\pi}{2}$

$\beta = \frac{4\pi}{6} = \frac{2\pi}{3}$

$\beta = \frac{5\pi}{6}$

$\beta = \frac{6\pi}{6} = \pi$

Convection

Figure 3: Comparison among different values of $\beta = \left[0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}, \pi\right]$: Upwind differences used for discretizing convection terms $\partial_x u$ and $\partial_y u$; fixed $\varepsilon = 10^{-1}$.

2 Analysis: 2D Eigenfuctions of Laplace operator

Recall 1.

→ We know how to expand $f(x)$ into EF

- case of BC: $u(x) = 0$ ✓ HW9, Ex1
- case of BC: $u(x) = g(x)$ ✓ HW9, Ex2

→ How about in 2D?

Example 2. *Examine the following problem where the squared unit domain defined as* $\Omega = [0,1]^2$ *and the function* $f(x,y)$ *given as follows*

$$\boxed{f(x,y) = xy(1-x)(1-y).}$$

Expand the given function $f(x,y)$ *to eigenfunctions derived from*

$$\begin{cases} -\Delta\phi = \lambda\phi, \ \ in\ \Omega, \\ \phi = 0, \ \ on\ \partial\Omega. \end{cases}$$

@ 2D

@ 1D: $\boxed{\phi_k(x) = C_1 \sin(k\pi x)}$

2D $\boxed{\text{How ?}}$

Approach: The Eigenfunctions take the form

$$\boxed{\phi_{j,k}(x,y) = C_1 \sin(j\pi x)\sin(k\pi y).}$$

Normalization of eigenfunctions goes as follows

$$\boxed{\begin{aligned} ||\phi_{j,k}(x,y)||_2^2 \stackrel{!}{=} 1 &\Leftrightarrow \int_\Omega \phi_{j,k}^2(x,y)\, d\Omega \stackrel{!}{=} 1 \\ &\Leftrightarrow \int_0^1\int_0^1 C_1^2 \sin^2(j\pi x)\sin^2(k\pi y)\ dxdy \stackrel{!}{=} 1 \Rightarrow C_1 = 2. \end{aligned}}$$

Derivation of eigenfunctions $\phi_{j,k}(x,y)$

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega \quad (*) \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

$$(*) \Leftrightarrow -\nabla\cdot\nabla\phi = \lambda\phi \Leftrightarrow \begin{pmatrix} -\partial_x & -\partial_y \end{pmatrix}\cdot\begin{pmatrix} \partial_x\phi \\ \partial_y\phi \end{pmatrix} = \lambda\phi$$

$$\Leftrightarrow -\partial_{xx}\phi - \partial_{yy}\phi = \lambda\phi \quad (**)$$

(Separation of variables)

$$\phi(x,y) = X(x)\,Y(y)$$

$$\begin{cases} \partial_{xx}\phi = X''(x)\,Y(y) \\ \partial_{yy}\phi = X(x)\,Y''(y) \end{cases}$$

$$(*)/(**) \Leftrightarrow -X''(x)\,Y(y) - X(x)\,Y''(y) = \lambda X(x)\,Y(y)$$

$$\Leftrightarrow \underbrace{-\frac{X''(x)}{X(x)}}_{\alpha} - \underbrace{\frac{Y''(y)}{Y(y)}}_{\beta} = \lambda$$

$$\boxed{\alpha + \beta = \lambda}$$

Ansatz

$$X(x) = e^{rx}$$

$$X''(x) = e^{rx}r^2$$

$$-\frac{X''(x)}{X(x)} = \alpha \Leftrightarrow +X''(x) + \alpha X(x) = 0$$

$$\Leftrightarrow r^2 e^{rx} + \alpha e^{rx} = 0$$

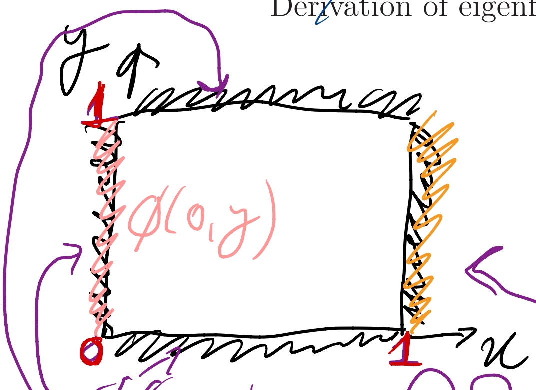
$$\Leftrightarrow \underbrace{e^{rx}}_{\neq 0}(r^2+\alpha) = 0 \Leftrightarrow \boxed{r = \pm i\sqrt{\alpha}}$$

$$X(x) = Ae^{i\sqrt{\alpha}x} + Be^{-i\sqrt{\alpha}x}$$

Euler formular

$$X(x) = \tilde{A}\cos(\sqrt{\alpha}\,x) + \tilde{B}\sin(\sqrt{\alpha}\,x)$$

Derivation of eigenfunctions $\phi_{j,k}(x,y)$ (cont.)



$\phi(x,y) = 0$ on $\partial\Omega$

$$\phi(x,y) = X(x)\,Y(y)$$

$$\phi(0,y) = X(0)\,Y(y) \overset{!}{=} 0 \Rightarrow \boxed{X(0) = 0}$$

$$\phi(1,y) = X(1)\,Y(y) \overset{!}{=} 0 \quad \boxed{X(1) = 0}$$

$$X(0) = 0 \Leftrightarrow \tilde{A}\cos(0) + \tilde{B}\cdot 0 = 0 \Rightarrow \tilde{A} = 0$$

$$\boxed{X(x) = \tilde{B}\sin(\sqrt{\alpha}\,x)} \Rightarrow X(1) = 0 \Leftrightarrow \tilde{B}\sin(\sqrt{\alpha}) = 0$$

$$\Rightarrow \sqrt{\alpha} = j\pi \Rightarrow \alpha = (j\pi)^2$$

Similarly for $\boxed{Y(y) = \tilde{D}\sin(\sqrt{\beta}\,y)} \Rightarrow \sqrt{\beta} = k\pi \Rightarrow \beta = (k\pi)^2$

$$\boxed{\phi(x,y) = c_1 \sin(j\pi x)\,\sin(k\pi y)} \qquad \lambda = \alpha + \beta = \pi^2(j^2 + k^2)$$

Not yet normalized

look at page 5/16 $\Rightarrow$ $c_1 \overset{!}{=} 2$

Eigen function for 2D case

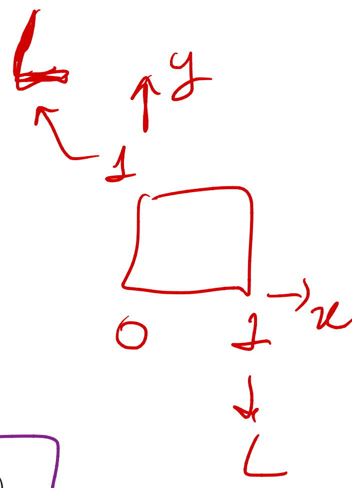
Coefficient

Expansion

$$\boxed{f(x,y) = xy(1-x)(1-y)} \stackrel{!}{=} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \alpha_{j,k}\phi_{j,k}(x,y)$$

EF@2D

where the coefficient $\alpha_{j,k}$ is computed as follows



$$\begin{aligned}
\alpha_{j,k} &= \int_{\Omega} f(x,y)\phi_{j,k}\, d\Omega \\
&= \int_0^1\int_0^1 xy(1-x)(1-y)\, 2\sin(j\pi x)\sin(k\pi y)\; dxdy \\
&= 2\int_0^1 x(1-x)\sin(j\pi x)\; dx \int_0^1 y(1-y)\sin(k\pi y)\; dy
\end{aligned}$$

$\langle f, \phi_{j,k} \rangle$

which is computed firstly w.r.t. dx step-by-step as follows

$$\int_0^1 x(1-x)\sin(j\pi x)\; dx = \frac{2}{j^3\pi^3}(1-\cos(j\pi)) - \frac{1}{j^2\pi^2}\sin(j\pi)$$

The term goes with sinus function gets vanished, while the term goes with cosinus function takes the form $\cos(j\pi) = (-1)^j$. Similarly, it goes the same for dy. Finally, we obtain

$$\begin{aligned}
\alpha_{j,k} &= 2\frac{2}{j^3\pi^3}\left(1-(-1)^j\right)\frac{2}{k^3\pi^3}\left(1-(-1)^k\right) \\
&= \begin{cases} \dfrac{32}{j^3k^3\pi^6}, & \text{if } j,k \text{ odd}, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

which leads to the final form of expansion of $f(x,y)$ as follows

$$\therefore \quad \boxed{f(x,y) = \sum_{j,k \text{ odd}} \frac{32}{j^3k^3\pi^6} 2\sin(j\pi x)\sin(k\pi y).}$$

EF@2D

3 Analysis: Distributional derivative

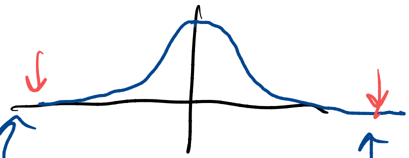
in the sense of distribution

Example 3. *Examine the following problem*

$\to \phi \quad [1,2]$ $\quad |x|$

$$f: \begin{cases}\mathbb{R}\to\mathbb{R},\\ x\mapsto f(x)=\boxed{|x+1|}.\end{cases}$$

$$= \begin{cases} x+1, & x\geq -1\\ -x-1, & x<-1\end{cases}$$



$\boxed{\phi \in C_0^\infty(\mathbb{R})}$

$\phi(\infty)=0$

$\phi(-\infty)=0$

compact support

$C_c^\infty \quad C_0^\infty$

(Derivation)

$$T_f'(\phi) = \langle f',\phi\rangle = \int_{\mathbb{R}} f'(x)\phi(x)\,dx$$

Integration by part: $= \underbrace{\left[f(x)\phi(x)\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,dx$

$$= -\int_{\mathbb{R}} f(x)\phi'(x)\,dx = -\langle f,\phi'\rangle$$

$$T_f'(\phi)=\langle f',\phi\rangle = -\langle f,\phi'\rangle$$

$$= -\int_{\mathbb{R}} f(x)\phi'(x)\,dx = -\int_{-\infty}^{-1}(-x-1)\phi'(x)\,dx - \int_{-1}^{+\infty}(x+1)\phi'(x)\,dx$$

$$= \int_{-\infty}^{-1}(x+1)\phi'(x)\,dx - \int_{-1}^{+\infty}(x+1)\phi'(x)\,dx$$

$$= \int_{-\infty}^{-1} x\phi'(x)\,dx + \int_{-\infty}^{-1}\phi'(x)\,dx - \int_{-1}^{+\infty} x\phi'(x)\,dx - \int_{-1}^{+\infty}\phi'(x)\,dx$$

Integration by part Integration by part

$$= \left[x\phi(x)\right]_{-\infty}^{-1} - \int_{-\infty}^{-1}(+1)\phi(x)\,dx + \phi(x)\Big|_{-\infty}^{-1} - \left[x\phi(x)\right]_{-1}^{+\infty} + \int_{-1}^{+\infty}(+1)\phi(x)\,dx - \phi(x)\Big|_{-1}^{+\infty}$$

$$= \cancel{-\phi(-1)} - \int_{-\infty}^{-1}(+1)\phi(x)\,dx + \cancel{\phi(-1)} - \cancel{\phi(-1)} + \int_{-1}^{+\infty}(+1)\phi(x)\,dx + \cancel{\phi(-1)}$$

$$= -\int_{-\infty}^{-1}(+1)\phi(x)\,dx - \int_{-1}^{+\infty}(-1)\phi'(x)\,dx = -\int_{\mathbb{R}} f'(x)\phi'(x)\,dx$$

$$\boxed{f'(x)=\begin{cases}+1 & x\geq -1\\ -1 & x<-1\end{cases}}$$

$\therefore$ therefore $\therefore$

4 Numerics: Ghost points in FDM

points which are not existing on domain Ω

Example 4. *Examine the following FDM problem:*

$$u''(x) - u'(x) + u(x) = -x^2 + 2x - 1, \quad \textit{for} \;\; x \in \Omega = (0,1),$$

where the boundary conditions are defined as follows

Dirichlet BC

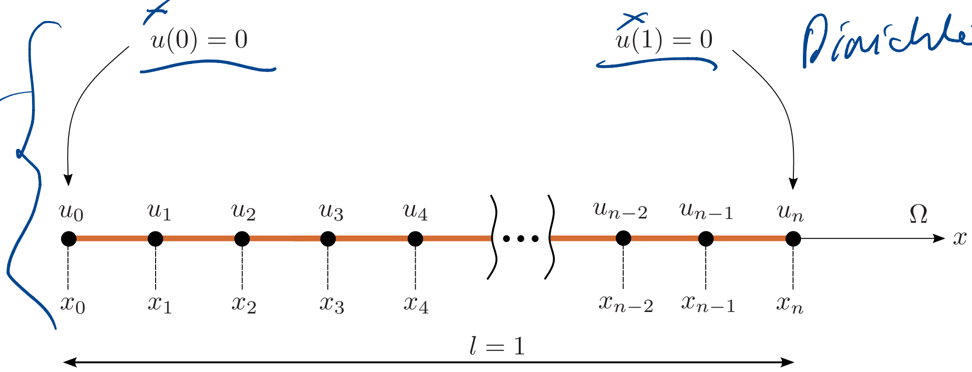


Figure 4: Discretization of domain Ω with $n+1$ points: From x_0 to x_n.

$$u(0) = 0, \quad u(1) = 0.$$

There is No ghost point in this problem

Determine the resulting system of linear equations $A_h u_h = b_h$ by using second order FDM scheme for both $u''(x)$ and $u'(x)$.

Approach: This problem exists no ghost point 👻

The second order FDM scheme for both $u''(x)$ and $u'(x)$ take the following forms

$$\boxed{\begin{aligned} u''(x) &\approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \\ u'(x) &\approx \frac{u(x+h) - u(x-h)}{2h}, \end{aligned}}$$

We think about how to insert the given BC into our system

$$\boxed{A_h u_h = f_h}$$

$$u_h = A_h^{-1} f_h$$

Example 5. *Examine the following FDM problem:*

$$u''(x) - u'(x) + u(x) = -x^2 + 2x - 1, \quad \text{for} \quad x \in \Omega = (0, 1),$$

where the boundary conditions are defined as follows

Neumann BC

$u(0) = 0$

Dirichlet BC

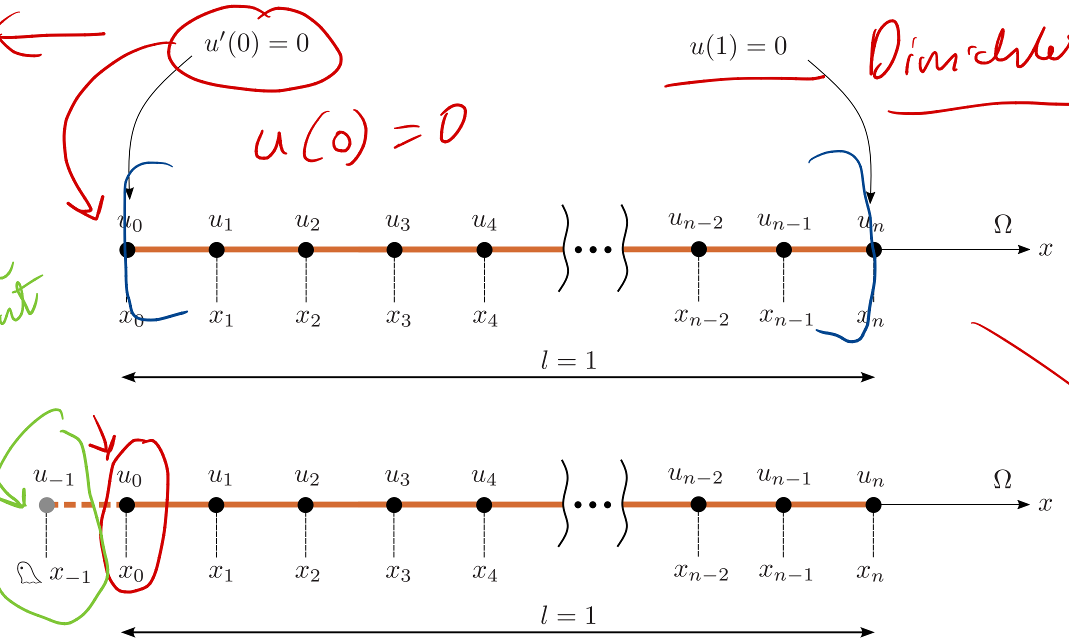


Figure 5: Discretization of domain Ω with $n+1$ points: From x_0 to x_n. Ghost point x_{n+1} is located on the left boundary $\partial\Omega_L$.

$$u'(0) = 0, \quad u(1) = 0.$$

Determine the resulting system of linear equations $A_h u_h = b_h$ by using second order FDM scheme for both $u''(x)$ and $u'(x)$.

Approach: This problem exists a ghost point on the LHS boundary $\partial\Omega_L$

The second order FDM scheme for both $u''(x)$ and $u'(x)$ take the following forms

$$\boxed{\begin{aligned} u''(x) &\approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \\ u'(x) &\approx \frac{u(x+h) - u(x-h)}{2h}, \end{aligned}}$$

2nd order

$$u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u''(x) + \frac{h^3}{6} u'''(x) + \frac{h^4}{24} u^{(4)}(x) + O(h^5)$$

$$u(x-h) = u(x) - h u'(x) + \frac{h^2}{2} u''(x) - \frac{h^3}{6} u'''(x) + \frac{h^4}{24} u^{(4)}(x) + O(h^5)$$

Summation

$$u(x+h) + u(x-h) = 2u(x) + h^2 u''(x) + O(h^4)$$

$$\Rightarrow \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + \frac{O(h^4)}{h^2}$$

at least order 2nd $O(h^2)$

Central method

Subtraction $u(x+h) - u(x-h) = 2h\,u'(x) + O(h^3)$

(**) $\dfrac{u(x+h) - u(x-h)}{2h} = \boxed{u'(x)} + \dfrac{O(h^3)}{2h}$ at least order 2 $O(h^2)$

Derivation the system of equations

$$\boxed{(2+h)u_{n-1} + (2h^2-4)u_n + (2-h)u_{n+1} = 2h^2(2nh-1-(nh)^2)} \quad (☆)$$

$-1 \qquad 0 \qquad 1$

$$\underbrace{\begin{bmatrix} \end{bmatrix}}_{A_h} \underbrace{\begin{bmatrix} \end{bmatrix}}_{u_h} = \underbrace{\begin{bmatrix} \end{bmatrix}}_{f_h}$$

2. We now take into consideration of BCs

LHS $u'(0) = 0$ b/c of the given boundary condition

approx. $\simeq$

u_0: $\dfrac{u_1 - u_{-1}}{2h} = 0$

$$\boxed{u_{-1} = u_1}$$

$$(☆): \boxed{(2h^2-4)u_0 + 4u_1 = -2h^2}$$

RHS $u(1) = 0$

No term is specified

just Dirichlet

Derivation the system of equations (cont.)

Summary

BC BC

0 1 N-1 N

1. Discretization for $n = 1, .., N-1$

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - \frac{u_{n+1} - u_{n-1}}{2h} + u_n = 2x_n - 1 - x_n^2,$$

which leads to the following grouped form

$$\boxed{(2+h)u_{n-1} + (2h^2 - 4)u_n + (2-h)u_{n+1} = 2h^2\left(2nh - 1 - (nh)^2\right).}$$

2. Left boundary exists **ghost** point u_{-1} taking the following expression

$$u'(0) = 0 \quad \Rightarrow \quad \frac{u_1 - u_{-1}}{2h} = 0 \quad \Rightarrow \quad u_1 = u_{-1}$$

which yields the first equation for the case $n = 0$ as follows

$$\boxed{(2h^2 - 4)u_0 + 4u_1 = -2h^2.}$$

from here

3. The right boundary where $u(1) = 0$, meaning $u_N = 0$, is for the case of $n = N-1$

$$\boxed{(2+h)u_{N-2} + (2h^2 - 4)u_{N-1} = 2h^2(2(1-h) - 1 - (1-h)^2).}$$

4. The final system of equations takes the following form

$$\boxed{\therefore \quad A_h = \begin{pmatrix} 2h^2-4 & 4 & & & \\ 2+h & 2h^2-4 & 2-h & & \\ & 2+h & 2h^2-4 & 2-h & \\ & & \ddots & \ddots & \ddots \\ & & & 2+h & 2h^2-4 \end{pmatrix}_{N\times N},}$$

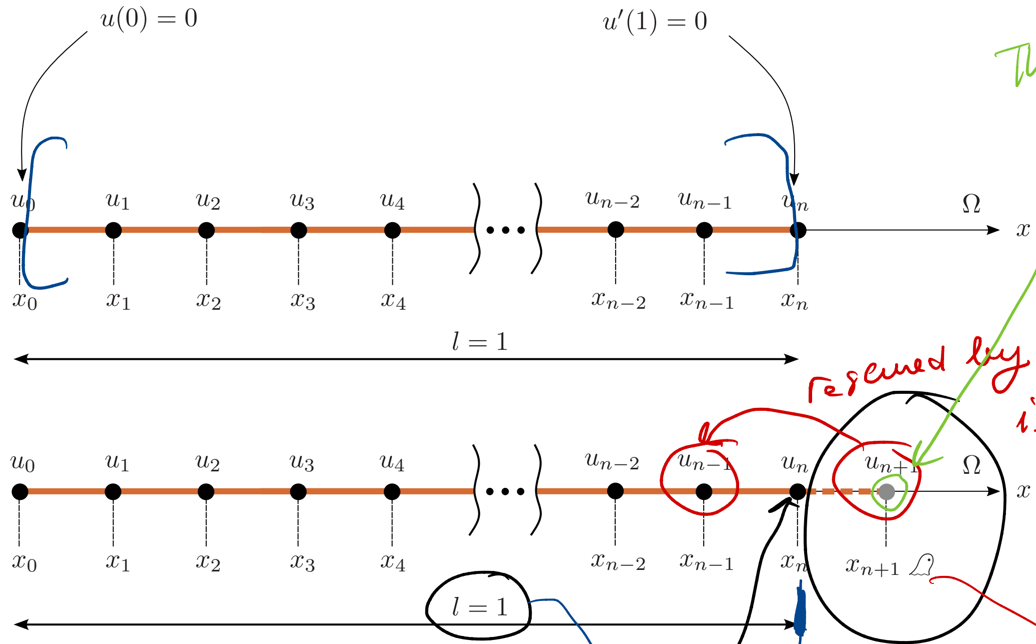
$$\boxed{\therefore \quad u_h = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}_{N\times 1}, \quad b_h = 2h^2 \begin{pmatrix} -1 \\ 2h - 1 - h^2 \\ 4h - 1 - (2h)^2 \\ \vdots \\ 2(N-1)h - 1 - ((N-1)h)^2 \end{pmatrix}_{N\times 1}.}$$

f_h RHS

Example 6. *Examine the following FDM problem:*

$$u''(x) - u'(x) + u(x) = -x^2 + 2x - 1, \quad \textit{for} \quad x \in \Omega = (0,1),$$

where the boundary conditions are defined as follows



This is the ghost point

represented by u_{n-1} i.e $u_{n-1} = u_{n+1}$

$u_{-1} = u_1$

Figure 6: Discretization of domain Ω with $n+1$ points: From x_0 to x_n. Ghost point x_{n+1} is located on the right boundary $\partial\Omega_R$.

$$\boxed{u(0) = 0, \quad u'(1) = 0.}$$

Determine the resulting system of linear equations $A_h u_h = b_h$ by using second order FDM scheme for both $u''(x)$ and $u'(x)$.

Approach: This problem exists a ghost point 🕭 on the RHS boundary $\partial\Omega_R$.

A_n The second order FDM scheme for both $u''(x)$ and $u'(x)$ take the following forms

2 sym.

$$\boxed{\begin{aligned} u''(x) &\approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \\ u'(x) &\approx \frac{u(x+h) - u(x-h)}{2h}, \end{aligned}}$$

sysm

5 Evaluation

Figure 7: QR Code for evaluation of Global exercises.

Alternative link: here