

Global Exercise - Gue10

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Content covered:

✓ Programming exercise 02: Explanation.

✓ Analysis:

* Distributional derivative.

✓ Numerics:

1 Programming exercise 02: Explanation

Example 1. *Examine the following Convection-diffusion problem*

$$\begin{aligned} -\varepsilon \Delta u(x, y) + \cos \beta \frac{\partial u}{\partial x}(x, y) + \sin \beta \frac{\partial u}{\partial y} &= f(x, y), \quad (x, y) \in \Omega = (0, 1)^2, \\ u(x, y) &= g(x, y), \quad (x, y) \text{ on } \partial\Omega, \end{aligned}$$

2 Review Example 3 · Gue08

Approach:

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}(h^3) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \mathcal{O}((2h)^3) \end{cases} \quad (1)$$

which leads to

$$\alpha f(x) = \alpha f(x) \quad (2)$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}(h^3) \quad (3)$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}((2h)^3) \quad (4)$$

Summation of (2), (3) and (4) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= (\alpha + \beta + \gamma) f(x) \\ &\quad + (\beta h + 2\gamma h) f'(x) \\ &\quad + \left(\frac{\beta h^2}{2} + 2\gamma h^2 \right) f''(x) \\ &\quad + \beta \mathcal{O}(h^3) \\ &\quad + \gamma \mathcal{O}((2h)^3) \end{aligned} \quad (5)$$

Since we would like to approximate $f'(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f'(x) \quad (6)$$

Comparison (6) with (5) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \stackrel{!}{=} 0 \\ \beta h + 2\gamma h \stackrel{!}{=} 1 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \stackrel{!}{=} 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \quad (7)$$

which leads the expression (5) to the following equality

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= f'(x) + \beta \mathcal{O}(h^3) + \gamma \mathcal{O}((2h)^3) \\ &= f'(x) + \frac{2}{h} \mathcal{O}(h^3) + \frac{-1}{2h} \mathcal{O}((2h)^3) \\ &= f'(x) + \mathcal{O}(h^2). \end{aligned} \quad (8)$$

Therefore, $f'(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (7), with a consistency order of at least order 2, as shown in (8).

2. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach from $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns for 4 equations. Therefore, the highest consistency order we may obtain is of order 2.