Global Exercise - Gue08

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Content covered:

✓ Analysis: Eigenvalues/Eigenfunction of Laplace operators. How to

* treat the inhomogeneous problem $\partial_t u(x,t) = \partial_{xx} u(x,t)$ with $u(L,t) \neq 0$.
* treat the $\partial_{tt} u(x,t)$ term in $\partial_{tt} u(x,t) = \partial_{xx} u(x,t) + a$.

✓ Numerics: FDM consistency order.

1 Analysis

Example 1. *Examine the following problem:*

$$\partial_t u(x,t) = \partial_{xx} u(x,t), \quad \textit{for } x \in \Omega = (0,L), \textit{ and } t > 0, \tag{1}$$

where the boundary conditions (BCs) at the left-most and right-most points for all time t with a constant $\mu \in \mathbb{R}$ are defined, respectively, as follows

$$\begin{cases} u(0,t) &= 0, \\ u(L,t) &= \mu t \exp(-\mu t), \end{cases} \tag{2}$$

and the initial condition (IC) with a constant $u_0 \in \mathbb{R}$ is given as

$$u(x,0) = u_0 \; x(L-x). \tag{3}$$

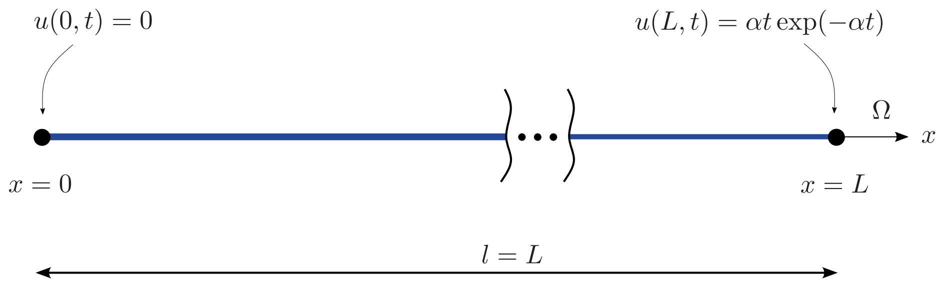

Figure 1: Domain Ω with given BCs.

Approach:

1. Homogenization of the BCs: We search for $u_{inh}(x,t)$ such that

$$\partial_{xx} u_{inh} = 0, \quad \text{with BCs: } \begin{cases} u_{inh}(0,t) = 0, \\ u_{inh}(L,t) = \mu t \exp(-\mu t), \end{cases} \tag{4}$$

which leads to

$$\boxed{u_{inh}(x,t) = \frac{x}{L}\mu t \exp(-\mu t).} \tag{5}$$

Likewise, we also obtain the following expression

$$\partial_t u_{inh} = \frac{x}{L}\mu \exp(-\mu t) - \frac{x}{L}\mu^2 t \exp(-\mu t) = \frac{x}{L}(1-\mu t)\mu \exp(-\mu t) \tag{6}$$

Note in passing that we have used $u_{inh}(x,t) = ax + b, \forall a, b \in \mathbb{R}$ to probe the solution mentioned at (4), since it satisfies $\partial_{xx}(ax+b) = 0$. The solution (5) is obtained by considering BCs given in (4).

2. We define

$$\tilde{u}(x,t) := u(x,t) - u_{inh}(x,t) \Leftrightarrow u(x,t) = \tilde{u}(x,t) + u_{inh}(x,t) \tag{7}$$

By substituting (7) into (1) we obtain the following equality

$$\begin{aligned}
\partial_t(\tilde{u} + u_{inh}) = \partial_{xx}(\tilde{u} + u_{inh}) &\Leftrightarrow \partial_t\tilde{u} - \partial_{xx}\tilde{u} = \cancelto{0;\,\text{cf.}(4)}{\partial_{xx}u_{inh}} - \partial_t u_{inh} \\
&\Leftrightarrow \partial_t\tilde{u} - \partial_{xx}\tilde{u} = -\partial_t u_{inh} \\
&\Leftrightarrow \partial_t\tilde{u} - \partial_{xx}\tilde{u} \overset{(6)}{=} -\frac{x}{L}(1-\mu t)\mu \exp(-\mu t)
\end{aligned} \tag{8}$$

Then, the homogenized problem reads

$$\boxed{\partial_t\tilde{u}(x,t) - \partial_{xx}\tilde{u}(x,t) = \tilde{f}(x,t), \quad \text{with } \begin{cases} \tilde{u}(0,t) = \tilde{u}(L,t) = 0, \\ \tilde{u}(x,0) = u_0\ x(L-x), \end{cases}} \tag{9}$$

where the source term $\tilde{f}(x,t)$ defined as $\tilde{f}(x,t) := -\frac{x}{L}(1-\mu t)\mu \exp(-\mu)$.

3. We now need to find out the Eigenfunctions $\varphi(x)$ satisfying

$$-\partial_{xx}\varphi(x) = \lambda\varphi(x), \quad \text{with } \varphi(0) = \varphi(L) = 0. \tag{10}$$

By taking into consideration the Ansatz

$$\begin{cases} \varphi(x) = \exp(rx) \\ \varphi''(x) = r^2 \exp(rx) \end{cases} \tag{11}$$

We then obtain the following expressions

$$(10) \overset{(11)}{\Leftrightarrow} \exp(rx)(r^2 + \lambda) = 0 \Leftrightarrow r = \pm i\sqrt{\lambda} \tag{12}$$

The solution to (10) $\forall A, B, C_1, C_2 \in \mathbb{R}$ takes the following form

$$\begin{aligned}
\varphi(x) &= A \exp\left(i\sqrt{\lambda}x\right) + B \exp\left(-i\sqrt{\lambda}x\right) \\
&= A\left(\cos\left(\sqrt{\lambda}x\right) + i\sin\left(\sqrt{\lambda}x\right)\right) + B\left(\cos\left(\sqrt{\lambda}x\right) - i\sin\left(\sqrt{\lambda}x\right)\right) \\
&= (A+B)\cos\left(\sqrt{\lambda}x\right) + (A-B)i\sin\left(\sqrt{\lambda}x\right) \\
&= C_1\cos\left(\sqrt{\lambda}x\right) + iC_2\sin\left(\sqrt{\lambda}x\right)
\end{aligned} \tag{13}$$

Note in passing that Euler's formula $\exp(ix) = \cos(x) + i\sin(x)$ has been used in (13). Next, by applying BCs from (10) we obtain the explicit values for C_1 and eigenvalues λ_k for (13), as follows

$$\varphi(0) = 0 \overset{(13)}{\Leftrightarrow} C_1 = 0. \tag{14}$$

$$\varphi(L) = 0 \overset{(13)(14)}{\Leftrightarrow} iC_2 \sin\left(\sqrt{\lambda}L\right) = 0 \Leftrightarrow \lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad \forall\ k \in \mathbb{Z}. \tag{15}$$

Till this point we have obtained so far only **non-normalized** eigenfunctions $\varphi_k(x)$ and their eigenvalues λ_k as follows

$$\begin{cases} \varphi_k(x) = iC_2 \sin\left(\frac{k\pi}{L}x\right), \\ \lambda_k = \left(\frac{k\pi}{L}\right)^2, \end{cases} \quad \forall\ k \in \mathbb{Z}. \tag{16}$$

Next, the eigenfunction $\varphi_k(x)$ in (16) needs to be normalized. It goes as follows

$$\begin{aligned} ||\varphi_k(x)||_2 \overset{!}{=} 1 \Leftrightarrow & \sqrt{\int_0^L (\varphi_k(x))^2 \ dx} = 1 \\ & \overset{(16)}{\Leftrightarrow} \int_0^L \left(iC_2 \sin\left(\frac{k\pi}{L}x\right)\right)^2 \ dx = 1 \\ & \Leftrightarrow \int_0^L \sin^2\left(\frac{k\pi}{L}x\right) \ dx = \frac{-1}{(C_2)^2} \\ & \Leftrightarrow \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2k\pi}{L}x\right)\right) \ dx = \frac{-1}{(C_2)^2} \\ & \Leftrightarrow \frac{1}{2}x\Big|_{x=0}^{x=L} - \frac{1}{2}\frac{L}{2k\pi}\sin\left(\frac{2k\pi}{L}x\right)\Big|_{x=0}^{x=L} = \frac{-1}{(C_2)^2} \\ & \Leftrightarrow \frac{1}{2}L = \frac{-1}{(C_2)^2} \Leftrightarrow C_2 = \pm i\sqrt{\frac{2}{L}} \end{aligned} \tag{17}$$

- $C_2 = i\sqrt{\frac{2}{L}} \overset{(16)}{\Rightarrow} \varphi_k(x) = -\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{h\pi}{L}x\right) \forall\ k, h \in \mathbb{Z}.$
- $C_2 = -i\sqrt{\frac{2}{L}} \overset{(16)}{\Rightarrow} \varphi_k(x) = +\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) \forall\ k \in \mathbb{Z}.$

Therefore, the **normalized** eigenfunctions $\varphi_k(x)$ and their eigenvalues λ_k take the following forms

$$\therefore \boxed{\begin{aligned} \varphi_k(x) &= \sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right) \\ \lambda_k &= \left(\frac{k\pi}{L}\right)^2, \forall\ k \in \mathbb{Z} \end{aligned}}$$

4. Expand the source term $\tilde{f}(x,t)$ by using the normalized eigenfunction $\varphi_k(x)$. The source term $\tilde{f}(x,t)$ takes the following form

$$\tilde{f}(x,t) = -\frac{x}{L}(1-\mu t)\mu\exp(-\mu t), \tag{18}$$

which can be written in an expansion of eigenfuctions

$$\boxed{\tilde{f}(x,t) = \sum_{k=1}^{\infty}\beta_k(t)\varphi_k(x) = \sum_{k=1}^{\infty}\beta_k(t)\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)} \tag{19}$$

where coefficient $\beta_k(t)$ is computed by projecting $\tilde{f}(x,t)$ on eigenfunctions $\varphi_k(x)$, as follows

$$\begin{aligned}
\beta_k(t) &= \langle \tilde{f}(x,t), \varphi_k(x)\rangle \\
&= \int_0^L \left(-\frac{x}{L}(1-\mu t)\mu\exp(-\mu t)\right)\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)\,dx \\
&= \frac{-(1-\mu t)\mu\exp(-\mu t)}{L}\sqrt{\frac{2}{L}}\int_0^L x\sin\left(\frac{k\pi}{L}x\right)\,dx \\
&= \frac{-(1-\mu t)\mu\exp(-\mu t)}{L}\sqrt{\frac{2}{L}}\frac{(-1)^k}{k\pi}L^2 \\
&= (-1)^k\frac{\sqrt{2L}}{k\pi}(1-\mu t)\mu\exp(-\mu t).
\end{aligned} \tag{20}$$

5. Ansatz for solution $\tilde{u}(x,t)$ take the following form

$$\tilde{u}(x,t) = \sum_{k=1}^{\infty}\alpha_k(t)\varphi_k(x). \tag{21}$$

By substituting (19) and (21) into (9) we obtain

$$\begin{aligned}
&\Leftrightarrow \sum_{k=1}^{\infty}\left(\partial_t\alpha_k(t)\varphi_k(x) - \alpha_k(t)\partial_{xx}\varphi_k(x) - \beta_k(t)\varphi_k(x)\right) = 0 \\
&\overset{(10)}{\Leftrightarrow} \sum_{k=1}^{\infty}\left(\alpha_k'(t)\varphi_k(x) + \alpha_k(t)\lambda_k\varphi_k(x) - \beta_k(t)\varphi_k(x)\right) = 0 \\
&\Leftrightarrow \sum_{k=1}^{\infty}\left(\alpha_k'(t) + \lambda_k\alpha_k(t) - \beta_k(t)\right)\varphi_k(x) = 0.
\end{aligned} \tag{22}$$

Since the eigenfunctions $\varphi_k(x)$ are orthonormal to each other, we induce from (22) and arrive at the following equality

$$\boxed{\alpha_k'(t) + \lambda_k\alpha_k(t) - \beta_k(t) = 0.} \tag{23}$$

This is the 1st order ODE with unknown $\alpha_k(t)$. The solution to $\alpha_k(t)$ takes the form

$$\therefore \quad \boxed{\alpha_k(t) = \alpha_k(0)\exp(-\lambda_k t) + \exp(-\lambda_k t)\int_0^t \exp(\lambda_k\tau)\beta_k(\tau)\,d\tau,}$$

whose derivation goes as follows

- Solving for homogeneous part:

$$\alpha_k'(t) + \lambda_k \alpha_k(t) = 0 \Leftrightarrow \alpha_k^{hom}(t) = \alpha_k(0)\exp(-\lambda_k t). \tag{24}$$

- Inhomogeneous part: By using the method *variation of constant* we exploit the ansatz

$$\alpha_k^{inh}(t) = \alpha_k(t)\exp(-\lambda_k t). \tag{25}$$

Insertion of (25) into (23) leads to

$$\alpha_k'(t)\exp(-\lambda_k t) - \lambda_k\alpha_k(t)\exp(-\lambda_k t) + \lambda_k\alpha_k(t)\exp(-\lambda_k t) - \beta_k(t) = 0$$

where the two terms in the middle have been canceled out with each other. We then obtain the equality

$$\begin{aligned}
&\alpha_k'(t)\exp(-\lambda_k t) = \beta_k(t)\\
&\qquad\Leftrightarrow \alpha_k'(t) = \exp(\lambda_k t)\beta_k(t)\\
&\qquad\Leftrightarrow \alpha_k(t) = \int_0^t \exp(\lambda_k \tau)\beta_k(\tau)\, d\tau\\
&\qquad\Leftrightarrow \alpha_k(t) \overset{(20)}{=} \int_0^t \exp(\lambda_k\tau)(-1)^k\frac{\sqrt{2L}}{k\pi}(1-\mu\tau)\mu\exp(-\mu\tau)\, d\tau\\
&\qquad\Leftrightarrow \alpha_k(t) = (-1)^k\frac{\sqrt{2L}}{k\pi}\mu\int_0^t \exp(\lambda_k\tau)(1-\mu\tau)\exp(-\mu\tau)\, d\tau\\
&\qquad\Leftrightarrow \alpha_k(t) = (-1)^k\frac{\sqrt{2L}}{k\pi}\mu\left(\int_0^t \exp(\lambda_k\tau-\mu\tau)\, d\tau - \mu\int_0^t \tau\exp(\lambda_k\tau-\mu\tau)\, d\tau\right)\\
&\qquad\Leftrightarrow \alpha_k(t) = (-1)^k\frac{\sqrt{2L}}{k\pi}\mu\frac{(\lambda_k-(\lambda_k-\mu)\mu t)\exp(\lambda_k-\mu t)-\lambda_k}{(\lambda_k-\mu)^2}\\
&\qquad\Leftrightarrow \alpha_k(t) = (-1)^k\frac{\sqrt{2L}}{k\pi}\mu\frac{(\lambda_k-(\lambda_k-\mu)\mu t)\exp(-\mu t)-\lambda_k\exp(-\lambda_k t)}{(\lambda_k-\mu)^2}
\end{aligned} \tag{26}$$

6. Find $\alpha_k(0)$ by taking into consideration the IC of $\tilde{u}(x,0)$, as follows

$$\tilde{u}(x,0) = \sum_{k=1}^{\infty}\alpha_k(0)\varphi_k(x) \overset{(9)}{\Leftrightarrow} u_0\, x(L-x) \overset{!}{=} \sum_{k=1}^{\infty}\alpha_k(0)\varphi_k(x) \tag{27}$$

where the coefficient $\alpha_k(0)$ is computed as follows

$$\begin{aligned}
\alpha_k(0) &= \langle \tilde{u}(x,0), \varphi_k(x)\rangle\\
&= \int_0^L u_0\, x(L-x)\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)\, dx = \begin{cases} u_0\sqrt{2L}\frac{4L^2}{(k\pi)^3}, & k \text{ odd},\\ 0, & k \text{ even}.\end{cases}
\end{aligned} \tag{28}$$

The solution to $\tilde{u}(x,t)$ then reads

$$\boxed{\begin{aligned}\tilde{u}(x,t) &= \sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x) \\ &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{8u_0L^2}{(k\pi)^3} \exp(-\lambda_k t) \sin\left(\frac{k\pi}{L}x\right) \\ &\quad + \sum_{k=1}^{\infty} (-1)^k \frac{2a}{k\pi} \frac{(\lambda_k - (\lambda_k - \mu)\mu t)\exp(-\mu t) - \lambda_k \exp(-\lambda_k t)}{(\lambda_k - \mu)^2} \sin\left(\frac{k\pi}{L}x\right)\end{aligned}} \tag{29}$$

7. Finally, by substituting (6) and (29) into (7) the solution to $u(x,t)$ reads

$$\therefore \quad \boxed{u(x,t) = \tilde{u}(x,t) + \frac{x}{L}\mu t \exp(-\mu t).}$$

Example 2. *Examine the term $\partial_{tt}u(x,t)$ in the following problem*

$$\partial_{tt}u(x,t) = \partial_{xx}u(x,t) + a, \quad \textit{for } x \in \Omega = (0,1),\ t > 0. \tag{30}$$

Similarly to steps approaching to (23), which is shown in Example 1, we obtain a 2nd order ODE, instead of the 1st order ODE, as follows

$$\therefore \quad \boxed{\alpha_k''(t) + \lambda_k \alpha_k(t) - \beta_k = 0.} \tag{31}$$

Essentially, the equation (31) has the form

$$y''(x) = -cy(x) + b, \quad \forall\ c, b \in \mathbb{R}, \tag{32}$$

whose solution takes the following form

$$y(x) = \frac{b}{c} + C_1 \sin\left(\sqrt{c}x\right) + C_2 \cos\left(\sqrt{c}x\right), \quad \forall\ C_1, C_2 \in \mathbb{R}. \tag{33}$$

2 Numerics: FDM consistency order

Example 3. *Examine the following approximation problem for $f'(x)$ given as follows*

$$f'(x) \approx \alpha f(x) + \beta f(x+h) + \gamma f(x+2h), \tag{34}$$

where constants $\alpha, \beta, \gamma \in \mathbb{R}$. Find α, β, γ such that the FDM used to approximate f at point x gains consistency order as high as possible.

Approach: By using Taylor expansion for $f(x+h)$ and $f(x+2h)$ we obtain the following expressions

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \dfrac{h^1}{1!} f'(x) + \dfrac{h^2}{2!} f''(x) + \dfrac{h^3}{3!} f'''(x) + \mathcal{O}(h^4) \\ f(x+2h) = f(x) + \dfrac{(2h)^1}{1!} f'(x) + \dfrac{(2h)^2}{2!} f''(x) + \dfrac{(2h)^3}{3!} f'''(x) + \mathcal{O}(h^4) \end{cases} \tag{35}$$

which leads to

$$\alpha f(x) = \alpha f(x) \tag{36}$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \frac{\beta h^3}{6} f'''(x) + \mathcal{O}(h^4) \tag{37}$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \frac{4\gamma h^3}{3} f'''(x) + \mathcal{O}(h^4) \tag{38}$$

Summation of (36), (37) and (38) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) = {} & (\alpha + \beta + \gamma) f(x) \\ & + (\beta h + 2\gamma h) f'(x) \\ & + \left(\frac{\beta h^2}{2} + 2\gamma h^2 \right) f''(x) \\ & + \left(\frac{\beta h^3}{6} + \frac{4\gamma h^3}{3} \right) f'''(x) + \mathcal{O}(h^4) \end{aligned} \tag{39}$$

Comparison of (39) with (34) yields the following equalities

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \beta h + 2\gamma h = 1 \\ \dfrac{\beta h^2}{2} + 2\gamma h^2 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \tag{40}$$

which leads to

$$af(x) + bf(x+h) + cf(x+2h) \approx f'(x) + f'''(x) \underbrace{\left(\frac{2}{h} \frac{h^3}{6} + \frac{-1}{2h} \frac{4h^3}{3} \right)}_{= -h^2/3} + \mathcal{O}(h^4). \tag{41}$$

Since the term $-h^2/3$ from (41) is nonzero, we only need to expand those two Taylor expansions from (35) till $\mathcal{O}(h^3)$. Therefore, the consistency order, due to the factor h coming along with f'(x), is guaranteed till at least order 2.