

Global Exercise - Gue10

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Content to be covered:

- ✓ Programming exercise 02: Explanation.
- ✓ [Q&A] Insight review: Example 3 · Gue08
- ✓ Euler scheme $\begin{cases} \text{Explicit} \\ \text{Implicit} \end{cases} : \Sigma \longrightarrow \text{Crank-Nicolson scheme}$
- ✓ Iterative solution method
- ✓ [Review HW10] Consistency error

1 Programming exercise 02: Explanation

Example 1. *Examine the following Convection-diffusion problem*

$$\begin{aligned} -\varepsilon \Delta u(x, y) + \cos \beta \frac{\partial u}{\partial x}(x, y) + \sin \beta \frac{\partial u}{\partial y} &= f(x, y), \quad (x, y) \in \Omega = (0, 1)^2, \\ u(x, y) &= g(x, y), \quad (x, y) \text{ on } \partial\Omega, \end{aligned}$$

Main points of discretization:

$$-\varepsilon \partial_{xx} u(x, y) - \varepsilon \partial_{yy} u(x, y) + \cos \beta \partial_x u(x, y) + \sin \beta \partial_y u(x, y) = 1$$

Since $\beta = 5\pi/6$, it goes with $\cos \beta < 0$ and $\sin \beta > 0$. Therefore, we need Upwind for $\partial_x u(x, y)$ and Downwind for $\partial_y u(x, y)$, as follows

1. Central difference method applied for $\partial_{xx} u(x, y)$,
2. Central difference method applied for $\partial_{yy} u(x, y)$,
3. Upwind method applied for $\partial_x u(x, y)$,
4. Downwind method applied for $\partial_y u(x, y)$.

$$\begin{aligned} -\varepsilon \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \varepsilon \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \\ + \cos \beta \frac{u_{i+1,j} - u_{i,j}}{h} + \sin \beta \frac{u_{i,j} - u_{i,j-1}}{h} = 1 \end{aligned}$$

... [further steps is followed by Problem 2.41 in the lecture note.]

2 [Q&A] Insight review: Example 3 · Gue08

Example 2. *Examine the consistency order of the following problem*

$$f'(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach:

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}(h^3) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \mathcal{O}((2h)^3) \end{cases} \quad (1)$$

which leads to

$$\alpha f(x) = \alpha f(x) \quad (2)$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}(h^3) \quad (3)$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}((2h)^3) \quad (4)$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= (\alpha + \beta + \gamma) f(x) \\ &\quad + (\beta h + 2\gamma h) f'(x) \\ &\quad + \left(\frac{\beta h^2}{2} + 2\gamma h^2 \right) f''(x) \\ &\quad + \beta \mathcal{O}(h^3) \\ &\quad + \gamma \mathcal{O}((2h)^3) \end{aligned} \quad (5)$$

Since we would like to approximate $f'(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f'(x) \quad (6)$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \stackrel{!}{=} 0 \\ \beta h + 2\gamma h \stackrel{!}{=} 1 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \stackrel{!}{=} 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \quad (7)$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \quad (8)$$

which leads the expression (14) to the following equality

$$\begin{aligned}\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= f'(x) + \beta \mathcal{O}(h^3) + \gamma \mathcal{O}((2h)^3) \\ &= f'(x) + \frac{2}{h} \mathcal{O}(h^3) + \frac{-1}{2h} \mathcal{O}((2h)^3) \\ &= f'(x) + \mathcal{O}(h^2) .\end{aligned}\tag{9}$$

Therefore, $f'(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (16), with a consistency order of at least order 2, as shown in (18).

2. Let's now expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach for $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns $\{\alpha, \beta, \gamma\}$ for 4 equations.
3. Therefore, the highest consistency order we may obtain is of order 2.

Example 3. Examine the consistency order of the following problem

$$f''(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach: The procedure goes as same as Example 2

1. Let's expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}(h^3) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \mathcal{O}((2h)^3) \end{cases} \quad (10)$$

which leads to

$$\alpha f(x) = \alpha f(x) \quad (11)$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}(h^3) \quad (12)$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}((2h)^3) \quad (13)$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= (\alpha + \beta + \gamma) f(x) \\ &\quad + (\beta h + 2\gamma h) f'(x) \\ &\quad + \left(\frac{\beta h^2}{2} + 2\gamma h^2 \right) f''(x) \\ &\quad + \beta \mathcal{O}(h^3) \\ &\quad + \gamma \mathcal{O}((2h)^3) \end{aligned} \quad (14)$$

Since we would like to approximate $f''(x)$ by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f''(x) \quad (15)$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \stackrel{!}{=} 0 \\ \beta h + 2\gamma h \stackrel{!}{=} 0 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \stackrel{!}{=} 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/h^2 \end{pmatrix} \quad (16)$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} ?/h^2 \\ ??/h^2 \\ ???/h^2 \end{pmatrix} \quad (17)$$

which leads the expression (14) to the following equality

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= f''(x) + \beta \mathcal{O}(h^3) + \gamma \mathcal{O}((2h)^3) \\ &= f''(x) + \frac{??}{h^2} \mathcal{O}(h^3) + \frac{???}{h^2} \mathcal{O}((2h)^3) \\ &= f''(x) + \mathcal{O}(h). \end{aligned} \quad (18)$$

Therefore, $f''(x)$ can be approximated by $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$, where the values of $\{\alpha, \beta, \gamma\}$ taken from (16), with a consistency order of at least order 1, as shown in (18).

2. Let's now expand $f(x+h)$ and $f(x+2h)$ by using Taylor expansion till $\mathcal{O}(h^4)$. The approach is the same as the above approach for $\mathcal{O}(h^3)$. However, this will lead to contradictory conditions, i.e. 3 unknowns $\{\alpha, \beta, \gamma\}$ for 4 equations.
3. Therefore, the highest consistency order we may obtain is of order 1.

3 Euler scheme · Crank-Nicolson scheme

Example 4. *Examine the following problem*

abc

4 Iterative solution method

Example 5. *Examine the following problem: The iteration scheme*

$$\begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix} x^{k+1} = \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta b, \quad \delta \in \mathbb{R}.$$

is used to solve the following problem

$$\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} x = b, \quad x, b \in \mathbb{R}^2,$$

1. Determine the matrix C from the iteration matrix $I - CA$
2. Which value of $a \in \mathbb{R}$ can help this method to be converged in case of $\delta = 1$?

Approach: Followed by Bemerkung 3.3. from lecture note.

1. The iteration scheme takes the form

$$\boxed{x^{k+1} = (I - CA)x^k + Cb.}$$

which is derived directly from the following iteration setting function

$$\Phi : \begin{cases} \mathbb{R}^n \rightarrow \mathbb{R}^n, \\ x \mapsto \Phi(x) := x + C(b - Ax). \end{cases} \quad (19)$$

Moreover, when x^* is the solution of the system $Ax = b$, we should obtain the following expression

$$\begin{aligned} \Phi(x^*) &= x^* + C \underbrace{(b - Ax^*)}_{=0} \\ &= x^*. \end{aligned}$$

Back to the main problem, we now examine the following derivation

$$x^{k+1} = \begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta \begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix}^{-1} b \quad (20)$$

$$= \begin{pmatrix} 1 & 0 \\ \delta a & 1 \end{pmatrix} \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta \begin{pmatrix} 1 & 0 \\ \delta a & 1 \end{pmatrix} b \quad (21)$$

$$= \begin{pmatrix} 1-\delta & \delta a \\ \delta a - \delta^2 a & \delta^2 a^2 + 1 - \delta \end{pmatrix} x^k + \begin{pmatrix} \delta & 0 \\ \delta^2 a & \delta \end{pmatrix} b \quad (22)$$

Moreover, we also expect

$$x^{k+1} = (I - CA)x^k + Cb. \quad (23)$$

Comparison of (23) with (20) yields the matrix C as follows

$$\therefore \boxed{C = \begin{pmatrix} \delta & 0 \\ \delta^2 a & \delta \end{pmatrix}} \quad (24)$$

2. For the case $\delta = 1$ we obtain the following iteration matrix $(I - CA)$

$$I - CA = \begin{pmatrix} 0 & a \\ 0 & a^2 \end{pmatrix}, \quad (25)$$

Observation: This matrix is of upper triangle. Hence, its eigenvalues are those on the main diagonal, i.e. we can recognize immediately the eigenvalues

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = a^2. \end{cases} \quad (26)$$

The iteration method is expected to be converged when the spectral radius of the iteration matrix $(I - CA)$ smaller than 1, i.e. we should have the following condition

$$\rho(I - CA) < 1 \Leftrightarrow a^2 < 1$$

which leads to

$$\therefore \quad \boxed{-1 < a < 1} \quad (27)$$

5 [Review HW10] Consistency error

Example 6. *Examine the following problem*

abc