

# Global Exercise - Gue10

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Content to be covered:

- ✓ Programming exercise 02: Explanation.
- ✓ [Q&A] Insight review: Example 3 · Gue08
- ✓ Euler scheme  $\begin{cases} \text{Explicit} \\ \text{Implicit} \end{cases} : \Sigma \longrightarrow \text{Crank-Nicolson scheme}$
- ✓ Iterative solution method
- ✓ [Review HW10] Consistency error

## 1 Programming exercise 02: Explanation

**Example 1.** *Examine the following Convection-diffusion problem*

$$\begin{aligned} -\varepsilon \Delta u(x, y) + \cos \beta \frac{\partial u}{\partial x}(x, y) + \sin \beta \frac{\partial u}{\partial y} &= f(x, y), \quad (x, y) \in \Omega = (0, 1)^2, \\ u(x, y) &= g(x, y), \quad (x, y) \text{ on } \partial\Omega, \end{aligned}$$

Main points of discretization:

$$-\varepsilon \partial_{xx} u(x, y) - \varepsilon \partial_{yy} u(x, y) + \cos \beta \partial_x u(x, y) + \sin \beta \partial_y u(x, y) = 1$$

Since  $\beta = 5\pi/6$ , it goes with  $\cos \beta < 0$  and  $\sin \beta > 0$ . Therefore, we need Upwind for  $\partial_x u(x, y)$  and Downwind for  $\partial_y u(x, y)$ , as follows

1. Central difference method applied for  $\partial_{xx} u(x, y)$ ,
2. Central difference method applied for  $\partial_{yy} u(x, y)$ ,
3. Upwind method applied for  $\partial_x u(x, y)$ ,
4. Downwind method applied for  $\partial_y u(x, y)$ .

$$\begin{aligned} -\varepsilon \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \varepsilon \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \\ + \cos \beta \frac{u_{i+1,j} - u_{i,j}}{h} + \sin \beta \frac{u_{i,j} - u_{i,j-1}}{h} = 1 \end{aligned}$$

... [further steps is followed by Problem 2.41 in the lecture note.]

## 2 [Q&A] Insight review: Example 3 · Gue08

**Example 2.** *Examine the consistency order of the following problem*

$$f'(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach:

1. Let's expand  $f(x+h)$  and  $f(x+2h)$  by using Taylor expansion till  $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}(h^3) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \mathcal{O}((2h)^3) \end{cases} \quad (1)$$

which leads to

$$\alpha f(x) = \alpha f(x) \quad (2)$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}(h^3) \quad (3)$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}((2h)^3) \quad (4)$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= (\alpha + \beta + \gamma) f(x) \\ &\quad + (\beta h + 2\gamma h) f'(x) \\ &\quad + \left( \frac{\beta h^2}{2} + 2\gamma h^2 \right) f''(x) \\ &\quad + \beta \mathcal{O}(h^3) \\ &\quad + \gamma \mathcal{O}((2h)^3) \end{aligned} \quad (5)$$

Since we would like to approximate  $f'(x)$  by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f'(x) \quad (6)$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \stackrel{!}{=} 0 \\ \beta h + 2\gamma h \stackrel{!}{=} 1 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \stackrel{!}{=} 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix} \quad (7)$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -3/(2h) \\ 2/h \\ -1/(2h) \end{pmatrix} \quad (8)$$

which leads the expression (14) to the following equality

$$\begin{aligned}\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= f'(x) + \beta \mathcal{O}(h^3) + \gamma \mathcal{O}((2h)^3) \\ &= f'(x) + \frac{2}{h} \mathcal{O}(h^3) + \frac{-1}{2h} \mathcal{O}((2h)^3) \\ &= f'(x) + \mathcal{O}(h^2) .\end{aligned}\tag{9}$$

Therefore,  $f'(x)$  can be approximated by  $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$ , where the values of  $\{\alpha, \beta, \gamma\}$  taken from (16), with a consistency order of at least order 2, as shown in (18).

2. Let's now expand  $f(x+h)$  and  $f(x+2h)$  by using Taylor expansion till  $\mathcal{O}(h^4)$ . The approach is the same as the above approach for  $\mathcal{O}(h^3)$ . However, this will lead to contradictory conditions, i.e. 3 unknowns  $\{\alpha, \beta, \gamma\}$  for 4 equations.
3. Therefore, the highest consistency order we may obtain is of order 2.

**Example 3.** Examine the consistency order of the following problem

$$f''(x) \approx f(x) + f(x+h) + f(x+2h).$$

(the consistency order is aimed as high as possible)

Approach: The procedure goes as same as Example 2

1. Let's expand  $f(x+h)$  and  $f(x+2h)$  by using Taylor expansion till  $\mathcal{O}(h^3)$

$$\begin{cases} f(x) = f(x) \\ f(x+h) = f(x) + \frac{h^1}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}(h^3) \\ f(x+2h) = f(x) + \frac{(2h)^1}{1!}f'(x) + \frac{(2h)^2}{2!}f''(x) + \mathcal{O}((2h)^3) \end{cases} \quad (10)$$

which leads to

$$\alpha f(x) = \alpha f(x) \quad (11)$$

$$\beta f(x+h) = \beta f(x) + \beta h f'(x) + \frac{\beta h^2}{2} f''(x) + \beta \mathcal{O}(h^3) \quad (12)$$

$$\gamma f(x+2h) = \gamma f(x) + 2\gamma h f'(x) + 2\gamma h^2 f''(x) + \gamma \mathcal{O}((2h)^3) \quad (13)$$

Summation of (11), (12) and (13) leads to

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= (\alpha + \beta + \gamma) f(x) \\ &\quad + (\beta h + 2\gamma h) f'(x) \\ &\quad + \left( \frac{\beta h^2}{2} + 2\gamma h^2 \right) f''(x) \\ &\quad + \beta \mathcal{O}(h^3) \\ &\quad + \gamma \mathcal{O}((2h)^3) \end{aligned} \quad (14)$$

Since we would like to approximate  $f''(x)$  by the following expression

$$\alpha f(x) + \beta f(x+h) + \gamma f(x+2h) \approx f''(x) \quad (15)$$

Comparison (15) with (14) leads to the following 3 conditions

$$\begin{cases} \alpha + \beta + \gamma \stackrel{!}{=} 0 \\ \beta h + 2\gamma h \stackrel{!}{=} 0 \\ \frac{\beta h^2}{2} + 2\gamma h^2 \stackrel{!}{=} 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/h^2 \end{pmatrix} \quad (16)$$

$$\Leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} ?/h^2 \\ ??/h^2 \\ ???/h^2 \end{pmatrix} \quad (17)$$

which leads the expression (14) to the following equality

$$\begin{aligned} \alpha f(x) + \beta f(x+h) + \gamma f(x+2h) &= f''(x) + \beta \mathcal{O}(h^3) + \gamma \mathcal{O}((2h)^3) \\ &= f''(x) + \frac{??}{h^2} \mathcal{O}(h^3) + \frac{???}{h^2} \mathcal{O}((2h)^3) \\ &= f''(x) + \mathcal{O}(h). \end{aligned} \quad (18)$$

Therefore,  $f''(x)$  can be approximated by  $\alpha f(x) + \beta f(x+h) + \gamma f(x+2h)$ , where the values of  $\{\alpha, \beta, \gamma\}$  taken from (16), with a consistency order of at least order 1, as shown in (18).

2. Let's now expand  $f(x+h)$  and  $f(x+2h)$  by using Taylor expansion till  $\mathcal{O}(h^4)$ . The approach is the same as the above approach for  $\mathcal{O}(h^3)$ . However, this will lead to contradictory conditions, i.e. 3 unknowns  $\{\alpha, \beta, \gamma\}$  for 4 equations.
3. Therefore, the highest consistency order we may obtain is of order 1.

### 3 Euler scheme · Crank-Nicolson scheme

**Example 4.** Examine the following problem for  $\alpha > 0$

$$\partial_t u(x, t) = \alpha \partial_{xx} u(x, t) + q(x, t), \quad t > 0, \quad x \in [0, 1], \quad (19)$$

with IC  $u(x, 0) = u_0(x)$ , and BC  $u(0, t) = u(1, t) = 0$ .

Approach: Vertical line method, i.e. we first discretize space  $x$ , we then obtain a system of ODEs w.r.t. time  $t$ .

1. Second order discretization in space

$$\partial_{xx} u(x_j, t) \approx \frac{u(x_{j+1}, t) - 2u(x_j, t) + u(x_{j-1}, t))}{h^2}$$

Assume  $y_j(t) \approx u(x_j, t)$  yields

$$y'_j(t) = \frac{\alpha}{h^2}(y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)) + q(x_j, t).$$

General form  $y' = Ay + b(t)$  with

$$A = -\frac{\alpha}{h_x^2} \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

and

$$b = \begin{pmatrix} q(x_1, t) \\ \vdots \\ q(x_{n-1}, t) \end{pmatrix}.$$

where the IC reads

$$y_i(0) = u_0(x_i), \quad i = 1, \dots, n-1$$

2. (a) Explicit Euler

$$\therefore \boxed{y^{i+1} = y^i + h_t (Ay^i + b^i)}.$$

- (b) Implicit Euler

$$\therefore \boxed{y^{i+1} = y^i + h_t (Ay^{i+1} + b^{i+1})}.$$

- (c) Averaging explicit Euler and implicit Euler leads to Crank-Nicolson

$$\therefore \boxed{\begin{aligned} y^{i+1} &= y^i + \frac{h_t}{2} (Ay^i + Ay^{i+1} + b^i + b^{i+1}) \\ &= y^i + \frac{h_t}{2} A(y^i + y^{i+1}) + \frac{h_t}{2} (b^i + b^{i+1}). \end{aligned}}$$

## 4 Iterative solution method

**Example 5.** *Examine the following problem: The iteration scheme*

$$\begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix} x^{k+1} = \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta b, \quad \delta \in \mathbb{R}.$$

*is used to solve the following problem*

$$\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} x = b, \quad x, b \in \mathbb{R}^2,$$

1. Determine the matrix  $C$  from the iteration matrix  $I - CA$
2. Which value of  $a \in \mathbb{R}$  can help this method to be converged in case of  $\delta = 1$ ?

Approach: Followed by Bemerkung 3.3. from lecture note.

1. The iteration scheme takes the form

$$\boxed{x^{k+1} = (I - CA)x^k + Cb.}$$

which is derived directly from the following iteration setting function

$$\Phi : \begin{cases} \mathbb{R}^n \rightarrow \mathbb{R}^n, \\ x \mapsto \Phi(x) := x + C(b - Ax). \end{cases} \quad (20)$$

Moreover, when  $x^*$  is the solution of the system  $Ax = b$ , we should obtain the following expression

$$\begin{aligned} \Phi(x^*) &= x^* + C \underbrace{(b - Ax^*)}_{=0} \\ &= x^*. \end{aligned}$$

Back to the main problem, we now examine the following derivation

$$x^{k+1} = \begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta \begin{pmatrix} 1 & 0 \\ -\delta a & 1 \end{pmatrix}^{-1} b \quad (21)$$

$$= \begin{pmatrix} 1 & 0 \\ \delta a & 1 \end{pmatrix} \begin{pmatrix} 1-\delta & \delta a \\ 0 & 1-\delta \end{pmatrix} x^k + \delta \begin{pmatrix} 1 & 0 \\ \delta a & 1 \end{pmatrix} b \quad (22)$$

$$= \begin{pmatrix} 1-\delta & \delta a \\ \delta a - \delta^2 a & \delta^2 a^2 + 1 - \delta \end{pmatrix} x^k + \begin{pmatrix} \delta & 0 \\ \delta^2 a & \delta \end{pmatrix} b \quad (23)$$

Moreover, we also expect

$$x^{k+1} = (I - CA)x^k + Cb. \quad (24)$$

Comparison of (24) with (21) yields the matrix  $C$  as follows

$$\therefore \quad \boxed{C = \begin{pmatrix} \delta & 0 \\ \delta^2 a & \delta \end{pmatrix}} \quad (25)$$

2. For the case  $\delta = 1$  we obtain the following iteration matrix  $(I - CA)$

$$I - CA = \begin{pmatrix} 0 & a \\ 0 & a^2 \end{pmatrix}, \quad (26)$$

Observation: This matrix is of upper triangle. Hence, its eigenvalues are those on the main diagonal, i.e. we can recognize immediately the eigenvalues

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = a^2. \end{cases} \quad (27)$$

The iteration method is expected to be converged when the spectral radius of the iteration matrix  $(I - CA)$  smaller than 1, i.e. we should have the following condition

$$\rho(I - CA) < 1 \Leftrightarrow a^2 < 1$$

which leads to

$$\therefore \boxed{-1 < a < 1} \quad (28)$$



## 5 [Review HW10] Consistency error

**Example 6.** *Examine the consistency error of the following problem*

$$u''(x) - u'(x) + u(x) = 2x - 1 - x^2$$

*with the exact solution is known, i.e.  $u(x) = 1 - x^2$ .*

Approach: The consistency error reads

$$\| -\Delta_h u|_{\bar{\Omega}_h} - f|_{\Omega_h} \|_{\infty} = \| A_h u|_{\Omega_h} - f|_{\Omega_h} \|_{\infty} \quad (29)$$

Observation: The consistency error can be foreseen (and indeed) with the value 0, i.e. the numerical solution resembles the exact/analytical solution, since we have been using the following numerical scheme + the given information about the exact solution:

1. **Second order** discretization scheme is used to approximate  $u''(x)$ .
2. **Second order** discretization scheme is used to approximate  $u'(x)$ .
3. The exact solution, which is given, is of **quadratics**.

$$\therefore \boxed{\| A_h u|_{\Omega_h} - f|_{\Omega_h} \|_{\infty} = 0}$$

Another way is to substitute the given exact solution  $u(x) = 1 - x^2$  into (29) and define it on grid point, together with known matrix  $A$  and known  $f$ , and then compute the consistency error (29) accordingly.