Global Exercise - Gue07

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1 Numerics: FDM · BC · Derivation of A_h & b_h

Example 1. *Discretization of the following convection-diffusion problem:*

$$-u''(x) - 5u'(x) = 3, \quad \text{for } x \in (0,1) \tag{1}$$

with boundary conditions $u(0) = u(1) = 1$.

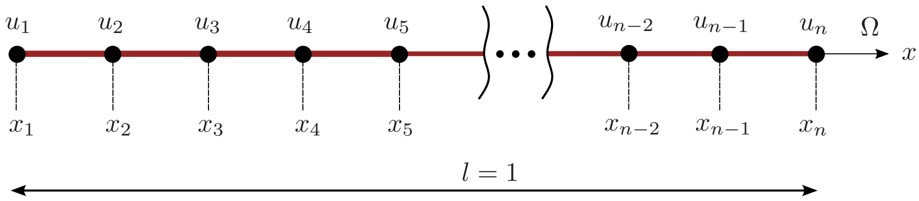

Figure 1: Discretization of domain Ω with n points.

Approach: Central method for $u''(x)$ is obtained as follows

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2). \tag{2}$$

Forward method for $u'(x)$ is obtained as follows

$$u'(x) = \frac{u(x+h) - u(x)}{h} + \mathcal{O}(h). \tag{3}$$

which leads to the system $A_h u_h = b_h$ where A_h and b_h are recognized as follows

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2+5h & -1-5h & & & & \\ -1 & 2+5h & -1-5h & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2+5h & -1-5h \\ & & & & -1 & 2+5h \end{pmatrix} \in \mathbb{R}^{(n-2)\times(n-2)}, \tag{4}$$

$$b_h = \begin{pmatrix} 3 + \dfrac{1}{h^2} \\ 3 \\ \vdots \\ 3 \\ 3 + \dfrac{1}{h^2} + \dfrac{5}{h} \end{pmatrix} \in \mathbb{R}^{(n-2)}. \tag{5}$$

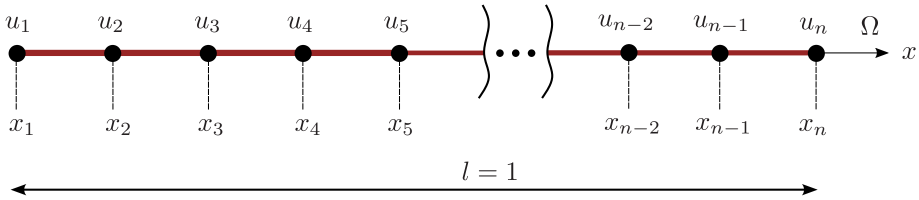

Figure 2: Discretization of domain Ω with n points.

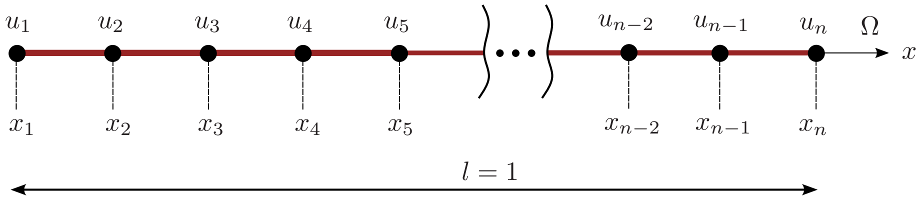

Figure 3: Discretization of domain Ω with n points.

2 Analysis: Spectral theory · Laplace operator application

Example 2. *Examine the following problem*

$$
\begin{aligned}
\partial_t u(x,t) &= \partial_{xx} u(x,t) + a, && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= \sin(\pi x), && x \in (0,1),
\end{aligned}
$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the source term* $a \in \mathbb{R}$ *in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 2 cont.)

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Example 3. *Examine the following problem*

$$
\begin{aligned}
\partial_t u(x,t) &= \partial_{xx} u(x,t) + \sin(\pi x), && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= a, && x \in (0,1),
\end{aligned}
$$

with constant IC $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from* ∂_{xx}.
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

Example 4. *Examine the following problem*

$$\begin{aligned}
\partial_{tt}u(x,t) &= \partial_{xx}u(x,t) + a, && x \in (0,1), t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(1,t) &= 0, && t > 0,\\
u(x,0) &= x, && x \in (0,1),\\
u_t(x,0) &= 0, && x \in (0,1),
\end{aligned}$$

with constant source term $a \in \mathbb{R}$.

1. *Compute the Eigenvalues and the normalized Eigenfunctions from ∂_{xx}.*
2. *Develop the IC and source term in these Eigenfunctions.*
3. *Compute the solution of the problem with the Eigenfunction ansatz.*

(Example 4 cont.)