

Global Exercise - Gue12

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Content covered:

- ✓ Programming exercise 3: Introduction + hints
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1 Programming exercise 3: Introduction + Hints

1.1 Task a: 1D problem

1.1.1 Implicit Euler

Discretization of the following given PDE

$$\partial_t u = \partial_{xx} u \quad (1)$$

by using implicit Euler for time t and second order FDM for space x leads to

$$\frac{u_j^{n+1} - u_j^n}{dt} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{dx^2}, \quad (2)$$

which yields the form used for numerical updating as follows

$$\therefore \boxed{u_j^{n+1} - \frac{dt}{dx^2} (u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) = u_j^n.} \quad (3)$$

Note in passing that all entries on the LHS are unknown since they are u based on time $(n+1)$. These unknowns are computed based on the term on the RHS, which is already known, i.e. u at time n . Then, we obtain the compact form

$$Mu_{(\cdot)}^{n+1} = u_j^n \Rightarrow u_{(\cdot)}^{n+1} = M \backslash u_j^n, \quad (4)$$

where matrix M is derived from the RHS of (3), as follows

$$\begin{pmatrix} 1 + 2dt/dx^2 & -dt/dx^2 & & & & \\ -dt/dx^2 & 1 + 2dt/dx^2 & -dt/dx^2 & & & \\ & -dt/dx^2 & 1 + 2dt/dx^2 & -dt/dx^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 + 2dt/dx^2 & -dt/dx^2 & \\ & & & -dt/dx^2 & 1 + 2dt/dx^2 \end{pmatrix} \quad (5)$$

which can be recast in MATLAB code

Listing 1: Matrix M.

```
1 e = ones(N,1);  
2 M = eye(N) - dt/dx^2 * spdiags([e -2*e e], -1:1, N, N);
```

1.1.2 Crank-Nicolson

Discretization of the following given PDE

$$\partial_t u = \partial_{xx} u \quad (6)$$

by using Crank-Nicolson for time t and second order FDM for space x leads to

$$\frac{u_j^{n+1} - u_j^n}{dt} = \frac{1}{2} \left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{dx^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{dx^2} \right), \quad (7)$$

which yields the form used for numerical updating as follows

$$\therefore \boxed{u_j^{n+1} - \frac{1}{2} \frac{dt}{dx^2} (u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) = u_j^n + \frac{1}{2} \frac{dt}{dx^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n)}. \quad (8)$$

Note in passing that all entries on the LHS are unknown since they are u based on time $(n+1)$. These unknowns are computed based on the term on the RHS, which is already known, i.e. u at time n . Then, we obtain the compact form

$$Mu_{(\cdot)}^{n+1} = Ku_{(\cdot)}^n \Rightarrow u_{(\cdot)}^{n+1} = M \backslash (Ku_{(\cdot)}^n), \quad (9)$$

where matrix M is derived from the RHS of (8), as follows

$$\begin{pmatrix} 1 + dt/dx^2 & -dt/dx^2/2 & & & & \\ -dt/dx^2/2 & 1 + dt/dx^2 & -dt/dx^2/2 & & & \\ & -dt/dx^2/2 & 1 + dt/dx^2 & -dt/dx^2/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 + dt/dx^2 & -dt/dx^2/2 \\ & & & & -dt/dx^2/2 & 1 + dt/dx^2 \end{pmatrix} \quad (10)$$

and matrix K , whose difference from M is only about the sign, takes the form

$$\begin{pmatrix} 1 - dt/dx^2 & -dt/dx^2/2 & & & & \\ -dt/dx^2/2 & 1 - dt/dx^2 & -dt/dx^2/2 & & & \\ & -dt/dx^2/2 & 1 - dt/dx^2 & -dt/dx^2/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 - dt/dx^2 & -dt/dx^2/2 \\ & & & & -dt/dx^2/2 & 1 - dt/dx^2 \end{pmatrix} \quad (11)$$

which can be recast in MATLAB code

Listing 2: Matrix M and K.

```

1 e = ones(N,1);
2 M = eye(N) - dt/dx^2/2 * spdiags([e -2*e e], -1:1, N, N);
3 K = eye(N) + dt/dx^2/2 * spdiags([e -2*e e], -1:1, N, N);

```

1.2 Task b: 2D problem + Melting a penny

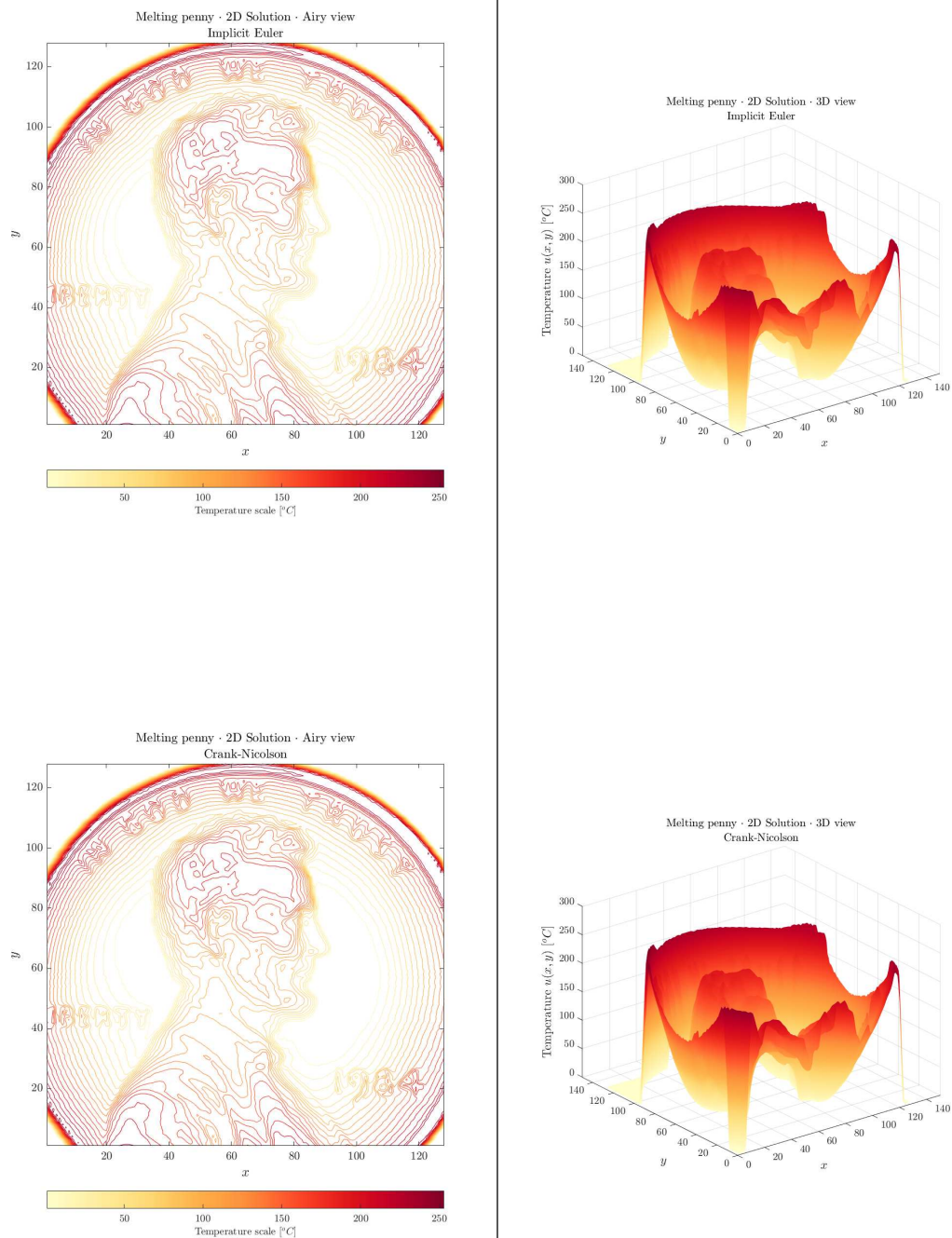


Figure 1: Melting penny: 2D solution; implicit Euler and Crank-Nicolson scheme.

2 Exam preparation