Global Exercise - Gue11

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Content covered:

- ✓ [Review HW10 - A2] Numerics: Consistency error (cont.)
- ✓ [Review HW11 - A1] Analysis: Fundamental solution Laplace-operator in 2D.
- ✓ Numerics: $\begin{cases} \text{Jacobi iteration solution} \\ \text{Gauss-Seidel iteration solution} \end{cases}$

1 [Review HW10 - A2] Consistency error (cont.)

Example 1. *Examine the consistency error of the following problem*

$$u''(x) - u'(x) + u(x) = 2x - 1 - x^2$$

with the Neumann and Dirichlet BC are given, respectively, as follows

$$u'(0) = 0, \quad u(1) = 0.$$

The exact solution is known, i.e. $u(x) = 1 - x^2$.

Approach: **Definition 2.19** in the lecture note.
The discretization error of FDM takes the form

$$e_h := u\big|_{\overline{\Omega}_h} - u_h \tag{1}$$

Furthermore, the discretization error is estimated as follows

$$\begin{aligned}
\|e_h\| &= \left\| u\big|_{\overline{\Omega}_h} - u_h \right\| \\
&= \left\| \Delta_h^{-1} \Delta_h \left(u\big|_{\overline{\Omega}_h} - u_h \right) \right\| \\
&= \left\| \Delta_h^{-1} \left(\Delta_h u\big|_{\overline{\Omega}_h} - \Delta_h u_h \right) \right\| \\
&= \left\| \Delta_h^{-1} \left(-\Delta_h u\big|_{\overline{\Omega}_h} + \Delta_h u_h \right) \right\| \\
&= \left\| \Delta_h^{-1} \left(-\Delta_h u\big|_{\overline{\Omega}_h} - f\big|_{\Omega_h} \right) \right\| \\
&\leq \underbrace{\left\| \Delta_h^{-1} \right\|}_{\text{Stability}} \underbrace{\left\| -\Delta_h u\big|_{\overline{\Omega}_h} - f\big|_{\Omega_h} \right\|}_{\text{Consistency}}
\end{aligned} \tag{2}$$

We need to estimate the consistency error

$$\boxed{\left\| -\Delta_h u\big|_{\overline{\Omega}_h} - f\big|_{\Omega_h} \right\|_\infty \to \left\| A_h u\big|_{\Omega_h} - f\big|_{\Omega_h} \right\|_\infty} \tag{3}$$

for the given problem.

By taking the discretization

$$u_n = u(x_n) \tag{4}$$
$$x_n = nh \tag{5}$$

where $n = 0, \dots, N-1$. The matrix A_h takes the form

$$\therefore \quad \boxed{A_h = \frac{1}{2h^2}\begin{pmatrix} 2h^2-4 & 4 & & & \\ 2+h & 2h^2-4 & 2-h & & \\ & 2+h & 2h^2-4 & 2-h & \\ & & \ddots & \ddots & \ddots \\ & & & 2+h & 2h^2-4 \end{pmatrix}_{N\times N},} \tag{6}$$

The exact solution $u(x) = 1 - x^2$ defined on grid Ω_h, i.e. $u\big|_{\Omega_h}$, takes the form

$$\therefore \quad \boxed{u\big|_{\Omega_h} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \dots \\ u_{N-2} \\ u_{N-1} \end{pmatrix}_{N\times 1} = \begin{pmatrix} 1-(0h)^2 \\ 1-(1h)^2 \\ 1-(2h)^2 \\ 1-(3h)^2 \\ \dots \\ 1-((N-2)h)^2 \\ 1-((N-1)h)^2 \end{pmatrix}_{N\times 1}} \tag{7}$$

The function $f(x) = 2x - 1 - x^2$ defined on grid Ω_h, i.e. $f\big|_{\Omega_h}$, takes the form

$$\therefore \quad \boxed{f\big|_{\Omega_h} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \dots \\ f_{N-2} \\ f_{N-1} \end{pmatrix}_{N\times 1} = \begin{pmatrix} 2(0h)-1-(0h)^2 \\ 2(1h)-1-(1h)^2 \\ 2(2h)-1-(2h)^2 \\ 2(3h)-1-(3h)^2 \\ \dots \\ 2((N-2)h)-1-((N-2)h)^2 \\ 2((N-1)h)-1-((N-1)h)^2 \end{pmatrix}_{N\times 1}} \tag{8}$$

Substitution of (6), (7) and (8) into (3) leads to

$$
\begin{aligned}
&\left\|A_h u\big|_{\Omega_h} - f\big|_{\Omega_h}\right\|_\infty \\
&= \left\| \frac{1}{2h^2} \begin{pmatrix} 2h^2-4 & 4 & & & \\ 2+h & 2h^2-4 & 2-h & & \\ & 2+h & 2h^2-4 & 2-h & \\ & & \ddots & \ddots & \ddots \\ & & & 2+h & 2h^2-4 \end{pmatrix} \begin{pmatrix} 1-(0h)^2 \\ 1-(1h)^2 \\ 1-(2h)^2 \\ 1-(3h)^2 \\ \dots \\ 1-((N-2)h)^2 \\ 1-((N-1)h)^2 \end{pmatrix} - f\big|_{\Omega_h} \right\|_\infty \\
&= \left\| \begin{pmatrix} -1 \\ -1+1(2h)-(1h)^2 \\ -1+2(2h)-(2h)^2 \\ -1+3(2h)-(3h)^2 \\ \dots \\ -1+(N-2)(2h)-((N-2)h)^2 \\ -1+(N-1)(2h)-((N-1)h)^2 \end{pmatrix} - f\big|_{\Omega_h} \right\|_\infty \\
&= \left\| \begin{pmatrix} -1 \\ -1+1(2h)-(1h)^2 \\ -1+2(2h)-(2h)^2 \\ -1+3(2h)-(3h)^2 \\ \dots \\ -1+(N-2)(2h)-((N-2)h)^2 \\ -1+(N-1)(2h)-((N-1)h)^2 \end{pmatrix} - \begin{pmatrix} 2(0h)-1-(0h)^2 \\ 2(1h)-1-(1h)^2 \\ 2(2h)-1-(2h)^2 \\ 2(3h)-1-(3h)^2 \\ \dots \\ 2((N-2)h)-1-((N-2)h)^2 \\ 2((N-1)h)-1-((N-1)h)^2 \end{pmatrix} \right\|_\infty \\
&= 0.
\end{aligned}
\tag{9}
$$

Finally, we obtain the consistency error

$$
\therefore \quad \boxed{\left\|A_h u\big|_{\Omega_h} - f\big|_{\Omega_h}\right\|_\infty = 0.} \tag{10}
$$

2 [Review HW10 - A1] Fundamental solution

Example 2. *Examine the following problem: Show that the γ function defined as*

$$\gamma : \begin{cases} \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}, \\ \boldsymbol{x} \mapsto \gamma(\boldsymbol{x}) = -\dfrac{1}{2\pi} \ln |\boldsymbol{x}|, \end{cases} \tag{11}$$

is a fundamental solution of Laplace equation $-\Delta\gamma(\boldsymbol{x}) = \delta_0$ in $\mathbb{R}^2$.

Approach:

Proof. For all test functions φ in the compactly supported, infinitely differentiable function space, i.e. $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we need to show the following equality

$$\begin{aligned} -\Delta\gamma(\boldsymbol{x}) = \delta_0 &\Leftrightarrow \langle -\Delta\gamma(\boldsymbol{x}), \varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle -\partial_{xx}\gamma(\boldsymbol{x}) - \partial_{yy}\gamma(\boldsymbol{x}), \varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow -\langle \partial_{xx}\gamma(\boldsymbol{x}), \varphi\rangle - \langle \partial_{yy}\gamma(\boldsymbol{x}), \varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow -(-1)^2\langle \gamma(\boldsymbol{x}), \partial_{xx}\varphi\rangle - (-1)^2\langle \gamma(\boldsymbol{x}), \partial_{yy}\varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\partial_{xx}\varphi\rangle + \langle \gamma(\boldsymbol{x}), -\partial_{yy}\varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\partial_{xx}\varphi - \partial_{yy}\varphi\rangle = \langle \delta_0, \varphi\rangle \\ &\Leftrightarrow \langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle = \langle \delta_0, \varphi\rangle \end{aligned} \tag{12}$$

yielding the following expression

$$\boxed{\langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle = \langle \delta_0, \varphi\rangle \Leftrightarrow \langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle \overset{!}{=} \varphi(0), \quad \forall\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).} \tag{13}$$

Setup: Let the test function $\varphi \in \mathcal{D}(\mathbb{R}^2) = \mathcal{C}_0^\infty(\mathbb{R}^2)$ be arbitrarily chosen with its support in a subset of a ball centered at O with radius R, i.e. $\mathrm{supp}(\varphi) \subset B_R(0)$. This setting leads to the fact

$$\begin{cases} \varphi(\boldsymbol{x}) \neq 0 \text{ in } B_R(0), \\ \varphi(\boldsymbol{x}) = 0 \text{ on } \partial B_R(0). \end{cases} \tag{14}$$

Besides, let another smaller ball centered at O with radius $\varepsilon \ll 1$ be defined $B_\varepsilon(0)$. Then, the domain to be considered is $\Omega_\varepsilon := B_R(0) \setminus B_\varepsilon(0)$, i.e. the *2D doughnut* as shown in Figure 1. By using Gauss divergence theorem twice we obtain the following equality:

$$\begin{aligned} \langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle &= -\int_{\mathbb{R}^2} \gamma(\boldsymbol{x})\Delta\varphi(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \lim_{\varepsilon\to 0^+} \left(-\int_{\Omega_\varepsilon} \gamma(\boldsymbol{x})\Delta\varphi(\boldsymbol{x})\, d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon\to 0^+} \left(-\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\, d\partial\Omega_\varepsilon + \int_{\Omega_\varepsilon} \nabla\gamma(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})\, d\Omega_\varepsilon \right) \\ &= \lim_{\varepsilon\to 0^+} \Bigg(-\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\, d\partial\Omega_\varepsilon \\ &\qquad + \left(\int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\, d\partial\Omega_\varepsilon - \int_{\Omega_\varepsilon} \cancelto{0}{\Delta\gamma(\boldsymbol{x})}\varphi(\boldsymbol{x})\, d\Omega_\varepsilon \right) \Bigg) \\ &= \lim_{\varepsilon\to 0^+} \left(-\int_{\partial\Omega_\varepsilon} \gamma(\boldsymbol{x}) \frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\, d\partial\Omega_\varepsilon + \int_{\partial\Omega_\varepsilon} \frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\, d\partial\Omega_\varepsilon \right) \end{aligned} \tag{15}$$

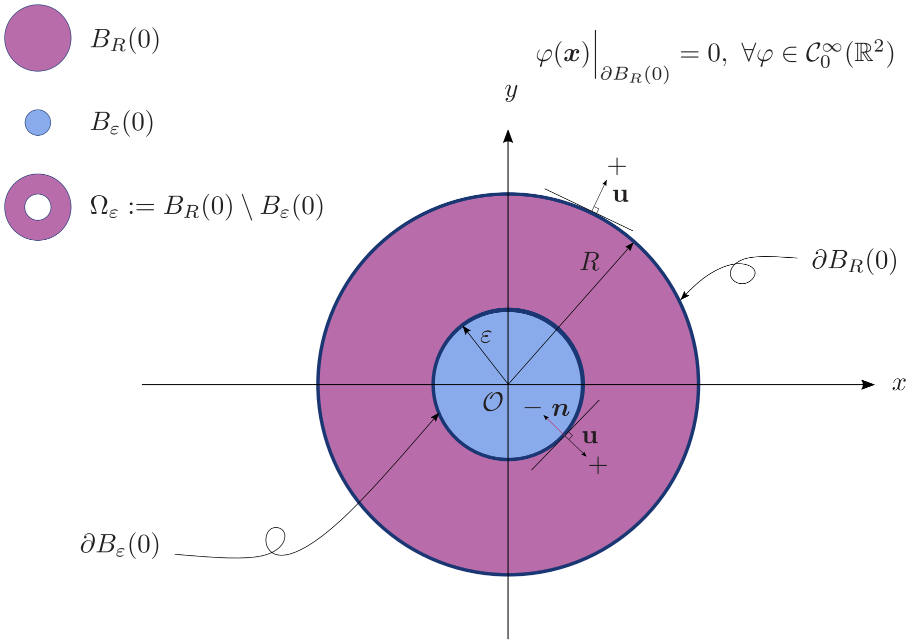

Figure 1: The *2D doughnut*: $\Omega_\varepsilon = B_R(0) \setminus B_\varepsilon(0)$.

Note in passing that $\Delta\gamma(\boldsymbol{x}) = 0$ from (15) due to the fact that

$$\boxed{\begin{aligned}
&\gamma(\boldsymbol{x}) = -\frac{1}{2\pi}\ln|\boldsymbol{x}| \Leftrightarrow \gamma(x,y) = -\frac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)\\
&\quad\text{which leads to}\\
&\qquad\partial_x\gamma(x,y) = \frac{-x}{2\pi(x^2+y^2)} = -\frac{1}{2\pi}\frac{x}{|\boldsymbol{x}|^2}\\
&\qquad\partial_y\gamma(x,y) = \frac{-y}{2\pi(x^2+y^2)} = -\frac{1}{2\pi}\frac{y}{|\boldsymbol{x}|^2}\\
&\qquad\text{and}\\
&\qquad\partial_{xx}\gamma(x,y) = \frac{2x^2-2y^2}{4\pi^2(x^2+y^2)^2}\\
&\qquad\partial_{yy}\gamma(x,y) = \frac{2y^2-2x^2}{4\pi^2(x^2+y^2)^2}\\
&\qquad\text{hence}\\
&\qquad\nabla\gamma(\boldsymbol{x}) = (\partial_x\gamma(x,y),\partial_y\gamma(x,y)) = -\frac{1}{2\pi}\frac{(x,y)}{|\boldsymbol{x}|^2} = -\frac{1}{2\pi}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}\\
&\qquad\Delta\gamma(\boldsymbol{x}) = \partial_{xx}\gamma(x,y) + \partial_{yy}\gamma(x,y) = 0
\end{aligned}} \tag{16}$$

Next, by decomposing the boundary $\partial\Omega_\varepsilon$ into $\partial B_R(0)$ and $\partial B_\varepsilon(0)$, i.e.

$$\partial\Omega_\varepsilon = \partial B_R(0) \cup \partial B_\varepsilon(0), \tag{17}$$

we consequently obtain from (15) the following expression

$$
\begin{aligned}
&\langle \gamma(\boldsymbol{x}), -\Delta\varphi\rangle \\
&= \lim_{\varepsilon\to 0^+}\left(-\int_{\partial\Omega_\varepsilon}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\,d\partial\Omega_\varepsilon + \int_{\partial\Omega_\varepsilon}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\,d\partial\Omega_\varepsilon\right) \\
&= \lim_{\varepsilon\to 0^+}\Bigg(-\left(\int_{\partial B_R(0)}\gamma(\boldsymbol{x})\cancelto{0}{\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}}\,d\partial B_R(0) + \int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}\,d\partial B_\varepsilon(0)\right) \\
&\qquad + \left(\int_{\partial B_R(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\cancelto{0}{\varphi(\boldsymbol{x})}\,d\partial B_R(0) + \int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\varphi(\boldsymbol{x})\,d\partial B_\varepsilon(0)\right)\Bigg) \\
&= \lim_{\varepsilon\to 0^+}\Bigg(-\left(\int_{\partial B_R(0)}\gamma(\boldsymbol{x})\cancelto{0}{\frac{\partial\varphi(\boldsymbol{x})}{\partial\nu}}\,d\partial B_R(0) - \int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right) \\
&\qquad + \left(\int_{\partial B_R(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial\nu}\cancelto{0}{\varphi(\boldsymbol{x})}\,d\partial B_R(0) - \int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\varphi(\boldsymbol{x})\,d\partial B_\varepsilon(0)\right)\Bigg) \\
&= \lim_{\varepsilon\to 0^+}\left(\int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) - \int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\varphi(\boldsymbol{x})\,d\partial B_\varepsilon(0)\right) \\
&= \lim_{\varepsilon\to 0^+}\Bigg(-\int_{\partial B_\varepsilon(0)}(\varphi(\boldsymbol{x})-\varphi(0))\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) \\
&\qquad -\varphi(0)\int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) + \int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\Bigg) \\
&= \underbrace{\lim_{\varepsilon\to 0^+}\left(-\int_{\partial B_\varepsilon(0)}(\varphi(\boldsymbol{x})-\varphi(0))\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right)}_{:=A} \\
&\quad + \underbrace{\lim_{\varepsilon\to 0^+}\left(-\varphi(0)\int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right)}_{:=B} \\
&\quad + \underbrace{\lim_{\varepsilon\to 0^+}\left(\int_{\partial B_\varepsilon(0)}\gamma(\boldsymbol{x})\frac{\partial\varphi(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0)\right)}_{:=C} = A+B+C.
\end{aligned}
\tag{18}
$$

Note in passing that the test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ vanishes on boundary $\partial B_R(0)$. Next, computation of the directional derivative of $\gamma(\boldsymbol{x})$ goes as follows

$$
\frac{\partial\gamma(\boldsymbol{x})}{\partial n} = \nabla\gamma(\boldsymbol{x})\cdot\boldsymbol{n} \overset{(16)}{=} -\frac{1}{2\pi}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}\cdot\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \frac{-1}{2\pi|\boldsymbol{x}|},
\tag{19}
$$

which leads to

$$
\begin{aligned}
\int_{\partial B_\varepsilon(0)}\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\,d\partial B_\varepsilon(0) &= \frac{-1}{2\pi}\int_{\partial B_\varepsilon(0)}\frac{1}{|\boldsymbol{x}|}\,d\partial B_\varepsilon(0) \\
&= \frac{-1}{2\pi}\int_{\partial B_\varepsilon(0)}\frac{1}{\varepsilon}\,d\partial B_\varepsilon(0) \\
&= \frac{-1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}d\partial B_\varepsilon(0) = \frac{-1}{2\pi\varepsilon}(2\pi\varepsilon) = -1
\end{aligned}
\tag{20}
$$

Therefore, term B in (18) yields

$$\therefore \quad \boxed{B = \lim_{\varepsilon \to 0^+} \left(-\varphi(0) \int_{\partial B_\varepsilon(0)} \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) \overset{(20)}{=} \varphi(0)} \tag{21}$$

Besides, the analysis of term A goes as follows

$$\begin{aligned}
|A| &= \lim_{\varepsilon \to 0^+} \left| - \int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x}) - \varphi(0)) \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right| \\
&\leq \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(0)} \Big| \varphi(\boldsymbol{x}) - \varphi(0) \Big| \left| \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \right| d\partial B_\varepsilon(0) \\
&\leq \lim_{\varepsilon \to 0^+} \max_{|\boldsymbol{x}| = \varepsilon} |\varphi(\boldsymbol{x}) - \varphi(0)| \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \right| d\partial B_\varepsilon(0) \\
&= \lim_{\varepsilon \to 0^+} \max_{|\boldsymbol{x}| = \varepsilon} |\nabla \varphi(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{0})| \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \right| d\partial B_\varepsilon(0) \\
&\leq \lim_{\varepsilon \to 0^+} \underbrace{\max_{|\boldsymbol{x}| = \varepsilon} |\nabla \varphi(\boldsymbol{x})|}_{=M_1 \in \mathbb{R}, \ \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)} \underbrace{|\boldsymbol{x}|}_{\varepsilon} \int_{\partial B_\varepsilon(0)} \left| \underbrace{\frac{\partial \gamma(\boldsymbol{x})}{\partial n}}_{\overset{(20)}{=} -1} \right| d\partial B_\varepsilon(0) \\
&= \lim_{\varepsilon \to 0^+} M_1 \varepsilon = 0
\end{aligned} \tag{22}$$

Note in passing that the Mean value theorem has been considered in the above analysis. Consequently, term A vanishes itself

$$\therefore \quad \boxed{A = \lim_{\varepsilon \to 0^+} \left(- \int_{\partial B_\varepsilon(0)} (\varphi(\boldsymbol{x}) - \varphi(0)) \frac{\partial \gamma(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) = 0} \tag{23}$$

Last but not least, consideration of term C leads to

$$\begin{aligned}
|C| &= \lim_{\varepsilon \to 0^+} \left| \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial \varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right| \\
&= \lim_{\varepsilon \to 0^+} \left| \int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \nabla \varphi(\boldsymbol{x}) \cdot \boldsymbol{n} \, d\partial B_\varepsilon(0) \right| \\
&\leq \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(0)} \Big| \gamma(\boldsymbol{x}) \Big| \Big| \nabla \varphi(\boldsymbol{x}) \Big| \Big| \boldsymbol{n} \Big| \, d\partial B_\varepsilon(0) \\
&\leq \lim_{\varepsilon \to 0^+} \underbrace{\max_{\boldsymbol{x} \in B_R(0)} |\nabla \varphi|}_{=: M_2 \in \mathbb{R}, \ \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)} \underbrace{|\boldsymbol{n}|}_{=1} \int_{\partial B_\varepsilon(0)} |\gamma(\boldsymbol{x})| \, d\partial B_\varepsilon(0) \\
&= \lim_{\varepsilon \to 0^+} M_2 \int_{\partial B_\varepsilon(0)} \frac{1}{2\pi} \ln |\boldsymbol{x}| \, d\partial B_\varepsilon(0) = \lim_{\varepsilon \to 0^+} \frac{M_2}{2\pi} \ln(\varepsilon) \underbrace{\int_{\partial B_\varepsilon(0)} d\partial B_\varepsilon(0)}_{=2\pi\varepsilon} \\
&= \lim_{\varepsilon \to 0^+} M_2 \varepsilon \ln(\varepsilon) = 0
\end{aligned} \tag{24}$$

As a consequence, term C also vanishes itself

$$\therefore \quad \boxed{C = \lim_{\varepsilon \to 0^+} \left(\int_{\partial B_\varepsilon(0)} \gamma(\boldsymbol{x}) \frac{\partial \varphi(\boldsymbol{x})}{\partial n} \, d\partial B_\varepsilon(0) \right) = 0} \tag{25}$$

Finally, substitution of (21), (23) and (25) into (18) we obtain

$$\therefore \quad \boxed{\langle \gamma(\boldsymbol{x}), -\Delta\varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).} \tag{26}$$

$\square$

3 Numerics: Jacobi iteration solution

Example 3. *Examine the following problem: $Ax = b$ with matrix A*

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \tag{27}$$

Perform the first step of Jacobi iteration given the initial value for the vector $x^{(0)} = (1, 0, 0)^T$ and the RHS vector $b = (3, 5, -1)^T$.

Approach: **Bemerkung 3.12** in the lecture note:

$$\boxed{Dx^{(k+1)} = (L + U)x^{(k)} + b.} \tag{28}$$

Note in passing that

$$A = D - L - U \tag{29}$$

where the matrix A is decomposed into

1. the diagonal matrix $D = \begin{pmatrix} 2 & & \\ & -4 & \\ & & 3 \end{pmatrix}$
2. the lower triangular matrix $L = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ -1 & -2 & 0 \end{pmatrix}$ with **sign changed**.
3. the upper triangular matrix $U = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 2 \\ & & 0 \end{pmatrix}$ with **sign changed**.

By using (28) the computation of the **first step** of iternation based on the **Jacobi** scheme goes as follows

$$Dx^{(1)} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}, \tag{30}$$

which leads to the solution of the first step of Jacobi iteration

$$\therefore \quad \boxed{x^{(1)} = \begin{pmatrix} 3/2 \\ -1 \\ -2/3 \end{pmatrix}.} \tag{31}$$

Remark 1. *The computation of* ***Jacobi*** *iteration requires the inverse of matrix D. Since matrix D is a diagonal matrix, its inverse is obtained by taking the inverse of every single entry on its diagonal, which is*

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{32}$$

and the inverse of D takes the form

$$D^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}, \tag{33}$$

where $a, b, c \neq 0$.

4 Numerics: Gauss-Seidel iteration solution

Example 4. *Examine the following problem: $Ax = b$ with matrix A*

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -4 & -2 \\ 1 & 2 & 3 \end{pmatrix}. \tag{34}$$

Perform the first step of Gauss-Seidel iteration given the initial value for the vector $x^{(0)} = (1, 0, 0)^T$ and the RHS vector $b = (3, 5, -1)^T$.

Approach: **Bemerkung 3.17** in the lecture note:
Either using

$$\boxed{x^{(k+1)} = (I - (D - L)^{-1}A)x^{(k)} + (D - L)^{-1}b} \tag{35}$$

or, equally, using

$$\boxed{(D - L)x^{(k+1)} = Ux^{(k)} + b} \tag{36}$$

Note in passing that

$$A = D - L - U \tag{37}$$

where the matrix A is decomposed into

1. the diagonal matrix $D = \begin{pmatrix} 2 & & \\ & -4 & \\ & & 3 \end{pmatrix}$
2. the lower triangular matrix $L = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ -1 & -2 & 0 \end{pmatrix}$ with **sign changed**.
3. the upper triangular matrix $U = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 2 \\ & & 0 \end{pmatrix}$ with **sign changed**.

By using (36) the computation of the **first step** of iternation based on the **Gauss-Seidel** scheme goes as follows

$$(D - L)x^{(k+1)} = Ux^{(k)} + b, \tag{38}$$

which leads to

$$(D - L)x^{(1)} = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 2 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}. \tag{39}$$

Therefore, we obtain the solution after the first step

$$\therefore \quad \boxed{x^{(1)} = (D - L)^{-1} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 & & \\ 1/8 & -1/4 & \\ -1/4 & 1/6 & 1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -7/8 \\ -1/4 \end{pmatrix}.} \tag{40}$$

Remark 2. *The computation of* ***Gauss-Seidel*** *iteration requires the inverse of matrix* $(D-L)$*. Since the matrix* $(D-L)$ *is a lower triangular matrix with non-zero entries on its diagonal, the matrix* $(D-L)$ *has the following form*

$$D-L=\begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}, \tag{41}$$

and the inverse $(D-L)^{-1}$ *takes the form*

$$(D-L)^{-1}=\begin{pmatrix} 1/a & 0 & 0 \\ -b/(ad) & 1/d & 0 \\ (be-cd)/(adf) & -e/(df) & 1/f \end{pmatrix}, \tag{42}$$

where $a,d,f\neq 0$.