

Global Exercise - 12

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12th January 2022

1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1: $W := R^{-1}U$ as the scheme shown in exercise

$$\begin{array}{l} U_t + AU_x = 0 \\ U_t + R\Lambda R^{-1}U_x = 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x = 0 \\ W_t + \Lambda W_x = 0 \end{array}$$

where the matrix A is diagonalizable with a transformation matrix $R \in \mathbb{R}^{N \times N}$ in the form

$$A = R\Lambda R^{-1}.$$

2. Case 2: $W := TU$ as the scheme shown in lecture note

$$\begin{array}{l} U_t + AU_x = 0 \\ U_t + T^{-1}\Lambda TU_x = 0 \\ TU_t + \Lambda TU_x = 0 \\ W_t + \Lambda W_x = 0 \end{array}$$

where the matrix A is diagonalizable with a transformation matrix $T \in \mathbb{R}^{N \times N}$ in the form

$$A = T^{-1}\Lambda T.$$

- Note in passing that both schemes result in the same solution.
→ We have just to be consistent with which scheme to follow.

2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 1. Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.

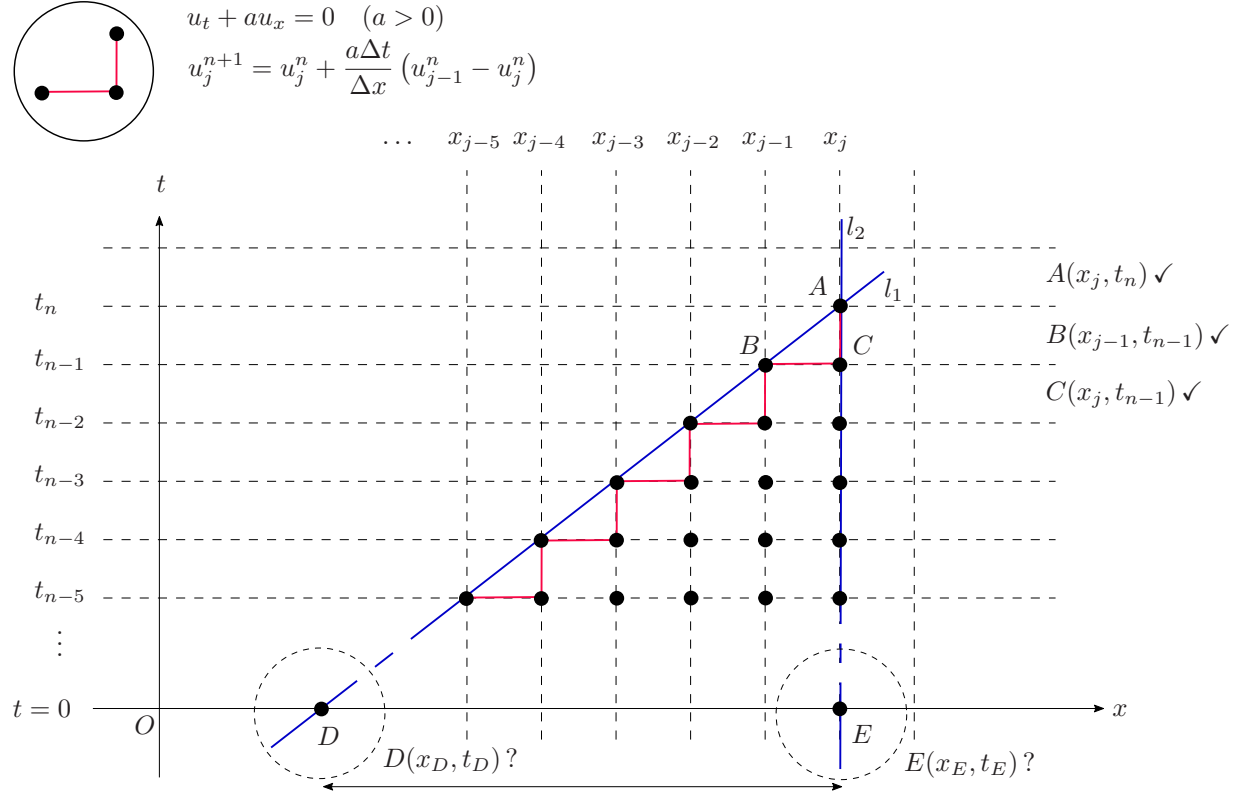


Figure 1: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 1, the numerical value computed at point A depends essentially on computed initial conditions laying between point D and E .

1. Perspective of indicial subscription:

Line (l_1) passing point $A(j, n)$ and $B(j-1, n-1)$ has the following form

$$\begin{aligned}
 (l_1) : \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A} (\xi - \xi_A) \\
 &\Leftrightarrow \tau = n + \frac{(n-1) - n}{(j-1) - j} (\xi - j) \\
 &\Leftrightarrow \tau = n + \frac{-1}{-1} (\xi - j), \tag{1}
 \end{aligned}$$

where τ is the indicial variable corresponding to t , and x the indicial variable to x . Hence, line (l_1) passing line x with index $\tau = 0$ at point D leads to the following relation

$$\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \tag{2}$$

Likewise, line (l_2) passing line x with index $\tau = 0$ at point E leads to the following relation

$$x_\xi - x_j = 0. \quad (3)$$

Therefore, by combining (2) and (3) we arrive at the numerical domain of dependence for the One-sided method in terms of indicial perspective

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x_\xi \mid -n\Delta x \leq x_\xi - x_j \leq 0 \right\}. \quad (4)$$

Next, by using the CFL number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \stackrel{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \quad (5)$$

Then, by substituting (5) into (4) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid -\frac{at_n}{\nu} \leq x - x_j \leq 0 \right\}}. \quad (6)$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (7)$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}(x_j, t_n) \subset \mathcal{D}_{\Delta t}(x_j, t_n), \quad (8)$$

which implies that characteristics should lie with the triangular zone under the line (l_1) and (l_2) , as shown in Figure 1. Therefore, substitution of (7) into (6) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \leq (x_j - at_n) - x_j \leq 0 \Leftrightarrow -\frac{at_n}{\nu} \leq -at_n \leq 0, \quad (9)$$

which, equally, leads to the CFL condition

$$\therefore \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}}. \quad (10)$$

Herein, the CFL condition (10) leads to constraint on the time step Δt for the case when $a > 0$. Note in passing that ν is non-negative.

2. Perspective of fixed-point value:

Line (l_1) passing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned} (l_1) : \quad t &= t_A + \frac{t_B - t_A}{x_B - x_A} (x - x_A) \Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x} (x - x_j). \end{aligned} \quad (11)$$

Hence, line (l_1) passing line $t = 0$ at point D leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \quad (12)$$

Likewise, line (l_2) passing line $t = 0$ at point E leads to the relation

$$x - x_j = 0. \quad (13)$$

Therefore, combination of (12) and (13) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \quad (14)$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (7), reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (15)$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

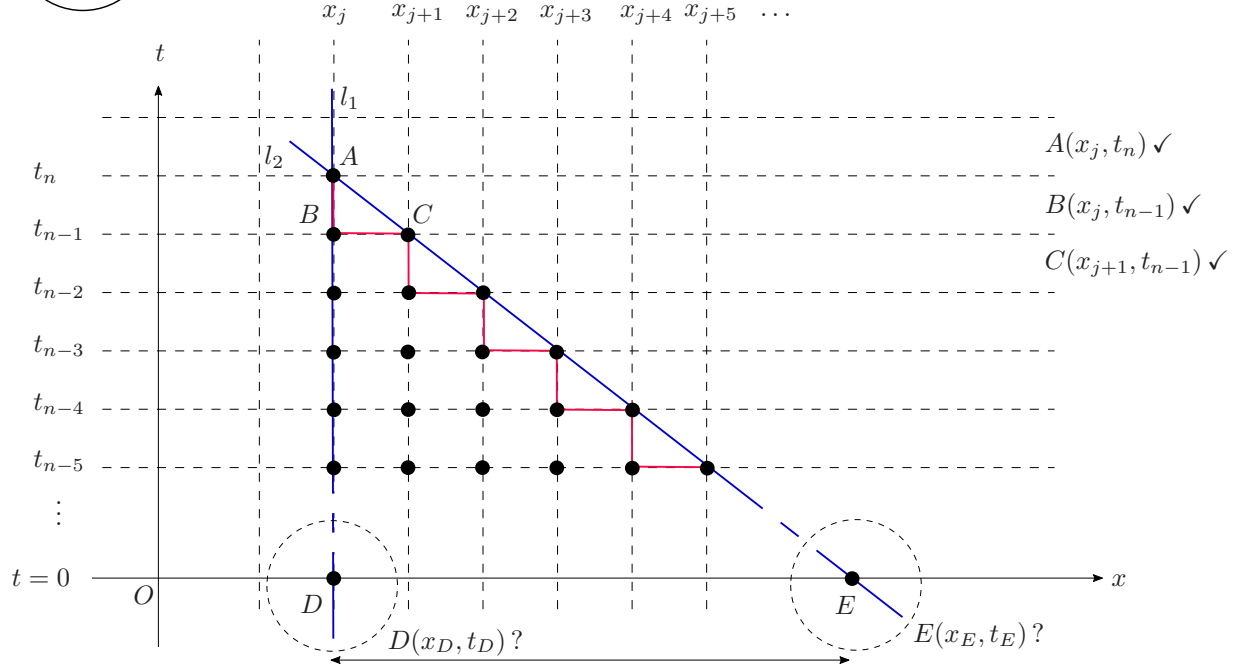
$$-\frac{t_n \Delta x}{\Delta t} \leq (x_j - at_n) - x_j \leq 0, \quad (16)$$

which we have substituted (15) into (14). Herein, the relation (16) enforcing CFL condition on the time step Δt

$$\therefore \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a}}, \quad (17)$$

which is similar to (10).

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x} (u_j^n - u_{j+1}^n)$$



Similarly, by following steps done in Example 1 we obtain the following summary:

- $$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (18)$$

- $$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid 0 \leq x - x_j \leq -\frac{at_n}{\nu} \right\}. \quad (19)$$

- $$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (20)$$

- $$\therefore \boxed{\Delta t \geq \frac{\Delta x}{a}}. \quad (21)$$

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Example 3. Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.

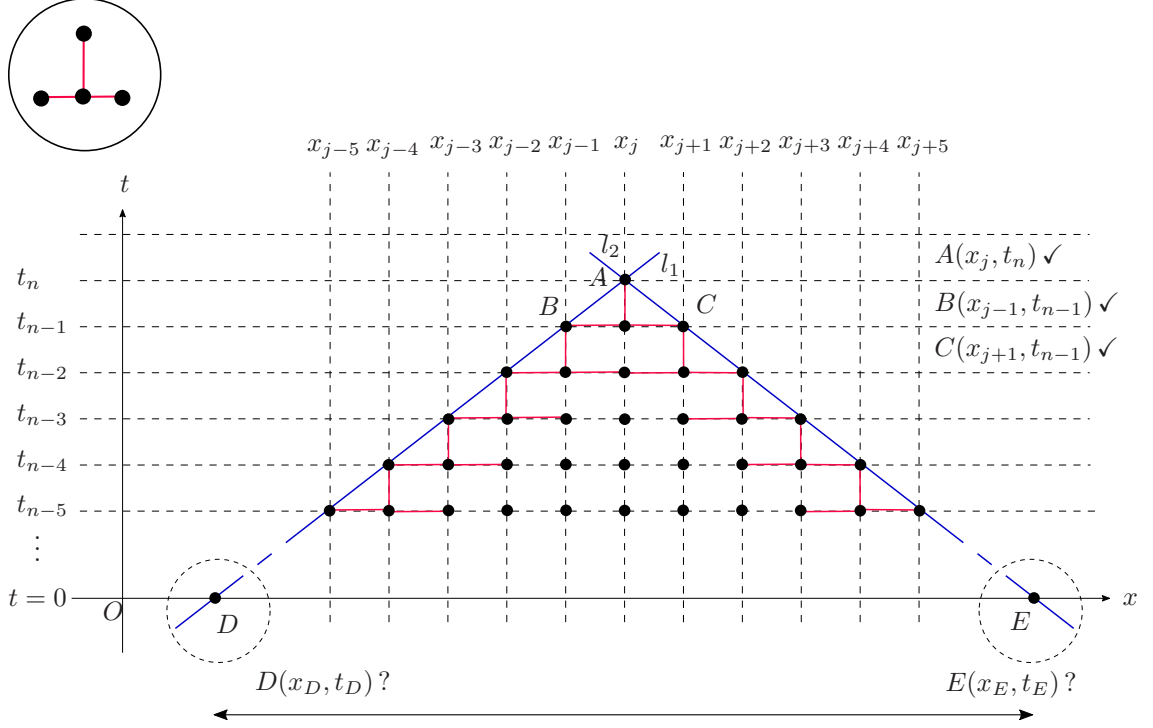


Figure 3: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \quad (22)$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (23)$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid |x - x_j| \leq \frac{at_n}{\nu} \right\}. \quad (24)$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (25)$$

4. CFL condition reads

$$\therefore \quad \left| \frac{a\Delta t}{\Delta x} \right| \leq 1. \quad (26)$$

Example 4. *Examine the numerical domain of dependence of Lax-Friedrichs method.*

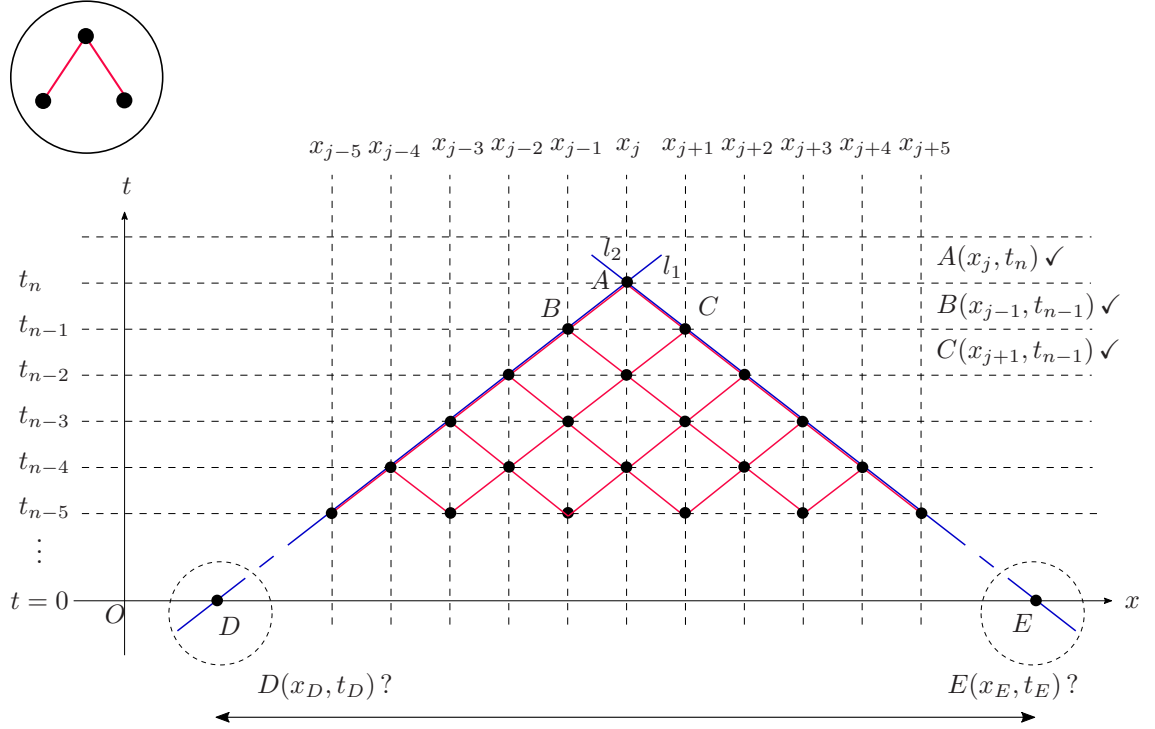


Figure 4: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1, or the same as 3 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \quad (27)$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (28)$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid |x - x_j| \leq \frac{at_n}{\nu} \right\}. \quad (29)$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (30)$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a\Delta t}{\Delta x} \right| \leq 1.} \quad (31)$$

3 von Neumann stability analysis

Example 5. *Perform the von Neumann stability analysis for Lax-Friedrichs scheme:*

$$u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\nu}{2} (u_{j+1}^n - u_{j-1}^n),$$

where $v := a\Delta t/\Delta x$ is the Courant number.

For the Lax - Friedrich scheme, the amplification factor is given by

$$g(\xi) = \cos \xi - \mathbf{i}\nu \sin \xi \quad (32)$$

which leads to

$$|g(\xi)| = \sqrt{\cos^2(\xi) + \nu^2 \sin^2(\xi)} \leq 1, \quad \text{for } 0 < \nu \leq 1. \quad (33)$$

Therefore, the *Lax-Friedrichs* scheme is *conditionally stable* for $0 < \nu \leq 1$.

Example 6. *von Neumann stability analysis for Upwind method.*

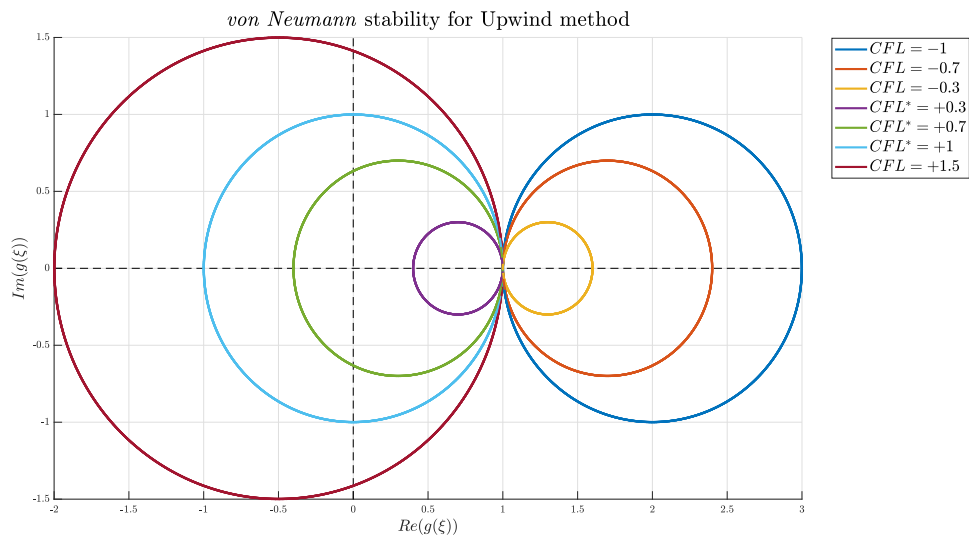


Figure 5: *von Neumann* stability analysis for Upwind method.

4 Conservative form - Finite Volume Method

Example 7. *Conservative form*