

# Global Exercise - 12

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## 1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1:  $W := R^{-1}U$  as the scheme shown in exercise

$$\begin{array}{l} U_t + AU_x = 0 \\ U_t + R\Lambda R^{-1}U_x = 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x = 0 \\ W_t + \Lambda W_x = 0 \end{array}$$

where the matrix  $A$  is diagonalizable with a transformation matrix  $R \in \mathbb{R}^{N \times N}$  in the form

$$A = R\Lambda R^{-1}.$$

2. Case 2:  $W := TU$  as the scheme shown in lecture note

$$\begin{array}{l} U_t + AU_x = 0 \\ U_t + T^{-1}\Lambda TU_x = 0 \\ TU_t + \Lambda TU_x = 0 \\ W_t + \Lambda W_x = 0 \end{array}$$

where the matrix  $A$  is diagonalizable with a transformation matrix  $T \in \mathbb{R}^{N \times N}$  in the form

$$A = T^{-1}\Lambda T.$$

Both schemes result in the same solution. We have just to be consistent with which scheme to follow.

## 2 Linearity - Oder of consistency - Stability

## 3 Time step methods: Comparison

1. Upwind left - One-sided method
2. Upwind right - One-sided method
3. Lax-Friedrichs
4. Lax-Wendroff
5. Leapfrog
- 6.

- 4 Courant-Friedrichs-Lewy (CFL) Condition
- 5 von Neumann stability
- 6 Conservative form
- 7 Comparison: Order of consistency
- 8 Godunov
- 9 TVD
- 10 Fourier transformation

**Example 1.** Consider

$$u_t + Au_x = 0 \quad \text{for } (x, t) \in \mathbb{R} \times \mathbb{R}^+,$$

where the unknown vector  $u(x, t)$  is defined, and the matrix  $A$  is given as follows

$$u(x, t) := \begin{pmatrix} u_1(x, t) \\ u_2(x, t) \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and the initial conditions are given by

$$\begin{cases} u_1(x, 0) = 1, \\ u_2(x, 0) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases} \end{cases}$$

Approach: Eigenvalue decomposition of matrix  $A$  is given as follows

$$A = R\Lambda R^{-1}, \quad R = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2}R, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Hence, we can now change variable  $u$  to a new variable  $w$  as follows

$$w := R^{-1}u = \frac{1}{2}Ru, \quad \text{so} \quad u = Rw.$$

Then, the initial problem can be rewritten as

$$w_t + \Lambda w_x = 0 \Leftrightarrow \begin{cases} (w_1)_t + \lambda_1(w_1)_x = 0, \\ (w_2)_t + \lambda_2(w_2)_x = 0, \end{cases}$$

with the diagonal matrix  $\Lambda$ . This step simply means that variables  $w_1$  and  $w_2$  are independent of each other, i.e. the original system has been fully decoupled. Next, let us find the initial condition in terms of  $w$  as follows

$$w(x, 0) = \frac{1}{2}Ru(x, 0) = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1(x, 0) \\ u_2(x, 0) \end{pmatrix} = \begin{pmatrix} -(u_1(x, 0) + u_2(x, 0))/2 \\ -(u_1(x, 0) - u_2(x, 0))/2 \end{pmatrix},$$

which yields the following relation

$$\begin{pmatrix} w_1(x, 0) \\ w_2(x, 0) \end{pmatrix} = \begin{pmatrix} -(u_1(x, 0) + u_2(x, 0))/2 \\ -(u_1(x, 0) - u_2(x, 0))/2 \end{pmatrix}.$$

Therefore, the initial conditions for the transformed variable  $w_1$  read

$$w_1(x, 0) = \begin{cases} -(1 + 1)/2 = -1, & x < 0, \\ -(1 - 1)/2 = 0, & x > 0, \end{cases}$$

and for the variable  $w_2$  read

$$w_2(x, 0) = \begin{cases} -(1 - 1)/2 = 0, & x < 0, \\ -(1 - (-1))/2 = -1, & x > 0. \end{cases}$$

Since we have the advection problem with speeds  $a_1 = 1$  and  $a_2 = 3$ , which are the diagonal values of the diagonal matrix  $\Lambda$ , for both  $w_1$  and  $w_2$ , we can derive explicit solution to each equation. The explicit solution  $w_1$  reads

$$w_1(x, t) = w_1(x - t, 0) = \begin{cases} -1, & x - t < 0, \\ 0, & x - t > 0, \end{cases} \Leftrightarrow w_1(x, t) = \begin{cases} -1, & x < t, \\ 0, & x > t, \end{cases}$$

and, likewise, the explicit solution  $w_2$  reads

$$w_2(x, t) = w_2(x - 3t, 0) = \begin{cases} 0, & x - 3t < 0, \\ -1, & x - 3t > 0, \end{cases} \Leftrightarrow w_2(x, t) = \begin{cases} 0, & x < 3t, \\ -1, & x > 3t. \end{cases}$$

Now we trace back to variable  $u$  from variable  $w$  by taking into the consideration of the relation

$$u(x, t) = Rw(x, t) = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1(x, t) \\ w_2(x, t) \end{pmatrix} = \begin{pmatrix} -(w_1(x, t) + w_2(x, t)) \\ -(w_1(x, t) - w_2(x, t)) \end{pmatrix},$$

which means the following relation holds

$$\begin{pmatrix} u_1(x, t) \\ u_2(x, t) \end{pmatrix} = \begin{pmatrix} -(w_1(x, t) + w_2(x, t)) \\ -(w_1(x, t) - w_2(x, t)) \end{pmatrix}.$$

Therefore, the solution for  $u_1(x, t)$  is computed as follows

$$u_1(x, t) = \begin{cases} -(-1 + 0) = 1, & x < t, \\ -(0 + 0) = 0, & t < x < 3t, \\ -(0 + (-1)) = 1, & x > 3t, \end{cases}$$

and, likewise, the solution for  $u_2(x, t)$  is

$$u_2(x, t) = \begin{cases} -(-1 - 0) = 1, & x < t, \\ -(0 - 0) = 0, & t < x < 3t, \\ -(0 - (-1)) = -1, & x > 3t. \end{cases}$$

Finally, the solution to  $u_1(x, t)$  and  $u_2(x, t)$  read

$$\therefore \begin{cases} u_1(x, t) = \begin{cases} 1, & x < t, \\ 0, & t < x < 3t, \\ 1, & x > 3t, \end{cases} \\ u_2(x, t) = \begin{cases} 1, & x < t, \\ 0, & t < x < 3t, \\ -1, & x > 3t. \end{cases} \end{cases}$$

## 11 Remark on step-by-step derivation: Weak formulation of conservation laws

**Example 2.** Consider the conservation law

$$u_t + f(u)_x = 0 \quad \text{for } (x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad (1)$$

with initial condition given by

$$u(x, 0) = u_0(x).$$

Find the weak solution.

Approach: The weak formulation is obtained by multiplying both sides of (1) with a test function  $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R})$ , and, then, integrating both sides over  $\mathbb{R} \times \mathbb{R}^+$

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \varphi (u_t + f(u)_x) dx dt = 0. \quad (2)$$

The derivation goes as follows

$$\begin{aligned}
 (2) &\Leftrightarrow \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi (u_t + f(u)_x) dx dt = 0 \\
 &\Leftrightarrow \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi (u_t) dx dt + \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi (f(u)_x) dx dt = 0 \\
 &\Leftrightarrow \int_{\mathcal{D}} \int_0^{\mathcal{T}} \varphi (u_t) dt dx + \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi (f(u)_x) dx dt = 0 \\
 &\Leftrightarrow \left[ \int_{\mathcal{D}} \varphi u \Big|_{t=0}^{t=+\infty} dx - \int_{\mathcal{D}} \int_0^{\mathcal{T}} \varphi_t u dt dx \right] \\
 &\quad + \left[ \int_0^{\mathcal{T}} \varphi f(u) \Big|_{x=-\infty}^{x=+\infty} dx dt - \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi_x f(u) dx dt \right] = 0 \\
 &\Leftrightarrow \left[ \int_{\mathcal{D}} \left( \varphi(x, \overset{t \rightarrow +\infty}{\mathcal{T}}) u(x, \overset{t \rightarrow +\infty}{\mathcal{T}}) - \varphi(x, 0) u(x, 0) \right) dx - \int_{\mathcal{D}} \int_0^{\mathcal{T}} \varphi_t u dt dx \right] \\
 &\quad + \left[ \int_0^{\mathcal{T}} \left( \varphi(\overset{x \rightarrow +\infty}{\mathcal{D}}, t) f(u(\overset{x \rightarrow +\infty}{\mathcal{D}}, t)) - \varphi(\overset{x \rightarrow -\infty}{\mathcal{D}}, t) f(u(\overset{x \rightarrow -\infty}{\mathcal{D}}, t)) \right) dt \right. \\
 &\quad \left. - \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi_x f(u) dx dt \right] = 0 \\
 &\Leftrightarrow \left[ \int_{\mathcal{D}} \left( \overset{0}{\varphi(x, \overset{t \rightarrow +\infty}{\mathcal{T}})} u(x, \overset{t \rightarrow +\infty}{\mathcal{T}}) - \varphi(x, 0) u(x, 0) \right) dx - \int_{\mathcal{D}} \int_0^{\mathcal{T}} \varphi_t u dt dx \right] \\
 &\quad + \left[ \int_0^{\mathcal{T}} \left( \overset{0}{\varphi(\overset{x \rightarrow +\infty}{\mathcal{D}}, t)} f(u(\overset{x \rightarrow +\infty}{\mathcal{D}}, t)) - \overset{0}{\varphi(\overset{x \rightarrow -\infty}{\mathcal{D}}, t)} f(u(\overset{x \rightarrow -\infty}{\mathcal{D}}, t)) \right) dt \right. \\
 &\quad \left. - \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi_x f(u) dx dt \right] = 0 \\
 &\Leftrightarrow - \int_{\mathcal{D}} \varphi(x, 0) u(x, 0) dx - \int_{\mathcal{D}} \int_0^{\mathcal{T}} \varphi_t u dt dx - \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi_x f(u) dx dt = 0 \\
 &\Leftrightarrow - \int_{\mathcal{D}} \varphi(x, 0) u_0(x) dx - \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi_t u dx dt - \int_0^{\mathcal{T}} \int_{\mathcal{D}} \varphi_x f(u) dx dt = 0.
 \end{aligned}$$

Therefore, the weak formulation reads

$$\therefore \boxed{\int_0^{\mathcal{T}} \int_{\mathcal{D}} (u\varphi_t + f(u)\varphi_x) dxdt + \int_{\mathcal{D}} \varphi(x, 0)u_0(x)dx = 0.}$$

Note in passing that the derivation above has taken into consideration the swapping order of integral limits between time  $t \in [0, +\infty]$  and space  $x \in [-\infty, +\infty]$ , which results in indifference as shown in Figure 1.

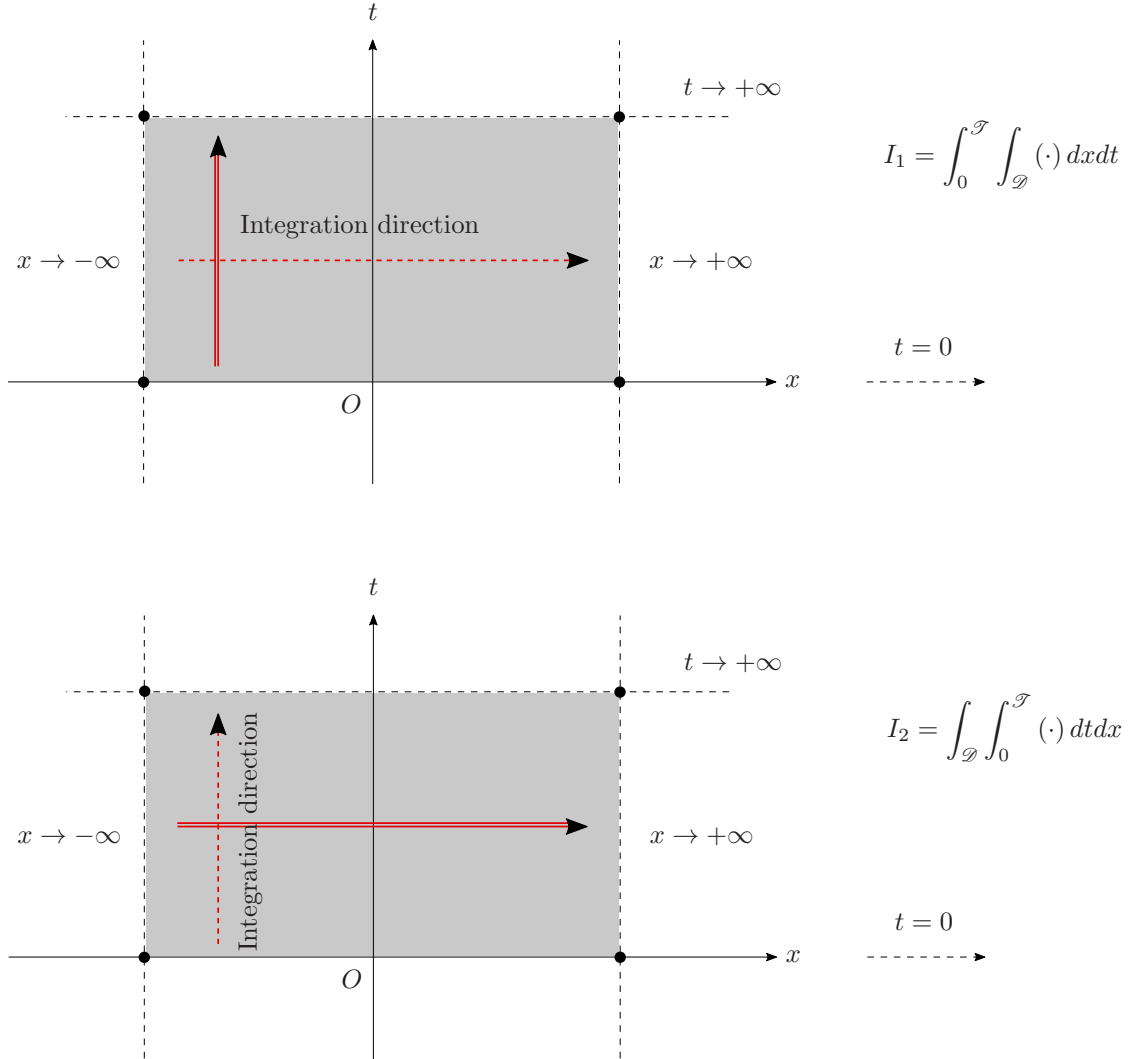


Figure 1: Indifference of limits when swapping order of integrals, i.e.  $I_1 = I_2$ .

## 12 Remark on step-by-step derivation: Weak solution of inviscid Burgers' equation in Riemann's problem

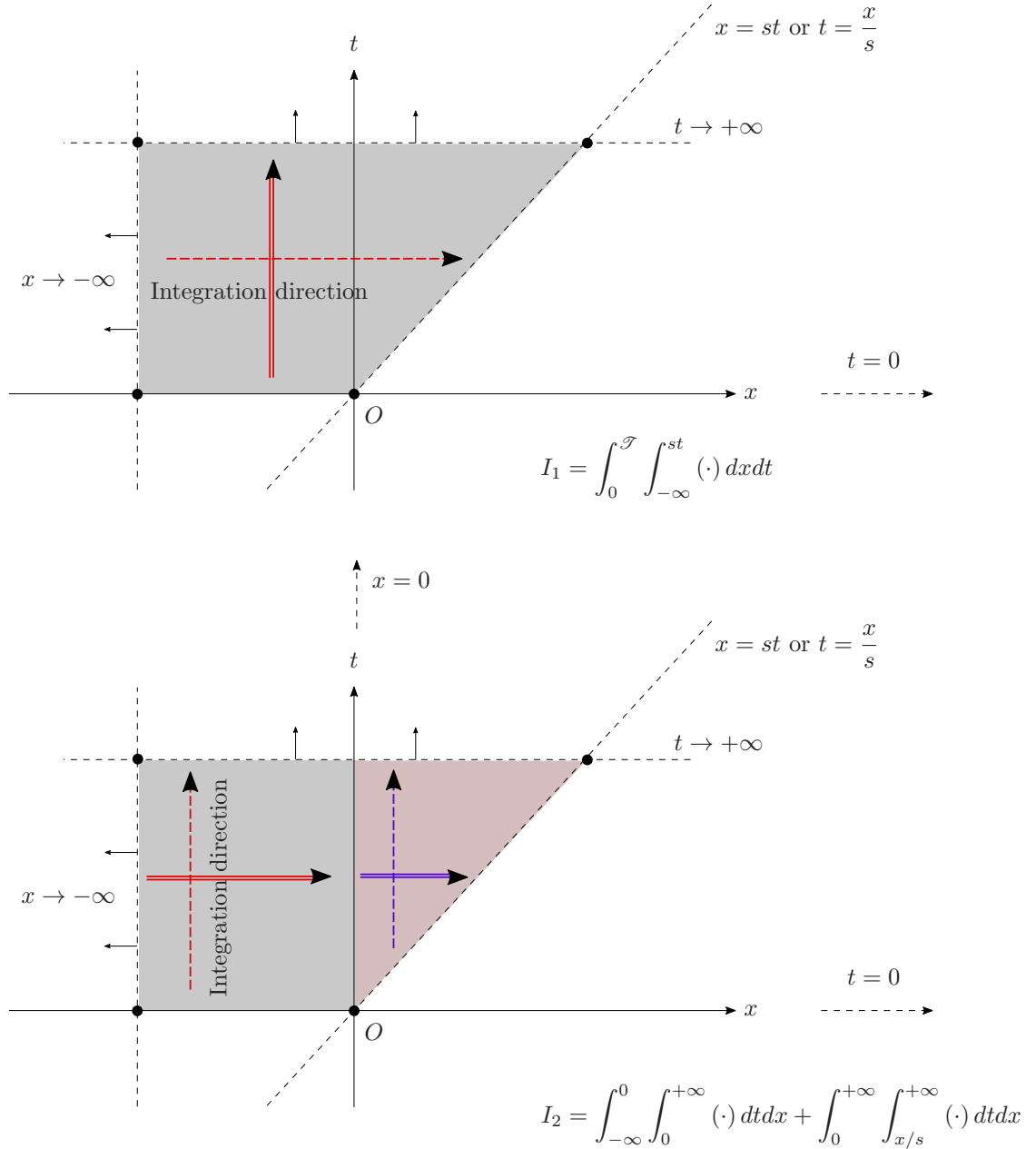


Figure 2: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee  $I_1 = I_2$ .

**Example 3.** Given Burgers' equation with initial condition

$$u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$

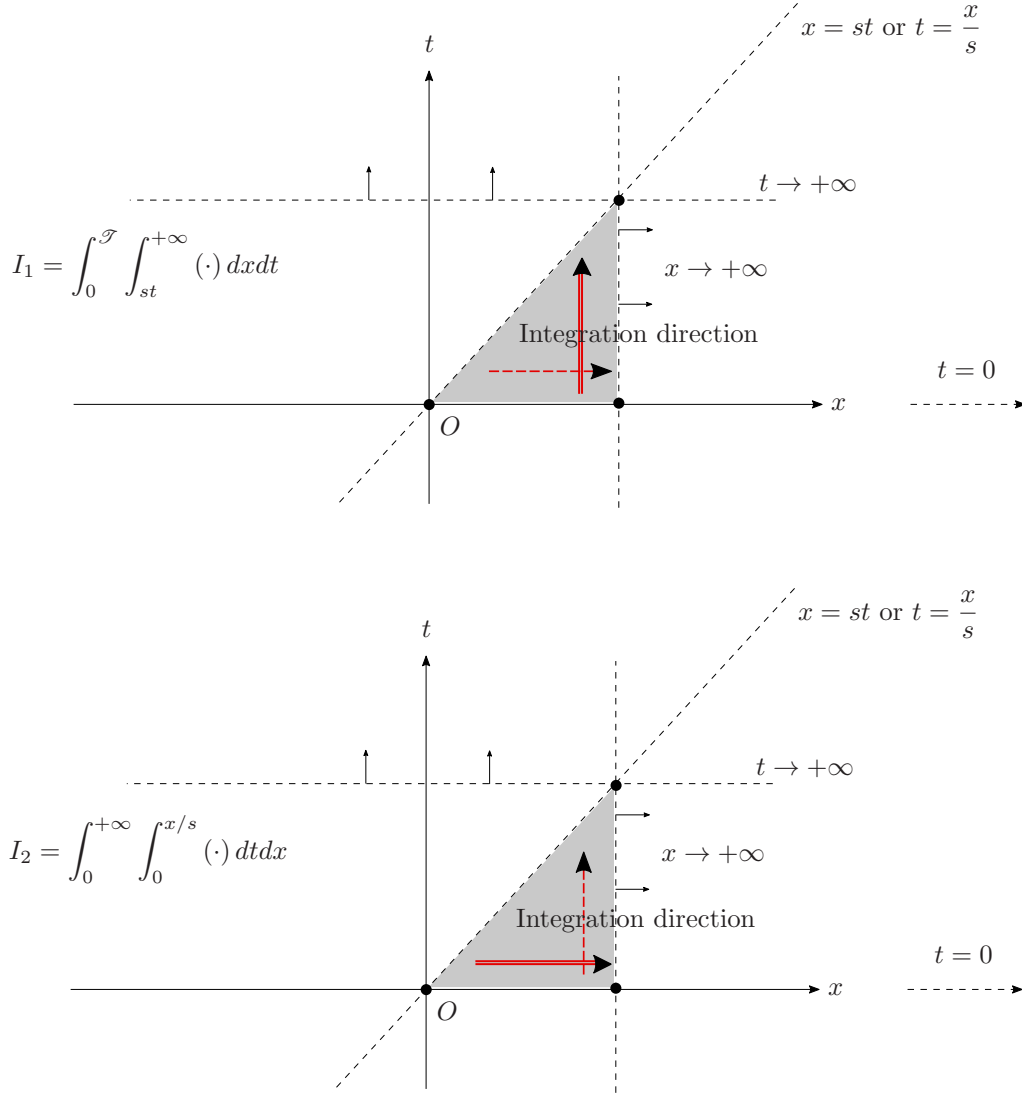


Figure 3: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee  $I_1 = I_2$ .

Then

$$u(x, t) = \begin{cases} u_L, & x < st, \\ u_R, & x > st, \end{cases}$$

is the weak solution to Burgers' equation.

*Proof.* RHS term

$$\begin{aligned} RHS &= - \int_{\mathcal{D}} \varphi(x, 0) u_0(x) dx \\ &= -u_L \int_{-\infty}^0 \varphi(x, 0) dx - u_R \int_0^{+\infty} \varphi(x, 0) dx \end{aligned}$$



LHS term

$$\begin{aligned}
LHS &= \int_0^{\mathcal{T}} \int_{-\infty}^{st} \varphi_t u \, dx dt + \int_0^{\mathcal{T}} \int_{st}^{+\infty} \varphi_t u \, dx dt \\
&\quad + \int_0^{\mathcal{T}} \int_{-\infty}^{st} \varphi_x f(u) \, dx dt + \int_0^{\mathcal{T}} \int_{st}^{+\infty} \varphi_x f(u) \, dx dt \\
&= u_L \int_0^{\mathcal{T}} \int_{-\infty}^{st} \varphi_t \, dx dt + u_R \int_0^{\mathcal{T}} \int_{st}^{+\infty} \varphi_t \, dx dt \\
&\quad + \frac{u_L^2}{2} \int_0^{\mathcal{T}} \int_{-\infty}^{st} \varphi_x \, dx dt + \frac{u_R^2}{2} \int_0^{\mathcal{T}} \int_{st}^{+\infty} \varphi_x \, dx dt \\
&= \left( u_L \int_{-\infty}^0 \int_0^{+\infty} \varphi_t \, dt dx + u_L \int_0^{+\infty} \int_{x/s}^{+\infty} \varphi_t \, dt dx \right) \\
&\quad + u_R \int_0^{+\infty} \int_0^{x/s} \varphi_t \, dt dx \\
&\quad + \frac{u_L^2}{2} \int_0^{\mathcal{T}} \int_{-\infty}^{st} \varphi_x \, dx dt + \frac{u_R^2}{2} \int_0^{\mathcal{T}} \int_{st}^{+\infty} \varphi_x \, dx dt \\
&= u_L \int_{-\infty}^0 \varphi|_{t=0}^{t=+\infty} dx + u_L \int_0^{+\infty} \varphi|_{t=x/s}^{t=+\infty} dx + u_R \int_0^{+\infty} \varphi|_{t=0}^{t=x/s} dx \\
&\quad + \frac{u_L^2}{2} \int_0^{+\infty} \varphi|_{x=-\infty}^{x=st} dt + \frac{u_R^2}{2} \int_0^{+\infty} \varphi|_{x=st}^{x=+\infty} dt \\
&= u_L \int_{-\infty}^0 (-\varphi(x, 0)) \, dx + u_L \int_0^{+\infty} (-\varphi(x, x/s)) \, dx \\
&\quad + u_R \int_0^{+\infty} (\varphi(x, x/s) - \varphi(x, 0)) \, dx \\
&\quad + \frac{u_L^2}{2} \int_0^{+\infty} \varphi(st, t) dt + \frac{u_R^2}{2} \int_0^{+\infty} (-\varphi(st, t)) \, dt \\
&= -u_L \int_{-\infty}^0 \varphi(x, 0) dx - u_R \int_0^{+\infty} \varphi(x, 0) dx \\
&\quad - (u_L - u_R) \int_0^{+\infty} \varphi(x, x/s) dx + (u_L - u_R) \int_0^{+\infty} \varphi(w, w/s) dw \\
&= -u_L \int_{-\infty}^0 \varphi(x, 0) dx - u_R \int_0^{+\infty} \varphi(x, 0) dx = RHS
\end{aligned}$$

Note in passing that we have used the following relation

$$\begin{aligned}
\frac{u_L^2}{2} \int_0^{+\infty} \varphi(st, t) dt + \frac{u_R^2}{2} \int_0^{+\infty} (-\varphi(st, t)) \, dt &= \frac{u_L^2 - u_R^2}{2} \int_0^{+\infty} \varphi(st, t) \, dt \\
&= \frac{u_L^2 - u_R^2}{2s} \int_0^{+\infty} \varphi(w, w/s) dw \\
&= (u_L - u_R) \int_0^{+\infty} \varphi(w, w/s) dw
\end{aligned}$$

where  $w := st$  and  $dw = s dt$ , and the RH condition for *Burgers'* equation  $s = 1/2(u_L + u_R)$ .  $\square$

## 13 Consistency order

**Example 4.** Leapfrog scheme for the linear advection equation reads

$$u_j^{n+1} = u_j^{n-1} + \frac{a\Delta t}{\Delta x} (u_{j-1}^n - u_{j+1}^n).$$

Show that the local consistency error  $L_{\Delta t}$

$$\|L_{\Delta t}(\cdot, t)\| \leq C(\Delta t)^2 \quad \text{for } \Delta t \rightarrow 0,$$

hold for a linear solution and  $\Delta t/\Delta x = \text{const.}$

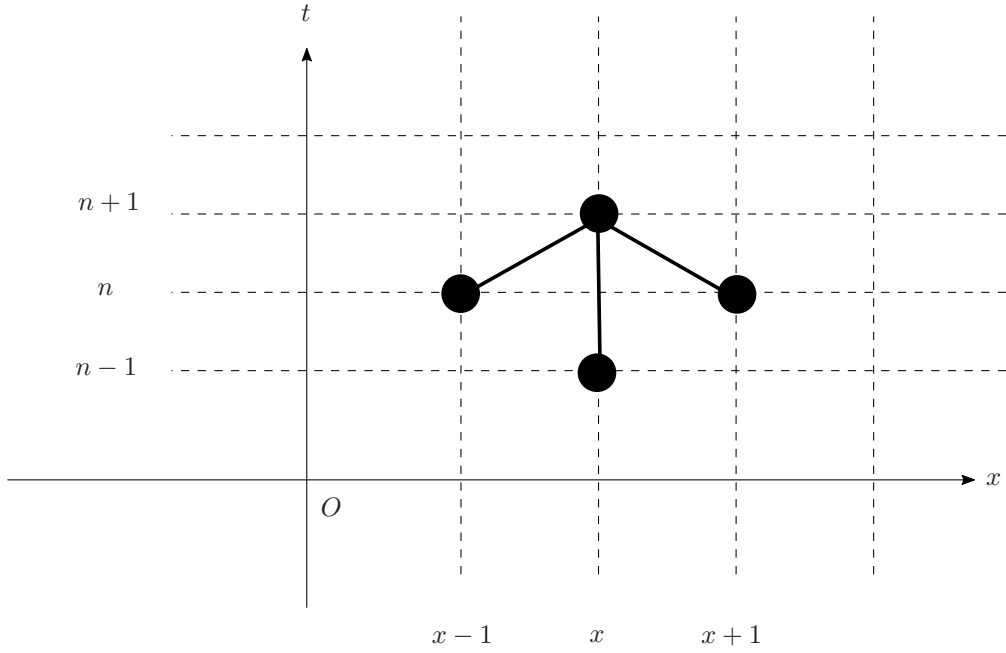


Figure 4: Leapfrog.

Consider the local truncation error

$$L_{\Delta t}(x, t) = \frac{1}{\Delta t} [u(x, t + \Delta t) - H_{\Delta t}(u(\cdot, t), x)],$$

which reads for the considered problem

$$L_{\Delta t}(x, t) = \frac{1}{\Delta t} \left[ u(x, t + \Delta t) - \left( u(x, t - \Delta t) + \frac{a\Delta t}{\Delta x} (u(x - \Delta x, t) - u(x + \Delta x, t)) \right) \right].$$

Applying Taylor's expansion for  $u(x, t + \Delta t)$

$$\begin{aligned} u(x, t + \Delta t) &= u(x, t) + u_t(x, t)\Delta t + u_{tt}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\ &= u(x, t) - au_x(x, t)\Delta t + a^2u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3). \end{aligned}$$

Applying Taylor's expansion for  $u(x, t - \Delta t)$

$$\begin{aligned} u(x, t - \Delta t) &= u(x, t) - u_t(x, t)\Delta t + u_{tt}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\ &= u(x, t) + au_x(x, t)\Delta t + a^2u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3). \end{aligned}$$

Furthermore, applying Taylor's expansion for  $u(x + \Delta x, t)$  and  $u(x - \Delta x, t)$

$$\begin{aligned} u(x + \Delta x, t) &= u(x, t) + u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3) \\ u(x - \Delta x, t) &= u(x, t) - u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3). \end{aligned}$$

Local truncation error

$$\begin{aligned} L_{\Delta t}(x, t) &= \frac{1}{\Delta t} \left[ u(x, t) - au_x(x, t)\Delta t + a^2u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \right. \\ &\quad - \left( u(x, t) + au_x(x, t)\Delta t + a^2u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \right) \\ &\quad + \frac{a\Delta t}{\Delta x} \left( \left( u(x, t) - u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3) \right) \right. \\ &\quad \left. \left. - \left( u(x, t) + u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3) \right) \right) \right] \\ &= \frac{1}{\Delta t} \left[ -2au_x(x, t)\Delta t + \mathcal{O}((\Delta t)^3) - \frac{a\Delta t}{\Delta x} \left( -2u_x(x, t)\Delta x + \mathcal{O}((\Delta x)^3) \right) \right] \\ &= \frac{1}{\Delta t} \mathcal{O}((\Delta t)^3) \\ &= \mathcal{O}((\Delta t)^2). \end{aligned}$$

Therefore, order of consistency for Leapfrog scheme is of order 2

$$\|L_{\Delta t}(\cdot, t)\| \leq C(\Delta t)^2 \quad \text{for } \Delta t \rightarrow 0.$$

Note in passing that  $u_t = -au_x$  and  $u_{tt} = a^2u_{xx}$  are from the linear advection equation.