Global Exercise - 9

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1 Conservation laws - Idea

Master balance principle: due to the fact that the mathematical structure of the fundamental balance relations, namely of mass, linear momentum, moment of momentum, energy and entropy, is in principle identical, they can be formulated within the concise shape of a master balance. To begin with, let Ψ and $\boldsymbol{\Psi}$ be volume specific scalar and vector value densities of a physical quantity to be balanced, respectively. Then, the general balance relations take the global **integral form** as follows

$$\frac{d}{dt}\int_{\Omega} \Psi \, d\Omega = -\int_{\partial\Omega} \boldsymbol{\phi}\cdot\boldsymbol{n}\, d\partial\Omega + \int_{\Omega} \sigma \, d\Omega + \int_{\Omega} \hat{\Psi}\, d\Omega \tag{1}$$

$$\frac{d}{dt}\int_{\Omega} \boldsymbol{\Psi} \, d\Omega = -\int_{\partial\Omega} \boldsymbol{\Phi}\cdot\boldsymbol{n}\, d\partial\Omega + \int_{\Omega} \boldsymbol{\sigma} \, d\Omega + \int_{\Omega} \hat{\boldsymbol{\Psi}}\, d\Omega \tag{2}$$

where $(\boldsymbol{\phi}\cdot\boldsymbol{n})$ and $(\boldsymbol{\Phi}\cdot\boldsymbol{n})$ describe action at the vicinity; σ and $\boldsymbol{\sigma}$ present for action from a distance; and $\hat{\Psi}$ and $\hat{\boldsymbol{\Psi}}$ for production or nucleation.

Example 1. *Conservation of mass: A derivation from integral to differential form.*

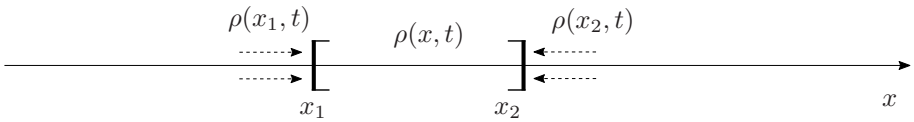

Figure 1: Conservation of mass in a one-dimensional gas dynamics problem.

Let x represent the distance along the tube and let $\rho(x,t)$ be the mass density of the fluid at point x and time t. Then, the total mass M in the section $[x_1, x_2]$ at time t is defined as follows

$$M := \int_{x_1}^{x_2} \rho(x,t)\, dx. \tag{3}$$

Assume that the walls of the tube are impermeable and mass is neither created nor destroyed, this total mass M in this interval $[x_1, x_2]$ varies over time only due to the fluid flowing across at the boundaries $x = x_1$ and $x = x_2$. Now let $v(x,t)$ be the measured velocity of flow at point x and time t. Then, the flow rate (also called *rate of flow*, or *flux*) passing the end points of the interval $[x_1, x_2]$ at time t is given by

$$\rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t). \tag{4}$$

The rate of change of mass in $[x_1, x_2]$ given by the difference of fluxes at x_1 and x_2 flow rate has to be equal with the time rate of change of total mass

$$\boxed{\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\, dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t)} \tag{5}$$

which is the **integral form** of the conservation law. Next, note that the following relation holds

$$\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(\rho)\, dx = f(\rho(x_2, t)) - f(\rho(x_1, t)). \tag{6}$$

Substitution of (6) into (5) leads to

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(x, t)\, dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(\rho)\, dx. \tag{7}$$

If $\rho(x, t)$ is sufficiently smooth, time derivative on the LHS of (7) can be brought inside the integral. Then, an arrangement of (7) yields

$$\int_{x_1}^{x_2} \left(\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} f(\rho) \right) dx = 0. \tag{8}$$

Since (8) holds for any choice of x_1 and x_2, the integrant must be vanished everywhere. Therefore, one obtains the **differential form** of the conservation law

$$\therefore \quad \boxed{\rho_t + f(\rho)_x = 0} \tag{9}$$

Example 2. *Different definitions of flux function yields different applications*

1. Advection equation: (9) with flux function $f(\rho) = a\rho,\ a \in \mathbb{R}^+$, taking form

$$\boxed{\rho_t + a\rho_x = 0} \tag{10}$$

2. *Inviscid* Burgers' equation: (9) with flux function $f(\rho) = \frac{1}{2}\rho^2$, yielding

$$\boxed{\rho_t + \rho\rho_x = 0} \tag{11}$$

   which is used to illustrate the distortion of waveform in simple waves. Meanwhile, *viscous* Burgers' equation is a scalar parabolic PDE, taking the form

$$\rho_t + \rho\rho_x = \varepsilon\rho_{xx}$$

   which is the simplest differential model for a fluid flow.

3. Lighthill-Whitham-Richards (LWR) equation: (9) with flux function

$$f(\rho) = u_{max}\, \rho \left(1 - \frac{\rho}{\rho_{max}} \right)$$

   which is used to model traffic flow; u_{max} is the given maximal speed of vehicle to be allowed, ρ_{max} the given maximal density of vehicle; taking the form

$$\rho_t + \left(u_{max} - \frac{2u_{max}}{\rho_{max}} \rho \right) \rho_x = 0 \tag{12}$$

4. (Typical) Buckley-Leverett petroleum equation: (9) with flux function

$$f(\rho) = \frac{\rho^2}{\rho^2 + (1-\rho)^2 \mu_{\text{water}}/\mu_{\text{oil}}}$$

   which is a one-dimensional model for a two-phase flow; ρ stands for water saturation and takes the value in the interval $[0, 1]$.

5. *Euler* equations of gas dynamics in 1D

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

$\rightarrow$ The pressure p should be specified as a given function of mass density ρ, linear momentum ρv and/or energy E in order to fulfill the *closure* problem. Such additional equation is called *equation of state* (normally in fluid), or *constitutive equation* (normally in solid).

6. *Euler* equations of gas dynamics in 2D

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

Definition 1. *Cauchy problem and Riemann problem*

Let u be a conserved unknown quantity to be modelled and defined as follows

$$u : \begin{cases} \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}, \\ (x,t) \mapsto u(x,t). \end{cases} \tag{15}$$

Cauchy's problem is simply the pure initial value problem (IVP), e.g. find a function u holding (15) that is a solution of $(16)_1$ satisfying the initial condition $(16)_2$:

$$\boxed{\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \\ u(x,0) &= u_0(x). \end{aligned}} \tag{16}$$

Riemann's problem is simply the conservation law together with particular initial data consisting of two constant states separated by a single discontinuity, e.g. find a function u holding (15) that is a solution of $(17)_1$ satisfying the initial condition $(17)_2$:

$$\boxed{\begin{aligned} &\partial_t u + \partial_x f(u) = 0, \\ &\qquad u(x,0) = u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \end{aligned}} \tag{17}$$

2 Characteristics

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function. We consider the *Cauchy* problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \qquad x \in \mathbb{R}, t > 0, \tag{18}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{19}$$

By setting $a(u) = f'(u)$, and letting u be a classical solution of (18), one obtains from (18) the *non-conservative* form as follows

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0. \tag{20}$$

The characteristic curves associated with (18) are defined as the integral curves of the differential equation

$$\frac{dx}{dt} = a(u(x(t), t)). \tag{21}$$

Proposition 1. *Assume that u is a smooth solution of the Cauchy problem. The characteristic curves are straight lines along which u is constant.*

Proof. Consider a characteristic curve passing through the point $(x_0, 0)$, i.e. a solution of the ordinary differential system

$$\begin{cases} \dfrac{dx}{dt} & = a(u(x(t), t)), \\ x(0) & = x_0. \end{cases} \tag{22}$$

It exists at least on a small time interval $[0, t_0)$. Along such a curve, u is constant since

$$\begin{aligned} \frac{d}{dt}u(x(t), t) &= \frac{\partial u}{\partial t}(x(t), t) + \frac{\partial u}{\partial x}(x(t), t)\frac{dx}{dt}(t) \\ &= \left(\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x}\right)(x(t), t) = 0. \end{aligned}$$

Therefore, the characteristic curves are straight lines whose constant slopes depend on the initial data. As a result, the characteristic straight line passing through the point $(x_0, 0)$ is defined by the equation

$$\boxed{x = x_0 + ta(u_0(x_0))} \tag{23}$$

□

Example 3. *Determine characteristics of the inviscid Burgers' equation with initial conditions given as*

$$u(x, 0) = u_0(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & x \geq 1. \end{cases} \quad x \in \mathbb{R}, \tag{24}$$

By using the method of characteristics, the solution can be solved up to the time when those characteristics first intersect with each other, i.e. so-called breaking time or shock. Since the $f'(u) = u$ in the inviscid Burgers' equation, it yields the characteristics passing through the point (x_0, t)

$$x = x_0 + t\, u_0(x_0) \tag{25}$$

which leads to

$$x(x_0, t) = \begin{cases} x_0 + t, & x_0 \leq 0, \\ x_0 + t(1 - x_0), & 0 \leq x_0 \leq 1, \\ x_0, & x_0 \geq 1. \end{cases} \tag{26}$$

(Sketch)