Global Exercise - 13

Tuan Vo

19$^{\text{th}}$ January 2022

1   Conservation form - Finite Volume Method (cont.)
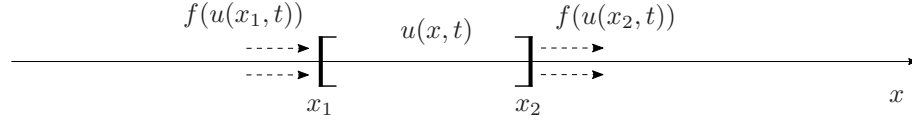
Example 1. *Derivation of conservation form.*

Figure 1: Conservation for a scalar conserved quantity $u(x,t)$ over domain $[x_1, x_2]$.

Starting point is the **integral form** of conservation law. Recall the integral formulation of conservation law for a scalar conserved quantity $u(x,t)$ over domain $[x_1, x_2]$, herein, reads

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x,t)\, dx = f(u(x_1,t)) - f(u(x_2,t)). \tag{1}$$

Integration both side of (1) over temporal interval $[t_1, t_2]$ yields

$$\left(\int_{x_1}^{x_2} u(x,t)\, dx \right) \bigg|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \left(f(u(x_1,t)) - f(u(x_2,t)) \right) dt, \tag{2}$$

which leads to ***another* integral form** of the conservation law, as follows

$$\int_{x_1}^{x_2} u(x,t_2)\, dx = \int_{x_1}^{x_2} u(x,t_1)\, dx + \int_{t_1}^{t_2} f(u(x_1,t))\, dt - \int_{t_1}^{t_2} f(u(x_2,t))\, dt. \tag{3}$$

The integral form of conservation law shown in (3) will be now used to derive the conservation form for finite volume method (FVM). Then, the derivation is performed by taking into consideration of the spatial interval $[x_{j-1/2}, x_{j+1/2}]$ and temporal interval $[t_n, t_{n+1}]$ instead of $[x_1, x_2]$ and $[t_1, t_2]$, respectively, as follows

$$[x_1, x_2] \rightarrow [x_{j-1/2}, x_{j+1/2}], \tag{4}$$
$$[t_1, t_2] \rightarrow [t_n, t_{n+1}]. \tag{5}$$

Hence, the integral form in (3) becomes

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})\, dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\, dx$$
$$+ \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt. \tag{6}$$
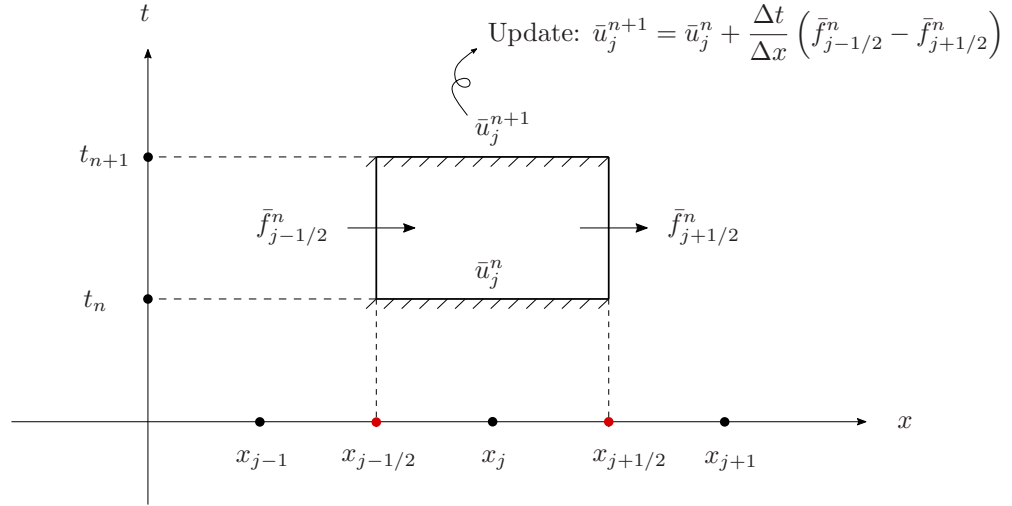
Figure 2: Finite volume update: conservation form.

Next, consideration of the cell average formulation, which, by definition, reads

$$\bar{u}_j^n := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\, dx, \tag{7}$$

and multiplication of both sides of (6) by $1/\Delta x$ yield the following relation

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{1}{\Delta x} \left(\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt \right). \tag{8}$$

Moreover, by defining the **numerical flux functions** as follows

$$\tilde{f}_{j-1/2}^n := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt, \tag{9}$$

$$\tilde{f}_{j+1/2}^n := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt, \tag{10}$$

the expression (8) finally becomes the **conservation form**

$$\therefore \quad \boxed{\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n \right).} \tag{11}$$

Likewise, a generalization to nonlinear *systems* takes the following form

$$\therefore \quad \boxed{\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n \right).} \tag{12}$$

Example 2. *Derivation of the conservative form of upwind scheme. Examine its numerical flux function: conservative? consistent?*

Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right). \tag{13}$$

Example 3. *Derivation of the conservative form of Lax-Friedrichs scheme. Examine its numerical flux function: conservative? consistent? Verify that the* Lax-Friedrichs *numerical flux function is consistent, including Lipschitz continuous.*

Approach: Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n \right).$$
(14)

The generalization of the *Lax-Friedrichs* scheme to nonlinear systems takes the following form

$$U_j^{n+1} = \frac{1}{2} \left(U_{j-1}^n + U_{j+1}^n \right) + \frac{\Delta t}{2\Delta x}$$
(15)

Example 4. *Derivation of the conservative form of Lax-Wendroff scheme. Examine its numerical flux function: conservative? consistent? Consider the nonlinear scalar conservation law given as follows*

$$u_t + f(u)_x = 0.$$

Write the Lax-Wendroff *scheme for the above equation in* conservation form*. Show that the numerical flux function is consistent.*

Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n \right). \tag{16}$$

Example 5. *Consider the two-step* Lax-Wendroff *scheme as follows*

Step-1: First, update the half-advanced-in-time point $u_{j+1/2}^{n+1/2}$ by using

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{\Delta t}{2\Delta x}\left(f\left(u_{j+1}^n\right) - f\left(u_j^n\right)\right), \qquad (17)$$

Step-2: Then, update the fully-advanced-in-time point u_j^{n+1} by using

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(f\left(u_{j+1/2}^{n+1/2}\right) - f\left(u_{j-1/2}^{n+1/2}\right)\right). \qquad (18)$$

Examine the two-step Lax-Wendroff scheme whether it can be written in the conservation form. *Examine also if its numerical flux function is consistent.*

Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right). \qquad (19)$$

Idea:

> One way to avoid using Jacobian matrix arising in flux function, when dealing with linear hyperbolic system $U_t + AU_x = 0$.   (20)

Approach:

1. Conservation form: A numerical scheme applied for the conservation law $u_t + f(u)_x = 0$ is said to be conservative if it can be written in the conservation form, as follows

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right), \qquad (21)$$

where $\tilde{f}_{j-1/2}^n$ and $\tilde{f}_{j+1/2}^n$ are called interfacial fluxes. These fluxes are given by

$$\tilde{f}_{j+1/2}^n = \tilde{f}\left(u_{j-m+1}^n, \ldots, u_{j+m}^n\right), \qquad (22)$$
$$\tilde{f}_{j-1/2}^n = \tilde{f}\left(u_{j-m}^n, \ldots, u_{j+m-1}^n\right). \qquad (23)$$

In case of the *Lax-Wendroff* scheme for the nonlinear conservation law equation, the conservation form is derived as follows

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) \qquad (24)$$
$$+ \frac{(\Delta t)^2}{2\left(\Delta x\right)^2}\left(a^2(u_j^n, u_{j+1}^n)(u_{j+1}^n - u_j^n) - a^2(u_{j-1}^n, u_j^n)(u_j^n - u_{j-1}^n)\right), \qquad (25)$$

which is rewritten by grouping relative terms together, as follows

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}\left\{f(u_j^n) + f(u_{j+1}^n)\right\} - \frac{\Delta t}{2\Delta x}a^2(u_j^n, u_{j+1}^n)(u_{j+1}^n - u_j^n)\right)$$
$$+ \frac{\Delta t}{\Delta x}\left(\frac{1}{2}\left\{f(u_{j-1}^n) + f(u_j^n)\right\} - \frac{\Delta t}{2\Delta x}a^2(u_{j-1}^n, u_j^n)(u_j^n - u_{j-1}^n)\right). \qquad (26)$$

Then, this form can be written as follows

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \left(f_{j+1/2}^n - f_{j-1/2}^n \right),$$

where

$$\tilde{f}_{j+1/2}^n = \tilde{f}(u_j^n, u_{j+1}^n) = \frac{1}{2} \left\{ f(u_j^n) + f(u_{j+1}^n) \right\} - \frac{\Delta t}{2\Delta x} a^2(u_j^n, u_{j+1}^n)(u_{j+1}^n - u_j^n),$$

$$\tilde{f}_{j-1/2}^n = \tilde{f}(u_{j-1}^n, u_j^n) = \frac{1}{2} \left\{ f(u_{j-1}^n) + f(u_j^n) \right\} - \frac{\Delta t}{2\Delta x} a^2(u_{j-1}^n, u_j^n)(u_j^n - u_{j-1}^n).$$

Finally, the conservative numerical flux function is then given by

$$f(u, v) = \frac{1}{2} \left\{ f(u) + f(v) \right\} - \frac{\Delta t}{2\Delta x} a^2(u, v)(u - v).$$

2. Consistent numerical flux function:
   When $v = u$, we obtain the relation

$$F(u, v) = F(u, u) = f(u).$$

This clearly shows that the numerical flux function is consistent with the continuous flux function.

Example 6. *Consider the* MacCormack *scheme given by*

$$u_j^{n+1/2} = u_j^n - \lambda \left(f\left(u_{j+1}^n\right) - f\left(u_j^n\right) \right)$$

$$u_j^{n+1} = \frac{1}{2} \left(u_j^n + u_j^{n+1/2} \right) - \frac{\lambda}{2} \left(f\left(u_j^{n+1/2}\right) - f\left(u_{j-1}^{n+1/2}\right) \right)$$

with $\lambda := \Delta t / \Delta x$. *Examine the MacCormack scheme whether it is conservative, i.e. the schemes can be written in the* conservation form. *Examine also if its numerical flux function is consistent.*

Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n \right). \tag{27}$$

Idea:

> Another way, beside the two-step Lax-Wendroff scheme, to avoid using Jacobian matrix arising in computing flux function, when dealing with linear hyperbolic system $U_t + AU_x = 0$. $\tag{28}$

Approach:

1. The two-step *Lax-Wendroff* scheme for the conservation law $u_t + f(u)_x = 0$ is given by

$$u_{j+1/2}^{n+1/2} = \frac{1}{2} \left(u_{j+1}^n + u_j^n \right) - \frac{\lambda}{2} \left(f\left(u_{j+1}^n\right) - f\left(u_j^n\right) \right),$$

$$u_j^{n+1} = u_j^n - \lambda \left(f\left(u_{j+1/2}^{n+1/2}\right) - f\left(u_{j-1/2}^{n+1/2}\right) \right).$$

This scheme can also be written in the conservation form

$$u_j^{n+1} = u_j^n - \lambda \left(f_{i+1/2}^n - f_{i-1/2}^n \right),$$

where

$$f_{j+1/2}^n = f\left(\frac{1}{2} \left(u_{j+1}^n + u_j^n \right) - \frac{\lambda}{2} \left(f\left(u_{j+1}^n\right) - f(u_j^n) \right) \right),$$

$$f_{j-1/2}^n = f\left(\frac{1}{2} \left(u_j^n + u_{j-1}^n \right) - \frac{\lambda}{2} \left(f(u_j^n) - f(u_{j-1}^n) \right) \right).$$

The conservative flux function is given by

$$F(u, v) = f\left(\frac{1}{2}(u + v) - \frac{\lambda}{2} \left(f(v) - f(u) \right) \right).$$

When $v = u$, we obtain the relation

$$F(u, u) = f(u),$$

which shows that the numerical flux function is consistent with the continuous flux function.

2. For the *MacCormack* scheme, the conservative flux function is given by

$$F(u, v) = \frac{1}{2} \left(f(v) + f\left(u - \lambda \left(f(v) - f(u) \right) \right) \right).$$

Finally, it can be trivially shown that the above flux function is consistent.

2 Godunov linear systems

3   Roe

4 Theory of high resolution