

# Global Exercise - 12

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## 1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1:  $W := R^{-1}U$  as the scheme shown in exercise

$$\begin{array}{l} U_t + AU_x = 0 \\ U_t + R\Lambda R^{-1}U_x = 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x = 0 \\ W_t + \Lambda W_x = 0 \end{array}$$

where the matrix  $A$  is diagonalizable with a transformation matrix  $R \in \mathbb{R}^{N \times N}$  in the form

$$A = R\Lambda R^{-1}.$$

2. Case 2:  $W := TU$  as the scheme shown in lecture note

$$\begin{array}{l} U_t + AU_x = 0 \\ U_t + T^{-1}\Lambda TU_x = 0 \\ TU_t + \Lambda TU_x = 0 \\ W_t + \Lambda W_x = 0 \end{array}$$

where the matrix  $A$  is diagonalizable with a transformation matrix  $T \in \mathbb{R}^{N \times N}$  in the form

$$A = T^{-1}\Lambda T.$$

- Note in passing that both schemes result in the same solution.  
→ We have just to be consistent with which scheme to follow.

## 2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

**Example 1.** *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*

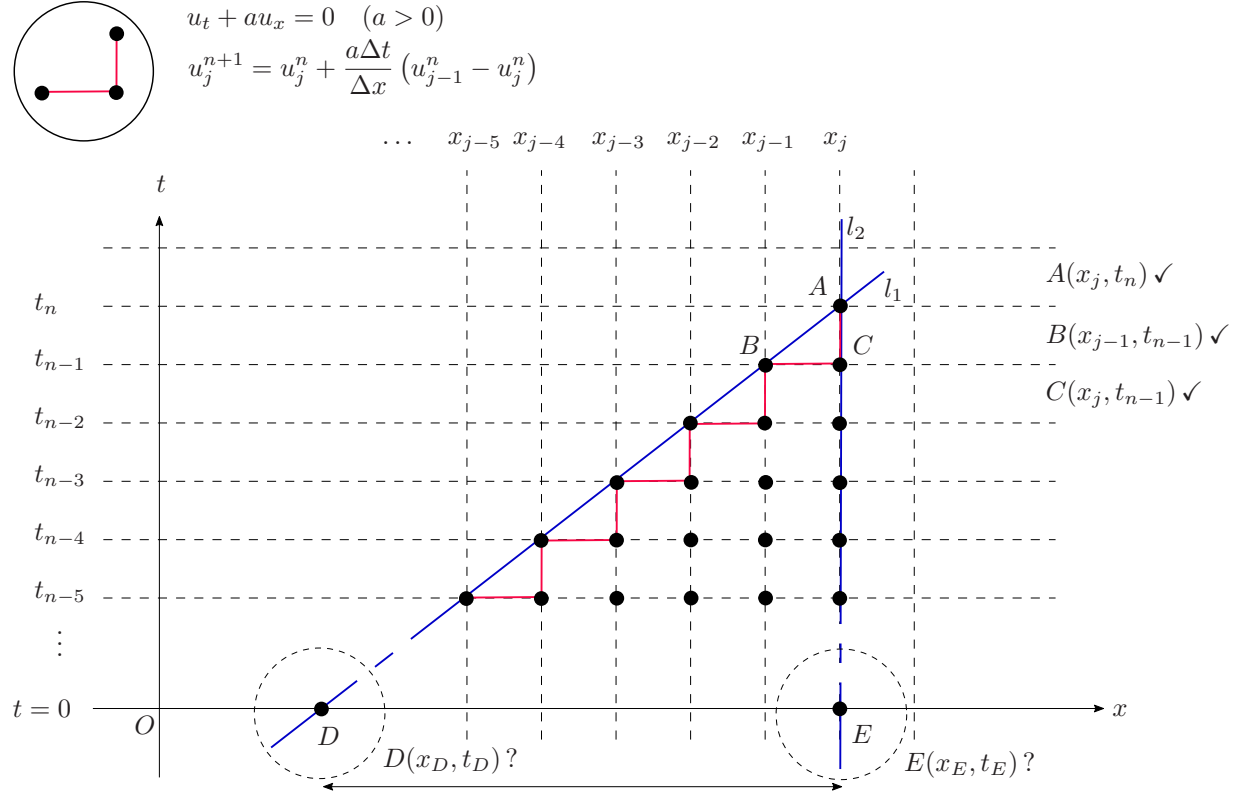


Figure 1: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 1, the numerical value computed at point  $A$  depends essentially on computed initial conditions laying between point  $D$  and  $E$ .

1. Perspective of indicial subscription:

Line  $(l_1)$  passing point  $A(j, n)$  and  $B(j-1, n-1)$  has the following form

$$\begin{aligned}
 (l_1) : \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A} (\xi - \xi_A) \\
 &\Leftrightarrow \tau = n + \frac{(n-1) - n}{(j-1) - j} (\xi - j) \\
 &\Leftrightarrow \tau = n + \frac{-1}{-1} (\xi - j), \tag{1}
 \end{aligned}$$

where  $\tau$  is the indicial variable corresponding to  $t$ , and  $x$  the indicial variable to  $x$ . Hence, line  $(l_1)$  passing line  $x$  with index  $\tau = 0$  at point  $D$  leads to the following relation

$$\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \tag{2}$$

Likewise, line  $(l_2)$  passing line  $x$  with index  $\tau = 0$  at point  $E$  leads to the following relation

$$x_\xi - x_j = 0. \quad (3)$$

Therefore, by combining (2) and (3) we arrive at the numerical domain of dependence for the One-sided method in terms of indicial perspective

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x_\xi \mid -n\Delta x \leq x_\xi - x_j \leq 0 \right\}. \quad (4)$$

Next, by using the CFL number  $\nu := a\Delta t/\Delta x$  we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \stackrel{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \quad (5)$$

Then, by substituting (5) into (4) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid -\frac{at_n}{\nu} \leq x - x_j \leq 0 \right\}}. \quad (6)$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (7)$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}(x_j, t_n) \subset \mathcal{D}_{\Delta t}(x_j, t_n), \quad (8)$$

which implies that characteristics should lie with the triangular zone under the line  $(l_1)$  and  $(l_2)$ , as shown in Figure 1. Therefore, substitution of (7) into (6) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \leq (x_j - at_n) - x_j \leq 0 \Leftrightarrow -\frac{at_n}{\nu} \leq -at_n \leq 0, \quad (9)$$

which, equally, leads to the CFL condition

$$\therefore \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}}. \quad (10)$$

Herein, the CFL condition (10) leads to constraint on the time step  $\Delta t$  for the case when  $a > 0$ . Note in passing that  $\nu$  is non-negative.

## 2. Perspective of fixed-point value:

Line  $(l_1)$  passing point  $A(x_j, t_n)$  and  $B(x_{j-1}, t_{n-1})$  has the following form

$$\begin{aligned} (l_1) : \quad t &= t_A + \frac{t_B - t_A}{x_B - x_A} (x - x_A) \Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x} (x - x_j). \end{aligned} \quad (11)$$

Hence, line  $(l_1)$  passing line  $t = 0$  at point  $D$  leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \quad (12)$$

Likewise, line  $(l_2)$  passing line  $t = 0$  at point  $E$  leads to the relation

$$x - x_j = 0. \quad (13)$$

Therefore, combination of (12) and (13) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \quad (14)$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (7), reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (15)$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

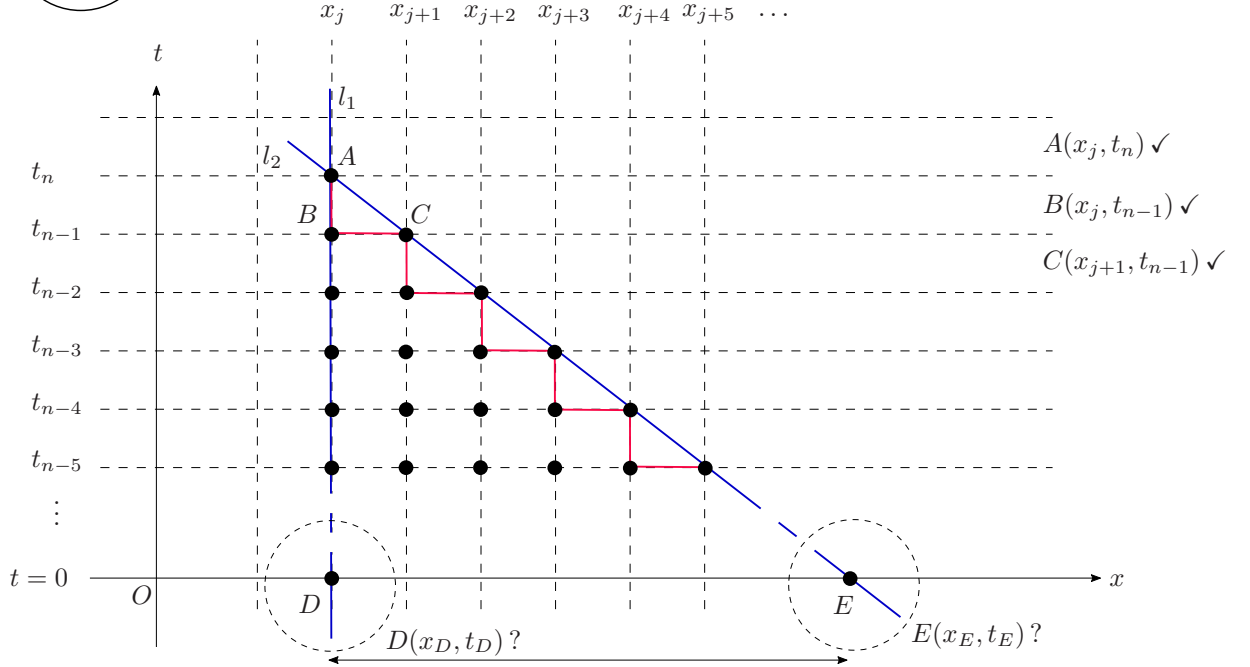
$$-\frac{t_n \Delta x}{\Delta t} \leq (x_j - at_n) - x_j \leq 0, \quad (16)$$

which we have substituted (15) into (14). Herein, the relation (16) enforcing CFL condition on the time step  $\Delta t$

$$\therefore \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a}}, \quad (17)$$

which is similar to (10).

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x} (u_j^n - u_{j+1}^n)$$



Similarly, by following steps done in Example 1 we obtain the following summary:

- $$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (18)$$

- $$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid 0 \leq x - x_j \leq -\frac{at_n}{\nu} \right\}. \quad (19)$$

- $$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (20)$$

- $$\therefore \quad \boxed{\Delta t \geq \frac{\Delta x}{a}}. \quad (21)$$

5/9

**Example 3.** Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.

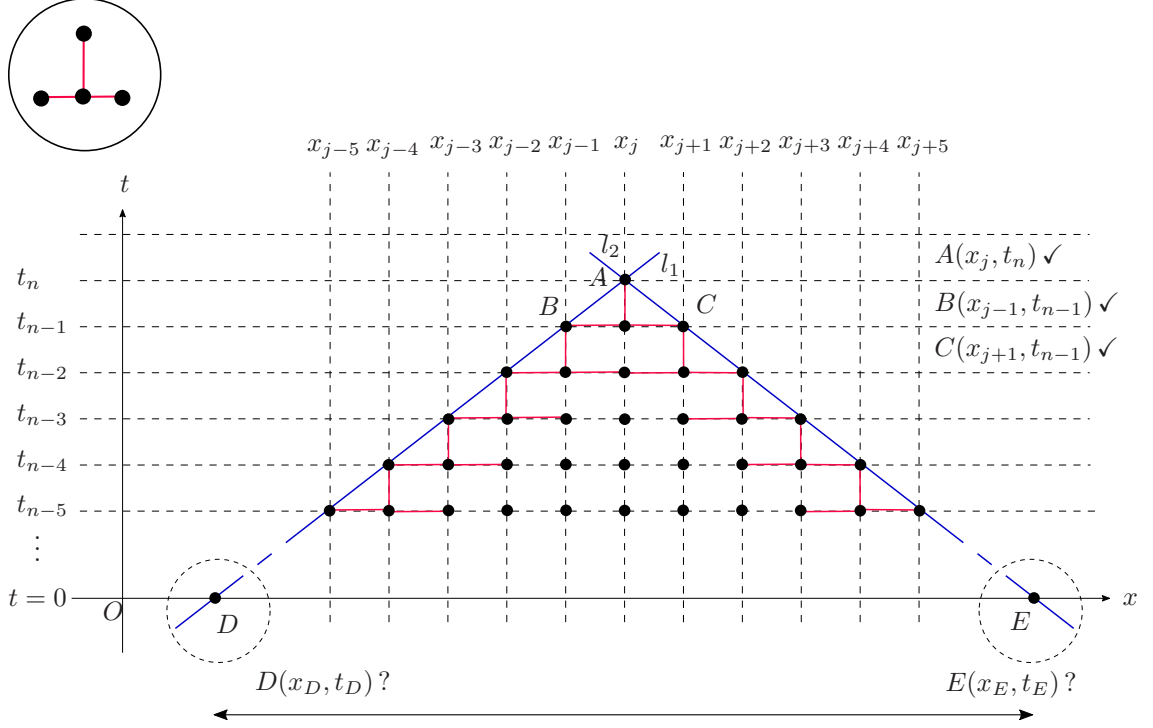


Figure 3: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point  $D$  and  $E$  in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \quad (22)$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (23)$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid |x - x_j| \leq \frac{at_n}{\nu} \right\}. \quad (24)$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (25)$$

4. CFL condition reads

$$\therefore \quad \left| \frac{a\Delta t}{\Delta x} \right| \leq 1. \quad (26)$$

**Example 4.** *Examine the numerical domain of dependence of Lax-Friedrichs method.*

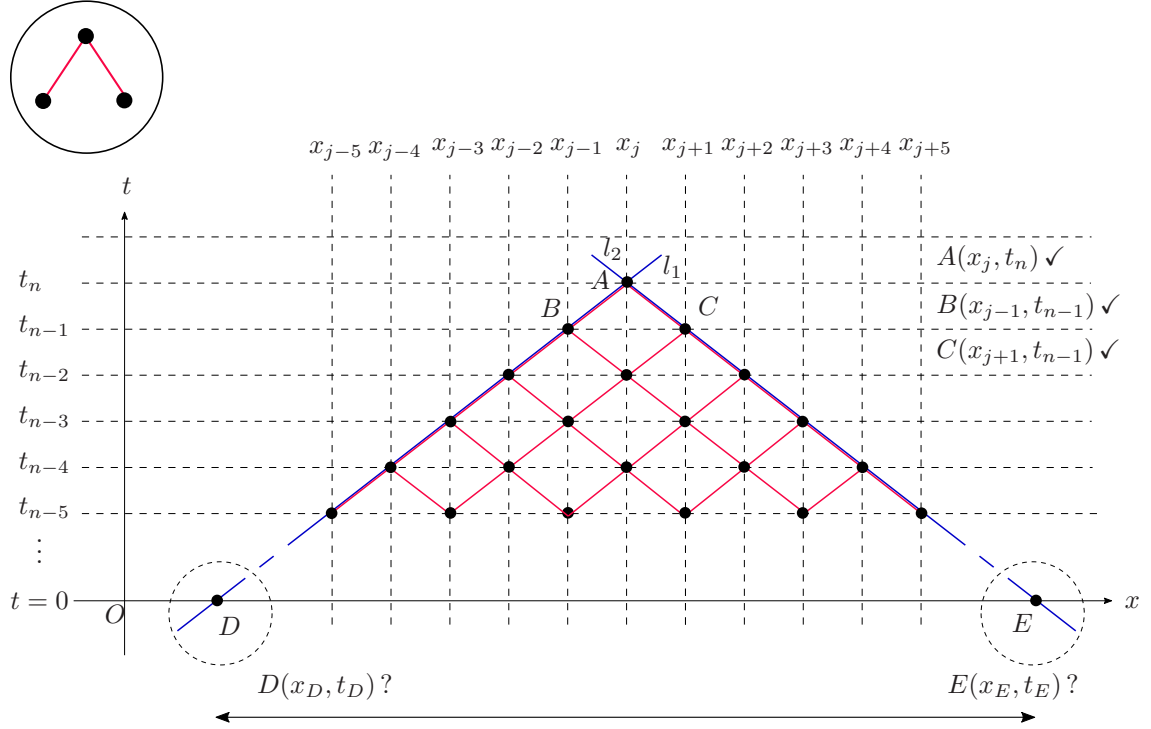


Figure 4: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1, or the same as 3 we obtain the following summary:

1. Point  $D$  and  $E$  in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \quad (27)$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (28)$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid |x - x_j| \leq \frac{at_n}{\nu} \right\}. \quad (29)$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (30)$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a\Delta t}{\Delta x} \right| \leq 1.} \quad (31)$$

### 3 von Neumann stability analysis

**Example 5.** *von Neumann stability analysis for Upwind method.*

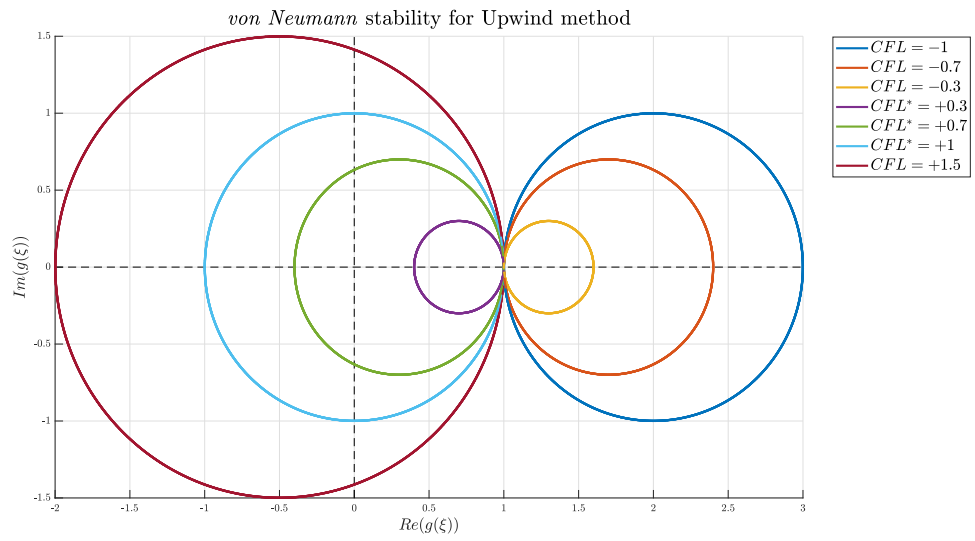


Figure 5: *von Neumann* stability analysis for Upwind method.



## 4 Conservative form - Finite Volume Method

**Example 6.** *Conservative form*