

1 A small remark for consistent numerical flux function used in Upwind scheme - FVM

Example 1. *Determine consistent numerical flux function for Upwind scheme.*

Recall the conservation form used for FVM reads

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n \right). \quad (1)$$

Meanwhile, the Upwind scheme with point-to-the-left stencils reads

$$\boxed{u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left(f(u_{j-1}^n) - f(u_j^n) \right).} \quad (2)$$

Next, we would like to write (2) in terms of (1). Since the two formulations are already identical, the numerical flux function $\tilde{f}_{j-1/2}^n$ and $\tilde{f}_{j+1/2}^n$ are recognized directly as follow

$$\tilde{f}_{j-1/2}^n = \tilde{f}_{j-1/2}^n(u_{j-1}^n, u_j^n) = f(u_{j-1}^n), \quad (3)$$

$$\tilde{f}_{j+1/2}^n = \tilde{f}_{j-1/2}^n(u_{j-1}^n, u_j^n) = f(u_{j+1}^n). \quad (4)$$

Then, the consistency of numerical flux functions are checked as follows

$$\tilde{f}_{j-1/2}^n(\beta, \beta) = f(\beta), \quad \checkmark \quad (5)$$

$$\tilde{f}_{j+1/2}^n(\beta, \beta) = f(\beta), \quad \checkmark \quad (6)$$

which is satisfied $\forall \beta \in \mathbb{R}$ in case of constant flow. Likewise, in case of the Upwind scheme with point-to-the-right stencils, we obtain the same structure, as follows

$$\boxed{u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left(f(u_j^n) - f(u_{j+1}^n) \right),} \quad (7)$$

which is the Upwind scheme with point-to-the-right stencils. Next, since the formulation (7) is already identical with (1), the numerical flux function $\tilde{f}_{j-1/2}^n$ and $\tilde{f}_{j+1/2}^n$ are recognized directly as follow

$$\tilde{f}_{j-1/2}^n = \tilde{f}_{j-1/2}^n(u_j^n, u_{j+1}^n) = f(u_{j-1}^n), \quad (8)$$

$$\tilde{f}_{j+1/2}^n = \tilde{f}_{j-1/2}^n(u_j^n, u_{j+1}^n) = f(u_{j+1}^n). \quad (9)$$

Checking the consistency of (8) and (9) is similar with (3) and (4), as follows

$$\tilde{f}_{j-1/2}^n(\beta, \beta) = f(\beta), \quad \checkmark \quad (10)$$

$$\tilde{f}_{j+1/2}^n(\beta, \beta) = f(\beta), \quad \checkmark \quad (11)$$

which is satisfied $\forall \beta \in \mathbb{R}$ in case of constant flow.

2 A small remark for consistent numerical flux function used in Lax-Friedrichs scheme - FVM

Example 2. Determine consistent numerical flux function for Lax-Friedrichs scheme.

Recall the conservation form used for FVM reads

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n \right). \quad (12)$$

Meanwhile, the Lax-Friedrichs scheme reads

$$u_j^{n+1} = \frac{1}{2} (u_{j-1}^n + u_{j+1}^n) - \frac{\Delta t}{2\Delta x} (f(u_{j+1}^n) - f(u_{j-1}^n)). \quad (13)$$

Next, we would like to write (13) in terms of (12). The derivation is done by some algebraic manipulations, as follows

$$\begin{aligned} u_j^{n+1} &= \frac{1}{2} (u_{j-1}^n + u_{j+1}^n) - \frac{\Delta t}{2\Delta x} (f(u_{j+1}^n) - f(u_{j-1}^n)) \\ &= u_j^n + \frac{1}{2}u_{j-1}^n - \frac{1}{2}u_j^n + \frac{1}{2}u_{j+1}^n - \frac{1}{2}u_j^n - \frac{\Delta t}{2\Delta x} f(u_{j+1}^n) + \frac{\Delta t}{2\Delta x} f(u_{j-1}^n) \\ &= u_j^n + \frac{\Delta t}{\Delta x} \left(\frac{\Delta x}{2\Delta t} u_{j-1}^n - \frac{\Delta x}{2\Delta t} u_j^n + \frac{\Delta x}{2\Delta t} u_{j+1}^n - \frac{\Delta x}{2\Delta t} u_j^n \right. \\ &\quad \left. - \frac{1}{2} f(u_{j+1}^n) + \frac{1}{2} f(u_{j-1}^n) \right) \\ &= u_j^n + \frac{\Delta t}{\Delta x} \left(\underbrace{\left(\frac{\Delta x}{2\Delta t} u_{j-1}^n - \frac{\Delta x}{2\Delta t} u_j^n + \frac{1}{2} f(u_{j-1}^n) \right)}_{=:\tilde{f}_{j-1/2}^n} \right. \\ &\quad \left. - \underbrace{\left(\frac{\Delta x}{2\Delta t} u_j^n - \frac{\Delta x}{2\Delta t} u_{j+1}^n + \frac{1}{2} f(u_{j+1}^n) \right)}_{=:\tilde{f}_{j+1/2}^n} \right) \\ &= u_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n \right), \end{aligned} \quad (14)$$

which confirms that the *Lax-Friedrichs* scheme given at (13) is able to be written in the *conservation form* with the numerical flux function recognized as follows

$$\tilde{f}_{j-1/2}^n = \tilde{f}_{j-1/2}^n(u_{j-1}^n, u_j^n) = \frac{\Delta x}{2\Delta t} (u_{j-1}^n - u_j^n) + \frac{1}{2} f(u_{j-1}^n) \quad (15)$$

$$\tilde{f}_{j+1/2}^n = \tilde{f}_{j+1/2}^n(u_j^n, u_{j+1}^n) = \frac{\Delta x}{2\Delta t} (u_j^n - u_{j+1}^n) + \frac{1}{2} f(u_{j+1}^n) \quad (16)$$

However, these numerical flux functions $\tilde{f}(\cdot, \cdot)$ are not consistent with the original flux function $f(\cdot)$, which can be checked for the case of constant flow, as follows

$$(15) \Leftrightarrow \tilde{f}_{j-1/2}^n(\beta, \beta) = \frac{\Delta x}{2\Delta t} (\beta - \beta) + \frac{1}{2} f(\beta) = \frac{1}{2} f(\beta) \quad \nless (17)$$

$$(16) \Leftrightarrow \tilde{f}_{j+1/2}^n(\beta, \beta) = \frac{\Delta x}{2\Delta t} (\beta - \beta) + \frac{1}{2} f(\beta) = \frac{1}{2} f(\beta) \quad \nless (18)$$

which have confirmed that $\tilde{f}_{j\pm 1/2}^n(\beta, \beta) \neq f(\beta)$, $\forall \beta \in \mathbb{R}$. Therefore, a modified version is required for these numerical flux functions, such that they become consistent with $f(\cdot)$, and simultaneously, the summation of these two fluxes do not lead to any change in the *Lax-Friedrichs* scheme. The consistent numerical flux functions for *Lax-Friedrichs* scheme reads

$$\tilde{f}_{j-1/2}^n = \tilde{f}_{j-1/2}^n(u_{j-1}^n, u_j^n) = \frac{\Delta x}{2\Delta t} (u_{j-1}^n - u_j^n) + \frac{1}{2} (f(u_{j-1}^n) + f(u_j^n)), \quad (19)$$

$$\tilde{f}_{j+1/2}^n = \tilde{f}_{j+1/2}^n(u_j^n, u_{j+1}^n) = \frac{\Delta x}{2\Delta t} (u_j^n - u_{j+1}^n) + \frac{1}{2} (f(u_{j+1}^n) + f(u_j^n)), \quad (20)$$

which are obtained by adding the term $1/2f(u_j^n)$ to both $\tilde{f}_{j-1/2}^n$ and $\tilde{f}_{j+1/2}^n$. Note in passing that the subtraction sign between these two fluxes will cancel out this extra term, as shown in (12). Next, the consistent property is checked as follows

$$(19) \Leftrightarrow \tilde{f}_{j-1/2}^n(\beta, \beta) = \frac{\Delta x}{2\Delta t} (\beta - \beta) + \frac{1}{2} (f(\beta) + f(\beta)) = f(\beta) \quad \checkmark \quad (21)$$

$$(20) \Leftrightarrow \tilde{f}_{j+1/2}^n(\beta, \beta) = \frac{\Delta x}{2\Delta t} (\beta - \beta) + \frac{1}{2} (f(\beta) + f(\beta)) = f(\beta) \quad \checkmark \quad (22)$$

which have confirmed that $\tilde{f}_{j\pm 1/2}^n(\beta, \beta) = f(\beta)$, $\forall \beta \in \mathbb{R}$ for constant flow. Hence, the numerical flux functions $\tilde{f}_{j\pm 1/2}^n$ for the *Lax-Friedrichs* scheme take the following formulation

$$\therefore \quad \boxed{\begin{aligned} \tilde{f}_{j-1/2}^n(u_{j-1}^n, u_j^n) &= \frac{\Delta x}{2\Delta t} (u_{j-1}^n - u_j^n) + \frac{1}{2} (f(u_{j-1}^n) + f(u_j^n)) \\ \tilde{f}_{j+1/2}^n(u_j^n, u_{j+1}^n) &= \frac{\Delta x}{2\Delta t} (u_j^n - u_{j+1}^n) + \frac{1}{2} (f(u_j^n) + f(u_{j+1}^n)) \end{aligned}} \quad (23)$$

or the generalized form used for FVM implementation reads

$$\therefore \quad \boxed{F(u_L, u_R) = \frac{1}{2} (f(u_L) + f(u_R)) + \frac{\Delta x}{2\Delta t} (u_L - u_R)} \quad (24)$$