Global Exercise - 11

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22^{nd} December 2021

1 Linear hyperbolic systems (cont.)

Example 1. *Let u be a scalar-valued function in $C^1(\mathbb{R} \times \mathbb{R}^+)$ and defined as follows*

$$u : \begin{cases} \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, \\ (x,t) \mapsto u(x,t). \end{cases} \tag{1}$$

Consider the natural 1D second-order wave equation

$$u_{tt} = \alpha^2 u_{xx}, \quad \forall \alpha \in \mathbb{R}, \tag{2}$$

with initial conditions given

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{3}$$

Find solution to (2) by means of linear hyperbolic system of conservation laws.

Approach: By introducing two new variables called v and w defined as follows

$$\begin{cases} v := u_x, \\ w := u_t, \end{cases} \tag{4}$$

we obtain the following relations

$$\begin{cases} v_x \overset{(4)}{=} u_{xx} \overset{(2)}{=} \dfrac{1}{\alpha^2} u_{tt} \overset{(4)}{=} \dfrac{1}{\alpha^2} w_t, \\ w_x \overset{(4)}{=} u_{tx} \overset{(4)}{=} u_{xt} \overset{(4)}{=} v_t, \end{cases} \Leftrightarrow \begin{cases} v_t - w_x = 0, \\ w_t - \alpha^2 v_x = 0. \end{cases} \tag{5}$$

Note in passing that $u_{tx} = u_{xt}$ since $u \in C^1(\mathbb{R} \times \mathbb{R}^+)$. The later system in (5) can be rewritten in form of a system, whose derivative term w.r.t. time t and space x follows with each other

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} -w \\ -\alpha^2 v \end{pmatrix}_x = 0. \tag{6}$$

In order to mimic explicitly the form of a system of conservation laws, the system in (6) is recast into the following system

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0. \tag{7}$$

The initial condition becomes

$$\begin{cases} v(x,0) = u_0'(x), \\ w(x,0) = u_1(x). \end{cases} \tag{8}$$

By defining the new variables

$$U := \begin{pmatrix} v \\ w \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} 0 & -1 \\ -\alpha^2 & 0 \end{pmatrix}, \tag{9}$$

we now write (7) as

$$U_t + AU_x = 0 \tag{10}$$

Next, we realize that the system (7) is not ready to solve since the two PDEs are interlocked with each other. The approach for solution to (10) goes as follows

1. Eigenvalue problem for matrix A yields

$$Av^A = \lambda v^A \Leftrightarrow (A - \lambda I)v^A = 0 \Leftrightarrow \begin{pmatrix} -\lambda & -1 \\ -\alpha^2 & -\lambda \end{pmatrix} \begin{pmatrix} v_1^A \\ v_2^A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

   where λ is eigenvalue and $v^A = (v_1^A, v_2^A)^\top$ eigenvector.

2. Characteristic polynomial

$$\det(A - \lambda I) = 0 \Leftrightarrow \lambda^2 - \alpha^2 = 0 \Leftrightarrow \begin{cases} \lambda_1 = +\alpha \\ \lambda_2 = -\alpha \end{cases} \tag{12}$$

   (a) For the 1^{st} eigenvalue $\lambda_1 = +\alpha$, we obtain its corresponding eigenvector

$$(11) \Leftrightarrow \begin{pmatrix} -\alpha & -1 \\ -\alpha^2 & -\alpha \end{pmatrix} \begin{pmatrix} v_1^A \\ v_2^A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

   Gaussian-elimination leads to

$$\begin{aligned} \begin{pmatrix} -\alpha & -1 \\ -\alpha^2 & -\alpha \end{pmatrix} &\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} -\alpha^2 & -\alpha \\ -\alpha & -1 \end{pmatrix} \\ &\xrightarrow{r_1 := -\frac{1}{\alpha^2} r_1} \begin{pmatrix} 1 & 1/\alpha \\ -\alpha & -1 \end{pmatrix} \xrightarrow{r_2 := \alpha r_1 + r_2} \begin{pmatrix} 1 & 1/\alpha \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{14}$$

   Hence, eigenvector for $\lambda_1 = +\alpha$ reads

$$v^{A(\lambda_1)} = \begin{pmatrix} v_1^{A(\lambda_1)} \\ v_2^{A(\lambda_1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \tag{15}$$

   (b) Similarly, for the 2^{nd} eigenvalue $\lambda_1 = -\alpha$, we obtain its corresponding eigenvector as follows

$$(11) \Leftrightarrow \begin{pmatrix} +\alpha & -1 \\ -\alpha^2 & +\alpha \end{pmatrix} \begin{pmatrix} v_1^A \\ v_2^A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{16}$$

   Gaussian-elimination leads to

$$\begin{aligned} \begin{pmatrix} +\alpha & -1 \\ -\alpha^2 & +\alpha \end{pmatrix} &\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} -\alpha^2 & +\alpha \\ +\alpha & -1 \end{pmatrix} \\ &\xrightarrow{r_1 := -\frac{1}{\alpha^2} r_1} \begin{pmatrix} 1 & -1/\alpha \\ +\alpha & -1 \end{pmatrix} \xrightarrow{r_2 := -\alpha r_1 + r_2} \begin{pmatrix} 1 & -1/\alpha \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{17}$$

   Hence, eigenvector for $\lambda_1 = -\alpha$ reads

$$v^{A(\lambda_2)} = \begin{pmatrix} v_1^{A(\lambda_2)} \\ v_2^{A(\lambda_2)} \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \tag{18}$$

3. Diagonalizable

$$
\begin{aligned}
A &= T\Lambda T^{-1} \\
&= \left(\; v^{A(\lambda_1)} \; \middle| \; v^{A(\lambda_2)} \; \right) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \left(\; v^{A(\lambda_1)} \; \middle| \; v^{A(\lambda_2)} \; \right)^{-1} \\
&= \left(\begin{array}{c|c} 1 & 1 \\ -\alpha & \alpha \end{array} \right) \begin{pmatrix} +\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \left(\begin{array}{c|c} 1 & 1 \\ -\alpha & \alpha \end{array} \right)^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ -\alpha & \alpha \end{pmatrix} \begin{pmatrix} +\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\alpha & \alpha \end{pmatrix}^{-1}
\end{aligned}
$$

4. Derivation

$$
\begin{aligned}
U_t + AU_x = 0 &\Leftrightarrow T^{-1}U_t + T^{-1}AU_x = 0 \\
&\Leftrightarrow T^{-1}U_t + T^{-1}T\Lambda T^{-1}U_x = 0 \\
&\Leftrightarrow T^{-1}U_t + \Lambda T^{-1}U_x = 0 \\
&\Leftrightarrow W_t + \Lambda W_x = 0 \\
&\Leftrightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix}_t + \Lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}_x = 0 \\
&\Leftrightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix}_t + \begin{pmatrix} +\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}_x = 0 \\
&\Leftrightarrow \begin{cases} \xi_t + \alpha \xi_x = 0 \\ \eta_t - \alpha \eta_x = 0 \end{cases} \Leftrightarrow \begin{cases} \xi(x,t) = \xi_0(x - \alpha t) \\ \eta(x,t) = \eta_0(x + \alpha t) \end{cases} \qquad (19)
\end{aligned}
$$

Note in passing that

$$
\begin{aligned}
\begin{pmatrix} \xi \\ \eta \end{pmatrix} = W := T^{-1}U &= \begin{pmatrix} 1 & 1 \\ -\alpha & \alpha \end{pmatrix}^{-1} \begin{pmatrix} v \\ w \end{pmatrix} \qquad (20) \\
&= \frac{1}{\alpha - (-\alpha)} \begin{pmatrix} \alpha & -1 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & -1/(2\alpha) \\ 1/2 & 1/(2\alpha) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \\
&= \begin{pmatrix} (1/2)v - 1/(2\alpha)w \\ (1/2)v + 1/(2\alpha)w \end{pmatrix}. \qquad (21)
\end{aligned}
$$

The initial conditions are as follows

$$
\xi_0(x) = \xi(x,0) = \frac{1}{2}v(x,0) - \frac{1}{2\alpha}w(x,0) = \frac{1}{2}u_0'(x) - \frac{1}{2\alpha}u_1(x) \qquad (22)
$$

$$
\eta_0(x) = \eta(x,0) = \frac{1}{2}v(x,0) + \frac{1}{2\alpha}w(x,0) = \frac{1}{2}u_0'(x) + \frac{1}{2\alpha}u_1(x) \qquad (23)
$$

Likewise,

$$
\begin{aligned}
U = TW &\Leftrightarrow \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\alpha & \alpha \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \\
&\Leftrightarrow \begin{cases} v(x,t) = \xi(x,t) + \eta(x,t) \\ w(x,t) = -\alpha\xi(x,t) + \alpha\eta(x,t) \end{cases} \qquad (24) \\
&\Leftrightarrow \begin{cases} v(x,t) = \xi_0(x - \alpha\, t) + \eta_0(x + \alpha\, t) \\ w(x,t) = -\alpha\xi_0(x - \alpha\, t) + \alpha\eta_0(x + \alpha\, t) \end{cases} \qquad (25)
\end{aligned}
$$

Therefore, the final solutions explicitly reads

$$\begin{cases} v(x,t) = \frac{1}{2}u_0'(x-\alpha\, t) - \frac{1}{2\alpha}u_1(x-\alpha\, t) + \frac{1}{2}u_0'(x+\alpha\, t) + \frac{1}{2\alpha}u_1(x+\alpha\, t) \\ w(x,t) = -\frac{\alpha}{2}u_0'(x-\alpha\, t) + \frac{1}{2}u_1(x-\alpha\, t) + \frac{\alpha}{2}u_0'(x+\alpha\, t) + \frac{1}{2}u_1(x+\alpha\, t) \end{cases} \tag{26}$$

Example 2. *Consider the* ***linearized shallow water equations***

$$\begin{pmatrix} u \\ \varphi \end{pmatrix}_t + \begin{pmatrix} \bar{u} & 1 \\ \bar{\varphi} & \bar{u} \end{pmatrix} \begin{pmatrix} u \\ \varphi \end{pmatrix}_x = 0, \tag{27}$$

where $\bar{u}, \bar{\varphi} \in \mathbb{R}$ are given, and sufficient initial conditions for u and φ are provided.

Summary

$$\boxed{\begin{aligned} q_t + Aq_x &= 0 \\ q_t + R\Lambda R^{-1}q_x &= 0 \\ R^{-1}q_t + \Lambda R^{-1}q_x &= 0 \\ w_t + \Lambda w_x &= 0 \end{aligned}}$$

Example 3. *Consider*

$$u_t + Au_x = 0 \qquad \textit{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

where the unknown vector $u(x,t)$ is defined, and the matrix A is given as follows

$$u(x,t) := \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}, \quad \textit{and} \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and the initial conditions are given by

$$\begin{cases} u_1(x,0) & = 1, \\ u_2(x,0) & = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases} \end{cases}$$

Approach: Eigenvalue decomposition of matrix A is given as follows

$$A = R\Lambda R^{-1}, \quad R = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2}R, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Hence, we can now change variable u to a new variable w as follows

$$w := R^{-1}u = \frac{1}{2}Ru, \quad \text{so} \quad u = Rw.$$

Then, the initial problem can be rewritten as

$$w_t + \Lambda w_x = 0 \Leftrightarrow \begin{cases} (w_1)_t + \lambda_1 (w_1)_x = 0, \\ (w_2)_t + \lambda_2 (w_2)_x = 0, \end{cases}$$

with the diagonal matrix Λ. This step simply means that variables w_1 and w_2 are independent of each other, i.e. the original system has been fully decoupled. Next, let us find the initial condition in terms of w as follows

$$w(x,0) = \frac{1}{2}Ru(x,0) = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} -\left(u_1(x,0) + u_2(x,0)\right)/2 \\ -\left(u_1(x,0) - u_2(x,0)\right)/2 \end{pmatrix},$$

which yields the following relation

$$\begin{pmatrix} w_1(x,0) \\ w_2(x,0) \end{pmatrix} = \begin{pmatrix} -\left(u_1(x,0) + u_2(x,0)\right)/2 \\ -\left(u_1(x,0) - u_2(x,0)\right)/2 \end{pmatrix}.$$

Therefore, the initial conditions for the transformed variable w_1 read

$$w_1(x,0) = \begin{cases} -(1+1)/2 = -1, & x < 0, \\ -(1-1)/2 = 0, & x > 0, \end{cases}$$

and for the variable w_2 read

$$w_2(x,0) = \begin{cases} -(1-1)/2 = 0, & x < 0, \\ -(1-(-1))/2 = -1, & x > 0. \end{cases}$$

Since we have the advection problem with speeds $a_1 = 1$ and $a_2 = 3$, which are the diagonal values of the diagonal matrix Λ, for both w_1 and w_2, we can derive explicit solution to each equation. The explicit solution w_1 reads

$$w_1(x,t) = w_1(x-t,0) = \begin{cases} -1, & x-t<0, \\ 0, & x-t>0, \end{cases} \Leftrightarrow w_1(x,t) = \begin{cases} -1, & x<t, \\ 0, & x>t, \end{cases}$$

and, likewise, the explicit solution w_2 reads

$$w_2(x,t) = w_2(x-3t,0) = \begin{cases} 0, & x-3t<0, \\ -1, & x-3t>0, \end{cases} \Leftrightarrow w_2(x,t) = \begin{cases} 0, & x<3t, \\ -1, & x>3t. \end{cases}$$

Now we trace back to variable u from variable w by taking into the consideration of the relation

$$u(x,t) = Rw(x,t) = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix} = \begin{pmatrix} -\left(w_1(x,t)+w_2(x,t)\right) \\ -\left(w_1(x,t)-w_2(x,t)\right) \end{pmatrix},$$

which means the following relation holds

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \begin{pmatrix} -\left(w_1(x,t)+w_2(x,t)\right) \\ -\left(w_1(x,t)-w_2(x,t)\right) \end{pmatrix}.$$

Therefore, the solution for $u_1(x,t)$ is computed as follows

$$u_1(x,t) = \begin{cases} -(-1+0) = 1, & x<t, \\ -(0+0) = 0, & t<x<3t, \\ -(0+(-1)) = 1, & x>3t, \end{cases}$$

and, likewise, the solution for $u_2(x,t)$ is

$$u_2(x,t) = \begin{cases} -(-1-0) = 1, & x<t, \\ -(0-0) = 0, & t<x<3t, \\ -(0-(-1)) = -1, & x>3t. \end{cases}$$

Finally, the solution to $u_1(x,t)$ and $u_2(x,t)$ read

$$\therefore \quad \boxed{\begin{cases} u_1(x,t) = \begin{cases} 1, & x<t, \\ 0, & t<x<3t, \\ 1, & x>3t, \end{cases} \\ u_2(x,t) = \begin{cases} 1, & x<t, \\ 0, & t<x<3t, \\ -1, & x>3t. \end{cases} \end{cases}}$$

2 Remark on step-by-step derivation: Weak formulation of conservation laws

Example 4. *Consider the conservation law*

$$u_t + f(u)_x = 0 \quad \textit{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{28}$$

with initial condition given by

$$u(x,0) = u_0(x).$$

Find the weak solution.

Approach: The weak formulation is obtained by multiplying both sides of (28) with a test function $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R})$, and, then, integrating both sides over $\mathbb{R} \times \mathbb{R}^+$

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \varphi \left(u_t + f(u)_x\right) dxdt = 0. \tag{29}$$

The derivation goes as follows

$$
\begin{aligned}
(29) \Leftrightarrow & \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi \left(u_t + f(u)_x\right) dxdt = 0 \\
\Leftrightarrow & \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi \left(u_t\right) dxdt + \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi \left(f(u)_x\right) dxdt = 0 \\
\Leftrightarrow & \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi \left(u_t\right) dtdx + \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi \left(f(u)_x\right) dxdt = 0 \\
\Leftrightarrow & \left[\int_{\mathscr{D}} \varphi\, u|_{t=0}^{t=+\infty}\, dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t\, u\, dtdx\right] \\
& + \left[\int_0^{\mathscr{T}} \varphi\, f(u)|_{x=-\infty}^{x=+\infty}\, dxdt - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt\right] = 0 \\
\Leftrightarrow & \left[\int_{\mathscr{D}} \left(\varphi(x, \overset{t\to+\infty}{\mathscr{T}})u(x, \overset{t\to+\infty}{\mathscr{T}}) - \varphi(x,0)u(x,0)\right) dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t\, u\, dtdx\right] \\
& + \left[\int_0^{\mathscr{T}} \left(\varphi(\overset{x\to+\infty}{\mathscr{D}}, t)\, f(u(\overset{x\to+\infty}{\mathscr{D}}, t)) - \varphi(\underset{x\to-\infty}{\mathscr{D}}, t)\, f(u(\underset{x\to-\infty}{\mathscr{D}}, t))\right) dt \right. \\
& \qquad \left. - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt\right] = 0 \\
\Leftrightarrow & \left[\int_{\mathscr{D}} \left(\cancelto{0}{\varphi(x, \overset{t\to+\infty}{\mathscr{T}})}u(x, \overset{t\to+\infty}{\mathscr{T}}) - \varphi(x,0)u(x,0)\right) dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t\, u\, dtdx\right] \\
& + \left[\int_0^{\mathscr{T}} \left(\cancelto{0}{\varphi(\overset{x\to+\infty}{\mathscr{D}}, t)}\, f(u(\overset{x\to+\infty}{\mathscr{D}}, t)) - \cancelto{0}{\varphi(\underset{x\to-\infty}{\mathscr{D}}, t)}\, f(u(\underset{x\to-\infty}{\mathscr{D}}, t))\right) dt \right. \\
& \qquad \left. - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt\right] = 0 \\
\Leftrightarrow & -\int_{\mathscr{D}} \varphi(x,0)u(x,0)dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t u\, dtdx - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt = 0 \\
\Leftrightarrow & -\int_{\mathscr{D}} \varphi(x,0)u_0(x)dx - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_t u\, dxdt - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt = 0.
\end{aligned}
$$

Therefore, the weak formulation reads

$$\therefore \quad \boxed{\int_0^{\mathscr{T}} \int_{\mathscr{D}} \left(u\varphi_t + f(u)\varphi_x\right) dxdt + \int_{\mathscr{D}} \varphi(x,0)u_0(x)dx = 0.}$$

Note in passing that the derivation above has taken into consideration the swapping order of integral limits between time $t \in [0, +\infty]$ and space $x \in [-\infty, +\infty]$, which results in indifference as shown in Figure 1.

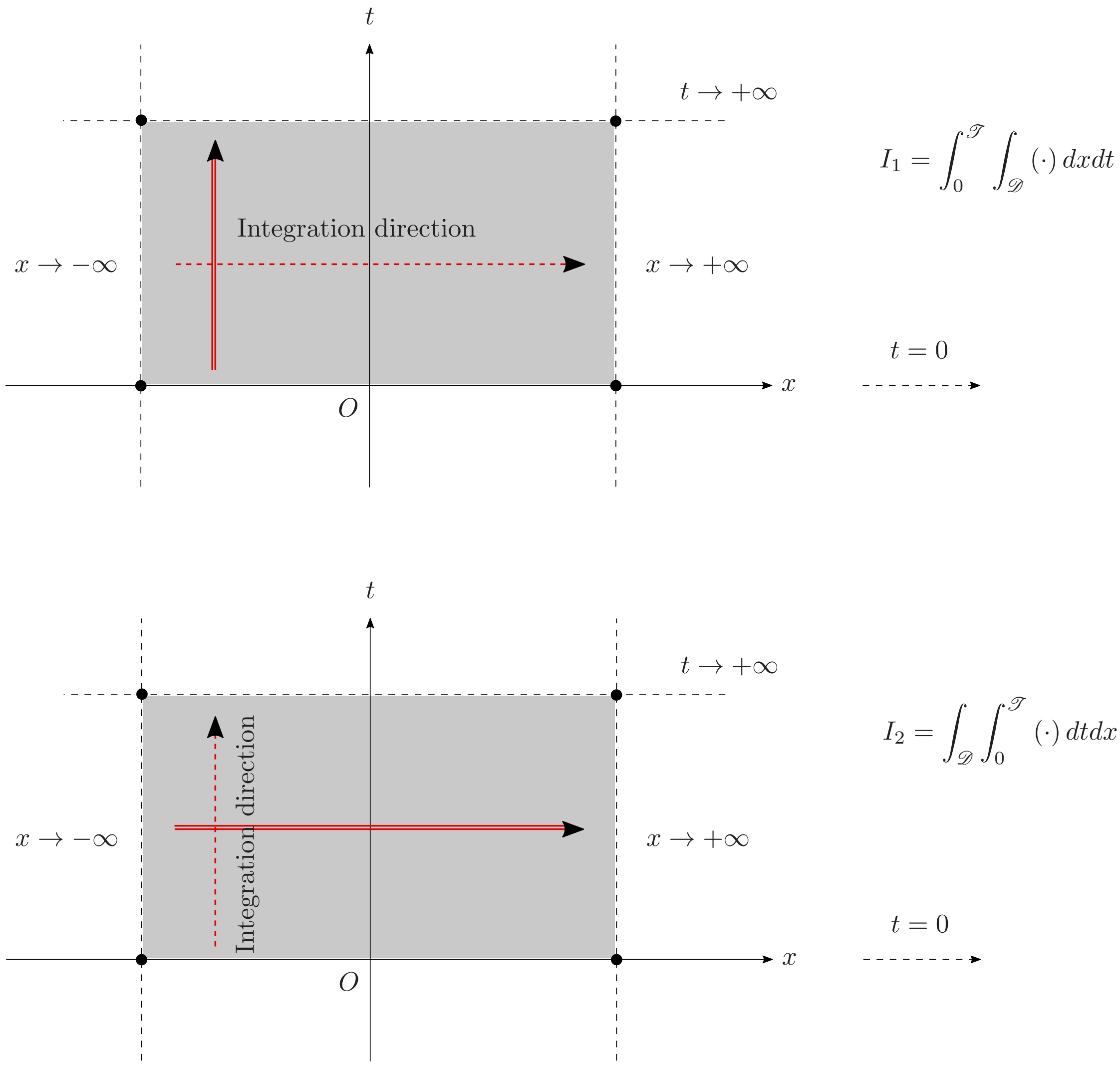

Figure 1: Indifference of limits when swapping order of integrals, i.e. $I_1 = I_2$.

3 Remark on step-by-step derivation: Weak solution of inviscid Burgers' equation in Riemann's problem

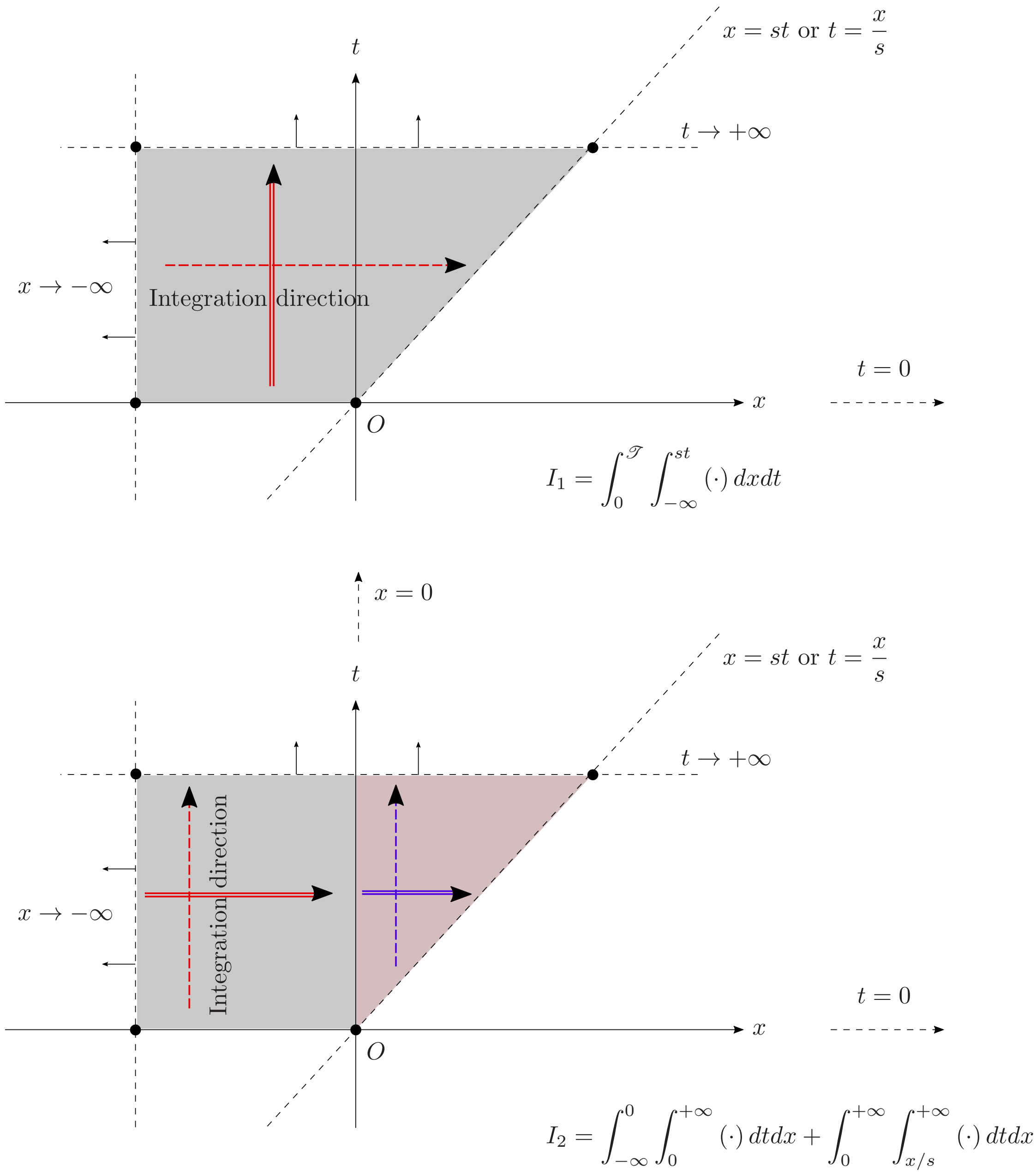

Figure 2: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee $I_1 = I_2$.

Example 5. *Given Burgers' equation with initial condition*

$$u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$

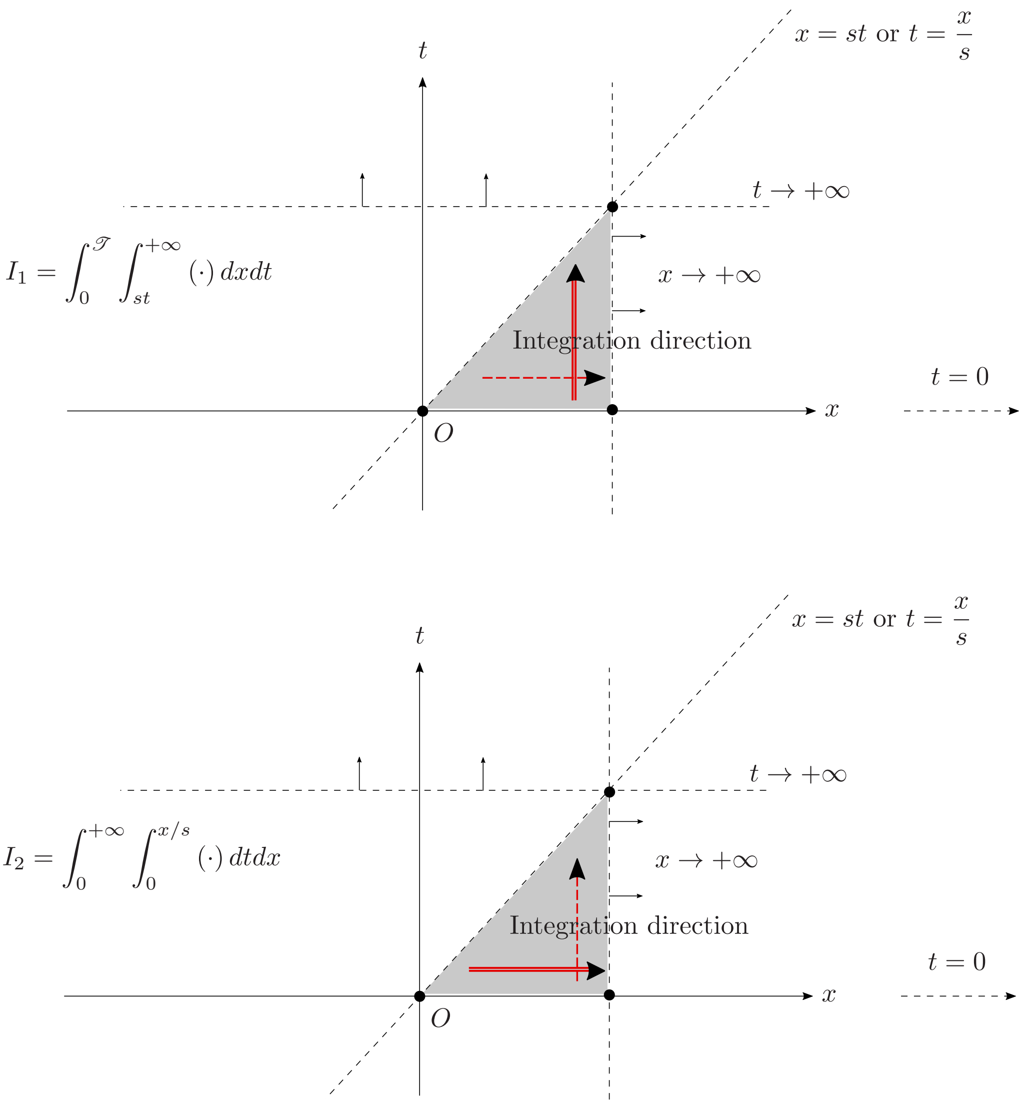

Figure 3: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee $I_1 = I_2$.

Then

$$u(x,t) = \begin{cases} u_L, & x < st, \\ u_R, & x > st, \end{cases}$$

is the weak solution to Burgers' equation.

Proof. RHS term

$$\begin{aligned} RHS &= -\int_{\mathscr{D}} \varphi(x,0) u_0(x) dx \\ &= -u_L \int_{-\infty}^{0} \varphi(x,0)dx - u_R \int_0^{+\infty} \varphi(x,0)dx \end{aligned}$$

LHS term

$$
\begin{aligned}
LHS &= \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_t u \, dxdt + \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_t u \, dxdt \\
&\quad + \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_x f(u) \, dxdt + \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_x f(u) \, dxdt \\
&= u_L \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_t \, dxdt + u_R \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_t \, dxdt \\
&\quad + \frac{u_L^2}{2} \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_x \, dxdt + \frac{u_R^2}{2} \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_x \, dxdt \\
&= \left(u_L \int_{-\infty}^{0} \int_0^{+\infty} \varphi_t \, dtdx + u_L \int_0^{+\infty} \int_{x/s}^{+\infty} \varphi_t \, dtdx \right) \\
&\quad + u_R \int_0^{+\infty} \int_0^{x/s} \varphi_t \, dtdx \\
&\quad + \frac{u_L^2}{2} \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_x \, dxdt + \frac{u_R^2}{2} \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_x \, dxdt \\
&= u_L \int_{-\infty}^{0} \varphi|_{t=0}^{t=+\infty} \, dx + u_L \int_0^{+\infty} \varphi|_{t=x/s}^{t=+\infty} \, dx + u_R \int_0^{+\infty} \varphi|_{t=0}^{t=x/s} \, dx \\
&\quad + \frac{u_L^2}{2} \int_0^{+\infty} \varphi|_{x=-\infty}^{x=st} \, dt + \frac{u_R^2}{2} \int_0^{+\infty} \varphi|_{x=st}^{x=+\infty} \, dt \\
&= u_L \int_{-\infty}^{0} (-\varphi(x,0)) \, dx + u_L \int_0^{+\infty} (-\varphi(x,x/s)) \, dx \\
&\quad + u_R \int_0^{+\infty} (\varphi(x,x/s) - \varphi(x,0)) \, dx \\
&\quad + \frac{u_L^2}{2} \int_0^{+\infty} \varphi(st,t) dt + \frac{u_R^2}{2} \int_0^{+\infty} (-\varphi(st,t)) \, dt \\
&= -u_L \int_{-\infty}^{0} \varphi(x,0) dx - u_R \int_0^{+\infty} \varphi(x,0) dx \\
&\quad - (u_L - u_R) \int_0^{+\infty} \varphi(x,x/s) dx + (u_L - u_R) \int_0^{+\infty} \varphi(w,w/s) dw \\
&= -u_L \int_{-\infty}^{0} \varphi(x,0) dx - u_R \int_0^{+\infty} \varphi(x,0) dx = RHS
\end{aligned}
$$

Note in passing that we have used the following relation

$$
\begin{aligned}
\frac{u_L^2}{2} \int_0^{+\infty} \varphi(st,t) dt + \frac{u_R^2}{2} \int_0^{+\infty} (-\varphi(st,t)) \, dt &= \frac{u_L^2 - u_R^2}{2} \int_0^{+\infty} \varphi(st,t) \, dt \\
&= \frac{u_L^2 - u_R^2}{2s} \int_0^{+\infty} \varphi(w,w/s) dw \\
&= (u_L - u_R) \int_0^{+\infty} \varphi(w,w/s) dw
\end{aligned}
$$

where $w := st$ and $dw = sdt$, and the RH condition for *Burgers'* equation $s = 1/2(u_L + u_R)$. □

4 Consistency order

Example 6. *Leapfrog scheme for the linear advection equation reads*

$$u_j^{n+1} = u_j^{n-1} + \frac{a\Delta t}{\Delta x}\left(u_{j-1}^n - u_{j+1}^n\right).$$

Show that the local consistency error $L_{\Delta t}$

$$\|L_{\Delta t}(\cdot, t)\| \leq C(\Delta t)^2 \qquad \textit{for } \Delta t \to 0,$$

hold for a linear solution and $\Delta t/\Delta x = const.$

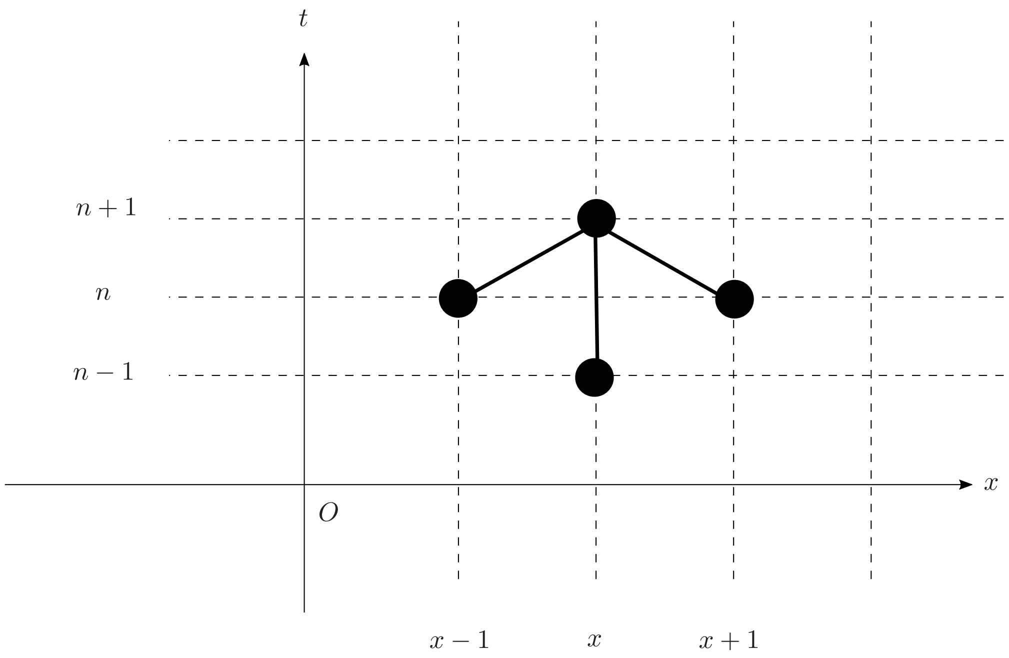

Figure 4: Leapfrog.

Consider the local truncation error

$$L_{\Delta t}(x,t) = \frac{1}{\Delta t}\left[u(x, t+\Delta t) - H_{\Delta t}(u(\cdot, t), x)\right],$$

which reads for the considered problem

$$L_{\Delta t}(x,t) = \frac{1}{\Delta t}\Big[u(x, t+\Delta t) - \Big(u(x, t-\Delta t) + \frac{a\Delta t}{\Delta x}\left(u(x-\Delta x, t) - u(x+\Delta x, t)\right)\Big)\Big].$$

Applying Taylor's expansion for $u(x, t+\Delta t)$

$$\begin{aligned} u(x, t+\Delta t) &= u(x,t) + u_t(x,t)\Delta t + u_{tt}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\ &= u(x,t) - a u_x(x,t)\Delta t + a^2 u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3). \end{aligned}$$

Applying Taylor's expansion for $u(x, t - \Delta t)$

$$\begin{aligned} u(x, t-\Delta t) &= u(x,t) - u_t(x,t)\Delta t + u_{tt}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\ &= u(x,t) + a u_x(x,t)\Delta t + a^2 u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3). \end{aligned}$$

Furthermore, applying Taylor's expansion for $u(x + \Delta x, t)$ and $u(x - \Delta x, t)$

$$\begin{aligned} u(x+\Delta x, t) &= u(x,t) + u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3) \\ u(x-\Delta x, t) &= u(x,t) - u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3). \end{aligned}$$

Local truncation error

$$\begin{aligned} L_{\Delta t}(x,t) = \frac{1}{\Delta t}\Big[& u(x,t) - a u_x(x,t)\Delta t + a^2 u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\ & - \Big(u(x,t) + a u_x(x,t)\Delta t + a^2 u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\ & + \frac{a\Delta t}{\Delta x}\Big(\Big(u(x,t) - u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3)\Big) \\ & \qquad - \Big(u(x,t) + u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3)\Big)\Big)\Big)\Big] \\ = \frac{1}{\Delta t}\Big[& -2a u_x(x,t)\Delta t + \mathcal{O}((\Delta t)^3) - \frac{a\Delta t}{\Delta x}\Big(-2u_x(x,t)\Delta x + \mathcal{O}((\Delta x)^3)\Big)\Big] \\ = & \frac{1}{\Delta t}\mathcal{O}((\Delta t)^3) \\ = & \mathcal{O}((\Delta t)^2). \end{aligned}$$

Therefore, order of consistency for Leapfrog scheme is of order 2

$$\|L_{\Delta t}(\cdot, t)\| \le C(\Delta t)^2 \qquad \text{for } \Delta t \to 0.$$

Note in passing that $u_t = -a u_x$ and $u_{tt} = a^2 u_{xx}$ are from the linear advection equation.