Global Exercise - 10

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1 Breaking time for an arbitrary flux function

Consider arbitrary convex scalar equation of conservation laws

$$u_t + f(u)_x = 0, \tag{1}$$

where $f(u)$ is convex, i.e. $f''(u) > 0$. Then, smooth flux function $f(u)$ leads to

$$u_t + f'(u)u_x = 0. \tag{2}$$

Arbitrarily consider one characteristic line passing the point with initial condition in space and time $x_0(t = 0)$. The first characteristic line passing point $(x_{0_1}, 0)$ reads

$$x = x_{0_1} + f'(u_0(x_{0_1}))\, t \tag{3}$$

The second characteristic line passing point $(x_{0_2}, 0)$ reads

$$x = x_{0_2} + f'(u_0(x_{0_2}))\, t \tag{4}$$

Equalizing the two characteristic lines from (3) and (4) yields

$$x_{0_1} + f'(u_0(x_{0_1}))\, t = x_{0_2} + f'(u_0(x_{0_2}))\, t \tag{5}$$

which leads to

$$t = -\frac{x_{0_1} - x_{0_2}}{f'(u_0(x_{0_1}) - f'(u_0(x_{0_2})} \tag{6}$$

The breaking time requires the following condition

$$\begin{aligned} t &= -\frac{1}{\dfrac{f'(u_0(x_{0_1}) - f'(u_0(x_{0_2})}{x_{0_1} - x_{0_2}}} \\ &= -\frac{1}{\dfrac{d}{dx} f'(u_0(x))} \end{aligned} \tag{7}$$

The *first* breaking time yields

$$\therefore \quad \boxed{T_b = -\frac{1}{\min\limits_{x \in \mathbb{R}} \dfrac{d}{dx} f'(u_0(x))}} \tag{8}$$

Example 1. *Compute the breaking time of Burgers' equation given different initial conditions as follows.*

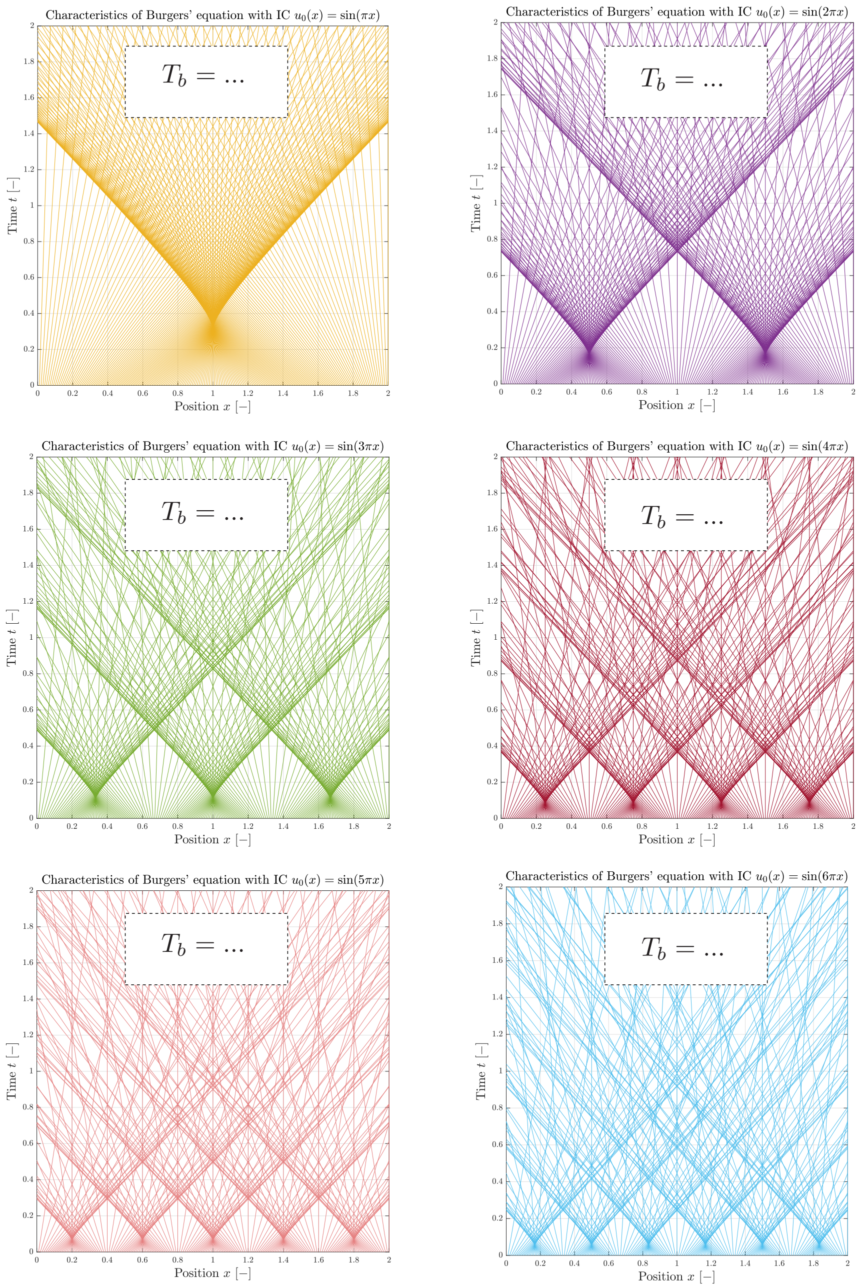

Figure 1: Characteristics of Burgers' equation with smooth initial conditions in form of $u(x, 0) = u_0(x) = \sin(k\pi x)$ for $k = 1, 2, 3, 4, 5, 6$.

2 Riemann's problem

Example 2. *Non-zero initial condition applied on u_L and u_R.*

Consider the two cases as follows

$$\begin{cases} u_L = \alpha, & x < 0, \\ u_R = \beta, & x > 0, \end{cases} \tag{9}$$

and

$$\begin{cases} u_L = \beta, & x < 0, \\ u_R = \alpha, & x > 0, \end{cases} \tag{10}$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$.

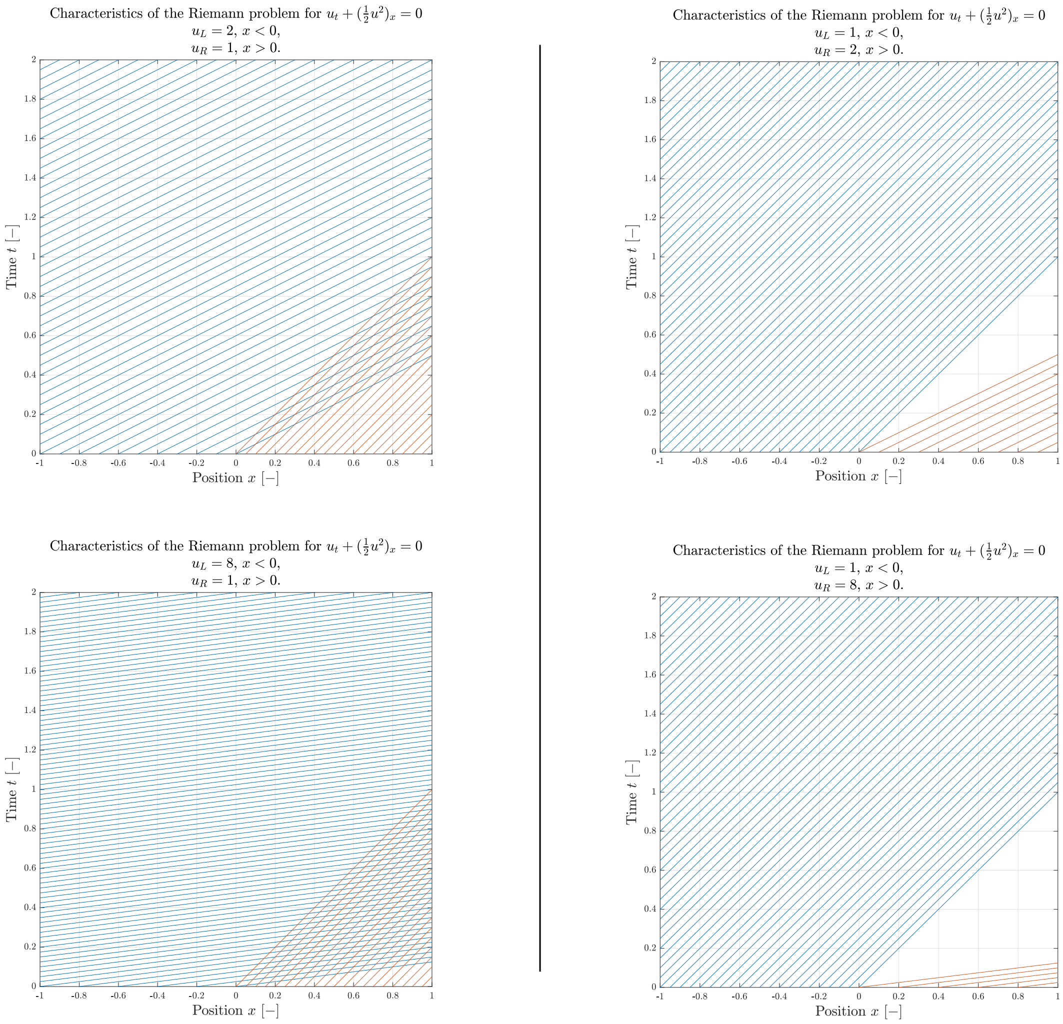

Figure 2: Characteristics of Riemann problem for Burgers' equation: non-zero initial condition applied on u_L and u_R.

Example 3. *Zero initial condition applied either on u_L or u_R.*

Consider the two cases as follows

$$\begin{cases} u_L = \alpha, & x < 0, \\ u_R = 0, & x > 0, \end{cases} \tag{11}$$

and

$$\begin{cases} u_L = 0, & x < 0, \\ u_R = \alpha, & x > 0, \end{cases} \tag{12}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$.

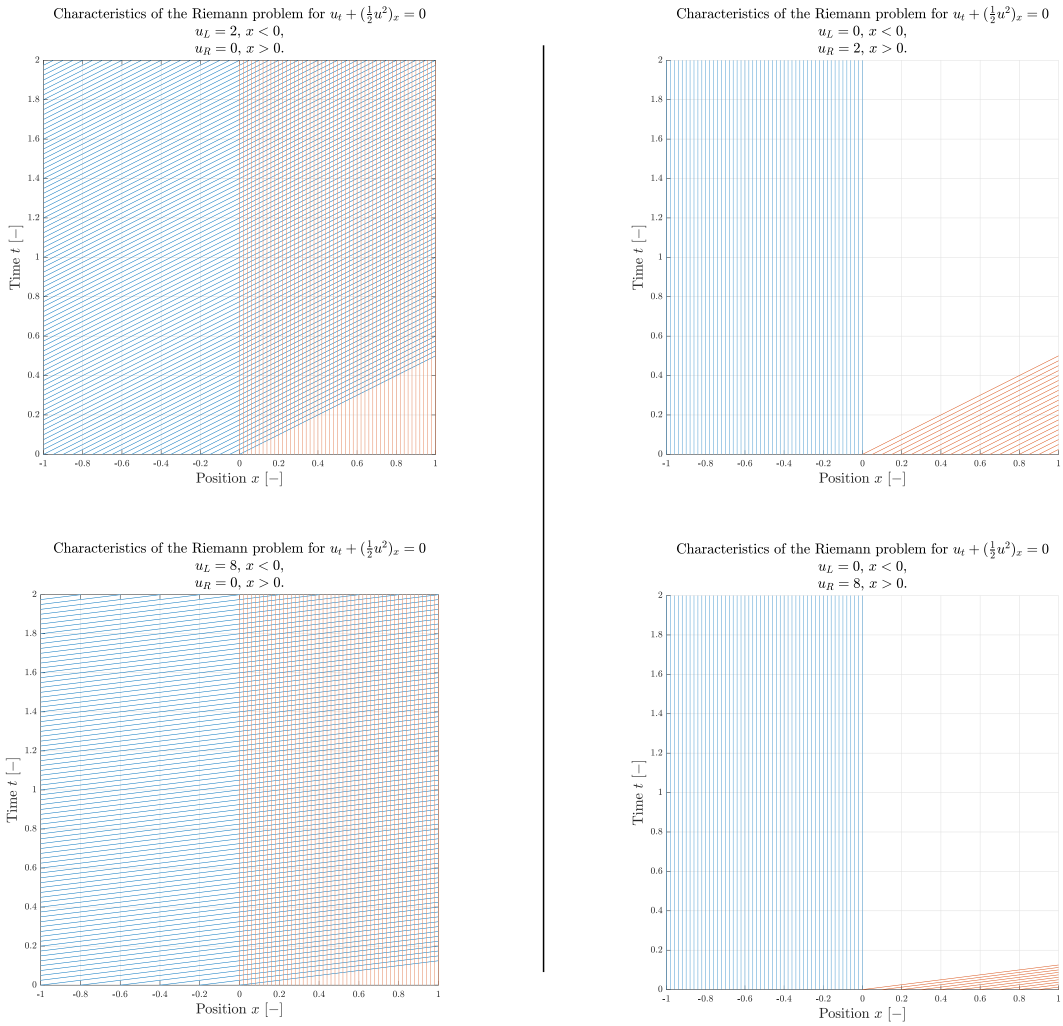

Figure 3: Characteristics of Riemann problem for Burgers' equation: zero initial condition applied either on u_L or on u_R.

3 Total variation (TV)

(Useful later on for structure of FVM, i.e. Total-variation-diminishing (TVD))

Example 4. *Let f be defined as follows*

$$f : \begin{cases} \mathbb{R} \to \mathbb{R}, \\ x \mapsto f(x) := \begin{cases} x, & 0 < x < 1, \\ 2, & 1 \leq x < 2, \\ 1 + x, & 2 \leq x < 4, \\ 0, & \textit{everywhere else.} \end{cases} \end{cases} \tag{13}$$

Find the total variation of f.

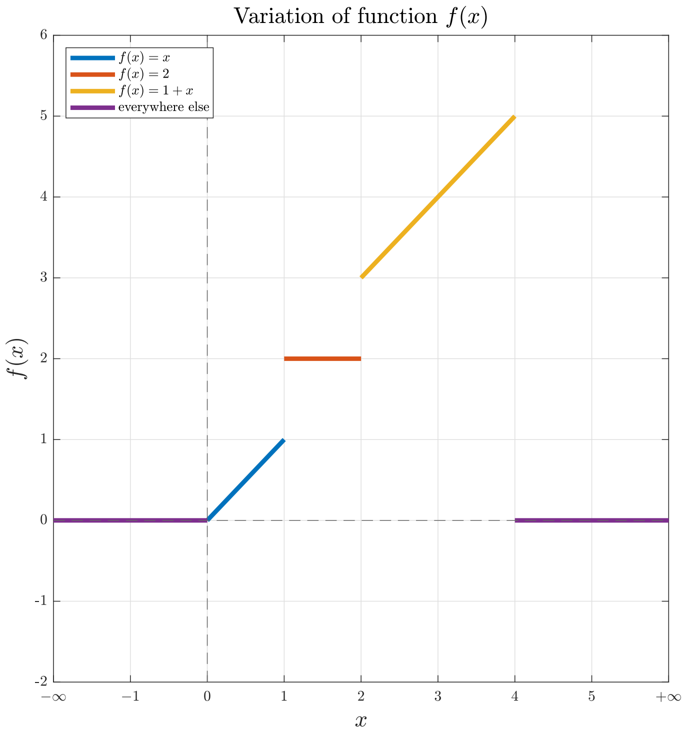

Figure 4: Variation of function $f(x)$.

The total variation of f is computed as follows

$$\begin{aligned} TV(f) = \int_0^1 |f'|dx + \lim_{h \to 0} \int_{1-h}^{1} \frac{|f(x+h) - f(x)|}{h} dx + \int_1^2 |f'|dx \\ + \lim_{h \to 0} \int_{2-h}^{2} \frac{|f(x+h) - f(x)|}{h} dx + \int_2^4 |f'|dx \\ + \lim_{h \to 0} \int_{4-h}^{4} \frac{|f(x+h) - f(x)|}{h} dx = 1 + 1 + 0 + 1 + 2 + 5 = 10 \end{aligned}$$

$$\therefore \quad \boxed{TV(f) = 10.} \tag{14}$$

Example 5. *Let f be defined as follows*

$$f : \begin{cases} \mathbb{R} \to \mathbb{R}, \\ x \mapsto f(x) := \begin{cases} -x+1, & 0 < x < 1, \\ 2, & 1 \leq x < 2, \\ -x+7, & 2 \leq x < 4, \\ 0, & \textit{everywhere else.} \end{cases} \end{cases} \tag{15}$$

Find the total variation of f.

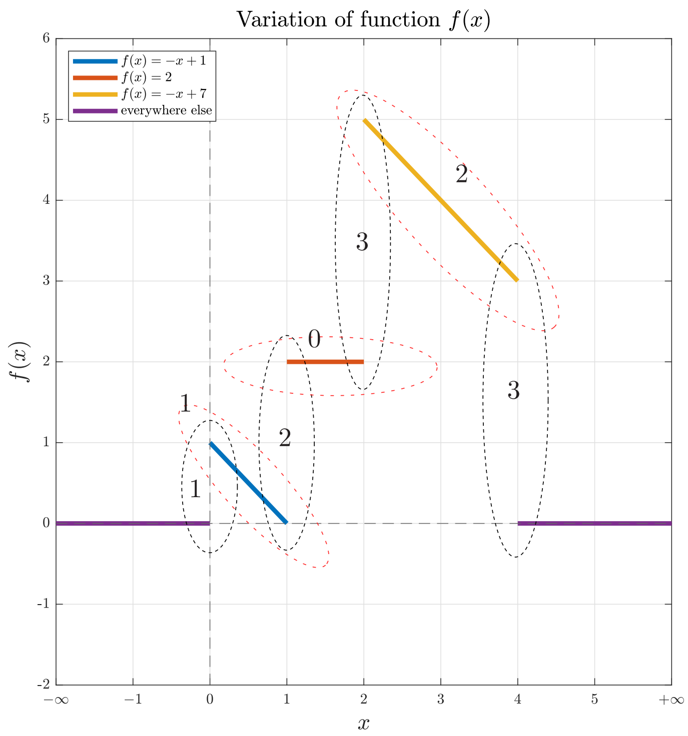

Figure 5: Variation of function $f(x)$.

The total variation of f is computed as follows

$$TV(f) = \dots \tag{16}$$

$$\therefore \quad \boxed{TV(f) = 12} \tag{17}$$

Example 6. *Let f be defined as follows*

$$f : \begin{cases} \mathbb{R} \to \mathbb{R}, \\ x \mapsto f(x) := \begin{cases} -x+1, & 0 < x < 1, \\ 2, & 1 \leq x < 2, \\ \sin(-\pi x) + 4, & 2 \leq x < 4, \\ 0, & \textit{everywhere else}. \end{cases} \end{cases} \tag{18}$$

Find the total variation of f.

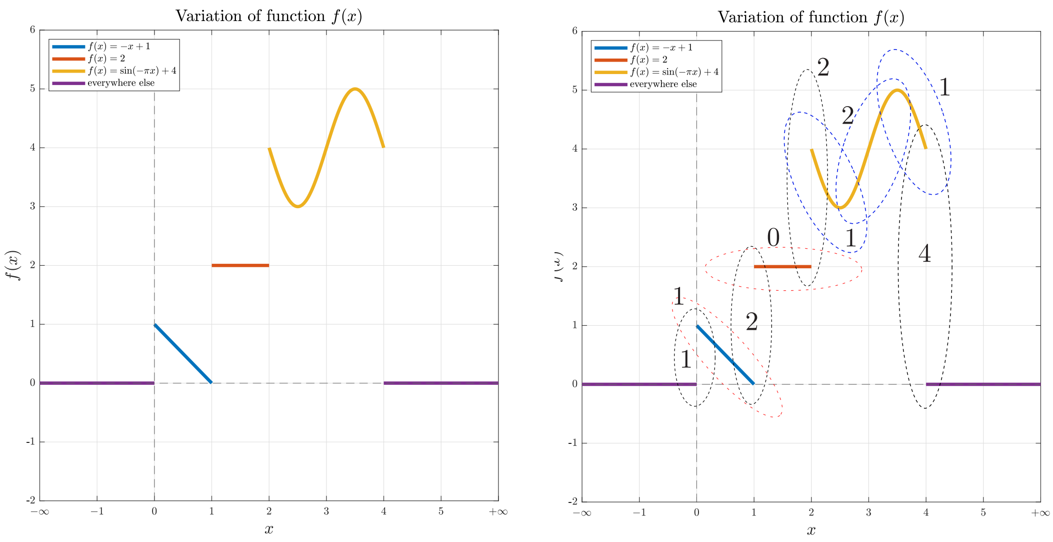

Figure 6: Variation of function $f(x)$.

The total variation of f is computed as follows

$$TV(f) = \dots \tag{19}$$

$$\therefore \quad \boxed{TV(f) = 14} \tag{20}$$

4 Manipulating conservation laws

Manipulation of conservation laws by transforming the differential form into another differential equation, which appears lately as equivalently as the original one, **may not** result in an equivalent differential equation with respect to weak solutions.

Example 7. *Manipulating conservation laws*

Consider the Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{21}$$

Manipulation of this PDE by multiplying (21) by $2u$ we obtain

$$2uu_t + 2u\left(\frac{1}{2}u^2\right)_x = 0, \tag{22}$$

which can be recast into

$$w_t + \left(\frac{2}{3}w^{\frac{3}{2}}\right)_x = 0 \tag{23}$$

where $w := u^2$, and the flux function become $f(w) = \dfrac{2}{3}w^{\frac{3}{2}}$. The solutions to (21) and (23), respectively, read

$$u(x,t) = g_0(x - ut), \tag{24}$$
$$w(x,t) = g_0(x - w^{1/2}t), \tag{25}$$

which are precisely the same. Note in passing that $w^{1/2} = u$. Although (21) and (23) have the same solution, their weak solutions are different from each other, i.e. by considering the *Riemann*'s problem with $u_L > u_R$ together with checking the *Rankine-Hugoniot* condition, we obtain unique weak solution with a shock travelling at speeds computed as follows

$$s_{\text{Burgers}} = \frac{[f(u)]}{[u]} = \dots \tag{26}$$

Likewise, for the *manipulated Burgers* we obtain

$$s_{\text{manipulated Burgers}} = \frac{[f(w)]}{[w]} = \dots \tag{27}$$

5 Linear hyperbolic systems

Example 8. *Consider the natural 1D second order wave equation*

$$u_{tt} = \alpha^2 u_{xx}, \quad x \in \mathbb{R}, \tag{28}$$

with initial condition

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{29}$$

Approach: By introducing

$$\begin{cases} v = u_x, \\ w = u_t, \end{cases} \tag{30}$$

one obtains

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} -w \\ -\alpha^2 v \end{pmatrix}_x = 0, \tag{31}$$

which can be recast into

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0. \tag{32}$$

The initial condition becomes

$$\begin{cases} v(x,0) = u_0'(x), \\ w(x,0) = u_1(x). \end{cases} \tag{33}$$

Example 9. *Consider the* ***linearized shallow water equations***

$$\begin{pmatrix} u \\ \varphi \end{pmatrix}_t + \begin{pmatrix} \bar{u} & 1 \\ \bar{\varphi} & \bar{u} \end{pmatrix} \begin{pmatrix} u \\ \varphi \end{pmatrix}_x = 0, \tag{34}$$

where $\bar{u}, \bar{\varphi} \in \mathbb{R}$ *are given, and sufficient initial conditions for* u *and* φ *are provided.*