# Global Exercise - 15

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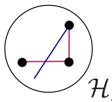
## 1 A small note about monotonicity: *Upwind* scheme

**Example 1.** *Examine monotonicity of $\mathcal{H}_{\Delta t}$ for conservation law $u_t + au_x = 0$.*



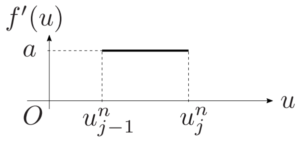
$(a > 0)$ CFL: $0 \le f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_{j-1}^n) - f(u_j^n)\Big)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \frac{\Delta t}{\Delta x}f'(u_{j-1}^n)$$

$$B_1 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 - \frac{\Delta t}{\Delta x}f'(u_j^n)$$

$$C_1 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = 0$$




1.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing?

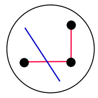
$A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0 \quad \checkmark$

$B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t \text{ (?)} \overset{\text{CFL}}{\to} \checkmark$

1.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$A_1 \le 0 \Leftrightarrow f'(u_{j-1}^n) \le 0 \quad \lightning$

$B_1 \le 0 \Leftrightarrow f'(u_j^n) \ge \Delta x/\Delta t \text{ (?)} \overset{\text{CFL}}{\to} \lightning$



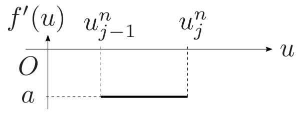
$(a < 0)$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_{j-1}^n) - f(u_j^n)\Big)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \frac{\Delta t}{\Delta x}f'(u_{j-1}^n)$$

$$B_1 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 - \frac{\Delta t}{\Delta x}f'(u_j^n)$$

$$C_1 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = 0$$




2.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing?

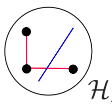
$A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0 \quad \lightning$

$B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t \quad \checkmark$

2.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$A_1 \le 0 \Leftrightarrow f'(u_{j-1}^n) \le 0 \quad \checkmark$

$B_1 \le 0 \Leftrightarrow f'(u_j^n) \ge \Delta x/\Delta t \quad \lightning$



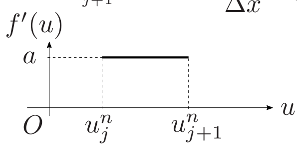
$(a > 0)$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_j^n) - f(u_{j+1}^n)\Big)$$

$$A_2 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = 0$$

$$B_2 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 + \frac{\Delta t}{\Delta x}f'(u_j^n)$$

$$C_2 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = -\frac{\Delta t}{\Delta x}f'(u_{j+1}^n)$$




3.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing?

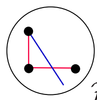
$B_2 \ge 0 \Leftrightarrow f'(u_j^n) \ge -\Delta x/\Delta t \quad \checkmark$

$C_2 \ge 0 \Leftrightarrow f'(u_{j+1}^n) \le 0 \quad \lightning$

3.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$B_2 \le 0 \Leftrightarrow f'(u_j^n) \le -\Delta x/\Delta t \quad \lightning$

$C_2 \le 0 \Leftrightarrow f'(u_{j+1}^n) \ge 0 \quad \checkmark$



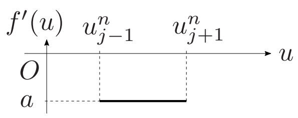
$(a < 0)$ CFL: $0 \le -f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_j^n) - f(u_{j+1}^n)\Big)$$

$$A_2 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = 0$$

$$B_2 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 + \frac{\Delta t}{\Delta x}f'(u_j^n)$$

$$C_2 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = -\frac{\Delta t}{\Delta x}f'(u_{j+1}^n)$$




4.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing?

$B_2 \ge 0 \Leftrightarrow f'(u_j^n) \ge -\Delta x/\Delta t \text{ (?)} \overset{\text{CFL}}{\to} \checkmark$

$C_2 \ge 0 \Leftrightarrow f'(u_{j+1}^n) \le 0 \quad \checkmark$

4.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$B_2 \le 0 \Leftrightarrow f'(u_j^n) \le -\Delta x/\Delta t \text{ (?)} \overset{\text{CFL}}{\to} \lightning$

$C_2 \le 0 \Leftrightarrow f'(u_{j+1}^n) \ge 0 \quad \lightning$

Figure 1: Monotonicity of *Upwind* scheme: 8 different cases.

## 2 Structure of FVM

**Example 2.** *Monotone → $L_1$-Contracting → TVD → Monotonicity-Preserving.*

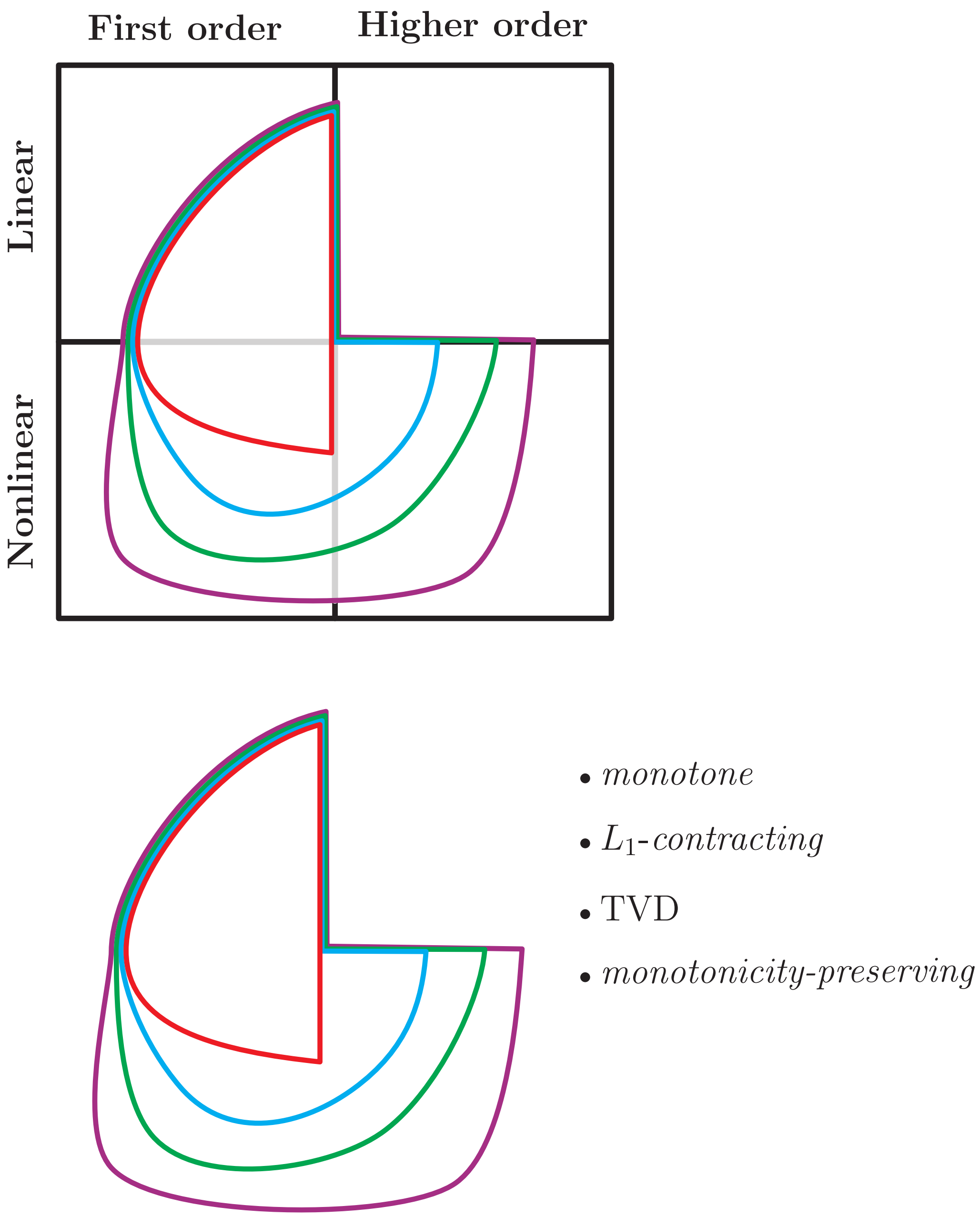



Figure 2: Structure of Finite Volume Method [Lecture note page 115].

# 3 Limiter

**Example 3.** *Examine the $1^{st}$-order-accurate LF and the $2^{nd}$-order-accurate LW.*

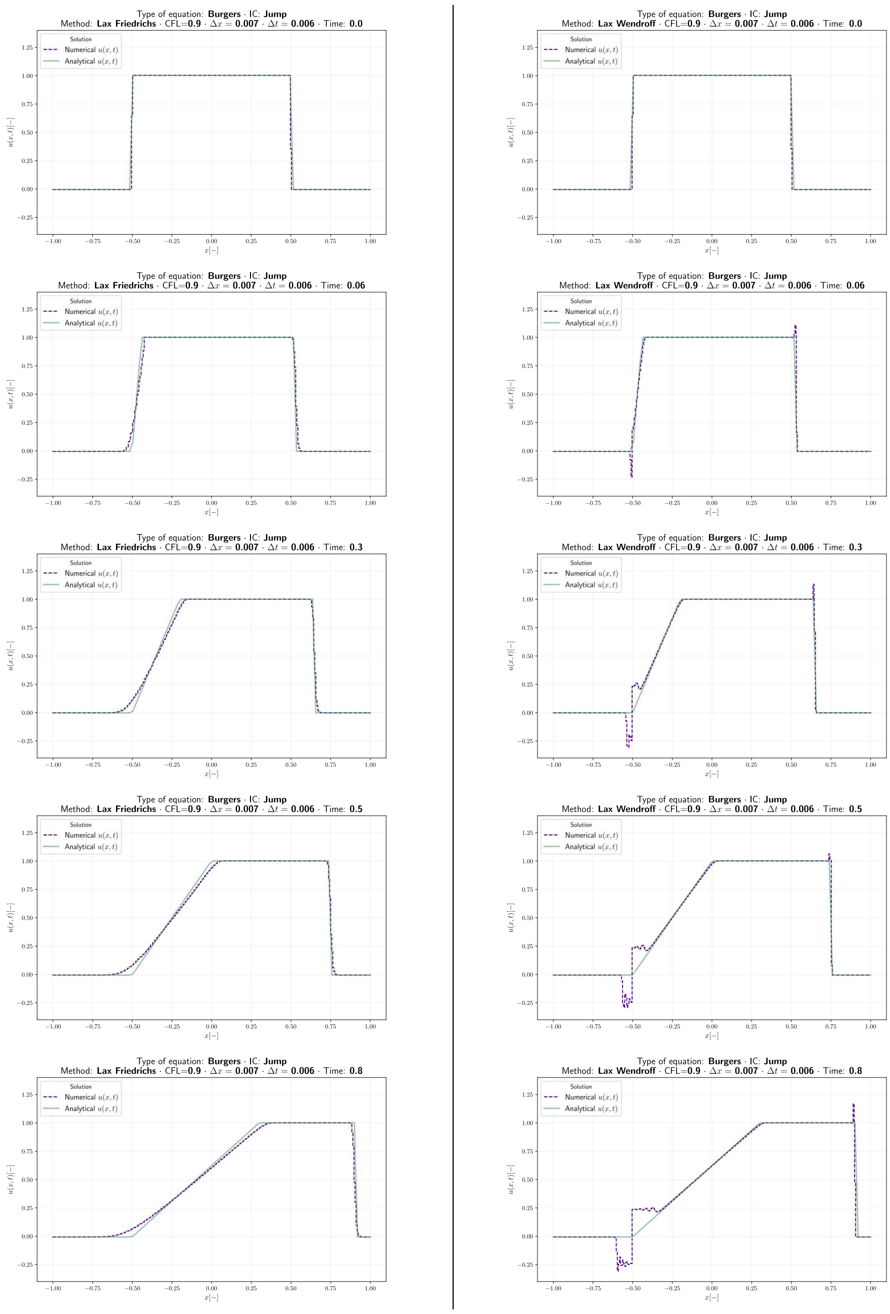



Figure 3: Oscillatory phenomena around discontinuity: (left) none oscillation founded in *Lax-Friedrichs*; (right) oscillation observed in *Lax-Wendroff*.

**Example 4.** *Assume the linear advection problem given by*

$$u_t + au_x = 0 \quad \textit{with} \quad a > 0,$$

*and the numerical scheme as follows*

$$u_j^{n+1} = u_j^n + \nu\left(u_{j-1}^n - u_j^n\right) + \frac{\nu(1-\nu)}{2}\Delta x\left(\sigma_{j-1}^n - \sigma_j^n\right),$$

*with limited slopes*

$$\sigma_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x}\phi(\theta_j).$$

*Prove that numerical scheme is TVD with limiters $\phi(\theta)$ given as follows*

1. $\phi(\theta) = \max(0, \min(\theta, 2))$,
2. $\phi(\theta) = \max(0, \min(2\theta, 2))$,
3. $\phi(\theta) = \dfrac{\theta + |\theta|}{\alpha + |\theta|}$ *for any fixed* $\alpha \geq 1$
4. $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$.

*Denote which of them are second order schemes.*

According to the theorem on TVD region of *Upwind* numerical scheme with limiter presented in the lecture, we know that limiter gives us TVD scheme if the following relations

$$\begin{aligned} 0 \leq \phi(\theta) &\leq 2\theta, \\ |\phi(\theta)| &\leq 2, \end{aligned}$$

holds. Besides, the scheme is of 2nd-order if and only if $\phi(1) = 1$ holds. Therefore, we can simply check lower and upper bounds of limiters, as follows

1. $\phi(\theta) = \max(0, \min(\theta, 2))$ satisfies bounds and goes through $\phi(1) = 1$.
   Hence, it corresponds to 2nd-order TVD method.

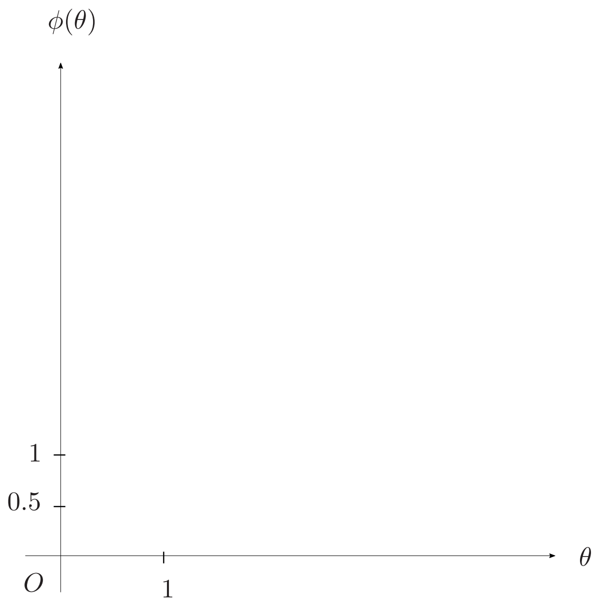
2. $\phi(\theta) = \max(0, \min(2\theta, 2))$ satisfies bounds but $\phi(1) = 2$.
   Hence, it is of 1st-order TVD scheme.

3. $\phi(\theta) = \dfrac{\theta + |\theta|}{\alpha + |\theta|}$ satisfies bounds for any fixed $\alpha \geq 1$, but $\phi(1) = 1$ holds only for $\alpha = 1$.
   Hence, it is of 2nd-order only if $\alpha = 1$ and of 1st-order otherwise.

4. $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$ satisfies bounds and $\phi(1) = 1$ holds.
   Hence, it is TVD scheme of 2nd-order.

**Example 5.** *Examine and sketch the following* Limiter *function.*

$$\phi(\theta) = \min\left(\delta\theta, \frac{1+\theta}{2}, \gamma\right), \ \textit{for } \delta, \gamma \geq 1.$$



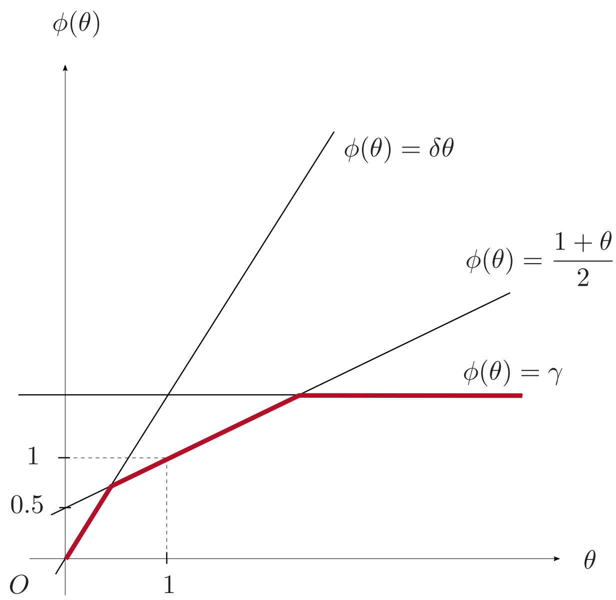




Figure 4: Limiter function $\phi(\theta) = \min\left(\delta\theta, \frac{1+\theta}{2}, \gamma\right)$, for $\delta, \gamma \geq 1$.

# 4 A bit insight into Godunov's solver

**Example 6.** *Verify the Godunov numerical flux function*

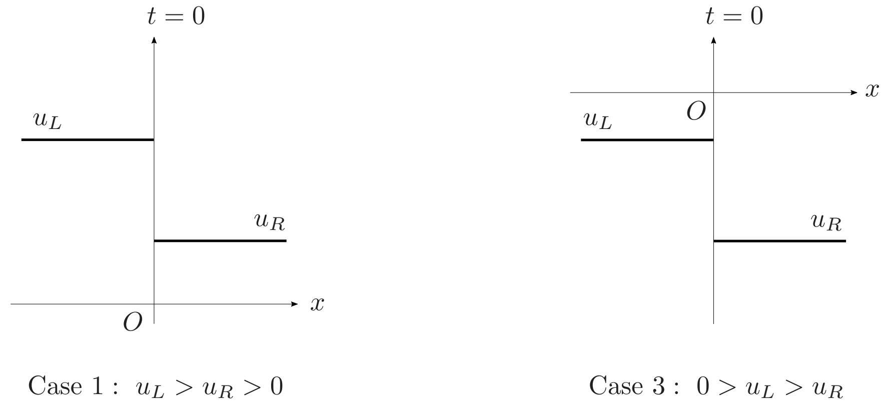
$$g(u_L, u_R) = \begin{cases} \min\limits_{u_L \le u \le u_R} f(u) & \text{if } u_L \le u_R, \\ \max\limits_{u_R \le u \le u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

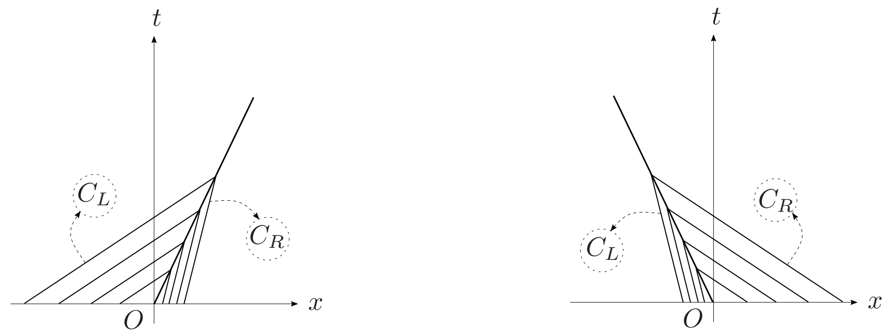
*by considering all possible cases.*

All possible cases read

1. $f'(u_L), f'(u_R) \ge 0 \Rightarrow u* = u_L$
2. $f'(u_L), f'(u_R) \le 0 \Rightarrow u* = u_R$
3. $f'(u_L) \ge 0 \ge f'(u_R) \Rightarrow u* = u_L$ if $[f]/[u] > 0$ or $u* = u_R$ if $[f]/[u] < 0$
4. $f'(u_L) < 0 < f'(u_R) \Rightarrow u* = u_{\mathrm{s}}$ where $f'(u_s) = 0$

**Example 7.** *Examine $u_L > u_R$.*




$$(C_L):\ t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad (C_R):\ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

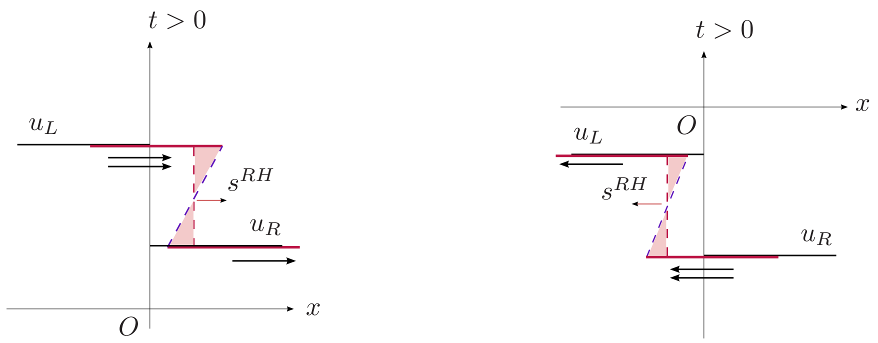
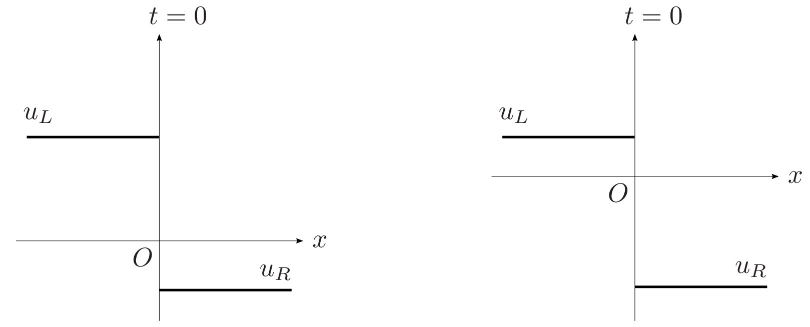


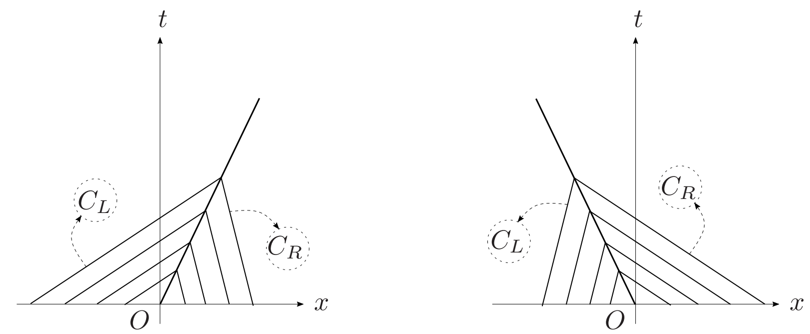

Figure 5: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

Schock solution

**Example 8.** *Examine $u_L > u_R$.*




Case 2.1 : $(u_L > 0 > u_R) \wedge |u_L| > |u_R|$ Case 2.2 : $(u_L > 0 > u_R) \wedge |u_L| < |u_R|$




$$(C_L): \ t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad (C_R): \ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

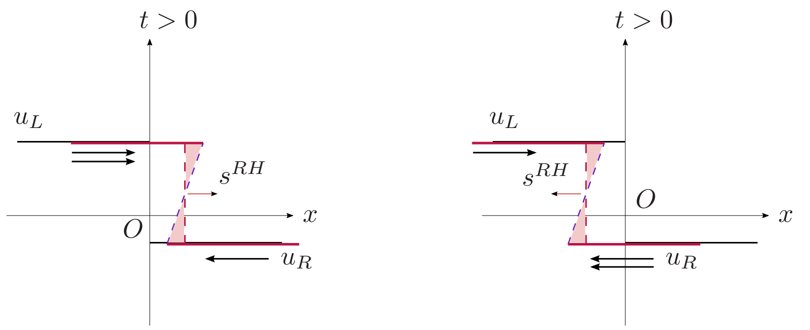
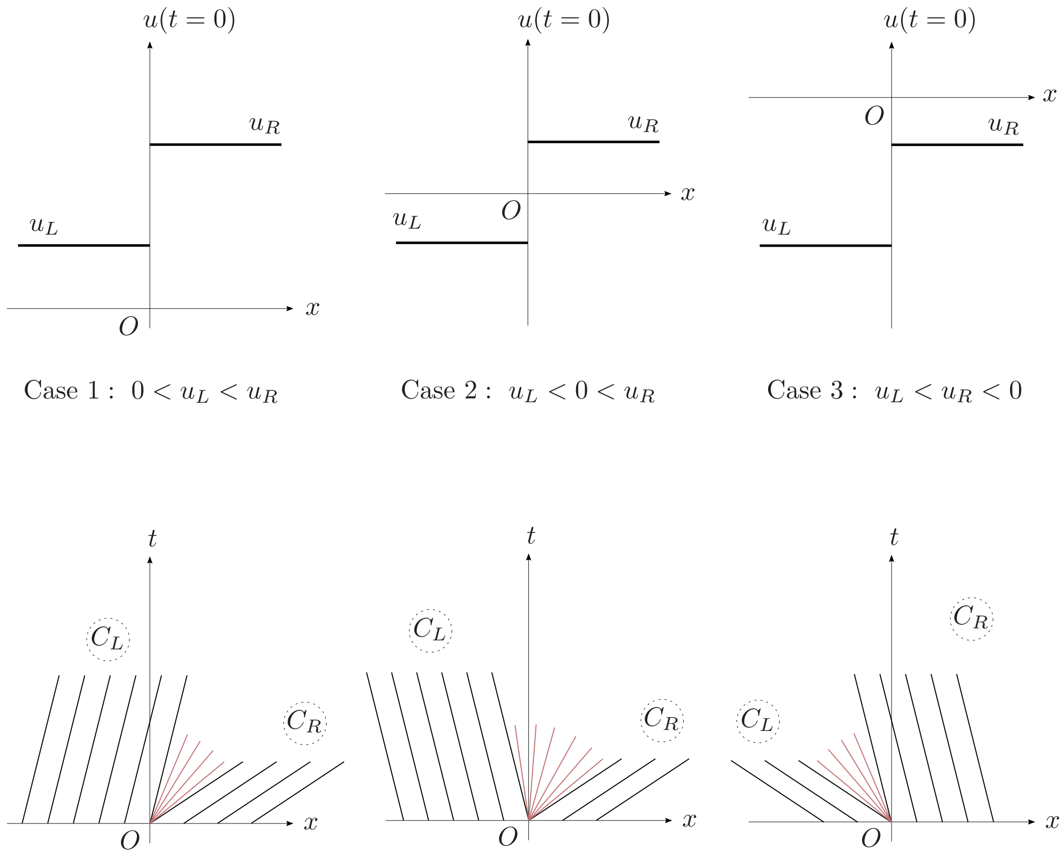



Figure 6: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

Schock solution

**Example 9.** *Examine $u_L < u_R$.*




$$(C_L): \ t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad (C_R): \ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

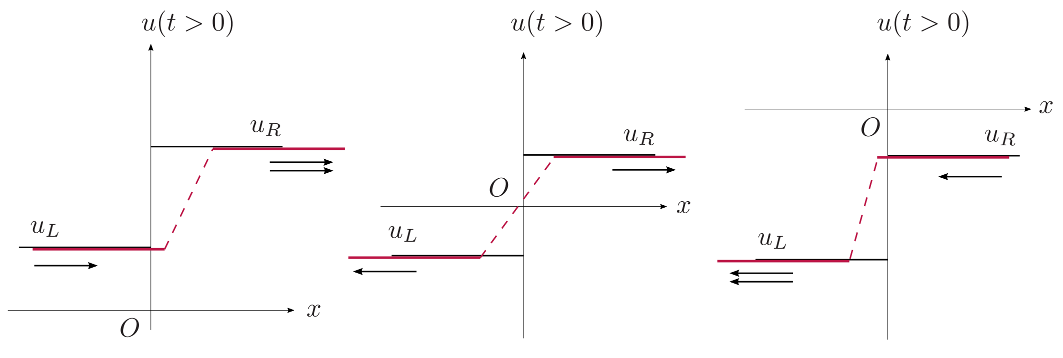



Figure 7: Riemann problem with $u_L < u_R$: IC, Characteristics, Solution.

Rarefaction solution

**Example 10.** *Consider the* Godunov*'s solver for the conservation law* $u_t + f(u)_x = 0$ *with convex and nonlinear flux function* $f$*, e.g. the flux function in* Burgers*' equation, where*

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(g(u_{j-1}^n, u_j^n) - g(u_j^n, u_{j+1}^n)\right),$$

*and the numerical flux function*

$$g(u_L, u_R) = \begin{cases} \min\limits_{u_L \le u \le u_R} f(u) & \text{if } u_L \le u_R, \\ \max\limits_{u_R \le u \le u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

1. *Show that the numerical flux function is* monotone*.*
2. *Rewrite the scheme in the* incremental form*.*
3. *Show that the scheme has the* total-variation-diminishing *(TVD) property.*

Approach:

1. *Proof.* In order to show the monotonicity of the numerical flux function, it is sufficient to show the following relations

$$\boxed{\begin{cases} \partial_u g(u,v) \ge 0, \\ \partial_v g(u,v) \le 0. \end{cases}} \tag{1}$$

Hence, in case of *Burgers*' equation, where $f'(u) = u$, we proceed as follows

$$g(u_L, u_R) = \begin{cases} f(u_L), & \text{if } \left[\begin{array}{l} f'(u_L) \ge 0 \text{ and } f'(u_R) \ge 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\ \\ f(u_R), & \text{if } \left[\begin{array}{l} f'(u_L) \le 0 \text{ and } f'(u_R) \le 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\ \\ f(u_s), & \text{if } \quad f'(u_L) < 0 < f'(u_R), \end{cases}$$

where $u_s$ denotes the sonic point where $f'(u_s) = 0$. Then, by taking the partial derivative of $g(u_L, u_R)$ w.r.t. $u_L$ and $u_R$ we arrive at the following expressions

$$\partial_{u_L} g(u_L, u_R) = \begin{cases} f'(u_L), & \text{if } \left[\begin{array}{l} f'(u_L) \ge 0 \text{ and } f'(u_R) \ge 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_L} g(u_L, u_R) \ge 0.} \tag{2}$$

Likewise, the partial derivative of $g(u_L, u_R)$ w.r.t. $u_R$ goes as follows

$$\partial_{u_R} g(u_L, u_R) = \begin{cases} f'(u_R), & \text{if } \left[\begin{array}{l} f'(u_L) \le 0 \text{ and } f'(u_R) \le 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_R} g(u_L, u_R) \leq 0.} \tag{3}$$

Therefore, by combining (2) and (3), and comparing them with the condition given at (1), we arrive at *q.e.d.* $\square$

2. The incremental form reads

$$\boxed{u_j^{n+1} = u_j^n + C_{j+1/2}^n (u_{j+1}^n - u_j^n) - D_{j-1/2}^n (u_j^n - u_{j-1}^n).}$$

For any conservative Finite Volume scheme we obtain

$$C_{j+1/2} = -\lambda \frac{g_{j+1/2} - f_j}{u_{j+1} - u_j}$$
$$D_{j-1/2} = \lambda \frac{f_j - g_{j-1/2}}{u_j - u_{j-1}}$$

where $\lambda = \Delta t / \Delta x$. Herein, for the case $C_{j+1/2}$, $u_L$ is $u_j$ and $u_R$ is $u_{j+1}$, as follows

$$C_{j+1/2} = \begin{cases} 0, \quad \text{if} \begin{bmatrix} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{bmatrix} \\ -\lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \quad \text{if} \begin{bmatrix} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{bmatrix} \\ -\lambda \dfrac{f(u_s) - f(u_L)}{u_R - u_L}, \quad \text{if} \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

Likewise, it goes for the case $D_{j+1/2}$ as follows

$$D_{j+1/2} = \begin{cases} \lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \quad \text{if} \begin{bmatrix} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{bmatrix} \\ 0, \quad \text{if} \begin{bmatrix} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{bmatrix} \\ \lambda \dfrac{f(u_R) - f(u_s)}{u_R - u_L}, \quad \text{if} \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

3. In order to show the scheme has TVD property, according to Theorem II.23 (*Harten* [1980]) from the lecture note, the following three conditions

$$\boxed{\begin{aligned} C_{j+1/2} &\geq 0, \\ D_{j+1/2} &\geq 0, \\ C_{j+1/2} + D_{j+1/2} &\leq 1, \end{aligned}}$$

must hold. Herein, the first two conditions $C_{j+1/2} \geq 0,\ D_{j+1/2} \geq 0$ can be shown by using convexity of flux function $f$. Besides, the third condition $C_{j+1/2} + D_{j+1/2} \leq 1$ is shown as follows

$$C_{j+1/2} + D_{j+1/2} = \begin{cases} \lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \text{ if } \left[\begin{array}{l} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\ \\ -\lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \text{ if } \left[\begin{array}{l} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\ \\ \lambda \dfrac{f(u_R) + f(u_L) - 2f(u_s)}{u_R - u_L}, \text{ if } \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

The $C_{j+1/2} + D_{j+1/2} \leq 1$ leads to the CFL condition for *Godunov*'s scheme. For $\lambda$ that satisfies $C_{j+1/2} + D_{j+1/2} \leq 1$, the scheme is TVD.