Global Exercise - 12

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1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1: $W := R^{-1}U$ as the scheme shown in exercise

$$
\boxed{
\begin{aligned}
U_t + AU_x &= 0 \\
U_t + R\Lambda R^{-1}U_x &= 0 \\
R^{-1}U_t + \Lambda R^{-1}U_x &= 0 \\
W_t + \Lambda W_x &= 0
\end{aligned}
}
$$

   where the matrix A is diagonalizable with a transformation matrix $R \in \mathbb{R}^{N \times N}$ in the form
$$A = R\Lambda R^{-1}.$$

2. Case 2: $W := TU$ as the scheme shown in lecture note

$$
\boxed{
\begin{aligned}
U_t + AU_x &= 0 \\
U_t + T^{-1}\Lambda TU_x &= 0 \\
TU_t + \Lambda TU_x &= 0 \\
W_t + \Lambda W_x &= 0
\end{aligned}
}
$$

   where the matrix A is diagonalizable with a transformation matrix $T \in \mathbb{R}^{N \times N}$ in the form
$$A = T^{-1}\Lambda T.$$

Both schemes result in the same solution. We have just to be consistent with which scheme to follow.

2 Linearity - Oder of consistency - Stability

3 Time step methods: Comparison

1. Upwind left - One-sided method

2. Upwind right - One-sided method

3. Lax-Friedrichs

4. Lax-Wendroff

5. Leapfrog

6.

Example 1. *Show that for a general convex scalar problem*

$$u_t + f(u)_x = 0$$

with initial condition in form of Riemann*'s problem*

$$u(x, t = 0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \qquad where \quad u_L < u_R,$$

the rarefaction wave solution is given by

$$u(x,t) = \begin{cases} u_L, & x/t < f'(u_L), \\ v(x/t), & f'(u_L) \le x/t \le f'(u_R), \\ u_R, & x/t > f'(u_R), \end{cases}$$

where $v(\xi)$ is the solution to $f'(v(\xi)) = \xi$.

Example 2. *Consider*

$$u_t + Au_x = 0 \qquad \textit{for} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+,$$

where the unknown vector $u(x, t)$ is defined, and the matrix A is given as follows

$$u(x, t) := \begin{pmatrix} u_1(x, t) \\ u_2(x, t) \end{pmatrix}, \quad \textit{and} \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and the initial conditions are given by

$$\begin{cases} u_1(x, 0) & = 1, \\ u_2(x, 0) & = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases} \end{cases}$$

Approach: Eigenvalue decomposition of matrix A is given as follows

$$A = R\Lambda R^{-1}, \quad R = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2}R, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Hence, we can now change variable u to a new variable w as follows

$$w := R^{-1}u = \frac{1}{2}Ru, \quad \text{so} \quad u = Rw.$$

Then, the initial problem can be rewritten as

$$w_t + \Lambda w_x = 0 \Leftrightarrow \begin{cases} (w_1)_t + \lambda_1(w_1)_x = 0, \\ (w_2)_t + \lambda_2(w_2)_x = 0, \end{cases}$$

with the diagonal matrix Λ. This step simply means that variables w_1 and w_2 are independent of each other, i.e. the original system has been fully decoupled. Next, let us find the initial condition in terms of w as follows

$$w(x, 0) = \frac{1}{2}Ru(x, 0) = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1(x, 0) \\ u_2(x, 0) \end{pmatrix} = \begin{pmatrix} -(u_1(x, 0) + u_2(x, 0))/2 \\ -(u_1(x, 0) - u_2(x, 0))/2 \end{pmatrix},$$

which yields the following relation

$$\begin{pmatrix} w_1(x, 0) \\ w_2(x, 0) \end{pmatrix} = \begin{pmatrix} -(u_1(x, 0) + u_2(x, 0))/2 \\ -(u_1(x, 0) - u_2(x, 0))/2 \end{pmatrix}.$$

Therefore, the initial conditions for the transformed variable w_1 read

$$w_1(x, 0) = \begin{cases} -(1 + 1)/2 = -1, & x < 0, \\ -(1 - 1)/2 = 0, & x > 0, \end{cases}$$

and for the variable w_2 read

$$w_2(x, 0) = \begin{cases} -(1 - 1)/2 = 0, & x < 0, \\ -(1 - (-1))/2 = -1, & x > 0. \end{cases}$$

Since we have the advection problem with speeds $a_1 = 1$ and $a_2 = 3$, which are the diagonal values of the diagonal matrix Λ, for both w_1 and w_2, we can derive explicit solution to each equation. The explicit solution w_1 reads

$$w_1(x,t) = w_1(x-t,0) = \begin{cases} -1, & x-t < 0, \\ 0, & x-t > 0, \end{cases} \Leftrightarrow w_1(x,t) = \begin{cases} -1, & x < t, \\ 0, & x > t, \end{cases}$$

and, likewise, the explicit solution w_2 reads

$$w_2(x,t) = w_2(x-3t,0) = \begin{cases} 0, & x-3t < 0, \\ -1, & x-3t > 0, \end{cases} \Leftrightarrow w_2(x,t) = \begin{cases} 0, & x < 3t, \\ -1, & x > 3t. \end{cases}$$

Now we trace back to variable u from variable w by taking into the consideration of the relation

$$u(x,t) = Rw(x,t) = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix} = \begin{pmatrix} -(w_1(x,t) + w_2(x,t)) \\ -(w_1(x,t) - w_2(x,t)) \end{pmatrix},$$

which means the following relation holds

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \begin{pmatrix} -(w_1(x,t) + w_2(x,t)) \\ -(w_1(x,t) - w_2(x,t)) \end{pmatrix}.$$

Therefore, the solution for $u_1(x,t)$ is computed as follows

$$u_1(x,t) = \begin{cases} -(-1+0) = 1, & x < t, \\ -(0+0) = 0, & t < x < 3t, \\ -(0+(-1)) = 1, & x > 3t, \end{cases}$$

and, likewise, the solution for $u_2(x,t)$ is

$$u_2(x,t) = \begin{cases} -(-1-0) = 1, & x < t, \\ -(0-0) = 0, & t < x < 3t, \\ -(0-(-1)) = -1, & x > 3t. \end{cases}$$

Finally, the solution to $u_1(x,t)$ and $u_2(x,t)$ read

$$\therefore \boxed{ \begin{cases} u_1(x,t) = \begin{cases} 1, & x < t, \\ 0, & t < x < 3t, \\ 1, & x > 3t, \end{cases} \\ u_2(x,t) = \begin{cases} 1, & x < t, \\ 0, & t < x < 3t, \\ -1, & x > 3t. \end{cases} \end{cases} }$$

12 Remark on step-by-step derivation: Weak formulation of conservation laws

Example 3. *Consider the conservation law*

$$u_t + f(u)_x = 0 \quad for \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \qquad (1)$$

with initial condition given by

$$u(x,0) = u_0(x).$$

Find the weak solution.

Approach: The weak formulation is obtained by multiplying both sides of (1) with a test function $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R})$, and, then, integrating both sides over $\mathbb{R} \times \mathbb{R}^+$

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \varphi\left(u_t + f(u)_x\right) dxdt = 0. \qquad (2)$$

The derivation goes as follows

$$(2) \Leftrightarrow \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi\left(u_t + f(u)_x\right) dxdt = 0$$

$$\Leftrightarrow \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi\left(u_t\right) dxdt + \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi\left(f(u)_x\right) dxdt = 0$$

$$\Leftrightarrow \int_{\mathscr{D}} \int_0^{\mathcal{T}} \varphi\left(u_t\right) dtdx + \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi\left(f(u)_x\right) dxdt = 0$$

$$\Leftrightarrow \left[\int_{\mathscr{D}} \varphi\, u\big|_{t=0}^{t=+\infty}\, dx - \int_{\mathscr{D}} \int_0^{\mathcal{T}} \varphi_t\, u\, dtdx \right]$$

$$+ \left[\int_0^{\mathcal{T}} \varphi\, f(u)\big|_{x=-\infty}^{x=+\infty}\, dxdt - \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt \right] = 0$$

$$\Leftrightarrow \left[\int_{\mathscr{D}} \left(\varphi(x, \overset{t\to+\infty}{\mathcal{T}})u(x, \overset{t\to+\infty}{\mathcal{T}}) - \varphi(x,0)u(x,0) \right) dx - \int_{\mathscr{D}} \int_0^{\mathcal{T}} \varphi_t\, u\, dtdx \right]$$

$$+ \left[\int_0^{\mathcal{T}} \left(\varphi(\overset{x\to+\infty}{\mathscr{D}},t)\, f(u(\overset{x\to+\infty}{\mathscr{D}},t)) - \varphi(\underset{x\to-\infty}{\mathscr{D}},t)\, f(u(\underset{x\to-\infty}{\mathscr{D}},t)) \right) dt \right.$$

$$\left. - \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt \right] = 0$$

$$\Leftrightarrow \left[\int_{\mathscr{D}} \left(\cancel{\varphi(x, \overset{t\to+\infty}{\mathcal{T}})}^{\,0} u(x, \overset{t\to+\infty}{\mathcal{T}}) - \varphi(x,0)u(x,0) \right) dx - \int_{\mathscr{D}} \int_0^{\mathcal{T}} \varphi_t\, u\, dtdx \right]$$

$$+ \left[\int_0^{\mathcal{T}} \left(\cancel{\varphi(\overset{x\to+\infty}{\mathscr{D}},t)}^{\,0} f(u(\overset{x\to+\infty}{\mathscr{D}},t)) - \cancel{\varphi(\underset{x\to-\infty}{\mathscr{D}},t)}^{\,0} f(u(\underset{x\to-\infty}{\mathscr{D}},t)) \right) dt \right.$$

$$\left. - \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt \right] = 0$$

$$\Leftrightarrow -\int_{\mathscr{D}} \varphi(x,0)u(x,0)dx - \int_{\mathscr{D}} \int_0^{\mathcal{T}} \varphi_t u\, dtdx - \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt = 0$$

$$\Leftrightarrow -\int_{\mathscr{D}} \varphi(x,0)u_0(x)dx - \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi_t u\, dxdt - \int_0^{\mathcal{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt = 0.$$

Therefore, the weak formulation reads

$$\therefore \quad \boxed{\int_0^{\mathscr{T}} \int_{\mathscr{D}} \left(u\varphi_t + f(u)\varphi_x \right) dxdt + \int_{\mathscr{D}} \varphi(x,0)u_0(x)dx = 0.}$$

Note in passing that the derivation above has taken into consideration the swapping order of integral limits between time $t \in [0, +\infty]$ and space $x \in [-\infty, +\infty]$, which results in indifference as shown in Figure 1.
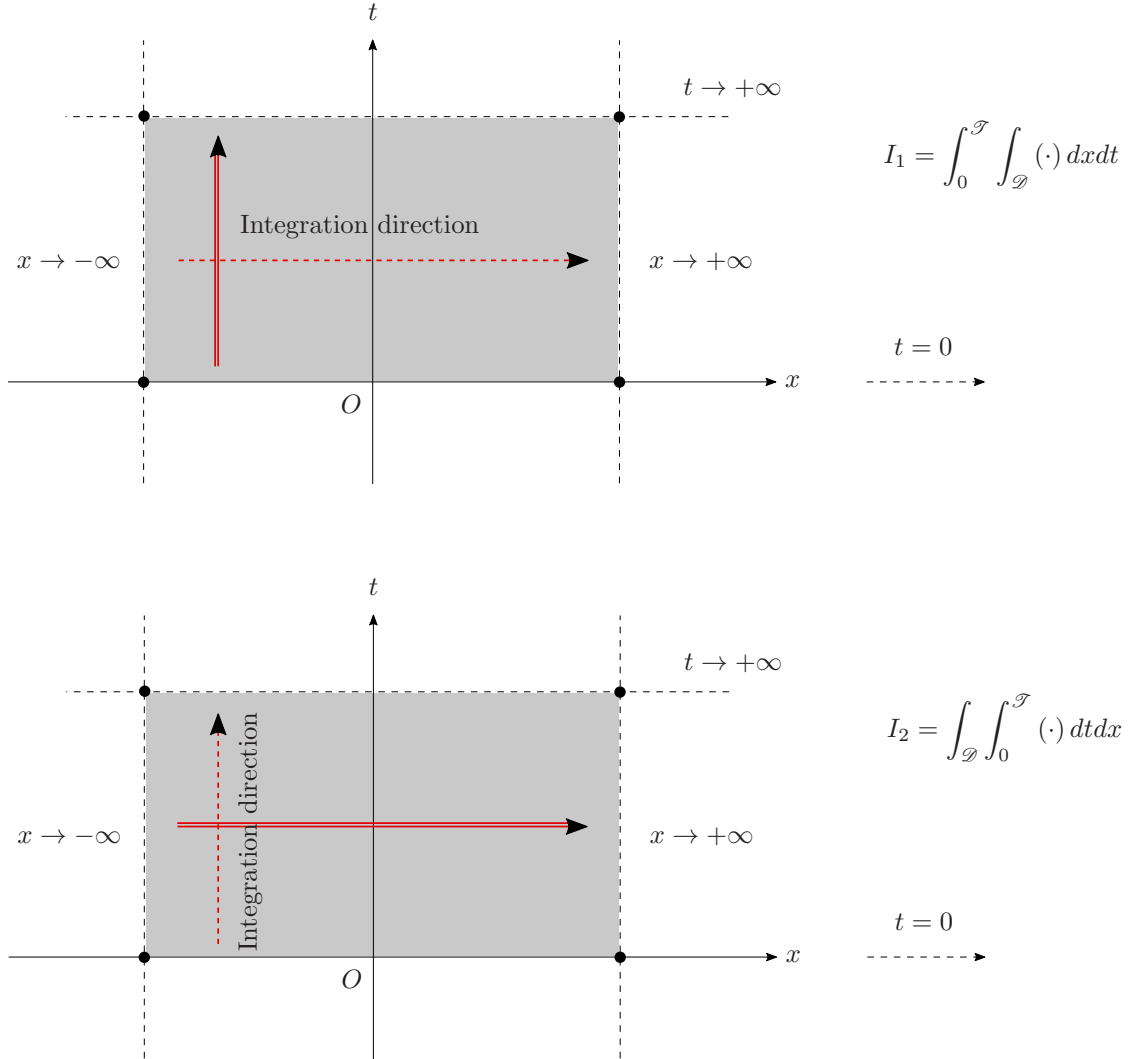

Figure 1: Indifference of limits when swapping order of integrals, i.e. $I_1 = I_2$.

13 Remark on step-by-step derivation: Weak solution of inviscid Burgers' equation in Riemann's problem

$$I_1 = \int_0^{\mathscr{T}} \int_{-\infty}^{st} (\cdot)\, dx\, dt$$

$$I_2 = \int_{-\infty}^{0} \int_0^{+\infty} (\cdot)\, dt\, dx + \int_0^{+\infty} \int_{x/s}^{+\infty} (\cdot)\, dt\, dx$$
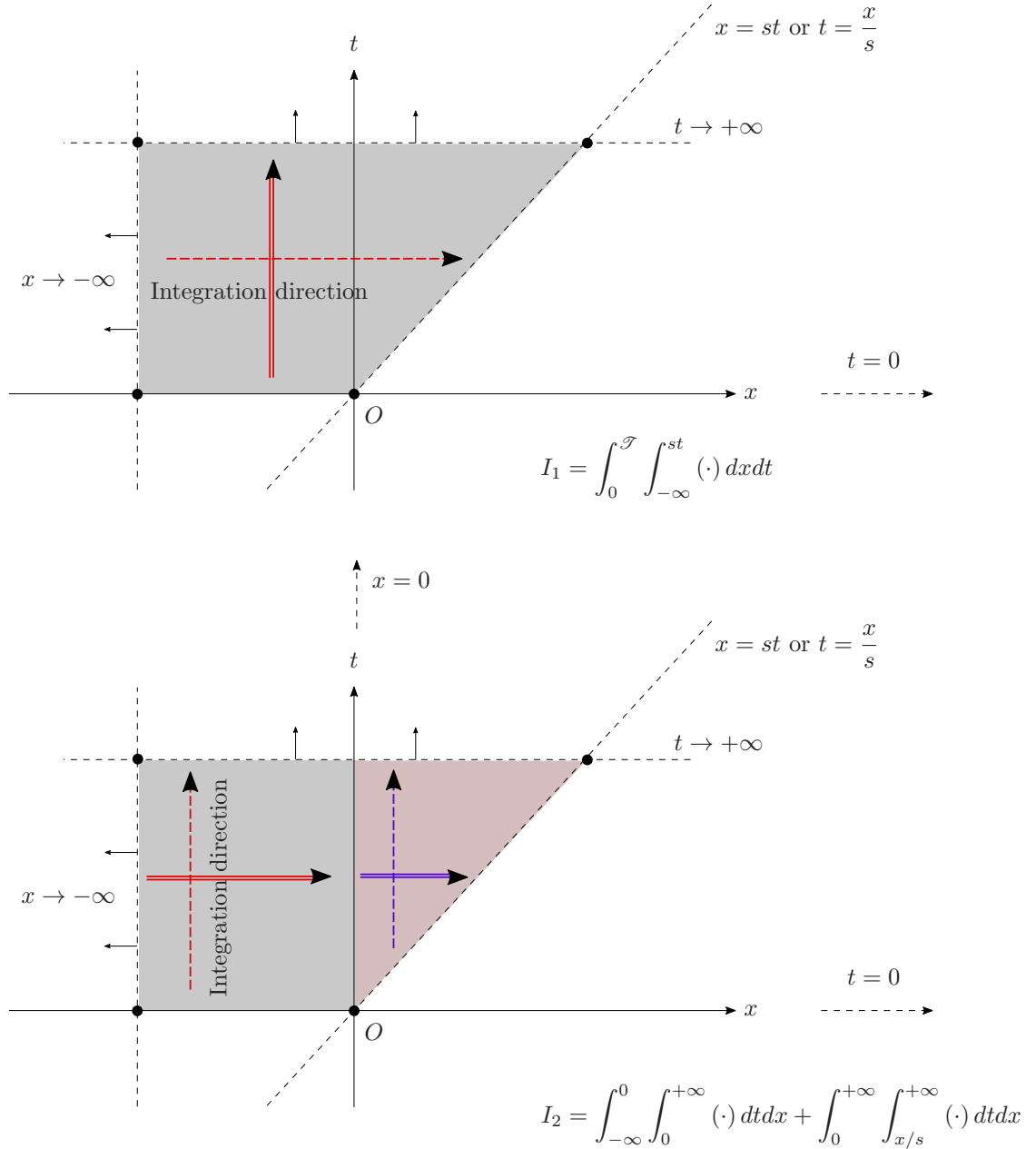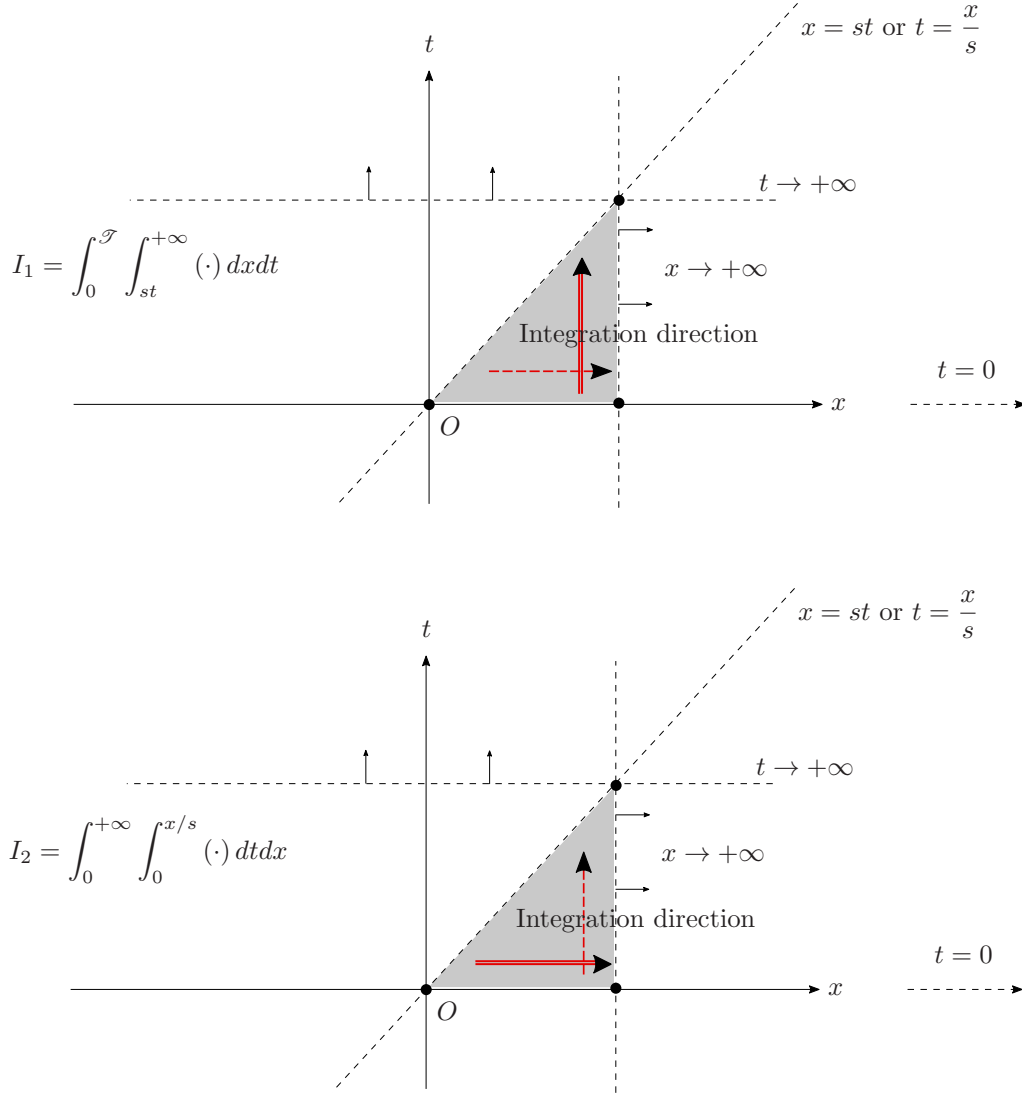
Figure 2: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee $I_1 = I_2$.

Example 4. *Given Burgers' equation with initial condition*

$$u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$

$$I_1 = \int_0^{\mathscr{T}} \int_{st}^{+\infty} (\cdot)\, dx\, dt$$

$$I_2 = \int_0^{+\infty} \int_0^{x/s} (\cdot)\, dt\, dx$$

Figure 3: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee $I_1 = I_2$.

Then

$$u(x,t) = \begin{cases} u_L, & x < st, \\ u_R, & x > st, \end{cases}$$

is the weak solution to Burgers' equation.

Proof. RHS term

$$RHS = -\int_{\mathscr{D}} \varphi(x,0) u_0(x)\, dx$$

$$= -u_L \int_{-\infty}^0 \varphi(x,0)\, dx - u_R \int_0^{+\infty} \varphi(x,0)\, dx$$

LHS term

$$LHS = \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_t u \, dxdt + \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_t u \, dxdt$$

$$+ \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_x f(u) \, dxdt + \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_x f(u) \, dxdt$$

$$= u_L \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_t \, dxdt + u_R \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_t \, dxdt$$

$$+ \frac{u_L^2}{2} \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_x \, dxdt + \frac{u_R^2}{2} \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_x \, dxdt$$

$$= \left(u_L \int_{-\infty}^{0} \int_0^{+\infty} \varphi_t \, dtdx + u_L \int_0^{+\infty} \int_{x/s}^{+\infty} \varphi_t \, dtdx \right)$$

$$+ u_R \int_0^{+\infty} \int_0^{x/s} \varphi_t \, dtdx$$

$$+ \frac{u_L^2}{2} \int_0^{\mathscr{T}} \int_{-\infty}^{st} \varphi_x \, dxdt + \frac{u_R^2}{2} \int_0^{\mathscr{T}} \int_{st}^{+\infty} \varphi_x \, dxdt$$

$$= u_L \int_{-\infty}^{0} \varphi|_{t=0}^{t=+\infty} \, dx + u_L \int_0^{+\infty} \varphi|_{t=x/s}^{t=+\infty} \, dx + u_R \int_0^{+\infty} \varphi|_{t=0}^{t=x/s} \, dx$$

$$+ \frac{u_L^2}{2} \int_0^{+\infty} \varphi|_{x=-\infty}^{x=st} \, dt + \frac{u_R^2}{2} \int_0^{+\infty} \varphi|_{x=st}^{x=+\infty} \, dt$$

$$= u_L \int_{-\infty}^{0} (-\varphi(x,0)) \, dx + u_L \int_0^{+\infty} (-\varphi(x,x/s)) \, dx$$

$$+ u_R \int_0^{+\infty} (\varphi(x,x/s) - \varphi(x,0)) \, dx$$

$$+ \frac{u_L^2}{2} \int_0^{+\infty} \varphi(st,t) dt + \frac{u_R^2}{2} \int_0^{+\infty} (-\varphi(st,t)) \, dt$$

$$= -u_L \int_{-\infty}^{0} \varphi(x,0) dx - u_R \int_0^{+\infty} \varphi(x,0) dx$$

$$- (u_L - u_R) \int_0^{+\infty} \varphi(x,x/s) dx + (u_L - u_R) \int_0^{+\infty} \varphi(w,w/s) dw$$

$$= -u_L \int_{-\infty}^{0} \varphi(x,0) dx - u_R \int_0^{+\infty} \varphi(x,0) dx = RHS$$

Note in passing that we have used the following relation

$$\frac{u_L^2}{2} \int_0^{+\infty} \varphi(st,t) dt + \frac{u_R^2}{2} \int_0^{+\infty} (-\varphi(st,t)) \, dt = \frac{u_L^2 - u_R^2}{2} \int_0^{+\infty} \varphi(st,t) \, dt$$

$$= \frac{u_L^2 - u_R^2}{2s} \int_0^{+\infty} \varphi(w,w/s) dw$$

$$= (u_L - u_R) \int_0^{+\infty} \varphi(w,w/s) dw$$

where $w := st$ and $dw = sdt$, and the RH condition for *Burgers'* equation $s = 1/2(u_L + u_R)$. $\qquad\square$

14   Consistency order

Example 5. *Leapfrog scheme for the linear advection equation reads*

$$u_j^{n+1} = u_j^{n-1} + \frac{a\Delta t}{\Delta x}\left(u_{j-1}^n - u_{j+1}^n\right).$$

Show that the local consistency error $L_{\Delta t}$

$$\|L_{\Delta t}(\cdot, t)\| \leq C(\Delta t)^2 \qquad \text{for } \Delta t \to 0,$$

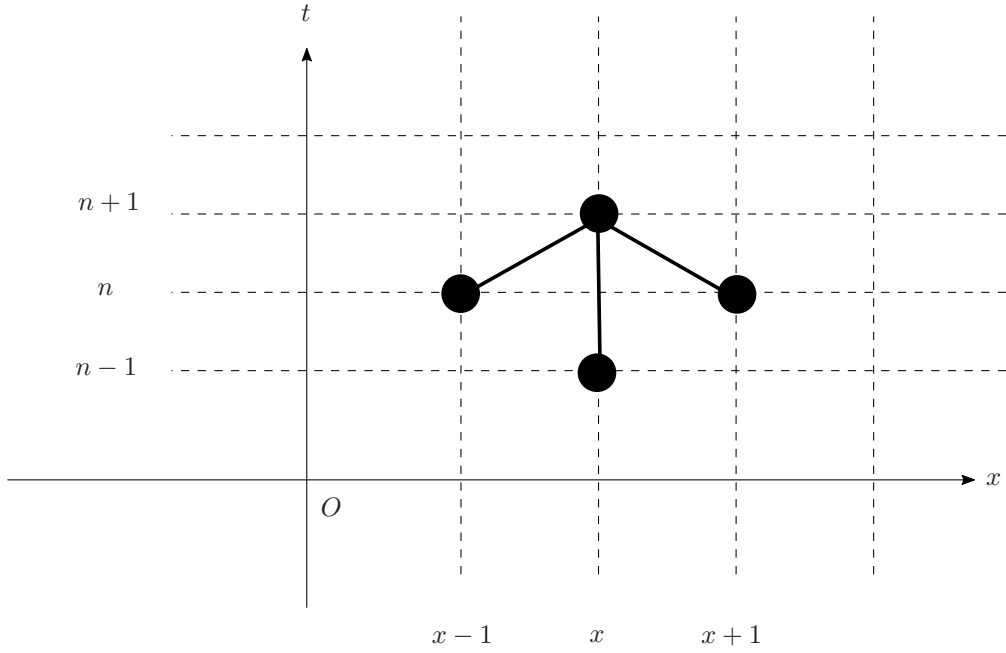*hold for a linear solution and $\Delta t/\Delta x = const.$*

Figure 4: Leapfrog.

Consider the local truncation error

$$L_{\Delta t}(x, t) = \frac{1}{\Delta t}\left[u(x, t + \Delta t) - H_{\Delta t}(u(\cdot, t), x)\right],$$

which reads for the considered problem

$$L_{\Delta t}(x, t) = \frac{1}{\Delta t}\left[u(x, t + \Delta t) - \left(u(x, t - \Delta t) + \frac{a\Delta t}{\Delta x}\left(u(x - \Delta x, t) - u(x + \Delta x, t)\right)\right)\right].$$

Applying Taylor's expansion for $u(x, t + \Delta t)$

$$u(x, t + \Delta t) = u(x, t) + u_t(x, t)\Delta t + u_{tt}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3)$$

$$= u(x, t) - au_x(x, t)\Delta t + a^2 u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3).$$

Applying Taylor's expansion for $u(x, t - \Delta t)$

$$u(x, t - \Delta t) = u(x, t) - u_t(x, t)\Delta t + u_{tt}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3)$$

$$= u(x, t) + au_x(x, t)\Delta t + a^2 u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3).$$

Furthermore, applying Taylor's expansion for $u(x + \Delta x, t)$ and $u(x - \Delta x, t)$

$$u(x + \Delta x, t) = u(x, t) + u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3)$$

$$u(x - \Delta x, t) = u(x, t) - u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3).$$

Local truncation error

$$\begin{aligned}
L_{\Delta t}(x, t) = \frac{1}{\Delta t}\Big[& u(x, t) - au_x(x, t)\Delta t + a^2 u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\
& - \Big(u(x, t) + au_x(x, t)\Delta t + a^2 u_{xx}(x, t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\
& + \frac{a\Delta t}{\Delta x}\Big(\Big(u(x, t) - u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3)\Big) \\
& - \Big(u(x, t) + u_x(x, t)\Delta x + u_{xx}(x, t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3)\Big)\Big)\Big)\Big] \\
= \frac{1}{\Delta t}\Big[& - 2au_x(x, t)\Delta t + \mathcal{O}((\Delta t)^3) - \frac{a\Delta t}{\Delta x}\Big(-2u_x(x, t)\Delta x + \mathcal{O}((\Delta x)^3)\Big)\Big] \\
= \frac{1}{\Delta t}& \mathcal{O}((\Delta t)^3) \\
= \mathcal{O}& ((\Delta t)^2).
\end{aligned}$$

Therefore, order of consistency for Leapfrog scheme is of order 2

$$\|L_{\Delta t}(\cdot, t)\| \leq C(\Delta t)^2 \qquad \text{for } \Delta t \to 0.$$

Note in passing that $u_t = -au_x$ and $u_{tt} = a^2 u_{xx}$ are from the linear advection equation.