Global Exercise - 14

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1 A small remark for consistent numerical flux function used in Upwind scheme - FVM

Example 1. *Determine consistent numerical flux function for Upwind scheme.*

Recall the conservation form used for FVM reads

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{f}_{j-1/2}^n - \widetilde{f}_{j+1/2}^n\right). \tag{1}$$

Meanwhile, the Upwind scheme with point-to-the-left stencils reads

$$\boxed{u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(f(u_{j-1}^n) - f(u_j^n)\right).} \tag{2}$$

Next, we would like to write (2) in terms of (1). Since the two formulations are already identical, the numerical flux function $\widetilde{f}_{j-1/2}^n$ and $\widetilde{f}_{j+1/2}^n$ are recognized directly as follow

$$\widetilde{f}_{j-1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_{j-1}^n, u_j^n\right) = f(u_{j-1}^n), \tag{3}$$
$$\widetilde{f}_{j+1/2}^n = \widetilde{f}_{j+1/2}^n\left(u_j^n, u_{j+1}^n\right) = f(u_j^n). \tag{4}$$

Then, the consistency of numerical flux functions are checked as follows

$$\widetilde{f}_{j-1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{5}$$
$$\widetilde{f}_{j+1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{6}$$

which is automatically satisfied $\forall \beta \in \mathbb{R}$ in case of constant flow. Likewise, in case of the Upwind scheme with point-to-the-right stencils, we obtain the same structure, as follows

$$\boxed{u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(f(u_j^n) - f(u_{j+1}^n)\right),} \tag{7}$$

which is the Upwind scheme with point-to-the-right stencils. Next, since the formulation (7) is already identical with (1), the numerical flux function $\widetilde{f}_{j-1/2}^n$ and $\widetilde{f}_{j+1/2}^n$ are recognized directly as follow

$$\widetilde{f}_{j-1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_{j-1}^n, u_j^n\right) = f(u_j^n), \tag{8}$$
$$\widetilde{f}_{j+1/2}^n = \widetilde{f}_{j+1/2}^n\left(u_j^n, u_{j+1}^n\right) = f(u_{j+1}^n). \tag{9}$$

Checking the consistency of (8) and (9) is similar with (3) and (4), as follows

$$\widetilde{f}_{j-1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{10}$$
$$\widetilde{f}_{j+1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{11}$$

which is automatically satisfied $\forall \beta \in \mathbb{R}$ in case of constant flow.

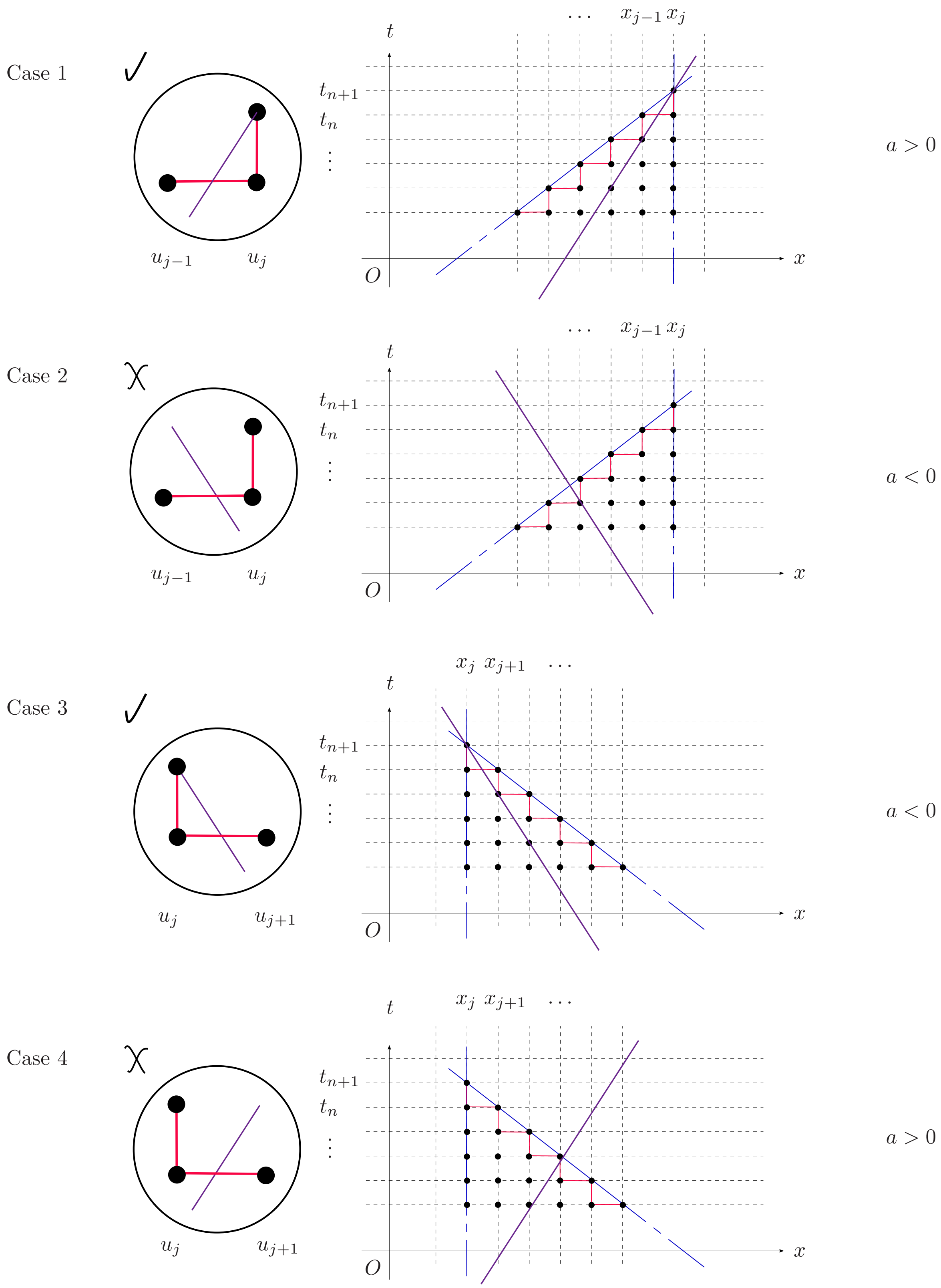

Figure 1: Correlation between coefficient a, point-to-the-left stencil, and point-to-the-right stencil of upwind scheme.

Example 2. *Comparison of numerical solutions between left-pointing and right-pointing stencils of Upwind scheme in case of $a > 0$.*

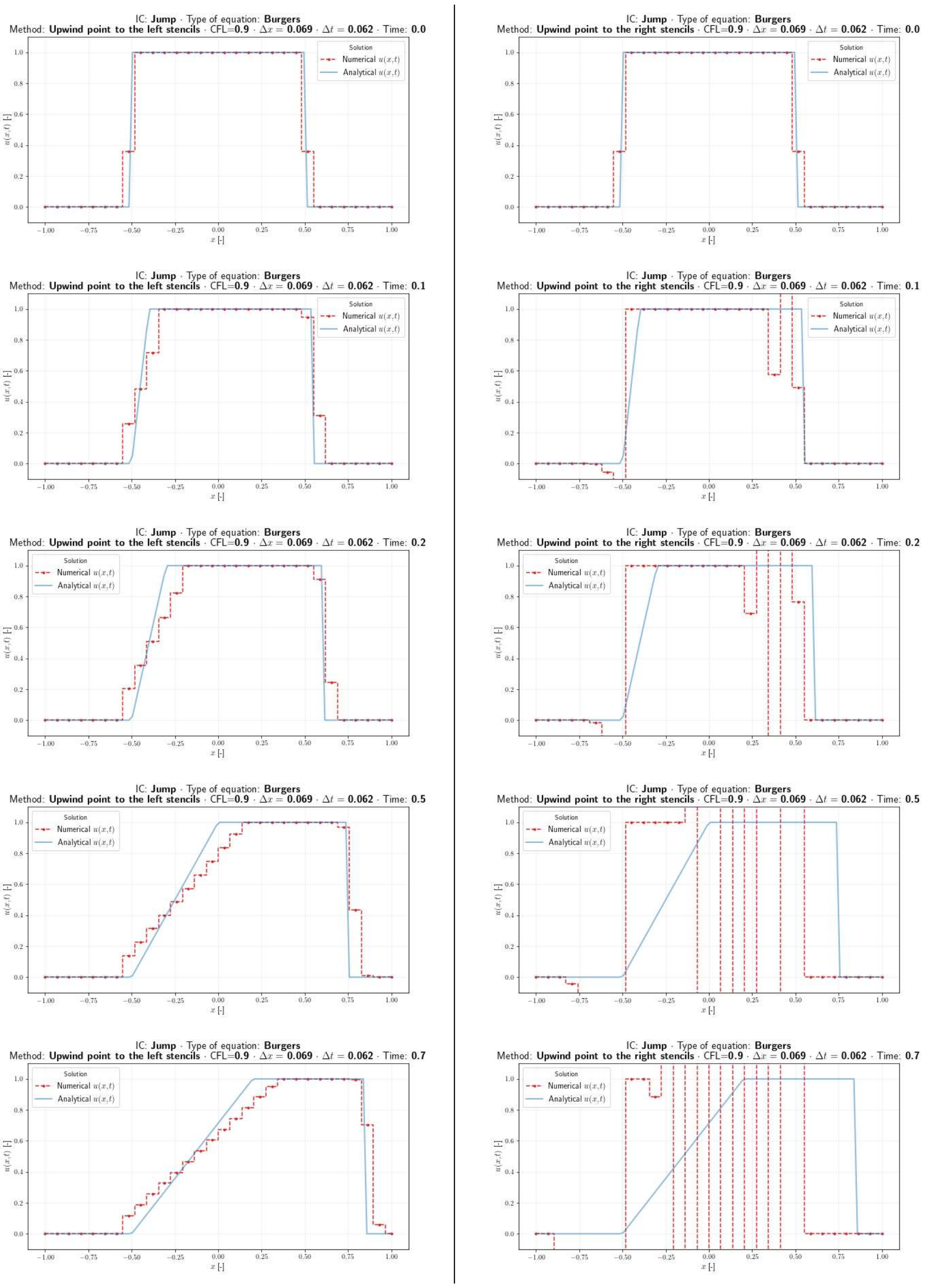

Figure 2: *Upwind* scheme: In case of $a > 0$ the scheme whose stencils are pointing to the left (Case 1 in Figure 1) is the suitable choice, compared to the case whose stencils are pointing to the right (Case 4 in Figure 1).

2 A small remark for consistent numerical flux function used in Lax-Friedrichs scheme - FVM

Example 3. *Determine consistent numerical flux function for Lax-Friedrichs scheme.*

Recall the conservation form used for FVM reads

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{f}^n_{j-1/2} - \widetilde{f}^n_{j+1/2}\right). \tag{12}$$

Meanwhile, the Lax-Friedrichs scheme reads

$$u_j^{n+1} = \frac{1}{2}\left(u^n_{j-1} + u^n_{j+1}\right) - \frac{\Delta t}{2\Delta x}\left(f(u^n_{j+1}) - f(u^n_{j-1})\right). \tag{13}$$

Next, we would like to write (13) in terms of (12). The derivation is done by some algebraic manipulations, as follows

$$\begin{aligned}
u_j^{n+1} &= \frac{1}{2}\left(u^n_{j-1} + u^n_{j+1}\right) - \frac{\Delta t}{2\Delta x}\left(f(u^n_{j+1}) - f(u^n_{j-1})\right) \\
&= u^n_j + \frac{1}{2}u^n_{j-1} - \frac{1}{2}u^n_j + \frac{1}{2}u^n_{j+1} - \frac{1}{2}u^n_j - \frac{\Delta t}{2\Delta x}f(u^n_{j+1}) + \frac{\Delta t}{2\Delta x}f(u^n_{j-1}) \\
&= u^n_j + \frac{\Delta t}{\Delta x}\left(\frac{\Delta x}{2\Delta t}u^n_{j-1} - \frac{\Delta x}{2\Delta t}u^n_j + \frac{\Delta x}{2\Delta t}u^n_{j+1} - \frac{\Delta x}{2\Delta t}u^n_j \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. - \frac{1}{2}f(u^n_{j+1}) + \frac{1}{2}f(u^n_{j-1})\right) \\
&= u^n_j + \frac{\Delta t}{\Delta x}\left(\underbrace{\left(\frac{\Delta x}{2\Delta t}u^n_{j-1} - \frac{\Delta x}{2\Delta t}u^n_j + \frac{1}{2}f(u^n_{j-1})\right)}_{=:\widetilde{f}^n_{j-1/2}}\right. \\
&\qquad\qquad\qquad \left. - \underbrace{\left(\frac{\Delta x}{2\Delta t}u^n_j - \frac{\Delta x}{2\Delta t}u^n_{j+1} + \frac{1}{2}f(u^n_{j+1})\right)}_{=:\widetilde{f}^n_{j+1/2}}\right) \\
&= u^n_j + \frac{\Delta t}{\Delta x}\left(\widetilde{f}^n_{j-1/2} - \widetilde{f}^n_{j+1/2}\right),
\end{aligned} \tag{14}$$

which confirms that the *Lax-Friedrichs* scheme given at (13) is able to be written in the *conservation form* with the numerical flux function recognized as follows

$$\widetilde{f}^n_{j-1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_{j-1}, u^n_j\right) = \frac{\Delta x}{2\Delta t}\left(u^n_{j-1} - u^n_j\right) + \frac{1}{2}f(u^n_{j-1}) \tag{15}$$

$$\widetilde{f}^n_{j+1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_j, u^n_{j+1}\right) = \frac{\Delta x}{2\Delta t}\left(u^n_j - u^n_{j+1}\right) + \frac{1}{2}f(u^n_{j+1}) \tag{16}$$

However, these numerical flux functions $\widetilde{f}(\cdot,\cdot)$ are not consistent with the original flux function $f(\cdot)$, which can be checked for the case of constant flow, as follows

$$(15) \Leftrightarrow \widetilde{f}^n_{j-1/2}(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta - \beta) + \frac{1}{2}f(\beta) = \frac{1}{2}f(\beta) \quad ↯ \tag{17}$$

$$(16) \Leftrightarrow \widetilde{f}^n_{j+1/2}(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta - \beta) + \frac{1}{2}f(\beta) = \frac{1}{2}f(\beta) \quad ↯ \tag{18}$$

which have confirmed that $\widetilde{f}^n_{j\pm1/2}(\beta,\beta) \neq f(\beta)$, $\forall \beta \in \mathbb{R}$. Therefore, a modified version is required for these numerical flux functions, such that they become consistent with $f(\cdot)$, and simultaneously, the summation of these two fluxes do not lead to any change in the *Lax-Friedrichs* scheme. The consistent numerical flux functions for *Lax-Friedrichs* scheme reads

$$\widetilde{f}^n_{j-1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_{j-1}, u^n_j\right) = \frac{\Delta x}{2\Delta t}\left(u^n_{j-1} - u^n_j\right) + \frac{1}{2}\left(f(u^n_{j-1}) + f(u^n_j)\right), \tag{19}$$

$$\widetilde{f}^n_{j+1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_j, u^n_{j+1}\right) = \frac{\Delta x}{2\Delta t}\left(u^n_j - u^n_{j+1}\right) + \frac{1}{2}\left(f(u^n_{j+1}) + f(u^n_j)\right), \tag{20}$$

which are obtained by adding the term $1/2 f(u^n_j)$ to both $\widetilde{f}^n_{j-1/2}$ and $\widetilde{f}^n_{j+1/2}$. Note in passing that the subtraction sign between these two fluxes will cancel out this extra term, as shown in (12). Next, the consistent property is checked as follows

$$(19) \Leftrightarrow \widetilde{f}^n_{j-1/2}(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta - \beta) + \frac{1}{2}(f(\beta) + f(\beta)) = f(\beta) \quad \checkmark \tag{21}$$

$$(20) \Leftrightarrow \widetilde{f}^n_{j+1/2}(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta - \beta) + \frac{1}{2}(f(\beta) + f(\beta)) = f(\beta) \quad \checkmark \tag{22}$$

which have confirmed that $\widetilde{f}^n_{j\pm1/2}(\beta,\beta) = f(\beta)$, $\forall \beta \in \mathbb{R}$ for constant flow. Hence, the numerical flux functions $\widetilde{f}^n_{j\pm1/2}$ for the *Lax-Friedrichs* scheme take the following formulation

$$\therefore \quad \boxed{\begin{aligned} \widetilde{f}^n_{j-1/2}\left(u^n_{j-1}, u^n_j\right) &= \frac{\Delta x}{2\Delta t}\left(u^n_{j-1} - u^n_j\right) + \frac{1}{2}\left(f(u^n_{j-1}) + f(u^n_j)\right) \\ \widetilde{f}^n_{j+1/2}\left(u^n_j, u^n_{j+1}\right) &= \frac{\Delta x}{2\Delta t}\left(u^n_j - u^n_{j+1}\right) + \frac{1}{2}\left(f(u^n_j) + f(u^n_{j+1})\right) \end{aligned}} \tag{23}$$

or the generalized form used for FVM implementation reads

$$\therefore \quad \boxed{F\left(u_L, u_R\right) = \frac{1}{2}\left(f(u_L) + f(u_R)\right) + \frac{\Delta x}{2\Delta t}\left(u_L - u_R\right)} \tag{24}$$

Example 4. *Crucialness of conservation form and consistent numerical flux fcn.*

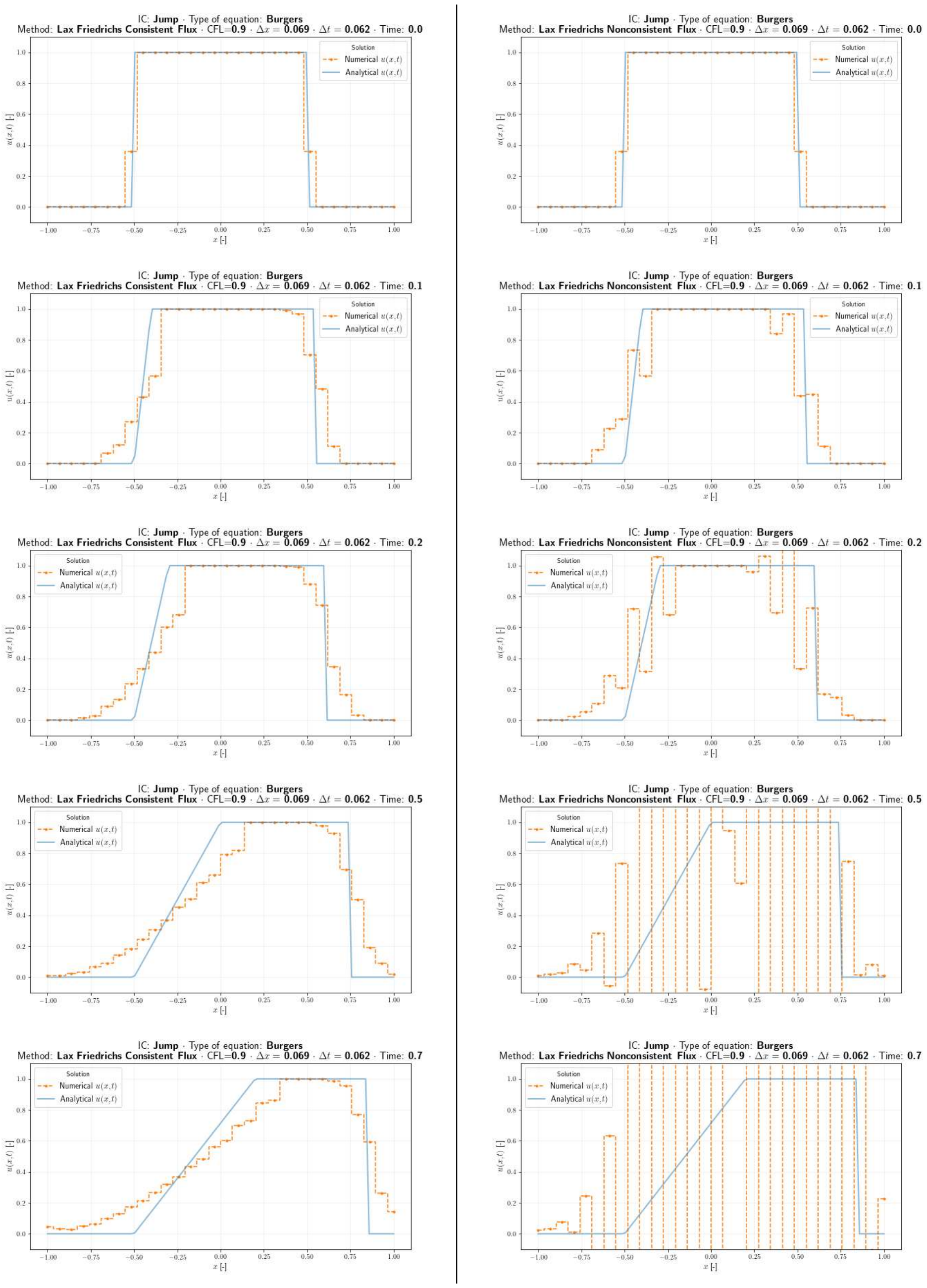

Figure 3: *Lax-Friedrichs* scheme: Consistent numerical flux function (left; based on the derived formulation in (19) and (20)) versus Non-consistent numerical flux function (right; based on the derived formulation in (15) and (16)).

3 Approximate Riemann solvers (cont.)

Example 5. *Roe's solver derivation.*

Example 6. *Roe's solver (R) versus Local Lax-Friedrichs (LLF) flux function.*

Recall that the *Roe*'s solver takes the following forms of numerical flux functions

$$\begin{aligned}\widetilde{F}_{j+1/2}\left(U_L, U_R\right) &= F(U_L) + \widehat{A}\left(U_L, U_R\right)^{-}\left(U_R - U_L\right),\\ \widetilde{F}_{j+1/2}\left(U_L, U_R\right) &= F(U_R) - \widehat{A}\left(U_L, U_R\right)^{+}\left(U_R - U_L\right).\end{aligned}$$

By summing and taking the average we obtain another form of numerical flux function in *Roe*'s solver as follows

$$\therefore \quad \boxed{\widetilde{F}^{(R)}_{j+1/2}(U_L, U_R) = \frac{1}{2}\left(F(u_L) + F(u_R)\right) - \frac{1}{2}\left|\widehat{A}\right|\left(U_R - U_L\right)} \tag{25}$$

Note in passing that we have used the following equality in the expression (25)

$$\widehat{A}^{-}(U_L, U_R) - \widehat{A}^{+}(U_L, U_R) = -\left|\widehat{A}\right|(U_L, U_R). \tag{26}$$

Besides, the local Lax-Friedrichs flux function take the following form

$$\therefore \quad \boxed{\widetilde{F}^{(LLF)}_{j+1/2}(U_L, U_R) = \frac{1}{2}\left(F(u_L) + F(u_R)\right) - \frac{1}{2}\lambda^{\max}\left(U_R - U_L\right)} \tag{27}$$

Note in passing that (27)

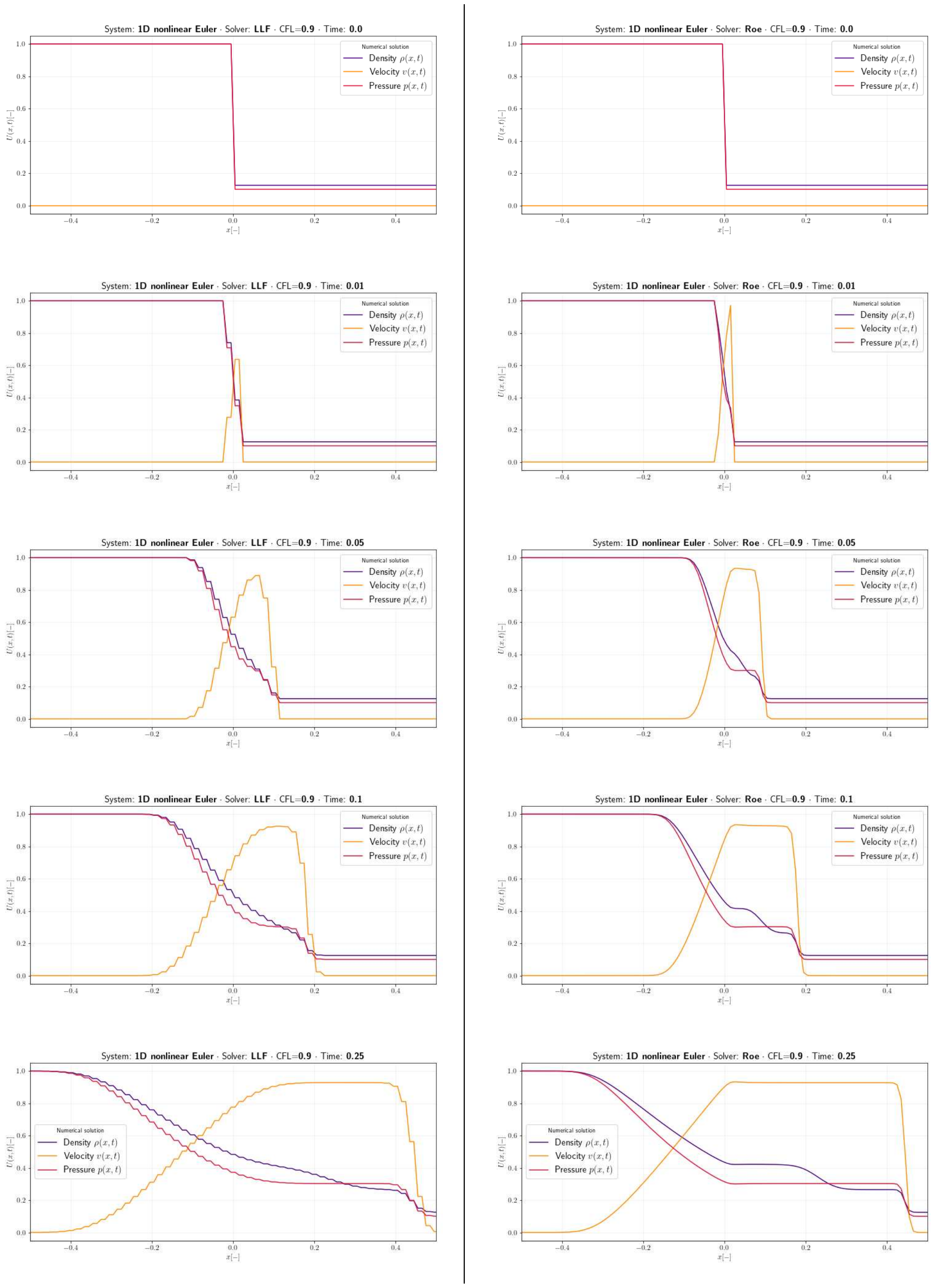

Figure 4: Comparison between LLF solver and *Roe*'s solver: Same initial conditions and parameters applied, seeking for numerical solution to 1D *Euler* equation.

Example 7. *Harten-Lax-van Leer.*

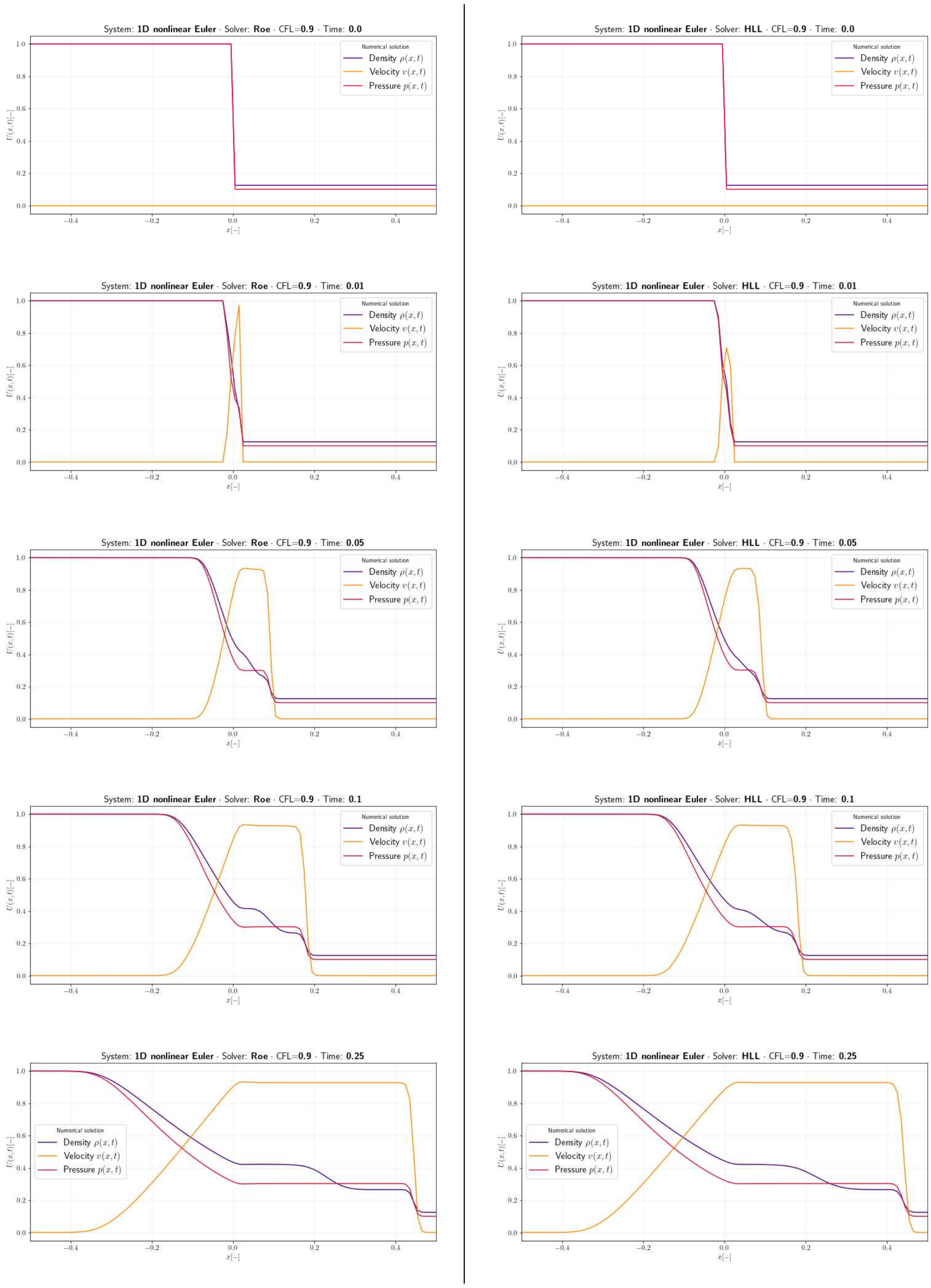

Figure 5: Comparison between *Roe*'s solver and HLL solver: Same initial conditions and parameters applied, seeking for numerical solution to 1D *Euler* equation.