

Global Exercise - 15

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1 A small note about monotonicity: *Upwind* scheme

Example 1. Examine monotonicity of $\mathcal{H}_{\Delta t}$ for conservation law $u_t + au_x = 0$.

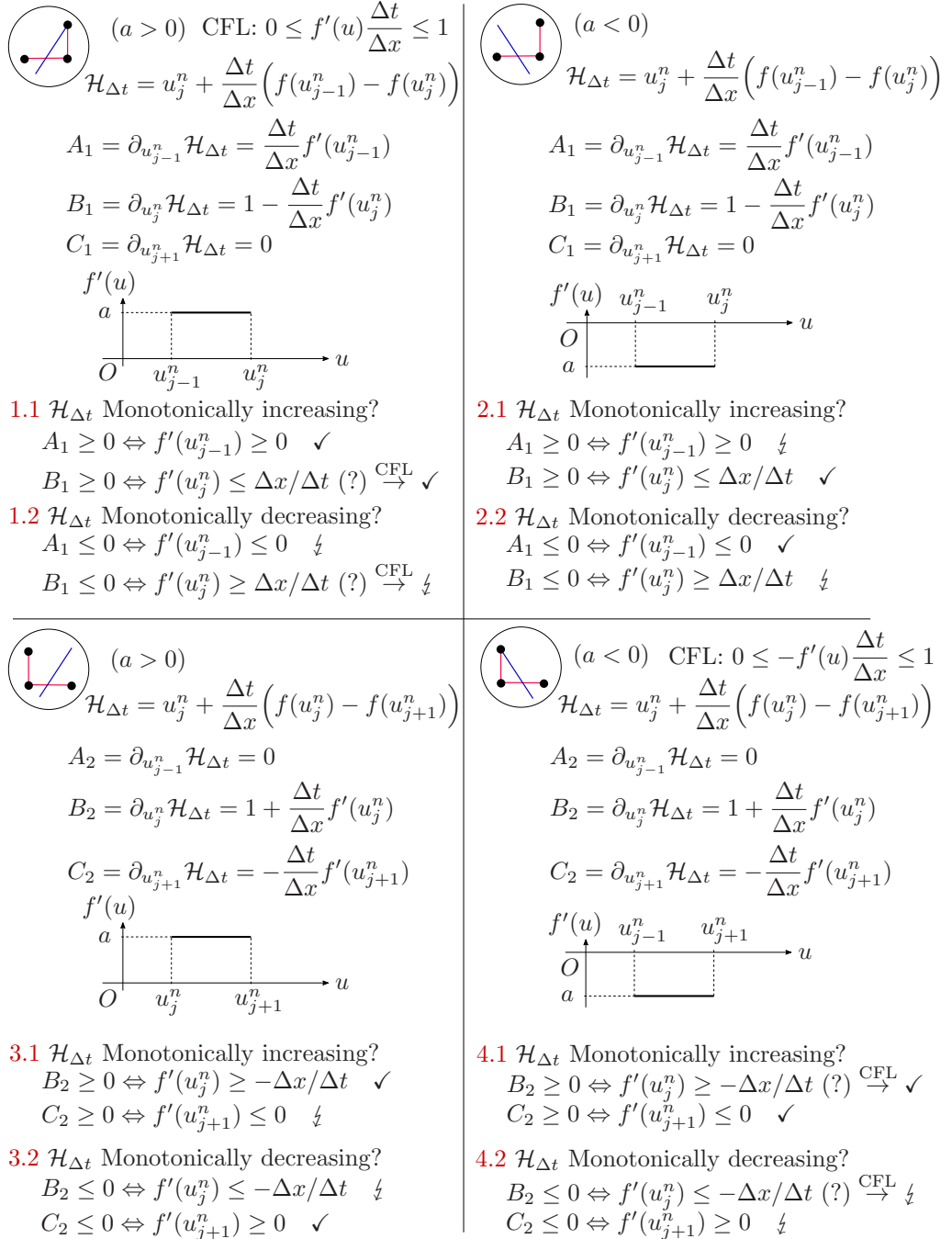


Figure 1: Monotonicity of *Upwind* scheme: 8 different cases.

2 Structure of FVM

Example 2. *Monotone $\rightarrow L_1$ -Contracting $\rightarrow TVD \rightarrow Monotonicity-Preserving$.*

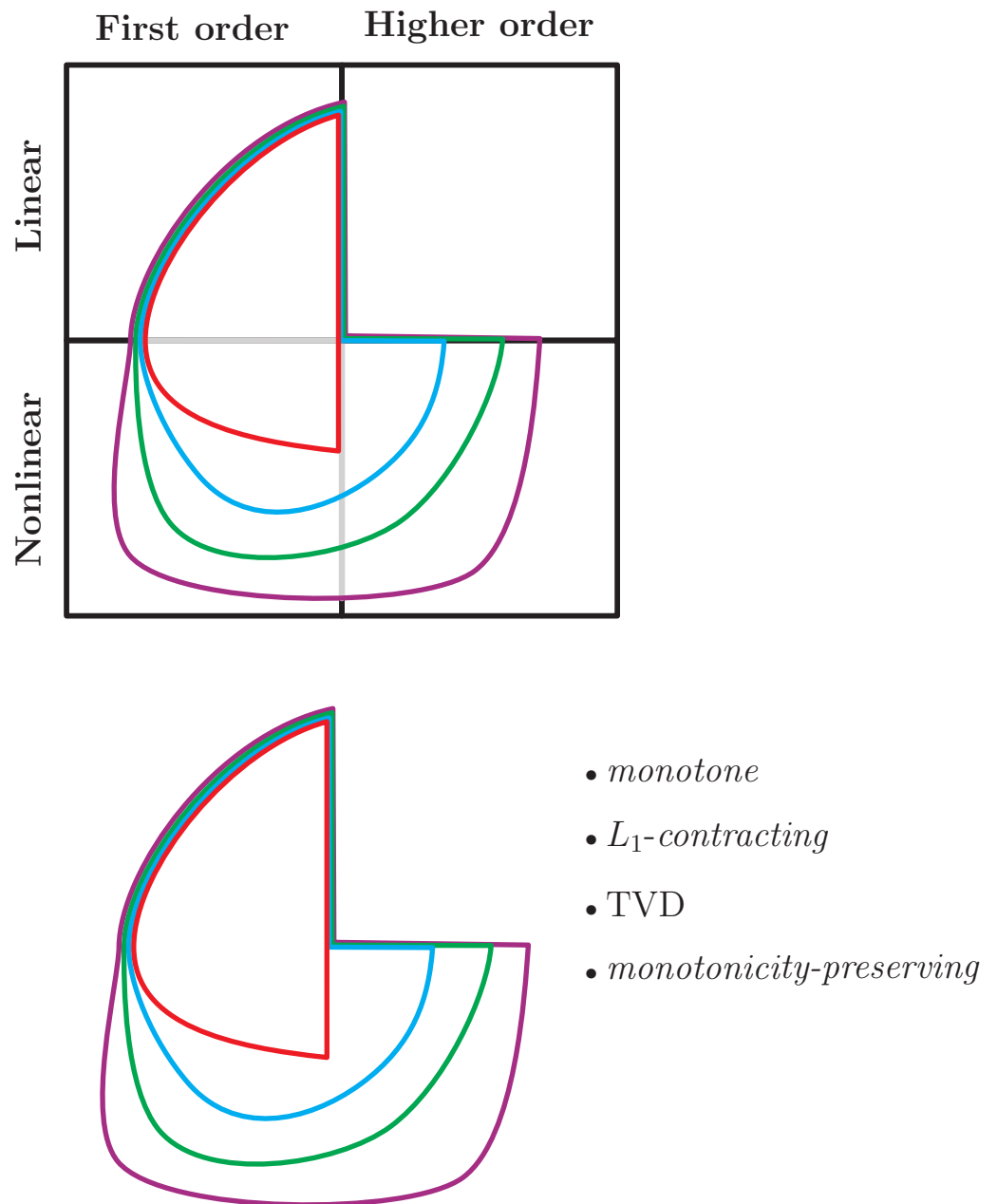


Figure 2: Structure of Finite Volume Method [Lecture note page 115].

3 Limiter

Example 3. Examine the 1st-order-accurate *LF* and the 2nd-order-accurate *LW*.

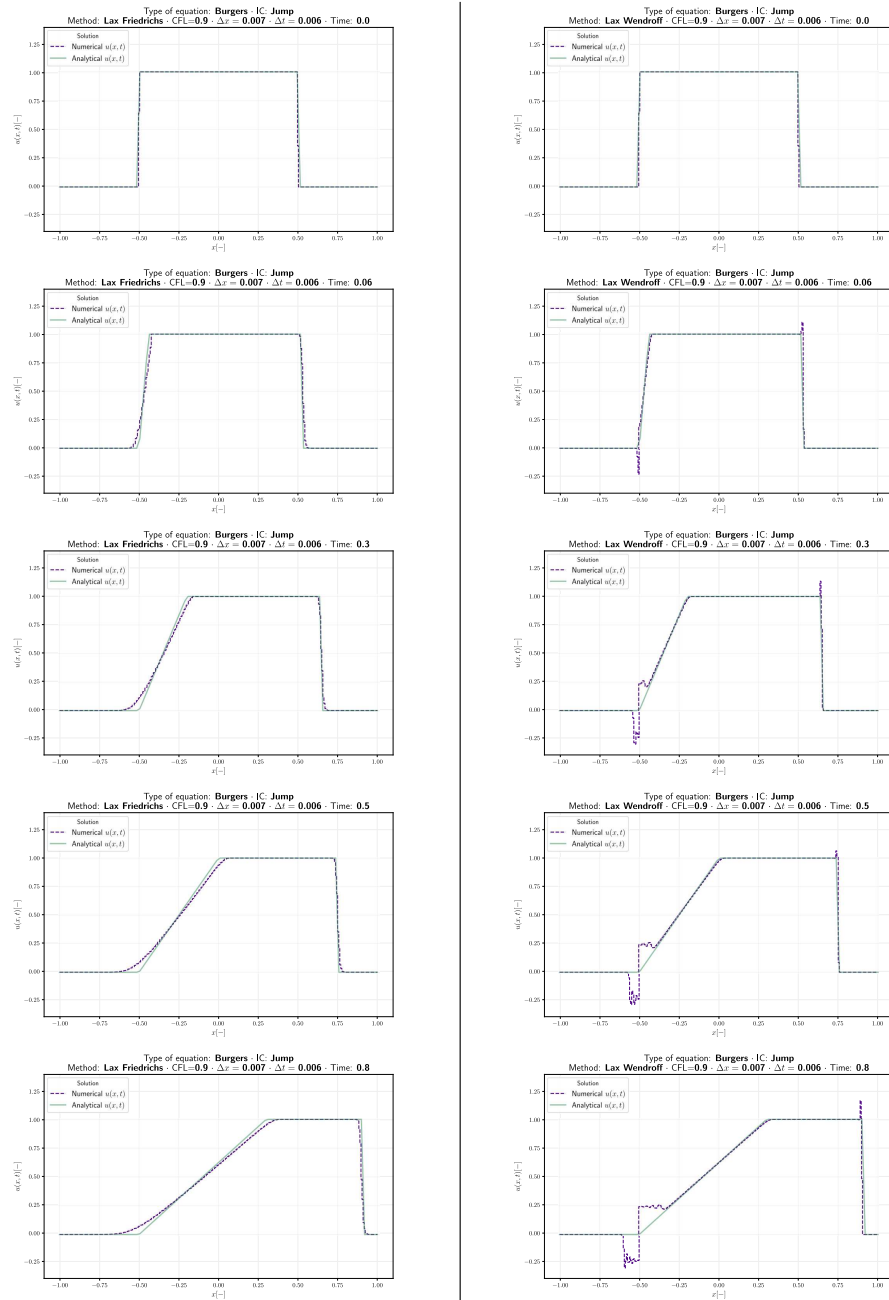


Figure 3: Oscillatory phenomena around discontinuity: (left) none oscillation founded in *Lax-Friedrichs*; (right) oscillation observed in *Lax-Wendroff*.

Example 4. Assume the linear advection problem given by

$$u_t + au_x = 0 \quad \text{with} \quad a > 0,$$

and the numerical scheme as follows

$$u_j^{n+1} = u_j^n + \nu (u_{j-1}^n - u_j^n) + \frac{\nu(1-\nu)}{2} \Delta x (\sigma_{j-1}^n - \sigma_j^n),$$

with limited slopes

$$\sigma_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} \phi(\theta_j).$$

Prove that numerical scheme is TVD with limiters $\phi(\theta)$ given as follows

1. $\phi(\theta) = \max(0, \min(\theta, 2))$,
2. $\phi(\theta) = \max(0, \min(2\theta, 2))$,
3. $\phi(\theta) = \frac{\theta + |\theta|}{\alpha + |\theta|}$ for any fixed $\alpha \geq 1$
4. $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$.

Denote which of them are second order schemes.

According to the theorem on TVD region of *Upwind* numerical scheme with limiter presented in the lecture, we know that limiter gives us TVD scheme if the following relations

$$\begin{aligned} 0 &\leq \phi(\theta) \leq 2\theta, \\ |\phi(\theta)| &\leq 2, \end{aligned}$$

holds. Besides, the scheme is of 2nd-order if and only if $\phi(1) = 1$ holds. Therefore, we can simply check lower and upper bounds of limiters, as follows

1. $\phi(\theta) = \max(0, \min(\theta, 2))$ satisfies bounds and goes through $\phi(1) = 1$.
Hence, it corresponds to 2nd-order TVD method.
2. $\phi(\theta) = \max(0, \min(2\theta, 2))$ satisfies bounds but $\phi(1) = 2$.
Hence, it is of 1st-order TVD scheme.
3. $\phi(\theta) = \frac{\theta + |\theta|}{\alpha + |\theta|}$ satisfies bounds for any fixed $\alpha \geq 1$, but $\phi(1) = 1$ holds only for $\alpha = 1$.
Hence, it is of 2nd-order only if $\alpha = 1$ and of 1st-order otherwise.
4. $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$ satisfies bounds and $\phi(1) = 1$ holds.
Hence, it is TVD scheme of 2nd-order.

Example 5. *Examine and sketch the following Limiter function.*

$$\phi(\theta) = \min \left(\delta\theta, \frac{1+\theta}{2}, \gamma \right), \text{ for } \delta, \gamma \geq 1.$$

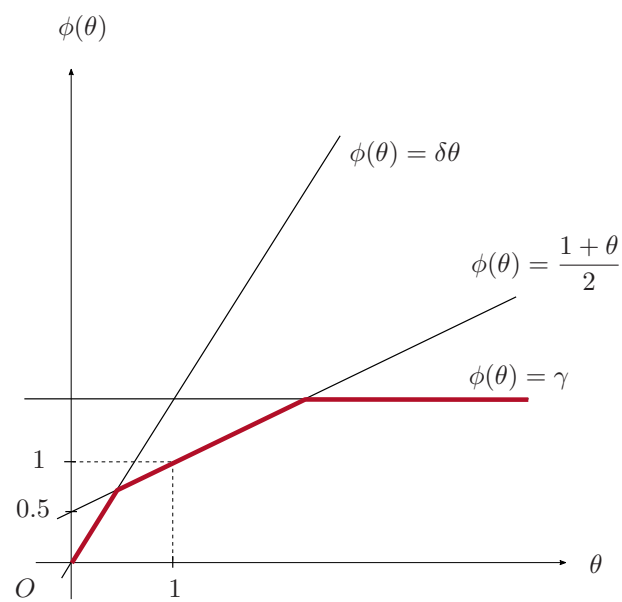
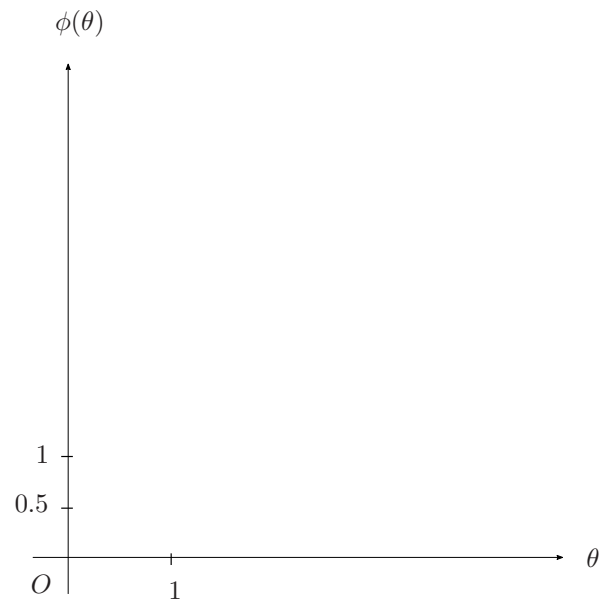


Figure 4: Limiter function $\phi(\theta) = \min \left(\delta\theta, \frac{1+\theta}{2}, \gamma \right)$, for $\delta, \gamma \geq 1$.

4 A bit insight into Godunov's solver

Example 6. *Verify the Godunov numerical flux function*

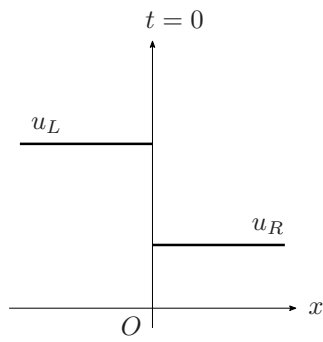
$$g(u_L, u_R) = \begin{cases} \min_{u_L \leq u \leq u_R} f(u) & \text{if } u_L \leq u_R, \\ \max_{u_R \leq u \leq u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

by considering all possible cases.

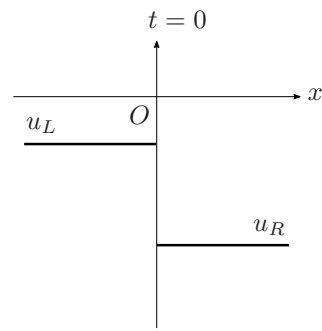
All possible cases read

1. $f'(u_L), f'(u_R) \geq 0 \Rightarrow u^* = u_L$
2. $f'(u_L), f'(u_R) \leq 0 \Rightarrow u^* = u_R$
3. $f'(u_L) \geq 0 \geq f'(u_R) \Rightarrow u^* = u_L$ if $[f]/[u] > 0$ or $u^* = u_R$ if $[f]/[u] < 0$
4. $f'(u_L) < 0 < f'(u_R) \Rightarrow u^* = u_s$ where $f'(u_s) = 0$

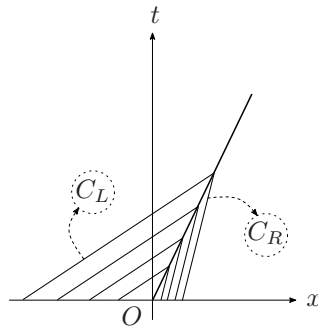
Example 7. Examine $u_L > u_R$.



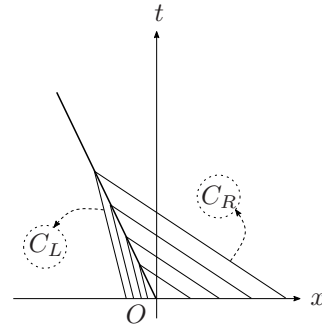
Case 1 : $u_L > u_R > 0$



Case 3 : $0 > u_L > u_R$



$$(C_L) : t = \frac{1}{u_L}x - \frac{x_0}{u_L}$$



$$(C_R) : t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

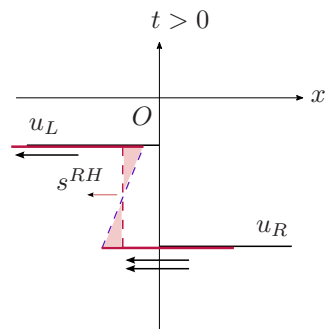
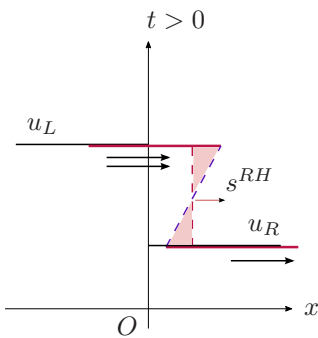
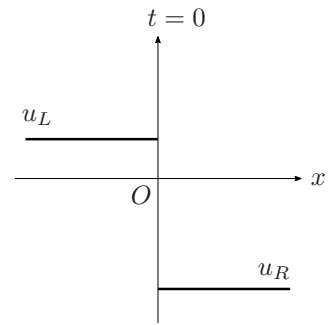
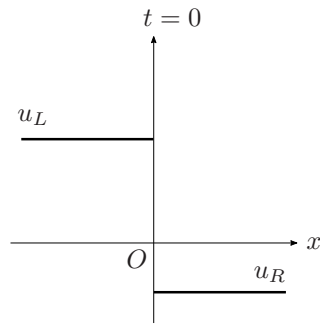


Figure 5: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

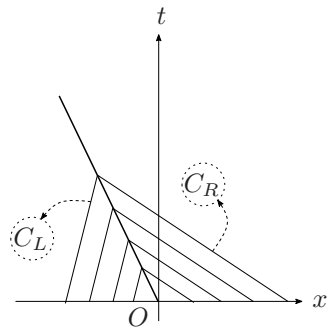
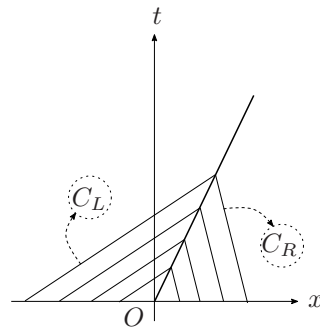
Schock solution

Example 8. Examine $u_L > u_R$.



Case 2.1 : $(u_L > 0 > u_R) \wedge |u_L| > |u_R|$

Case 2.2 : $(u_L > 0 > u_R) \wedge |u_L| < |u_R|$



$$(C_L) : t = \frac{1}{u_L}x - \frac{x_0}{u_L}$$

$$(C_R) : t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

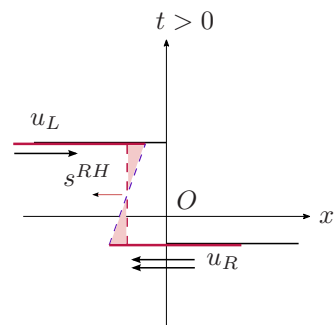
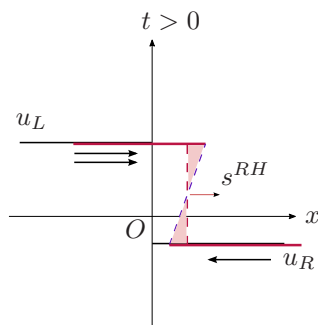
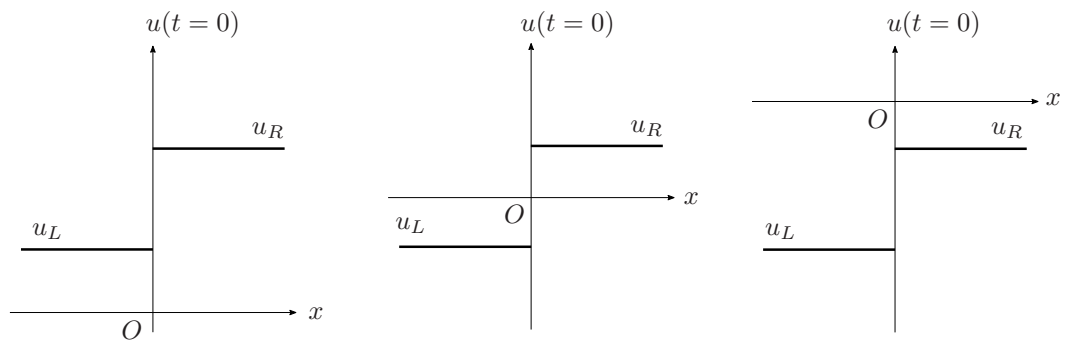


Figure 6: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

Schock solution

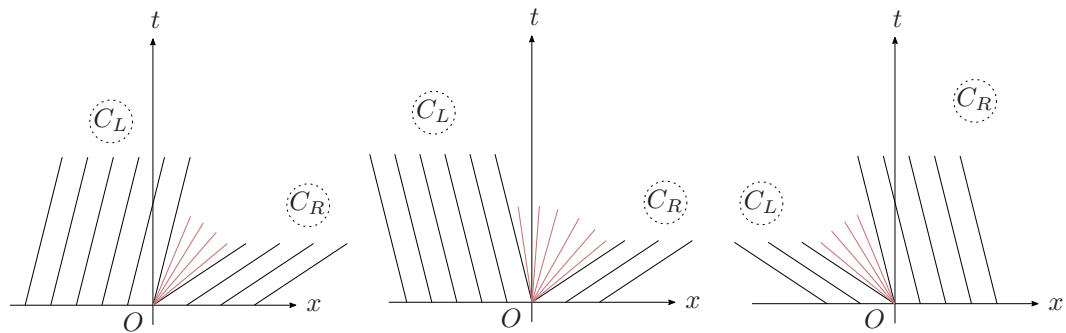
Example 9. Examine $u_L < u_R$.



Case 1 : $0 < u_L < u_R$

Case 2 : $u_L < 0 < u_R$

Case 3 : $u_L < u_R < 0$



$$(C_L) : t = \frac{1}{u_L}x - \frac{x_0}{u_L}$$

$$(C_R) : t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

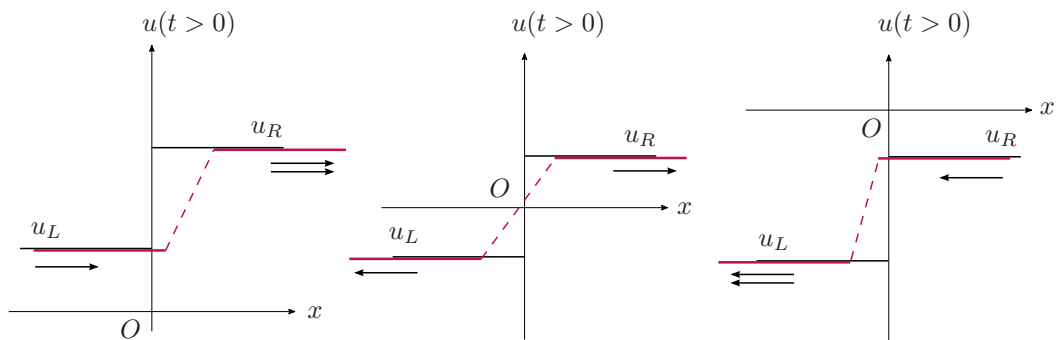


Figure 7: Riemann problem with $u_L < u_R$: IC, Characteristics, Solution.

Rarefaction solution

Example 10. Consider the Godunov's solver for the conservation law $u_t + f(u)_x = 0$ with convex and nonlinear flux function f , e.g. the flux function in Burgers' equation, where

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} (g(u_{j-1}^n, u_j^n) - g(u_j^n, u_{j+1}^n)),$$

and the numerical flux function

$$g(u_L, u_R) = \begin{cases} \min_{u_L \leq u \leq u_R} f(u) & \text{if } u_L \leq u_R, \\ \max_{u_R \leq u \leq u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

1. Show that the numerical flux function is monotone.
2. Rewrite the scheme in the incremental form.
3. Show that the scheme has the total-variation-diminishing (TVD) property.

Approach:

1. *Proof.* In order to show the monotonicity of the numerical flux function, it is sufficient to show the following relations

$$\boxed{\begin{cases} \partial_u g(u, v) \geq 0, \\ \partial_v g(u, v) \leq 0. \end{cases}} \quad (1)$$

Hence, in case of Burgers' equation, where $f'(u) = u$, we proceed as follows

$$g(u_L, u_R) = \begin{cases} f(u_L), & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ f(u_R), & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ f(u_s), & \text{if } f'(u_L) < 0 < f'(u_R), \end{cases}$$

where u_s denotes the sonic point where $f'(u_s) = 0$. Then, by taking the partial derivative of $g(u_L, u_R)$ w.r.t. u_L and u_R we arrive at the following expressions

$$\partial_{u_L} g(u_L, u_R) = \begin{cases} f'(u_L), & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_L} g(u_L, u_R) \geq 0.} \quad (2)$$

Likewise, the partial derivative of $g(u_L, u_R)$ w.r.t. u_R goes as follows

$$\partial_{u_R} g(u_L, u_R) = \begin{cases} f'(u_R), & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_R} g(u_L, u_R) \leq 0.} \quad (3)$$

Therefore, by combining (2) and (3), and comparing them with the condition given at (1), we arrive at *q.e.d.* \square

2. The incremental form reads

$$\boxed{u_j^{n+1} = u_j^n + C_{j+1/2}^n (u_{j+1}^n - u_j^n) - D_{j-1/2}^n (u_j^n - u_{j-1}^n).}$$

For any conservative Finite Volume scheme we obtain

$$C_{j+1/2} = -\lambda \frac{g_{j+1/2} - f_j}{u_{j+1} - u_j}$$

$$D_{j-1/2} = \lambda \frac{f_j - g_{j-1/2}}{u_j - u_{j-1}}$$

where $\lambda = \Delta t / \Delta x$. Herein, for the case $C_{j+1/2}$, u_L is u_j and u_R is u_{j+1} , as follows

$$C_{j+1/2} = \begin{cases} 0, & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ -\lambda \frac{f(u_s) - f(u_L)}{u_R - u_L}, & \text{if } f'(u_L) < 0 < f'(u_R). \end{cases}$$

Likewise, it goes for the case $D_{j+1/2}$ as follows

$$D_{j+1/2} = \begin{cases} \lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ 0, & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ \lambda \frac{f(u_R) - f(u_s)}{u_R - u_L}, & \text{if } f'(u_L) < 0 < f'(u_R). \end{cases}$$

3. In order to show the scheme has TVD property, according to Theorem II.23 (Harten [1980]) from the lecture note, the following three conditions

$$\boxed{\begin{aligned} C_{j+1/2} &\geq 0, \\ D_{j+1/2} &\geq 0, \\ C_{j+1/2} + D_{j+1/2} &\leq 1, \end{aligned}}$$

must hold. Herein, the first two conditions $C_{j+1/2} \geq 0$, $D_{j+1/2} \geq 0$ can be shown by using convexity of flux function f . Besides, the third condition $C_{j+1/2} + D_{j+1/2} \leq 1$ is shown as follows

$$C_{j+1/2} + D_{j+1/2} = \begin{cases} \lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ \lambda \frac{f(u_R) + f(u_L) - 2f(u_s)}{u_R - u_L}, & \text{if } f'(u_L) < 0 < f'(u_R). \end{cases}$$

The $C_{j+1/2} + D_{j+1/2} \leq 1$ leads to the CFL condition for *Godunov's* scheme. For λ that satisfies $C_{j+1/2} + D_{j+1/2} \leq 1$, the scheme is TVD.