

# Global Exercise - 15-extra

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## 1 How the convexity property of $f(u)$ guarantees both $C_{j+1/2} \geq 0$ and $D_{j-1/2} \geq 0$ at the same time

(Continued from Example 10 · GE15 · 02<sup>nd</sup> February 2022)

In order to show the scheme has TVD property, according to Theorem II.23 (*Harten* [1980]) from the lecture note, the following three conditions

$$\boxed{\begin{array}{l} C_{j+1/2} \geq 0, \\ D_{j-1/2} \geq 0, \\ C_{j+1/2} + D_{j-1/2} \leq 1, \end{array}}$$

must hold. Herein, the first two conditions  $C_{j+1/2} \geq 0$ ,  $D_{j-1/2} \geq 0$  can be shown by using convexity of flux function  $f$ . We now examine  $C_{j+1/2}$ . The analysis is as follows

1. Case 1: According to Fig. 1 and Fig. 2, if  $\left(f'(u_L) \geq 0 \wedge f'(u_R) \geq 0\right) \vee \left(f'(u_L) \geq 0 \geq f'(u_R) \wedge [f]/[u] > 0\right)$  then

$$C_{j+1/2} = 0 \quad \checkmark$$

which satisfies  $C_{j+1/2} \geq 0$ .

2. Case 2.1: According to Fig. 3, if  $f'(u_L) \leq 0 \wedge f'(u_R) \leq 0$  ( $\star$ ) then

$$C_{j+1/2} = -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L} = -\lambda\alpha.$$

- $f(u)$  convex with ( $\star$ ) leads to  $f(u)$  decreasing at both  $u_L$  and  $u_R$ 
  - If  $u_L > u_R$  then  $\alpha < 0$ . Hence,  $C_{j+1/2} > 0$   $\checkmark$
  - If  $u_L < u_R$  then  $\alpha < 0$ . Hence,  $C_{j+1/2} > 0$   $\checkmark$
- $f(u)$  concave with ( $\star$ ) leads to  $f(u)$  decreasing at both  $u_L$  and  $u_R$ 
  - If  $u_L > u_R$  then  $\alpha < 0$ . Hence,  $C_{j+1/2} > 0$   $\checkmark$
  - If  $u_L < u_R$  then  $\alpha < 0$ . Hence,  $C_{j+1/2} > 0$   $\checkmark$

3. Case 2.2: According to Fig. 4, if  $\left(f'(u_L) \geq 0 \geq f'(u_R)\right) \wedge \left([f]/[u] < 0\right)$  ( $\star\star$ ) then

$$C_{j+1/2} = -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L} = -\lambda\alpha.$$

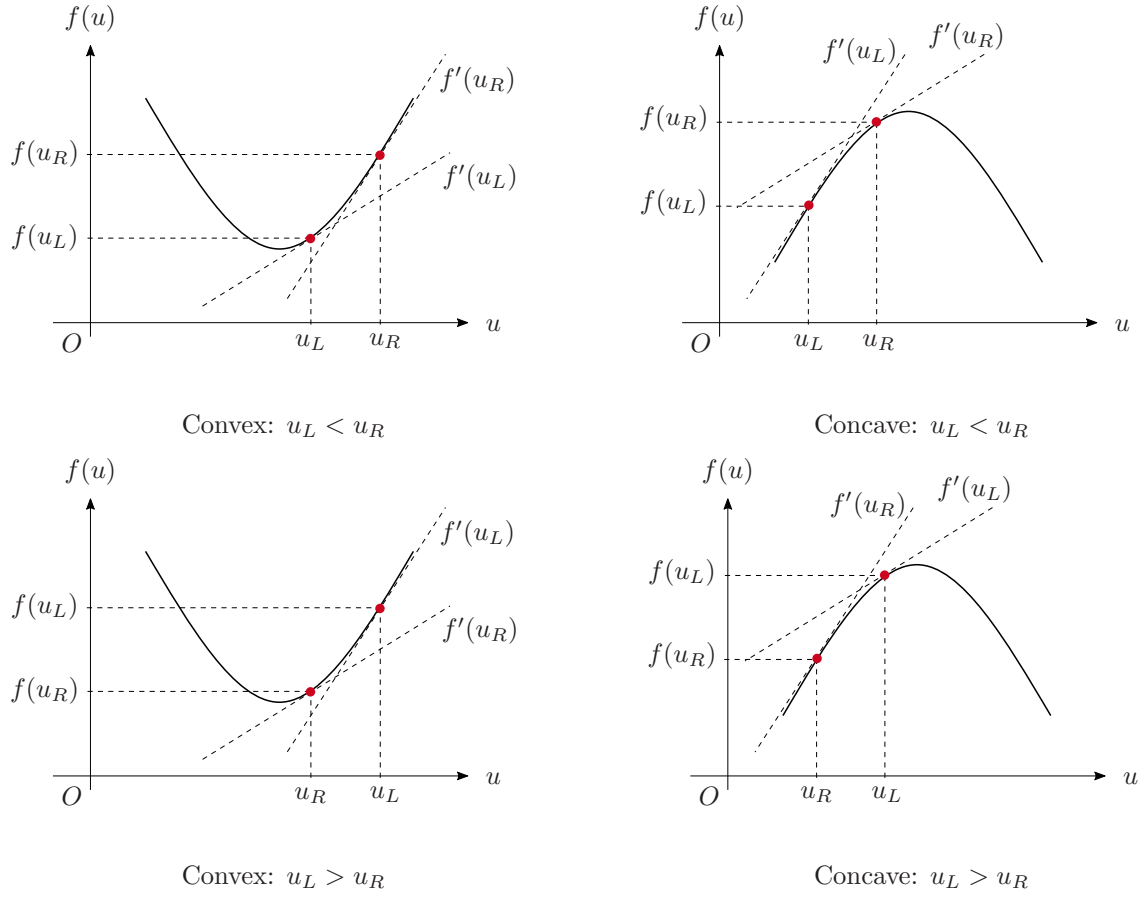


Figure 1: Case 1.1: Convexity and concavity for  $u_L < u_R$  and  $u_L > u_R$ , where the condition  $f'(u_L) \geq 0 \wedge f'(u_R) \geq 0$  holds.

- $f(u)$  convex with the first condition in  $(\star\star)$  leads to  $f(u)$  decreasing at  $u_R$  and increasing at  $u_L$ . Hence, it must be  $u_L > u_R$ , which implies a shock. Besides, the second condition  $[f]/[u] < 0$  implies also an existence of shock, which is moving to the left in this case.

→ If  $u_L > u_R$  and  $f(u_L) > f(u_R)$  then  $\alpha > 0$ . Hence,  $C_{j+1/2} < 0$  ✗  
→ If  $u_L > u_R$  and  $f(u_L) < f(u_R)$  then  $\alpha < 0$ . Hence,  $C_{j+1/2} > 0$  ✓

Therefore, the convexity of  $f(u)$  gives us the possibility of  $C_{j+1/2} > 0$ .

- $f(u)$  concave with the first condition in  $(\star\star)$  leads to  $f(u)$  decreasing at  $u_R$  and increasing at  $u_L$ . Hence, it must be  $u_L < u_R$ , which does not imply a shock, but rather a rarefaction solution. Besides, the second condition  $[f]/[u] < 0$  implies an existence of shock, which is moving to the left in this case. This is a **contradiction** for the case of  $f(u)$  concave. Therefore, the concavity of  $f(u)$  does not lead us to the possibility of  $C_{j+1/2} > 0$ .

4. Case 3: According to Fig. 5, if  $f'(u_L) < 0 < f'(u_R)$   $(\star\star\star)$  then

$$C_{j+1/2} = -\lambda \frac{f(u_s) - f(u_L)}{u_R - u_L} = -\lambda\alpha.$$

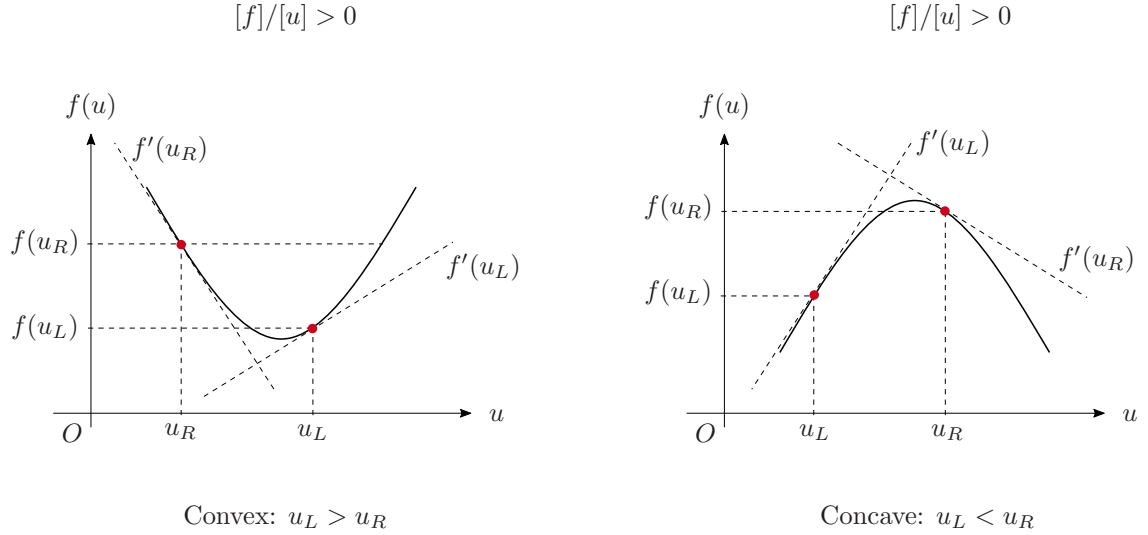
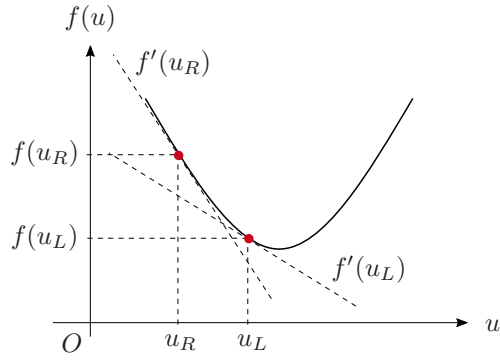


Figure 2: Case 1.2: Convexity and concavity for  $u_L < u_R$  and  $u_L > u_R$ , where the condition  $f'(u_L) \geq 0 \geq f'(u_R) \wedge [f]/[u] > 0$  holds.

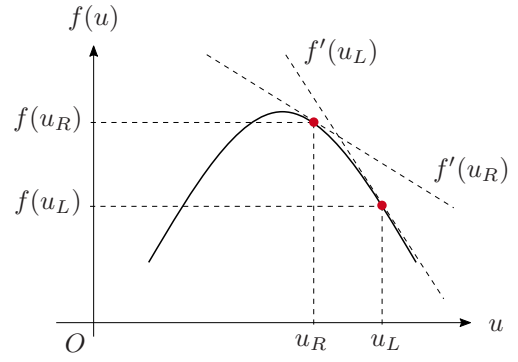
- $f(u)$  convex with the condition  $(\star \star \star)$  leads to  $f(u)$  decreasing at  $u_L$  and increasing at  $u_R$ . Hence, it must be  $u_R > u_L$ , which also implies a rarefaction. Therefore, we obtain  $u_s \in [u_L, u_R]$  which leads to  $f(u_s) < f(u_{R,L})$ . As a consequence,  $\alpha < 0$ . Finally, the convexity of  $f(u)$  do lead us to the possibility of  $C_{j+1/2} > 0$ .
- $f(u)$  concave with the condition  $(\star \star \star)$  leads to  $f(u)$  decreasing at  $u_L$  and increasing at  $u_R$ . Hence, it must be  $u_R < u_L$ , which also implies a shock. Therefore, we obtain  $u_s \in [u_R, u_L]$  which leads to  $f(u_s) > f(u_{R,L})$ . As a consequence,  $\alpha < 0$ . Finally, the convexity of  $f(u)$  do lead us also to the possibility of  $C_{j+1/2} > 0$ .

Therefore, by combining case 1, case 2.1, case 2.2, and case 3 altogether, we recognize that only the convexity of  $f$  does lead us to the possibility of  $C_{j+1/2} > 0$ , while the concavity of  $f$  has failed at case 2.2

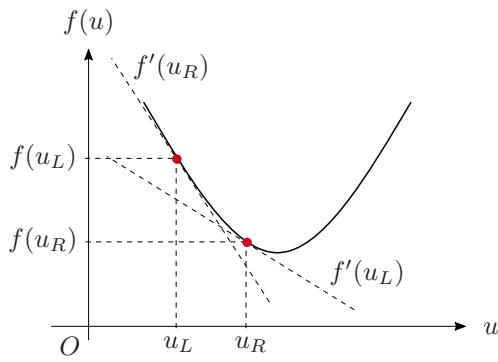
In case of  $D_{j-1/2}$  the argument will be proceeded the same as for case  $C_{j+1/2}$ , meaning that the convexity of  $f$  does lead us to the possibility of  $D_{j-1/2} > 0$ , while the concavity is not so.



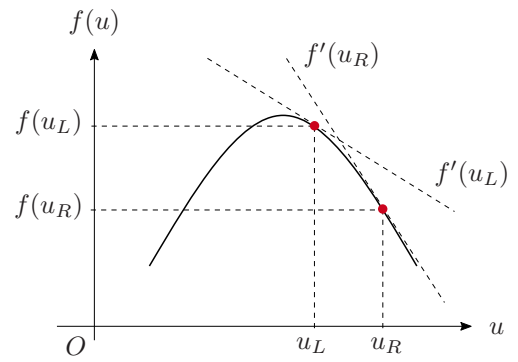
Convex:  $u_L > u_R$



Concave:  $u_L > u_R$



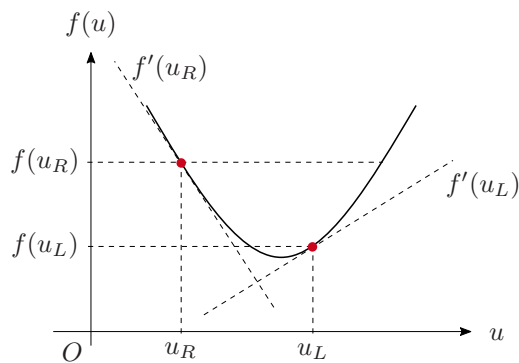
Convex:  $u_L < u_R$



Concave:  $u_L < u_R$

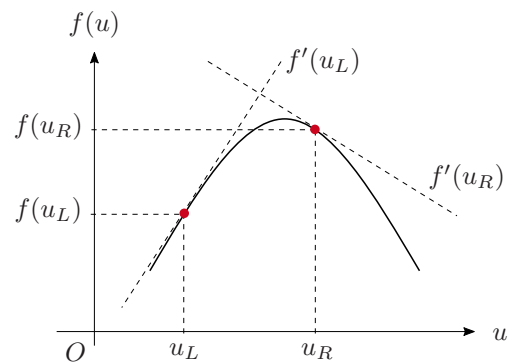
Figure 3: Case 2.1: Convexity and concavity for  $u_L < u_R$  and  $u_L > u_R$ , where the condition  $f'(u_L) \leq 0 \wedge f'(u_R) \leq 0$  holds.

$$[f]/[u] < 0$$



Convex:  $u_L > u_R$

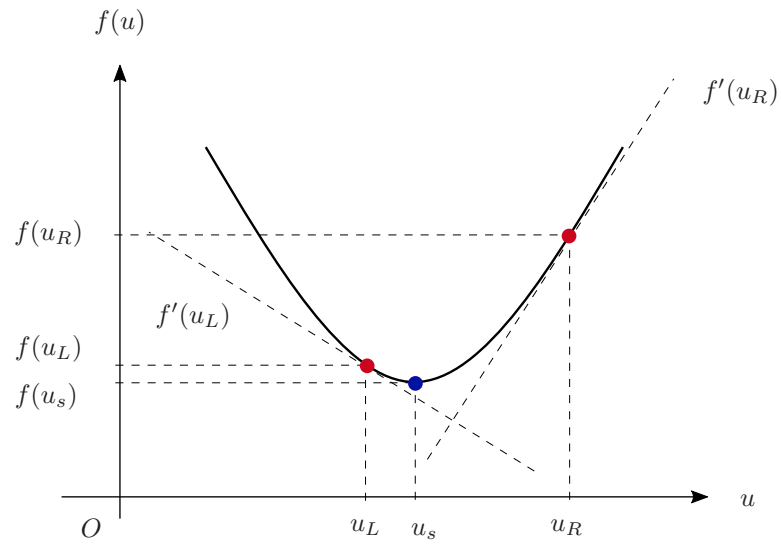
$$[f]/[u] < 0$$



Concave:  $u_L < u_R$

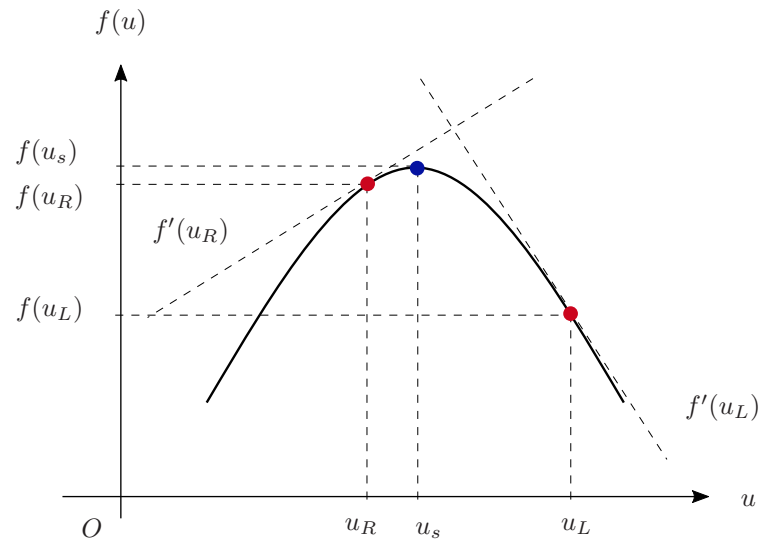
Figure 4: Case 2.2: Convexity and concavity for  $u_L < u_R$  and  $u_L > u_R$ , where the condition  $f'(u_L) \geq 0 \geq f'(u_R) \wedge [f]/[u] < 0$  holds.

$$f'(u_s) = 0$$



Convex:  $u_L < u_R$

$$f'(u_s) = 0$$



Concave:  $u_L > u_R$

Figure 5: Case 3: Convexity and concavity for  $u_L < u_R$  and  $u_L > u_R$ , where the condition  $f'(u_L) < 0 < f'(u_R)$  holds.