→ Remark about monotonicity → Upwind < UWL / UWR $a>0 \mid f'(u)>0$ increasing, $a<0 \mid f'(u)<0$ decreasing | 8 cases

→ Structure of FVM (Figure @ 115 lecture note)

→ limiter (structure of FVM → development)

→ Godunov (visualization / sketch) ↳ insight

Global Exercise - 15

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1 A small note about monotonicity: *Upwind* scheme

Example 1. *Examine monotonicity of $\mathcal{H}_{\Delta t}$ for conservation law $u_t + au_x = 0$.* (scalar / constant)

1st – UWL (characteristics)

$(a > 0)$ CFL: $0 \le f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_{j-1}^n) - f(u_j^n)\Big)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \boxed{\frac{\Delta t}{\Delta x} f'(u_{j-1}^n)}$$

$$B_1 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = \boxed{1 - \frac{\Delta t}{\Delta x} f'(u_j^n)}$$

$$C_1 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = 0$$

$f'(u)$, a, O, u_{j-1}^n, u_j^n, u

monotone ✓ — with CFL okay

1.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing?
- $A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0$ ✓
- $B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ ✓

1.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing? ✗
- $A_1 \le 0 \Leftrightarrow f'(u_{j-1}^n) \le 0$ ↯
- $B_1 \le 0 \Leftrightarrow f'(u_j^n) \ge \Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ ↯

2nd – UWL

$(a < 0)$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_{j-1}^n) - f(u_j^n)\Big)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \frac{\Delta t}{\Delta x} f'(u_{j-1}^n)$$

$$B_1 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 - \frac{\Delta t}{\Delta x} f'(u_j^n)$$

$$C_1 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = 0$$

slope — NOT monotone

$f'(u)$, u_{j-1}^n, u_j^n, u, O, a

2.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing? ✗
- $A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0$ ↯
- $B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t$ ✓

2.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing?
- $A_1 \le 0 \Leftrightarrow f'(u_{j-1}^n) \le 0$ ✓
- $B_1 \le 0 \Leftrightarrow f'(u_j^n) \ge \Delta x/\Delta t$ ↯

non-positive — $a \ge \frac{\Delta x}{\Delta t} \Leftrightarrow \frac{a\Delta t}{\Delta x} \ge 1$

3rd – UWR

$(a > 0)$ ✗ CFL

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_j^n) - f(u_{j+1}^n)\Big)$$

$$A_2 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = 0$$

$$B_2 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = \boxed{1 + \frac{\Delta t}{\Delta x} f'(u_j^n)}$$

$$C_2 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = \boxed{-\frac{\Delta t}{\Delta x} f'(u_{j+1}^n)}$$

slope — NOT monotone !

(Burgers) convexity $f(u_{j+1}^n) > f(u_j^n)$, $f(u)$

$f'(u)$, a, O, u_j^n, u_{j+1}^n, u

3.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing? ✗
- $B_2 \ge 0 \Leftrightarrow f'(u_j^n) \ge -\Delta x/\Delta t$ ✓
- $C_2 \ge 0 \Leftrightarrow f'(u_{j+1}^n) \le 0$ ↯

3.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing? ✗
- $B_2 \le 0 \Leftrightarrow f'(u_j^n) \le -\Delta x/\Delta t$ ↯
- $C_2 \le 0 \Leftrightarrow f'(u_{j+1}^n) \ge 0$ ✓

4th – UWR

$(a < 0)$ CFL: $0 \le -f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_j^n) - f(u_{j+1}^n)\Big)$$

$$A_2 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = 0$$

$$B_2 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = \boxed{1 + \frac{\Delta t}{\Delta x} f'(u_j^n)}$$

$$C_2 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = \boxed{-\frac{\Delta t}{\Delta x} f'(u_{j+1}^n)}$$

$f'(u)$, u_{j-1}^n, u_{j+1}^n, u, O, $a < 0$

$a \ge -\frac{\Delta x}{\Delta t}$, $-\frac{a\Delta t}{\Delta x} \le 1$ ✓ ok

4.1 $\mathcal{H}_{\Delta t}$ Monotonically increasing? ok
- $B_2 \ge 0 \Leftrightarrow f'(u_j^n) \ge -\Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ ✓
- $C_2 \ge 0 \Leftrightarrow f'(u_{j+1}^n) \le 0$ ✓

4.2 $\mathcal{H}_{\Delta t}$ Monotonically decreasing?
- $B_2 \le 0 \Leftrightarrow f'(u_j^n) \le -\Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ ↯
- $C_2 \le 0 \Leftrightarrow f'(u_{j+1}^n) \ge 0$ ↯ ($a \ge 0$ ✗)

Figure 1: Monotonicity of *Upwind* scheme: 8 different cases.

OK monotone ✓

Total variation diminishing

2 Structure of FVM

Example 2. *Monotone* → L_1*-Contracting* → *TVD* → *Monotonicity-Preserving.*

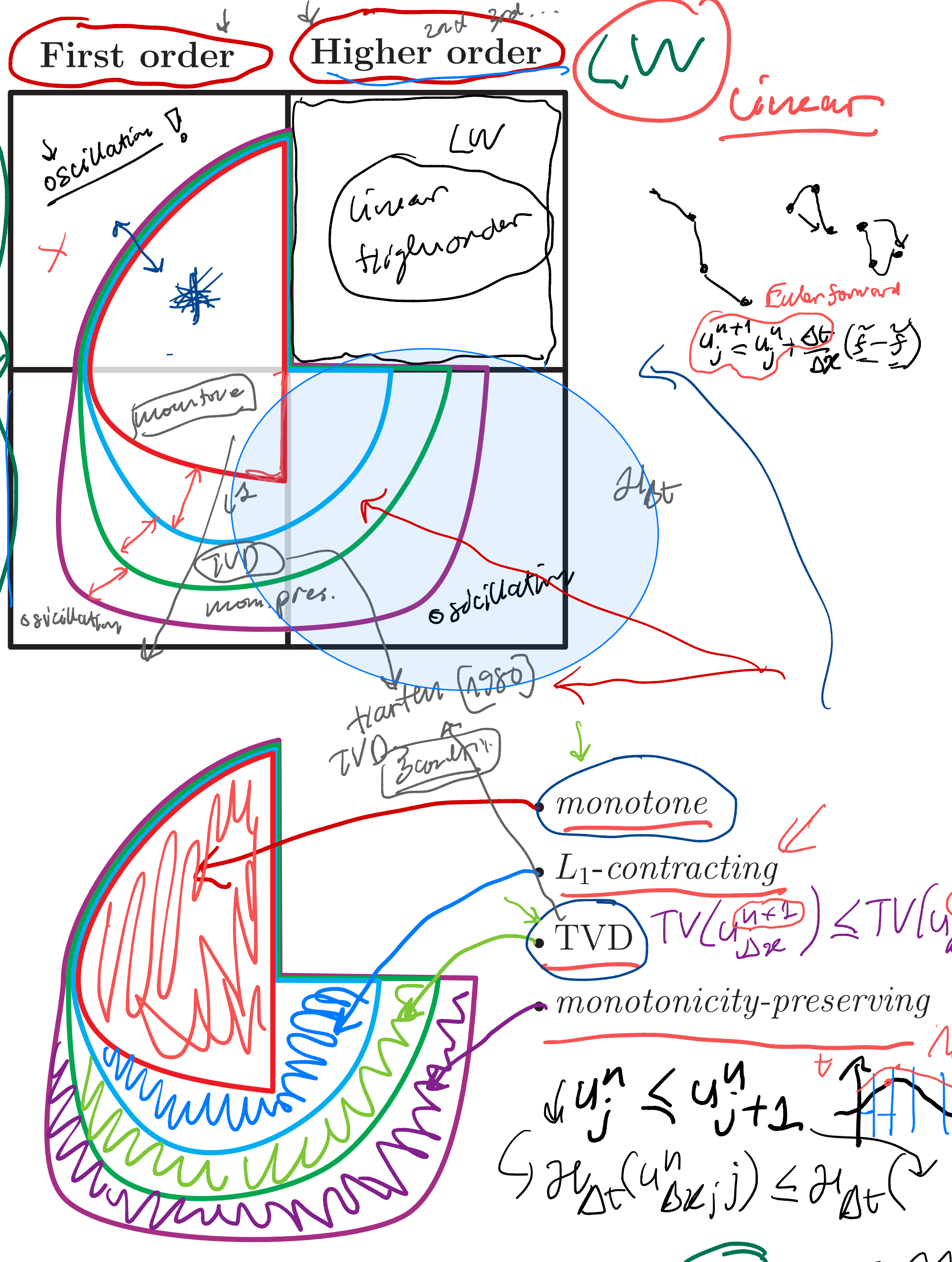
Linear LF UWB UWL 1st

LW linear

$\mathcal{H}_{\Delta t}(\alpha x+\beta y) = \alpha \mathcal{H}_{\Delta t}(x)+\beta \mathcal{H}_{\Delta t}(y)$

Euler forward

$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}(\tilde f - \tilde f)$



$TV(u^{n+1}_{\Delta x}) \le TV(u^n_{\Delta x})$

$u_j^n \le u_{j+1}^n$

$\mathcal{H}_{\Delta t}(u^n_{\Delta x;j}) \le \mathcal{H}_{\Delta t}(\;)$

Figure 2: Structure of Finite Volume Method [Lecture note page 115].

for Higher order + Nonlinear

suppress oscillation

(PE3)

3 Limiter

Lax Friedrichs Lax Wendroff

Example 3. *Examine the* 1^{st}*-order-accurate LF and the* 2^{nd}*-order-accurate LW.*

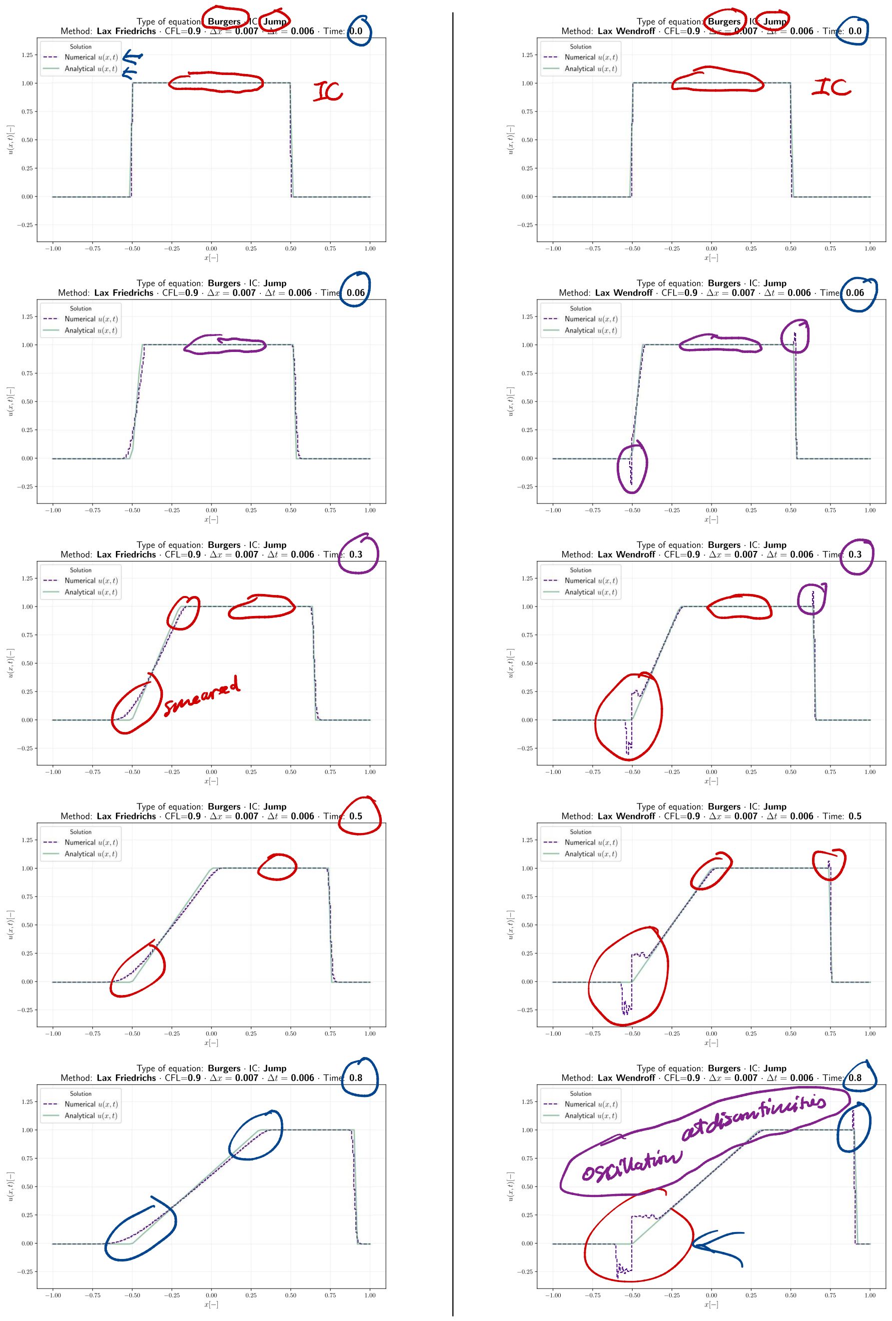


Figure 3: Oscillatory phenomena around discontinuity: (left) none oscillation founded in *Lax-Friedrichs*; (right) oscillation observed in *Lax-Wendroff*.

→ Numerical scheme with Higher order + Nonlinear

Development

TVD → no oscillation

Example 4. *Assume the linear advection problem given by*

$$u_t + au_x = 0 \quad \textit{with} \quad a > 0,$$

and the numerical scheme as follows

$$u_j^{n+1} = u_j^n + \nu\left(u_{j-1}^n - u_j^n\right) + \frac{\nu(1-\nu)}{2}\Delta x\left(\sigma_{j-1}^n - \sigma_j^n\right),$$

with limited slopes

→ suppress oscillation

$+\varepsilon$

$$\sigma_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x}\phi(\theta_j).$$

ε

PEG

Prove that numerical scheme is TVD with limiters $\phi(\theta)$ given as follows

HLL

Harten-Lax-van Leer

1. $\phi(\theta) = \max(0, \min(\theta, 2))$,
2. $\phi(\theta) = \max(0, \min(2\theta, 2))$,
3. $\phi(\theta) = \dfrac{\theta + |\theta|}{\alpha + |\theta|}$ *for any fixed $\alpha \geq 1$,*
4. $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$.

Van Leer

Denote which of them are second order schemes.

Harten (1980) 125 page

According to the theorem on TVD region of *Upwind* numerical scheme with limiter presented in the lecture, we know that limiter gives us TVD scheme if the following relations

Harten

$$0 \leq \phi(\theta) \leq 2\theta,$$
$$|\phi(\theta)| \leq 2,$$

TVD + 2nd

holds. Besides, the scheme is of 2nd-order if and only if $\phi(1) = 1$ holds. Therefore, we can simply check lower and upper bounds of limiters, as follows

1. $\phi(\theta) = \max(0, \min(\theta, 2))$ satisfies bounds and goes through $\phi(1) = 1$. Hence, it corresponds to 2nd-order TVD method.

2. $\phi(\theta) = \max(0, \min(2\theta, 2))$ satisfies bounds but $\phi(1) = 2$. Hence, it is of 1st-order TVD scheme.

3. $\phi(\theta) = \dfrac{\theta + |\theta|}{\alpha + |\theta|}$ satisfies bounds for any fixed $\alpha \geq 1$, but $\phi(1) = 1$ holds only for $\alpha = 1$.
   Hence, it is of 2nd-order only if $\alpha = 1$ and of 1st-order otherwise.

4. $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$ satisfies bounds and $\phi(1) = 1$ holds. Hence, it is TVD scheme of 2nd-order.

Example 5. *Examine and sketch the following* Limiter *function.*

$$\phi(\theta) = \min\left(\delta\theta, \left|\frac{1+\theta}{2}\right|, \gamma\right), \quad \textit{for } \delta, \gamma \geq 1.$$

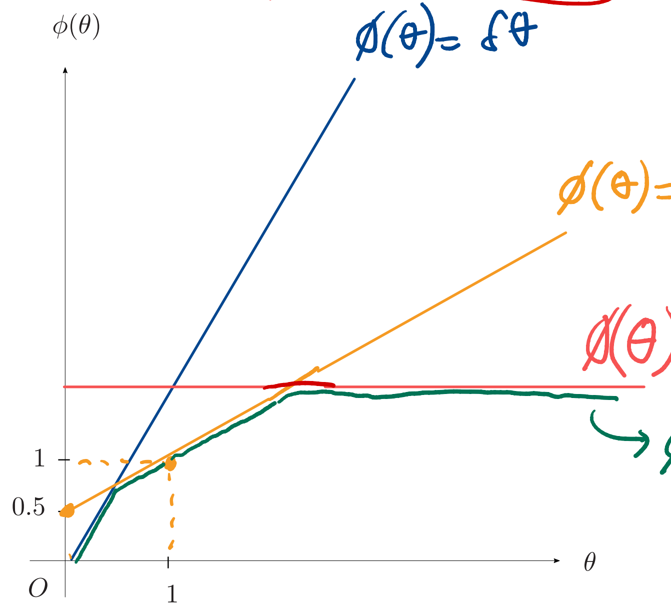
$\phi(\theta) = \delta\theta$ $\quad \boxed{\delta = 3}$

$\phi(\theta) = \frac{1+\theta}{2}$

$\theta = 0 \rightarrow \phi = \frac{1}{2}$

$\theta = 1 \rightarrow \phi = 1$

sketch



real problem

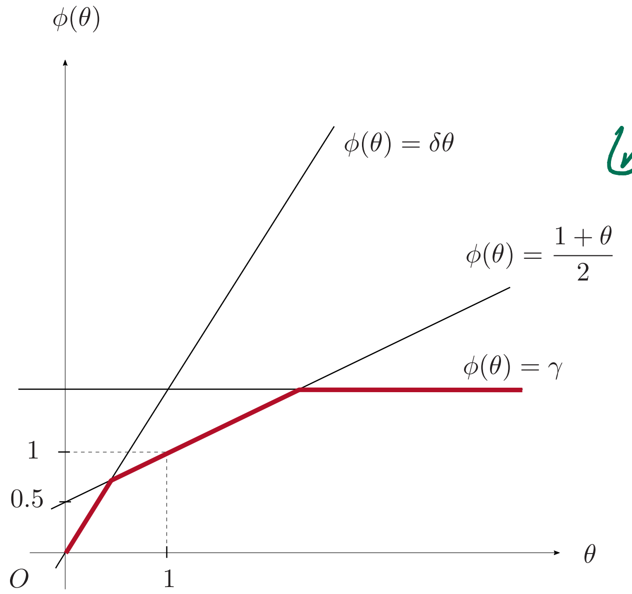


Figure 4: Limiter function $\phi(\theta) = \min\left(\delta\theta, \frac{1+\theta}{2}, \gamma\right)$, for $\delta, \gamma \geq 1$.

Čada Schmitzmann

$\gamma \in \mathbb{R}$

$\gamma \geq 1$

$$\phi(\theta) = \max\left(0, \min\left(\phi_3(\theta), \max\left(\theta, \min\left(2\theta, \phi_3(\theta), \gamma\right)\right)\right)\right)$$

$\min(2\theta, \phi_3(\theta), \gamma)$

inside → outside

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(g(u_{j-1}^n, u_j^n) - g(u_j^n, u_{j+1}^n) \right)$$

f — f

4 A bit insight into Godunov's solver

Example 6. *Verify the Godunov numerical flux function*

$$g(u_L, u_R) = \begin{cases} \min\limits_{u_L \le u \le u_R} f(u) & \text{if } u_L \le u_R, \\ \max\limits_{u_R \le u \le u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

compact

by considering all possible cases.

Riemann selection

All possible cases read

1. $f'(u_L), f'(u_R) \ge 0 \Rightarrow u* = u_L$
2. $f'(u_L), f'(u_R) \le 0 \Rightarrow u* = u_R$
3. $f'(u_L) \ge 0 \ge f'(u_R) \Rightarrow u* = u_L$ if $[f]/[u] > 0$ or $u* = u_R$ if $[f]/[u] < 0$
4. $f'(u_L) < 0 < f'(u_R) \Rightarrow u* = u_s$ where $f'(u_s) = 0$

u_{sonic}

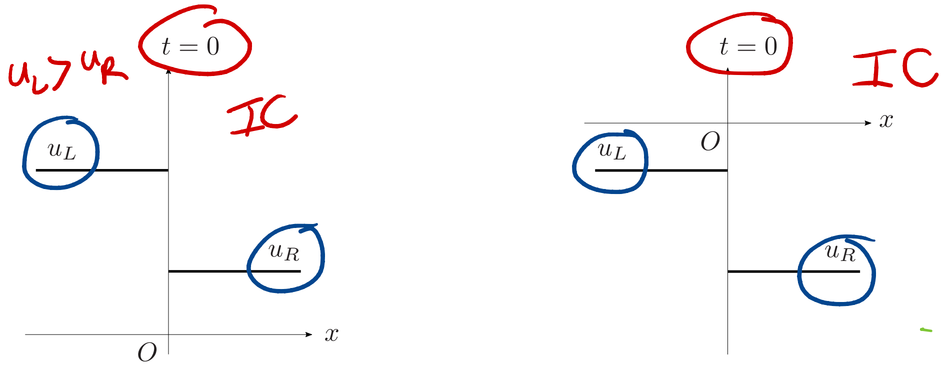
$f'(u_s) = 0$

$f'(u_s) = u_s = 0$

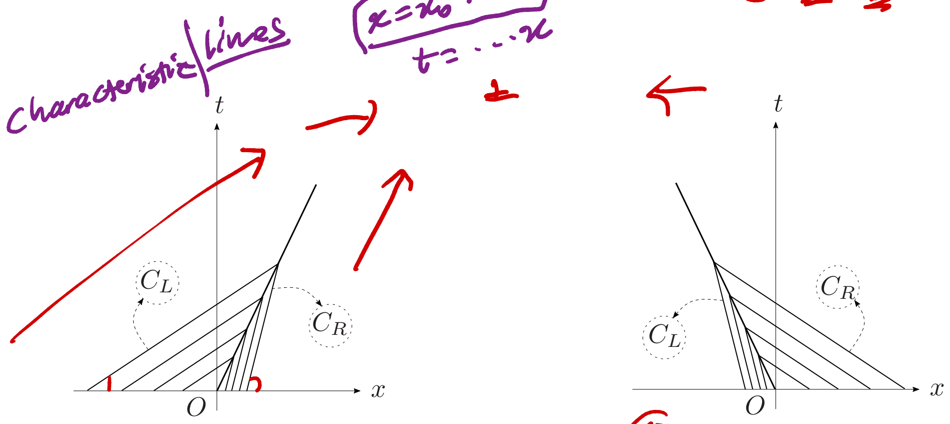
$u_s = 0$

@ Burgers

Example 7. *Examine* $u_L > u_R$



Case 1 : $u_L > u_R > 0$ Case 3 : $0 > u_L > u_R$



$(C_L):\ t = \frac{1}{u_L}x - \frac{x_0}{u_L}$ $(C_R):\ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$

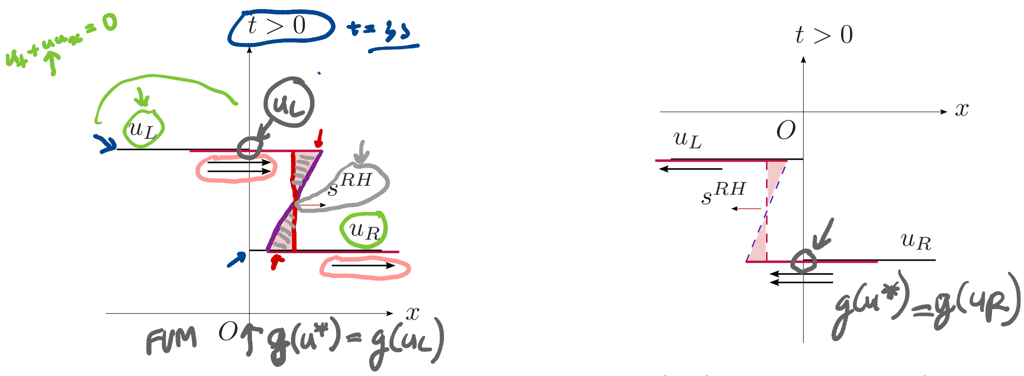
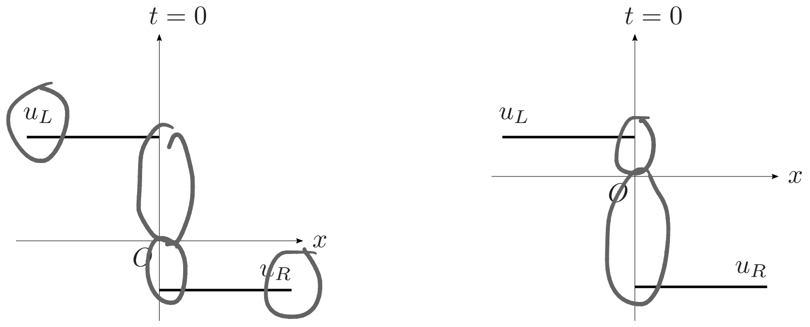


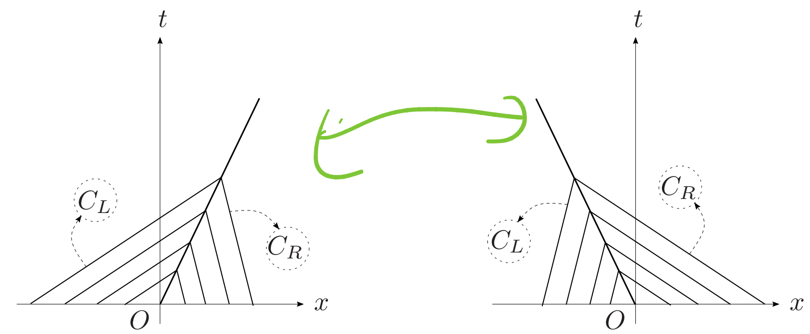
Figure 5: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

Schock solution

Example 8. *Examine $u_L > u_R$.*



Case 2.1 : $(u_L > 0 > u_R) \wedge |u_L| > |u_R|$ Case 2.2 : $(u_L > 0 > u_R) \wedge |u_L| < |u_R|$



$$(C_L): \ t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad (C_R): \ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

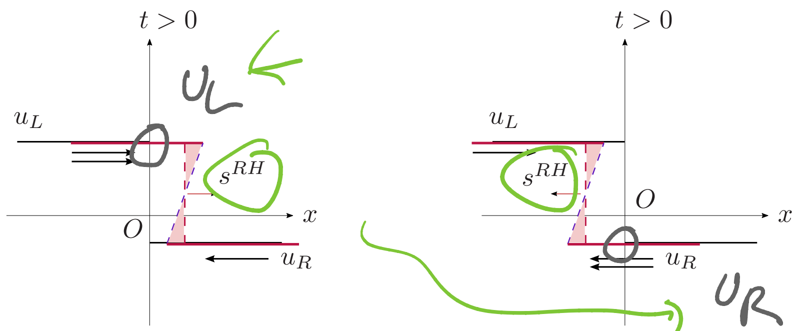
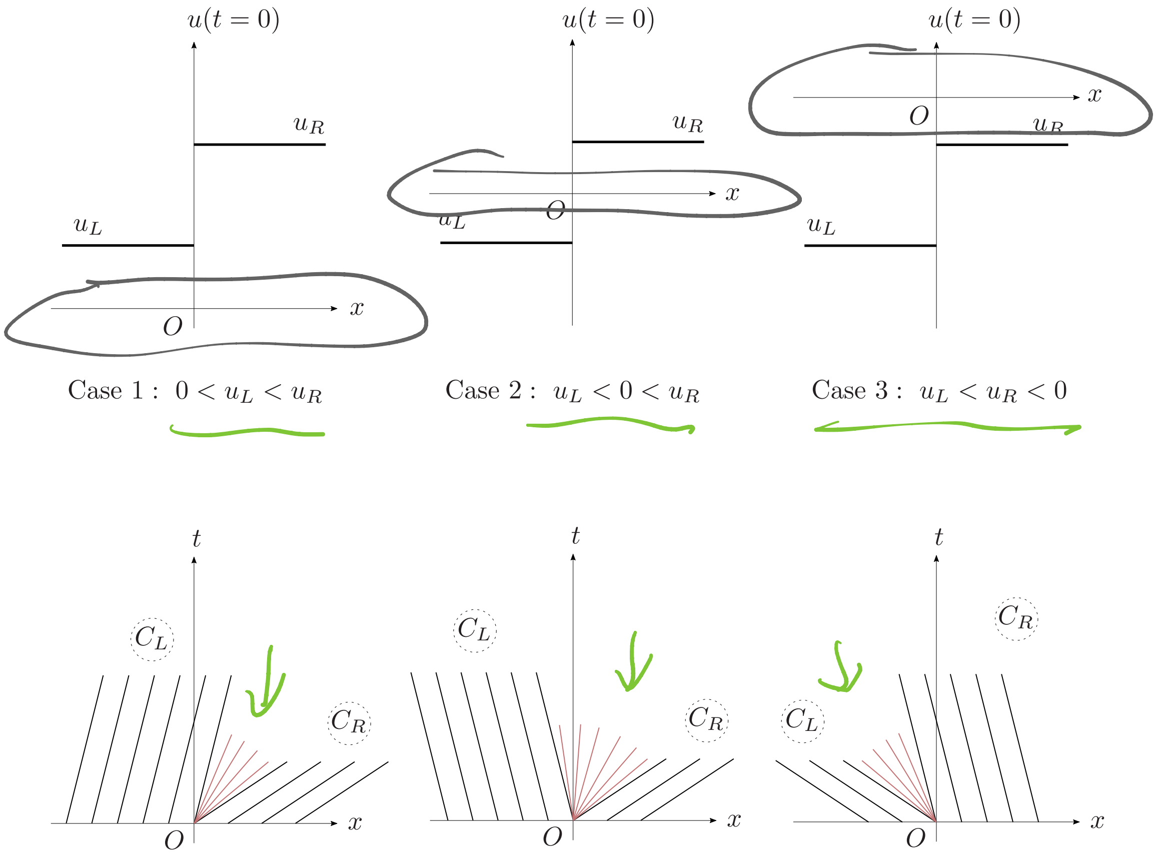


Figure 6: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

Schock solution

Example 9. *Examine* $u_L < u_R$.



$$(C_L): \; t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad (C_R): \; t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

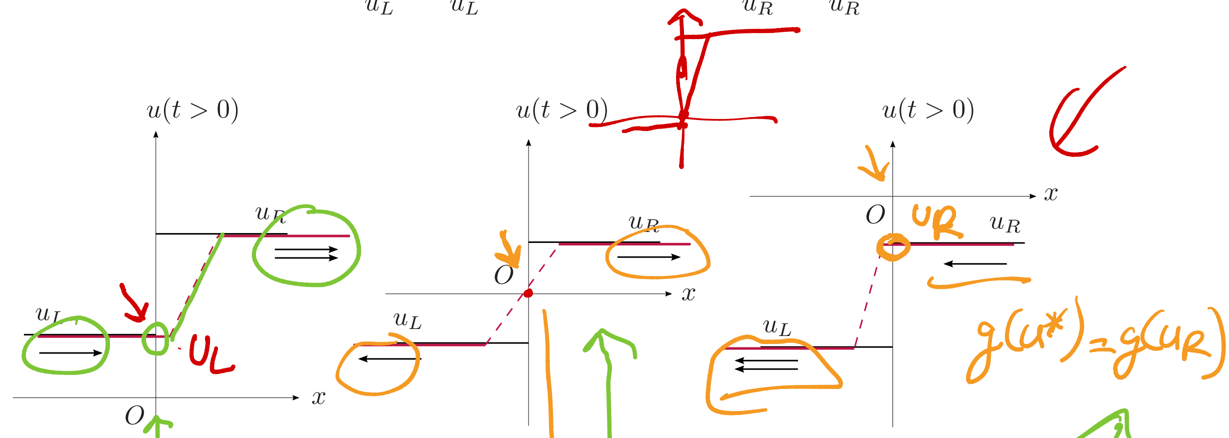


Figure 7: Riemann problem with $u_L < u_R$: IC, Characteristics, Solution.

Rarefaction solution

u_{sonic}

Example 10. *Consider the* Godunov*'s solver for the conservation law* $u_t + f(u)_x = 0$ *with convex and nonlinear flux function* f*, e.g. the flux function in* Burgers*' equation, where*

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(g(u_{j-1}^n, u_j^n) - g(u_j^n, u_{j+1}^n)\right),$$

and the numerical flux function

$$g(u_L, u_R) = \begin{cases} \min\limits_{u_L \le u \le u_R} f(u) & \text{if } u_L \le u_R, \\ \max\limits_{u_R \le u \le u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

1. *Show that the numerical flux function is* monotone.
2. *Rewrite the scheme in the* incremental form.

Harten theorem

3. *Show that the scheme has the* total-variation-diminishing *(TVD) property.*

Approach:

1. *Proof.* In order to show the monotonicity of the numerical flux function, it is sufficient to show the following relations

$$\boxed{\begin{cases} \partial_u g(u,v) \ge 0, \\ \partial_v g(u,v) \le 0. \end{cases}} \tag{1}$$

Hence, in case of *Burgers*' equation, where $f'(u) = u$, we proceed as follows

$$g(u_L, u_R) = \begin{cases} f(u_L), & \text{if } \begin{bmatrix} f'(u_L) \ge 0 \text{ and } f'(u_R) \ge 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] > 0, \end{bmatrix} \\ \\ f(u_R), & \text{if } \begin{bmatrix} f'(u_L) \le 0 \text{ and } f'(u_R) \le 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] < 0, \end{bmatrix} \\ \\ f(u_s), & \text{if } \quad f'(u_L) < 0 < f'(u_R), \end{cases}$$

where u_s denotes the sonic point where $f'(u_s) = 0$. Then, by taking the partial derivative of $g(u_L, u_R)$ w.r.t. u_L and u_R we arrive at the following expressions

$$\partial_{u_L} g(u_L, u_R) = \begin{cases} f'(u_L), & \text{if } \begin{bmatrix} f'(u_L) \ge 0 \text{ and } f'(u_R) \ge 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] > 0, \end{bmatrix} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_L} g(u_L, u_R) \ge 0.} \tag{2}$$

Likewise, the partial derivative of $g(u_L, u_R)$ w.r.t. u_R goes as follows

$$\partial_{u_R} g(u_L, u_R) = \begin{cases} f'(u_R), & \text{if } \begin{bmatrix} f'(u_L) \le 0 \text{ and } f'(u_R) \le 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] < 0, \end{bmatrix} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_R} g(u_L, u_R) \leq 0.} \tag{3}$$

Therefore, by combining (2) and (3), and comparing them with the condition given at (1), we arrive at *q.e.d.* $\square$

2. The incremental form reads

UW +

$$\boxed{u_j^{n+1} = u_j^n + C^n_{j+1/2}(u^n_{j+1} - u^n_j) - D^n_{j-1/2}(u^n_j - u^n_{j-1}).}$$

For any conservative Finite Volume scheme we obtain

$$C_{j+1/2} = -\lambda \frac{g_{j+1/2} - f_j}{u_{j+1} - u_j}$$

$$D_{j-1/2} = \lambda \frac{f_j - g_{j-1/2}}{u_j - u_{j-1}}$$

where $\lambda = \Delta t / \Delta x$. Herein, for the case $C_{j+1/2}$, u_L is u_j and u_R is u_{j+1}, as follows

$$C_{j+1/2} = \begin{cases} 0, & \text{if } \left[\begin{array}{l} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\[2ex] -\lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \left[\begin{array}{l} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\[2ex] -\lambda \dfrac{f(u_s) - f(u_L)}{u_R - u_L}, & \text{if } \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

Likewise, it goes for the case $D_{j+1/2}$ as follows

$$D_{j+1/2} = \begin{cases} \lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \left[\begin{array}{l} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\[2ex] 0, & \text{if } \left[\begin{array}{l} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\[2ex] \lambda \dfrac{f(u_R) - f(u_s)}{u_R - u_L}, & \text{if } \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

3. In order to show the scheme has TVD property, according to Theorem II.23 (*Harten* [1980]) from the lecture note, the following three conditions

$$\boxed{\begin{array}{r} C_{j+1/2} \geq 0, \\ D_{j+1/2} \geq 0, \\ C_{j+1/2} + D_{j+1/2} \leq 1, \end{array}}$$

must hold. Herein, the first two conditions $C_{j+1/2} \geq 0$, $D_{j+1/2} \geq 0$ can be shown by using convexity of flux function f. Besides, the third condition $C_{j+1/2} + D_{j+1/2} \leq 1$ is shown as follows

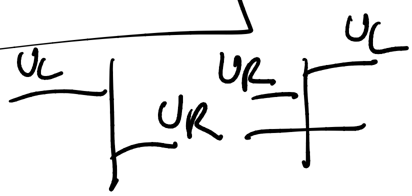
$$C_{j+1/2} + D_{j+1/2} = \begin{cases} \lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{bmatrix} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{bmatrix} \\[2ex] -\lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{bmatrix} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{bmatrix} \\[2ex] \lambda \dfrac{f(u_R) + f(u_L) - 2f(u_s)}{u_R - u_L}, & \text{if } \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

The $C_{j+1/2} + D_{j+1/2} \leq 1$ leads to the CFL condition for *Godunov*'s scheme. For λ that satisfies $C_{j+1/2} + D_{j+1/2} \leq 1$, the scheme is TVD.

$$\rightarrow \quad U_t + A U_x = 0$$

$$A = R^{-1} \Lambda R$$

$$\frac{1}{2}u^2$$



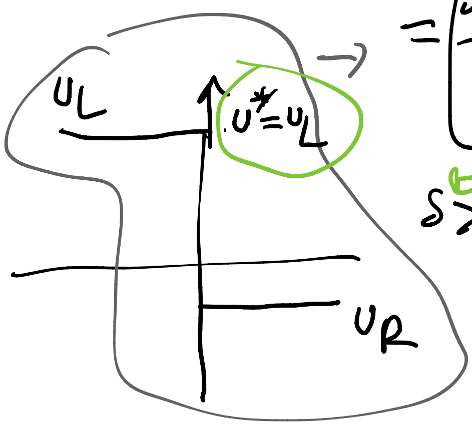
Rankine Hugoniote

RH

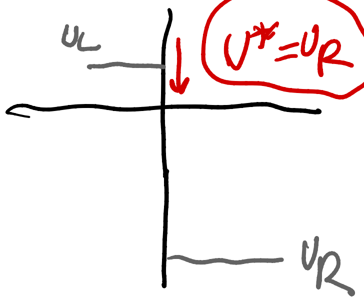
$$s = \frac{[f]}{[u]} = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \boxed{\frac{f(u_R) - f(u_L)}{u_R - u_L}} = \frac{\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2}{u_R - u_L} = \frac{1}{2}(u_L + u_R)$$

$f(u) = \frac{1}{2}u^2$

$|u_L| \leq |u_R|$



$s > 0$



$s \leq 0$