Global Exercise - 12

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1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1: $W := R^{-1}U$ as the scheme shown in exercise

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + R\Lambda R^{-1}U_x &= 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

where the matrix A is diagonalizable with a transformation matrix $R \in \mathbb{R}^{N\times N}$ in the form

$$A = R\Lambda R^{-1}.$$

2. Case 2: $W := TU$ as the scheme shown in lecture note

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + T^{-1}\Lambda TU_x &= 0 \\ TU_t + \Lambda TU_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

where the matrix A is diagonalizable with a transformation matrix $T \in \mathbb{R}^{N\times N}$ in the form

$$A = T^{-1}\Lambda T.$$

Check 1. *Let matrix A be given as*

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

then the diagonalization with transformation matrix R and T goes as follows

$$A = R\Lambda R^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 1. *Consider*

$$u_t + Au_x = 0 \qquad \textit{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

where the unknown vector $u(x,t)$ is defined, and the matrix A is given as follows

$$u(x,t) := \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}, \quad \textit{and} \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and the initial conditions are given by

$$\begin{cases} u_1(x,0) & = 1, \\ u_2(x,0) & = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases} \end{cases}$$

Approach: Eigenvalue decomposition of matrix A is given as follows

$$A = R\Lambda R^{-1}, \quad R = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2}R, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Hence, we can now change variable u to a new variable w as follows

$$w := R^{-1}u = \frac{1}{2}Ru, \quad \text{so} \quad u = Rw.$$

Then, the initial problem can be rewritten as

$$w_t + \Lambda w_x = 0 \Leftrightarrow \begin{cases} (w_1)_t + \lambda_1 (w_1)_x = 0, \\ (w_2)_t + \lambda_2 (w_2)_x = 0, \end{cases}$$

with the diagonal matrix Λ. This step simply means that variables w_1 and w_2 are independent of each other, i.e. the original system has been fully decoupled. Next, let us find the initial condition in terms of w as follows

$$w(x,0) = \frac{1}{2}Ru(x,0) = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} -\left(u_1(x,0) + u_2(x,0)\right)/2 \\ -\left(u_1(x,0) - u_2(x,0)\right)/2 \end{pmatrix},$$

which yields the following relation

$$\begin{pmatrix} w_1(x,0) \\ w_2(x,0) \end{pmatrix} = \begin{pmatrix} -\left(u_1(x,0) + u_2(x,0)\right)/2 \\ -\left(u_1(x,0) - u_2(x,0)\right)/2 \end{pmatrix}.$$

Therefore, the initial conditions for the transformed variable w_1 read

$$w_1(x,0) = \begin{cases} -(1+1)/2 = -1, & x < 0, \\ -(1-1)/2 = 0, & x > 0, \end{cases}$$

and for the variable w_2 read

$$w_2(x,0) = \begin{cases} -(1-1)/2 = 0, & x < 0, \\ -(1-(-1))/2 = -1, & x > 0. \end{cases}$$

Since we have the advection problem with speeds $a_1 = 1$ and $a_2 = 3$, which are the diagonal values of the diagonal matrix Λ, for both w_1 and w_2, we can derive explicit solution to each equation. The explicit solution w_1 reads

$$w_1(x,t) = w_1(x-t,0) = \begin{cases} -1, & x-t<0, \\ 0, & x-t>0, \end{cases} \Leftrightarrow w_1(x,t) = \begin{cases} -1, & x<t, \\ 0, & x>t, \end{cases}$$

and, likewise, the explicit solution w_2 reads

$$w_2(x,t) = w_2(x-3t,0) = \begin{cases} 0, & x-3t<0, \\ -1, & x-3t>0, \end{cases} \Leftrightarrow w_2(x,t) = \begin{cases} 0, & x<3t, \\ -1, & x>3t. \end{cases}$$

Now we trace back to variable u from variable w by taking into the consideration of the relation

$$u(x,t) = Rw(x,t) = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix} = \begin{pmatrix} -\left(w_1(x,t) + w_2(x,t)\right) \\ -\left(w_1(x,t) - w_2(x,t)\right) \end{pmatrix},$$

which means the following relation holds

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \begin{pmatrix} -\left(w_1(x,t) + w_2(x,t)\right) \\ -\left(w_1(x,t) - w_2(x,t)\right) \end{pmatrix}.$$

Therefore, the solution for $u_1(x,t)$ is computed as follows

$$u_1(x,t) = \begin{cases} -(-1+0) = 1, & x<t, \\ -(0+0) = 0, & t<x<3t, \\ -(0+(-1)) = 1, & x>3t, \end{cases}$$

and, likewise, the solution for $u_2(x,t)$ is

$$u_2(x,t) = \begin{cases} -(-1-0) = 1, & x<t, \\ -(0-0) = 0, & t<x<3t, \\ -(0-(-1)) = -1, & x>3t. \end{cases}$$

Finally, the solution to $u_1(x,t)$ and $u_2(x,t)$ read

$$\therefore \quad \boxed{\begin{cases} u_1(x,t) = \begin{cases} 1, & x<t, \\ 0, & t<x<3t, \\ 1, & x>3t, \end{cases} \\ u_2(x,t) = \begin{cases} 1, & x<t, \\ 0, & t<x<3t, \\ -1, & x>3t. \end{cases} \end{cases}}$$

2 Remark on step-by-step derivation: Weak formulation of conservation laws

Example 2. *Consider the conservation law*

$$u_t + f(u)_x = 0 \quad \textit{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1}$$

with initial condition given by

$$u(x,0) = u_0(x).$$

Find the weak solution.

Approach: The weak formulation is obtained by multiplying both sides of (1) with a test function $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R})$, and, then, integrating both sides over $\mathbb{R} \times \mathbb{R}^+$

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \varphi\,(u_t + f(u)_x)\, dxdt = 0. \tag{2}$$

The derivation goes as follows

$$\begin{aligned}
(2) \Leftrightarrow & \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi\,(u_t + f(u)_x)\, dxdt = 0 \\
\Leftrightarrow & \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi\,(u_t)\, dxdt + \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi\,(f(u)_x)\, dxdt = 0 \\
\Leftrightarrow & \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi\,(u_t)\, dtdx + \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi\,(f(u)_x)\, dxdt = 0 \\
\Leftrightarrow & \left[\int_{\mathscr{D}} \varphi\, u|_{t=0}^{t=+\infty}\, dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t\, u\, dtdx \right] \\
& + \left[\int_0^{\mathscr{T}} \varphi\, f(u)|_{x=-\infty}^{x=+\infty}\, dxdt - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt \right] = 0 \\
\Leftrightarrow & \left[\int_{\mathscr{D}} \Big(\varphi(x, \overset{t\to+\infty}{\mathscr{T}}) u(x, \overset{t\to+\infty}{\mathscr{T}}) - \varphi(x,0)u(x,0) \Big)\, dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t\, u\, dtdx \right] \\
& + \left[\int_0^{\mathscr{T}} \Big(\varphi(\overset{x\to+\infty}{\mathscr{D}}, t)\, f(u(\overset{x\to+\infty}{\mathscr{D}}, t)) - \varphi(\underset{x\to-\infty}{\mathscr{D}}, t)\, f(u(\underset{x\to-\infty}{\mathscr{D}}, t)) \Big)\, dt \right. \\
& \left. \qquad - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt \right] = 0 \\
\Leftrightarrow & \left[\int_{\mathscr{D}} \Big(\cancelto{0}{\varphi(x, \overset{t\to+\infty}{\mathscr{T}})} u(x, \overset{t\to+\infty}{\mathscr{T}}) - \varphi(x,0)u(x,0) \Big)\, dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t\, u\, dtdx \right] \\
& + \left[\int_0^{\mathscr{T}} \Big(\cancelto{0}{\varphi(\overset{x\to+\infty}{\mathscr{D}}, t)}\, f(u(\overset{x\to+\infty}{\mathscr{D}}, t)) - \cancelto{0}{\varphi(\underset{x\to-\infty}{\mathscr{D}}, t)}\, f(u(\underset{x\to-\infty}{\mathscr{D}}, t)) \Big)\, dt \right. \\
& \left. \qquad - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt \right] = 0 \\
\Leftrightarrow & - \int_{\mathscr{D}} \varphi(x,0)u(x,0) dx - \int_{\mathscr{D}} \int_0^{\mathscr{T}} \varphi_t u\, dtdx - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt = 0 \\
\Leftrightarrow & - \int_{\mathscr{D}} \varphi(x,0)u_0(x) dx - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_t u\, dxdt - \int_0^{\mathscr{T}} \int_{\mathscr{D}} \varphi_x\, f(u)\, dxdt = 0.
\end{aligned}$$

Therefore, the weak formulation reads

$$\therefore \quad \boxed{\int_0^{\mathscr{T}} \int_{\mathscr{D}} (u\varphi_t + f(u)\varphi_x)\, dxdt + \int_{\mathscr{D}} \varphi(x,0)u_0(x)dx = 0.}$$

Note in passing that the derivation above has taken into consideration the swapping order of integral limits between time $t \in [0, +\infty]$ and space $x \in [-\infty, +\infty]$, which results in indifference as shown in Figure 1.

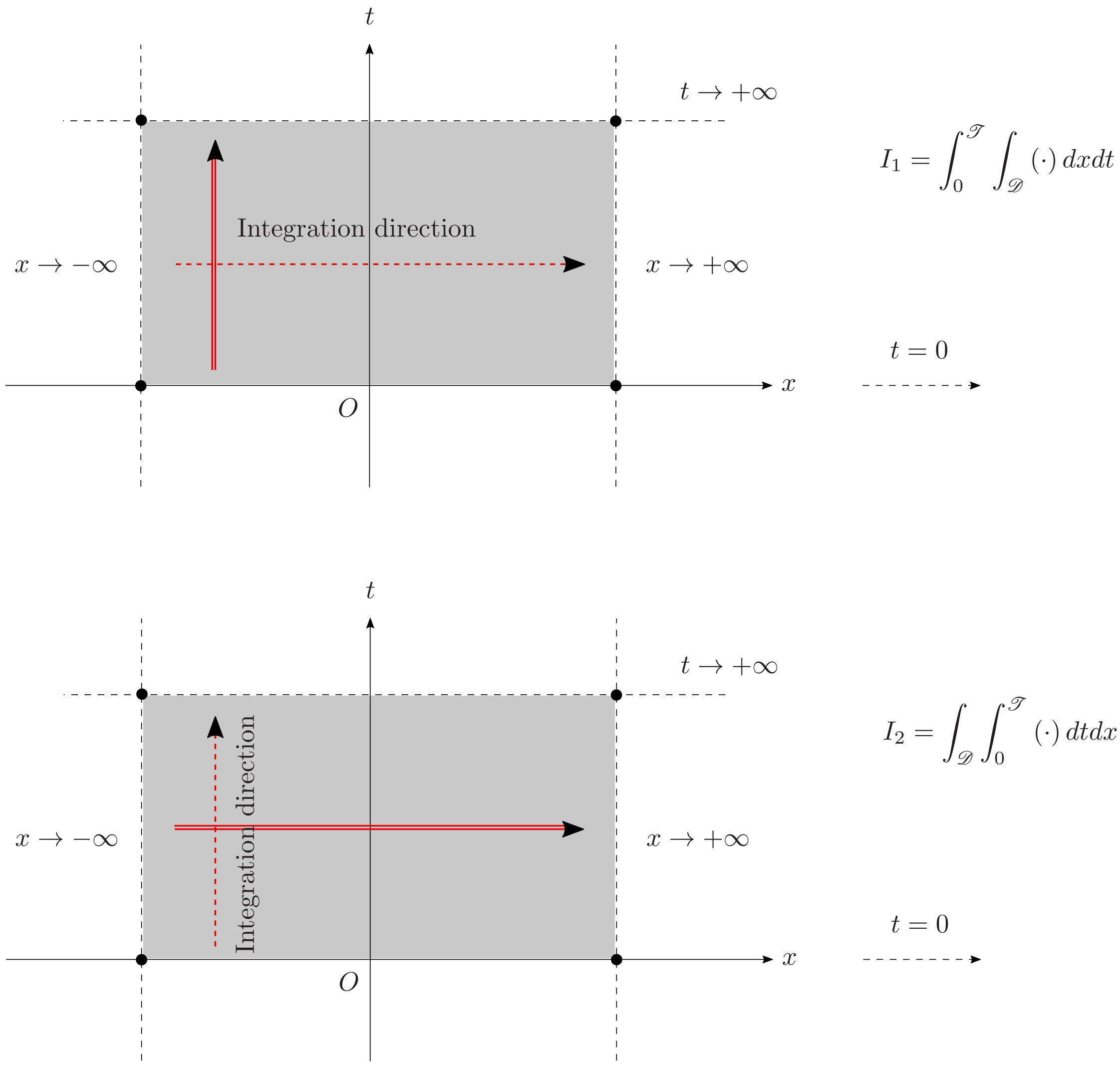


Figure 1: Indifference of limits when swapping order of integrals, i.e. $I_1 = I_2$.

3 Remark on step-by-step derivation: Weak solution of inviscid Burgers' equation in Riemann's problem

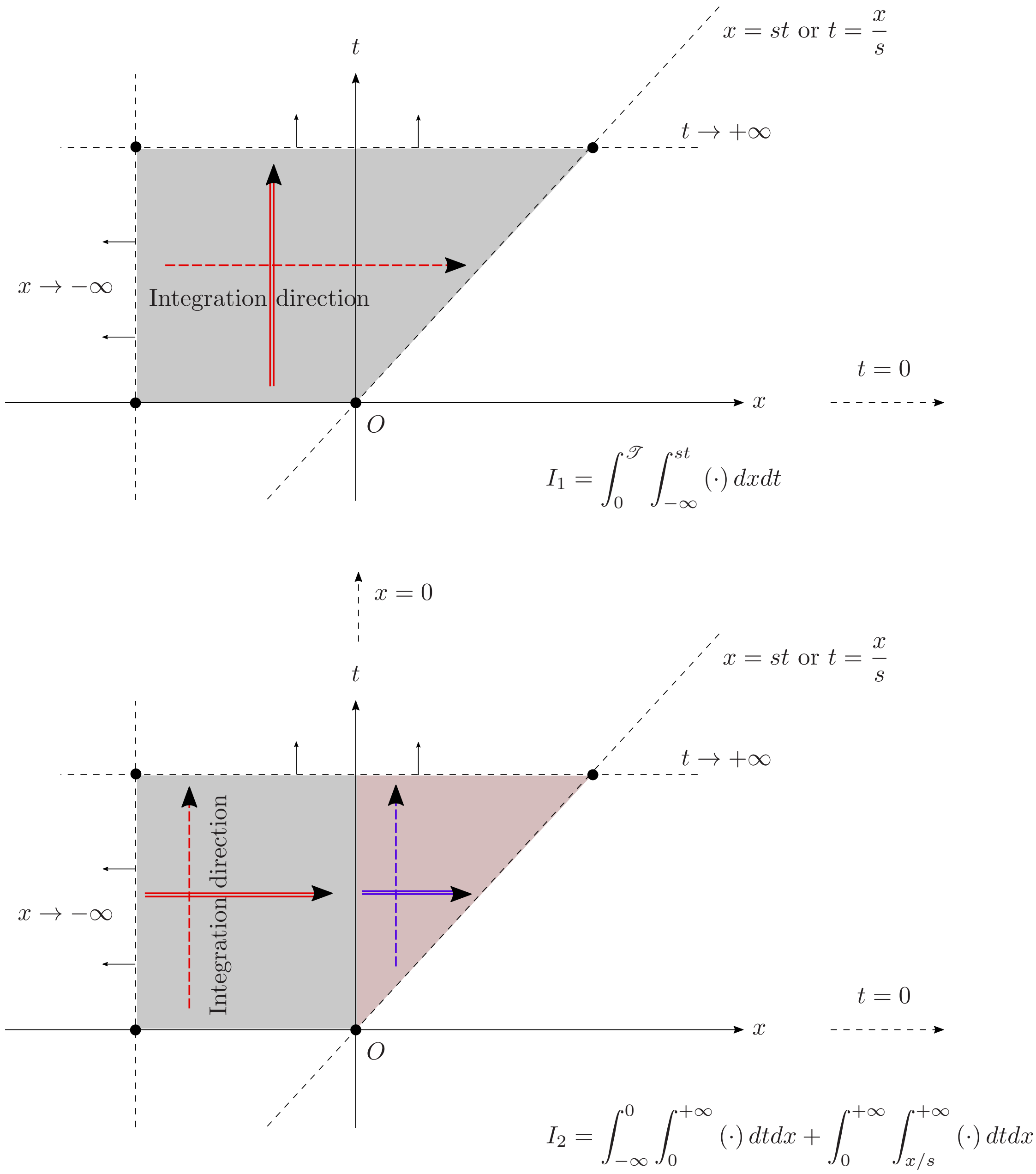
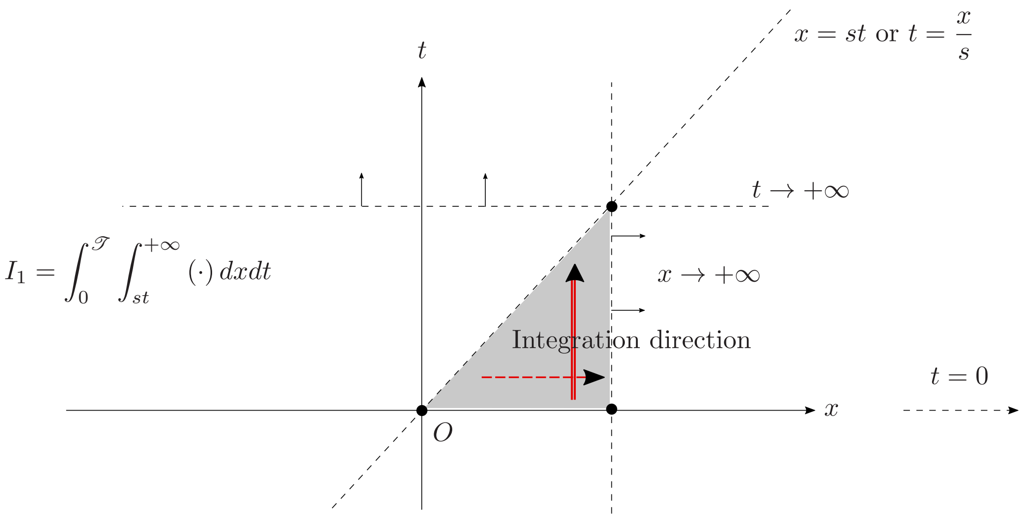


Figure 2: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee $I_1 = I_2$.

Example 3. *Given Burgers' equation with initial condition*

$$u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$



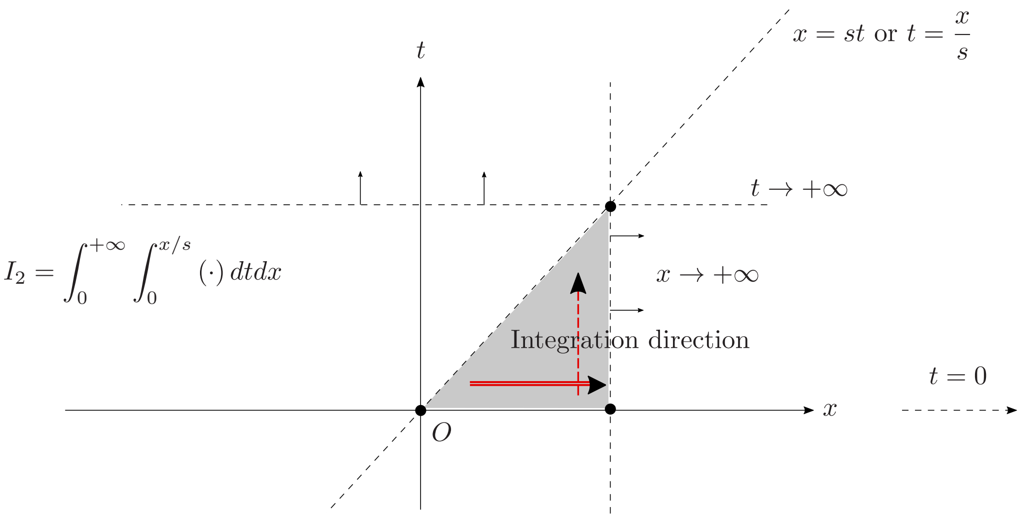


Figure 3: Difference of limits when swapping order of integrals is observed, i.e. the corresponding limits have to be adjusted in order to guarantee $I_1 = I_2$.

Then

$$u(x,t) = \begin{cases} u_L, & x < st, \\ u_R, & x > st, \end{cases}$$

is the weak solution to Burgers' equation.

Proof. RHS term

$$\begin{aligned} RHS &= -\int_{\mathscr{D}} \varphi(x,0) u_0(x) dx \\ &= -u_L \int_{-\infty}^{0} \varphi(x,0) dx - u_R \int_0^{+\infty} \varphi(x,0) dx \end{aligned}$$

LHS term

$$
\begin{aligned}
LHS = &\int_0^{\mathscr{T}}\int_{-\infty}^{st}\varphi_t u\,dxdt + \int_0^{\mathscr{T}}\int_{st}^{+\infty}\varphi_t u\,dxdt \\
&+ \int_0^{\mathscr{T}}\int_{-\infty}^{st}\varphi_x f(u)\,dxdt + \int_0^{\mathscr{T}}\int_{st}^{+\infty}\varphi_x f(u)\,dxdt \\
= &\, u_L\int_0^{\mathscr{T}}\int_{-\infty}^{st}\varphi_t\,dxdt + u_R\int_0^{\mathscr{T}}\int_{st}^{+\infty}\varphi_t\,dxdt \\
&+ \frac{u_L^2}{2}\int_0^{\mathscr{T}}\int_{-\infty}^{st}\varphi_x\,dxdt + \frac{u_R^2}{2}\int_0^{\mathscr{T}}\int_{st}^{+\infty}\varphi_x\,dxdt \\
= &\left(u_L\int_{-\infty}^{0}\int_0^{+\infty}\varphi_t\,dtdx + u_L\int_0^{+\infty}\int_{x/s}^{+\infty}\varphi_t\,dtdx\right) \\
&+ u_R\int_0^{+\infty}\int_0^{x/s}\varphi_t\,dtdx \\
&+ \frac{u_L^2}{2}\int_0^{\mathscr{T}}\int_{-\infty}^{st}\varphi_x\,dxdt + \frac{u_R^2}{2}\int_0^{\mathscr{T}}\int_{st}^{+\infty}\varphi_x\,dxdt \\
= &\, u_L\int_{-\infty}^{0}\varphi|_{t=0}^{t=+\infty}\,dx + u_L\int_0^{+\infty}\varphi|_{t=x/s}^{t=+\infty}\,dx + u_R\int_0^{+\infty}\varphi|_{t=0}^{t=x/s}\,dx \\
&+ \frac{u_L^2}{2}\int_0^{+\infty}\varphi|_{x=-\infty}^{x=st}\,dt + \frac{u_R^2}{2}\int_0^{+\infty}\varphi|_{x=st}^{x=+\infty}\,dt \\
= &\, u_L\int_{-\infty}^{0}(-\varphi(x,0))\,dx + u_L\int_0^{+\infty}(-\varphi(x,x/s))\,dx \\
&+ u_R\int_0^{+\infty}(\varphi(x,x/s)-\varphi(x,0))\,dx \\
&+ \frac{u_L^2}{2}\int_0^{+\infty}\varphi(st,t)dt + \frac{u_R^2}{2}\int_0^{+\infty}(-\varphi(st,t))\,dt \\
= &-u_L\int_{-\infty}^{0}\varphi(x,0)dx - u_R\int_0^{+\infty}\varphi(x,0)dx \\
&-(u_L-u_R)\int_0^{+\infty}\varphi(x,x/s)dx + (u_L-u_R)\int_0^{+\infty}\varphi(w,w/s)dw \\
= &-u_L\int_{-\infty}^{0}\varphi(x,0)dx - u_R\int_0^{+\infty}\varphi(x,0)dx = RHS
\end{aligned}
$$

Note in passing that we have used the following relation

$$
\begin{aligned}
\frac{u_L^2}{2}\int_0^{+\infty}\varphi(st,t)dt + \frac{u_R^2}{2}\int_0^{+\infty}(-\varphi(st,t))\,dt &= \frac{u_L^2-u_R^2}{2}\int_0^{+\infty}\varphi(st,t)\,dt \\
&= \frac{u_L^2-u_R^2}{2s}\int_0^{+\infty}\varphi(w,w/s)dw \\
&= (u_L-u_R)\int_0^{+\infty}\varphi(w,w/s)dw
\end{aligned}
$$

where $w := st$ and $dw = sdt$, and the RH condition for *Burgers*' equation $s = 1/2(u_L+u_R)$. □

4 Consistency order

Example 4. *Leapfrog scheme for the linear advection equation reads*

$$u_j^{n+1} = u_j^{n-1} + \frac{a\Delta t}{\Delta x}\left(u_{j-1}^n - u_{j+1}^n\right).$$

Show that the local consistency error $L_{\Delta t}$

$$\|L_{\Delta t}(\cdot, t)\| \leq C(\Delta t)^2 \qquad \text{for } \Delta t \to 0,$$

hold for a linear solution and $\Delta t/\Delta x = const.$

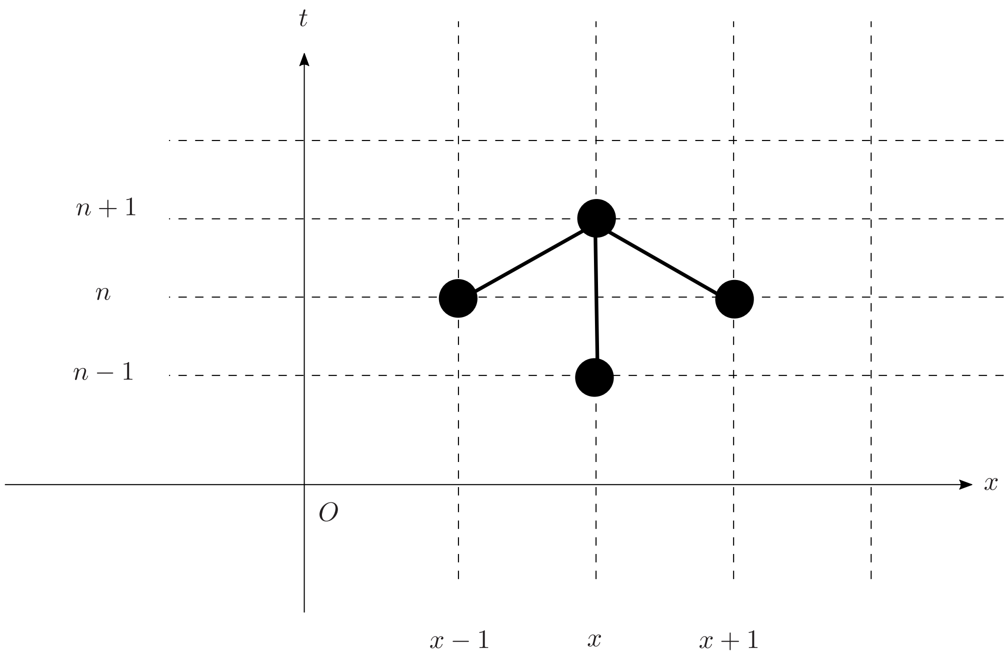


Figure 4: Leapfrog.

Consider the local truncation error

$$L_{\Delta t}(x,t) = \frac{1}{\Delta t}\left[u(x, t+\Delta t) - H_{\Delta t}(u(\cdot, t), x)\right],$$

which reads for the considered problem

$$L_{\Delta t}(x,t) = \frac{1}{\Delta t}\Big[u(x, t+\Delta t) - \Big(u(x, t-\Delta t) + \frac{a\Delta t}{\Delta x}\left(u(x-\Delta x, t) - u(x+\Delta x, t)\right)\Big)\Big].$$

Applying Taylor's expansion for $u(x, t+\Delta t)$

$$\begin{aligned} u(x, t+\Delta t) &= u(x,t) + u_t(x,t)\Delta t + u_{tt}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\ &= u(x,t) - a u_x(x,t)\Delta t + a^2 u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3). \end{aligned}$$

Applying Taylor's expansion for $u(x, t - \Delta t)$

$$\begin{aligned}
u(x, t-\Delta t) &= u(x,t) - u_t(x,t)\Delta t + u_{tt}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\
&= u(x,t) + au_x(x,t)\Delta t + a^2 u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3).
\end{aligned}$$

Furthermore, applying Taylor's expansion for $u(x + \Delta x, t)$ and $u(x - \Delta x, t)$

$$\begin{aligned}
u(x+\Delta x, t) &= u(x,t) + u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3) \\
u(x-\Delta x, t) &= u(x,t) - u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3).
\end{aligned}$$

Local truncation error

$$\begin{aligned}
L_{\Delta t}(x,t) = \frac{1}{\Delta t}\Big[& u(x,t) - au_x(x,t)\Delta t + a^2u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\
& - \Big(u(x,t) + au_x(x,t)\Delta t + a^2u_{xx}(x,t)\frac{(\Delta t)^2}{2} + \mathcal{O}((\Delta t)^3) \\
& + \frac{a\Delta t}{\Delta x}\Big(\Big(u(x,t) - u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3)\Big) \\
& \qquad - \Big(u(x,t) + u_x(x,t)\Delta x + u_{xx}(x,t)\frac{(\Delta x)^2}{2} + \mathcal{O}((\Delta x)^3)\Big)\Big)\Big)\Big] \\
= \frac{1}{\Delta t}\Big[& -2au_x(x,t)\Delta t + \mathcal{O}((\Delta t)^3) - \frac{a\Delta t}{\Delta x}\Big(-2u_x(x,t)\Delta x + \mathcal{O}((\Delta x)^3)\Big)\Big] \\
&= \frac{1}{\Delta t}\mathcal{O}((\Delta t)^3) \\
&= \mathcal{O}((\Delta t)^2).
\end{aligned}$$

Therefore, order of consistency for Leapfrog scheme is of order 2

$$\|L_{\Delta t}(\cdot, t)\| \leq C(\Delta t)^2 \qquad \text{for } \Delta t \to 0.$$

Note in passing that $u_t = -au_x$ and $u_{tt} = a^2u_{xx}$ are from the linear advection equation.