Global Exercise - 15

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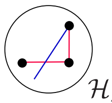
02$^{\text{nd}}$ February 2022

1 A small note about monotonicity: *Upwind* scheme

Example 1. *Examine monotonicity of* $\mathcal{H}_{\Delta t}$ *for conservation law* $u_t + f(u)_x = 0$.



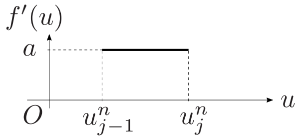
$(a > 0)$ CFL: $0 \le f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_{j-1}^n) - f(u_j^n)\Big)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \frac{\Delta t}{\Delta x} f'(u_{j-1}^n)$$

$$B_1 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 - \frac{\Delta t}{\Delta x} f'(u_j^n)$$

$$C_1 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = 0$$



$\mathcal{H}_{\Delta t}$ Monotonically increasing?

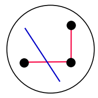
$A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0$ ✓

$B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ ✓

$\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$A_1 \le 0 \Leftrightarrow f'(u_{j-1}^n) \le 0$ $\lightning$

$B_1 \le 0 \Leftrightarrow f'(u_j^n) \ge \Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ $\lightning$



$(a < 0)$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_{j-1}^n) - f(u_j^n)\Big)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \frac{\Delta t}{\Delta x} f'(u_{j-1}^n)$$

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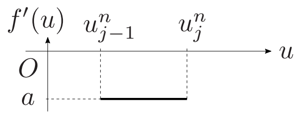

$\mathcal{H}_{\Delta t}$ Monotonically increasing?

$A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0$ $\lightning$

$B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t$ ✓

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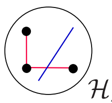


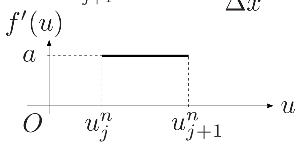
$(a > 0)$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_j^n) - f(u_{j+1}^n)\Big)$$

$$A_2 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = 0$$

$$B_2 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 + \frac{\Delta t}{\Delta x} f'(u_j^n)$$

$$C_2 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = -\frac{\Delta t}{\Delta x} f'(u_{j+1}^n)$$



$\mathcal{H}_{\Delta t}$ Monotonically increasing?

$B_2 \ge 0 \Leftrightarrow f'(u_j^n) \ge -\Delta x/\Delta t$ ✓

$C_2 \ge 0 \Leftrightarrow f'(u_{j+1}^n) \le 0$ $\lightning$

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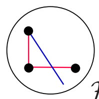

$(a < 0)$ CFL: $0 \le -f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_j^n) - f(u_{j+1}^n)\Big)$$

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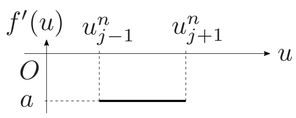

$\mathcal{H}_{\Delta t}$ Monotonically increasing?

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$C_2 \ge 0 \Leftrightarrow f'(u_{j+1}^n) \le 0$ ✓

$\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$B_2 \le 0 \Leftrightarrow f'(u_j^n) \le -\Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ $\lightning$

$C_2 \le 0 \Leftrightarrow f'(u_{j+1}^n) \ge 0$ $\lightning$

Figure 1: Monotonicity of *Upwind* scheme: 8 different cases.

2 Structure of FVM

Example 2. *Monotone* → L_1*-Contracting* → *TVD* → *Monotonicity-Preserving.*

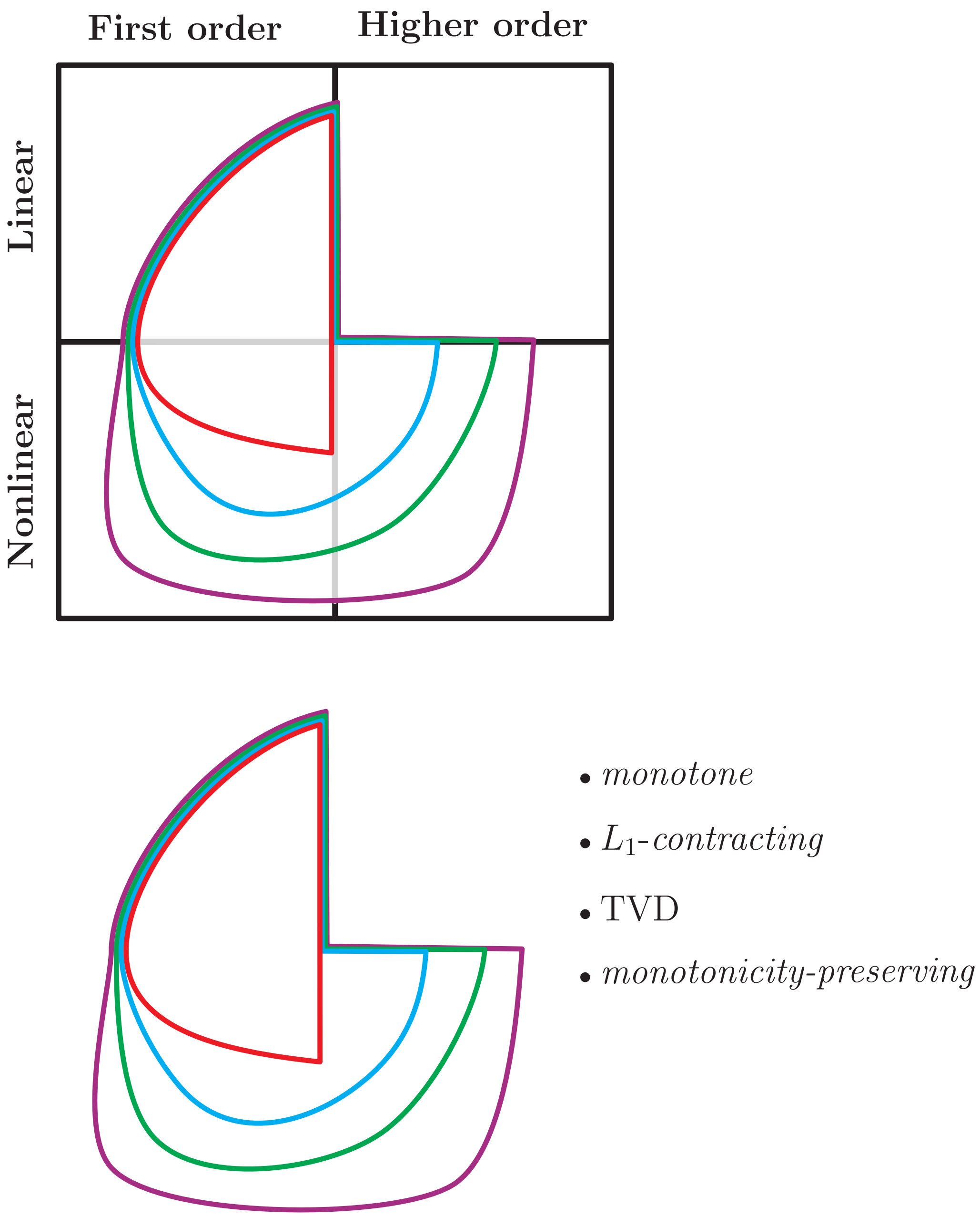


Figure 2: Structure of Finite Volume Method [Lecture note page 115].

3 Limiter

Example 3. *Examine the* 1^{st}*-order-converged LF and the* 2^{nd}*-order-converged LW.*

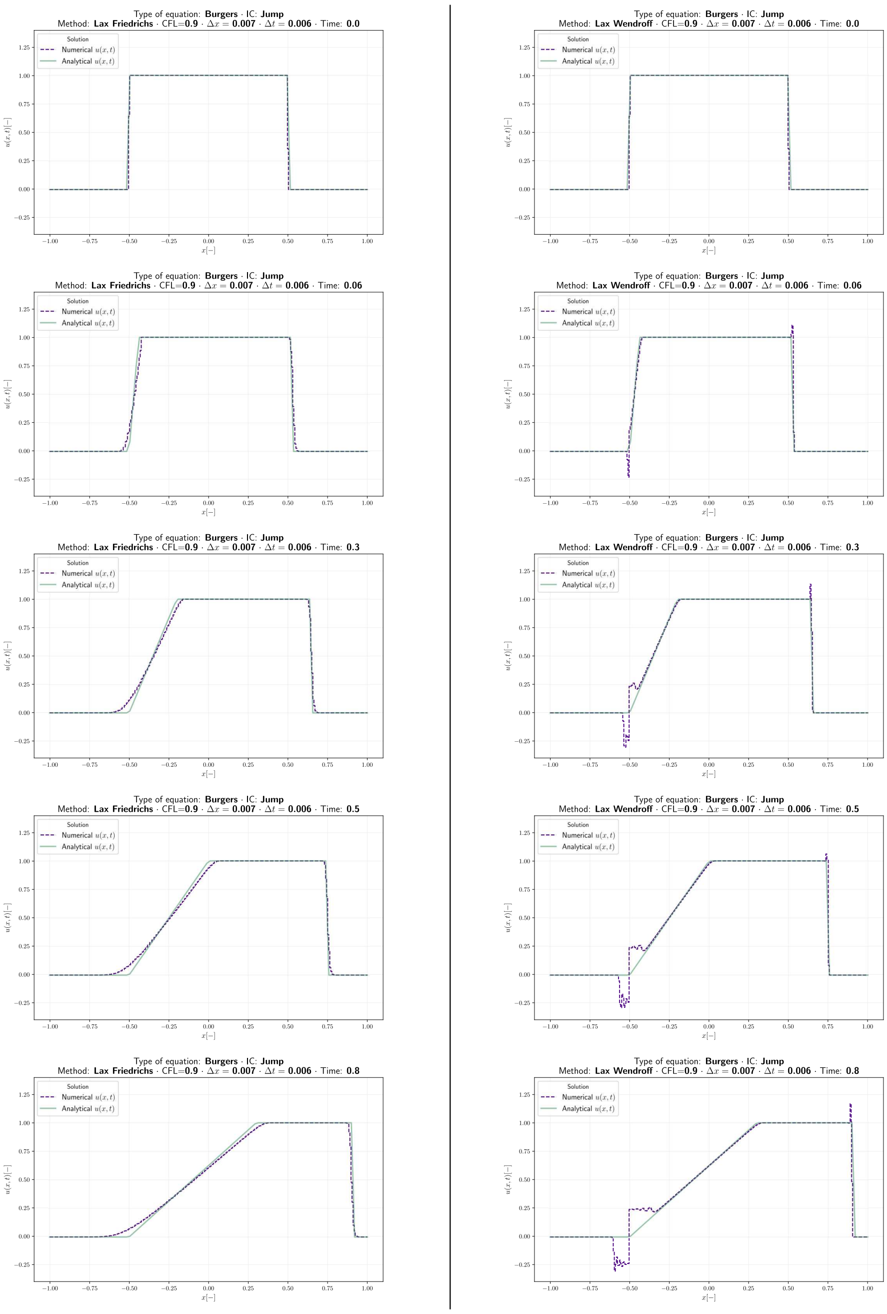


Figure 3: Oscillatory phenomena around discontinuity: (left) none oscillation founded in *Lax-Friedrichs*; (right) oscillation observed in *Lax-Wendroff*.

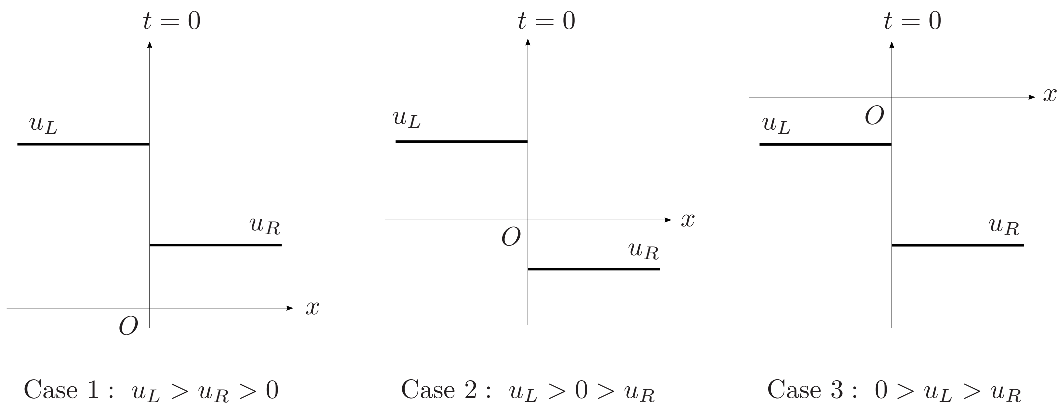
4 Insight of Godunov's solver

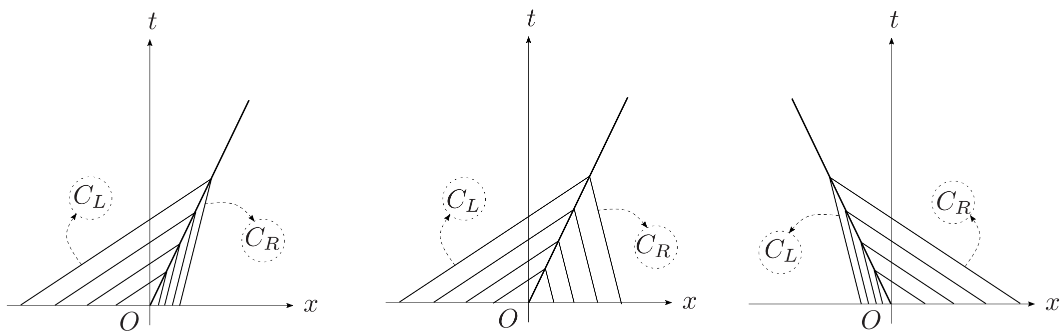
Example 4. *Examine the Godunov numerical flux function*

$$g(u_L, u_R) = \begin{cases} \min\limits_{u_L \le u \le u_R} f(u) & \text{if } u_L \le u_R, \\ \max\limits_{u_R \le u \le u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

by considering all possible cases.

Example 5. *Examine* $u_L > u_R$.





$$(C_L): \ t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad\qquad (C_R): \ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

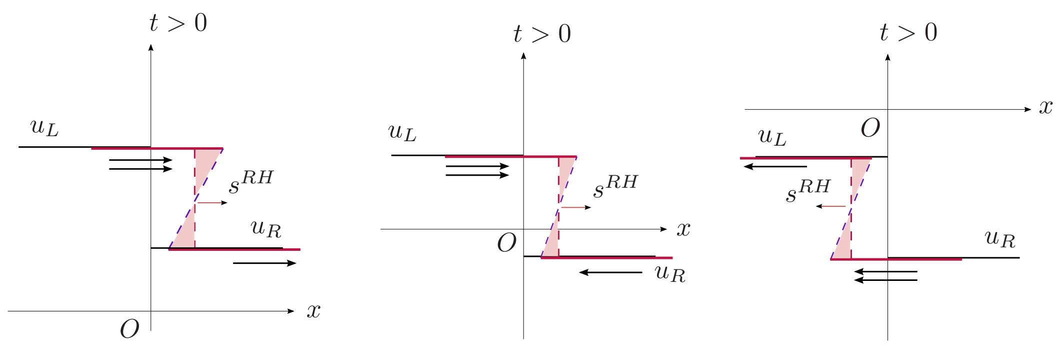
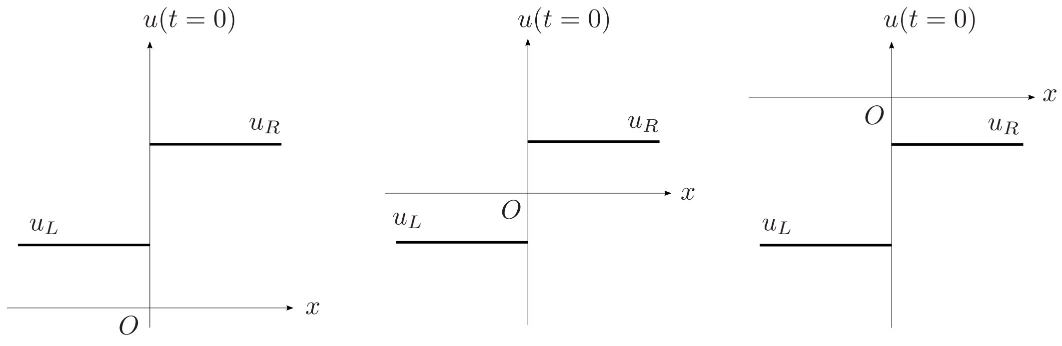


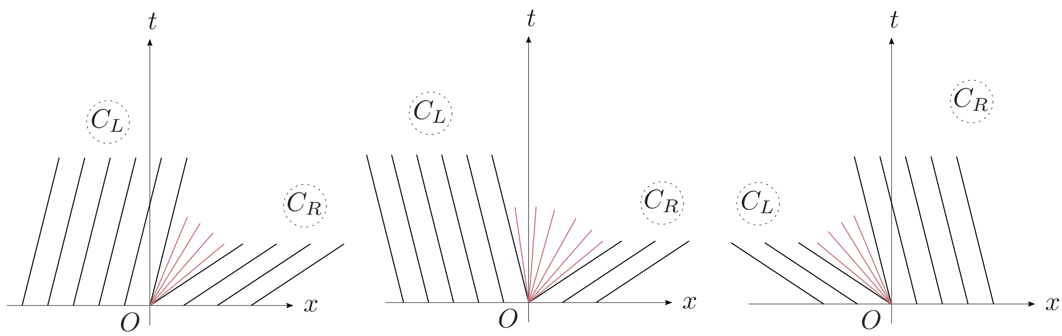
Figure 4: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

Schock solution

Example 6. *Examine* $u_L < u_R$.



Case 1 : $0 < u_L < u_R$ Case 2 : $u_L < 0 < u_R$ Case 3 : $u_L < u_R < 0$



$$(C_L): \ t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad (C_R): \ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

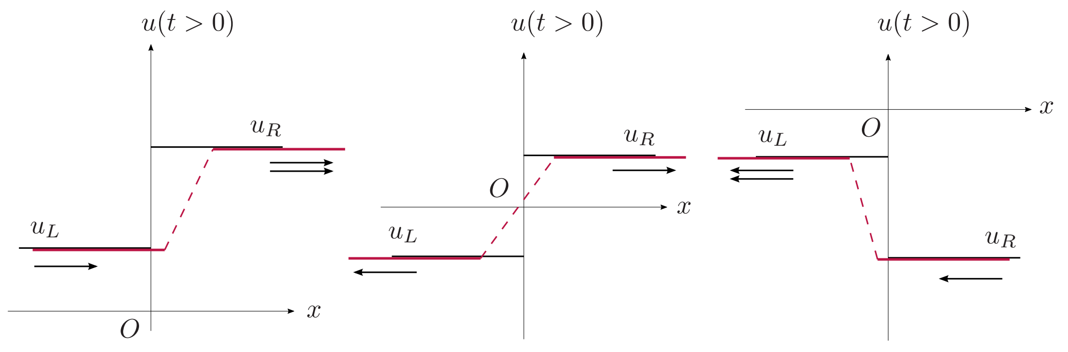


Figure 5: Riemann problem with $u_L < u_R$: IC, Characteristics, Solution.

Rarefaction solution

Example 7. *Consider the* Godunov*'s solver for the conservation law $u_t + f(u)_x = 0$ with convex and nonlinear flux function f, e.g. the flux function in* Burgers*' equation, where*

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(g(u_{j-1}^n, u_j^n) - g(u_j^n, u_{j+1}^n)\right),$$

and the numerical flux function

$$g(u_L, u_R) = \begin{cases} \min\limits_{u_L \le u \le u_R} f(u) & \text{if } u_L \le u_R, \\ \max\limits_{u_R \le u \le u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

1. *Show that the numerical flux function is* monotone.
2. *Rewrite the scheme in the* incremental form.
3. *Show that the scheme has the* total-variation-diminishing *(TVD) property.*

Approach:

1. *Proof.* In order to show the monotonicity of the numerical flux function, it is sufficient to show the following relations

$$\boxed{\begin{cases} \partial_u g(u,v) \ge 0, \\ \partial_v g(u,v) \le 0. \end{cases}} \tag{1}$$

Hence, in case of *Burgers*' equation, where $f'(u) = u$, we proceed as follows

$$g(u_L, u_R) = \begin{cases} f(u_L), & \text{if } \begin{bmatrix} f'(u_L) \ge 0 \text{ and } f'(u_R) \ge 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] > 0, \end{bmatrix} \\ \\ f(u_R), & \text{if } \begin{bmatrix} f'(u_L) \le 0 \text{ and } f'(u_R) \le 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] < 0, \end{bmatrix} \\ \\ f(u_s), & \text{if } \quad f'(u_L) < 0 < f'(u_R), \end{cases}$$

where u_s denotes the sonic point where $f'(u_s) = 0$. Then, by taking the partial derivative of $g(u_L, u_R)$ w.r.t. u_L and u_R we arrive at the following expressions

$$\partial_{u_L} g(u_L, u_R) = \begin{cases} f'(u_L), & \text{if } \begin{bmatrix} f'(u_L) \ge 0 \text{ and } f'(u_R) \ge 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] > 0, \end{bmatrix} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_L} g(u_L, u_R) \ge 0.} \tag{2}$$

Likewise, the partial derivative of $g(u_L, u_R)$ w.r.t. u_R goes as follows

$$\partial_{u_R} g(u_L, u_R) = \begin{cases} f'(u_R), & \text{if } \begin{bmatrix} f'(u_L) \le 0 \text{ and } f'(u_R) \le 0, \\ f'(u_L) \ge 0 \ge f'(u_R) \text{ and } [f]/[u] < 0, \end{bmatrix} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_R} g(u_L, u_R) \leq 0.} \tag{3}$$

Therefore, by combining (2) and (3), and comparing them with the condition given at (1), we arrive at *q.e.d.* □

2. The incremental form reads

$$\boxed{u_j^{n+1} = u_j^n + C_{j+1/2}^n (u_{j+1}^n - u_j^n) - D_{j-1/2}^n (u_j^n - u_{j-1}^n).}$$

For any conservative Finite Volume scheme we obtain

$$C_{j+1/2} = -\lambda \frac{g_{j+1/2} - f_j}{u_{j+1} - u_j}$$
$$D_{j-1/2} = \lambda \frac{f_j - g_{j-1/2}}{u_j - u_{j-1}}$$

where $\lambda = \Delta t / \Delta x$. Herein, for the case $C_{j+1/2}$, u_L is u_j and u_R is u_{j+1}, as follows

$$C_{j+1/2} = \begin{cases} 0, \quad \text{if} \left[\begin{array}{l} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\ \\ -\lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \quad \text{if} \left[\begin{array}{l} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\ \\ -\lambda \dfrac{f(u_s) - f(u_L)}{u_R - u_L}, \quad \text{if} \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

Likewise, it goes for the case $D_{j+1/2}$ as follows

$$D_{j+1/2} = \begin{cases} \lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \quad \text{if} \left[\begin{array}{l} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\ \\ 0, \quad \text{if} \left[\begin{array}{l} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\ \\ \lambda \dfrac{f(u_R) - f(u_s)}{u_R - u_L}, \quad \text{if} \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

3. In order to show the scheme has TVD property, according to Theorem II.23 (*Harten* [1980]) from the lecture note, the following three conditions

$$\boxed{\begin{aligned} C_{j+1/2} &\geq 0, \\ D_{j+1/2} &\geq 0, \\ C_{j+1/2} + D_{j+1/2} &\leq 1, \end{aligned}}$$

must hold. Herein, the first two conditions $C_{j+1/2} \geq 0,\ D_{j+1/2} \geq 0$ can be shown by using convexity of flux function f. Besides, the third condition $C_{j+1/2} + D_{j+1/2} \leq 1$ is shown as follows

$$C_{j+1/2} + D_{j+1/2} = \begin{cases} \lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \text{ if } \left[\begin{array}{l} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{array}\right. \\ \\ -\lambda \dfrac{f(u_R) - f(u_L)}{u_R - u_L}, \text{ if } \left[\begin{array}{l} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{array}\right. \\ \\ \lambda \dfrac{f(u_R) + f(u_L) - 2f(u_s)}{u_R - u_L}, \text{ if } \quad f'(u_L) < 0 < f'(u_R). \end{cases}$$

The $C_{j+1/2} + D_{j+1/2} \leq 1$ leads to the CFL condition for *Godunov*'s scheme. For λ that satisfies $C_{j+1/2} + D_{j+1/2} \leq 1$, the scheme is TVD.