Global Exercise - 14

Tuan Vo

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1 A small remark for consistent numerical flux function used in Upwind scheme - FVM

Example 1. *Determine consistent numerical flux function for Upwind scheme.*

Recall the conservation form used for FVM reads

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{f}_{j-1/2}^n - \widetilde{f}_{j+1/2}^n\right). \tag{1}$$

Meanwhile, the Upwind scheme with point-to-the-left stencils reads

$$\boxed{u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(f(u_{j-1}^n) - f(u_j^n)\right).} \tag{2}$$

Next, we would like to write (2) in terms of (1). Since the two formulations are already identical, the numerical flux function $\widetilde{f}_{j-1/2}^n$ and $\widetilde{f}_{j+1/2}^n$ are recognized directly as follow

$$\widetilde{f}_{j-1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_{j-1}^n, u_j^n\right) = f(u_{j-1}^n), \tag{3}$$
$$\widetilde{f}_{j+1/2}^n = \widetilde{f}_{j+1/2}^n\left(u_j^n, u_{j+1}^n\right) = f(u_j^n). \tag{4}$$

Then, the consistency of numerical flux functions are checked as follows

$$\widetilde{f}_{j-1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{5}$$
$$\widetilde{f}_{j+1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{6}$$

which is automatically satisfied $\forall \beta \in \mathbb{R}$ in case of constant flow. Likewise, in case of the Upwind scheme with point-to-the-right stencils, we obtain the same structure, as follows

$$\boxed{u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(f(u_j^n) - f(u_{j+1}^n)\right),} \tag{7}$$

which is the Upwind scheme with point-to-the-right stencils. Next, since the formulation (7) is already identical with (1), the numerical flux function $\widetilde{f}_{j-1/2}^n$ and $\widetilde{f}_{j+1/2}^n$ are recognized directly as follow

$$\widetilde{f}_{j-1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_{j-1}^n, u_j^n\right) = f(u_j^n), \tag{8}$$
$$\widetilde{f}_{j+1/2}^n = \widetilde{f}_{j+1/2}^n\left(u_j^n, u_{j+1}^n\right) = f(u_{j+1}^n). \tag{9}$$

Checking the consistency of (8) and (9) is similar with (3) and (4), as follows

$$\widetilde{f}_{j-1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{10}$$
$$\widetilde{f}_{j+1/2}^n(\beta,\beta) = f(\beta), \quad \checkmark \tag{11}$$

which is automatically satisfied $\forall \beta \in \mathbb{R}$ in case of constant flow.

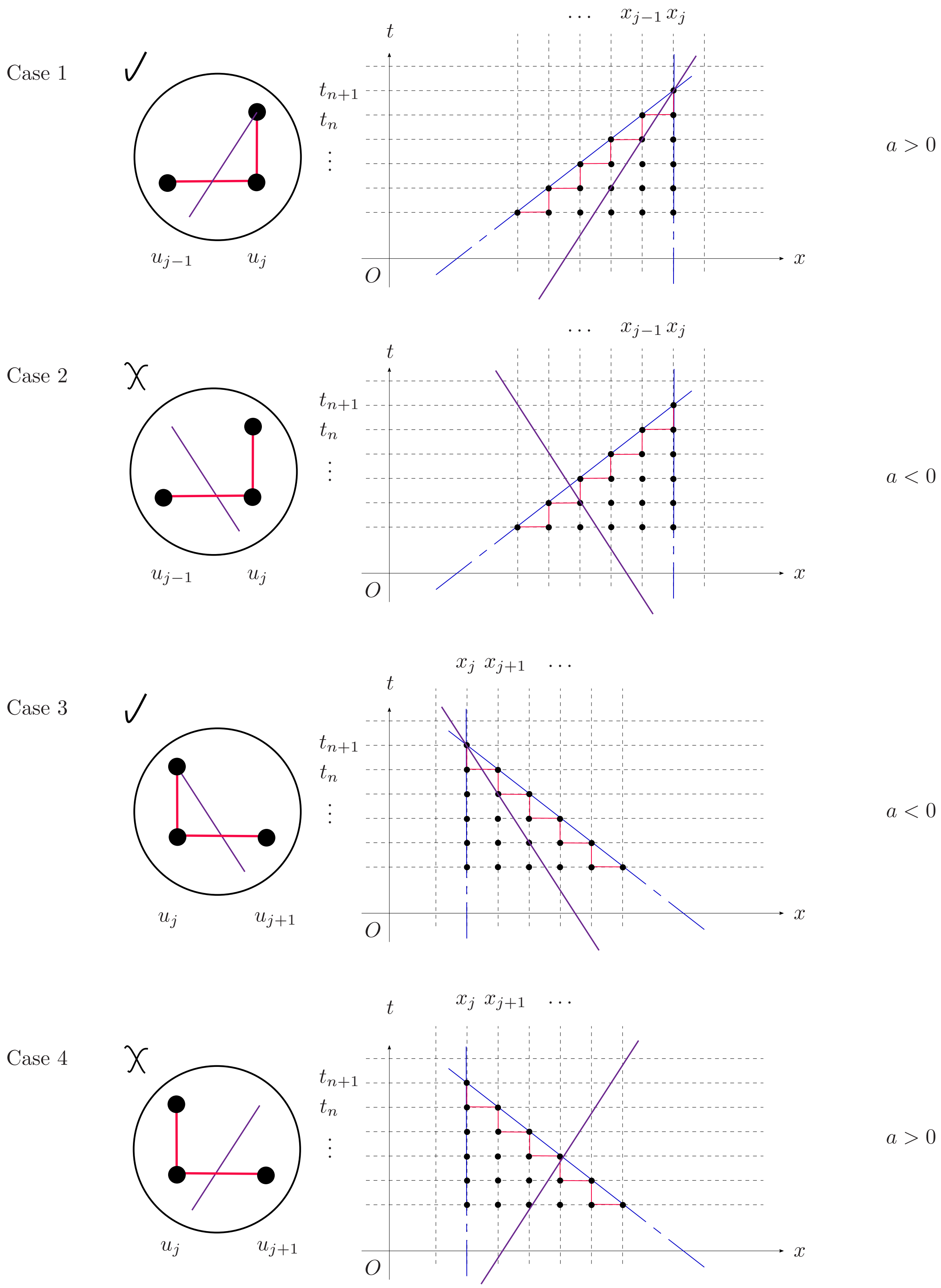


Figure 1: Correlation between coefficient a, point-to-the-left stencil, and point-to-the-right stencil of upwind scheme.

Example 2. *Comparison of numerical solutions between left-pointing and right-pointing stencils of Upwind scheme in case of $a > 0$.*

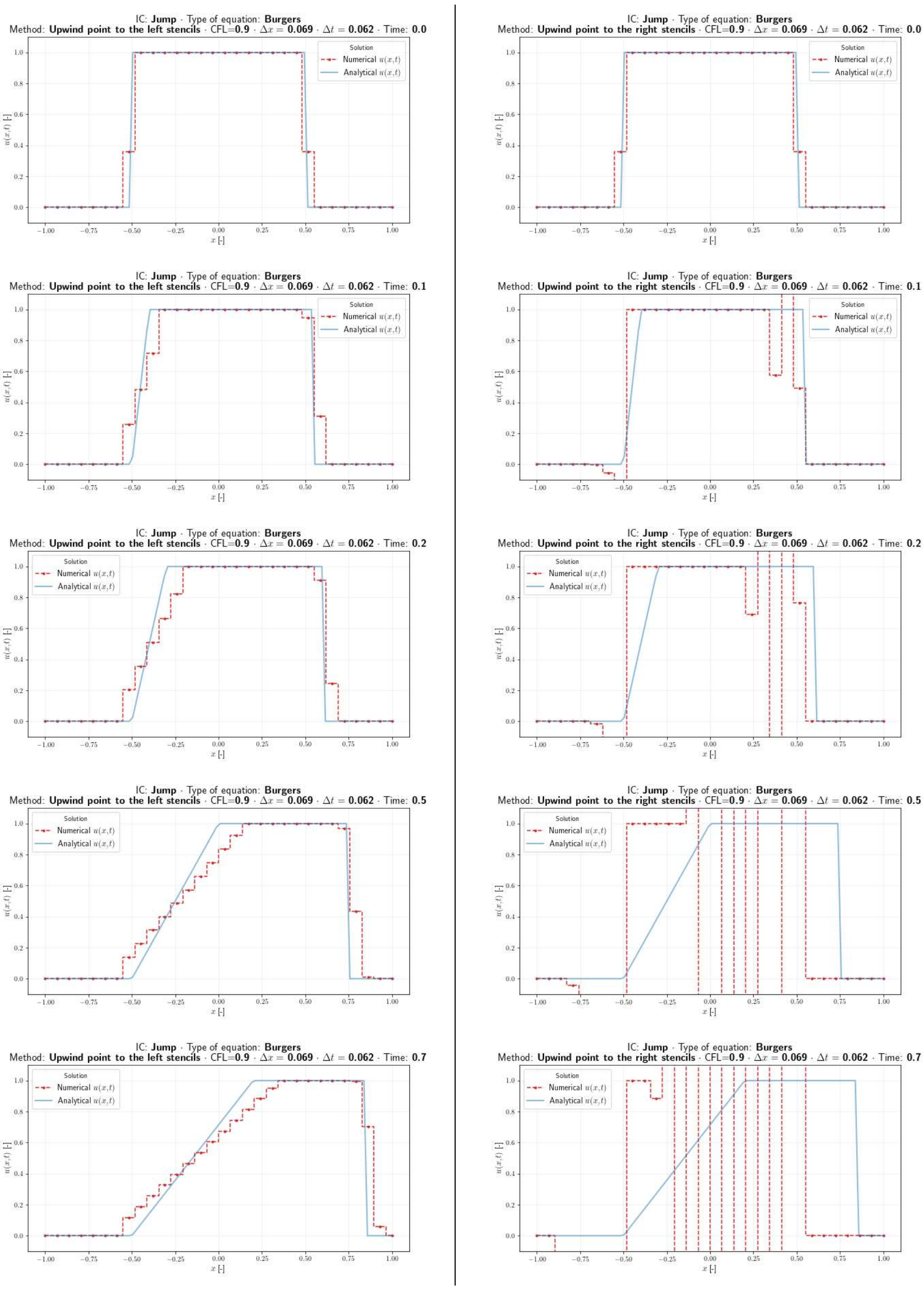


Figure 2: *Upwind* scheme: In case of $a > 0$ the scheme whose stencils are pointing to the left (Case 1 in Figure 1) is the suitable choice, compared to the case whose stencils are pointing to the right (Case 4 in Figure 1).

2 A small remark for consistent numerical flux function used in Lax-Friedrichs scheme - FVM

Example 3. *Determine consistent numerical flux function for Lax-Friedrichs scheme.*

Recall the conservation form used for FVM reads

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{f}_{j-1/2}^n - \widetilde{f}_{j+1/2}^n\right). \tag{12}$$

Meanwhile, the Lax-Friedrichs scheme reads

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right). \tag{13}$$

Next, we would like to write (13) in terms of (12). The derivation is done by some algebraic manipulations, as follows

$$\begin{aligned}
u_j^{n+1} &= \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) \\
&= u_j^n + \frac{1}{2}u_{j-1}^n - \frac{1}{2}u_j^n + \frac{1}{2}u_{j+1}^n - \frac{1}{2}u_j^n - \frac{\Delta t}{2\Delta x}f(u_{j+1}^n) + \frac{\Delta t}{2\Delta x}f(u_{j-1}^n) \\
&= u_j^n + \frac{\Delta t}{\Delta x}\Bigg(\frac{\Delta x}{2\Delta t}u_{j-1}^n - \frac{\Delta x}{2\Delta t}u_j^n + \frac{\Delta x}{2\Delta t}u_{j+1}^n - \frac{\Delta x}{2\Delta t}u_j^n \\
&\qquad\qquad -\frac{1}{2}f(u_{j+1}^n) + \frac{1}{2}f(u_{j-1}^n)\Bigg) \\
&= u_j^n + \frac{\Delta t}{\Delta x}\Bigg(\underbrace{\left(\frac{\Delta x}{2\Delta t}u_{j-1}^n - \frac{\Delta x}{2\Delta t}u_j^n + \frac{1}{2}f(u_{j-1}^n)\right)}_{=:\widetilde{f}_{j-1/2}^n} \\
&\qquad\qquad - \underbrace{\left(\frac{\Delta x}{2\Delta t}u_j^n - \frac{\Delta x}{2\Delta t}u_{j+1}^n + \frac{1}{2}f(u_{j+1}^n)\right)}_{=:\widetilde{f}_{j+1/2}^n}\Bigg) \\
&= u_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{f}_{j-1/2}^n - \widetilde{f}_{j+1/2}^n\right),
\end{aligned} \tag{14}$$

which confirms that the *Lax-Friedrichs* scheme given at (13) is able to be written in the *conservation form* with the numerical flux function recognized as follows

$$\widetilde{f}_{j-1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_{j-1}^n, u_j^n\right) = \frac{\Delta x}{2\Delta t}\left(u_{j-1}^n - u_j^n\right) + \frac{1}{2}f(u_{j-1}^n) \tag{15}$$

$$\widetilde{f}_{j+1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_j^n, u_{j+1}^n\right) = \frac{\Delta x}{2\Delta t}\left(u_j^n - u_{j+1}^n\right) + \frac{1}{2}f(u_{j+1}^n) \tag{16}$$

However, these numerical flux functions $\widetilde{f}(\cdot,\cdot)$ are not consistent with the original flux function $f(\cdot)$, which can be checked for the case of constant flow, as follows

$$(15) \Leftrightarrow \widetilde{f}_{j-1/2}^n(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta - \beta) + \frac{1}{2}f(\beta) = \frac{1}{2}f(\beta) \quad ↯ \tag{17}$$

$$(16) \Leftrightarrow \widetilde{f}_{j+1/2}^n(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta - \beta) + \frac{1}{2}f(\beta) = \frac{1}{2}f(\beta) \quad ↯ \tag{18}$$

which have confirmed that $\widetilde{f}^n_{j\pm1/2}\,(\beta,\beta) \neq f(\beta)$, $\forall\beta \in \mathbb{R}$. Therefore, a modified version is required for these numerical flux functions, such that they become consistent with $f(\cdot)$, and simultaneously, the summation of these two fluxes do not lead to any change in the *Lax-Friedrichs* scheme. The consistent numerical flux functions for *Lax-Friedrichs* scheme reads

$$\widetilde{f}^n_{j-1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_{j-1}, u^n_j\right) = \frac{\Delta x}{2\Delta t}\left(u^n_{j-1} - u^n_j\right) + \frac{1}{2}\left(f(u^n_{j-1}) + f(u^n_j)\right), \tag{19}$$

$$\widetilde{f}^n_{j+1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_j, u^n_{j+1}\right) = \frac{\Delta x}{2\Delta t}\left(u^n_j - u^n_{j+1}\right) + \frac{1}{2}\left(f(u^n_{j+1}) + f(u^n_j)\right), \tag{20}$$

which are obtained by adding the term $1/2f(u^n_j)$ to both $\widetilde{f}^n_{j-1/2}$ and $\widetilde{f}^n_{j+1/2}$. Note in passing that the subtraction sign between these two fluxes will cancel out this extra term, as shown in (12). Next, the consistent property is checked as follows

$$(19) \Leftrightarrow \widetilde{f}^n_{j-1/2}\,(\beta,\beta) = \frac{\Delta x}{2\Delta t}\,(\beta-\beta) + \frac{1}{2}\,(f(\beta) + f(\beta)) = f(\beta) \quad \checkmark \tag{21}$$

$$(20) \Leftrightarrow \widetilde{f}^n_{j+1/2}\,(\beta,\beta) = \frac{\Delta x}{2\Delta t}\,(\beta-\beta) + \frac{1}{2}\,(f(\beta) + f(\beta)) = f(\beta) \quad \checkmark \tag{22}$$

which have confirmed that $\widetilde{f}^n_{j\pm1/2}\,(\beta,\beta) = f(\beta)$, $\forall\beta \in \mathbb{R}$ for constant flow. Hence, the numerical flux functions $\widetilde{f}^n_{j\pm1/2}$ for the *Lax-Friedrichs* scheme take the following formulation

$$\therefore \quad \boxed{\begin{aligned} \widetilde{f}^n_{j-1/2}\left(u^n_{j-1}, u^n_j\right) &= \frac{\Delta x}{2\Delta t}\left(u^n_{j-1} - u^n_j\right) + \frac{1}{2}\left(f(u^n_{j-1}) + f(u^n_j)\right) \\ \widetilde{f}^n_{j+1/2}\left(u^n_j, u^n_{j+1}\right) &= \frac{\Delta x}{2\Delta t}\left(u^n_j - u^n_{j+1}\right) + \frac{1}{2}\left(f(u^n_j) + f(u^n_{j+1})\right) \end{aligned}} \tag{23}$$

or the generalized form used for FVM implementation reads

$$\therefore \quad \boxed{F\left(u_L, u_R\right) = \frac{1}{2}\left(f(u_L) + f(u_R)\right) + \frac{\Delta x}{2\Delta t}\left(u_L - u_R\right)} \tag{24}$$

Example 4. *Crucialness of conservation form and consistent numerical flux fcn.*

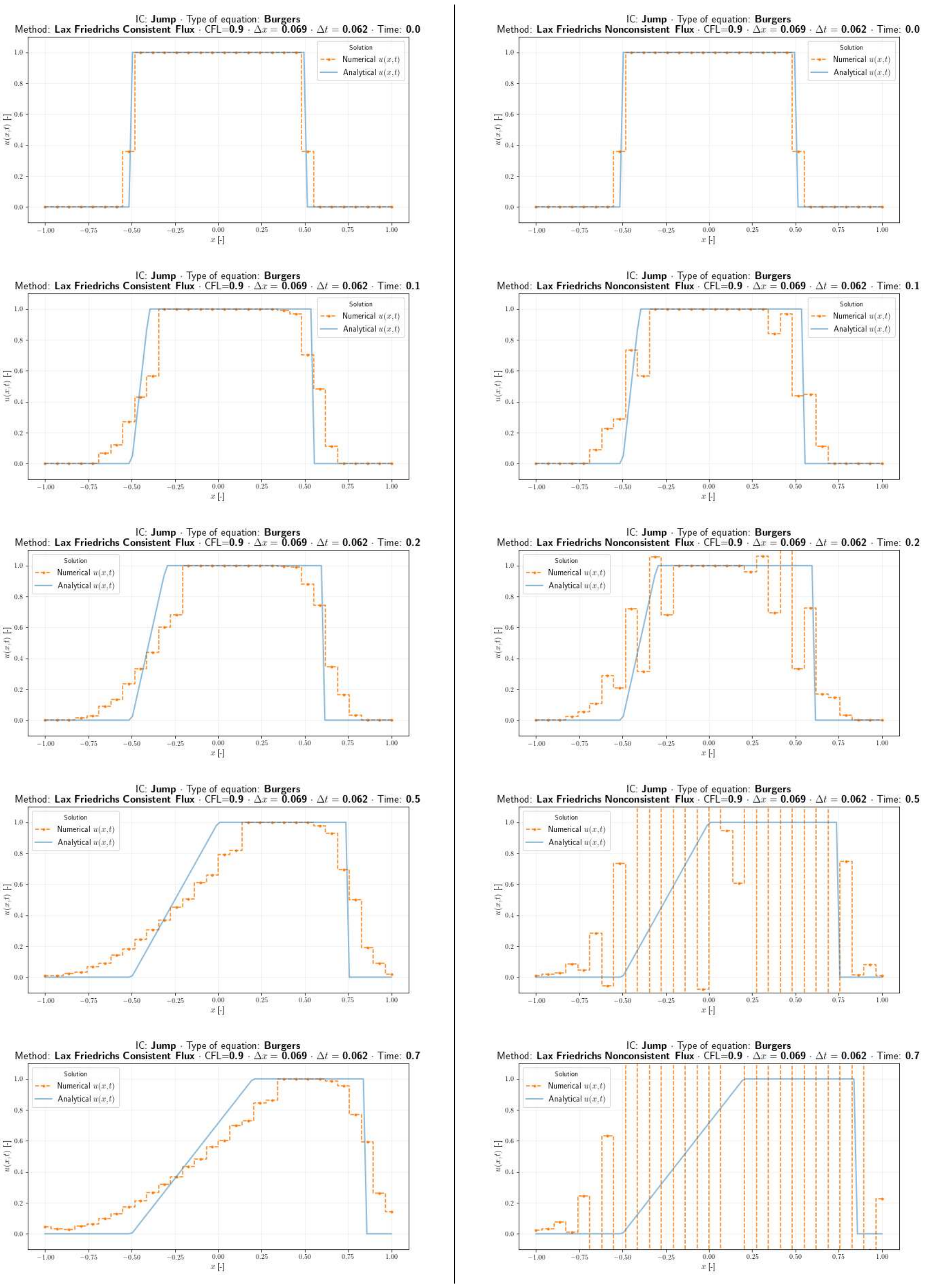


Figure 3: *Lax-Friedrichs* scheme: Consistent numerical flux function (left; based on the derived formulation in (19) and (20)) versus Non-consistent numerical flux function (right; based on the derived formulation in (15) and (16)).

3 Approximate Riemann solvers (cont.)

Example 5. *Roe's solver.*

Nonlinear *Riemann*'s problem

$$U_t + F(U)_x = 0, \quad \text{with} \quad U(x,0) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0, \end{cases} \tag{25}$$

which is written in quasi-linear form as follows

$$U_t + A(U)U_x = 0, \quad \text{with} \quad U(x,0) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0, \end{cases} \tag{26}$$

where $A(U)$ is the non-constant Jacobian matrix.

Recall the conservation form used for FVM reads

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{F}_{j-1/2}^n - \widetilde{F}_{j+1/2}^n\right). \tag{27}$$

Godunov's solver takes the intercell numerical flux function

$$\widetilde{F}_{j-1/2}^n = \widetilde{F}_{j-1/2}^n(U_L, U_R) = F_{j-1/2}^n(U^*(U_L, U_R)), \tag{28}$$

$$\widetilde{F}_{j+1/2}^n = \widetilde{F}_{j+1/2}^n(U_L, U_R) = F_{j+1/2}^n(U^*(U_L, U_R)), \tag{29}$$

where $U^*(U_L, U_R)$ is the exact *Riemann*'s solution at the interface between cells.

Instead of solving exactly the nonlinear *Riemann*'s problem above for every single space-time increment in FVM, which turns out to be costly and not efficient in general, *Roe* proposed solving the following system

$$U_t + \widehat{A}(U_L, U_R)U_x = 0, \quad \text{with} \quad U(x,0) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0, \end{cases} \tag{30}$$

where $\widehat{A}(U_L, U_R)$ is essentially a constant Jacobian matrix. The linearized system (30) easily to be solved exactly is known as *Approximate Riemann Solver*. Besides, the correlation between (30) and (25) is guaranteed by three conditions applied on matrix $\widehat{A}(U_L, U_R)$ proposed by *Roe* as follows

1. Hyperbolicity: real eigenvalues $\widehat{\lambda}_p = \widehat{\lambda}_p(U_L, U_R)$ required.
2. Consistency with the exact Jacobian matrix $\widehat{A}(U, U) = A(U)$
3. Conservation across discontinuities $F(U_R) - F(U_L) = \widehat{A}(U_L, U_R)(U_L - U_R)$

Matrix $\widehat{A}(U_L, U_R) \rightarrow$ Eigenvalues $\widehat{\lambda}_p(U_L, U_R) \rightarrow$ Eigenvectors $\widehat{r}_p(U_L, U_R)$
Consider

$$U_R - U_L = \sum_{p=1}^{m} \widehat{\alpha}_p(U_L, U_R)\widehat{\lambda}_p(U_L, U_R) \tag{31}$$

Next, interfacial values $U_{j+1/2}(x/t)$ along $t-$axis, i.e. $x/t = 0$, take the following equality

$$U_{j+1/2}(0) = U_L + \sum_{\widehat{\lambda}_p \leq 0} \widehat{\alpha}_p(U_L, U_R)\widehat{\lambda}_p(U_L, U_R) \tag{32}$$

$$U_{j+1/2}(0) = U_R - \sum_{\widehat{\lambda}_p \leq 0} \widehat{\alpha}_p(U_L, U_R)\widehat{\lambda}_p(U_L, U_R) \tag{33}$$

Besides, according to Definition II.21 from lecture note, the corresponding numerical flux function reads

Example 6. *Local Lax-Friedrichs (LLF) flux function.*

Recall that the *Roe*'s solver takes the following forms of numerical flux functions

$$
\begin{aligned}
\widetilde{F}_{j+1/2}\left(U_L, U_R\right) &= F(U_L) + \widehat{A}\left(U_L, U_R\right)^{-}\left(U_R - U_L\right), \\
\widetilde{F}_{j+1/2}\left(U_L, U_R\right) &= F(U_R) - \widehat{A}\left(U_L, U_R\right)^{+}\left(U_R - U_L\right).
\end{aligned}
$$

By summing and taking the average we obtain another form of numerical flux function in *Roe*'s solver as follows

$$
\therefore \quad \boxed{\widetilde{F}^{(R)}_{j+1/2}(U_L, U_R) = \frac{1}{2}\left(F(u_L) + F(u_R)\right) - \frac{1}{2}\left|\widehat{A}\right|\left(U_R - U_L\right)} \tag{34}
$$

Note in passing that we have used the following equality in the expression (34)

$$
\widehat{A}^{-}(U_L, U_R) - \widehat{A}^{+}(U_L, U_R) = -\left|\widehat{A}\right|(U_L, U_R). \tag{35}
$$

Besides, the local Lax-Friedrichs flux function take the following form

$$
\therefore \quad \boxed{\widetilde{F}^{(LLF)}_{j+1/2}(U_L, U_R) = \frac{1}{2}\left(F(u_L) + F(u_R)\right) - \frac{1}{2}\lambda^{\max}\left(U_R - U_L\right)} \tag{36}
$$

Note in passing that we have used an approximation for $\left|\widehat{A}\right|$ in (34) and applied it for $\lambda^{\max}$ in (36), as follows

Example 7. *Examine Local Lax-Friedrichs (LLF) versus* Roe*'s solver.*

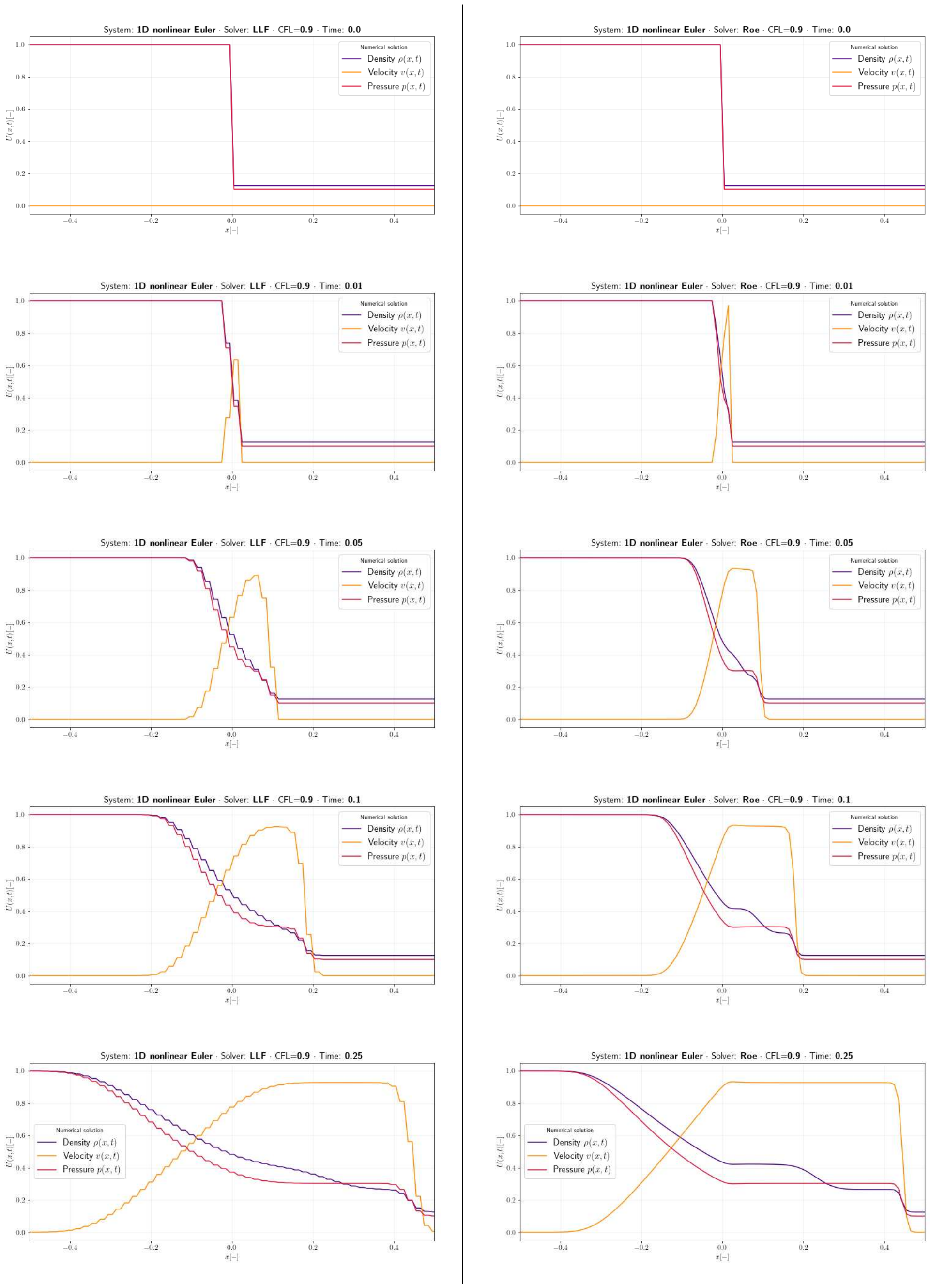


Figure 4: Comparison between LLF solver and *Roe*'s solver: Same initial conditions and parameters applied, seeking for numerical solution to 1D *Euler* equation.

Example 8. *Derivation of Harten-Lax-van Leer (HLL).*

1. Recall that the two left-and right-sided formulae for the numerical flux functions based on *Roe*'s solver are written as follows

$$\widetilde{F}_{j+1/2}\left(U_L, U_R\right) = F(U_L) + \widehat{A}\left(U_L, U_R\right)^{-}\left(U_R - U_L\right) \tag{37}$$
$$\widetilde{F}_{j+1/2}\left(U_L, U_R\right) = F(U_R) - \widehat{A}\left(U_L, U_R\right)^{+}\left(U_R - U_L\right) \tag{38}$$

   By first taking summation and then averaging the above two expressions we arrive at

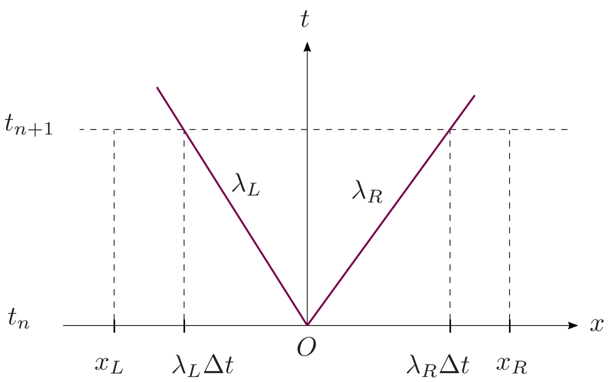
$$\widetilde{F}_{j+1/2}\left(U_L, U_R\right) = \frac{1}{2}\left(F(U_L) + F(U_R)\right) - \frac{1}{2}\left|\widehat{A}\right|\left(U_R - U_L\right). \tag{39}$$

   Note in passing that $\widehat{A}^{-}(U_L, U_R) - \widehat{A}^{+}(U_L, U_R) = -\left|\widehat{A}\right|(U_L, U_R)$.

2. The integral formulation applied to the domain $[x_L, x_R] \times [t_n, t_{n+1}]$ and together with the condition $x_L < 0 < x_R$ reads

$$\begin{aligned}\int_{x_L}^{x_R} U(x, t_{n+1})\, dx = &\int_{x_L}^{x_R} U(x, t_n)\, dx \\ &+ \int_{t_n}^{t_{n+1}} F(U(x_L, t))\, dt - \int_{t_n}^{t_{n+1}} F(U(x_R, t))\, dt.\end{aligned} \tag{40}$$

   Since the solution is piece-wise constant, these integrals are computed as



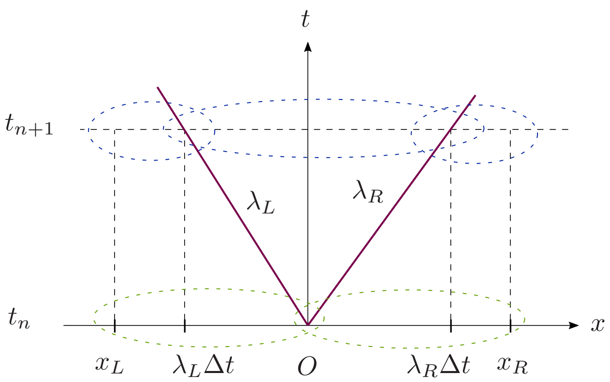


Figure 5: Integration path.

follows

$$\int_{t_n}^{t_{n+1}} F(U(x_L,t))\,dt = \cdots\,,$$
$$\int_{t_n}^{t_{n+1}} F(U(x_R,t))\,dt = \cdots\,,$$
$$\int_{x_L}^{x_R} U(x,t_n)\,dx = \cdots + \cdots\,,$$
$$\int_{x_L}^{x_R} U(x,t_{n+1})dx = \cdots + \cdots + \cdots\,.$$

Next, by substituting these four integrals back into (40) we obtain the following relation

$$U^{*} = \frac{|\lambda_L|\,U_L + \lambda_R U_R + F(U_L) - F(U_R)}{|\lambda_L| + \lambda_R}. \tag{41}$$

3. The decomposition with the two waves are given by

$$U_R - U_L = \sum_{p=1}^{N} \alpha_p \widehat{r}_p = (U_R - U^\star) + (U^\star - U_L)\,, \tag{42}$$

and hence we arrive at the following expression

$$\sum_{p=1}^{N} \alpha_p \left|\widehat{\lambda}_p\right| \widehat{r}_p = \lambda_R\,(U_R - U^\star) + |\lambda_L|\,(U^\star - U_L)\,, \tag{43}$$

which leads to the following expression

$$\widetilde{F}_{j+1/2}\,(U_L,U_R) = \frac{1}{2}\,(F(U_L)+F(U_R)) - \frac{\lambda_R}{2}\,(U_R - U^\star) - \frac{|\lambda_L|}{2}\,(U^\star - U_L)\,. \tag{44}$$

Then, by using (43), inserting (41) into (44), and applying some more algebraic manipulations we arrive at the HLL numerical flux function, where $\lambda_L < 0 < \lambda_R$, as follows

$$\widetilde{F}_{j+1/2}\,(U_L,U_R) = \frac{|\lambda_L|\,F(U_R) + \lambda_R F(U_L)}{|\lambda_L| + \lambda_R} + \frac{\lambda_R\,|\lambda_L|}{|\lambda_L| + \lambda_R}\,(U_L - U_R)\,. \tag{45}$$

4. When all waves travel to the right, we obtain

$$\lambda_{L,R} > 0 \to U^* = U_L \to F(U^*) = F(U_L). \tag{46}$$

When all waves travel to the left, we obtain

$$\lambda_{L,R} < 0 \to U^* = U_R \to F(U^*) = F(U_R). \tag{47}$$

Finally, we obtain a complete HLL numerical flux function as follows

$$\therefore \quad \boxed{\widetilde{F}^{(\mathrm{HLL})}_{j+1/2}\,(U_L,U_R) = \begin{cases} F(U_R), \quad (\lambda_{L,R} < 0), \\ \dfrac{|\lambda_L|\,F(U_R) + \lambda_R F(U_L)}{|\lambda_L| + \lambda_R} + \dfrac{\lambda_R\,|\lambda_L|}{|\lambda_L| + \lambda_R}\,(U_L - U_R)\,, \\ \hfill (\lambda_L < 0 < \lambda_R), \\ F(U_L), \quad (\lambda_{L,R} > 0). \end{cases}} \tag{48}$$

Example 9. *Examine* Roe*'s solver versus Harten-Lax-van Leer (HLL).*

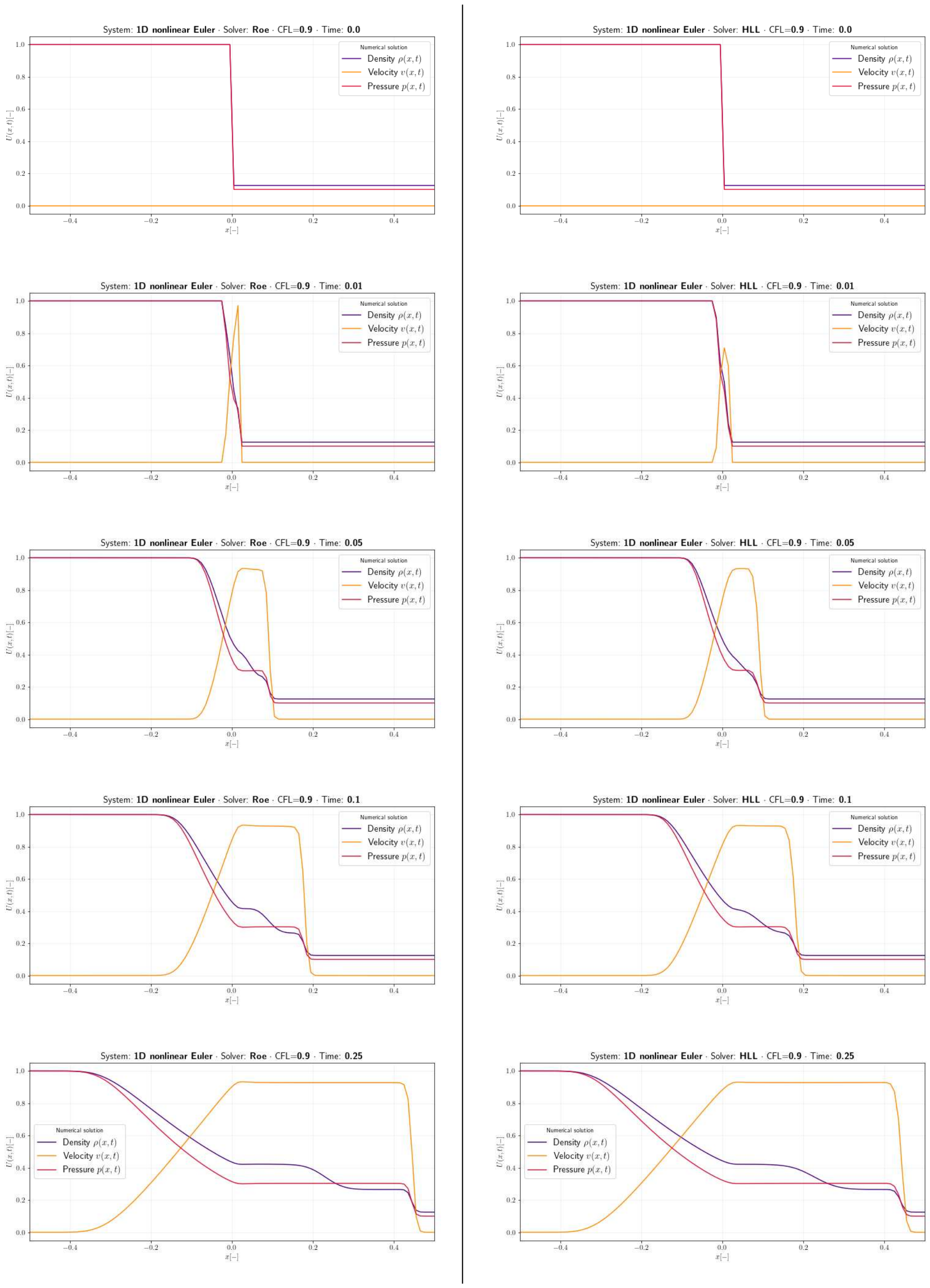


Figure 6: Comparison between *Roe*'s solver and HLL solver: Same initial conditions and parameters applied, seeking for numerical solution to 1D *Euler* equation.