Global Exercise - 13

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1 Conservation form - Finite Volume Method (cont.)

Example 1. *Derivation of conservation form.*

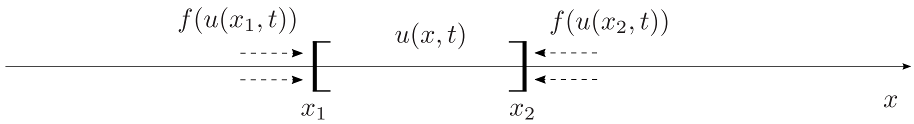


Figure 1: Conservation for a scalar conserved quantity $u(x,t)$ over domain $[x_1, x_2]$.

Starting point is the **integral form** of conservation law. Recall the integral formulation of conservation law for a scalar conserved quantity $u(x,t)$ over domain $[x_1, x_2]$, herein, reads

$$\boxed{\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t)).} \tag{1}$$

Integration both side of (1) over temporal interval $[t_1, t_2]$ yields

$$\left(\int_{x_1}^{x_2} u(x,t)\,dx\right)\Bigg|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \left(f(u(x_1,t)) - f(u(x_2,t))\right)dt, \tag{2}$$

which leads to **another integral form** of the conservation law, as follows

$$\boxed{\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt,} \tag{3}$$

which is useful to derive the conservation form for finite volume method. Then, instead of $[x_1, x_2]$ and $[t_1, t_2]$ we consider spatial interval $[x_{j-1/2}, x_{j+1/2}]$ and temporal interval $[t_n, t_{n+1}]$ as follows

$$[x_1, x_2] \to [x_{j-1/2}, x_{j+1/2}], \tag{4}$$
$$[t_1, t_2] \to [t_n, t_{n+1}]. \tag{5}$$

Hence, the integral form in (3) becomes

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,dx \\ + \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))\,dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,dt.\end{aligned} \tag{6}$$

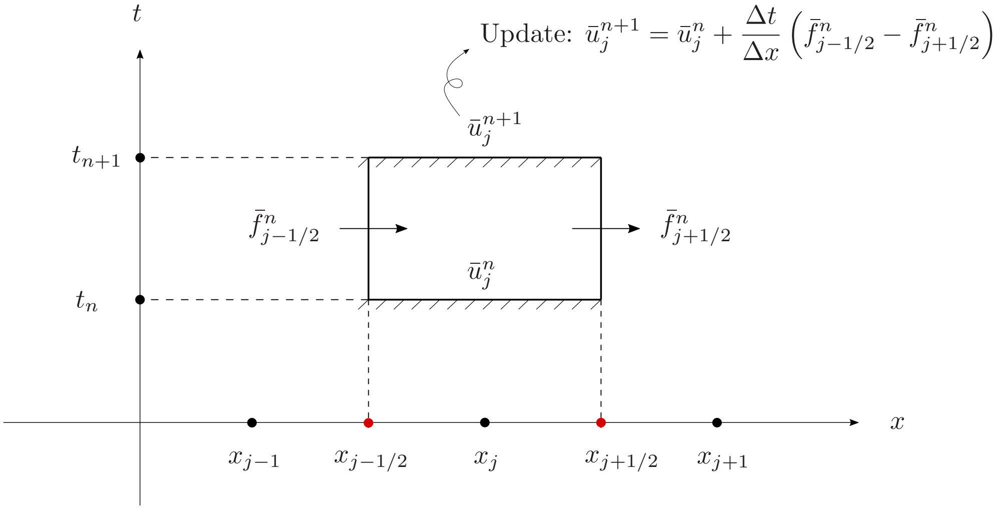


Figure 2: Finite volume update: conservation form.

Next, consideration of the cell average formulation, which, by definition, reads

$$\bar{u}_j^n := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\, dx, \tag{7}$$

and multiplication of both sides of (6) by $1/\Delta x$ yield the following relation

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{1}{\Delta x}\left(\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt\right). \tag{8}$$

Moreover, by defining the **numerical flux functions** as follows

$$\tilde{f}_{j-1/2}^n := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt, \tag{9}$$

$$\tilde{f}_{j+1/2}^n := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt, \tag{10}$$

the expression (8) now becomes the **conservation form**

$$\therefore \quad \boxed{\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right).} \tag{11}$$

Example 2. *Lipschitz continuity.*

Example 3. *Derivation of the conservative form of upwind scheme. Examine its numerical flux function: conservative? consistent?*

Example 4. *Derivation of the conservative form of Lax-Friedrichs scheme. Examine its numerical flux function: conservative? consistent?*

Example 5. *Derivation of the conservative form of Lax-Wendroff scheme. Examine its numerical flux function: conservative? consistent?*

Example 6. *Derivation of the conservative form of the two-step Lax-Wendroff scheme. Examine its numerical flux function: conservative? consistent?*

Example 7. *Derivation of the conservative form of MacCormack scheme. Examine its numerical flux function: conservative? consistent?*

6 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 8. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*

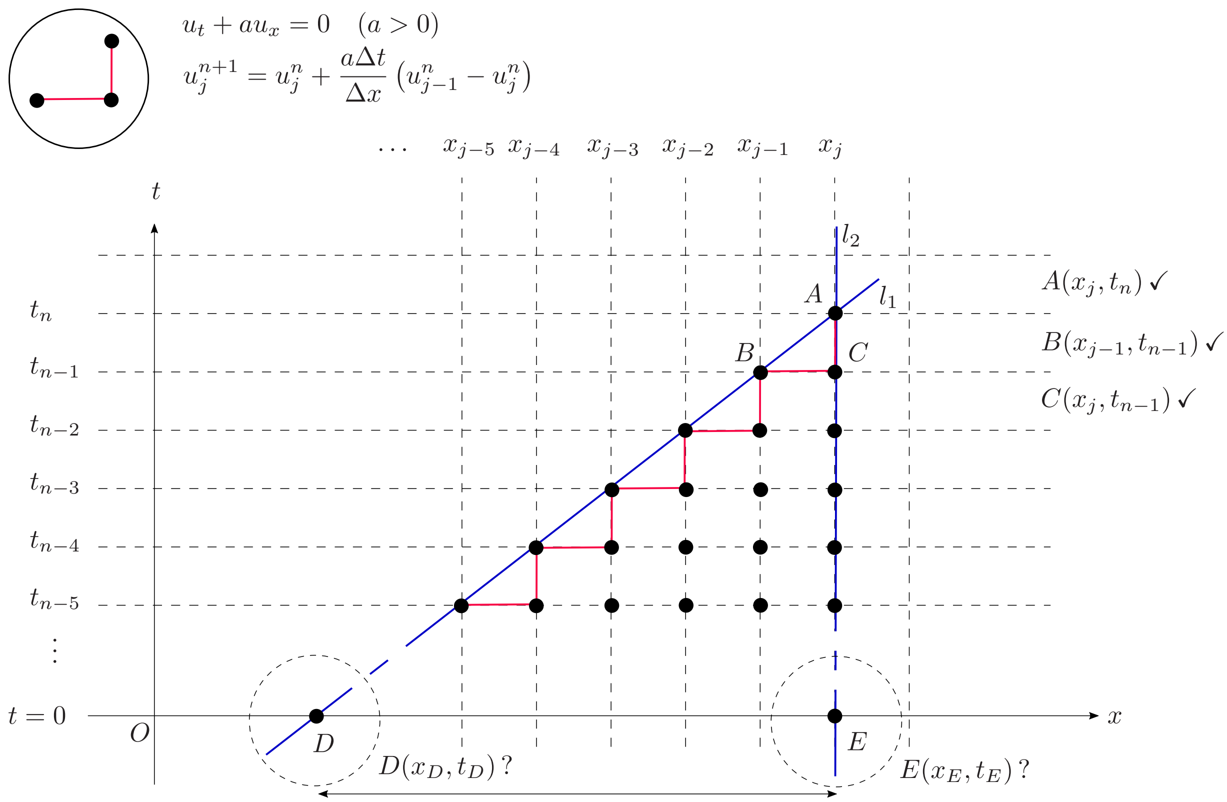


Figure 3: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 3, the numerical value computed at point A depends essentially on computed initial conditions laying between point D and E.

1. Perspective of indical subscription:
   Line (l_1) passsing point $A(j, n)$ and $B(j-1, n-1)$ has the following form

$$
\begin{aligned}
(l_1): \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A}(\xi - \xi_A) \\
\Leftrightarrow \tau &= n + \frac{(n-1) - n}{(j-1) - j}(\xi - j) \\
\Leftrightarrow \tau &= n + \frac{-1}{-1}(\xi - j)\,,
\end{aligned}
\tag{12}
$$

   where τ is the indical variable corresponding to t, and x the indical variable to x. Hence, line (l_1) passing line x with index $\tau = 0$ at point D leads to the following relation

$$
\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \tag{13}
$$

Likewise, line (l_2) passing line x with index $\tau = 0$ at point E leads to the following relation

$$x_\xi - x_j = 0. \tag{14}$$

Therefore, by combining (13) and (14) we arrive at the numerical domain of dependence for the One-sided method in terms of indical perspective

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x_\xi \,\middle|\, -n\Delta x \leq x_\xi - x_j \leq 0 \right\}. \tag{15}$$

Next, by using the CFL number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \overset{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \tag{16}$$

Then, by substituting (16) into (15) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \leq x - x_j \leq 0 \right\}}. \tag{17}$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{18}$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}(x_j, t_n) \subset \mathcal{D}_{\Delta t}(x_j, t_n), \tag{19}$$

which implies that characteristics should lie with the triangular zone under the line (l_1) and (l_2), as shown in Figure 3. Therefore, substitution of (18) into (17) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \leq (x_j - at_n) - x_j \leq 0 \Leftrightarrow -\frac{at_n}{\nu} \leq -at_n \leq 0, \tag{20}$$

which, equally, leads to the CFL condtion

$$\therefore \quad \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}}. \tag{21}$$

Herein, the CFL condition (21) leads to constraint on the time step Δt for the case when $a > 0$. Note in passing that ν is non-negative.

2. Perspective of fixed-point value:
Line (l_1) passsing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned}(l_1): \quad t = t_A + \frac{t_B - t_A}{x_B - x_A}(x - x_A) &\Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x}(x - x_j). \end{aligned} \tag{22}$$

Hence, line (l_1) passing line $t = 0$ at point D leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \tag{23}$$

Likewise, line (l_2) passing line $t = 0$ at point E leads to the relation

$$x - x_j = 0. \tag{24}$$

Therefore, combination of (23) and (24) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \tag{25}$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (18), reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{26}$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

$$-\frac{t_n \Delta x}{\Delta t} \leq \left(x_j - a t_n\right) - x_j \leq 0, \tag{27}$$

which we have substituted (26) into (25). Herein, the relation (27) enforcing CFL condition on the time step Δt

$$\therefore \quad \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a},} \tag{28}$$

which is similar to (21).

Example 9. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the right.*

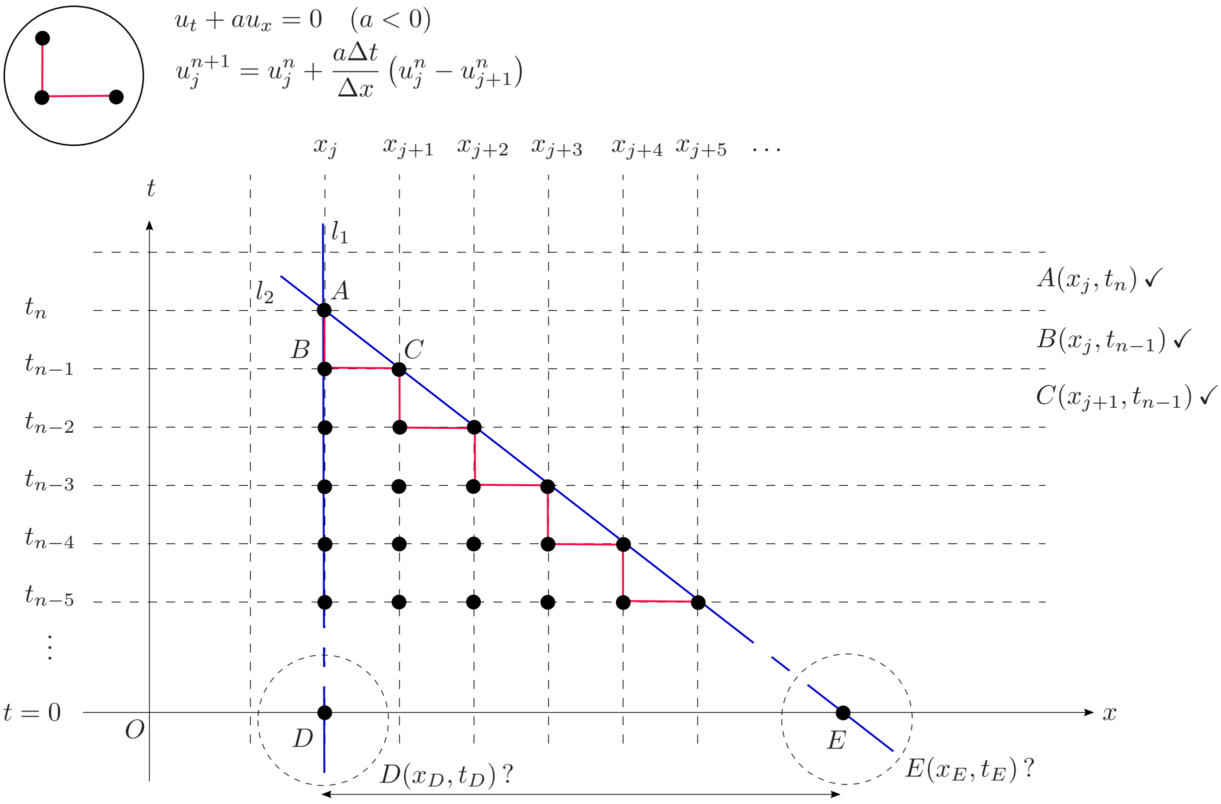


Figure 4: Numerical domain of dependence for One-sided method, where the stencils point to the right.

Similarly, by following steps done in Example 8 we obtain the following summary:

1. Point E in terms of fixed-point value satisfying

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\left(x - x_j\right) \tag{29}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, 0 \leq x - x_j \leq -\frac{a t_n}{\nu} \right\}. \tag{30}$$

3. Analytical domain of dependence reads

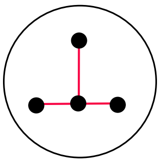
$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{31}$$

4. CFL condition reads

$$\therefore \quad \boxed{\Delta t \geq \frac{\Delta x}{a}.} \tag{32}$$

Note in passing that the advection velocity a in this Example 9 is $a < 0$.

Example 10. *Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.*



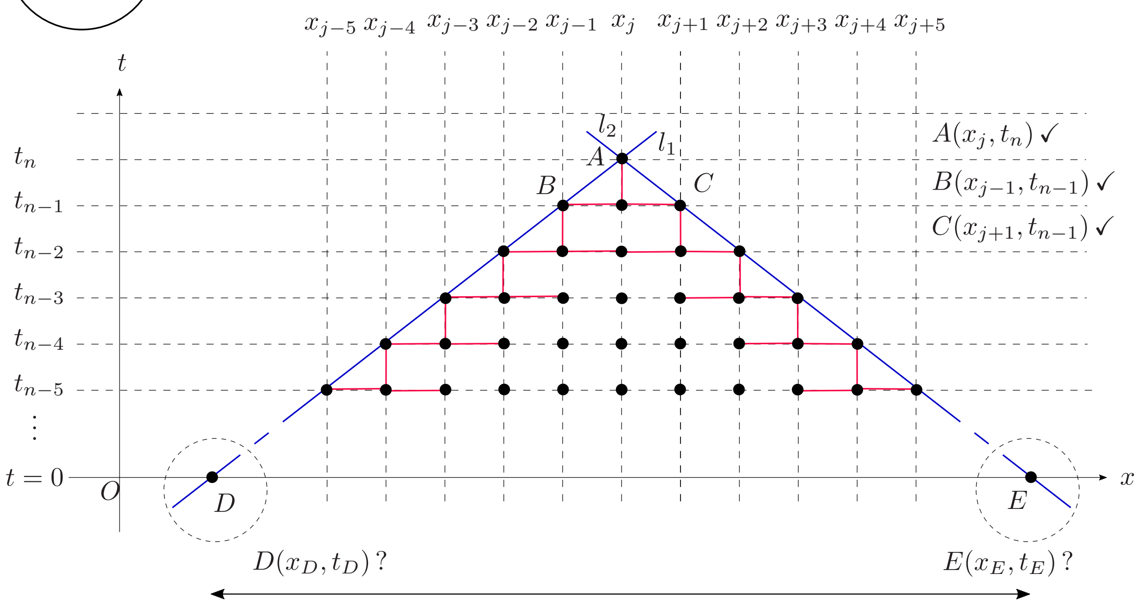


Figure 5: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 8 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}\,(x - x_j) \tag{33}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\,(x - x_j) \tag{34}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\,(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \le \frac{a t_n}{\nu} \right\}. \tag{35}$$

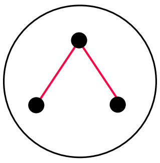
3. Analytical domain of dependence reads

$$\mathcal{D}\,(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{36}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left|\frac{a\Delta t}{\Delta x}\right| \le 1.} \tag{37}$$

Example 11. *Examine the numerical domain of dependence of* Lax-Friedrichs *method.*



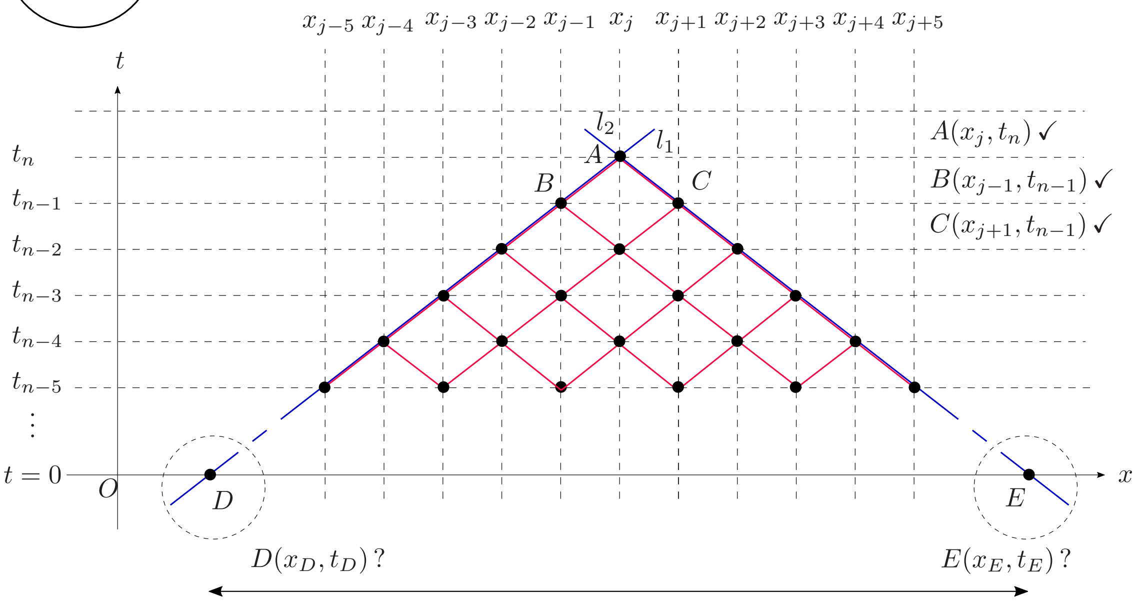


Figure 6: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 8, or the same as 10 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \tag{38}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}(x - x_j) \tag{39}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \le \frac{at_n}{\nu} \right\}. \tag{40}$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{41}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left|\frac{a\Delta t}{\Delta x}\right| \le 1.} \tag{42}$$

7 von Neumann stability analysis

Example 12. *von Neumann stability analysis for Upwind method.*

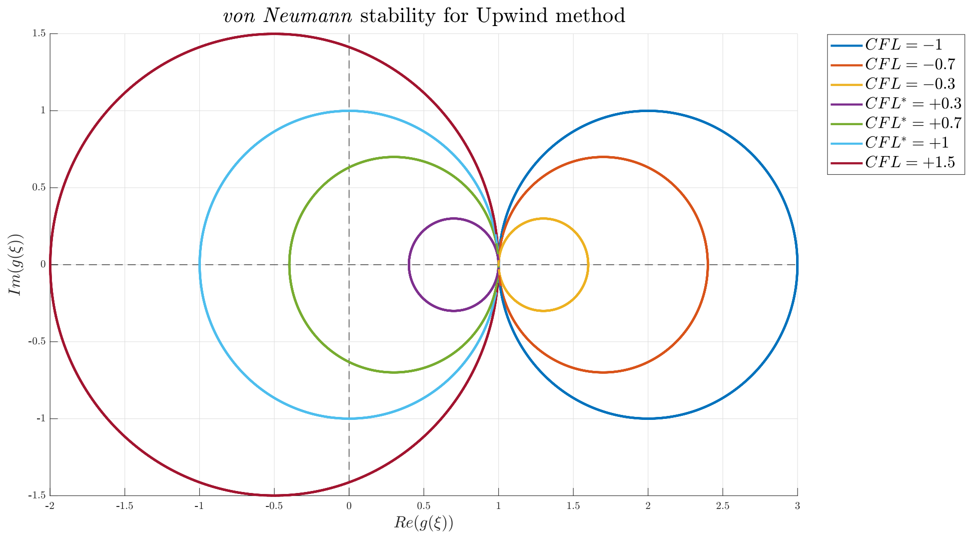


Figure 7: *von Neumann* stability analysis for Upwind method.

Example 13. *Summary consistency + stability $\Rightarrow$ Convergence*

8 Conservation form - Finite Volume Method

Example 14. *Derivation of conservation form.*