

→ Remark of conservation form (cont.)

Upwind scheme

LF
UN (2 steps) + (2 steps)

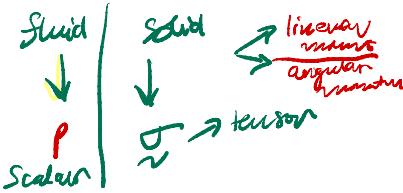
MaxCormack

consistent

Global Exercise - 13

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19th January 2022



1 Conservation form - Finite Volume Method (cont.)

→ Approximate Riemann solver

Example 1. Derivation of conservation form.

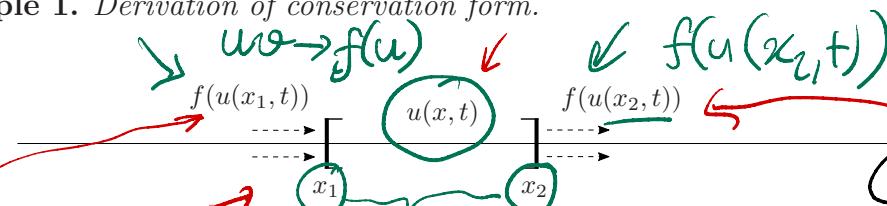


Figure 1: Conservation for a scalar conserved quantity $u(x, t)$ over domain $[x_1, x_2]$.

Starting point is the **integral form** of conservation law. Recall the integral formulation of conservation law for a scalar conserved quantity $u(x, t)$ over domain $[x_1, x_2]$, herein, reads

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x, t) dx = f(u(x_1, t)) - f(u(x_2, t)). \quad (1)$$

Integration both side of (1) over temporal interval $[t_1, t_2]$ yields

$$\left(\int_{x_1}^{x_2} u(x, t) dx \right) \Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} (f(u(x_1, t)) - f(u(x_2, t))) dt, \quad (2)$$

which leads to another integral form of the conservation law, as follows

$$\int_{x_1}^{x_2} u(x, t_2) dx = \int_{x_1}^{x_2} u(x, t_1) dx + \int_{t_1}^{t_2} f(u(x_1, t)) dt - \int_{t_1}^{t_2} f(u(x_2, t)) dt. \quad (3)$$

The integral form of conservation law shown in (3) will be now used to derive the conservation form for finite volume method (FVM). Then, the derivation is performed by taking into consideration of the spatial interval $[x_{j-1/2}, x_{j+1/2}]$ and temporal interval $[t_n, t_{n+1}]$ instead of $[x_1, x_2]$ and $[t_1, t_2]$, respectively, as follows

$$\rightarrow [x_1, x_2] \rightarrow [x_{j-1/2}, x_{j+1/2}], \quad (4)$$

$$[t_1, t_2] \rightarrow [t_n, t_{n+1}]. \quad (5)$$

Hence, the integral form in (3) becomes

$$\left(\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1}) dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n) dx + \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) dt. \right) \quad (6)$$

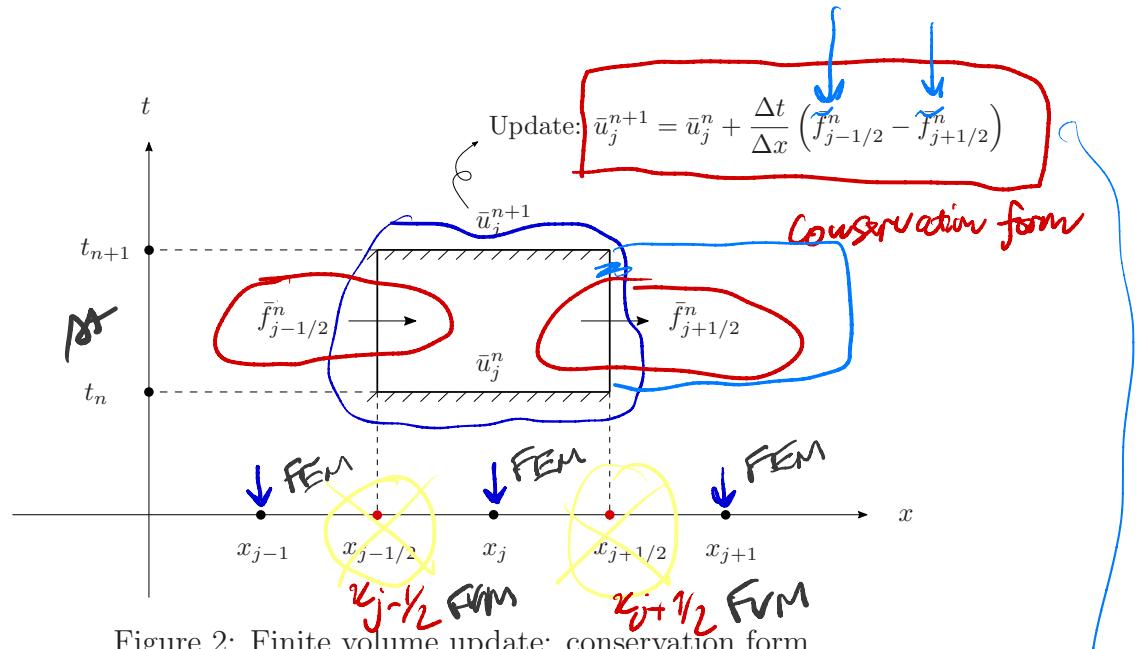


Figure 2: Finite volume update: conservation form.

Next, consideration of the cell average formulation, which, by definition, reads

$$\bar{u}_j^n := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n) dx, \quad (7)$$

and multiplication of both sides of (6) by $1/\Delta x$ yield the following relation

$$\rightarrow \bar{u}_j^{n+1} = \bar{u}_j^n + \frac{1}{\Delta x} \left(\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) dt \right). \quad (8)$$

Moreover, by defining the **numerical flux functions** as follows

$$\rightarrow \tilde{f}_{j-1/2}^n := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) dt, \quad (9)$$

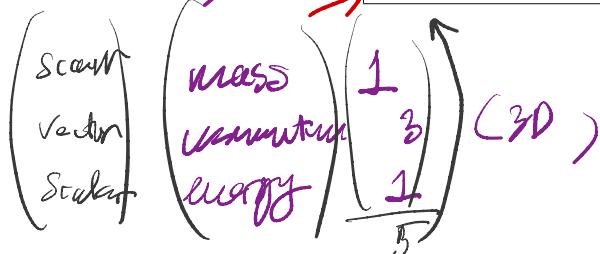
$$\rightarrow \tilde{f}_{j+1/2}^n := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) dt, \quad (10)$$

the expression (8) finally becomes the **conservation form**

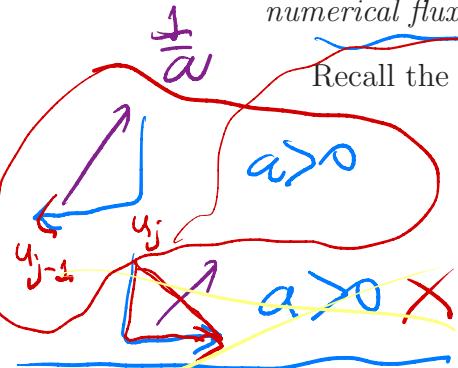
$$\rightarrow \therefore \bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} (\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n). \quad (11)$$

Likewise, a generalization to nonlinear *systems* takes the following form

$$\rightarrow \therefore \bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x} (\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n). \quad (12)$$



Example 2. Derivation of the conservative form of upwind scheme. Examine its numerical flux function: consistent?

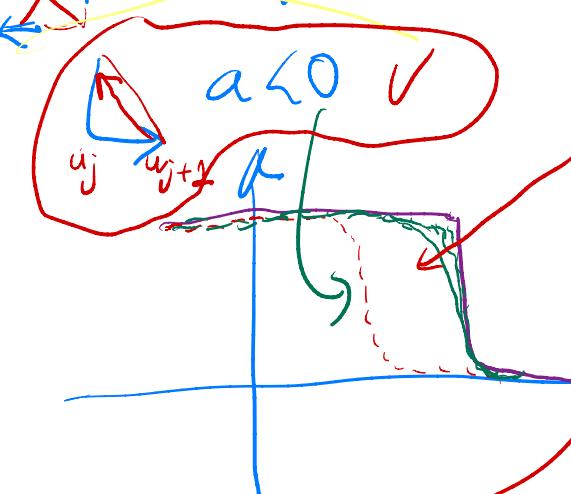


Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} (\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n). \quad (13)$$

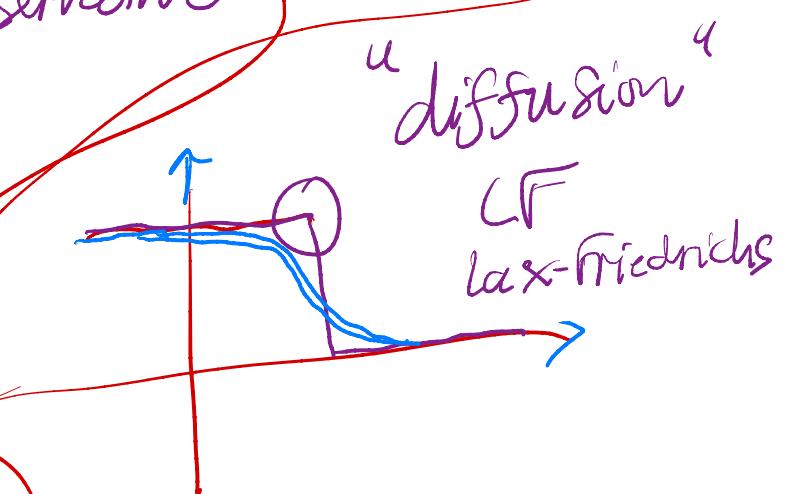
$$a > 0$$

→ conservative



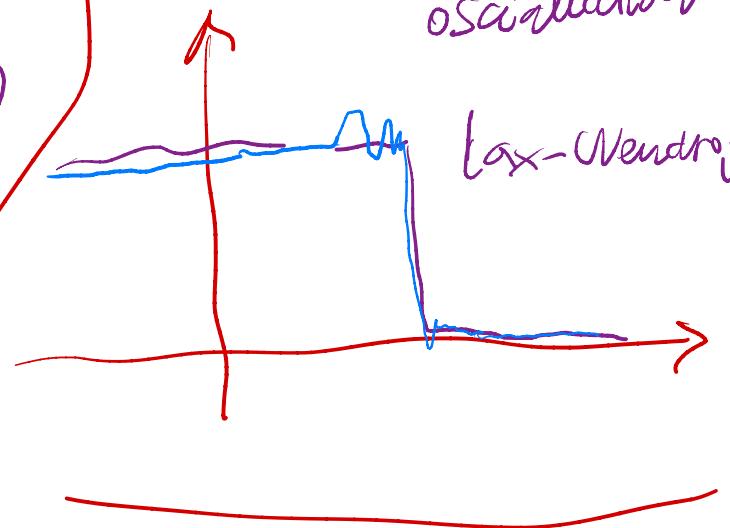
$$\tilde{f}_{j-\frac{1}{2}}^n(u_{j-1}, u_j) = a u_{j-1}$$

$$\tilde{f}_{j+\frac{1}{2}}^n(u_j, u_{j+1}) = a u_j$$



"oscillation"

Lax-Wendroff



$$\tilde{f}_{j-\frac{1}{2}}^n(\varepsilon, \varepsilon) = a \varepsilon = f(\varepsilon)$$

$$U_t + \vec{f}(U)_x = 0$$

Example 3. Derivation of the conservative form of Lax-Friedrichs scheme. Examine its numerical flux function: consistent?

Approach: Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} (\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n). \quad (14)$$

The generalization of the Lax-Friedrichs scheme to nonlinear systems takes the following form

$$\rightarrow u_j^{n+1} = \frac{1}{2} (u_{j-1}^n + u_{j+1}^n) + \frac{\Delta t}{2\Delta x} (f(u_{j+1}^n) - f(u_{j-1}^n)) \quad (15)$$

$$\begin{aligned} u_j^{n+1} &= u_j^n + \frac{1}{2} (u_{j-1}^n + u_{j+1}^n - 2u_j^n) + \frac{\Delta t}{2\Delta x} (f(u_{j+1}^n) - f(u_{j-1}^n)) \\ &= u_j^n + \frac{\Delta t}{\Delta x} \left[\frac{1}{2} \frac{\Delta x}{\Delta t} (u_{j+1}^n - u_j^n) - \frac{1}{2} \frac{\Delta x}{\Delta t} (u_j^n - u_{j-1}^n) \right] \\ &\quad \text{if } |f(x) - f(y)| \leq K|x-y| \\ &= u_j^n + \frac{\Delta t}{\Delta x} \left[\frac{1}{2} \frac{\Delta x}{\Delta t} (u_{j+1}^n - u_j^n) - \frac{1}{2} f(u_{j+1}^n) + \frac{1}{2} f(u_{j-1}^n) \right] \\ &\quad \text{if } |f(x) - f(y)| \leq K|x-y| \\ &= u_j^n + \frac{\Delta t}{\Delta x} \left[\frac{1}{2} \frac{\Delta x}{\Delta t} (u_{j+1}^n - u_j^n) - \frac{1}{2} f(u_{j+1}^n) + \frac{1}{2} f(u_{j-1}^n) \right] \\ &\quad \text{if } |f(x) - f(y)| \leq K|x-y| \\ \tilde{f}_{j+1/2}^n &= \frac{1}{2} \frac{\Delta x}{\Delta t} (u_{j+1}^n - u_j^n) + \frac{1}{2} f(u_{j+1}^n) - \frac{1}{2} f(u_j^n) \\ &= \frac{1}{2} \frac{\Delta x}{\Delta t} (u_{j+1}^n - u_j^n) + \frac{1}{2} f(u_{j+1}^n) - \frac{1}{2} f(u_j^n) \end{aligned}$$

$$F(u_j^n, u_{j+1}^n) = \frac{1}{2} \frac{\Delta x}{\Delta t} (u_{j+1}^n - u_j^n)$$

$$+ \left(\frac{1}{2} \right) \underbrace{f(u_{j+1}^n) + f(u_j^n)}_{}$$

$$F(u_{j-1}^n, u_j^n) = \frac{1}{2} \frac{\Delta x}{\Delta t} (u_j^n - u_{j-1}^n) -$$

$$+ \left(\frac{1}{2} \right) \underbrace{f(u_{j-1}^n) + f(u_j^n)}_{}$$

→ This is consistent.
 numerical
 flux function.

$$\partial F = \Delta \text{ Jacobian}$$

Example 4. Derivation of the conservative form of Lax-Wendroff scheme. Examine its numerical flux function: consistent? Consider the nonlinear scalar conservation law given as follows

$$U_t + F(U)_x = 0. \quad \rightarrow \quad F'(U) U_x$$

Write the Lax-Wendroff scheme for the above equation in conservation form. Show that the numerical flux function is consistent.

Approach:

1. Conservation form: A numerical scheme applied for the conservation law $U_t + F(U)_x = 0$ is said to be conservative if it can be written in the conservation form, as follows

$$\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x} \left(\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n \right), \quad (16)$$

where $\tilde{F}_{j-1/2}^n$ and $\tilde{F}_{j+1/2}^n$ are called interfacial fluxes. Next, by taking into consideration for the case of Lax-Wendroff scheme for the nonlinear conservation law equation, the conservation form is derived as follows

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{\Delta t}{2\Delta x} (F(U_{j+1}^n) - F(U_{j-1}^n)) \\ &\quad + \frac{(\Delta t)^2}{2(\Delta x)^2} (A^2(U_j^n, U_{j+1}^n)(U_{j+1}^n - U_j^n) - A^2(U_{j-1}^n, U_j^n)(U_j^n - U_{j-1}^n)), \end{aligned} \quad \checkmark \text{ Jacobian matrix} \quad (17)$$

which is rewritten by grouping relative terms together, as follows

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{\Delta t}{\Delta x} \left(\frac{1}{2} (F(U_j^n) + F(U_{j+1}^n)) - \frac{\Delta t}{2\Delta x} A^2(U_j^n, U_{j+1}^n)(U_{j+1}^n - U_j^n) \right) \\ &\quad + \frac{\Delta t}{\Delta x} \left(\frac{1}{2} (F(U_{j-1}^n) + F(U_j^n)) - \frac{\Delta t}{2\Delta x} A^2(U_{j-1}^n, U_j^n)(U_j^n - U_{j-1}^n) \right). \end{aligned} \quad (18)$$

Then, this form can be written in terms of conservation form, as follows

$$\rightarrow U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} \left(\tilde{F}_{j+1/2}^n - \tilde{F}_{j-1/2}^n \right), \quad (19)$$

where the numerical flux functions are recognized as

$$\begin{aligned} \tilde{F}_{j+1/2}^n(U_j^n, U_{j+1}^n) &= \frac{1}{2} (F(U_j^n) + F(U_{j+1}^n)) - \frac{\Delta t}{2\Delta x} A^2(U_j^n, U_{j+1}^n)(U_{j+1}^n - U_j^n), \\ \tilde{F}_{j-1/2}^n(U_{j-1}^n, U_j^n) &= \frac{1}{2} (F(U_{j-1}^n) + F(U_j^n)) - \frac{\Delta t}{2\Delta x} A^2(U_{j-1}^n, U_j^n)(U_j^n - U_{j-1}^n). \end{aligned}$$

2. Consistent numerical flux function:

The conservative numerical flux function is then given by

$$\tilde{F}(V, W) = \frac{1}{2} (\underline{F(V)} + \underline{F(W)}) - \frac{\Delta t}{2\Delta x} A^2(V, W)(V - W). \quad (20)$$

When $V = W$, we obtain the following relation

$$\tilde{F}(V, W) = \tilde{F}(W, W) = F(W), \quad (21)$$

which confirms that the numerical flux function \tilde{F} is consistent with the continuous flux function F .

Example 5. Consider the two-step Lax-Wendroff scheme as follows

Step-1: First, update the half-advanced-in-time point $U_{j+1/2}^{n+1/2}$ by computing

$$U_{j+1/2}^{n+1/2} = \frac{1}{2} (U_{j+1}^n + U_j^n) - \frac{\Delta t}{2\Delta x} (F(U_{j+1}^n) - F(U_j^n)), \quad (22)$$

Step-2: Then, update the fully-advanced-in-time point U_j^{n+1} by computing

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} (F(U_{j+1/2}^{n+1/2}) - F(U_{j-1/2}^{n+1/2})). \quad (23)$$

Examine the two-step Lax-Wendroff scheme whether it is conservative, i.e. the scheme can be written in the conservation form. In case it is conservative, determining the numerical flux function.

Idea:

One way to avoid using Jacobian matrix arising in flux function, when dealing with linear hyperbolic system $U_t + AU_x = 0$.

Recall the conservation form for vectorial unknown quantity:

$$\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x} (\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n). \quad (25)$$

Approach: The two-step Lax-Wendroff scheme for the conservation law $u_t + f(u)_x = 0$ is given by

$$\begin{aligned} u_{j+1/2}^{n+1/2} &= \frac{1}{2} (u_{j+1}^n + u_j^n) - \frac{\Delta t}{2\Delta x} (f(u_{j+1}^n) - f(u_j^n)), \\ u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x} (f(u_{j+1/2}^{n+1/2}) - f(u_{j-1/2}^{n+1/2})). \end{aligned}$$

This scheme can also be written in the conservation form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} (f_{i+1/2}^n - f_{i-1/2}^n),$$

where

$$\begin{aligned} \tilde{f}_{j+1/2}^n &= \tilde{F}\left(\frac{1}{2} (u_{j+1}^n + u_j^n) - \frac{\Delta t}{2\Delta x} (f(u_{j+1}^n) - f(u_j^n))\right), \\ \tilde{f}_{j-1/2}^n &= \tilde{F}\left(\frac{1}{2} (u_j^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x} (f(u_j^n) - f(u_{j-1}^n))\right). \end{aligned}$$

The conservative flux function is given by

$$\tilde{F}(u, v) = \tilde{F}\left(\frac{1}{2}(u+v) - \frac{\lambda}{2}(f(v) - f(u))\right).$$

When $v = u$, we obtain the relation

$$3.4 \quad 5.4 \dots \quad \tilde{F}(u, u) = \tilde{F}(u),$$

which shows that the numerical flux function is consistent with the continuous flux function.

2 step to improve / avoid for Jacobian

Example 6. Consider the MacCormack scheme (*) given by

Step-1: First, update the half-advanced-in-time point $u_j^{n+1/2}$ by computing

$$u_j^{n+1/2} = u_j^n - \frac{\Delta t}{\Delta x} (f(u_{j+1}^n) - f(u_j^n)), \quad (26)$$

Step-2: Then, update the fully-advanced-in-time point u_j^{n+1} by computing

$$u_j^{n+1} = \frac{1}{2} (u_j^n + u_j^{n+1/2}) - \frac{\Delta t}{2\Delta x} (f(u_j^{n+1/2}) - f(u_{j-1}^{n+1/2})). \quad (27)$$

Examine the MacCormack scheme whether it is conservative, i.e. the scheme can be written in the conservation form. Examine also if its numerical flux function is consistent.

→ (*) R.W.MacCormack [1969]: The effects of viscosity in hypervelocity impact cratering.

Idea:

Another way, beside the two-step Lax-Wendroff scheme, to avoid using Jacobian matrix arising in computing flux function, when dealing with linear hyperbolic system $U_t + AU_x = 0$.

Recall the conservation form for vectorial unknown quantity:

$$\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x} (\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n). \quad (29)$$

Approach:

In case of MacCormack scheme we arrive at the conservative flux function as follows

$$\rightarrow F(u, v) = \frac{1}{2} \left(\cancel{F}(v) + \cancel{F}\left(u - \frac{\Delta t}{\Delta x} (f(v) - f(u)) \right) \right). \quad (30)$$

The above flux function is consistent since

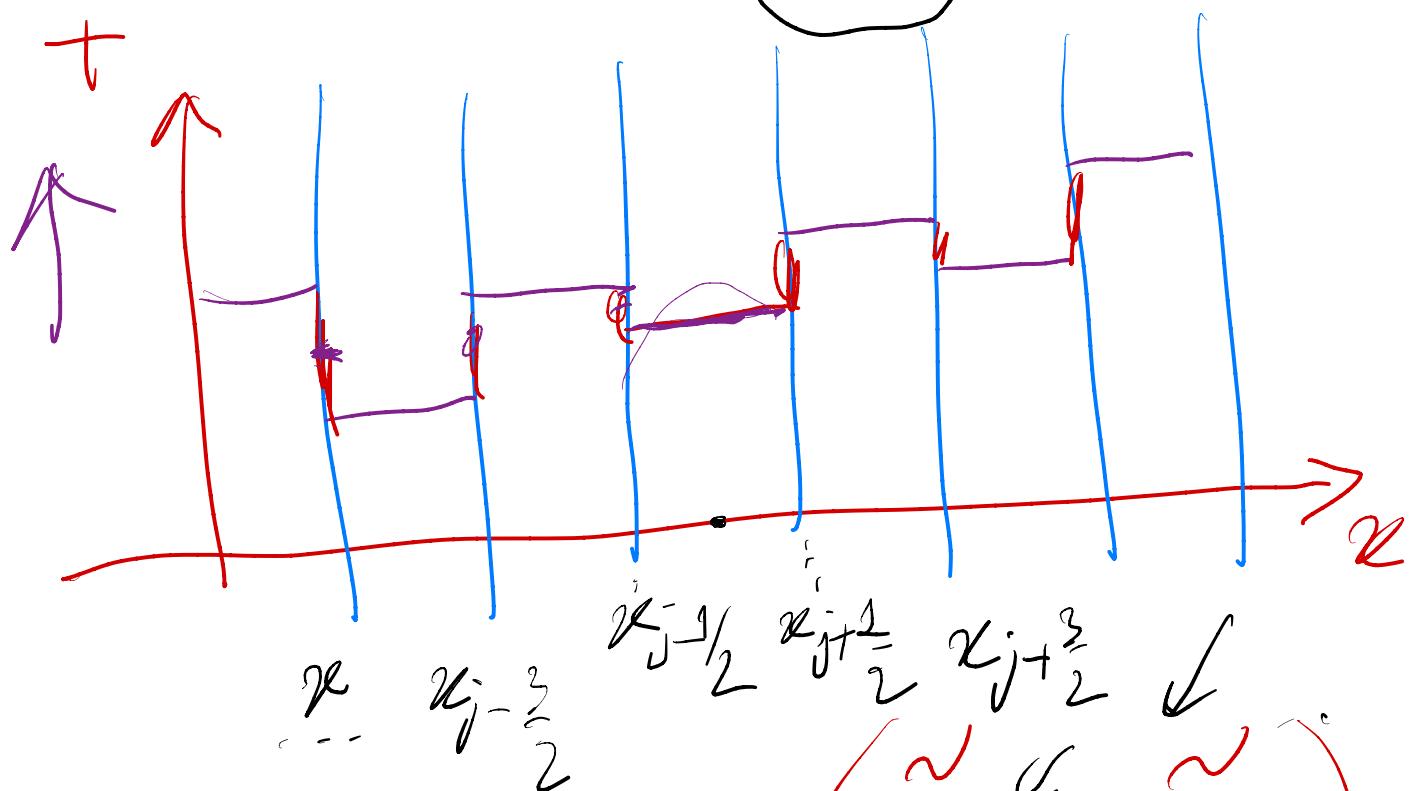
$$\tilde{F}(u, v) = \frac{1}{2} \left(\cancel{F}(v) + \cancel{F}\left(u - \frac{\Delta t}{\Delta x} (f(v) - f(u)) \right) \right) =$$

$$\frac{1}{2} f(\omega) + f(\omega) = f(\omega)$$

consistent

2 Approximate Riemann Solvers

Example 7. Linearized Riemann solvers - Roe Solver.



Riemann problem

{
 x space 100 cells
 + time 100 steps expensive
 costly

10 000 RPs

Example 8. Local Lax-Friedrichs flux.

(Tuesday)

→ check Exam registration @ RWTH online