Global Exercise - 12

Tuan Vo

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1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1: $W := R^{-1}U$ as the scheme shown in exercise

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + R\Lambda R^{-1}U_x &= 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

   where the matrix A is diagonalizable with a transformation matrix $R \in \mathbb{R}^{N\times N}$ in the form
$$A = R\Lambda R^{-1}.$$

2. Case 2: $W := TU$ as the scheme shown in lecture note

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + T^{-1}\Lambda TU_x &= 0 \\ TU_t + \Lambda TU_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

   where the matrix A is diagonalizable with a transformation matrix $T \in \mathbb{R}^{N\times N}$ in the form
$$A = T^{-1}\Lambda T.$$

$\rightarrow$ Note in passing that both schemes result in the same solution.
$\rightarrow$ We have just to be consistent with which scheme to follow.

2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 1. *Examine the numerical domain of dependence of the One-sided method (to the left).*

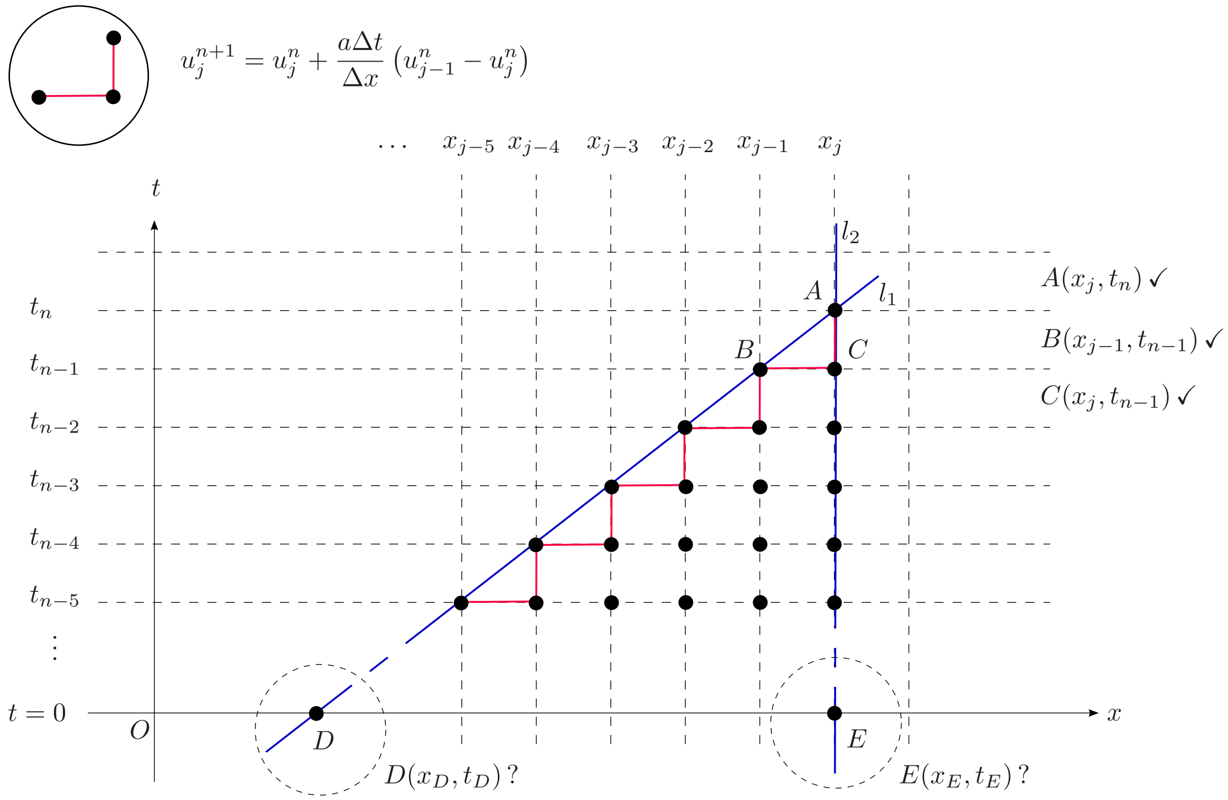


Figure 1: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 1, the numerical value computed at point A depends essentially on computed initial conditions laying between point D and E.

1. Perspective of indical subscription:
   Line (l_1) passsing point $A(j,n)$ and $B(j-1,n-1)$ has the following form

$$
\begin{aligned}
(l_1): \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A}(\xi - \xi_A) \\
\Leftrightarrow \tau &= n + \frac{(n-1)-n}{(j-1)-j}(\xi - j) \\
\Leftrightarrow \tau &= n + \frac{-1}{-1}(\xi - j), \qquad (1)
\end{aligned}
$$

   where τ is the indical variable corresponding to t, and x the indical variable to x. Hence, line (l_1) passing line x with index $\tau = 0$ at point D leads to the following relation

$$
\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \qquad (2)
$$

Likewise, line (l_2) passing line x with index $\tau = 0$ at point E leads to the following relation

$$x_\xi - x_j = 0. \tag{3}$$

Therefore, by combining (2) and (3) we arrive at the numerical domain of dependence for the One-sided method in terms of indical perspective

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x_\xi \,\middle|\, -n\Delta x \leq x_\xi - x_j \leq 0 \right\}. \tag{4}$$

Next, by using the CFL number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \overset{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \tag{5}$$

Then, by substituting (5) into (4) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \leq x - x_j \leq 0 \right\}}. \tag{6}$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{7}$$

Futhermore, the CFL condition enforces that

$$\mathcal{D}\left(x_j, t_n\right) \subset \mathcal{D}_{\Delta t}\left(x_j, t_n\right), \tag{8}$$

which implies that characteristics should lie with the triangular zone under line (l_1) and (l_2), as shown in Figure 1. Therefore, substitution of (7) into (6) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \leq \left(x_j - at_n\right) - x_j \leq 0 \Leftrightarrow -\frac{at_n}{\nu} \leq -at_n \leq 0, \tag{9}$$

which, equally, leads to the CFL condtion

$$\therefore \quad \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}.} \tag{10}$$

Herein, the CFL condition (10) leads to contraint on the time step Δt for the case when $a > 0$. Note in passing that ν is non-negative.

2. Perspective of fixed-point value:
Line (l_1) passsing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned} (l_1): \quad t = t_A + \frac{t_B - t_A}{x_B - x_A}\left(x - x_A\right) &\Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}\left(x - x_j\right) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x}\left(x - x_j\right). \end{aligned} \tag{11}$$

Hence, line (l_1) passing line $t = 0$ at point D leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \tag{12}$$

Likewise, line (l_2) passing line $t = 0$ at point E leads to the relation

$$x - x_j = 0. \tag{13}$$

Therefore, combination of (12) and (13) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \tag{14}$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (7), reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{15}$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

$$-\frac{t_n \Delta x}{\Delta t} \leq \left(x_j - a t_n\right) - x_j \leq 0, \tag{16}$$

which we have substituted (15) into (14). Herein, the relation (16) enforcing CFL condition on the time step Δt

$$\therefore \quad \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a},} \tag{17}$$

which is similar to (10).

Example 2. *Examine the numerical domain of dependence of the One-sided method (to the right).*

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j+1}^n\right)$$

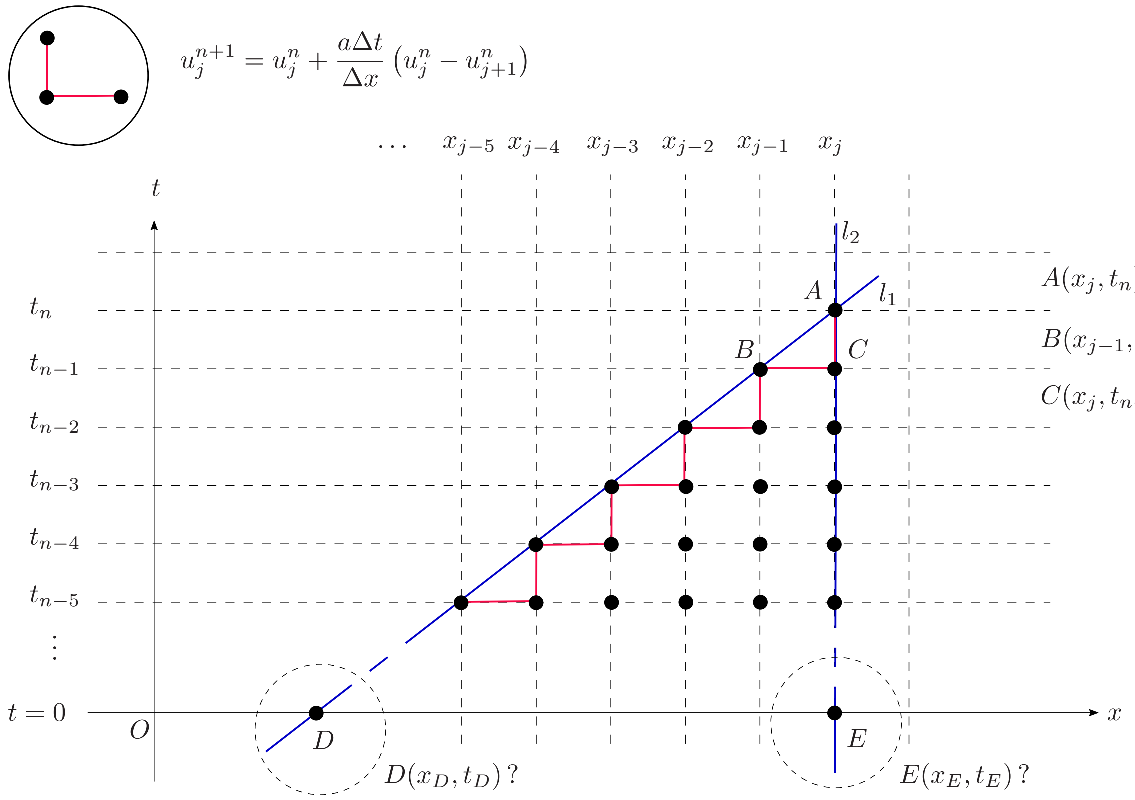


Figure 2: Numerical domain of dependence for One-sided method.

Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \leq x - x_j \leq 0 \right\}. \tag{18}$$

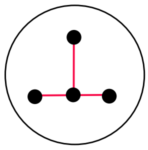
Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{19}$$

CFL condition reads

$$\therefore \quad \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}.} \tag{20}$$

Example 3. *Examine the numerical domain of dependence of the* Lax-Wendroff *method.*



$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x}\left(u_{j-1}^n - u_j^n\right)$$

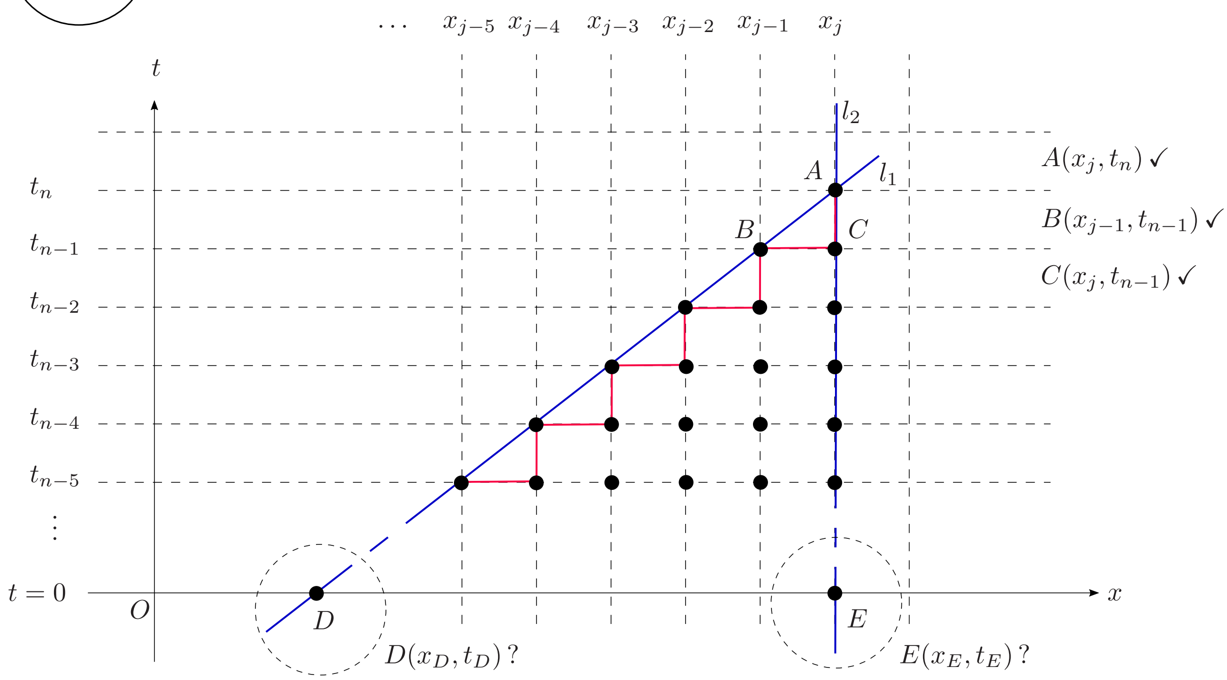


Figure 3: Numerical domain of dependence for One-sided method.

Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \le x - x_j \le 0 \right\}. \tag{21}$$

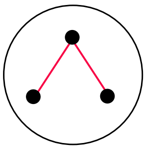
Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{22}$$

CFL condition reads

$$\therefore \quad \boxed{0 \le \nu \le 1 \Leftrightarrow 0 \le \Delta t \le \frac{\Delta x}{a}.} \tag{23}$$

Example 4. *Examine the numerical domain of dependence of the* Lax-Friedrichs *method.*



$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) + \frac{a\Delta t}{2\Delta x}\left(u_{j-1}^n - u_{j+1}^n\right)$$

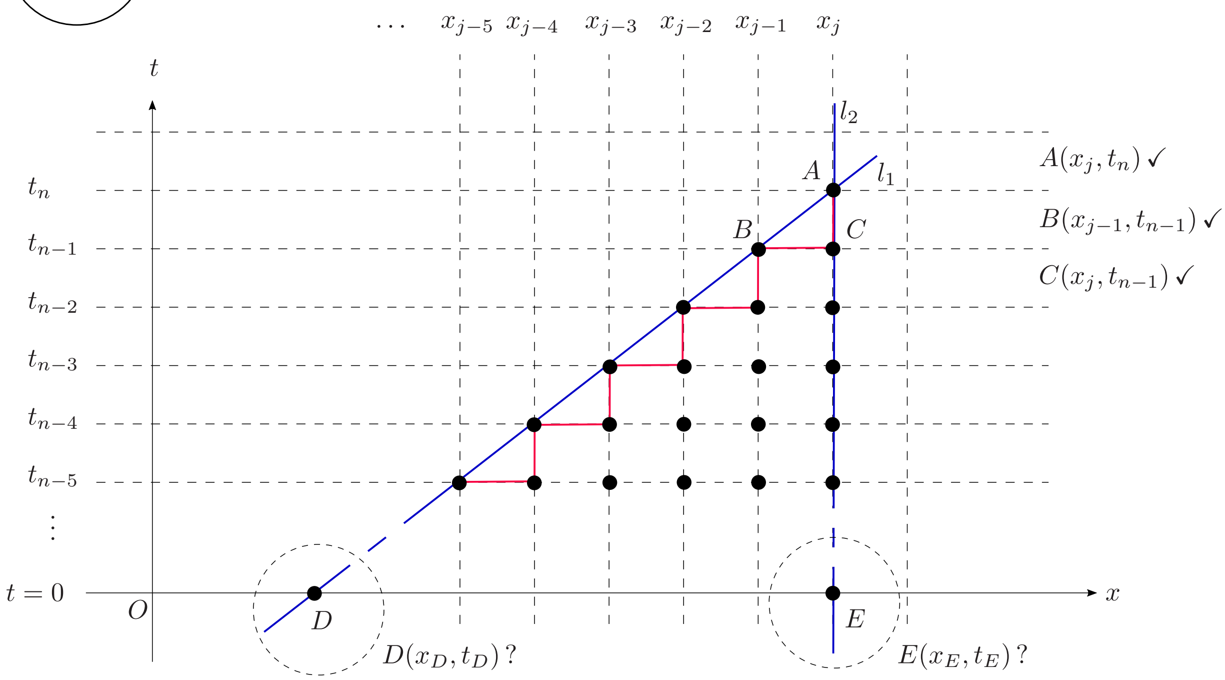


Figure 4: Numerical domain of dependence for One-sided method.

Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \leq x - x_j \leq 0 \right\}. \tag{24}$$

Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{25}$$

CFL condition reads

$$\therefore \quad \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}.} \tag{26}$$

3 von Neumann stability analysis

Example 5. *von Neumann stability*

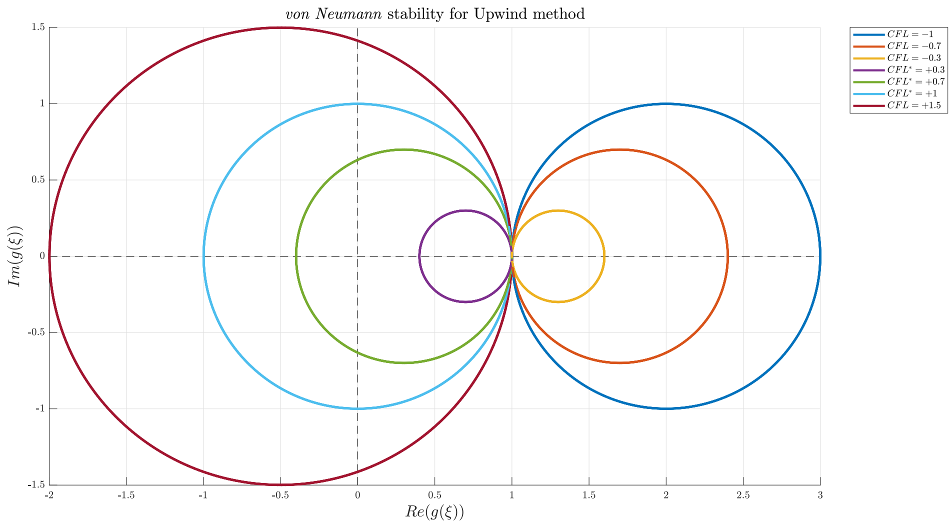


Figure 5: *von Neumann* stability analysis for Upwind method.

4 Conservative form - Finite Volume Method

Example 6. *Conservative form*