Global Exercise - 13

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1 Conservation form - Finite Volume Method (cont.)

Example 1. *Derivation of conservation form.*

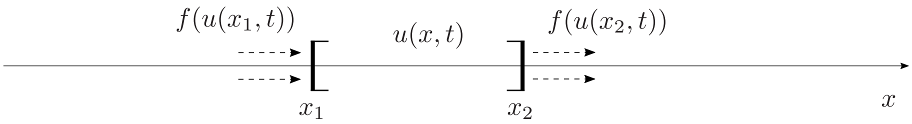

Figure 1: Conservation for a scalar conserved quantity $u(x,t)$ over domain $[x_1,x_2]$.

Starting point is the **integral form** of conservation law. Recall the integral formulation of conservation law for a scalar conserved quantity $u(x,t)$ over domain $[x_1,x_2]$, herein, reads

$$\boxed{\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t)).} \tag{1}$$

Integration both side of (1) over temporal interval $[t_1,t_2]$ yields

$$\left(\int_{x_1}^{x_2} u(x,t)\,dx\right)\Bigg|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \left(f(u(x_1,t)) - f(u(x_2,t))\right) dt, \tag{2}$$

which leads to ***another*** **integral form** of the conservation law, as follows

$$\boxed{\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt.} \tag{3}$$

The integral form of conservation law shown in (3) will be now used to derive the conservation form for finite volume method (FVM). Then, the derivation is performed by taking into consideration of the spatial interval $[x_{j-1/2}, x_{j+1/2}]$ and temporal interval $[t_n, t_{n+1}]$ instead of $[x_1,x_2]$ and $[t_1,t_2]$, respectively, as follows

$$[x_1,x_2] \to [x_{j-1/2}, x_{j+1/2}], \tag{4}$$
$$[t_1,t_2] \to [t_n, t_{n+1}]. \tag{5}$$

Hence, the integral form in (3) becomes

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,dx &= \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,dx \\ &\quad + \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))\,dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,dt.\end{aligned} \tag{6}$$

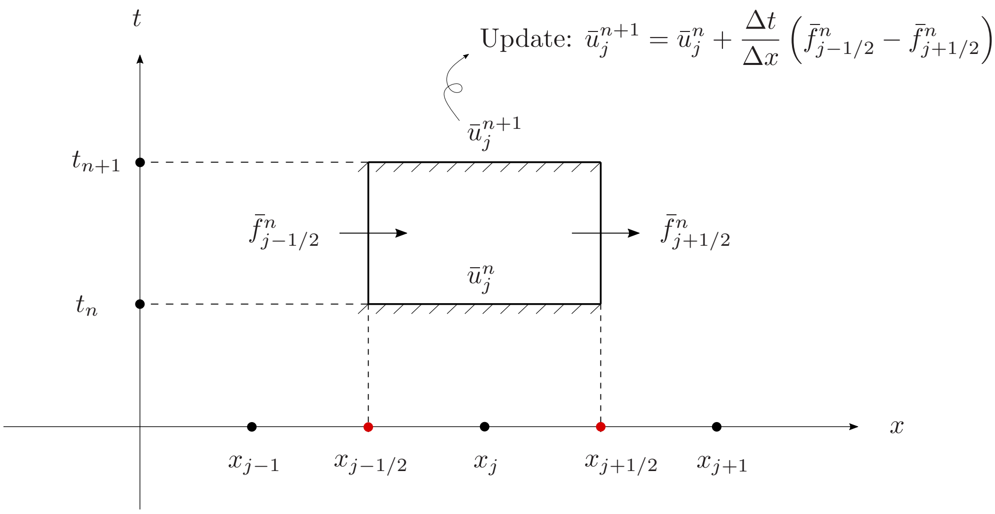

Figure 2: Finite volume update: conservation form.

Next, consideration of the cell average formulation, which, by definition, reads

$$\bar{u}_j^n := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\, dx, \tag{7}$$

and multiplication of both sides of (6) by $1/\Delta x$ yield the following relation

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{1}{\Delta x}\left(\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt\right). \tag{8}$$

Moreover, by defining the **numerical flux functions** as follows

$$\tilde{f}_{j-1/2}^n := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt, \tag{9}$$

$$\tilde{f}_{j+1/2}^n := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt, \tag{10}$$

the expression (8) finally becomes the **conservation form**

$$\therefore \quad \boxed{\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right).} \tag{11}$$

Likewise, a generalization to nonlinear *systems* takes the following form

$$\therefore \quad \boxed{\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n\right).} \tag{12}$$

Example 2. *Derivation of the conservative form of upwind scheme. Examine its numerical flux function: consistent?*

Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right). \tag{13}$$

Example 3. *Derivation of the conservative form of Lax-Friedrichs scheme. Examine its numerical flux function: consistent?*

Approach: Recall the conservation form

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right). \tag{14}$$

The generalization of the *Lax-Friedrichs* scheme to nonlinear systems takes the following form

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) + \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n - f(u_{j-1}^n)\right) \tag{15}$$

Example 4. *Derivation of the conservative form of Lax-Wendroff scheme. Examine its numerical flux function: consistent? Consider the nonlinear scalar conservation law given as follows*

$$U_t + F(U)_x = 0.$$

Write the Lax-Wendroff *scheme for the above equation in* conservation form. *Show that the numerical flux function is consistent.*

Approach:

1. Conservation form: A numerical scheme applied for the conservation law $U_t + F(U)_x = 0$ is said to be conservative if it can be written in the conservation form, as follows

$$\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n\right), \tag{16}$$

where $\tilde{F}_{j-1/2}^n$ and $\tilde{F}_{j+1/2}^n$ are called interfacial fluxes. Next, by taking into consideration for the case of *Lax-Wendroff* scheme for the nonlinear conservation law equation, the conservation form is derived as follows

$$\begin{aligned} U_j^{n+1} = U_j^n &- \frac{\Delta t}{2\Delta x}\left(F(U_{j+1}^n) - F(U_{j-1}^n)\right) \\ &+ \frac{(\Delta t)^2}{2\,(\Delta x)^2}\left(A^2(U_j^n, U_{j+1}^n)(U_{j+1}^n - U_j^n) - A^2(U_{j-1}^n, U_j^n)(U_j^n - U_{j-1}^n)\right), \end{aligned} \tag{17}$$

which is rewritten by grouping relative terms together, as follows

$$\begin{aligned} U_j^{n+1} = U_j^n &- \frac{\Delta t}{\Delta x}\left(\frac{1}{2}\left(F(U_j^n) + F(U_{j+1}^n)\right) - \frac{\Delta t}{2\Delta x}A^2(U_j^n, U_{j+1}^n)(U_{j+1}^n - U_j^n)\right) \\ &+ \frac{\Delta t}{\Delta x}\left(\frac{1}{2}\left(F(U_{j-1}^n) + F(U_j^n)\right) - \frac{\Delta t}{2\Delta x}A^2(U_{j-1}^n, U_j^n)(U_j^n - U_{j-1}^n)\right). \end{aligned} \tag{18}$$

Then, this form can be written in terms of conservation form, as follows

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{j+1/2}^n - \tilde{F}_{j-1/2}^n\right), \tag{19}$$

where the numerical flux functions are recognized as

$$\begin{aligned} \tilde{F}_{j+1/2}^n(U_j^n, U_{j+1}^n) &= \frac{1}{2}\left(F(U_j^n) + F(U_{j+1}^n)\right) - \frac{\Delta t}{2\Delta x}A^2(U_j^n, U_{j+1}^n)(U_{j+1}^n - U_j^n), \\ \tilde{F}_{j-1/2}^n(U_{j-1}^n, U_j^n) &= \frac{1}{2}\left(F(U_{j-1}^n) + F(U_j^n)\right) - \frac{\Delta t}{2\Delta x}A^2(U_{j-1}^n, U_j^n)(U_j^n - U_{j-1}^n). \end{aligned}$$

2. Consistent numerical flux function:
The conservative numerical flux function is then given by

$$\tilde{F}(V, W) = \frac{1}{2}\left(F(V) + F(W)\right) - \frac{\Delta t}{2\Delta x}A^2(V, W)(V - W). \tag{20}$$

When $V = W$, we obtain the following relation

$$\tilde{F}(V, W) = \tilde{F}(W, W) = F(W), \tag{21}$$

which confirms that the numerical flux function $\tilde{F}$ is consistent with the continuous flux function F.

Example 5. *Consider the two-step* Lax-Wendroff *scheme as follows*

Step-1: First, update the half-advanced-in-time point $U^{n+1/2}_{j+1/2}$ *by computing*

$$U^{n+1/2}_{j+1/2} = \frac{1}{2}\left(U^n_{j+1} + U^n_j\right) - \frac{\Delta t}{2\Delta x}\left(F\left(U^n_{j+1}\right) - F\left(U^n_j\right)\right), \tag{22}$$

Step-2: Then, update the fully-advanced-in-time point U^{n+1}_j *by computing*

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}\left(F\left(U^{n+1/2}_{j+1/2}\right) - F\left(U^{n+1/2}_{j-1/2}\right)\right). \tag{23}$$

Examine the two-step Lax-Wendroff scheme whether it is conservative, i.e. the scheme can be written in the conservation form. *In case it is conservative, determining the numerical flux function.*

Idea:

$$\boxed{\begin{array}{c}\text{One way to avoid using Jacobian matrix arising in flux function,}\\ \text{when dealing with linear hyperbolic system } U_t + AU_x = 0.\end{array}} \tag{24}$$

Recall the conservation form for vectorial unknown quantity:

$$\bar{U}^{n+1}_j = \bar{U}^n_j + \frac{\Delta t}{\Delta x}\left(\tilde{F}^n_{j-1/2} - \tilde{F}^n_{j+1/2}\right). \tag{25}$$

Approach: The two-step *Lax-Wendroff* scheme for the conservation law $u_t + f(u)_x = 0$ is given by

$$\begin{aligned} u^{n+1/2}_{j+1/2} &= \frac{1}{2}\left(u^n_{j+1} + u^n_j\right) - \frac{\Delta t}{2\Delta x}\left(f\left(u^n_{j+1}\right) - f\left(u^n_j\right)\right), \\ u^{n+1}_j &= u^n_j - \frac{\Delta t}{\Delta x}\left(f\left(u^{n+1/2}_{j+1/2}\right) - f\left(u^{n+1/2}_{j-1/2}\right)\right). \end{aligned}$$

This scheme can also be written in the conservation form

$$u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x}\left(f^n_{i+1/2} - f^n_{i-1/2}\right),$$

where

$$\begin{aligned} f^n_{j+1/2} &= f\left(\frac{1}{2}\left(u^n_{j+1} + u^n_j\right) - \frac{\Delta t}{2\Delta x}\left(f\left(u^n_{j+1}\right) - f(u^n_j)\right)\right), \\ f^n_{j-1/2} &= f\left(\frac{1}{2}\left(u^n_j + u^n_{j-1}\right) - \frac{\Delta t}{2\Delta x}\left(f(u^n_j) - f(u^n_{j-1})\right)\right). \end{aligned}$$

The conservative flux function is given by

$$F(u,v) = f\left(\frac{1}{2}(u+v) - \frac{\lambda}{2}\left(f(v) - f(u)\right)\right).$$

When $v = u$, we obtain the relation

$$F(u,u) = f(u),$$

which shows that the numerical flux function is consistent with the continuous flux function.

Example 6. *Consider the* MacCormack *scheme* $(*)$ *given by*

Step-1: First, update the half-advanced-in-time point $u_j^{n+1/2}$ *by computing*

$$u_j^{n+1/2} = u_j^n - \frac{\Delta t}{\Delta x}\left(f\left(u_{j+1}^n\right) - f\left(u_j^n\right)\right), \tag{26}$$

Step-2: Then, update the fully-advanced-in-time point u_j^{n+1} *by computing*

$$u_j^{n+1} = \frac{1}{2}\left(u_j^n + u_j^{n+1/2}\right) - \frac{\Delta t}{2\Delta x}\left(f\left(u_j^{n+1/2}\right) - f\left(u_{j-1}^{n+1/2}\right)\right). \tag{27}$$

Examine the MacCormack scheme whether it is conservative, i.e. the scheme can be written in the conservation form*. Examine also if its numerical flux function is consistent.*

$(*)$ R.W.MacCormack [1969]: The effects of viscosity in hypervelocity impact cratering.

Idea:

$$\boxed{\begin{array}{c}\text{Another way, beside the two-step Lax-Wendroff scheme,}\\ \text{to avoid using Jacobian matrix arising in computing flux function,}\\ \text{when dealing with linear hyperbolic system } U_t + AU_x = 0.\end{array}} \tag{28}$$

Recall the conservation form for vectorial unknown quantity:

$$\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{F}_{j-1/2}^n - \tilde{F}_{j+1/2}^n\right). \tag{29}$$

Approach:

In case of *MacCormack* scheme we arrive at the conservative flux function as follows

$$F(u,v) = \frac{1}{2}\left(f(v) + f\left(u - \frac{\Delta t}{\Delta x}\left(f(v) - f(u)\right)\right)\right). \tag{30}$$

The above flux function is consistent since

$$F(u,v) = \frac{1}{2}\left(f(v) + f\left(u - \frac{\Delta t}{\Delta x}\left(f(v) - f(u)\right)\right)\right) = \tag{31}$$

2 Approximate Riemann Solvers

Example 7. *Linearized Riemann solvers - Roe Solver.*

Example 8. *Local Lax-Friedrichs flux.*