Global Exercise - 14

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1 Conservation form - Finite Volume Method

Example 1. *Determine consistent numerical flux function for Lax-Friedrichs scheme.*

Recall the conservation form used for FVM reads

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{f}_{j-1/2}^n - \widetilde{f}_{j+1/2}^n\right). \tag{1}$$

Meanwhile, the Lax-Friedrichs scheme reads

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right). \tag{2}$$

Next, we would like to write (2) in terms of (1). The derivation is done by some algebraic manipulations, as follows

$$\begin{aligned}
u_j^{n+1} &= \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) \\
&= u_j^n + \frac{1}{2}u_{j-1}^n - \frac{1}{2}u_j^n + \frac{1}{2}u_{j+1}^n - \frac{1}{2}u_j^n - \frac{\Delta t}{2\Delta x}f(u_{j+1}^n) + \frac{\Delta t}{2\Delta x}f(u_{j-1}^n) \\
&= u_j^n + \frac{\Delta t}{\Delta x}\Bigg(\frac{\Delta x}{2\Delta t}u_{j-1}^n - \frac{\Delta x}{2\Delta t}u_j^n + \frac{\Delta x}{2\Delta t}u_{j+1}^n - \frac{\Delta x}{2\Delta t}u_j^n \\
&\qquad\qquad -\frac{1}{2}f(u_{j+1}^n) + \frac{1}{2}f(u_{j-1}^n)\Bigg) \\
&= u_j^n + \frac{\Delta t}{\Delta x}\Bigg(\underbrace{\left(\frac{\Delta x}{2\Delta t}u_{j-1}^n - \frac{\Delta x}{2\Delta t}u_j^n + \frac{1}{2}f(u_{j-1}^n)\right)}_{=:\widetilde{f}_{j-1/2}^n} \\
&\qquad\qquad - \underbrace{\left(\frac{\Delta x}{2\Delta t}u_j^n - \frac{\Delta x}{2\Delta t}u_{j+1}^n + \frac{1}{2}f(u_{j+1}^n)\right)}_{=:\widetilde{f}_{j+1/2}^n}\Bigg) \\
&= u_j^n + \frac{\Delta t}{\Delta x}\left(\widetilde{f}_{j-1/2}^n - \widetilde{f}_{j+1/2}^n\right),
\end{aligned} \tag{3}$$

which confirms that the *Lax-Friedrichs* scheme given at (2) is able to be written in the *conservation form* with the numerical flux function recognized as follows

$$\widetilde{f}_{j-1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_{j-1}^n, u_j^n\right) = \frac{\Delta x}{2\Delta t}\left(u_{j-1}^n - u_j^n\right) + \frac{1}{2}f(u_{j-1}^n) \tag{4}$$

$$\widetilde{f}_{j+1/2}^n = \widetilde{f}_{j-1/2}^n\left(u_j^n, u_{j+1}^n\right) = \frac{\Delta x}{2\Delta t}\left(u_j^n - u_{j+1}^n\right) + \frac{1}{2}f(u_{j+1}^n) \tag{5}$$

However, these numerical flux function $\widetilde{f}(\cdot,\cdot)$ is not consistent with the original flux function $f(\cdot)$, which can be checked for the case of constant flow, as follows

$$(4) \Leftrightarrow \widetilde{f}_{j-1/2}^n(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta-\beta) + \frac{1}{2}f(\beta) = \frac{1}{2}f(\beta) \quad ↯ \tag{6}$$

$$(5) \Leftrightarrow \widetilde{f}_{j+1/2}^n(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta-\beta) + \frac{1}{2}f(\beta) = \frac{1}{2}f(\beta) \quad ↯ \tag{7}$$

which have confirmed that $\widetilde{f}^n_{j\pm1/2}(\beta,\beta) \neq f(\beta)$, $\forall\beta\in\mathbb{R}$. Therefore, a modified version is required for these numerical flux functions, such that they become consistent with $f(\cdot)$, and simultaneously, the summation of these two fluxes do not lead to any change in the *Lax-Friedrichs* scheme. The consistent numerical flux functions for *Lax-Friedrichs* scheme reads

$$\widetilde{f}^n_{j-1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_{j-1}, u^n_j\right) = \frac{\Delta x}{2\Delta t}\left(u^n_{j-1} - u^n_j\right) + \frac{1}{2}\left(f(u^n_{j-1}) + f(u^n_j)\right), \tag{8}$$

$$\widetilde{f}^n_{j+1/2} = \widetilde{f}^n_{j-1/2}\left(u^n_j, u^n_{j+1}\right) = \frac{\Delta x}{2\Delta t}\left(u^n_j - u^n_{j+1}\right) + \frac{1}{2}\left(f(u^n_{j+1}) + f(u^n_j)\right), \tag{9}$$

whose consistent property is checked as follows

$$(8) \Leftrightarrow \widetilde{f}^n_{j-1/2}(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta-\beta) + \frac{1}{2}(f(\beta)+f(\beta)) = f(\beta) \quad \checkmark \tag{10}$$

$$(9) \Leftrightarrow \widetilde{f}^n_{j+1/2}(\beta,\beta) = \frac{\Delta x}{2\Delta t}(\beta-\beta) + \frac{1}{2}(f(\beta)+f(\beta)) = f(\beta) \quad \checkmark \tag{11}$$

which have confirmed that $\widetilde{f}^n_{j\pm1/2}(\beta,\beta) = f(\beta)$, $\forall\beta\in\mathbb{R}$ for constant flow. Hence, the numerical flux functions $\widetilde{f}^n_{j\pm1/2}$ for the *Lax-Friedrichs* scheme take the following formulation

$$\therefore \quad \boxed{\begin{aligned} \widetilde{f}^n_{j-1/2}\left(u^n_{j-1}, u^n_j\right) &= \frac{\Delta x}{2\Delta t}\left(u^n_{j-1} - u^n_j\right) + \frac{1}{2}\left(f(u^n_{j-1}) + f(u^n_j)\right) \\ \widetilde{f}^n_{j+1/2}\left(u^n_j, u^n_{j+1}\right) &= \frac{\Delta x}{2\Delta t}\left(u^n_j - u^n_{j+1}\right) + \frac{1}{2}\left(f(u^n_j) + f(u^n_{j+1})\right) \end{aligned}} \tag{12}$$

or the generalized form used for FVM implementation reads

$$\therefore \quad \boxed{F(u_L, u_R) = \frac{1}{2}(f(u_L) + f(u_R)) + \frac{\Delta x}{2\Delta t}(u_L - u_R)} \tag{13}$$

2 Godunov's method

Example 2. *Summary*

$\rightarrow$ One-sided method cannot be used for system, i.e. mixed sign of eigenvalue causes difficulty.
$\rightarrow$

3 Approximate Riemann Solvers

Example 3. *Linearized Riemann solvers - Roe Solver.*

Example 4. *Local Lax-Friedrichs flux.*

4 High resolution methods

Example 5. *Linearized Riemann solvers - Roe Solver.*