1 How the convexity property of $f(u)$ guarantees both $C_{j+1/2} \geq 0$ and $D_{j-1/2} \geq 0$ at the same time

In order to show the scheme has TVD property, according to Theorem II.23 (*Harten* [1980]) from the lecture note, the following three conditions

$$\boxed{\begin{aligned} C_{j+1/2} &\geq 0, \\ D_{j-1/2} &\geq 0, \\ C_{j+1/2} + D_{j-1/2} &\leq 1, \end{aligned}}$$

must hold. Herein, the first two conditions $C_{j+1/2} \geq 0$, $D_{j-1/2} \geq 0$ can be shown by using convexity of flux function f. We now examine $C_{j+1/2}$. The analysis is as follows

1. Case 1: According to Fig. 1 and Fig. 2, if $\Big(f'(u_L) \geq 0 \wedge f'(u_R) \geq 0\Big) \vee \Big(f'(u_L) \geq 0 \geq f'(u_R) \wedge [f]/[u] > 0\Big)$ then

$$C_{j+1/2} = 0 \quad \checkmark$$

which satisfies $C_{j+1/2} \geq 0$.

2. Case 2.1: According to Fig. 3, if $f'(u_L) \leq 0 \wedge f'(u_R) \leq 0$ $(\star)$ then

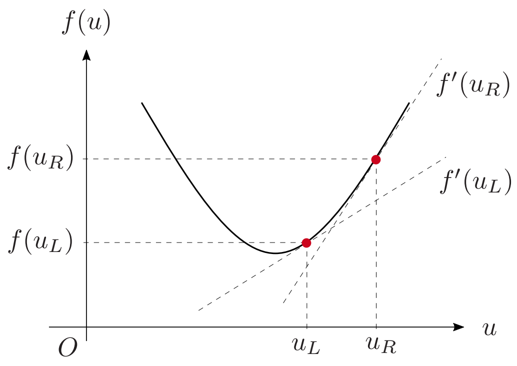
$$C_{j+1/2} = -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L} = -\lambda\alpha.$$

- $f(u)$ convex with $(\star)$ leads to $f(u)$ decreasing at both u_L and u_R
  - $\rightarrow$ If $u_L > u_R$ then $\alpha < 0$. Hence, $C_{j+1/2} > 0$ $\checkmark$
  - $\rightarrow$ If $u_L < u_R$ then $\alpha < 0$. Hence, $C_{j+1/2} > 0$ $\checkmark$
- $f(u)$ concave with $(\star)$ leads to $f(u)$ decreasing at both u_L and u_R
  - $\rightarrow$ If $u_L > u_R$ then $\alpha < 0$. Hence, $C_{j+1/2} > 0$ $\checkmark$
  - $\rightarrow$ If $u_L < u_R$ then $\alpha < 0$. Hence, $C_{j+1/2} > 0$ $\checkmark$

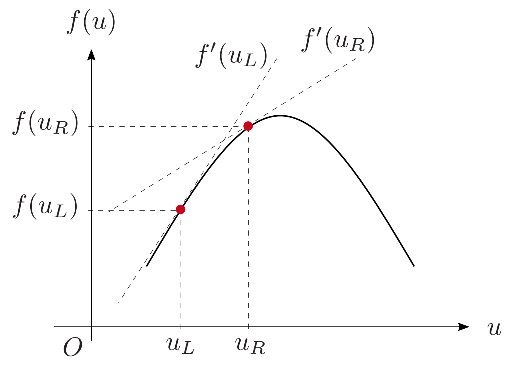
3. Case 2.2: According to Fig. 4, if $\Big(f'(u_L) \geq 0 \geq f'(u_R)\Big) \wedge \Big([f]/[u] < 0\Big)$ $(\star\star)$ then

$$C_{j+1/2} = -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L} = -\lambda\alpha.$$

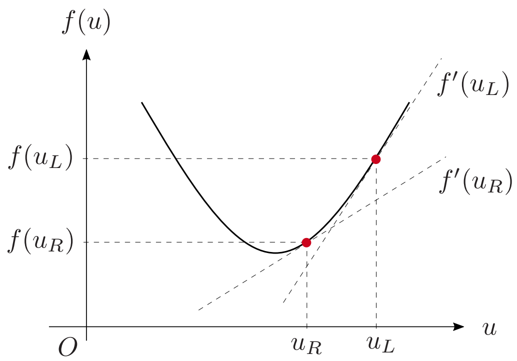
- $f(u)$ convex with the first condition in $(\star\star)$ leads to $f(u)$ decreasing at u_R and increasing at u_L. Hence, it must be $u_L > u_R$, which implies a shock. Besides, the second condition $[f]/[u] < 0$ implies also an existence of shock, which is moving to the left in this case.
  - $\rightarrow$ If $u_L > u_R$ and $f(u_L) > f(u_R)$ then $\alpha > 0$. Hence, $C_{j+1/2} < 0$ $\lightning$



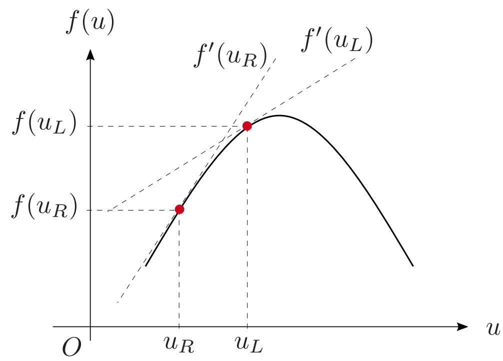
Convex: $u_L < u_R$



Concave: $u_L < u_R$



Convex: $u_L > u_R$



Concave: $u_L > u_R$

Figure 1: Case 1.1: Convexity and concavity for $u_L < u_R$ and $u_L > u_R$, where the condition $f'(u_L) \geq 0 \wedge f'(u_R) \geq 0$ holds.

$\rightarrow$ If $u_L > u_R$ and $f(u_L) < f(u_R)$ then $\alpha < 0$. Hence, $C_{j+1/2} > 0$ ✓

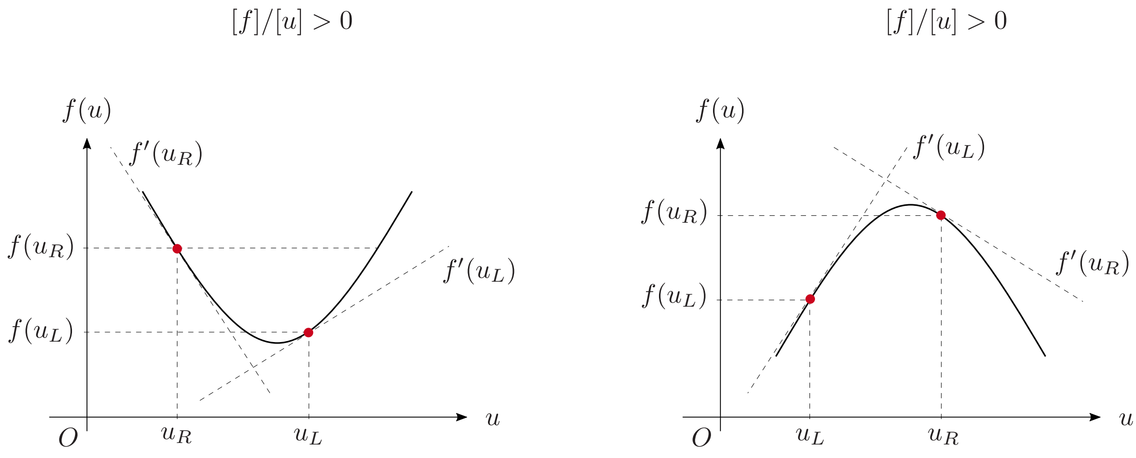
Therefore, the convexity of $f(u)$ gives us the possibility of $C_{j+1/2} > 0$.

- $f(u)$ concave with the first condition in $(\star\star)$ leads to $f(u)$ decreasing at u_R and increasing at u_L. Hence, it must be $u_L < u_R$, which does not imply a shock, but rather a rarefaction solution. Besides, the second condition $[f]/[u] < 0$ implies an existence of shock, which is moving to the left in this case. This is a **contradiction** for the case of $f(u)$ concave. Therefore, the concavity of $f(u)$ does not lead us to the possibility of $C_{j+1/2} > 0$.

4. Case 3: According to Fig. 5, if $f'(u_L) < 0 < f'(u_R)$ $(\star\star\star)$ then

$$C_{j+1/2} = -\lambda \frac{f(u_s) - f(u_L)}{u_R - u_L} = -\lambda\alpha.$$

- $f(u)$ convex with the condition $(\star\star\star)$ leads to $f(u)$ decreasing at u_L and increasing at u_R. Hence, it must be $u_R > u_L$, which also implies a rarefaction. Therefore, we obtain $u_s \in [u_L, u_R]$ which leads to $f(u_s) < f(u_{R,L})$. As a consequence, $\alpha < 0$. Finally, the convexity of $f(u)$ do lead us to the possibility of $C_{j+1/2} > 0$.



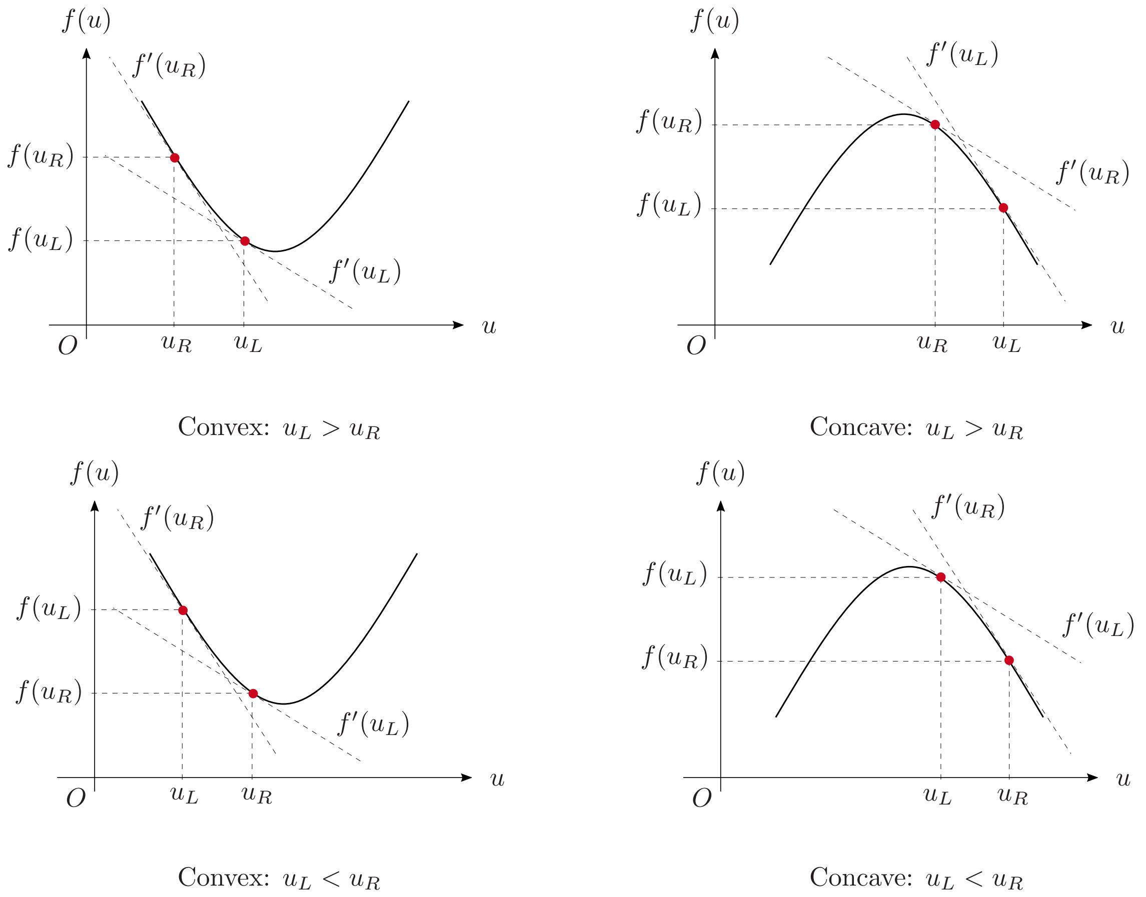
Convex: $u_L > u_R$ Concave: $u_L < u_R$

Figure 2: Case 1.2: Convexity and concavity for $u_L < u_R$ and $u_L > u_R$, where the condition $f'(u_L) \geq 0 \geq f'(u_R) \wedge [f]/[u] > 0$ holds.

- $f(u)$ concave with the condition $(\star\star\star)$ leads to $f(u)$ decreasing at u_L and increasing at u_R. Hence, it must be $u_R < u_L$, which also implies a shock. Therefore, we obtain $u_s \in [u_R, u_l]$ which leads to $f(u_s) > f(u_{R,L})$. As a consequence, $\alpha < 0$. Finally, the convexity of $f(u)$ do lead us also to the possibility of $C_{j+1/2} > 0$.

Therefore, by combining case 1, case 2.1, case 2.2, and case 3 altogether, we recognize that only the convexity of f does lead us to the possibility of $C_{j+1/2} > 0$, while the concavity of f has failed at case 2.2

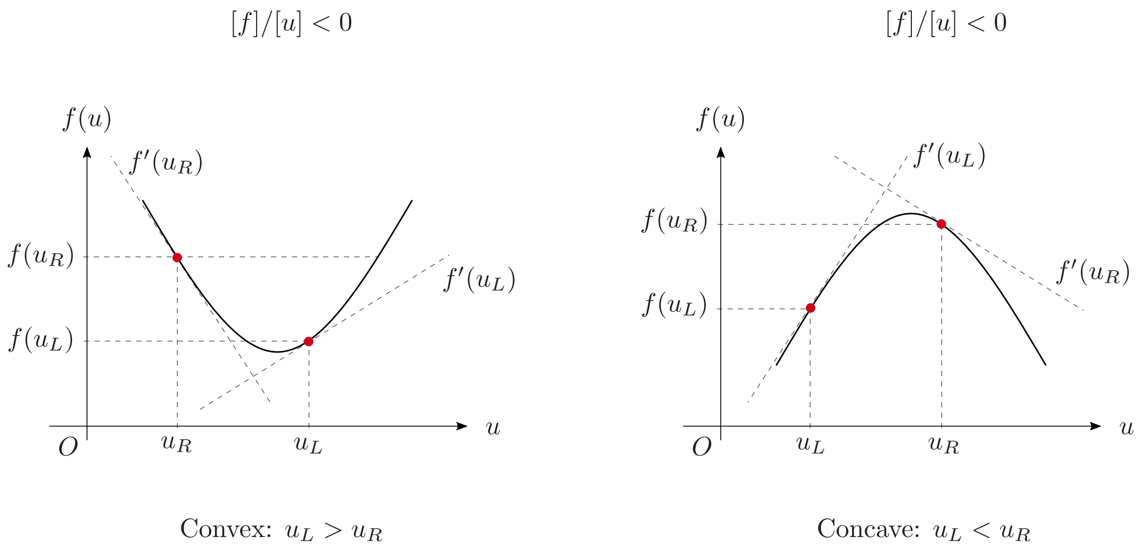
In case of $D_{j-1/2}$ the argument will be proceeded the same as for case $C_{j+1/2}$, meaning that the convexity of f does lead us to the possibility of $D_{j-1/2} > 0$, while the concavity is not so.



Convex: $u_L > u_R$ Concave: $u_L > u_R$

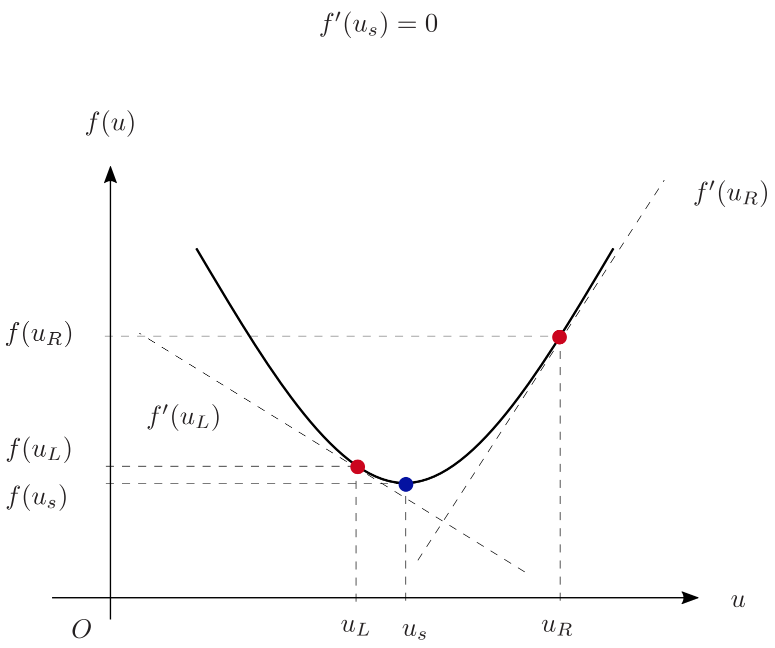
Convex: $u_L < u_R$ Concave: $u_L < u_R$

Figure 3: Case 2.1: Convexity and concavity for $u_L < u_R$ and $u_L > u_R$, where the condition $f'(u_L) \leq 0 \wedge f'(u_R) \leq 0$ holds.

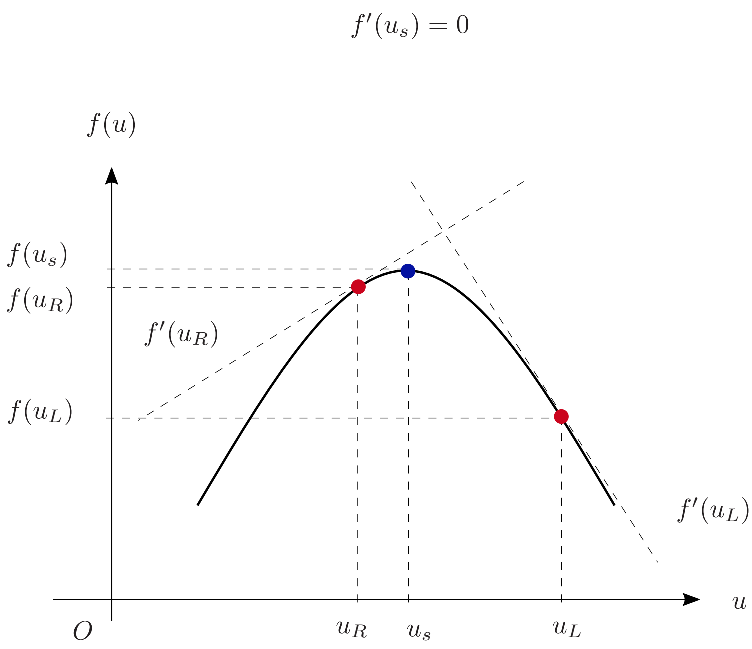


Convex: $u_L > u_R$ Concave: $u_L < u_R$

Figure 4: Case 2.2: Convexity and concavity for $u_L < u_R$ and $u_L > u_R$, where the condition $f'(u_L) \geq 0 \geq f'(u_R) \wedge [f]/[u] < 0$ holds.



Convex: $u_L < u_R$



Concave: $u_L > u_R$

Figure 5: Case 3: Convexity and concavity for $u_L < u_R$ and $u_L > u_R$, where the condition $f'(u_L) < 0 < f'(u_R)$ holds.