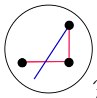
1 A small note about monotonicity: *Upwind* scheme



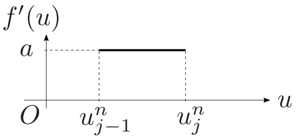
$(a > 0)$ CFL: $0 \le f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\left(f(u_{j-1}^n) - f(u_j^n)\right)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \frac{\Delta t}{\Delta x} f'(u_{j-1}^n)$$

$$B_1 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 - \frac{\Delta t}{\Delta x} f'(u_j^n)$$

$$C_1 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = 0$$



$\mathcal{H}_{\Delta t}$ Monotonically increasing?

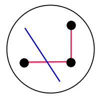
$A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0$ ✓

$B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ ✓

$\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$A_1 \le 0 \Leftrightarrow f'(u_{j-1}^n) \le 0$ $\lightning$

$B_1 \le 0 \Leftrightarrow f'(u_j^n) \ge \Delta x/\Delta t$ $\overset{\text{CFL}}{\to}$ $\lightning$



$(a < 0)$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\Big(f(u_{j-1}^n) - f(u_j^n)\Big)$$

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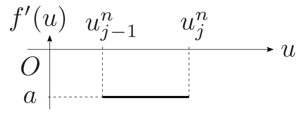

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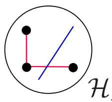

$(a > 0)$

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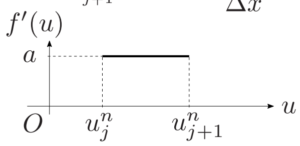

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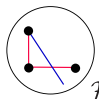

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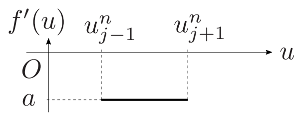

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$B_2 \le 0 \Leftrightarrow f'(u_j^n) \le -\Delta x/\Delta t$ $\overset{\text{CFL}}{\to}$ $\lightning$

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Figure 1: Monotonicity of *Upwind* scheme: 8 different cases.

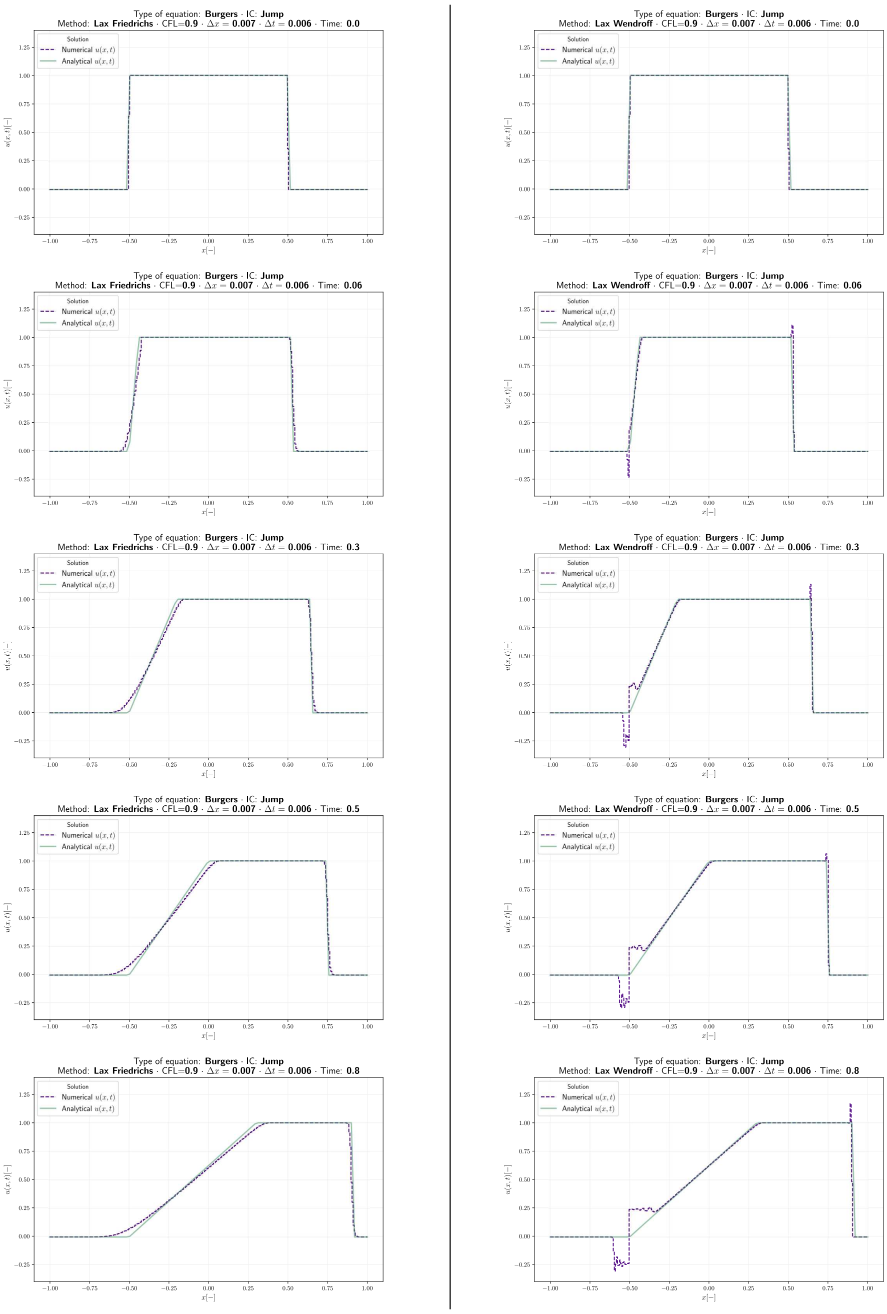

Figure 2: Oscillatory phenomena around discontinuity: (left) none oscillation founded in *Lax-Friedrichs*; (right) oscillation observed in *Lax-Wendroff*.