Global Exercise - 13

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1 Conservation form - Finite Volume Method (cont.)

Example 1. *Derivation of conservation form.*

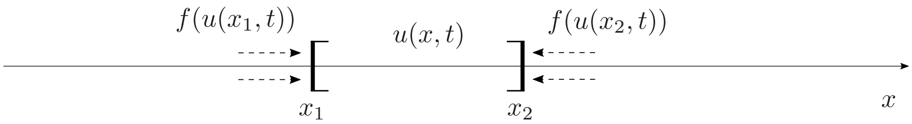


Figure 1: Conservation for a scalar conserved quantity $u(x,t)$ over domain $[x_1, x_2]$.

Starting point is the **integral form** of conservation law. Recall the integral formulation of conservation law for a scalar conserved quantity $u(x,t)$ over domain $[x_1, x_2]$, herein, reads

$$\boxed{\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t)).} \tag{1}$$

Integration both side of (1) over temporal interval $[t_1, t_2]$ yields

$$\left(\int_{x_1}^{x_2} u(x,t)\,dx\right)\Bigg|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \left(f(u(x_1,t)) - f(u(x_2,t))\right) dt, \tag{2}$$

which leads to ***another* integral form** of the conservation law, as follows

$$\boxed{\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt.} \tag{3}$$

The integral form of conservation law shown in (3) is useful to derive the conservation form for finite volume method (FVM). Then, the derivation is performed by taking into consideration of the spatial interval $[x_{j-1/2}, x_{j+1/2}]$ and temporal interval $[t_n, t_{n+1}]$ instead of $[x_1, x_2]$ and $[t_1, t_2]$, respectively, as follows

$$[x_1, x_2] \to [x_{j-1/2}, x_{j+1/2}], \tag{4}$$
$$[t_1, t_2] \to [t_n, t_{n+1}]. \tag{5}$$

Hence, the integral form in (3) becomes

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,dx \\ + \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))\,dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,dt.\end{aligned} \tag{6}$$

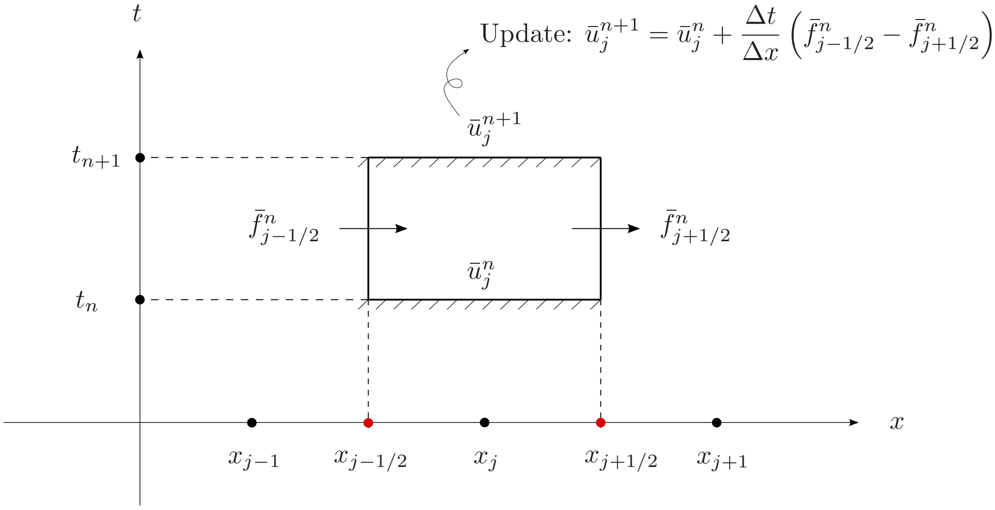


Figure 2: Finite volume update: conservation form.

Next, consideration of the cell average formulation, which, by definition, reads

$$\bar{u}_j^n := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\, dx, \tag{7}$$

and multiplication of both sides of (6) by $1/\Delta x$ yield the following relation

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{1}{\Delta x}\left(\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt\right). \tag{8}$$

Moreover, by defining the **numerical flux functions** as follows

$$\tilde{f}_{j-1/2}^n := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt, \tag{9}$$

$$\tilde{f}_{j+1/2}^n := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt, \tag{10}$$

the expression (8) now becomes the **conservation form**

$$\therefore \quad \boxed{\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n\right).} \tag{11}$$

Example 2. *Lipschitz continuity.*

Example 3. *Derivation of the conservative form of upwind scheme. Examine its numerical flux function: conservative? consistent?*

Example 4. *Derivation of the conservative form of Lax-Friedrichs scheme. Examine its numerical flux function: conservative? consistent?*

Example 5. *Derivation of the conservative form of Lax-Wendroff scheme. Examine its numerical flux function: conservative? consistent?*

Example 6. *Derivation of the conservative form of the two-step Lax-Wendroff scheme. Examine its numerical flux function: conservative? consistent?*

Example 7. *Derivation of the conservative form of MacCormack scheme. Examine its numerical flux function: conservative? consistent?*

6 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 8. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*

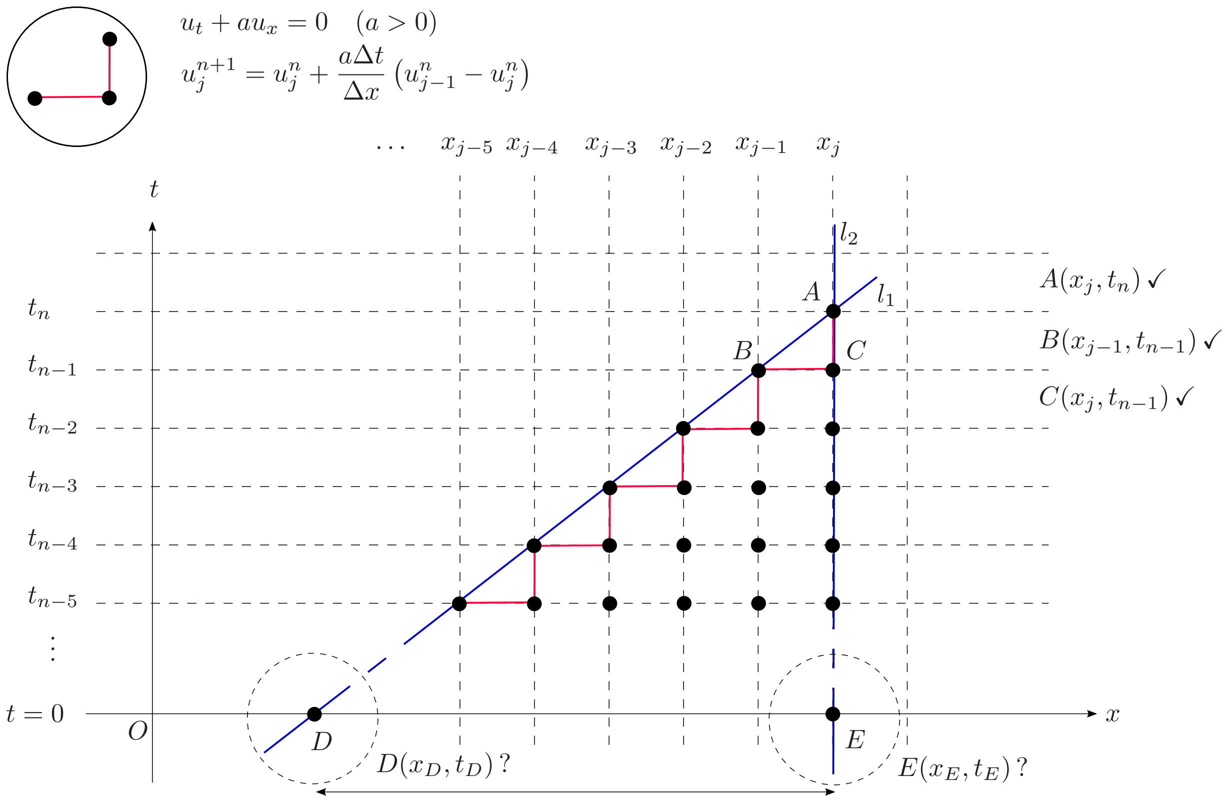


Figure 3: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 3, the numerical value computed at point A depends essentially on computed initial conditions laying between point D and E.

1. Perspective of indical subscription:
   Line (l_1) passsing point $A(j, n)$ and $B(j-1, n-1)$ has the following form

$$
\begin{aligned}
(l_1): \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A}(\xi - \xi_A) \\
\Leftrightarrow \tau &= n + \frac{(n-1) - n}{(j-1) - j}(\xi - j) \\
\Leftrightarrow \tau &= n + \frac{-1}{-1}(\xi - j), \qquad (12)
\end{aligned}
$$

   where τ is the indical variable corresponding to t, and x the indical variable to x. Hence, line (l_1) passing line x with index $\tau = 0$ at point D leads to the following relation

$$
\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \qquad (13)
$$

Likewise, line (l_2) passing line x with index $\tau = 0$ at point E leads to the following relation

$$x_\xi - x_j = 0. \tag{14}$$

Therefore, by combining (13) and (14) we arrive at the numerical domain of dependence for the One-sided method in terms of indical perspective

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x_\xi \,\middle|\, -n\Delta x \le x_\xi - x_j \le 0 \right\}. \tag{15}$$

Next, by using the CFL number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \overset{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \tag{16}$$

Then, by substituting (16) into (15) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \le x - x_j \le 0 \right\}}. \tag{17}$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{18}$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}(x_j, t_n) \subset \mathcal{D}_{\Delta t}(x_j, t_n), \tag{19}$$

which implies that characteristics should lie with the triangular zone under the line (l_1) and (l_2), as shown in Figure 3. Therefore, substitution of (18) into (17) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \le (x_j - at_n) - x_j \le 0 \Leftrightarrow -\frac{at_n}{\nu} \le -at_n \le 0, \tag{20}$$

which, equally, leads to the CFL condtion

$$\therefore \quad \boxed{0 \le \nu \le 1 \Leftrightarrow 0 \le \Delta t \le \frac{\Delta x}{a}}. \tag{21}$$

Herein, the CFL condition (21) leads to constraint on the time step Δt for the case when $a > 0$. Note in passing that ν is non-negative.

2. Perspective of fixed-point value:
   Line (l_1) passsing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned}(l_1): \quad t = t_A + \frac{t_B - t_A}{x_B - x_A}(x - x_A) &\Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x}(x - x_j). \end{aligned} \tag{22}$$

Hence, line (l_1) passing line $t = 0$ at point D leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \tag{23}$$

Likewise, line (l_2) passing line $t = 0$ at point E leads to the relation

$$x - x_j = 0. \tag{24}$$

Therefore, combination of (23) and (24) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \tag{25}$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (18), reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{26}$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

$$-\frac{t_n \Delta x}{\Delta t} \leq \left(x_j - a t_n\right) - x_j \leq 0, \tag{27}$$

which we have substituted (26) into (25). Herein, the relation (27) enforcing CFL condition on the time step Δt

$$\therefore \quad \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a},} \tag{28}$$

which is similar to (21).

Example 9. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the right.*

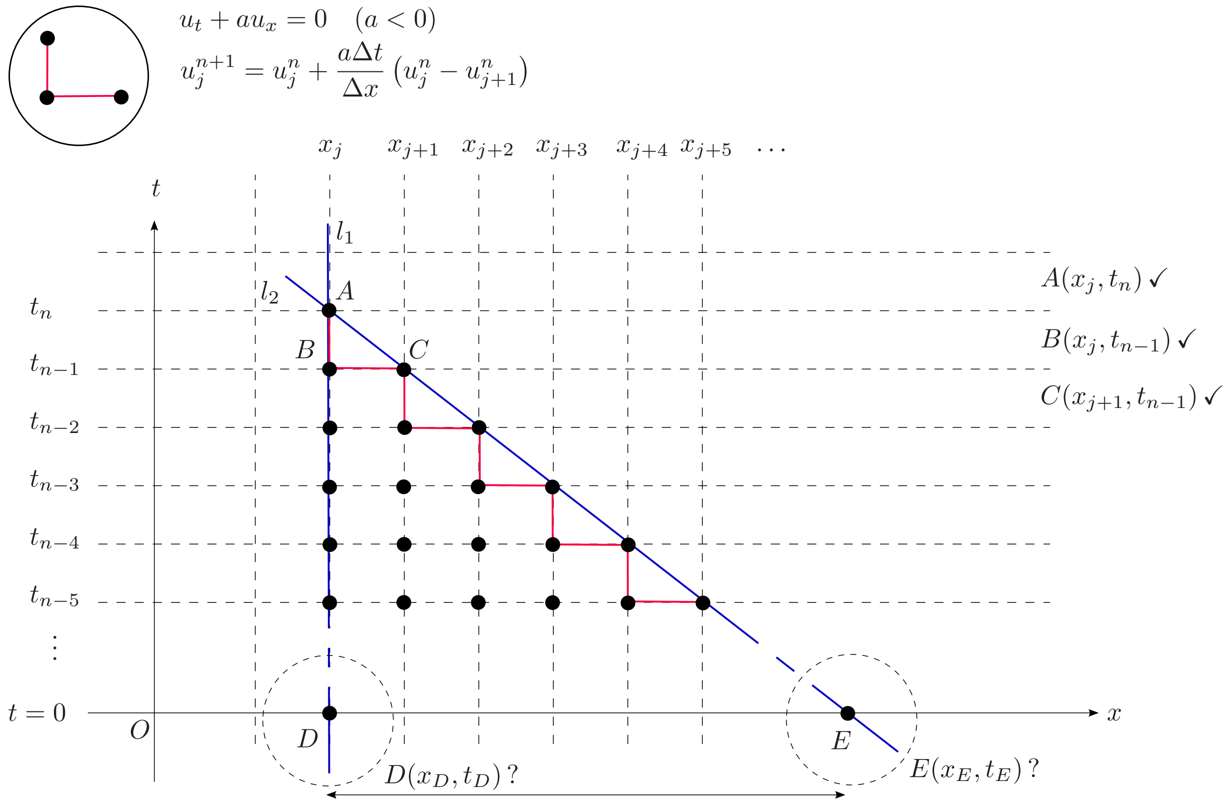


Figure 4: Numerical domain of dependence for One-sided method, where the stencils point to the right.

Similarly, by following steps done in Example 8 we obtain the following summary:

1. Point E in terms of fixed-point value satisfying

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\left(x - x_j\right) \tag{29}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, 0 \leq x - x_j \leq -\frac{a t_n}{\nu} \right\}. \tag{30}$$

3. Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{31}$$

4. CFL condition reads

$$\therefore \quad \boxed{\Delta t \geq \frac{\Delta x}{a}.} \tag{32}$$

Note in passing that the advection velocity a in this Example 9 is $a < 0$.

Example 10. *Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.*

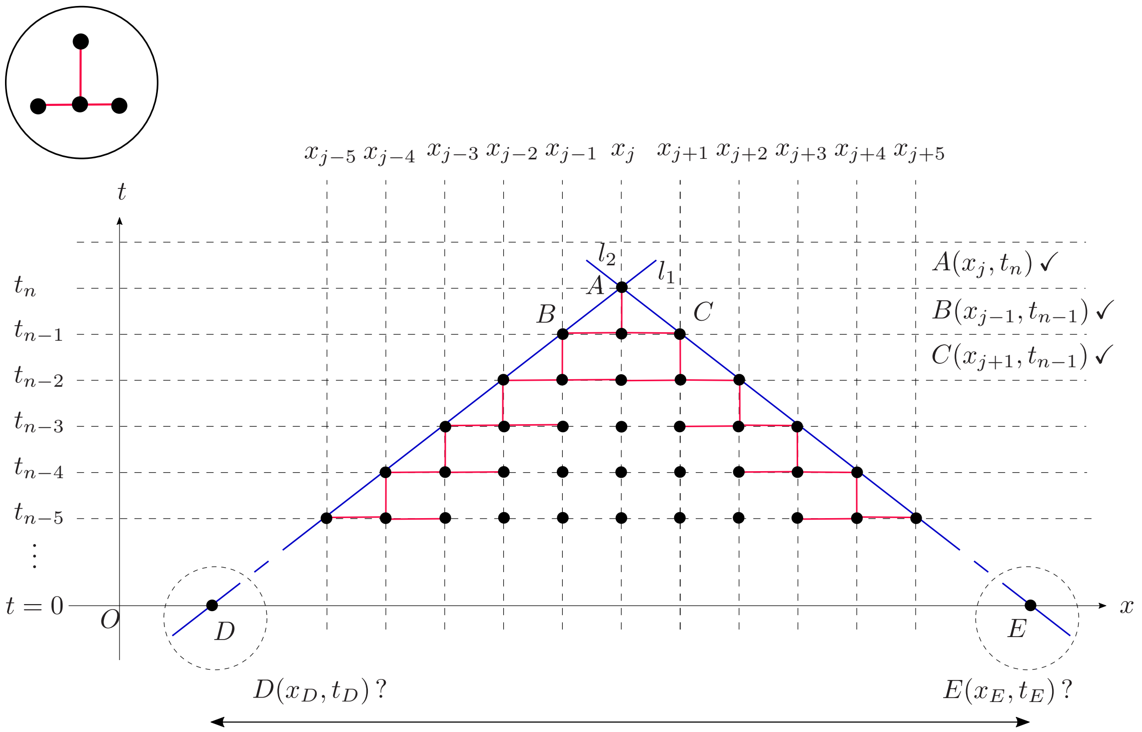


Figure 5: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 8 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}\left(x - x_j\right) \tag{33}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\left(x - x_j\right) \tag{34}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, |x - x_j| \leq \frac{a t_n}{\nu} \right\}. \tag{35}$$

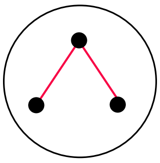
3. Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{36}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left|\frac{a \Delta t}{\Delta x}\right| \leq 1.} \tag{37}$$

Example 11. *Examine the numerical domain of dependence of* Lax-Friedrichs *method.*



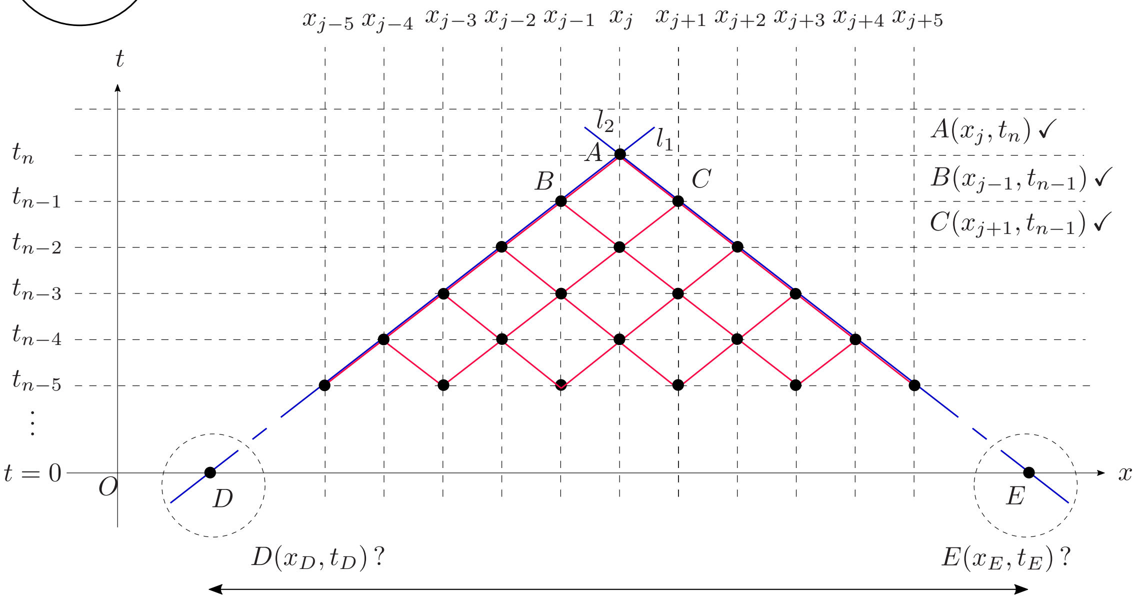


Figure 6: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 8, or the same as 10 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \tag{38}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}(x - x_j) \tag{39}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \le \frac{a t_n}{\nu} \right\}. \tag{40}$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{41}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a \Delta t}{\Delta x} \right| \le 1.} \tag{42}$$

7 von Neumann stability analysis

Example 12. *von Neumann stability analysis for Upwind method.*

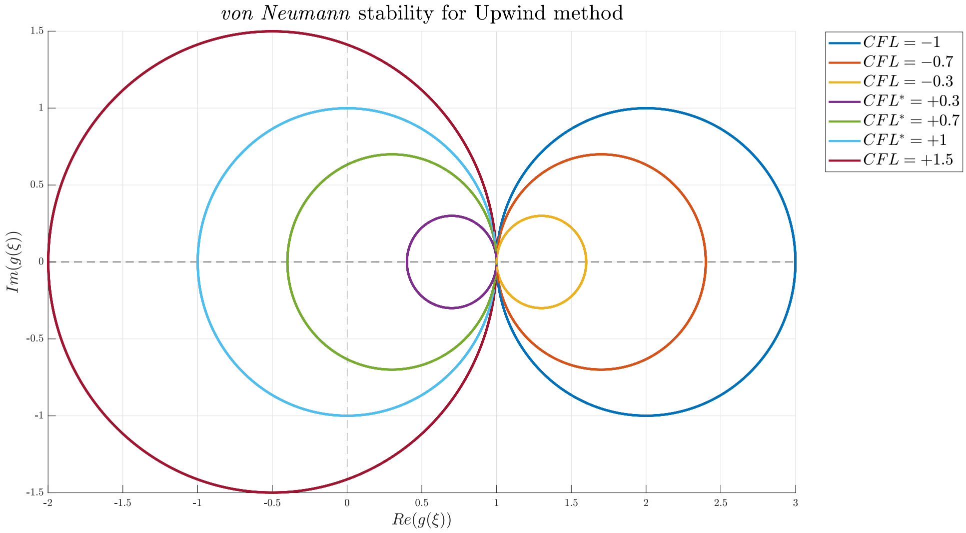


Figure 7: *von Neumann* stability analysis for Upwind method.

Example 13. *Summary consistency + stability* $\Rightarrow$ *Convergence*

8 Conservation form - Finite Volume Method

Example 14. *Derivation of conservation form.*