# Global Exercise - 12

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## 1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1: $W := R^{-1}U$ as the scheme shown in exercise

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + R\Lambda R^{-1}U_x &= 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

   where the matrix $A$ is diagonalizable with a transformation matrix $R \in \mathbb{R}^{N\times N}$ in the form

$$A = R\Lambda R^{-1}.$$

2. Case 2: $W := TU$ as the scheme shown in lecture note

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + T^{-1}\Lambda TU_x &= 0 \\ TU_t + \Lambda TU_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

   where the matrix $A$ is diagonalizable with a transformation matrix $T \in \mathbb{R}^{N\times N}$ in the form

$$A = T^{-1}\Lambda T.$$

→ Note in passing that both schemes result in the same solution.
→ We have just to be consistent with which scheme to follow.

## 2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

**Example 1.** *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*

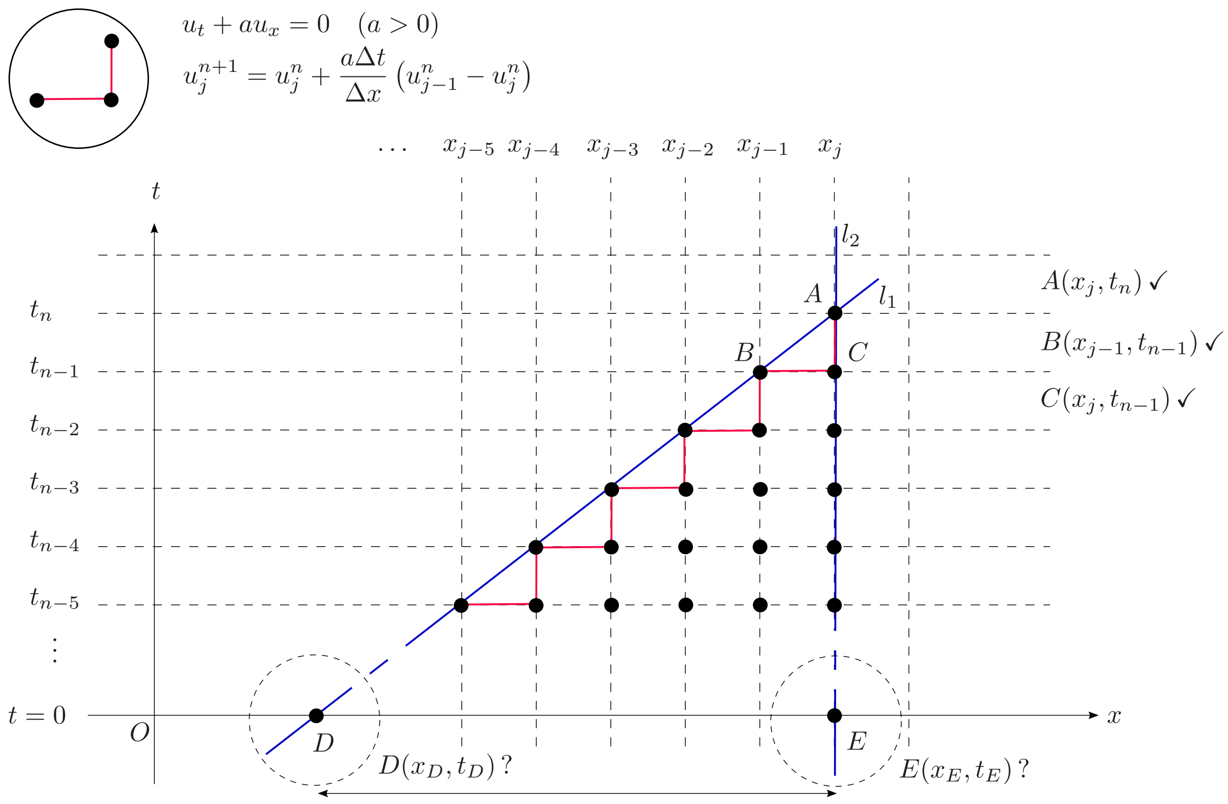



Figure 1: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 1, the numerical value computed at point $A$ depends essentially on computed initial conditions laying between point $D$ and $E$.

1. Perspective of indical subscription:
   Line $(l_1)$ passsing point $A(j, n)$ and $B(j-1, n-1)$ has the following form

$$
\begin{aligned}
(l_1): \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A}\left(\xi - \xi_A\right) \\
\Leftrightarrow \tau &= n + \frac{(n-1) - n}{(j-1) - j}\left(\xi - j\right) \\
\Leftrightarrow \tau &= n + \frac{-1}{-1}\left(\xi - j\right), \qquad (1)
\end{aligned}
$$

   where $\tau$ is the indical variable corresponding to $t$, and $x$ the indical variable to $x$. Hence, line $(l_1)$ passing line $x$ with index $\tau = 0$ at point $D$ leads to the following relation

$$
\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \qquad (2)
$$

Likewise, line $(l_2)$ passing line $x$ with index $\tau = 0$ at point $E$ leads to the following relation

$$x_\xi - x_j = 0. \tag{3}$$

Therefore, by combining (2) and (3) we arrive at the numerical domain of dependence for the One-sided method in terms of indical perspective

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x_\xi \,\middle|\, -n\Delta x \le x_\xi - x_j \le 0 \right\}. \tag{4}$$

Next, by using the $CFL$ number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \overset{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \tag{5}$$

Then, by substituting (5) into (4) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \le x - x_j \le 0 \right\}}. \tag{6}$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{7}$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}\left(x_j, t_n\right) \subset \mathcal{D}_{\Delta t}\left(x_j, t_n\right), \tag{8}$$

which implies that characteristics should lie with the triangular zone under the line $(l_1)$ and $(l_2)$, as shown in Figure 1. Therefore, substitution of (7) into (6) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \le \left(x_j - at_n\right) - x_j \le 0 \Leftrightarrow -\frac{at_n}{\nu} \le -at_n \le 0, \tag{9}$$

which, equally, leads to the CFL condtion

$$\therefore \quad \boxed{0 \le \nu \le 1 \Leftrightarrow 0 \le \Delta t \le \frac{\Delta x}{a}.} \tag{10}$$

Herein, the CFL condition (10) leads to constraint on the time step $\Delta t$ for the case when $a > 0$. Note in passing that $\nu$ is non-negative.

2. Perspective of fixed-point value:
   Line $(l_1)$ passsing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned}(l_1): \quad t = t_A + \frac{t_B - t_A}{x_B - x_A}\left(x - x_A\right) &\Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}\left(x - x_j\right) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x}\left(x - x_j\right).\end{aligned} \tag{11}$$

Hence, line $(l_1)$ passing line $t = 0$ at point $D$ leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \tag{12}$$

Likewise, line $(l_2)$ passing line $t = 0$ at point $E$ leads to the relation

$$x - x_j = 0. \tag{13}$$

Therefore, combination of (12) and (13) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \tag{14}$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (7), reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{15}$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

$$-\frac{t_n \Delta x}{\Delta t} \leq \left(x_j - a t_n\right) - x_j \leq 0, \tag{16}$$

which we have substituted (15) into (14). Herein, the relation (16) enforcing CFL condition on the time step $\Delta t$

$$\therefore \quad \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a},} \tag{17}$$

which is similar to (10).

**Example 2.** *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the right.*

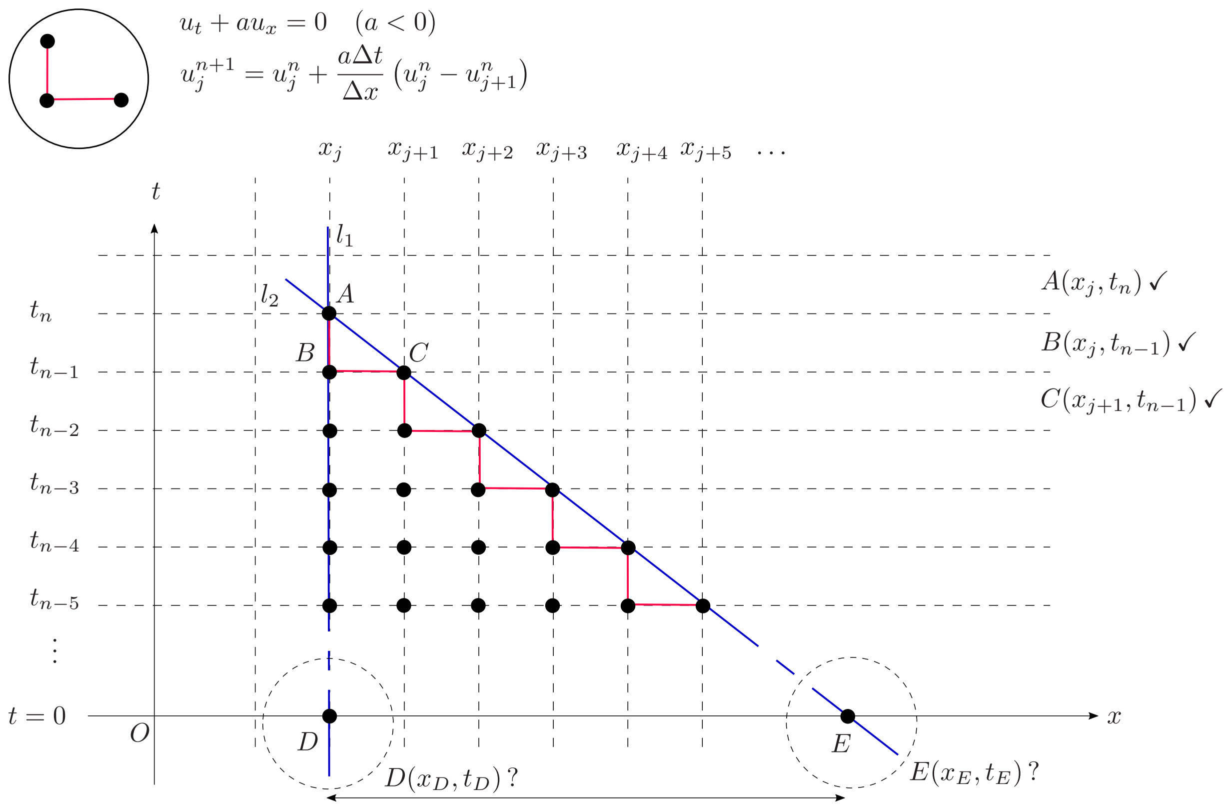



Figure 2: Numerical domain of dependence for One-sided method, where the stencils point to the right.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point $E$ in terms of fixed-point value satisfying

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\left(x - x_j\right) \tag{18}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, 0 \leq x - x_j \leq -\frac{at_n}{\nu} \right\}. \tag{19}$$

3. Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{20}$$

4. CFL condition reads

$$\therefore \quad \boxed{\Delta t \geq \frac{\Delta x}{a}.} \tag{21}$$

Note in passing that the advection velocity $a$ in this Example 2 is $a < 0$.

**Example 3.** *Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.*

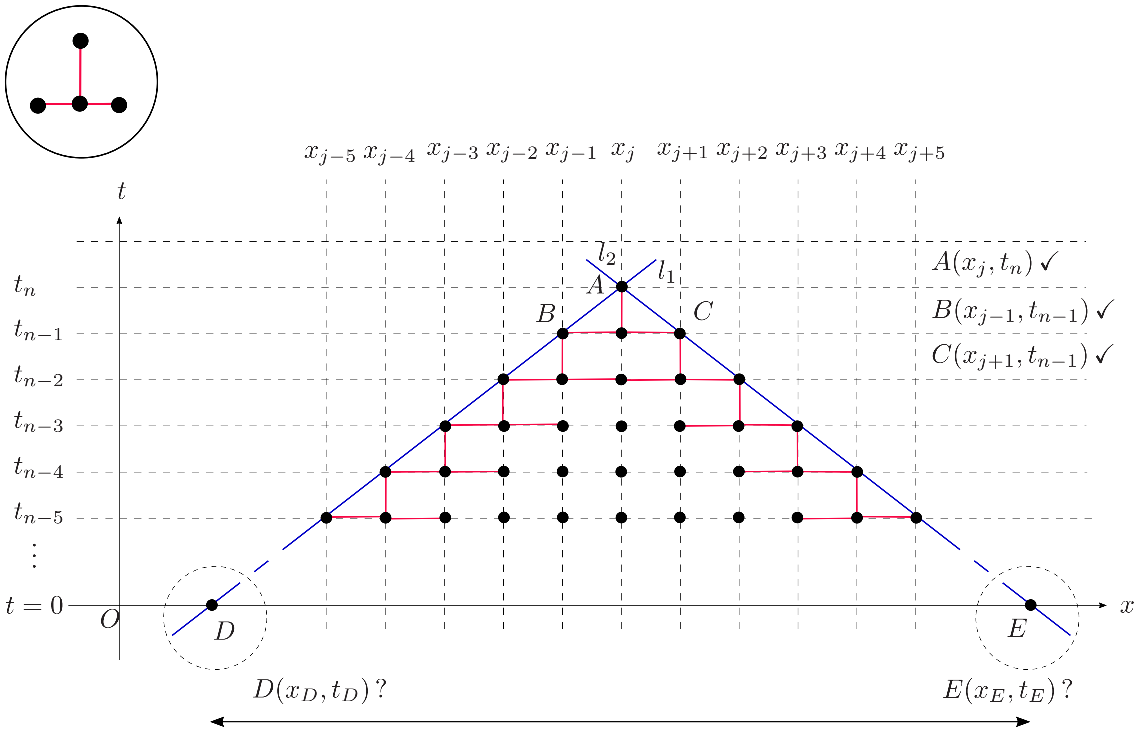



Figure 3: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point $D$ and $E$ in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \tag{22}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}(x - x_j) \tag{23}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \leq \frac{at_n}{\nu} \right\}. \tag{24}$$

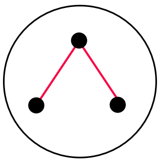
3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{25}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left|\frac{a\Delta t}{\Delta x}\right| \leq 1.} \tag{26}$$

**Example 4.** *Examine the numerical domain of dependence of* Lax-Friedrichs *method.*



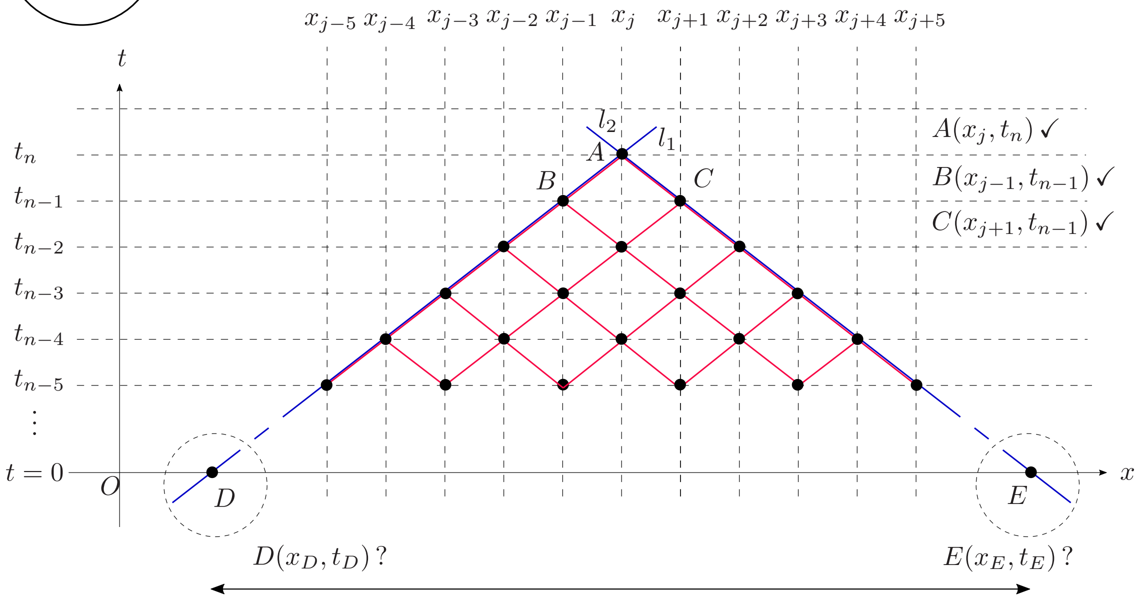



Figure 4: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1, or the same as 3 we obtain the following summary:

1. Point $D$ and $E$ in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \tag{27}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}(x - x_j) \tag{28}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \le \frac{at_n}{\nu} \right\}. \tag{29}$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{30}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a\Delta t}{\Delta x} \right| \le 1.} \tag{31}$$

# 3 Conservation form - Finite Volume Method

**Example 5.** *Consider the* Burgers' *equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0.$$

1. *Show that the upwind scheme for this version of* Burgers' *equation as follows*

   $$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_{j-1}^n)^2 - \frac{1}{2}(u_j^n)^2\right)$$

   can *be represented in* conservation form*.*

2. *If we rewrite the above* Burgers' *equation as*

   $$u_t + uu_x = 0,$$

   *we can assume it as a linear advection problem with speed $u$. Prove that the upwind scheme for the latter equation as*

   $$u_j^{n+1} = u_j^n + \frac{u_j^n \Delta t}{\Delta x}(u_{j-1}^n - u_j^n)$$

   cannot *be represented in* conservation form*.*

Approach:

1. If we take the numerical flux

   $$\tilde{F}_{j+1/2}(u_j^n, u_{j+1}^n) = f(u_j^n) = \frac{1}{2}(u_j^n)^2,$$

   then the corresponding numerical scheme can be rewritten in conservation form

   $$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}(\tilde{F}_{j-1/2}(u_{j-1}^n, u_j^n) - \tilde{F}_{j+1/2}(u_j^n, u_{j+1}^n)).$$

2. *Proof.* Let us take a look at the derivatives of $H_{\Delta t}$ by $u_{j-1}^n, u_j^n$ and $u_{j+1}^n$ as follows

   $$\begin{aligned} x &= \partial_{u_{j-1}^n} H_{\Delta t} = \frac{u_j^n \Delta t}{\Delta x}, \\ y &= \partial_{u_j^n} H_{\Delta t} = 1 + \frac{u_{j-1}^n \Delta t}{\Delta x} - \frac{2u_j^n \Delta t}{\Delta x}, \\ z &= \partial_{u_{j+1}^n} H_{\Delta t} = 0. \end{aligned}$$

   Summation of the term $x|_{j+1}$, the term $y|_j$, and the term $z|_{j-1}$ must be equal to 1 in order to satisfy the conservation form. However, we observe that the summation of these three terms does not satisfy the equality to 1, as follows

   $$\begin{aligned} x|_{j+1} + y|_j + z|_{j-1} &= 1 + \frac{\Delta t}{\Delta x}(u_{j+1}^n - 2u_j^n + u_{j-1}j^n) \\ &\neq 1. \end{aligned}$$

Besides, if we would like to check this property for the first scheme, we obtain the following relations

$$
\begin{aligned}
&x = \frac{\Delta t}{\Delta x} u_{j-1}^{n}, \\
&y = 1 - \frac{\Delta t}{\Delta x} u_{j}^{n}, \\
&z = 0,
\end{aligned}
$$

which leads to the summation of these three terms, i.e. the term $x|_{j+1}$, the term $y|_{j}$, and the term $z|_{j-1}$, equal to 1, as follows

$$
x|_{j+1} + y|_{j} + z|_{j-1} = 1.
$$

$\square$