

# Global Exercise - 15

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## 1 A small note about monotonicity: *Upwind* scheme

**Example 1.** Examine monotonicity of  $\mathcal{H}_{\Delta t}$  for conservation law  $u_t + au_x = 0$ .

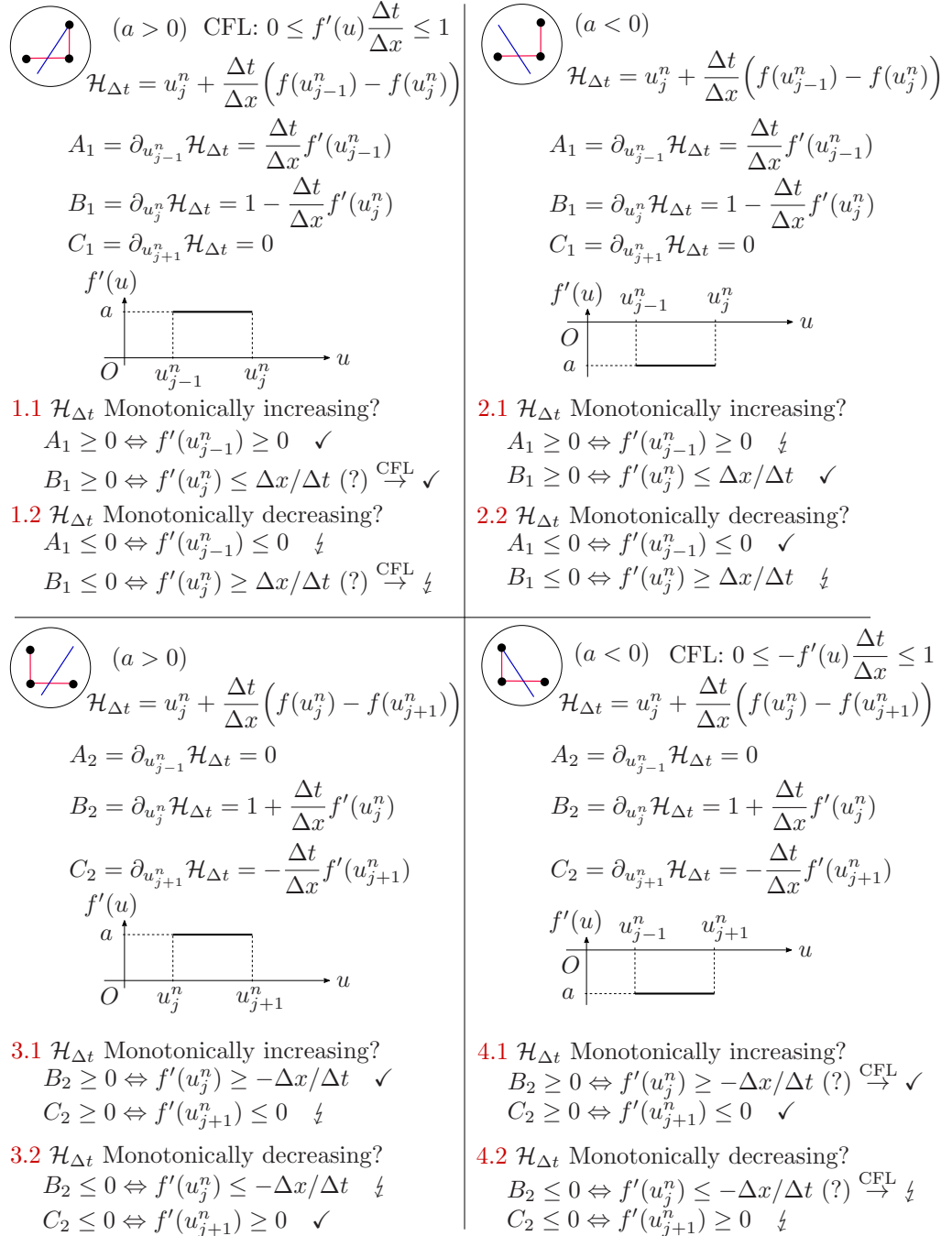


Figure 1: Monotonicity of *Upwind* scheme: 8 different cases.

## 2 Structure of FVM

**Example 2.** *Monotone  $\rightarrow L_1$ -Contracting  $\rightarrow TVD \rightarrow Monotonicity-Preserving$ .*

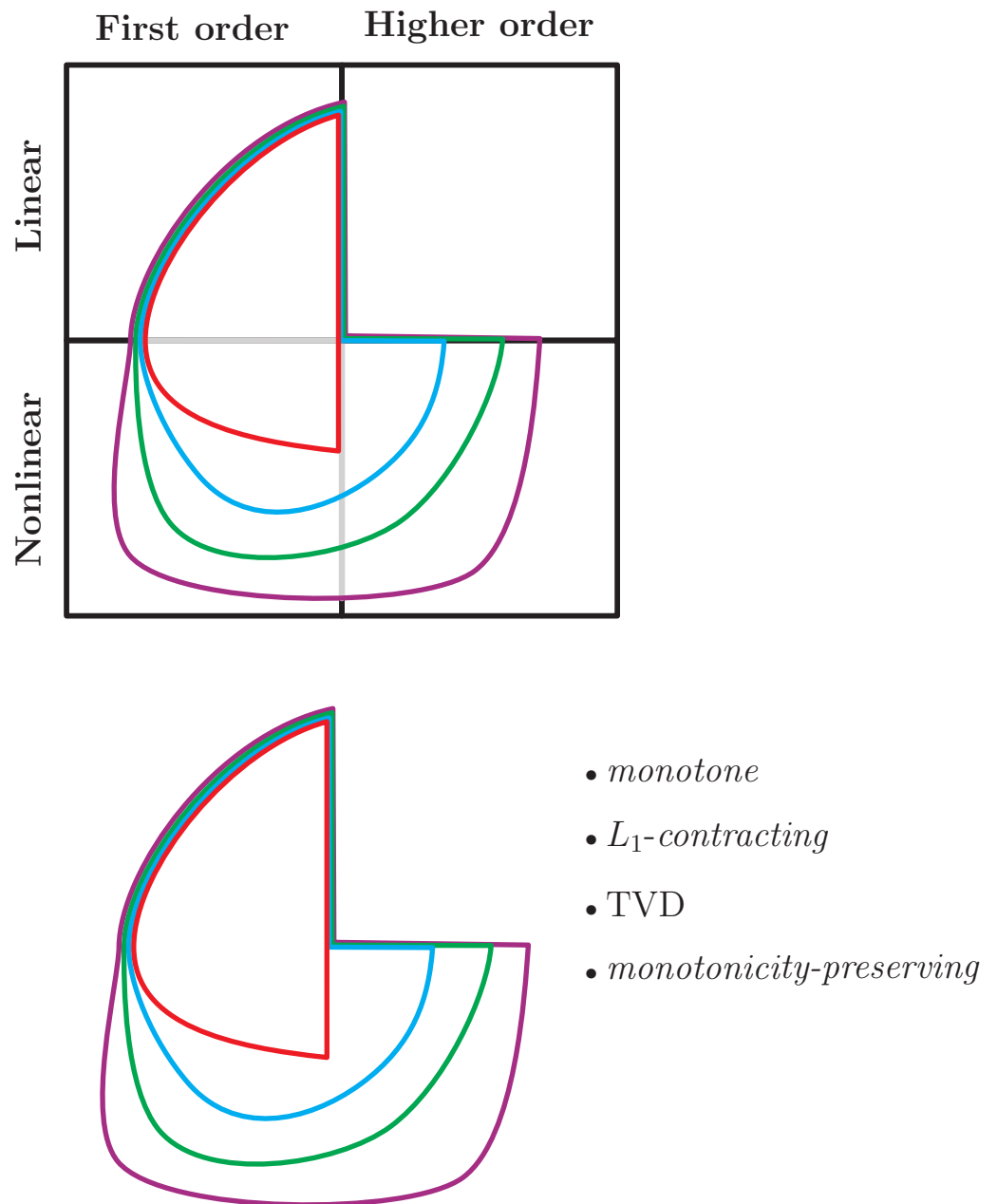


Figure 2: Structure of Finite Volume Method [Lecture note page 115].

### 3 Limiter

**Example 3.** Examine the 1<sup>st</sup>-order-accurate *LF* and the 2<sup>nd</sup>-order-accurate *LW*.

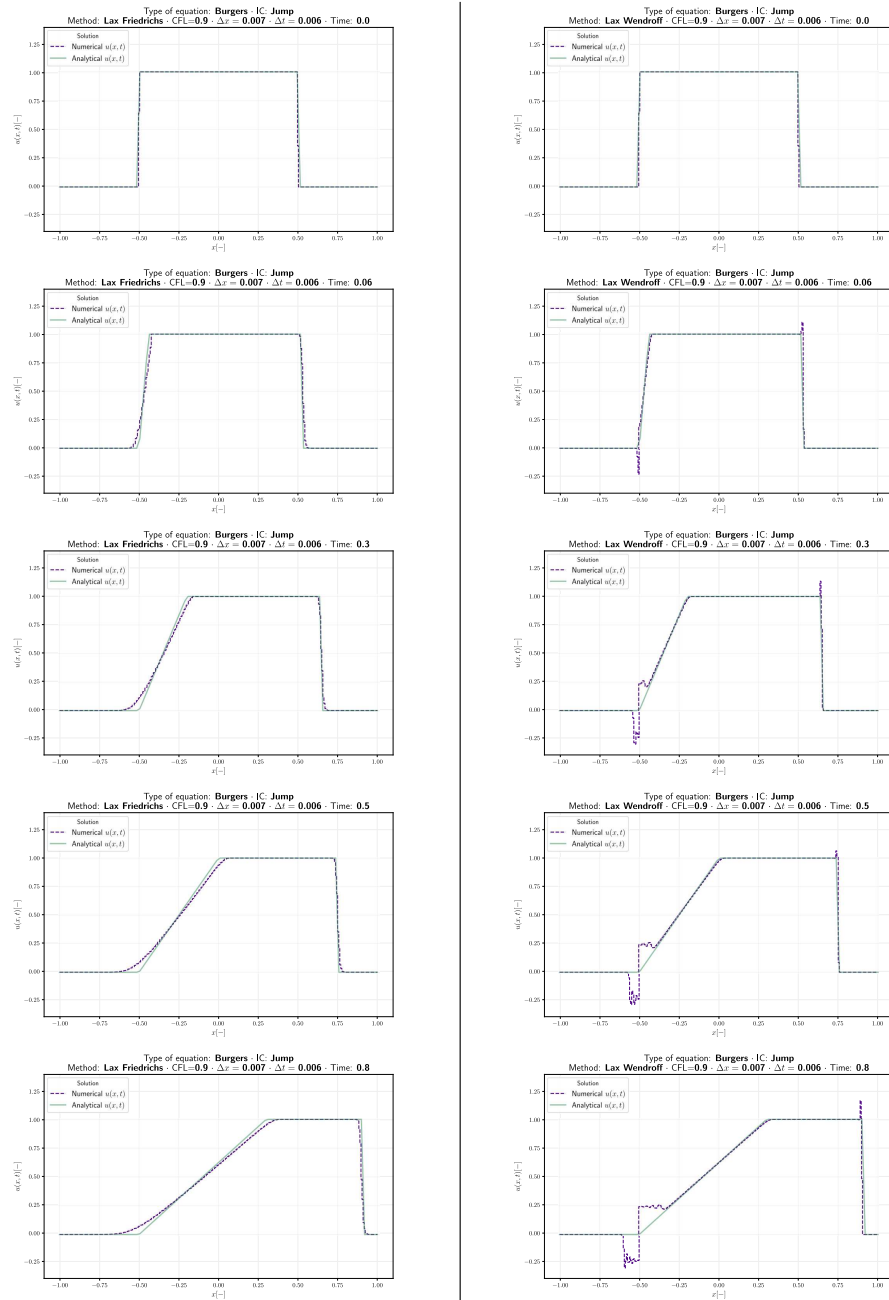


Figure 3: Oscillatory phenomena around discontinuity: (left) none oscillation founded in *Lax-Friedrichs*; (right) oscillation observed in *Lax-Wendroff*.

**Example 4.** Assume the linear advection problem given by

$$u_t + au_x = 0 \quad \text{with} \quad a > 0,$$

and the numerical scheme as follows

$$u_j^{n+1} = u_j^n + \nu (u_{j-1}^n - u_j^n) + \frac{\nu(1-\nu)}{2} \Delta x (\sigma_{j-1}^n - \sigma_j^n),$$

with limited slopes

$$\sigma_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} \phi(\theta_j).$$

Prove that numerical scheme is TVD with limiters  $\phi(\theta)$  given as follows

1.  $\phi(\theta) = \max(0, \min(\theta, 2))$ ,
2.  $\phi(\theta) = \max(0, \min(2\theta, 2))$ ,
3.  $\phi(\theta) = \frac{\theta + |\theta|}{\alpha + |\theta|}$  for any fixed  $\alpha \geq 1$
4.  $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$ .

Denote which of them are second order schemes.

According to the theorem on TVD region of *Upwind* numerical scheme with limiter presented in the lecture, we know that limiter gives us TVD scheme if the following relations

$$\begin{aligned} 0 &\leq \phi(\theta) \leq 2\theta, \\ |\phi(\theta)| &\leq 2, \end{aligned}$$

holds. Besides, the scheme is of 2nd-order if and only if  $\phi(1) = 1$  holds. Therefore, we can simply check lower and upper bounds of limiters, as follows

1.  $\phi(\theta) = \max(0, \min(\theta, 2))$  satisfies bounds and goes through  $\phi(1) = 1$ .  
Hence, it corresponds to 2nd-order TVD method.
2.  $\phi(\theta) = \max(0, \min(2\theta, 2))$  satisfies bounds but  $\phi(1) = 2$ .  
Hence, it is of 1st-order TVD scheme.
3.  $\phi(\theta) = \frac{\theta + |\theta|}{\alpha + |\theta|}$  satisfies bounds for any fixed  $\alpha \geq 1$ , but  $\phi(1) = 1$  holds only for  $\alpha = 1$ .  
Hence, it is of 2nd-order only if  $\alpha = 1$  and of 1st-order otherwise.
4.  $\phi(\theta) = \max(0, \min(2\theta, 1), \min(\theta, 2))$  satisfies bounds and  $\phi(1) = 1$  holds.  
Hence, it is TVD scheme of 2nd-order.

**Example 5.** *Examine and sketch the following Limiter function.*

$$\phi(\theta) = \min \left( \delta\theta, \frac{1+\theta}{2}, \gamma \right), \text{ for } \delta, \gamma \geq 1.$$

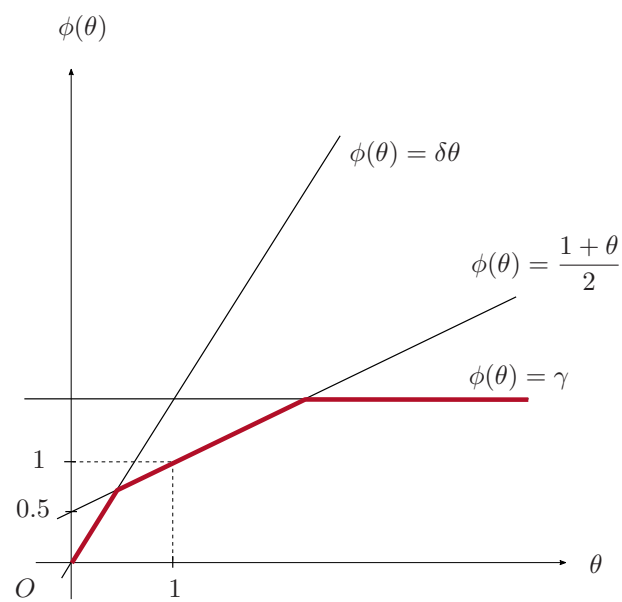
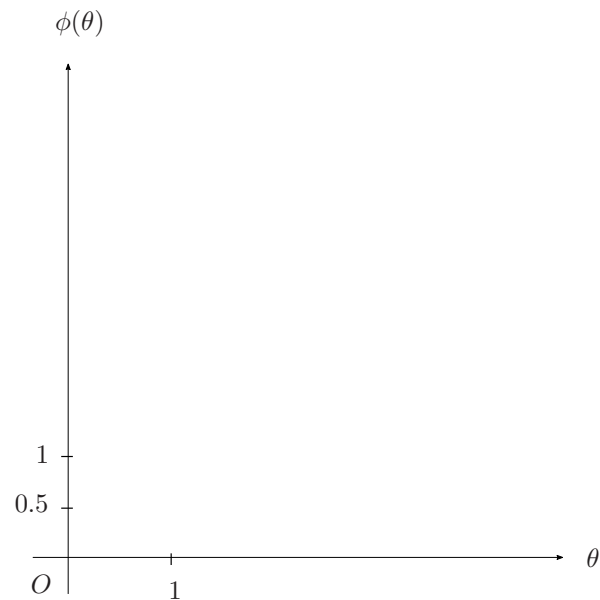


Figure 4: Limiter function  $\phi(\theta) = \min \left( \delta\theta, \frac{1+\theta}{2}, \gamma \right)$ , for  $\delta, \gamma \geq 1$ .

## 4 A bit insight into Godunov's solver

**Example 6.** *Verify the Godunov numerical flux function*

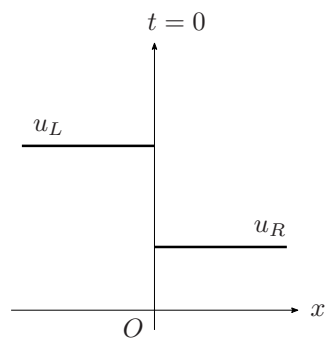
$$g(u_L, u_R) = \begin{cases} \min_{u_L \leq u \leq u_R} f(u) & \text{if } u_L \leq u_R, \\ \max_{u_R \leq u \leq u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

*by considering all possible cases.*

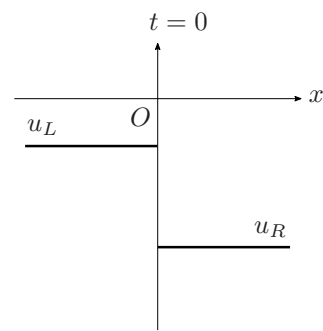
All possible cases read

1.  $f'(u_L), f'(u_R) \geq 0 \Rightarrow u^* = u_L$
2.  $f'(u_L), f'(u_R) \leq 0 \Rightarrow u^* = u_R$
3.  $f'(u_L) \geq 0 \geq f'(u_R) \Rightarrow u^* = u_L$  if  $[f]/[u] > 0$  or  $u^* = u_R$  if  $[f]/[u] < 0$
4.  $f'(u_L) < 0 < f'(u_R) \Rightarrow u^* = u_s$  where  $f'(u_s) = 0$

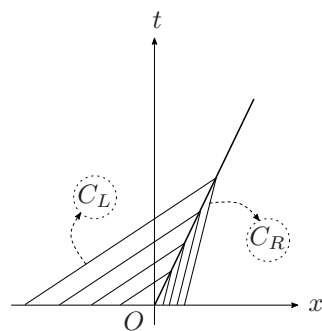
**Example 7.** Examine  $u_L > u_R$ .



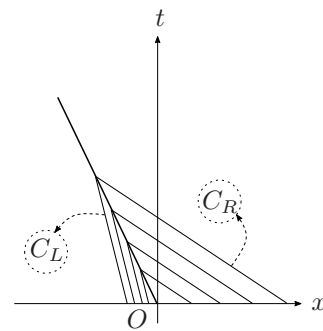
Case 1 :  $u_L > u_R > 0$



Case 3 :  $0 > u_L > u_R$



$$(C_L) : t = \frac{1}{u_L}x - \frac{x_0}{u_L}$$



$$(C_R) : t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

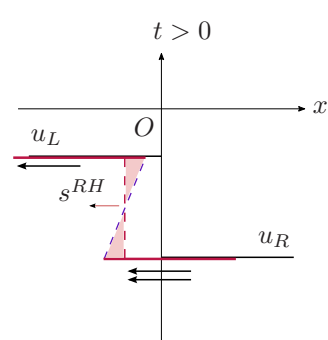
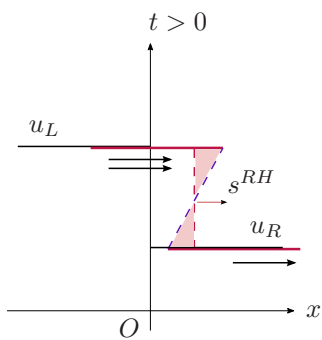
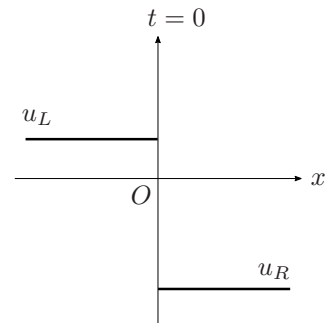
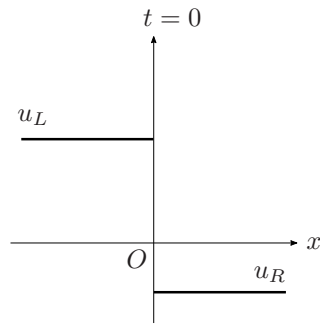


Figure 5: Riemann problem with  $u_L > u_R$ : IC, Characteristics, Solution.

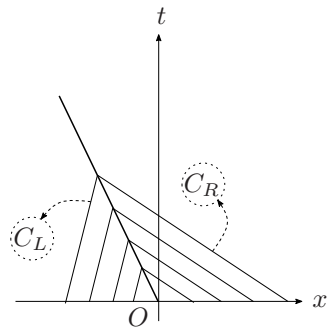
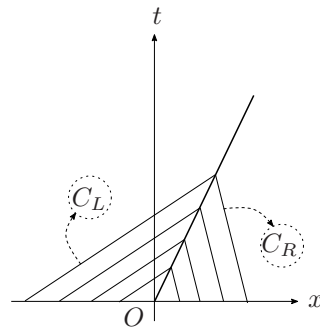
Schock solution

**Example 8.** Examine  $u_L > u_R$ .



Case 2.1 :  $(u_L > 0 > u_R) \wedge |u_L| > |u_R|$

Case 2.2 :  $(u_L > 0 > u_R) \wedge |u_L| < |u_R|$



$$(C_L) : t = \frac{1}{u_L}x - \frac{x_0}{u_L}$$

$$(C_R) : t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

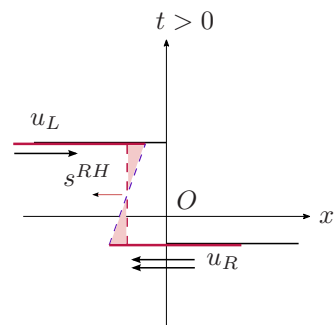
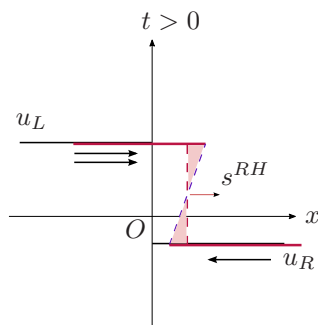


Figure 6: Riemann problem with  $u_L > u_R$ : IC, Characteristics, Solution.

Schock solution



**Example 9.** Examine  $u_L < u_R$ .

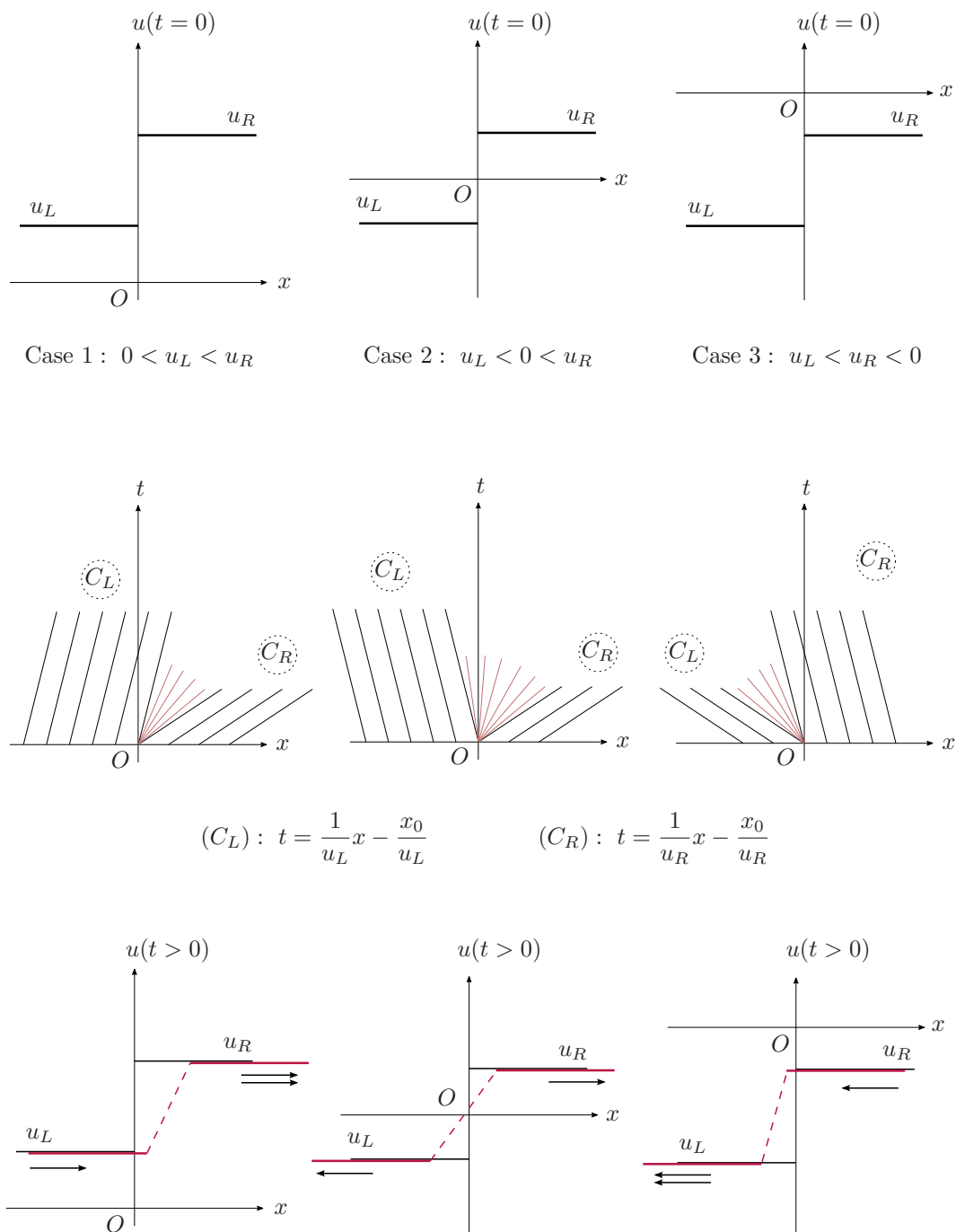


Figure 7: Riemann problem with  $u_L < u_R$ : IC, Characteristics, Solution.

Rarefaction solution

**Example 10.** Consider the Godunov's solver for the conservation law  $u_t + f(u)_x = 0$  with convex and nonlinear flux function  $f$ , e.g. the flux function in Burgers' equation, where

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} (g(u_{j-1}^n, u_j^n) - g(u_j^n, u_{j+1}^n)),$$

and the numerical flux function

$$g(u_L, u_R) = \begin{cases} \min_{u_L \leq u \leq u_R} f(u) & \text{if } u_L \leq u_R, \\ \max_{u_R \leq u \leq u_L} f(u) & \text{if } u_L > u_R. \end{cases}$$

1. Show that the numerical flux function is monotone.
2. Rewrite the scheme in the incremental form.
3. Show that the scheme has the total-variation-diminishing (TVD) property.

Approach:

1. *Proof.* In order to show the monotonicity of the numerical flux function, it is sufficient to show the following relations

$$\boxed{\begin{cases} \partial_u g(u, v) \geq 0, \\ \partial_v g(u, v) \leq 0. \end{cases}} \quad (1)$$

Hence, in case of Burgers' equation, where  $f'(u) = u$ , we proceed as follows

$$g(u_L, u_R) = \begin{cases} f(u_L), & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ f(u_R), & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ f(u_s), & \text{if } f'(u_L) < 0 < f'(u_R), \end{cases}$$

where  $u_s$  denotes the sonic point where  $f'(u_s) = 0$ . Then, by taking the partial derivative of  $g(u_L, u_R)$  w.r.t.  $u_L$  and  $u_R$  we arrive at the following expressions

$$\partial_{u_L} g(u_L, u_R) = \begin{cases} f'(u_L), & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_L} g(u_L, u_R) \geq 0.} \quad (2)$$

Likewise, the partial derivative of  $g(u_L, u_R)$  w.r.t.  $u_R$  goes as follows

$$\partial_{u_R} g(u_L, u_R) = \begin{cases} f'(u_R), & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to

$$\boxed{\partial_{u_R} g(u_L, u_R) \leq 0.} \quad (3)$$

Therefore, by combining (2) and (3), and comparing them with the condition given at (1), we arrive at *q.e.d.*  $\square$

2. The incremental form reads

$$\boxed{u_j^{n+1} = u_j^n + C_{j+1/2}^n (u_{j+1}^n - u_j^n) - D_{j-1/2}^n (u_j^n - u_{j-1}^n).}$$

For any conservative Finite Volume scheme we obtain

$$C_{j+1/2} = -\lambda \frac{g_{j+1/2} - f_j}{u_{j+1} - u_j}$$

$$D_{j-1/2} = \lambda \frac{f_j - g_{j-1/2}}{u_j - u_{j-1}}$$

where  $\lambda = \Delta t / \Delta x$ . Herein, for the case  $C_{j+1/2}$ ,  $u_L$  is  $u_j$  and  $u_R$  is  $u_{j+1}$ , as follows

$$C_{j+1/2} = \begin{cases} 0, & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ -\lambda \frac{f(u_s) - f(u_L)}{u_R - u_L}, & \text{if } f'(u_L) < 0 < f'(u_R). \end{cases}$$

Likewise, it goes for the case  $D_{j+1/2}$  as follows

$$D_{j+1/2} = \begin{cases} \lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ 0, & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ \lambda \frac{f(u_R) - f(u_s)}{u_R - u_L}, & \text{if } f'(u_L) < 0 < f'(u_R). \end{cases}$$

3. In order to show the scheme has TVD property, according to Theorem II.23 (Harten [1980]) from the lecture note, the following three conditions

$$\boxed{\begin{aligned} C_{j+1/2} &\geq 0, \\ D_{j+1/2} &\geq 0, \\ C_{j+1/2} + D_{j+1/2} &\leq 1, \end{aligned}}$$

must hold. Herein, the first two conditions  $C_{j+1/2} \geq 0$ ,  $D_{j+1/2} \geq 0$  can be shown by using convexity of flux function  $f$ . Besides, the third condition  $C_{j+1/2} + D_{j+1/2} \leq 1$  is shown as follows

$$C_{j+1/2} + D_{j+1/2} = \begin{cases} \lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \geq 0 \text{ and } f'(u_R) \geq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] > 0, \end{cases} \\ -\lambda \frac{f(u_R) - f(u_L)}{u_R - u_L}, & \text{if } \begin{cases} f'(u_L) \leq 0 \text{ and } f'(u_R) \leq 0, \\ f'(u_L) \geq 0 \geq f'(u_R) \text{ and } [f]/[u] < 0, \end{cases} \\ \lambda \frac{f(u_R) + f(u_L) - 2f(u_s)}{u_R - u_L}, & \text{if } f'(u_L) < 0 < f'(u_R). \end{cases}$$

The  $C_{j+1/2} + D_{j+1/2} \leq 1$  leads to the CFL condition for *Godunov's* scheme. For  $\lambda$  that satisfies  $C_{j+1/2} + D_{j+1/2} \leq 1$ , the scheme is TVD.