Global Exercise - 15

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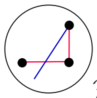
1 Monotone $\to$ L_1-Contracting $\to$ TVD $\to$ Mon.Pre.

Example 1. *abc*

Roe'solver:
P.L.Roe [1981], Approximate Riemann solvers, parameter vectors, and difference schemes.

2 A small note about monotonicity: *Upwind* scheme



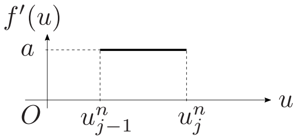
$(a>0)$ CFL: $0 \le f'(u)\dfrac{\Delta t}{\Delta x} \le 1$

$$\mathcal{H}_{\Delta t} = u_j^n + \frac{\Delta t}{\Delta x}\left(f(u_{j-1}^n) - f(u_j^n)\right)$$

$$A_1 = \partial_{u_{j-1}^n}\mathcal{H}_{\Delta t} = \frac{\Delta t}{\Delta x} f'(u_{j-1}^n)$$

$$B_1 = \partial_{u_j^n}\mathcal{H}_{\Delta t} = 1 - \frac{\Delta t}{\Delta x} f'(u_j^n)$$

$$C_1 = \partial_{u_{j+1}^n}\mathcal{H}_{\Delta t} = 0$$



$\mathcal{H}_{\Delta t}$ Monotonically increasing?

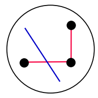
$A_1 \ge 0 \Leftrightarrow f'(u_{j-1}^n) \ge 0$ ✓

$B_1 \ge 0 \Leftrightarrow f'(u_j^n) \le \Delta x/\Delta t$ (?) $\overset{\text{CFL}}{\to}$ ✓

$\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$A_1 \le 0 \Leftrightarrow f'(u_{j-1}^n) \le 0$ $\lightning$

$B_1 \le 0 \Leftrightarrow f'(u_j^n) \ge \Delta x/\Delta t$ $\overset{\text{CFL}}{\to}$ $\lightning$



$(a<0)$

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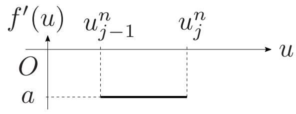

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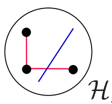

$(a>0)$

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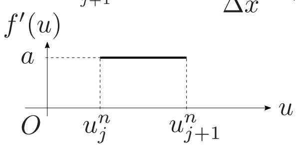

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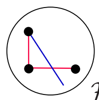

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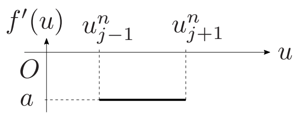

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$\mathcal{H}_{\Delta t}$ Monotonically decreasing?

$B_2 \le 0 \Leftrightarrow f'(u_j^n) \le -\Delta x/\Delta t$ $\overset{\text{CFL}}{\to}$ $\lightning$

$C_2 \le 0 \Leftrightarrow f'(u_{j+1}^n) \ge 0$ $\lightning$

Figure 1: Monotonicity of *Upwind* scheme: 8 different cases.

3 Total variation diminishing (TVD)

Example 2. *abc*

4 Limiter

Example 3. *Examine the 1^{st}-order-converged LF and the 2^{nd}-order-converged LW.*

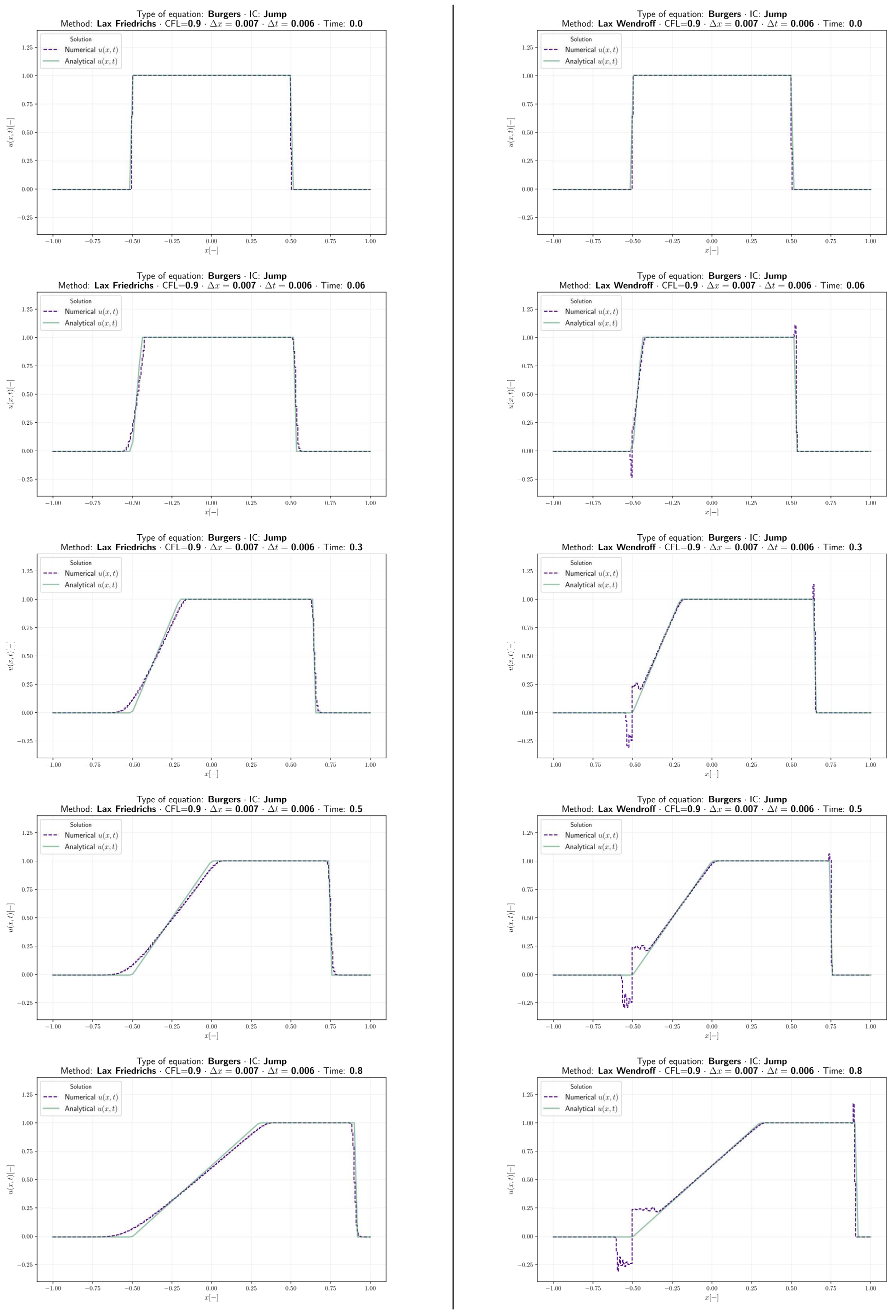
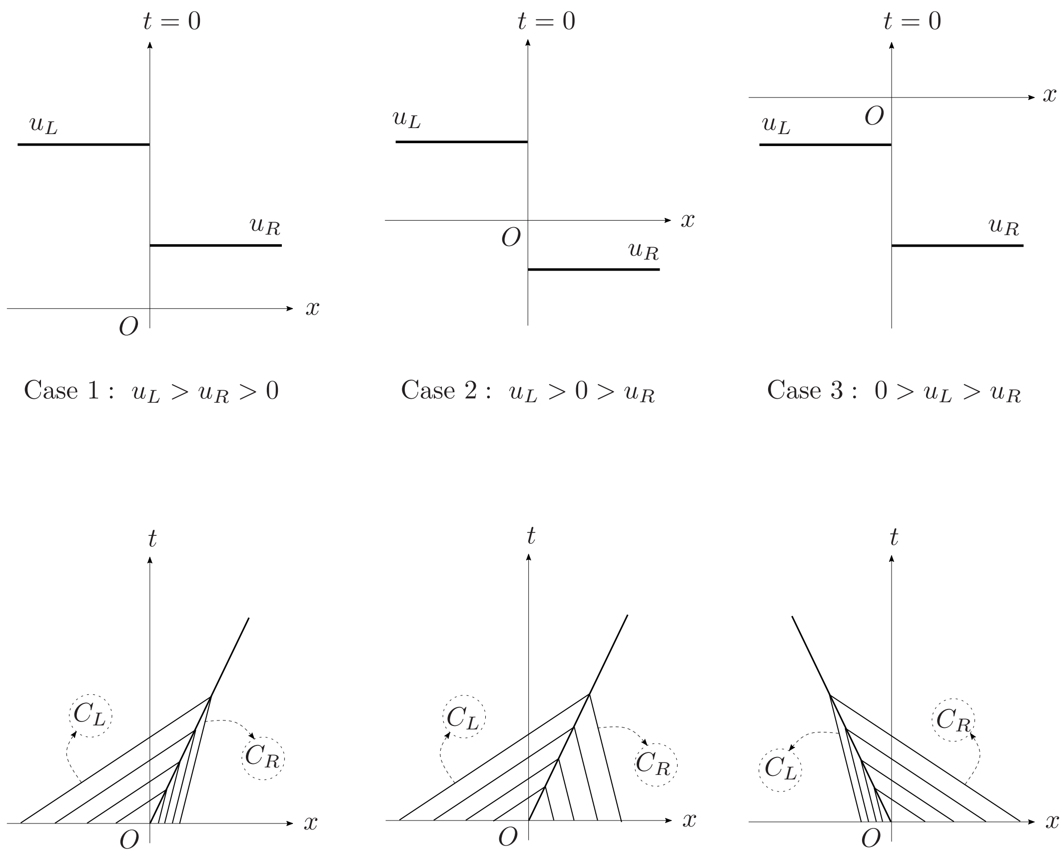


Figure 2: Oscillatory phenomena around discontinuity: (left) none oscillation founded in *Lax-Friedrichs*; (right) oscillation observed in *Lax-Wendroff*.

5 Review Riemann's problem and Godunov's solver

Example 4. *abc*

Example 5. *Examine $u_L > u_R$.*



$$(C_L): \ t = \frac{1}{u_L}x - \frac{x_0}{u_L} \qquad (C_R): \ t = \frac{1}{u_R}x - \frac{x_0}{u_R}$$

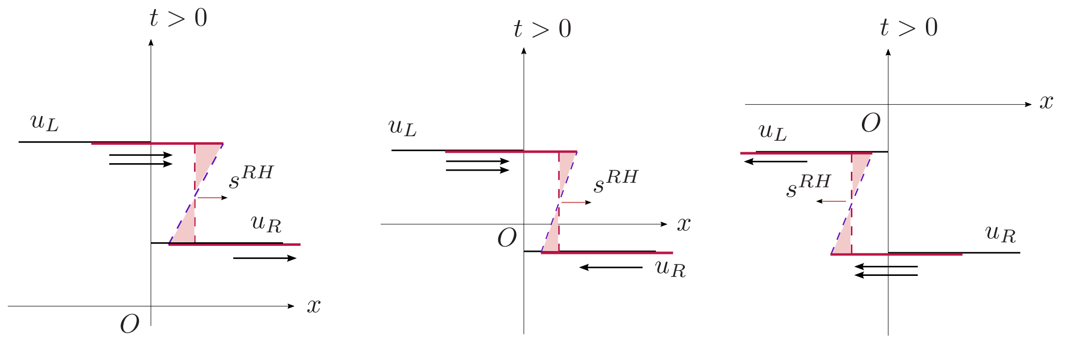
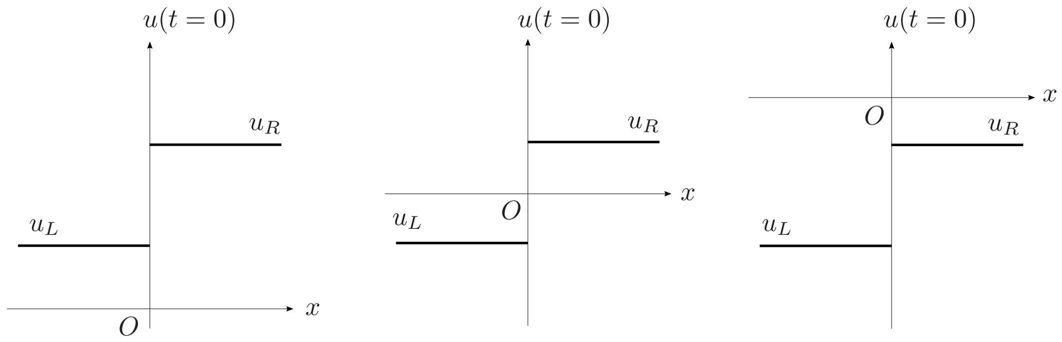


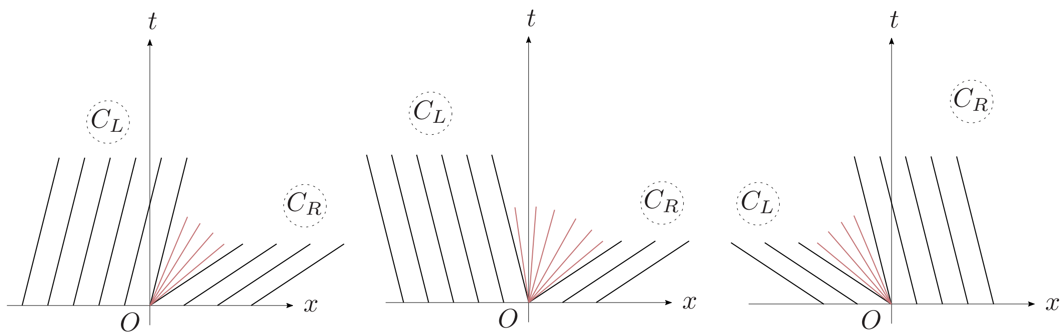
Figure 3: Riemann problem with $u_L > u_R$: IC, Characteristics, Solution.

Schock solution

Example 6. *Examine $u_L < u_R$.*



Case 1 : $0 < u_L < u_R$ Case 2 : $u_L < 0 < u_R$ Case 3 : $u_L < u_R < 0$



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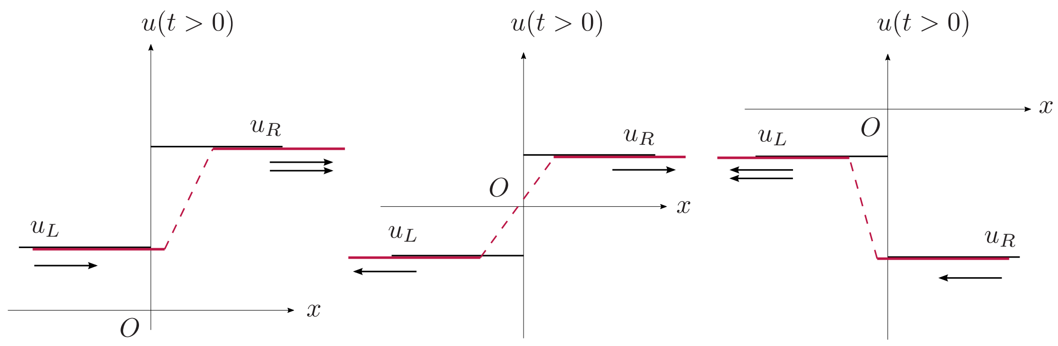

Figure 4: Riemann problem with $u_L < u_R$: IC, Characteristics, Solution.

Rarefaction solution