→ Remark
→ CFL + domain of dependence (numerical + analytical (exact))
→ consistency + stability → convergence
→ FEM vs FVM
(derivation of FVM)
→ (conservation form)

Global Exercise - 12

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1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1: $W := R^{-1}U$ as the scheme shown in exercise

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + R\Lambda R^{-1}U_x &= 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

$U_t + AU_x = 0$
$R^{-1}U_t + \cancel{R^{-1}R}\Lambda R^{-1}U_x = 0$
$(R^{-1}U)_t + \Lambda(R^{-1}U)_x = 0$

where the matrix A is diagonalizable with a transformation matrix $R \in \mathbb{R}^{N\times N}$ in the form

$$A = R\Lambda R^{-1}.$$

2. Case 2: $W := TU$ as the scheme shown in lecture note

$W = TU$

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + T^{-1}\Lambda TU_x &= 0 \\ TU_t + \Lambda TU_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

$U_t + AU_x = 0$
$TU_t + \cancel{TT^{-1}}\Lambda TU_x = 0$
$(TU)_t + \Lambda(TU)_x = 0$

where the matrix A is diagonalizable with a transformation matrix $T \in \mathbb{R}^{N\times N}$ in the form

$$A = T^{-1}\Lambda T.$$

→ Note in passing that both schemes result in the same solution.
→ We have just to be consistent with which scheme to follow.

2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 1. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*

$u_t + a u_x = 0 \quad (a > 0)$

$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x}(u_{j-1}^n - u_j^n)$

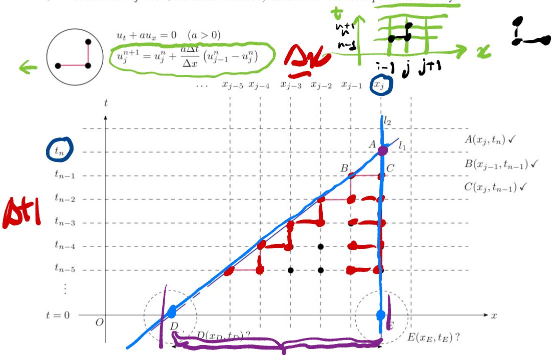


Figure 1: Numerical domain of dependence for One-sided method.

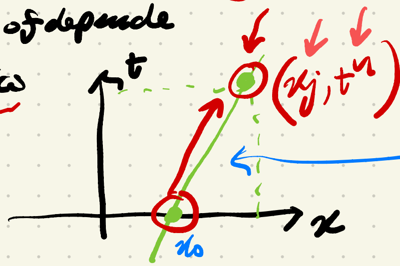
interval

$$\frac{a\Delta t}{\Delta x} = CFL$$

→ Numerical domain of dependence : DE

→ Analytical domain of depende

↳ characteristics



characteristics line

$x = x_0 + at$

↳ $x_0 = x - at$

(1)

$$\boxed{\text{Ana DoD}: \quad D(x_j, t_n) = \{x \mid x = x_j - a t_n\}}$$

(2) Num DoD: DE

How can recognize point D & point E?

i) Perspective of indical supscription.

(2) Num DoD: DE

How can we recognize point D & point E?

i) Perspective of indicial subscription.

2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 1. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*

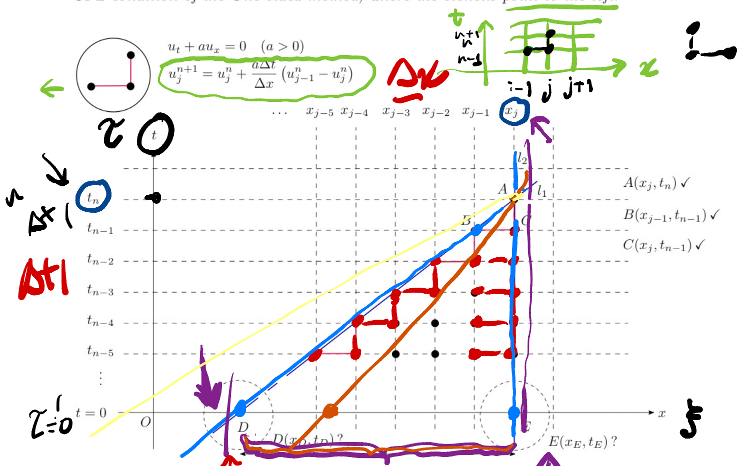


Figure 1: Numerical domain of dependence for One-sided method.

value $A(x_j, t_n)$

value $B(x_{j-1}, t_{n-1})$

index $A(j, n)$

index $B(j-1, n-1)$

$$y = y_A + \frac{y_B - y_A}{x_B - x_A}(x - x_A)$$

$$\tau = n + \frac{(n-1) - n}{(j-1) - j}(\xi - j)$$

$$\tau = n + \frac{-1}{-1}(\xi - j)$$

$$\hookrightarrow 0 = n + \xi - j$$

$$\hookrightarrow \xi = j - n \quad \text{index!}$$

$$x_\xi = x_{j-n}$$

$$x_\xi = x_j - n\Delta x$$

$$\boxed{x_\xi - x_j = -n\Delta x}$$

RHS @ Point E

$x_\xi = x_j \qquad \Delta t / \Delta x$

$$\boxed{D_{\Delta t}(x_j, t_n) = \{x_\xi \mid -n\Delta x \le x_\xi - x_j \le 0\}}$$

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} = -n\Delta t \frac{a}{\nu} = \frac{-at_n}{\nu}$$

$$\nu := \frac{a\Delta t}{\Delta x} \qquad t_n = n\Delta t$$

$$D_{\Delta t}(x_j, t_n) = \left\{x \;\middle|\; -\frac{at_n}{\nu} \le x - x_j \le 0\right\}$$

Ana DoD: $D(x_j, t_n) = \{x \mid x = x_j - at_n\} \subseteq D_{\Delta t}(x_j, t_n) = \{x_\xi \mid -n\Delta x \le x_\xi - x_j \le 0\}$!

→ CFL condition

$$x = (x_j - at_n)$$

$$-n\Delta x \le x - x_j \le 0$$

$$-n\Delta x \le \cancel{x_j} - at_n - \cancel{x_j} \le 0 \rightarrow$$

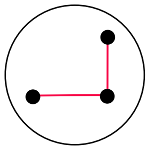
$$-n\Delta x \le -at_n \le 0$$

$$\boxed{0 \le \nu \le 1} \qquad \frac{a\Delta t}{\Delta x}$$

$$\hookrightarrow \boxed{0 \le \Delta t \le \frac{\Delta x}{a}}$$

2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 1. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*



$$u_t + au_x = 0 \quad (a > 0)$$
$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x}\left(u_{j-1}^n - u_j^n\right)$$

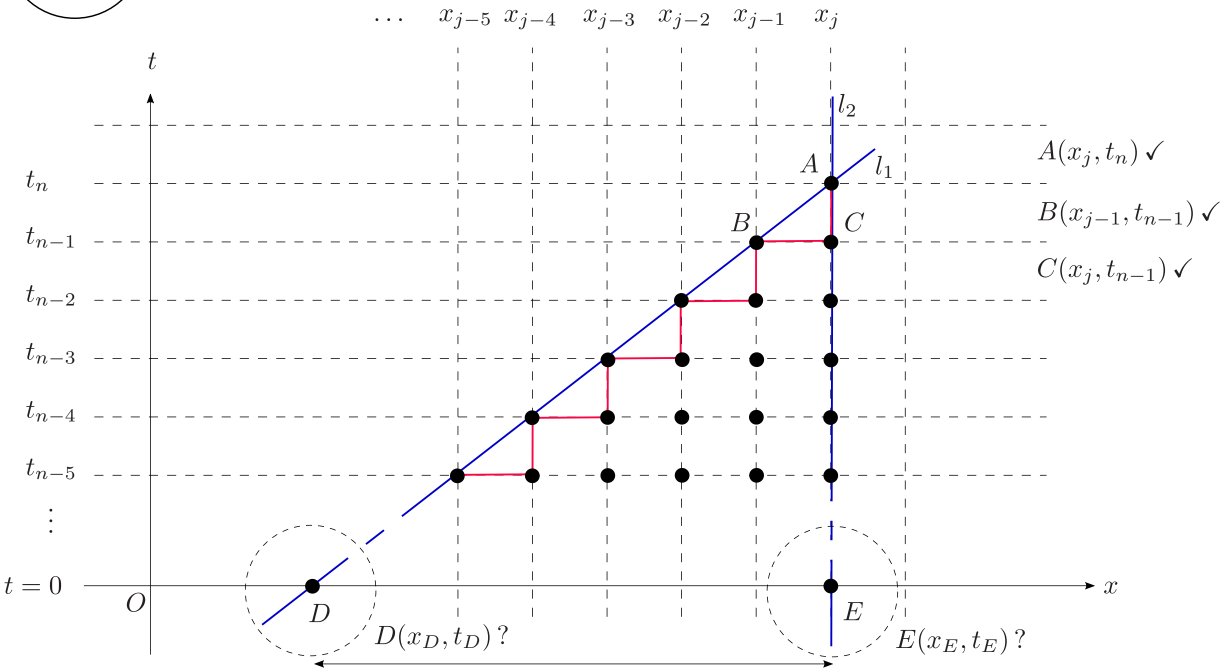


Figure 1: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 1, the numerical value computed at point A depends essentially on computed initial conditions laying between point D and E.

1. Perspective of indical subscription:
   Line (l_1) passsing point $A(j, n)$ and $B(j-1, n-1)$ has the following form

$$\begin{aligned}
(l_1): \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A}(\xi - \xi_A) \\
\Leftrightarrow \tau &= n + \frac{(n-1) - n}{(j-1) - j}(\xi - j) \\
\Leftrightarrow \tau &= n + \frac{-1}{-1}(\xi - j), \qquad (1)
\end{aligned}$$

   where τ is the indical variable corresponding to t, and x the indical variable to x. Hence, line (l_1) passing line x with index $\tau = 0$ at point D leads to the following relation

$$\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \qquad (2)$$

Likewise, line (l_2) passing line x with index $\tau = 0$ at point E leads to the following relation

$$x_\xi - x_j = 0. \tag{3}$$

Therefore, by combining (2) and (3) we arrive at the numerical domain of dependence for the One-sided method in terms of indical perspective

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{x_\xi \,\middle|\, -n\Delta x \leq x_\xi - x_j \leq 0\right\}. \tag{4}$$

Next, by using the CFL number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t\frac{a\Delta x}{a\Delta t} \overset{(CFL)}{=} -n\Delta t\frac{a}{\nu} = -\frac{at_n}{\nu}. \tag{5}$$

Then, by substituting (5) into (4) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{x \,\middle|\, -\frac{at_n}{\nu} \leq x - x_j \leq 0\right\}}. \tag{6}$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{x \,\middle|\, x = x_j - at_n\right\}. \tag{7}$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}\left(x_j, t_n\right) \subset \mathcal{D}_{\Delta t}\left(x_j, t_n\right), \tag{8}$$

which implies that characteristics should lie with the triangular zone under the line (l_1) and (l_2), as shown in Figure 1. Therefore, substitution of (7) into (6) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \leq (x_j - at_n) - x_j \leq 0 \Leftrightarrow -\frac{at_n}{\nu} \leq -at_n \leq 0, \tag{9}$$

which, equally, leads to the CFL condtion

$$\therefore \quad \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}.} \tag{10}$$

Herein, the CFL condition (10) leads to constraint on the time step Δt for the case when $a > 0$. Note in passing that ν is non-negative.

2. Perspective of fixed-point value:
Line (l_1) passsing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned}(l_1): \quad t = t_A + \frac{t_B - t_A}{x_B - x_A}\left(x - x_A\right) &\Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}\left(x - x_j\right) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x}\left(x - x_j\right).\end{aligned} \tag{11}$$

Hence, line (l_1) passing line $t=0$ at point D leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \tag{12}$$

Likewise, line (l_2) passing line $t=0$ at point E leads to the relation

$$x - x_j = 0. \tag{13}$$

Therefore, combination of (12) and (13) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, -\frac{t_n \Delta x}{\Delta t} \le x - x_j \le 0 \right\}.} \tag{14}$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (7), reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{15}$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

$$-\frac{t_n \Delta x}{\Delta t} \le \left(x_j - a t_n\right) - x_j \le 0, \tag{16}$$

which we have substituted (15) into (14). Herein, the relation (16) enforcing CFL condition on the time step Δt

$$\therefore \quad \boxed{0 \le \Delta t \le \frac{\Delta x}{a},} \tag{17}$$

which is similar to (10).

Example 2. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the right.*

$$u_t + au_x = 0 \quad (a < 0)$$
$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j+1}^n\right)$$

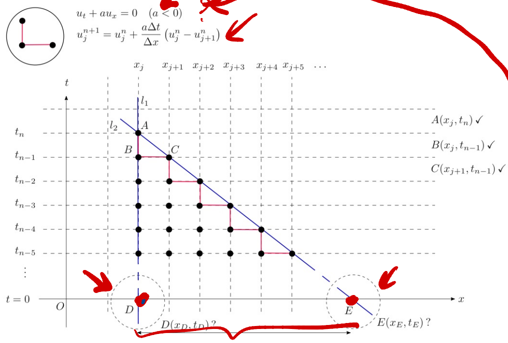


Figure 2: Numerical domain of dependence for One-sided method, where the stencils point to the right.

DE

$$D_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, 0 \le x - x_j \le -\frac{at_n}{v} \right\}$$

$$D(x_j, t_n) = \{ x \mid x = x_j - at_n \}$$

CFL: $\Delta t \geq \dfrac{\Delta x}{a}$ $\Leftarrow$ $(a<0)$

$a = -2$

e.g $\Delta t \le \dfrac{\Delta x}{2}$

Example 2. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the right.*

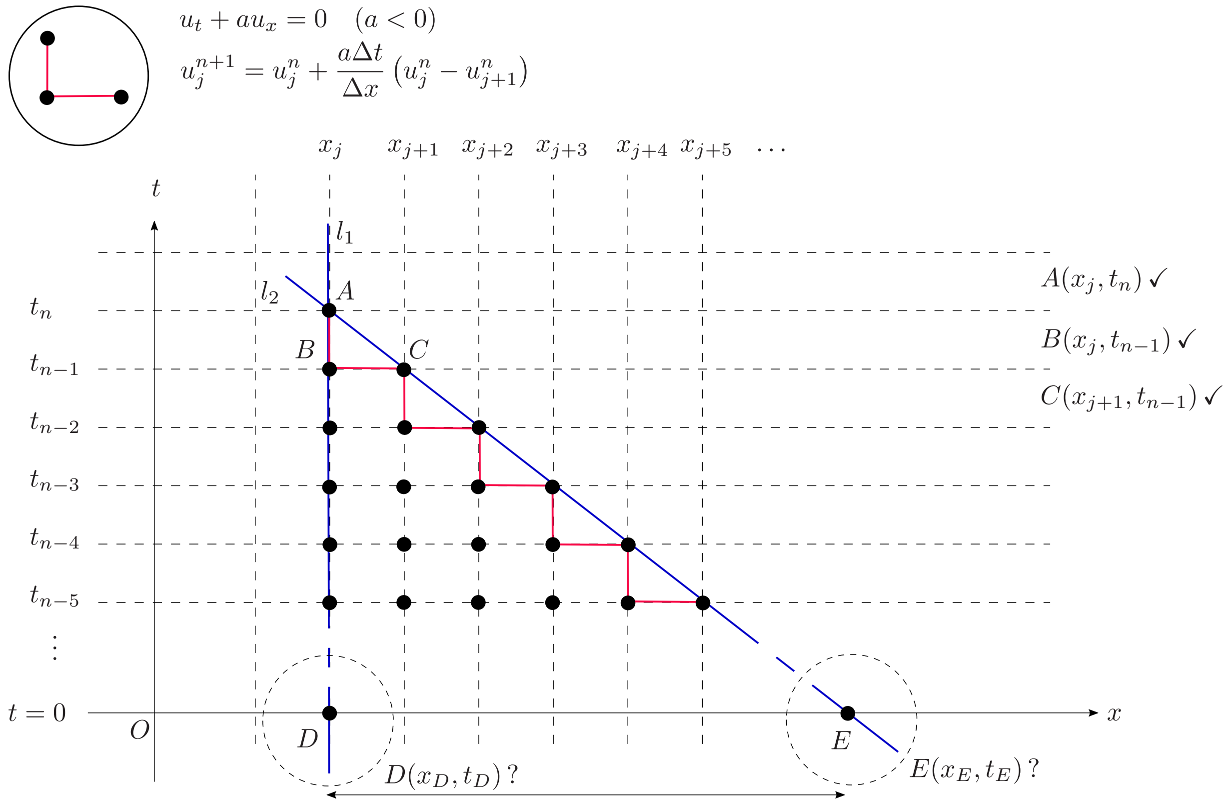


Figure 2: Numerical domain of dependence for One-sided method, where the stencils point to the right.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point E in terms of fixed-point value satisfying

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\left(x - x_j\right) \tag{18}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, 0 \leq x - x_j \leq -\frac{at_n}{\nu} \right\}. \tag{19}$$

3. Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{20}$$

4. CFL condition reads

$$\therefore \quad \boxed{\Delta t \geq \frac{\Delta x}{a}.} \tag{21}$$

Note in passing that the advection velocity a in this Example 2 is $a < 0$.

Example 3. *Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.*

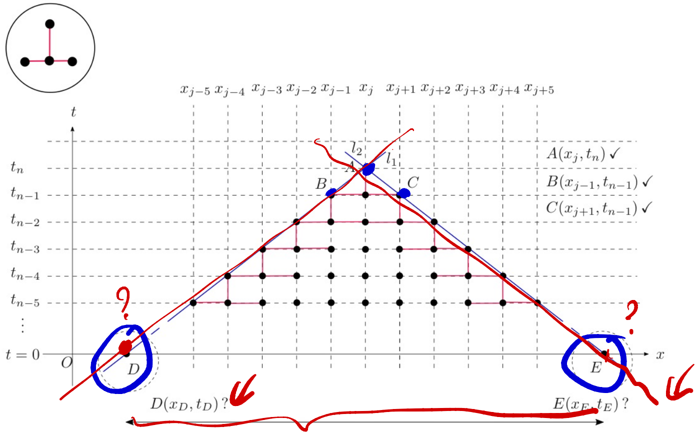
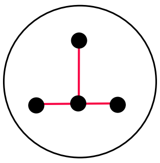


Figure 3: General numerical domain of dependence for a three-point scheme.

$D_{\Delta t}$ $\qquad |x - x_j| \le \dfrac{a t_n}{\nu}$

CFL $\qquad \left| \dfrac{a \Delta t}{\Delta x} \right| \le 1$

Example 3. *Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.*



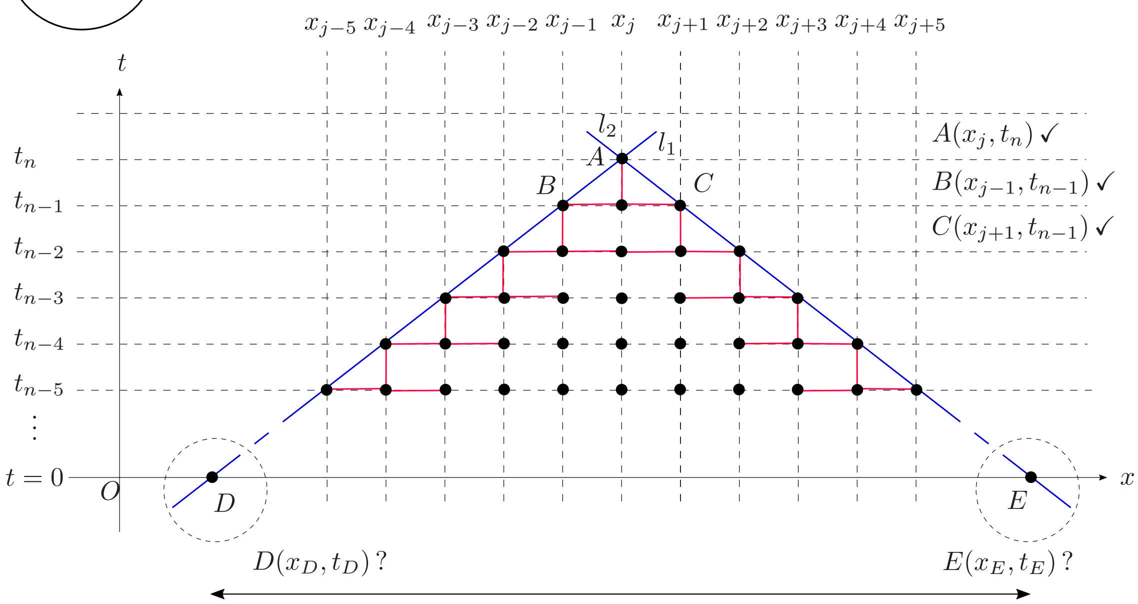


Figure 3: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \tag{22}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}(x - x_j) \tag{23}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \leq \frac{a t_n}{\nu} \right\}. \tag{24}$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{25}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a \Delta t}{\Delta x} \right| \leq 1.} \tag{26}$$

Example 4. *Examine the numerical domain of dependence of* Lax-Friedrichs *method.*

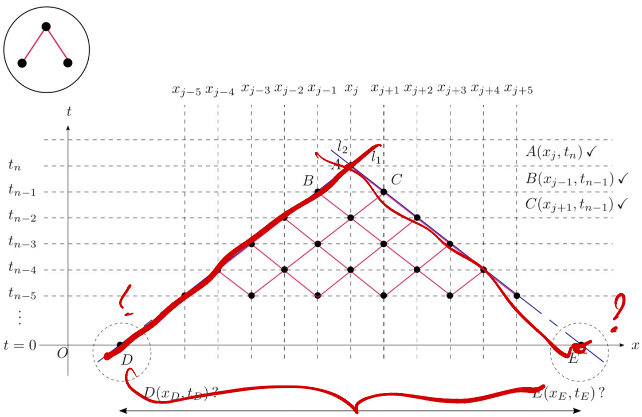
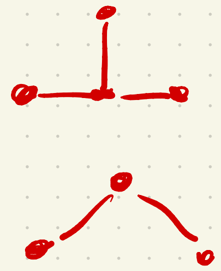


Figure 4: General numerical domain of dependence for a three-point scheme.



CFL $\left|\dfrac{a\Delta t}{\Delta x}\right| \leq 1$

Example 4. *Examine the numerical domain of dependence of* Lax-Friedrichs *method.*

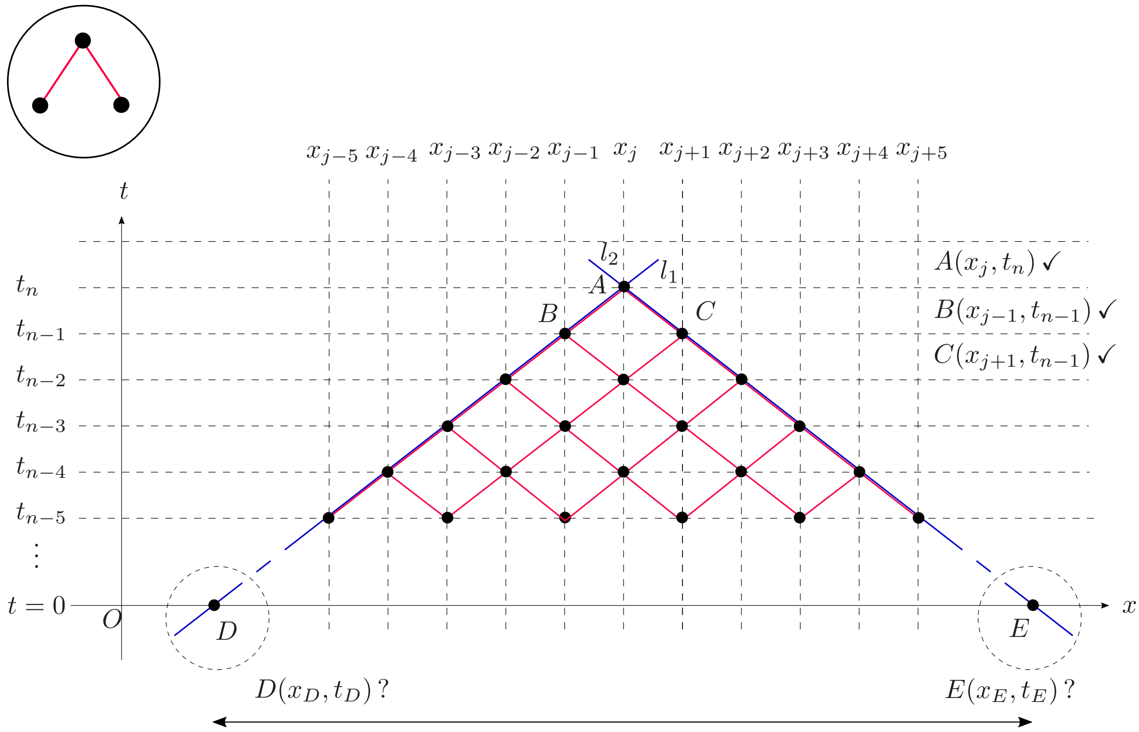


Figure 4: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1, or the same as 3 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \tag{27}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}(x - x_j) \tag{28}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \leq \frac{at_n}{\nu} \right\}. \tag{29}$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{30}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a\Delta t}{\Delta x} \right| \leq 1.} \tag{31}$$

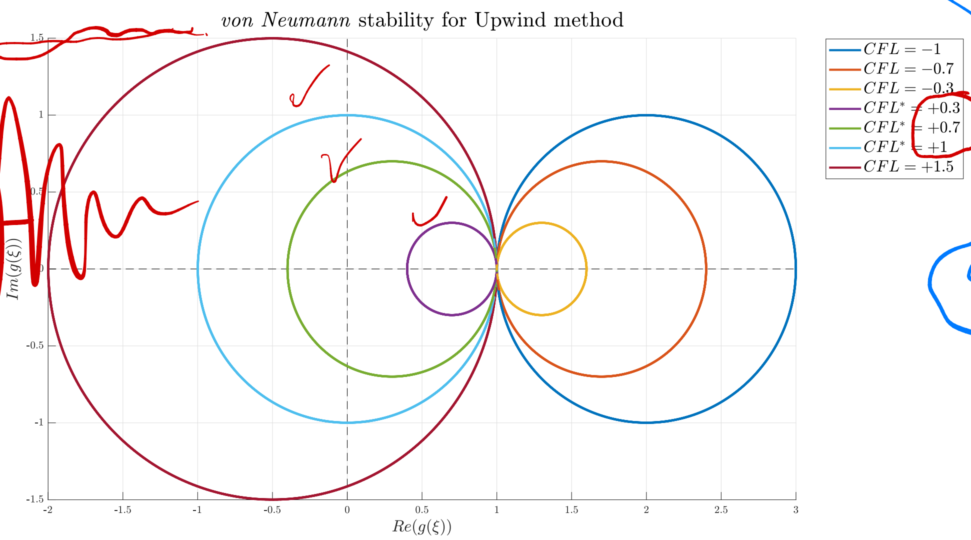
3 von Neumann stability analysis

one-sided method

Ex 1

Ex 2

Example 5. *von Neumann stability analysis for Upwind method.*



$0 \le CFL \le 1$

Figure 5: *von Neumann* stability analysis for Upwind method.

Example 6. *Summary consistency + stability ⇒ Convergence*

(I) consistency

error → local consistency error

$E_{\Delta t}(x,t)$ $\quad$ $L_{\Delta t}(x,t)$

- consistent
- consistency order

$$\|L_{\Delta t}(\cdot,t)\| \le C_L \Delta t^{p}$$

(II) extra smoother solution:

global error

same order p iff stability

stability $\|u^n_{\Delta x}\| \le C_S \|u^0_{\Delta x}\|$

(III) stability ⟺ Convergence

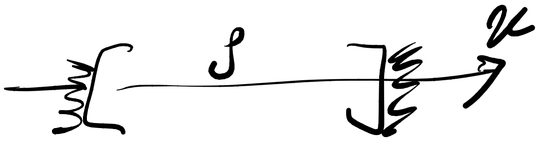
$u_t + a u_x = \varepsilon u_{xx}$

εu_{xxx}

$\frac{d}{dt}\int(\;)\,dx$ = surface + ~~volume~~ + ~~unsteady~~

4 Conservation form - Finite Volume Method

Example 7. *Derivation of conservation form.*

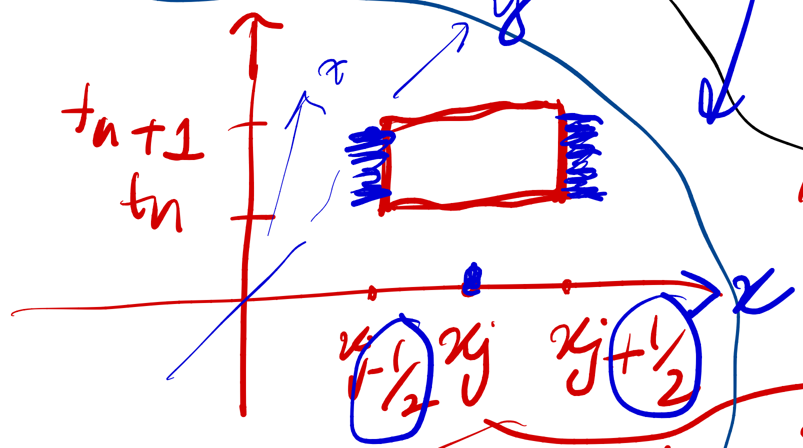


$$\frac{d}{dt}\int_{x_1}^{x_2}\rho(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

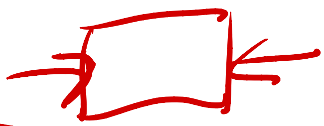
integration $\int$ over time t

$$\frac{1}{\Delta x}\left[\int_{x_1}^{x_2}\rho(x,t_2)\,dx - \int_{x_1}^{x_2}\rho(x,t_1)\,dx\right] = \int_{t_1}^{t_2} f(\rho(x_1,t))\,dt - \int_{t_1}^{t_2} f(\rho(x_2,t))\,dt$$

(1) ✓ (2) ✓



FVM



cell average: $\bar{u}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,dx$

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x}(RHS)$$

$$\tilde{F}(u(x_{j+1/2},t)) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,dt$$

$$\boxed{\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{x_{j+1/2}} - \tilde{F}_{x_{j-1/2}}\right)}$$