Global Exercise - 12

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1 A remark about the derivation from coupled to decoupled form of linear hyperbolic systems

1. Case 1: $W := R^{-1}U$ as the scheme shown in exercise

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + R\Lambda R^{-1}U_x &= 0 \\ R^{-1}U_t + \Lambda R^{-1}U_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

   where the matrix A is diagonalizable with a transformation matrix $R \in \mathbb{R}^{N\times N}$ in the form

$$A = R\Lambda R^{-1}.$$

2. Case 2: $W := TU$ as the scheme shown in lecture note

$$\boxed{\begin{aligned} U_t + AU_x &= 0 \\ U_t + T^{-1}\Lambda TU_x &= 0 \\ TU_t + \Lambda TU_x &= 0 \\ W_t + \Lambda W_x &= 0 \end{aligned}}$$

   where the matrix A is diagonalizable with a transformation matrix $T \in \mathbb{R}^{N\times N}$ in the form

$$A = T^{-1}\Lambda T.$$

$\rightarrow$ Note in passing that both schemes result in the same solution.
$\rightarrow$ We have just to be consistent with which scheme to follow.

2 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 1. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.*

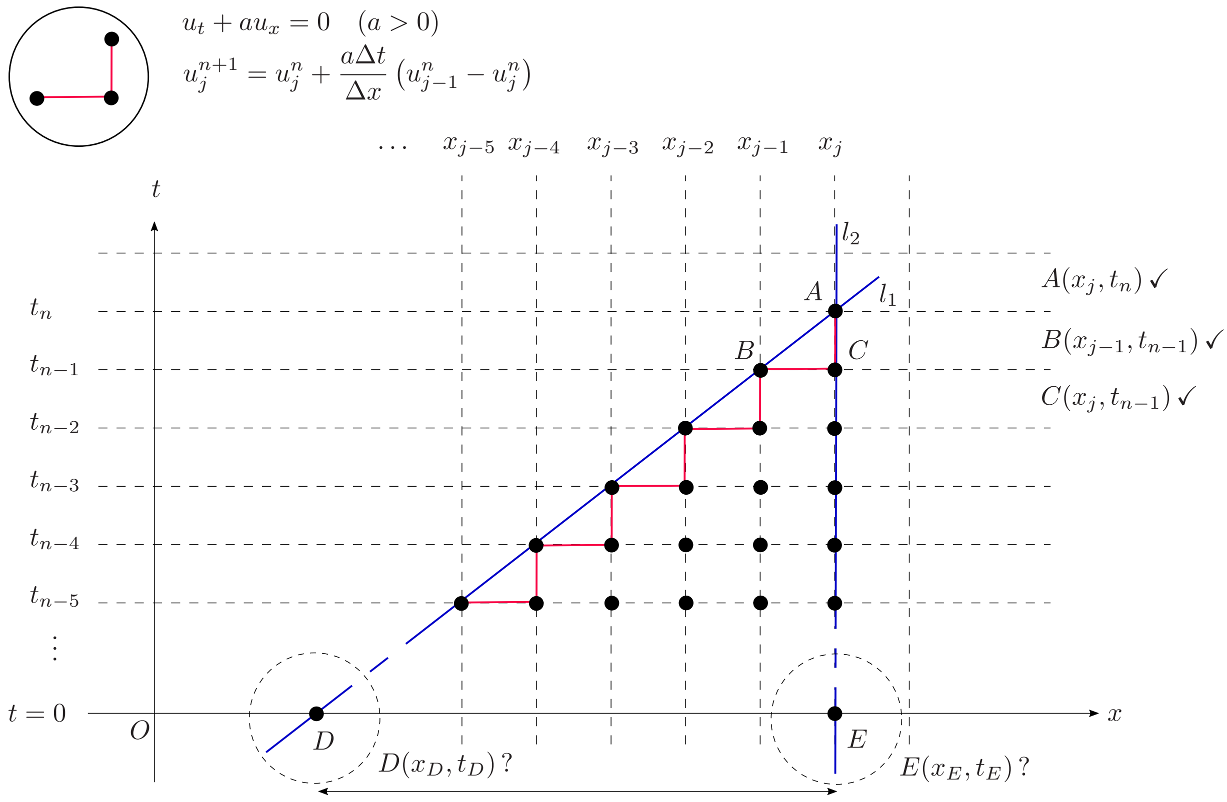


Figure 1: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 1, the numerical value computed at point A depends essentially on computed initial conditions laying between point D and E.

1. Perspective of indical subscription:
   Line (l_1) passsing point $A(j, n)$ and $B(j-1, n-1)$ has the following form

$$\begin{aligned} (l_1): \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A}(\xi - \xi_A) \\ \Leftrightarrow \tau &= n + \frac{(n-1) - n}{(j-1) - j}(\xi - j) \\ \Leftrightarrow \tau &= n + \frac{-1}{-1}(\xi - j), \end{aligned} \tag{1}$$

   where τ is the indical variable corresponding to t, and x the indical variable to x. Hence, line (l_1) passing line x with index $\tau = 0$ at point D leads to the following relation

$$\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \tag{2}$$

Likewise, line (l_2) passing line x with index $\tau = 0$ at point E leads to the following relation

$$x_\xi - x_j = 0. \tag{3}$$

Therefore, by combining (2) and (3) we arrive at the numerical domain of dependence for the One-sided method in terms of indical perspective

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x_\xi \,\middle|\, -n\Delta x \le x_\xi - x_j \le 0 \right\}. \tag{4}$$

Next, by using the CFL number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \overset{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \tag{5}$$

Then, by substituting (5) into (4) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, -\frac{at_n}{\nu} \le x - x_j \le 0 \right\}}. \tag{6}$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{7}$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}(x_j, t_n) \subset \mathcal{D}_{\Delta t}(x_j, t_n), \tag{8}$$

which implies that characteristics should lie with the triangular zone under the line (l_1) and (l_2), as shown in Figure 1. Therefore, substitution of (7) into (6) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \le (x_j - at_n) - x_j \le 0 \Leftrightarrow -\frac{at_n}{\nu} \le -at_n \le 0, \tag{9}$$

which, equally, leads to the CFL condtion

$$\therefore \quad \boxed{0 \le \nu \le 1 \Leftrightarrow 0 \le \Delta t \le \frac{\Delta x}{a}.} \tag{10}$$

Herein, the CFL condition (10) leads to constraint on the time step Δt for the case when $a > 0$. Note in passing that ν is non-negative.

2. Perspective of fixed-point value:
Line (l_1) passsing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned}(l_1): \quad t = t_A + \frac{t_B - t_A}{x_B - x_A}(x - x_A) &\Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x}(x - x_j). \end{aligned} \tag{11}$$

Hence, line (l_1) passing line $t = 0$ at point D leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \tag{12}$$

Likewise, line (l_2) passing line $t = 0$ at point E leads to the relation

$$x - x_j = 0. \tag{13}$$

Therefore, combination of (12) and (13) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \tag{14}$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (7), reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{15}$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

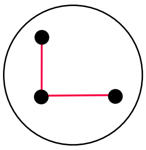
$$-\frac{t_n \Delta x}{\Delta t} \leq (x_j - a t_n) - x_j \leq 0, \tag{16}$$

which we have substituted (15) into (14). Herein, the relation (16) enforcing CFL condition on the time step Δt

$$\therefore \quad \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a},} \tag{17}$$

which is similar to (10).

Example 2. *Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the right.*



$$u_t + a u_x = 0 \quad (a < 0)$$

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j+1}^n\right)$$

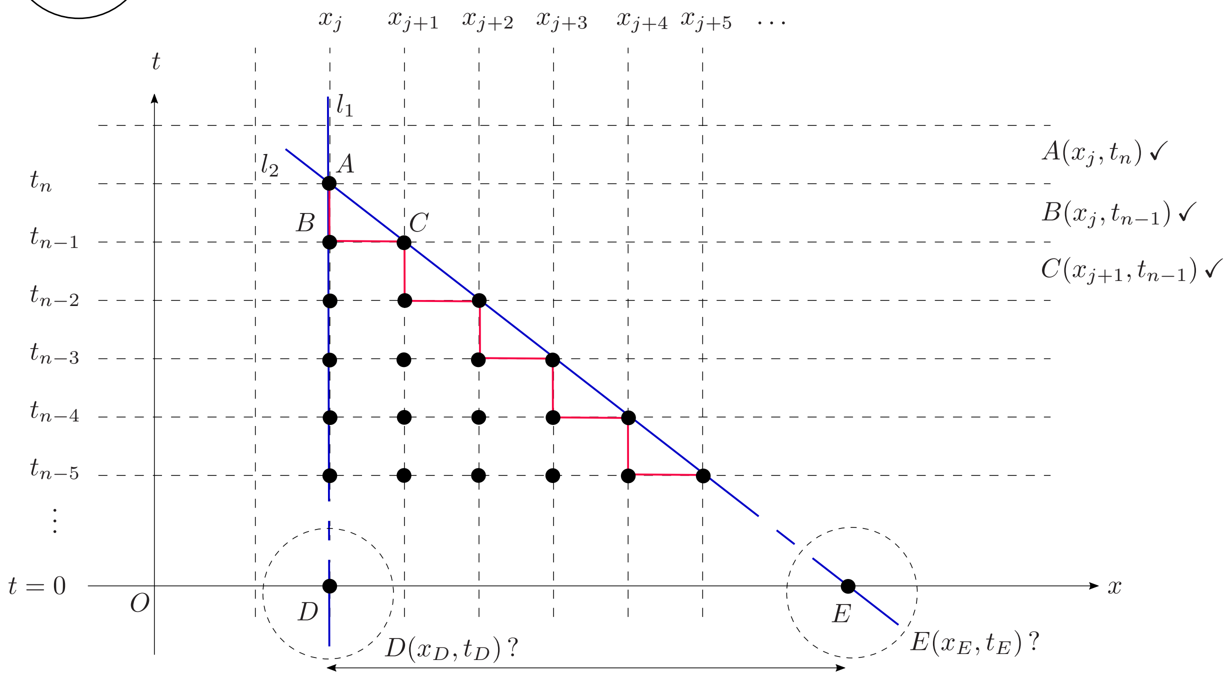


Figure 2: Numerical domain of dependence for One-sided method, where the stencils point to the right.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point E in terms of fixed-point value satisfying

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\left(x - x_j\right) \tag{18}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\left(x_j, t_n\right) = \left\{ x \,\middle|\, 0 \leq x - x_j \leq -\frac{a t_n}{\nu} \right\}. \tag{19}$$

3. Analytical domain of dependence reads

$$\mathcal{D}\left(x_j, t_n\right) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{20}$$

4. CFL condition reads

$$\therefore \quad \boxed{\Delta t \geq \frac{\Delta x}{a}.} \tag{21}$$

Note in passing that the advection velocity a in this Example 2 is $a < 0$.

Example 3. *Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.*

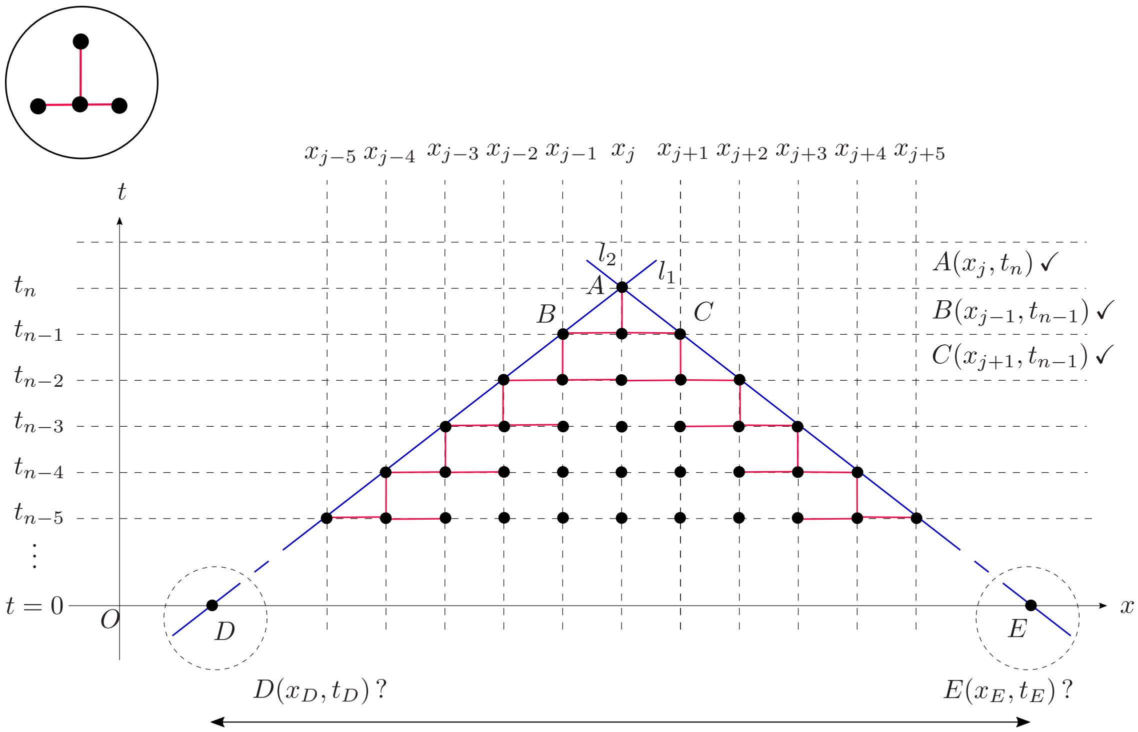


Figure 3: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}\,(x - x_j) \tag{22}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}\,(x - x_j) \tag{23}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}\,(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \le \frac{a t_n}{\nu} \right\}. \tag{24}$$

3. Analytical domain of dependence reads

$$\mathcal{D}\,(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - a t_n \right\}. \tag{25}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a\Delta t}{\Delta x} \right| \le 1.} \tag{26}$$

Example 4. *Examine the numerical domain of dependence of* Lax-Friedrichs *method.*

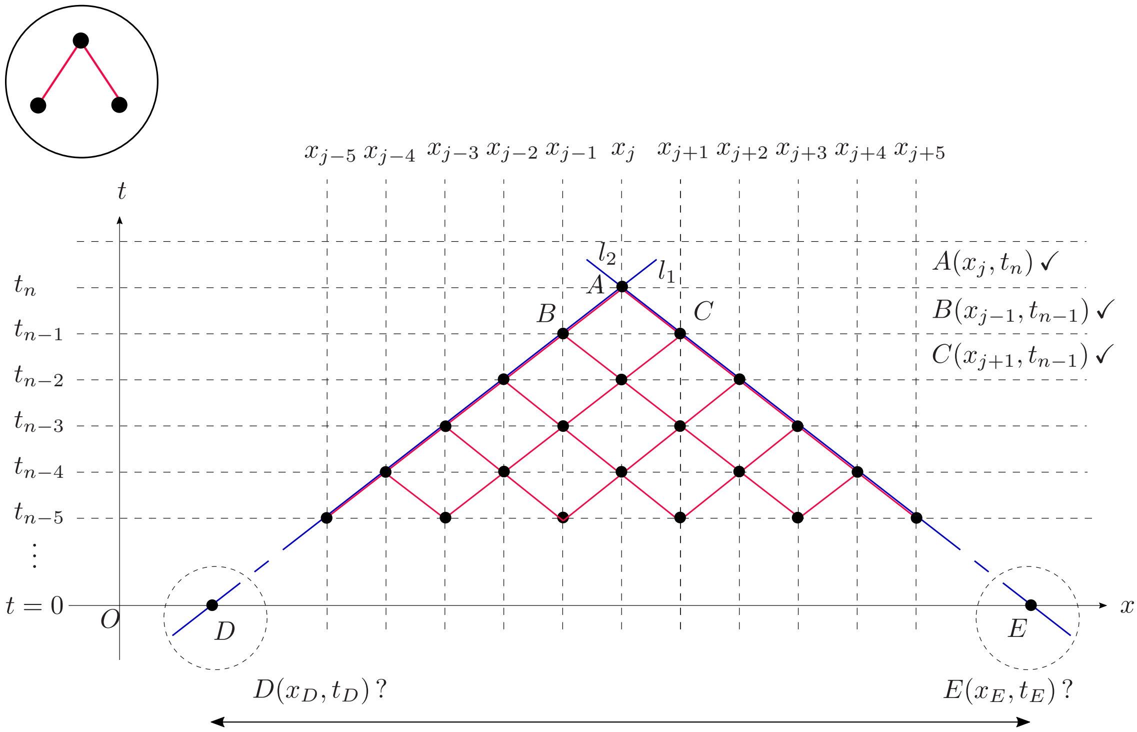


Figure 4: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 1, or the same as 3 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j}(x - x_j) \tag{27}$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j}(x - x_j) \tag{28}$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \,\middle|\, |x - x_j| \le \frac{at_n}{\nu} \right\}. \tag{29}$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \,\middle|\, x = x_j - at_n \right\}. \tag{30}$$

4. CFL condition reads

$$\therefore \quad \boxed{\left| \frac{a\Delta t}{\Delta x} \right| \le 1.} \tag{31}$$

3 von Neumann stability analysis

Example 5. *Perform von Neumann stability analysis for* Lax-Friedrichs *scheme.*

$$u_i^{j+1} = \frac{1}{2}\left(u_{i+1}^j + u_{i-1}^j\right) - \frac{c}{2}\left(u_{i+1}^j - u_{i-1}^j\right)$$

For the Lax - Friedrich scheme, the amplification factor is given by

$$g(\xi) = \cos\xi - \mathbf{i}\nu\sin\xi \tag{32}$$

which leads to

$$|g(\xi)| = \sqrt{\cos^2 + \nu^2\sin^2(\xi)} \leq 1, \quad \text{for} \quad 0 < \nu \leq 1. \tag{33}$$

Therefore, the *Lax-Friedrichs* scheme is *conditionally stable* for $0 < \nu \leq 1$.

Example 6. *von Neumann stability analysis for Upwind method.*

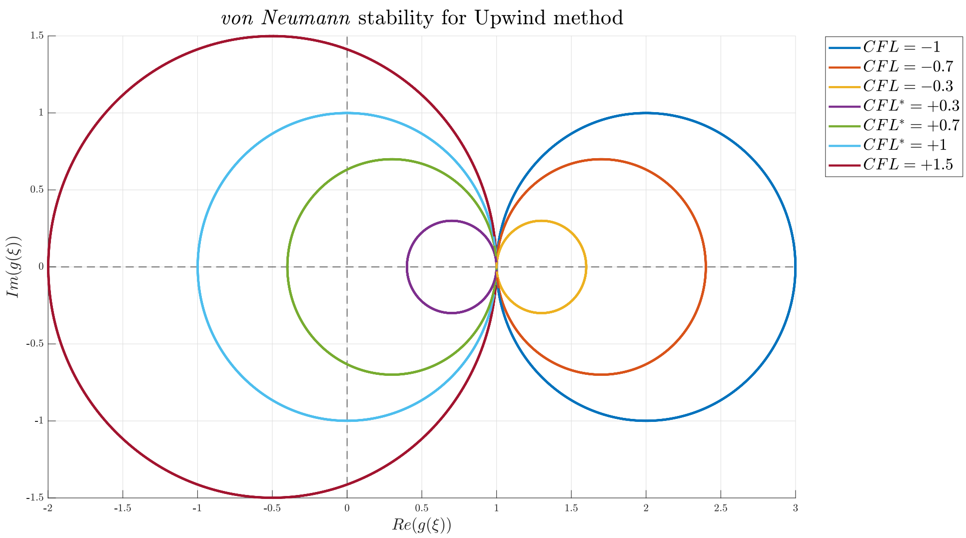


Figure 5: *von Neumann* stability analysis for Upwind method.

4 Conservative form - Finite Volume Method

Example 7. *Conservative form*