

Global Exercise - 13

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19th January 2022

1 Conservation form - Finite Volume Method (cont.)

Example 1. *Derivation of conservation form.*

Starting point is the **integral form** of conservation law. Recall example 1 in the global exercise 9, the integral formulation of conservation law for a scalar conserved quantity $u(x, t)$ over real line $[x_1, x_2]$, herein, reads

$$\boxed{\frac{d}{dt} \int_{x_1}^{x_2} u(x, t) dx = f(u(x_1, t)) - f(u(x_2, t))}. \quad (1)$$

Integration both side of (1) over temporal interval $[t_1, t_2]$ yields

$$\left(\int_{x_1}^{x_2} u(x, t) dx \right) \Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} (f(u(x_1, t)) - f(u(x_2, t))) dt, \quad (2)$$

which leads to **another integral form** of the conservation law, as follows

$$\boxed{\int_{x_1}^{x_2} u(x, t_2) dx = \int_{x_1}^{x_2} u(x, t_1) dx + \int_{t_1}^{t_2} f(u(x_1, t)) dt - \int_{t_1}^{t_2} f(u(x_2, t)) dt}, \quad (3)$$

which is useful to derive the conservation form for finite volume method. Then, instead of $[x_1, x_2]$ and $[t_1, t_2]$ we consider spatial interval $[x_{j-1/2}, x_{j+1/2}]$ and temporal interval $[t_n, t_{n+1}]$ as follows

$$[x_1, x_2] \rightarrow [x_{j-1/2}, x_{j+1/2}], \quad (4)$$

$$[t_1, t_2] \rightarrow [t_n, t_{n+1}]. \quad (5)$$

Hence, the integral form in (3) becomes

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1}) dx &= \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n) dx \\ &+ \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) dt. \end{aligned} \quad (6)$$

Next, consideration of the cell average formulation, which, by definition, reads

$$\bar{u}_j^n := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n) dx, \quad (7)$$

and multiplication of both sides of (6) by $1/\Delta x$ yield the following relation

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{1}{\Delta x} \left(\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) dt - \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) dt \right). \quad (8)$$

Moreover, by defining the **numerical flux functions** as follows

$$\tilde{f}_{j-1/2}^n := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) dt, \quad (9)$$

$$\tilde{f}_{j+1/2}^n := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) dt, \quad (10)$$

the expression (8) now becomes the **conservation form**

$$\boxed{\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x} (\tilde{f}_{j-1/2}^n - \tilde{f}_{j+1/2}^n)}. \quad (11)$$

Example 2. *Lipschitz continuity.*

Example 3. *Derivation of the conservative form of upwind scheme. Examine its numerical flux function: conservative? consistent?*

Example 4. *Derivation of the conservative form of Lax-Friedrichs scheme. Examine its numerical flux function: conservative? consistent?*

Example 5. *Derivation of the conservative form of Lax-Wendroff scheme. Examine its numerical flux function: conservative? consistent?*

Example 6. *Derivation of the conservative form of the two-step Lax-Wendroff scheme. Examine its numerical flux function: conservative? consistent?*

Example 7. *Derivation of the conservative form of MacCormack scheme. Examine its numerical flux function: conservative? consistent?*

2 Godunov linear systems

3 Roe

4 Theory of high resolution

5 Discontinuous solution

6 Correlation between Domain of dependence (DoD) and Courant-Friedrichs-Lewy (CFL) condition

Example 8. Examine the numerical domain of dependence and the corresponding CFL condition of the One-sided method, where the stencils point to the left.

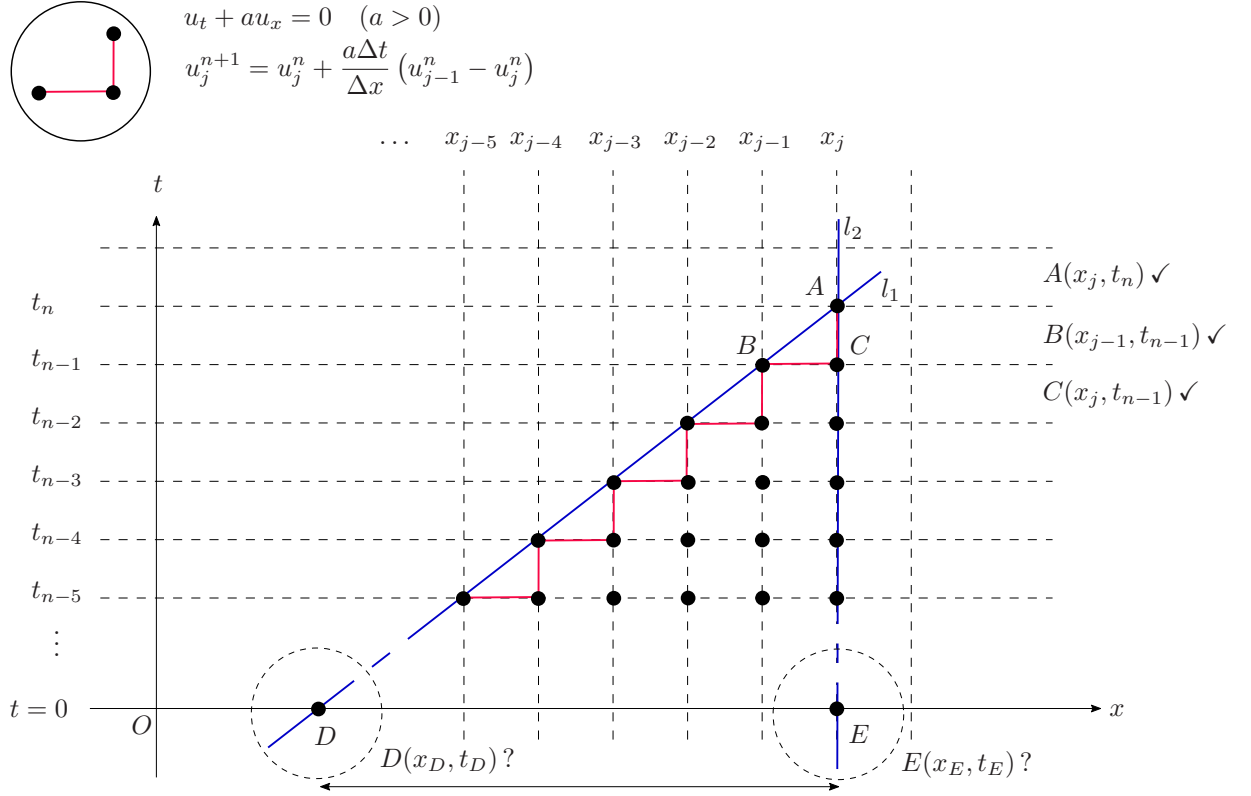


Figure 1: Numerical domain of dependence for One-sided method.

As it can be seen from Figure 1, the numerical value computed at point A depends essentially on computed initial conditions laying between point D and E .

1. Perspective of indicial subscription:

Line (l_1) passing point $A(j, n)$ and $B(j-1, n-1)$ has the following form

$$\begin{aligned}
 (l_1) : \quad \tau &= \tau_A + \frac{\tau_B - \tau_A}{\xi_B - \xi_A} (\xi - \xi_A) \\
 &\Leftrightarrow \tau = n + \frac{(n-1) - n}{(j-1) - j} (\xi - j) \\
 &\Leftrightarrow \tau = n + \frac{-1}{-1} (\xi - j), \tag{12}
 \end{aligned}$$

where τ is the indicial variable corresponding to t , and x the indicial variable to x . Hence, line (l_1) passing line x with index $\tau = 0$ at point D leads to the following relation

$$\xi = j - n \Leftrightarrow x_\xi = x_{j-n} \Leftrightarrow x_\xi = x_j - n\Delta x \Leftrightarrow x_\xi - x_j = -n\Delta x. \tag{13}$$

Likewise, line (l_2) passing line x with index $\tau = 0$ at point E leads to the following relation

$$x_\xi - x_j = 0. \quad (14)$$

Therefore, by combining (13) and (14) we arrive at the numerical domain of dependence for the One-sided method in terms of indicial perspective

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x_\xi \mid -n\Delta x \leq x_\xi - x_j \leq 0 \right\}. \quad (15)$$

Next, by using the *CFL* number $\nu := a\Delta t/\Delta x$ we obtain the following equality

$$-n\Delta x = -n\Delta t \frac{a\Delta x}{a\Delta t} \stackrel{(CFL)}{=} -n\Delta t \frac{a}{\nu} = -\frac{at_n}{\nu}. \quad (16)$$

Then, by substituting (16) into (15) with limit consideration we obtain the entire set of the numerical domain of dependence, as follows

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid -\frac{at_n}{\nu} \leq x - x_j \leq 0 \right\}}. \quad (17)$$

Besides, the analytical domain of dependence for the linear advection PDE reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (18)$$

Furthermore, the CFL condition enforces that

$$\mathcal{D}(x_j, t_n) \subset \mathcal{D}_{\Delta t}(x_j, t_n), \quad (19)$$

which implies that characteristics should lie with the triangular zone under the line (l_1) and (l_2) , as shown in Figure 1. Therefore, substitution of (18) into (17) yields the CFL condition applied on the linear advection equation, as follows

$$-\frac{at_n}{\nu} \leq (x_j - at_n) - x_j \leq 0 \Leftrightarrow -\frac{at_n}{\nu} \leq -at_n \leq 0, \quad (20)$$

which, equally, leads to the CFL condition

$$\therefore \boxed{0 \leq \nu \leq 1 \Leftrightarrow 0 \leq \Delta t \leq \frac{\Delta x}{a}}. \quad (21)$$

Herein, the CFL condition (21) leads to constraint on the time step Δt for the case when $a > 0$. Note in passing that ν is non-negative.

2. Perspective of fixed-point value:

Line (l_1) passing point $A(x_j, t_n)$ and $B(x_{j-1}, t_{n-1})$ has the following form

$$\begin{aligned} (l_1) : \quad t &= t_A + \frac{t_B - t_A}{x_B - x_A} (x - x_A) \Leftrightarrow t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \\ &\Leftrightarrow t = t_n + \frac{-\Delta t}{-\Delta x} (x - x_j). \end{aligned} \quad (22)$$

Hence, line (l_1) passing line $t = 0$ at point D leads to the relation

$$x = x_j - \frac{t_n \Delta x}{\Delta t} \Leftrightarrow x - x_j = -\frac{t_n \Delta x}{\Delta t}. \quad (23)$$

Likewise, line (l_2) passing line $t = 0$ at point E leads to the relation

$$x - x_j = 0. \quad (24)$$

Therefore, combination of (23) and (24) leads to the numerical domain of dependence for the One-sided method in terms of fixed-point value

$$\boxed{\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid -\frac{t_n \Delta x}{\Delta t} \leq x - x_j \leq 0 \right\}.} \quad (25)$$

Besides, the analytical domain of dependence for the linear advection PDE, as given by (18), reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (26)$$

Then, by taking into consideration of requirement of the CFL condition, we obtain the following relation

$$-\frac{t_n \Delta x}{\Delta t} \leq (x_j - at_n) - x_j \leq 0, \quad (27)$$

which we have substituted (26) into (25). Herein, the relation (27) enforcing CFL condition on the time step Δt

$$\therefore \boxed{0 \leq \Delta t \leq \frac{\Delta x}{a}}, \quad (28)$$

which is similar to (21).

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{\Delta x} (u_j^n - u_{j+1}^n)$$



1. Point E in terms of fixed-point value satisfying

2. Numerical domain of dependence reads

3. Analytical domain of dependence reads

4. CFL condition reads

Note in passing that the advection velocity a in this Example 9 is $a < 0$.

Example 10. Examine the numerical domain of dependence and the corresponding CFL condition for a three-point scheme.

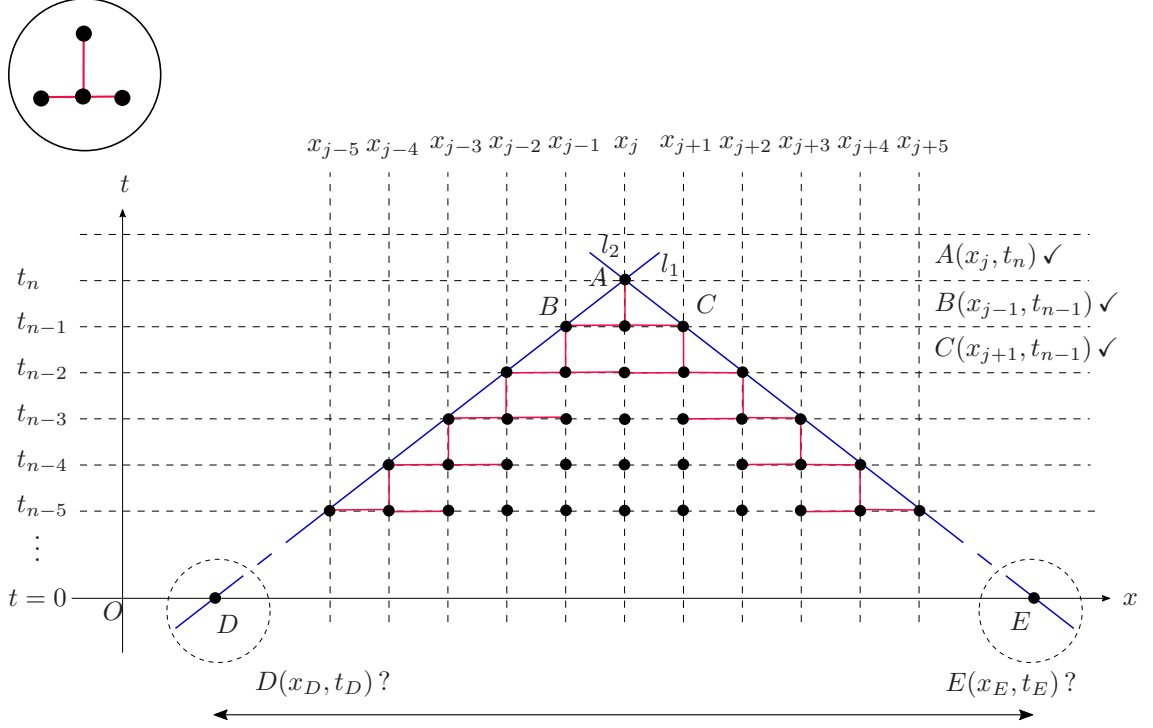


Figure 3: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 8 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1) : \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \quad (33)$$

$$(l_2) : \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (34)$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid |x - x_j| \leq \frac{at_n}{\nu} \right\}. \quad (35)$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (36)$$

4. CFL condition reads

$$\therefore \quad \left| \frac{a\Delta t}{\Delta x} \right| \leq 1. \quad (37)$$

Example 11. *Examine the numerical domain of dependence of Lax-Friedrichs method.*

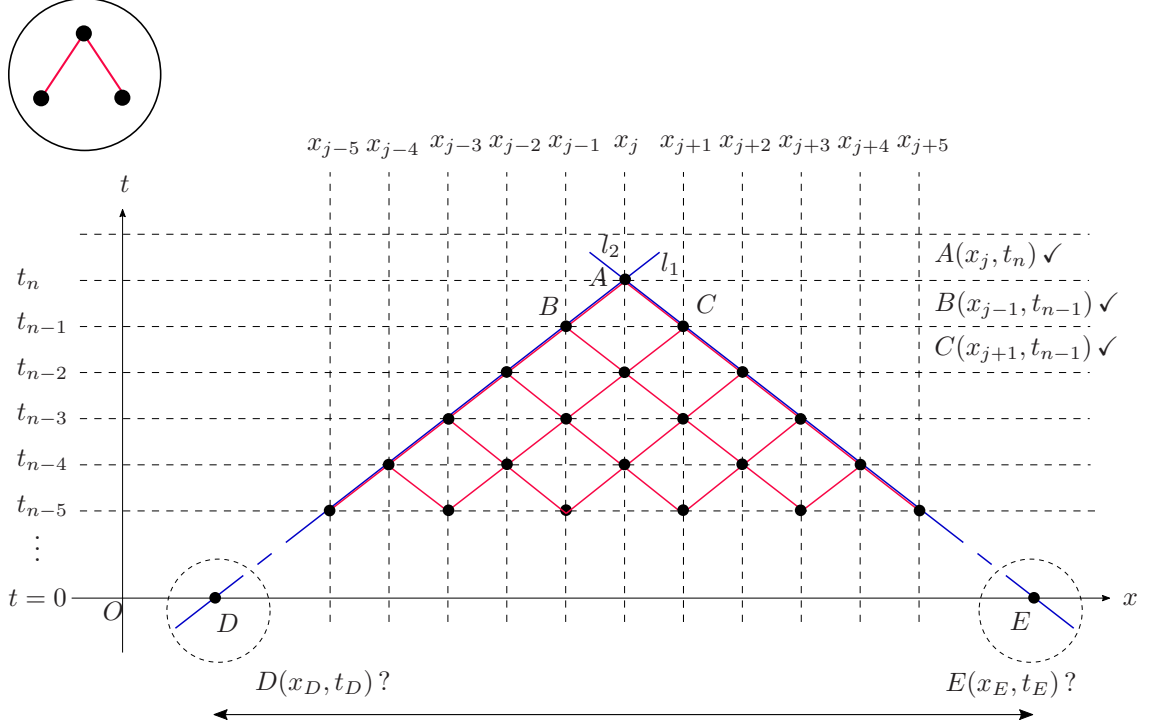


Figure 4: General numerical domain of dependence for a three-point scheme.

Similarly, by following steps done in Example 8, or the same as 10 we obtain the following summary:

1. Point D and E in terms of fixed-point value satisfying

$$(l_1): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j-1} - x_j} (x - x_j) \quad (38)$$

$$(l_2): \quad t = t_n + \frac{t_{n-1} - t_n}{x_{j+1} - x_j} (x - x_j) \quad (39)$$

2. Numerical domain of dependence reads

$$\mathcal{D}_{\Delta t}(x_j, t_n) = \left\{ x \mid |x - x_j| \leq \frac{at_n}{\nu} \right\}. \quad (40)$$

3. Analytical domain of dependence reads

$$\mathcal{D}(x_j, t_n) = \left\{ x \mid x = x_j - at_n \right\}. \quad (41)$$

4. CFL condition reads

$$\therefore \quad \left| \frac{a\Delta t}{\Delta x} \right| \leq 1. \quad (42)$$

7 von Neumann stability analysis

Example 12. *von Neumann stability analysis for Upwind method.*

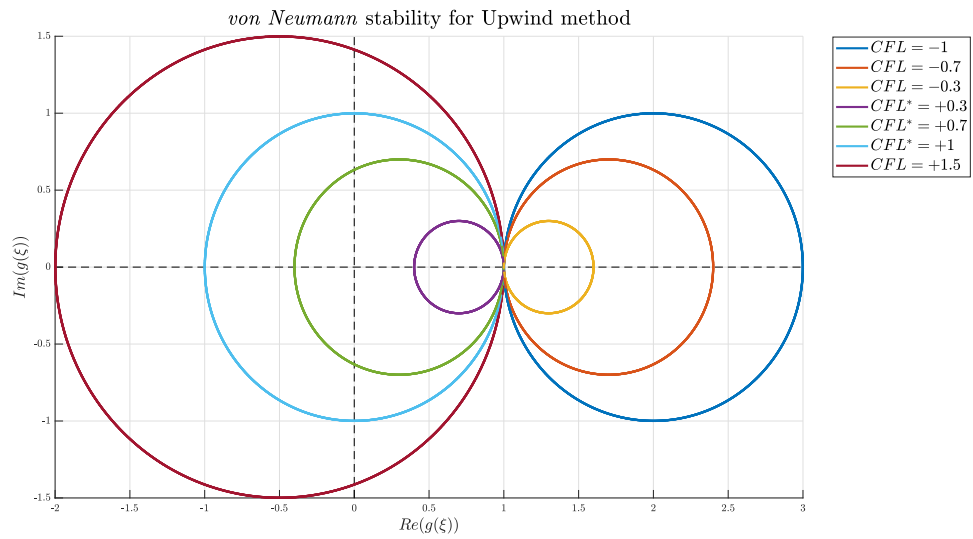


Figure 5: *von Neumann* stability analysis for Upwind method.

Example 13. *Summary consistency + stability \Rightarrow Convergence*

8 Conservation form - Finite Volume Method

Example 14. *Derivation of conservation form.*