Forward Algorithm in Hidden Markov Models for Credit Risk Progression

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1. Introduction

The forward algorithm is a dynamic programming technique used to compute the probability of an observed sequence in a Hidden Markov Model (HMM). In this context, we apply it to model the likelihood of credit score transitions over time.

2. Hidden Markov Model Definition

We define the HMM with states $S = \{\text{Low}, \text{Medium}, \text{High}\}$ and transition probabilities:

$$\begin{aligned} P(\text{Low} \to \text{Low}) = 0.7, \quad & P(\text{Low} \to \text{Medium}) = 0.3, \\ P(\text{Medium} \to \text{Medium}) = 0.6, \quad & P(\text{Medium} \to \text{High}) = 0.4, \\ P(\text{High} \to \text{High}) = 0.8, \quad & P(\text{High} \to \text{Medium}) = 0.2. \end{aligned}$$

The emission probabilities are defined based on credit scores corresponding to each state.

3. Forward Algorithm Computation

Given the observation sequence $O = [705, 645]$, we initialize the forward probabilities:

3.1. Initialization Step

Let $\alpha_1(i)$ represent the probability of being in state S_i at time $t = 1$ and observing O_1. Assuming equal initial probabilities $P(S_i) = \frac{1}{3}$:

$$\begin{aligned}\alpha_1(\text{Low}) &= P(\text{Low})P(705|\text{Low}),\\ \alpha_1(\text{Medium}) &= P(\text{Medium})P(705|\text{Medium}),\\ \alpha_1(\text{High}) &= P(\text{High})P(705|\text{High}).\end{aligned}$$

3.2. Recursion Step

For $t = 2$, we compute:

$$\begin{aligned}\alpha_2(\text{Low}) &= \sum_j \alpha_1(j)P(S_j \to \text{Low})P(645|\text{Low}),\\ \alpha_2(\text{Medium}) &= \sum_j \alpha_1(j)P(S_j \to \text{Medium})P(645|\text{Medium}),\\ \alpha_2(\text{High}) &= \sum_j \alpha_1(j)P(S_j \to \text{High})P(645|\text{High}).\end{aligned}$$

3.3. Termination Step

The probability of observing the sequence O is:

$$P(O) = \sum_i \alpha_2(i).$$