VEHICLE PLANAR DYNAMICS – BICYCLE MODEL

Assumptions

- 2-DoF,
  - Lateral, y (measured from instantaneous center of rotation O)
  - Yaw, ψ (wrt Global Axis)
- **Longitudinal velocity v_x is assumed to be constant.**
- Small slip angles, i.e. tires operate in the linear region.
- No rear wheel steering.
- No aligning moment in both tires.
- No road gradient or bank angle.
- There are only two wheels, one in the front and one in the rear.
- No lateral and longitudinal load transfer
- No rolling and pitching motion
- No chassis or suspension compliance effects

Notations

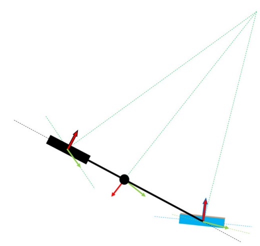
• Lateral Acceleration of CoG in the G frame	a_y	(calculated)
• Lateral Acceleration of CoG in the B frame.	$\ddot{y}$	(assumed to be measured)
• Yaw Rate	$\dot{\psi}$	(assumed to be measured)
• Longitudinal Velocity of Vehicle	v_x	(assumed to be known)
• Front/Rear Tire Cornering Stiffness	$C_{\alpha f}, C_{\alpha r}$	(assumed to be known)
• Front/Rear Tire Slip Angle	α_f, α_r	(calculated)
• Front Steering Angle	δ_f	(assumed to be measured)
• Front/Rear Wheel Velocity Angle	θ_{vf}, θ_{vr}	(calculated)
• Distance from CoG to Rear/Front Axels	L_f, L_r	(assumed to be known)
• Vehicle Length	$L = L_f + L_r$	(calculated)
• Road Radius	R	

System Model

- Lateral Force Equilibrium $\qquad ma_y = F_{yf} + F_{yr}$
- Moment Equilibrium $\qquad I_z\ddot{\psi} = l_f F_{yf} - l_r F_{yr}$

Lateral Acceleration in the G frame $\qquad a_y = \ddot{y} + v_x\dot{\psi}$

	Rear Tire	Front Tire
Lateral Forces in the B frame	$F_{yr} = F_{cr}\cos\delta_r$	$F_{yf} = F_{cf}\cos\delta_f$
Lateral Forces in the W frame	$F_{cr} = C_{\alpha r}\alpha_r$	$F_{cf} = C_{\alpha f}\alpha_f$
Slip Angle	$\alpha_r = \delta_r - \theta_{vr}$	$\alpha_f = \delta_f - \theta_{vf}$
Velocity Angle	$\theta_{vr} = \tan^{-1}\left(\dfrac{\dot{y} - L_r\dot{\psi}}{v_x}\right)$	$\theta_{vf} = \tan^{-1}\left(\dfrac{\dot{y} + L_f\dot{\psi}}{v_x}\right)$



Exact Lateral Force Balance

$$\begin{aligned}
ma_y &= F_{yr} + F_{yf} \\
&= F_{cr}\cos\delta_r + F_{cf}\cos\delta_f \\
&= C_{\alpha r}\alpha_r\cos\delta_r + C_{\alpha f}\alpha_f\cos\delta_f \\
&= C_{\alpha r}(\delta_r - \theta_{vr})\cos\delta_r + C_{\alpha f}(\delta_f - \theta_{vf})\cos\delta_f
\end{aligned}$$

$$m(\ddot{y} + v_x\dot{\psi}) = C_{\alpha r}\left(\delta_r - \tan^{-1}\left(\frac{\dot{y} - L_r\dot{\psi}}{v_x}\right)\right)\cos\delta_r + C_{\alpha f}\left(\delta_f - \tan^{-1}\left(\frac{\dot{y} + L_f\dot{\psi}}{v_x}\right)\right)\cos\delta_f$$

Approximate Lateral Force Balance

$$m(\ddot{y}+v_x\dot{\psi}) \cong C_{\alpha r}\left(\delta_r - \frac{\dot{y}-L_r\,\dot{\psi}}{v_x}\right)+C_{\alpha f}\left(\delta_f - \frac{\dot{y}+L_f\dot{\psi}}{v_x}\right)$$

$$=\left(C_{\alpha r}\delta_r - C_{\alpha r}\frac{\dot{y}}{v_x} - C_{\alpha r}\frac{-L_r\,\dot{\psi}}{v_x}\right)+\left(C_{\alpha f}\delta_f - C_{\alpha f}\frac{\dot{y}}{v_x} - C_{\alpha f}\frac{L_f\dot{\psi}}{v_x}\right)$$

$$=-C_{\alpha r}\frac{\dot{y}}{v_x} - C_{\alpha f}\frac{\dot{y}}{v_x} - C_{\alpha r}\frac{-L_r\,\dot{\psi}}{v_x} - C_{\alpha f}\frac{L_f\dot{\psi}}{v_x} + C_{\alpha r}0 + C_{\alpha f}\delta_f$$

$$=\left(-C_{\alpha r}\frac{1}{v_x} - C_{\alpha f}\frac{1}{v_x}\right)\dot{y}+\left(-C_{\alpha r}\frac{-L_r}{v_x} - C_{\alpha f}\frac{L_f}{v_x}\right)\dot{\psi} + C_{\alpha f}\delta_f$$

$$m(\ddot{y}+v_x\dot{\psi}) = \left(-\frac{C_{\alpha r}+C_{\alpha f}}{v_x}\right)\dot{y}+\left(\frac{C_{\alpha r}L_r - C_{\alpha f}L_f}{v_x}\right)\dot{\psi} + C_{\alpha f}\delta_f$$

$$\ddot{y} = \left(-\frac{C_{\alpha r}+C_{\alpha f}}{mv_x}\right)\dot{y}+\left(\frac{C_{\alpha r}L_r - C_{\alpha f}L_f}{mv_x} - v_x\right)\dot{\psi} + \frac{C_{\alpha f}}{m}\delta_f$$

Approximate Moment Balance

$$I_z\ddot{\psi} \cong -C_{\alpha r}\left(\delta_r - \frac{\dot{y}-L_r\dot{\psi}}{v_x}\right)L_r + C_{\alpha f}\left(\delta_f - \frac{\dot{y}+L_f\dot{\psi}}{v_x}\right)L_f$$

State Space Representation (Jazar Chapter 10 page 612 - Equation 10.184)

$$\begin{Bmatrix}\ddot{y}\\ \ddot{\psi}\end{Bmatrix} = \begin{bmatrix} -\dfrac{C_{\alpha r}+C_{\alpha f}}{mv_x} & \dfrac{L_rC_{\alpha r}-L_fC_{\alpha f}}{mv_x} - v_x \\ \dfrac{L_rC_{\alpha r}-L_fC_{\alpha f}}{I_zv_x} & -\dfrac{L_r^2C_{\alpha r}+L_f^2C_{\alpha f}}{I_zv_x}\end{bmatrix}\begin{Bmatrix}\dot{y}\\ \dot{\psi}\end{Bmatrix} + \begin{bmatrix}\dfrac{C_{\alpha f}}{m}\\ \dfrac{C_{\alpha f}L_f}{I_z}\end{bmatrix}\delta_f \qquad \delta_r = 0$$

Centrifugal force and tire forces also have a longitudinal component in the BODY axis.

$$
\begin{aligned}
\dot{\beta} &= \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \\
&= \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\dot{x}^2} \\
&= \frac{\ddot{y}\,\dot{x} - 0\,\dot{y}}{\dot{x}^2} \\
&= \frac{\ddot{y}}{\dot{x}}
\end{aligned}
$$

The Other Version:

$$
\begin{Bmatrix} \dot{\beta} \\ \ddot{\psi} \end{Bmatrix} = \begin{bmatrix} -\dfrac{C_{\alpha r} + C_{\alpha f}}{m v_x} & \dfrac{L_r C_{\alpha r} - L_f C_{\alpha f}}{m v_x^2} - 1 \\ \dfrac{L_r C_{\alpha r} - L_f C_{\alpha f}}{I_z} & -\dfrac{L_r^2 C_{\alpha r} + L_f^2 C_{\alpha f}}{I_z v_x} \end{bmatrix} \begin{Bmatrix} \beta \\ \dot{\psi} \end{Bmatrix} + \begin{bmatrix} \dfrac{C_{\alpha f}}{m v_x} \\ \dfrac{C_{\alpha f} L_f}{I_z} \end{bmatrix} \delta_f
$$

$$
\dot{\hat{\beta}} = -\frac{a_x}{v}\sin\hat{\beta} + \frac{a_y}{v}\cos\hat{\beta} - \dot{\psi}
$$

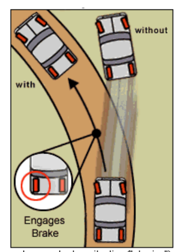
Steady State Steering Angle

$\delta_{ss} = \delta_f \cong \dfrac{L}{R} + \alpha_f - \alpha_r$ GEOMETRY

- $F_{yf} + F_{yr} = m\dfrac{v_x^2}{R} = ma_y$ CONSTANT Centripetal/Lateral Acc.
- $L_f F_{yf} - L_r F_{yr} = I_z \ddot{\psi} = 0$ ZERO Angular Acc.
- $F_{yf} = m\dfrac{L_r}{L}a_y$ $F_{yr} = m\dfrac{L_f}{L}a_y$ CONSTANT
- $\alpha_f = \dfrac{F_{yf}}{C_{\alpha f}} = \dfrac{L_r}{C_{\alpha f}}\dfrac{ma_y}{L}$ $\alpha_r = \dfrac{F_{yr}}{C_{\alpha r}} = \dfrac{L_f}{C_{\alpha r}}\dfrac{ma_y}{L}$ CONSTANT

$$\delta_{ss} = \frac{L}{R} + \left(\frac{L_r}{C_{\alpha r}} - \frac{L_f}{C_{\alpha f}}\right)\frac{ma_y}{L}$$

$$= \frac{L}{R} + K_v a_y$$



In an understeer situation ("plowing"), the stability system gently brakes the inside rear wheel to help the front end follow the path of the curve.

- Under-steer gradient $K_v = \left(\dfrac{L_r}{C_{\alpha r}} - \dfrac{L_f}{C_{\alpha f}}\right)\dfrac{m}{L}$
  - Under Steer $K_v > 0$ $\dfrac{L_r}{C_{\alpha r}} > \dfrac{L_f}{C_{\alpha f}}$ $\alpha_f > \alpha_r$
  - Neutral Steer $K_v = 0$ $\dfrac{L_f}{C_{\alpha f}} = \dfrac{L_r}{C_{\alpha r}}$ $\alpha_f = \alpha_r$
  - Over Steer $K_v < 0$ $\dfrac{L_r}{C_{\alpha r}} < \dfrac{L_f}{C_{\alpha f}}$ $\alpha_f < \alpha_r$

Vehicle Side Slip Angle Estimation

$$\dot{\hat{\beta}} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)$$
$$= \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\dot{x}^2}$$

This looks okay, but we would like to use sensor measurements.

$$\dot{\hat{\beta}} = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\dot{x}^2} \cong \frac{\dot{x}^2}{v^2}\frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\dot{x}^2}$$
$$= \frac{1}{v^2}\left(\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}\right)$$
$$= \frac{1}{v}\left(\ddot{y}\frac{\dot{x}}{v} - \ddot{x}\frac{\dot{y}}{v}\right)$$
$$= \frac{1}{v}\left(\ddot{y}\cos\beta - \ddot{x}\sin\beta\right)$$
$$= -\frac{\ddot{x}}{v}\sin\beta + \frac{\ddot{y}}{v}\cos\beta + \dot{\psi} - \dot{\psi}$$
$$= -\frac{\ddot{x}}{v}\sin\beta + \frac{\ddot{y}}{v}\cos\beta + \dot{\psi}\left(\sin^2\beta + \cos^2\beta\right) - \dot{\psi}$$
$$= -\frac{\ddot{x}}{v}\sin\beta + \frac{\ddot{y}}{v}\cos\beta + \left(\dot{\psi}\sin^2\beta + \dot{\psi}\cos^2\beta\right) - \dot{\psi}$$
$$= -\frac{\ddot{x}}{v}\sin\beta + \dot{\psi}\sin^2\beta + \frac{\ddot{y}}{v}\cos\beta + \dot{\psi}\cos^2\beta - \dot{\psi}$$
$$= -\frac{\ddot{x}}{v}\sin\beta + \frac{\dot{y}\dot{\psi}}{v}\sin\beta + \frac{\ddot{y}}{v}\cos\beta + \frac{\dot{x}\dot{\psi}}{v}\cos\beta - \dot{\psi}$$
$$= -\frac{\ddot{x} - \dot{y}\dot{\psi}}{v}\sin\beta + \frac{\ddot{y} + \dot{x}\dot{\psi}}{v}\cos\beta - \dot{\psi}$$
$$= -\frac{a_x}{v}\sin\beta + \frac{a_y}{v}\cos\beta - \dot{\psi}$$

Now we can use yaw rate and acceleration sensors to estimate the yaw rate as below.

$$\boxed{\dot{\hat{\beta}} = -\frac{a_x}{v}\sin\hat{\beta} + \frac{a_y}{v}\cos\hat{\beta} - \dot{\psi}}$$

Maximum Vehicle Side Slip Angle

It is a saturating function of the characteristic velocity.

$$v_x < v_{ch} \Rightarrow \beta_{\max} = 2\left(k_1 - k_2\right)\frac{v_x^3}{v_{ch}^3} - 3\left(k_1 - k_2\right)\frac{v_x^2}{v_{ch}^2} + k_1$$
$$v_x > v_{ch} \Rightarrow \beta_{\max} = k_2$$

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Yaw Rate Estimation

This is actually the steady state response, i.e. the gain, of the transfer function between the steering input and the yaw rate output.

$$\dot{\psi} = v_x \frac{\delta_f}{L\left(1 + \frac{v_x^2}{v_{ch}^2}\right)}$$

Derivation of the above equation and the Characteristic Velocity

$$\begin{aligned}\delta_{ss} &= \frac{L}{R} + K_v a_y \\ &= \frac{L}{R} + K_v \frac{v_x^2}{R} \\ &= \frac{L}{v_x}\frac{v_x}{R} + K_v v_x \frac{v_x}{R} \\ &= \frac{L}{v_x}\dot{\psi} + K_v v_x \dot{\psi}\end{aligned}$$

$$\dot{\psi} = \frac{\delta_{ss}}{\frac{L}{v_x} + K_v v_x}$$

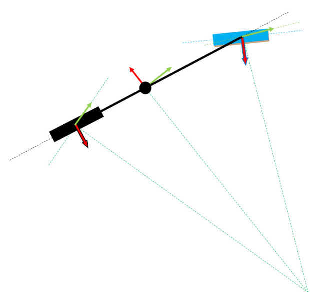
$$\dot{\psi} = \frac{\delta_{ss}}{\frac{L}{v_x} + \frac{L}{v_x}\frac{v_x}{L}K_v v_x}$$

$$\dot{\psi} = \frac{\delta_{ss}}{\frac{L}{v_x}\left(1 + K_v \frac{v_x^2}{L}\right)} = v_x \frac{\delta_{ss}}{L\left(1 + \frac{v_x^2}{v_{ch}^2}\right)}$$

$$\boxed{v_{ch}^2 = \frac{L}{K_v}}$$

Maximum Yaw Rate

$$\begin{aligned}a_{rad\ meas} &= a_y \cos\beta + a_x \sin\beta \\ &= \left(\ddot{y} + v_x\dot{\psi}\right)\cos\beta + a_x \sin\beta \\ a_{rad\ meas}^{\max} &\cong \left(0 + v_x\dot{\psi}_{\max}\right)1 + a_x \sin\beta \\ \dot{\psi}_{\max} &= \frac{1}{v_x}\left(a_{rad\ meas}^{\max} - a_x \sin\hat{\beta}\right)\end{aligned}$$



We assume that

- CoG draws a perfect circle during a maneuver.
- The radial acceleration is in the direction of the radial force acting on the CoG.
- The radial acceleration is measured as the car travels around a circular trajectory.
- The longitudinal acceleration of the car measured along the longitudinal axis of the vehicle also contributes to the radial acceleration of the CoG.
- All the acceleration is spent for the yaw motion, no lateral translation exist which also means that the CoG stays on the circular trajectory.
- The max radial acceleration corresponds to a maximum yaw rate.

Note that radial and lateral accelerometers do not point at the same direction.

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3 tane yerde yaw rate hesapladik

1.Bicycle model steady state

2.Max yawrate Steering Pad deneylerinden

1 ile 2 yi karistirip gaini bulduk sonra bicycle modelin steering input/yaw rate output (gaini 1 olan) transfer fonksiyonundan gecirdik

(bir adet daha saturation function var en son islem olarak)

3.4W modelden gelen gercege daha yakin yaw rate var birde

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```
clear all

close all
clc

% What is the sampling time in [s]?
dt  = 0.1;      % [sec]

% What is the longitudinal velocity of the vehicle CoG in [m/s]?
vx  = 10;       % [m/s]

% What are the vehicle parameters?
% Alfa Romeo Parameter Set
Caf = 42200;    % [N/rad]
Car = 28567;    % [N/rad]
Lf  = 1.18;     % [m]
Lr  = 1.52;     % [m]
m   = 1582;     % [kg]
Iz  = 2430;     % [kgm^2]


function [Kv,sysd] = bicycle(Caf,Car,Lf,Lr,m,Iz,vx,dt)

a11 =  -(Caf+Car)/(m*vx);
a12 =  -(Lf*Caf-Lr*Car)/(m*vx^2)-1;
a21 =  -(Lf*Caf-Lr*Car)/Iz;
a22 =  -(Lf^2*Caf+Lr^2*Car)/(Iz*vx);
Ac  = [a11 a12; a21 a22];

b11 =   Caf/(m*vx);
b21 =   Lf*Caf/Iz;
Bc  = [b11; b21];

Cc  = eye(2);

Dc  = zeros(2,1);

[Ad,Bd,Cd,Dd] = c2dm(Ac,Bc,Cc,Dc,dt);

sysd = ss(Ad,Bd,Cd,Dd);

set(sysd,'Name','BICYCLE MODEL')
set(sysd,'InputName',{'delta_f'})
set(sysd,'StateName',{'beta','psi_dot'})
set(sysd,'OutputName',{'beta','psi_dot'})
set(sysd,'Ts',dt)

Kv = m*Lr/(Lr+Lf)/Caf-m*Lf/(Lr+Lf)/Car;
```