# Core-Stable Kidney Exchange via Altruistic Donors

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**Abstract**

Kidney exchange programs—connecting hospitals in the U.S. or countries in Europe—aim to improve efficiency by pooling donors and patients on a centralized platform. Core-stable allocations are essential for sustaining cooperation and preventing defections. However, when the core is empty, incentive problems emerge as hospitals or countries may withhold easily matched pairs for internal use, reducing the overall efficiency and scope of the exchange. We propose a solution to restore core stability by supplementing the platform with additional altruistic donors. Although the worst-case requirement for the number of such donors may be large, we show that in realistic settings, only a few are sufficient. We analyze two models of the compatibility graph: one based on random graphs, and the other on compatibility types. When only pairwise exchanges are allowed, the number of required altruists is bounded by the maximum number of independent odd cycles—disjoint odd cycles with no edges between them. This parameter grows logarithmically with market size in the random graph model, and is at most one-third the number of compatibility types in the type-based model. When small exchange cycles are allowed, it suffices for each participating organization to receive a number of altruists proportional to the number of compatibility types. Finally, simulations show that far fewer altruists are needed in practice than theory suggests.

## 1 Introduction

TODO: switch to players or hospitals from countries in the text The kidney exchange market in the U.S. facilitates hundreds of transplants annually for patients with a willing but incompatible live donor. However, the market is fragmented and operates inefficiently (Agarwal et al. [1]). A key reason is that exchange platforms use inferior mechanisms and hospitals have little incentive to submit patients and donors with a high likelihood of compatible exchange. A similar challenge exists in Europe, where kidney exchange programs are often smaller in scale and face coordination difficulties across national borders. The fragmentation of exchange pools and the varying institutional incentives further limit the efficiency of matches (Biró et al. [10]).

What makes an exchange platform effective? Economic theory offers a clear benchmark: the allocation generated by the platform should lie in the core—that is, no group of hospitals or countries should be able to achieve better outcomes by deviating from the centralized solution and conducting their own exchange. When the deviating group consists of a single hospital or country, this condition ensures that it does not conduct internal exchanges while submitting only its hard-to-match patient-donor pairs to the platform, a current common practice in both U.S and European programs, that undermines the overall efficiency of the exchange (Agarwal et al. [1], Biró et al. [10] [2]).

However, a fundamental question remains: under what conditions does a core allocation exist? And when it does not, what alternative solutions can prevent the exchange platform from unraveling? This paper investigates these questions within a well-studied graph-theoretic model of kidney exchange, establishing both impossibility results and potential remedies.

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[2]Agarwal et al. [1] attribute inefficiency to the individual trade-off faced by hospitals between the costs and benefits of submitting donor–patient pairs. However, even in the absence of such submission costs, exchange platforms continue to face challenges in preventing market unraveling.

We show that in general compatibility graphs, a weak core solution may fail to exist under both pairwise and longer cycle exchanges. To address this issue, we introduce the concept of a *supplemented core*. The key idea is to use altruistic donors—individuals willing to donate an organ without receiving one in return—to help stabilize the market. A supplemented core consists of exchanges among the original patient-donor pairs together with a few additional altruistic donors provided by the centralized platform, such that no coalition of organizations can benefit by forming an alternative exchange on their own.

In theory, with a sufficient number of altruistic donors, a centralized platform can match all patients and achieve a core outcome. However, the number of altruistic donors required for this is unrealistically large. We show that while general compatibility graphs may require a large number of additional donors, the number needed is significantly smaller for graphs that reflect realistic settings. To model such graphs, we consider two frameworks: one based on a random graph process, as studied in the literature (Ashlagi and Roth [5], Delorme et al. [13]), and another based on type presentation (Dickerson et al. [14]). We present three main results.

First, without any assumptions on the compatibility graph with $|V|$ donor-patient pairs, obtaining a core solution requires at least $\Omega(|V|)$ additional donors for either pairwise or cyclic exchange.

Second, for pairwise exchange, the required altruistic donor is at most the maximum number of *independent odd cycles* in the mutual compatibility graph, where a set of odd cycles is considered independent if no donor-patient vertex in one cycle has mutual compatibility with any donor-patient vertex in another cycle. This result has two implications.

- When the compatibility graph is generated using a standard random graph model, the maximum number of independent odd cycles is $\mathcal{O}(\log |V|)$. In the class of compatibility graphs based on type presentation, we show that this number is at most one-third of the number of types. Thus, in these graphs, only a small number of additional donors are needed to stabilize the exchange.
- In the special case where the underlying compatibility graph is bipartite, our result implies that a weak-core allocation always exists. Moreover, we show that bipartite graphs capture a broad class of utility functions in one-sided exchange economies, which we term *binary assignment valuations*. This framework generalizes the results of Echenique et al. [17].

Third, we show that for small cyclic exchanges, the required number of additional altruistic donors is at most the number of types in the type-representation multiplied by the cycle length bound per participating organization. Thus, when the number of compatibility types are fixed, regardless of the overall market size, in the worst case, it suffices to have at most a constant number of altruistic donors per organization and a constant number of donors for the whole instance, if the number of organizations is also a constant - for example in Europe, most programs consist of at most 3 or 4 countries.

We complement our theoretical findings with empirical analysis using data and simulations. We find that the number of additional altruistic donors required in practice is significantly smaller than the theoretical worst-case bounds. In all simulated cases, consisting of 100-150 donor-recipient pairs, at most one altruistic donor was always sufficient to guarantee a supplemented core solution. Furthermore, we show that even a much stronger notion of core stability, which we refer to as the transferable utility (TU) core, can be achieved with a similarly small number of altruistic donors.

Our results highlight the crucial role of altruistic donors—not only in increasing match rates and enabling long exchange chains, but also in promoting market stability. In the United States, approximately 300-400 altruistic donors participate annually, compared to about 6,000 patient-donor pairs in the UNOS database, and 164 altruistic donors versus 1,265 patient-donor pairs in the NKR dataset (Agarwal et al. [1]). Despite their significantly smaller numbers, our findings show that altruistic donors can have a disproportionately large impact on the efficiency and robustness of the exchange. These insights highlight the need to more efficiently recognize and incorporate altruistic donors in the design of kidney exchange platforms.

The paper is organized as follows. After discussing related work, we introduce the model, the main solution concept and provide preliminaries on the methodology used to establish our general results. Section 3 studies pairwise exchange in general graphs. Section 4 focuses on an application to one-sided markets. Section 5 analyzes cyclic exchange. Section 6 presents simulation results. Section 7 concludes. Many proofs are given in Appendix due to space constraints.

**Related Literature**

The success of kidney exchange has inspired a rich literature on market design challenges, beginning with Roth et al. [25]. Research spans various exchange structures—such as pairwise and cyclic exchanges—and participant models, from individual agents to coordinating organizations. While individual-level incentives are relatively well understood (Roth et al. [25], Sönmez and Ünver [29]), multi-hospital and international exchanges present growing challenges (Ashlagi and Roth [4, 5]). Hospitals, as key decision-makers, face more complex choices than individual donor-patient pairs, and their control over multiple pairs introduces strategic considerations that can significantly affect efficiency (Agarwal et al. [1]).

The key distinction between our work and the growing literature on kidney exchange among organizations lies in our positive result for core-stable solutions. In contrast, existing studies either emphasize negative results regarding core stability or adopt weaker solution concepts, such as individual rationality. For example, Ashlagi and Roth [5] focus on individual rationality rather than addressing core stability. Similarly, Ashlagi et al. [6] analyze incentive issues and provide multiplicative efficiency bounds, but their framework does not incorporate core stability constraints.

Our model builds on the frameworks developed in Csáji et al. [11] and Kern et al. [18]. However, these studies primarily focus on negative results regarding core stability. To the best of our knowledge, no existing work has established a positive result on the existence of (near-feasible) core-stable outcomes in the kidney exchange setting. Even in the restricted case of bipartite graphs, our result on the non-emptiness of the weak core is novel and constitutes a strict generalization of Echenique et al. [17]. Our approach differs from this literature in its emphasis on core stability and in its use of an additive error to measure near feasibility. Our key insight is that even a few directed donors, deployed strategically by a centralized platform, can stabilize the market.

There is also a large literature on the dynamic aspects of kidney exchange, including Anderson et al. [3], Dickerson and Sandholm [15], Akbarpour et al. [2], which primarily focuses on trade-offs between efficiency, waiting time, and fairness, while largely overlooking more complex issues such as core-stability—the focus of our paper. Related to dynamic settings, Klimentova et al. [19] and Kern et al. [18] proposed a credit system for international kidney exchange programs (IKEPs) in Europe, using cooperative game theory to assign fair transplant targets each round and carrying deviations as credits to promote long-term fairness. We instead propose round-by-round core allocations to ensure fairness and incentive compatibility without relying on future adjustments.

From a practical operational perspective, the literature has explored various computational methods for finding good solutions to kidney exchange programs (e.g., Mincu et al. [21], Ashlagi et al. [7], Druzsin et al. [16]). However, these approaches are primarily based on simulations and generally lack theoretical guarantees. To obtain theoretical results, we adopt the Relax-and-Round method of Nguyen and Vohra [23], Nguyen et al. [22], that uses Scarf's lemma to relax stability while preserving feasibility. Our approach differs in two key ways: we incorporate endowments and introduce a new rounding procedure. Unlike prior work that limits bundle sizes, we consider hospitals endowed with multiple pairs, enabling large-scale exchanges. Our rounding must therefore respect the structure of the compatibility graph and constraints on allowable exchanges.

Orthogonal to our work, Agarwal et al. [1], maybo other papers.. examine the additional costs associated with submitting applications and propose a reimbursement mechanism to mitigate this issue. Even if such costs are effectively addressed, the remaining challenge—addressed in this paper—is to design an algorithm capable of identifying matchings that satisfy additional stability constraints.

## 2 Model and Preliminaries

In this Section we describe our mathematical model for Kidney Exchange, motivated by the works of Csáji et al. [11] and Kern et al. [18]. Unlike Roth et al. [25], which models each agent as a donor-patient pair, we consider agents as organizations, such as hospitals in the U.S. exchange system or countries in the international exchange system. We consider an exchange involving a set of $n$ countries (our results naturally extends to other kidney exchange settings), denoted by $N = \{1, \ldots, n\}$. Each country $i \in N$ is endowed with a set $V^i$ consisting of donor-recipient pairs, unpaired patients, and altruistic donors. The sets $V^i$ are disjoint across countries. Within each $V^i$, let $U^i \subset V^i$ denote the set of non-altruistic donor-patient pairs and unpaired patients. The remaining elements, $V^i \setminus U^i$, are altruistic donors, who are willing to donate without requiring a kidney in return.

**Compatibility graphs.** Compatibility is modeled via a directed graph, denoted by $\mathcal{G} = (V, E)$, where $V = V^1 \cup .. \cup V^n$, and the set of edges, $E$, captures the compatibility of exchange. Specifically,

- For two non-altruistic donors—patient-donor pairs $u, v$, a directed edge $(u, v) \in E$ indicates that the donor in pair $u$ is compatible with the patient in pair $v$, meaning a transplant from $u$ to $v$ is feasible.
- To simplify notation and the description of feasible exchanges, we can model altruistic donors as special donor–patient pairs by introducing a dummy patient who is compatible with any donor. This allows us to define directed edges following the same rules as before.
- Similarly, we can model unpaired patients by adding a dummy donor who is only compatible with the dummy patient associated with the altruistic donors.

The bounds we derive on the number of altruistic donors required depend on two key parameter of the compatibility graph: the maximum number of independent odd cycles and the number of types in an optimal type-based representation. As discussed in the introduction, both parameters are typically small in practice, especially in kidney exchange applications. To formalize this, we examine two important models of compatibility graphs.

*Random graphs:* Using random graphs is a common framework for modeling compatibility of kidney exchange. Mathematically, donor-recipient pairs are partitioned into a finite set of groups $\Phi$, based on characteristics such as blood type and the sensitization levels of both the donor and the patient. For any two groups $i, j \in \Phi$ (not necessarily distinct), let $p_{ij}$ denote the probability that a donor from group $i$ is compatible with a patient from group $j$. If $p_{ij} = 0$, then compatibility is ruled out—for instance, due to incompatible blood types. Otherwise, $p_{ij}$ may depend on additional clinical factors, such as the patient's calculated Panel Reactive Antibody (cPRA) level. A random graph is then generated by assigning donor-recipient pairs to groups according to a distribution calibrated to match empirical data. For any two donor-recipient pairs $u$ from group $i$ and $v$ from group $j$, a directed compatibility edge from $u$ to $v$ is added independently with probability $p_{ij}$.

*Type-representation of graphs:* Another related but conceptually distinct approach is the deterministic type-based model. In contrast to the probabilistic model, compatibility here is deterministic: whether one pair is compatible with another is fully determined by their assigned types. However, unlike in the random graph model above where types reflect observable donor and patient characteristics (e.g., blood type, cPRA), the "type" in this setting is an abstract construct introduced solely to parametrize the compatibility graph. It does not necessarily correspond to any specific medical or demographic attributes. Specifically, we say that a (di)graph $\mathcal{G} = (V, E)$ *can be represented by a set of types* $T$, if it holds that there exists a labeling $f : V \to T$, such that for any two types $t_1, t_2 \in T$, either $\{(u, v) \mid u \in f^{-1}(t_1), v \in f^{-1}(t_2)\} \subseteq E$ or $\{(u, v) \mid u \in f^{-1}(t_1), v \in f^{-1}(t_2)\} \cap E = \emptyset$. That is, the existence of a (directed) edge only depends on the types of the endpoints. Dickerson et al. [14] conducted simulations showing that kidney exchange graphs can usually be represented by only a few types. We may assume that for any type $t_i$, there is no edge $(u, v)$ with $f(u) = f(v) = t_i$, as donor-recipient pairs are deleted, if they are compatible with each other.

**Exchanges.** We consider two models of kidney exchange: **pairwise exchange** and **cyclic exchange**. In pairwise exchange, only mutually compatible donor-patient pairs can trade kidneys. Cyclic exchange allows longer cycles, where each donor gives to the next patient and receives a kidney from the previous donor. We assume a maximum cycle length of $\Delta$, with $\Delta = 2$ corresponding to pairwise exchange. In practice, $\Delta = 3$ is typical, while values above 5 are rare due to the logistical and medical challenges of coordinating multiple simultaneous surgeries.[3]

Let $\mathcal{C}_\Delta$ denote the cycles of length at most $\Delta$. Given a subset of vertices $W \subseteq V$, denote the induced compatibility graph on $W$ as $\mathcal{G}[W]$. We call $\mathcal{E}$ an exchange among $W$ if $\mathcal{E}$ is an union of disjoint cycles of length at most $\Delta$ in the induced compatibility graph $\mathcal{G}[W]$. We say that $y^* \in [0, 1]^{\mathcal{C}_\Delta}$ is a *fractional exchange* if it holds that for any vertex $v \in V$, $\sum\limits_{c \in \mathcal{C}_\Delta | v \in c} y_c^* \leq 1$.

Given an exchange $\mathcal{E} = (V_\mathcal{E}, E_\mathcal{E})$, the utility of a country $i$ is given by the number of their patients receiving a kidney, that is

$$u_i(\mathcal{E}) := |V_\mathcal{E} \cap U^i|.$$

Overall, we refer to the economy as a *partition exchange economy*, represented by the tuple $(\mathcal{G}, \Delta, \{V^i, U^i\}_{i=1}^n)$.

Definition 2.1. *Given a partition exchange economy, an exchange $\mathcal{E}$ is* individually rational, *if $u_i(\mathcal{E}) \geq u_i(\mathcal{E}_i)$ for any $i \in N$ and exchange $\mathcal{E}_i \subseteq \mathcal{G}[U^i]$. That is, no player can improve its utility with an exchange of its own.*

Definition 2.2. *Given a partition exchange economy, an exchange $\mathcal{E}$ among $V$, and a coalition $P \subseteq N = \{1, .., n\}$, we say that $P$ is a* (strong) blocking coalition to $\mathcal{E}$, *if there exists an exchange $\mathcal{E}'$ in the graph $\mathcal{G}[\cup_{i \in P} V_i]$ such that $u_i(\mathcal{E}') > u_i(\mathcal{E})$ for all $i \in P$. We say that $\mathcal{E}$ is in the (weak) core, if there is no blocking coalition to $\mathcal{E}$.*

Note that in the definition above, some papers in the literature refer to such a block as a strong block and the corresponding concept as the weak core. However, since our paper does not consider the stronger notion of the core—namely, the strong core, for reasons outlined below—we will, without loss of clarity, omit the qualifiers strong and weak from our terminology.

As we will show in this paper, the core can be empty in both pairwise and cyclic exchanges. To obtain a positive result, we either impose a restriction on the compatibility graph—discussed in Section 4—or relax the notion of the core by introducing what we call the supplemented core. To this end, we introduce a special agent, indexed by 0, representing the centralized market designer, with $V^0$ denoting the designer's endowment of *altruistic* donors. Let $\mathcal{G}^{+V^0}$ be the extended compatibility graph that incorporates the additional donors in $V^0$. The designer *is not a player of the game and is not allowed to*

[3]Even in the case of altruistic donors, cycles become chains, and operational difficulties with chains are less severe. Longer chains are also rare in practice (Agarwal et al. [1]). Moreover, chains of size 10–20 typically result from the accumulation of exchanges over an extended period. For example, the National Kidney Registry reported a 25-transplant chain that was built up over several months. Our model aims to capture the clearing mechanism over a relatively shorter period, during which longer chains can be broken into smaller segments. The donor at the end of a chain in one period can then be viewed as an altruistic donor initiating a new chain in the next period.

*participate in any coalition.* Their sole role is to stabilize the exchange by providing additional resources when necessary.

Definition 2.3. *Given additional altruistic donors $V^0$, an exchange $\mathcal{E}^*$ in the extended compatibility graph $\mathcal{G}^{+V^0}$ is a $V^0$-supplemented core if it is not blocked by any coalition $P \subseteq N = \{1, .., n\}$. If $|V^0| \leq d$, we also refer to $\mathcal{E}^*$ as a $d$-supplemented core exchange.*

Definition 2.4. *Given a set of additional altruistic donor $V^0$, an exchange $\mathcal{E}^*$ in the extended compatibility graph $\mathcal{G}^{+V^0}$ is* Pareto optimal *if there does not exist another exchange $\mathcal{E}'$ in $\mathcal{G}^{+V^0}$ such that all countries are weakly better off and at least one country is strictly better off.*

Remark 2.1. *An alternative solution concept is the strong core, which rules out weakly blocking coalitions—that is, coalitions where all members are weakly better off and at least one member is strictly better off. However, it is well known that in exchange economies with satiated utilities, the strong core is an unreasonably demanding solution concept. Consider the following example: Country $A$ has $k$ altruistic donors who are compatible with any patient. There are $k+1$ other countries, $B_1, \ldots, B_{k+1}$ with $k$ donor-recipient pairs, each of which could fully utilize all $k$ donors of $A$ to meet their own needs. Moreover, no exchange is possible among the countries $B_1, \ldots, B_{k+1}$ without the help of additional donors. No matter how country $A$ distributes the donors among the other countries, there will always be a weakly blocking coalition between $A$ and one of the $B_i$ countries that has none of its patients matched. To achieve a strong core allocation, we would need to guarantee that all $B_i$ countries match all their donors, otherwise, $B_i$ and $A$ would still weakly block. Hence, we need at least $k^2$ altruistic donors to ensure a strong core solution, which is roughly the number of donor-recipient pairs in the original instance (which is $k^2 + 2k$), while for the weak core, 1 is sufficient.*

The requirement in Remark 2.1 is clearly unrealistic. Therefore, we focus on the weak core and Pareto optimality, rather than the strong core. Remark 2.1 also shows that to have a strong core solution, we may need to add $|V^i|$ many additional donors for some of the countries $i$ – note that adding $|V^i|$ many additional donors for $i$ ensures that $i$ cannot strictly improve anymore.

TODO: Core or core exchange?

### 2.1 Scarf's Lemma, Rounding

For a matrix $Q$, let $Q_{ij}$ denote the $j$-th element of the $i$-th row of $Q$, $Q_i$ denote the $i$-th row of $Q$ and $Q_{\cdot j}$ denote the $j$-th column of $Q$.

The tool that we use to establish our result is Scarf's lemma and rounding (Nguyen and Vohra [23]).

Lemma 2.1 (Scarf [27]). *Let $Q$ be an $n \times m$ nonnegative matrix, such that every column of $Q$ has a nonzero element and let $q \in \mathbb{R}^n_+$. Suppose that every row $i$ has a strict ordering $\succ_i$ on those columns $j$ for which $Q_{ij} > 0$. Then there is an extreme point of $\{x \in \mathbb{R}^m \mid Qx \leq q,\ x \geq 0\}$, that dominates every column in some row, where we say that $x \geq 0$ dominates column $j$ in row $i$, if $Q_{ij} > 0$, $Q_i x = q_i$ and $k \succ_i j$ for all $j \neq k \in \{1, \ldots, m\}$, such that $Q_{ik} x_k > 0$. Also, this extreme point can be found algorithmically.*

Let $\gamma^i_c = |U^i \cap c|$ for a cycle $c$ and country $i$. Recall that $\mathcal{C}_\Delta$ denotes the set of cycles of length at most $\Delta$. The following key Lemma – building on Scarf's Lemma – provides an existential guarantee for a special fractional solution: one that we can round off to find a $V^0$-supplemented core.

Lemma 2.2 (Rounding Lemma). *Given a partition exchange economy with cycles of size at most $\Delta$, there exists a fractional exchange $y^*$ in $\mathcal{G}$ such that: for any set $V^0$ of altruistic donors and any (deterministic) exchange $\mathcal{E}^*$ in $\mathcal{G}^{+V^0}$, the exchange $\mathcal{E}^*$ belongs to the $V^0$-supplemented core if*

$$u_i(\mathcal{E}^*) \geq \lfloor \sum_{c \in \mathcal{C}_\Delta} \gamma^i_c y^*_c \rfloor \text{ for all } i \in N.$$

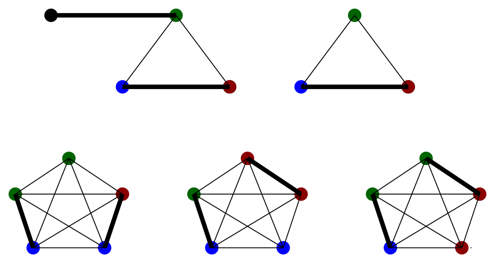

**Figure 1:** A partition exchange economy with pairwise exchanges without a core, but with a 1-supplemented core. The red, green and blue vertices denote the countries, while the black vertex in the upper left corner is $V^0$. The bold matching shows a $V^0$-supplemented core.

## 3 Pair-wise Exchange

In the case of pairwise exchange, we can discard all compatibility arcs $(u, v)$ such that $(v, u)$ is not a compatibility arc, and only consider the undirected graph $\mathcal{G}$ of the mutual compatibilities. Hence, in this case, we replace the set $E$ of the edges of $\mathcal{G}$ by the possible pairwise exchanges $\mathcal{C}_2$. Therefore, the feasible exchanges are now assumed to be the matchings of the graph $\mathcal{G}$.

We show in a simple example how an additional altruistic donor can guarantee the existence of a core exchange.

Example 3.1. *Consider the partition economy illustrated in Figure 1, featuring three countries represented by red, green, and blue vertices, respectively. This instance, from Csáji et al. [11], demonstrates an empty core. However, adding a single altruistic donor (the black vertex) makes the bold matching $\mathcal{E}$ a core outcome. To see this, observe that any country alone can cover all but 3 of its vertices, any two countries can cover all but 2 of their vertices and the three countries together can cover all but 5 of their vertices. Since in the bold $V^0$-supplemented exchange $\mathcal{E}$, one country has 2 vertices unmatched, and the others only one, no single country, nor the 3 countries together block $\mathcal{E}$. Furthermore, no 2 countries block either, as then they could leave at most $(2-1)+(1-1)=1$ of their vertices unmatched, which is impossible.*

We extend this example to show that a constant fraction of all patients may need an additional altruistic donor for the core in the worst case.

Theorem 3.1. *For arbitrarily large values of $|V|$, there exists a partition exchange economy with pairwise exchanges such that the $(\frac{|V|}{18}-3)$-supplemented core is empty, even if the number of countries is $n=3$.*

Our main result on the existence of the supplemented core relies on the following concept of independent odd cycles.

Definition 3.1. *Given an undirected graph $\mathcal{G}=(V,E)$, a set of odd-length cycles $\mathcal{C}=\{c_1,\dots,c_l\}$ is called* independent, *if $c_i \cap c_j = \emptyset$ for $i \neq j$ and there exists no $u \in c_i, v \in c_j, j \neq i$ such that $(u,v) \in E$. Let $\nu(\mathcal{G})$ denote the maximum number of independent odd cycles of $\mathcal{G}$.*

Theorem 3.2. *In a partition exchange economy with pairwise exchanges, the $\nu(\mathcal{G})$-supplemented core is always nonempty and contains a Pareto-optimal solution. As a special case, if $\mathcal{G}$ is bipartite, then the core is nonempty and contains a Pareto-optimal solution.*

Remark 3.1. *The $\nu(\mathcal{G})$-supplemented core exchange in Theorem 3.2 can be chosen in such a way that each country $i$ requires at most $\frac{|U^i|}{3}$ additional donors.*

Next, we show that in a random compatibility graph, the expected maximum number of independent odd cycles grows logarithmically with the size of the graph. This shows that in a typical random compatibility graph, compared with the market size, only a small number of altruistic donors are needed to stabilize the exchange.

Lemma 3.1. *Assuming a constant number $g$ of groups, for any distribution of the vertices to the groups, in a random kidney exchange graph $\mathcal{G}$, the expected value $\mathbb{E}[\nu(\mathcal{G})] = \mathcal{O}(\log |V|)$ and as the size of graph increases, the probability that $\nu(\mathcal{G}) = \mathcal{O}(\log |V|)$ approaches 1.*

On the other hand, if the compatibility graph can be represented using types, we show that the maximum number of independent odd cycles is at most one-third the number of types.

Lemma 3.2. *If a graph $\mathcal{G}$ can be represented with $t$ types, then $\nu(\mathcal{G}) \leq \frac{t}{3}$.*

*Proof.* We show that there exists an independent set of odd cycles of size $\nu(\mathcal{G})$ that contains each type at most once. As odd cycles have at least 3 vertices, the statement will follow.

Take any maximum size set of independent odd cycles $\mathcal{C}$. Suppose for the contrary that $t_i$ appears in cycles $c_{j_1}, c_{j_2}$. Then the neighbors of the type $t_i$ vertex in $c_{j_1}$ are connected to the type $t_i$ vertex in $c_{j_2}$ too with an edge, which contradicts the independence of the cycles. Similarly, if a type $t_i$ appears twice in the same odd cycle $c_j$, then there exists an edge between two non-adjacent (as a type is never compatible with itself) vertices in the cycle. Hence, we can obtain a smaller odd cycle $c'_j$ and replace $c_j$ with $c'_j$ in $\mathcal{C}$. This still gives an independent set of odd cycles, and after a finite number of steps, no type will appear twice within an odd cycle either. □

Thus, we obtain the following corollary establishing the existence of the supplemented core under both models of compatibility.

Corollary 3.1. *In a partition exchange economy with a random compatibility graph, the $\mathcal{O}(\log |V|)$-supplemented core is nonempty with probability approaching 1 as the market becomes large. In a partition exchange economy representable by $t$ types, the $\frac{t}{3}$-supplemented core is always nonempty.*

## 4 Applications to One-Sided Exchange Economy

In this section, we apply Theorem 3.2 in the special case where $\mathcal{G}$ is a bipartite graph, to a one-sided market setting. In this setting, our result extends the main result of Echenique et al. [17].

There is a set of $m$ goods, denoted by $M = M_1 \cup \cdots \cup M_n$, where $M_i$ represents the set of goods originally owned by agent $i$. Agents have preferences over bundles of goods, and we denote by $v_i(X)$ the utility that agent $i$ derives from consuming bundle $X \subset M$.

Definition 4.1. *A partition $M = M_1^* \cup \cdots \cup M_n^*$ is in the weak core if there is no strongly blocking coalition. A strongly blocking coalition is a subset of agents $S \subset N$ and a partition of $\cup_{i \in S} M_i$ into $\cup_{i \in S} M'_i$ such that all agents in $S$ are strictly better off, i.e., for all $i \in S$, $\quad v_i(M'_i) > v_i(M_i^*)$.*

Echenique et al. [17] provide conditions on the economy and utility functions under which the weak core exists. Specifically, they assume that goods are partitioned into categories, that is, $M = \bigcup_{k=1}^{K} O^k$, where each $O^k$ represents a set of goods that belong to the same category. We call such a market a *categorical economy.*

Each agent $i$ has a set of acceptable objects $G_i$, for which they obtain a utility of 1, while objects outside $G_i$ have a utility of 0. The utility of agent $i$ for a bundle $X$ is defined as

$$v_i(X) = \sum_{k=1}^{K} \min\{|X \cap O^k \cap G_i|, 1\}.$$

That is, each agent can consume at most one good from each category and derive a utility of 1 from any acceptable good. This class of utility functions is called *additively separable and dichotomous preferences.*

Theorem 4.1 (Echenique et al. [17]). *A categorical economy with additively separable and dichotomous preferences has a nonempty weak core.*

We next show that Theorem 3.2 implies a more general result compared to Theorem 4.1. In particular, when the utilities of the agents are in the class of binary assignment valuation, defined below, the core is nonempty.

Definition 4.2 (Binary Assignment Valuation). *A valuation function $v(.)$ over a set $M$ of objects is binary assignment valuation if there exists a set of positions $J$ and a $\{0,1\}$ matrix $\alpha$ of dimension $|M| \times |J|$ such that for any set $X \subseteq M$, $v(X) = \max_z \sum_{i \in X} \sum_{j \in J} \alpha_{ij} z_{ij}$, where $z$ varies over all possible assignments of elements in $X$ to elements in $J$.*

The key difference between binary assignment valuation and the assignment valuation in Milgrom [20] is that $\alpha_{ij}$ is restricted to binary values (0 or 1). This class is strictly more general than the assumption in Echenique et al. [17]. Specifically, the following construction corresponds to a categorical economy with additively separable and dichotomous preferences.

Fix an agent $i$, and let $J^i$ be a set of $K$ positions, where each position corresponds to a good category. The matrix $\alpha$ is defined as follows: for $j \in J^i$ and $g \in M$, $\alpha_{gj} = 1$ if and only if good $g$ belongs to category $j$ and is acceptable to the agent. It is straightforward to see that this construction precisely characterizes a categorical economy with additively separable and dichotomous preferences.

The following example shows that binary assignment valuation is strictly more general than additively separable and dichotomous preferences of a categorical economy. Consider three goods: $a, b$, and $c$. The valuation function is given by:

$$v(a,b,c) = v(a,b) = v(b,c) = v(a,c) = 2, \quad v(a) = v(b) = v(c) = 1.$$

This valuation does not correspond to any additively separable and dichotomous preferences of a categorical economy. However, it corresponds to two positions, $J = \{1,2\}$, with the matrix $\alpha$ having entries equal to 1 for all pairs of positions and goods.

Theorem 4.2. *If valuations of the agents are binary assignment valuation, then the weak core is nonempty.*

*Proof.* This is a corollary of Theorem 3.2. We need to construct a compatible graph for kidney exchange that corresponds to the one-sided market with binary assignment valuation.

Let $J^i$ be the set of positions that describe the valuation of agent $i$, and let $\alpha^i$ be the 0-1 matrix of dimension $|M| \times |J^i|$ representing that valuation description. The construction of the corresponding kidney exchange instance is as follows.

The goods $M$ correspond to altruistic donors, where $M^i$ is endowed by agent $i$. The set $J^i$ corresponds to the set of patient-donor pairs endowed by agent $i$. The compatible graph is a bipartite graph between $M$ and $\cup_{i \in \mathbb{N}} J^i$. There is an edge between $j \in J^i$ and $g \in M$ if and only if $\alpha^i_{gj} = 1$.

Because all the goods in $M$ correspond to altruistic donors, each agent's valuation only depends on how many positions can be matched. This is exactly the valuation of the agent under the binary assignment valuation assumption. □

## 5 Cyclic Exchange

We will consider a general case where exchanges involve cycles of size at most $\Delta$. Recall that in this exchange model, each exchange corresponds to a subgraph $\mathcal{E}$, which is a union of directed cycles of size at most $\Delta$.

We first show that the core may be empty, and show that in the worst case, a linear number of additional donors is required to achieve a weak core.

Example 5.1. *The following is a simple example to show that the core can be empty for cycles of length 3. (For pairwise exchanges, a no-instance was given by Csáji et al. [11]). We have 5 countries* $1, 2, 3, 4, 5$ *with* $V^i = U^i = \{v_i\}$ *for* $i \in \{1, .., 5\}$ *along a cycle of pairwise exchanges of length 5, in this order. Then, we subdivide the arcs* $(v_i, v_{i+1})$ *(*$i + 1$ *taken modulo* $5$*) with a new vertex* $v_{i,i+1}$ *that belongs to the country* $i + 1$*. It is easy to see that any feasible exchange after this subdivision corresponds to one or two pairwise exchanges before it. So, there will always be a country* $i$ *such that* $v_i$ *is unmatched. Then,* $i$ *and* $i + 1$ *block, as there is a 3-cycle* $(v_i, v_{i,i+1}, v_{i+1})$*, where* $i + 1$ *has 2 vertices covered and* $i$ *has 1, instead of 1 and 0 respectively.*

Using disjoint copies of Example 5.1, we get the following result.

Theorem 5.1. *In a partition exchange economy with* $\Delta = 3$*, the* $(\frac{|V|}{10} - 1)$*-supplemented core can be empty for arbitrarily large instances.*

In fact, a similar result holds even if there exists no bound on the exchange cycles at all.

Theorem 5.2. *In a partition exchange economy with unbounded exchanges, the* $(\frac{|V|}{15} - 2)$*-supplemented core can be empty for arbitrarily large instances, even with* $n = 5$ *countries.*

*Proof.* We construct a partition exchange economy similar to Example 5.1. We have 5 countries $1, 2, 3, 4, 5$. Each of them has $3k - 1$ vertices. The exchanges are not bounded in the instance, i.e., $\Delta = \infty$.

The instance is again obtained by subdividing edges of a cycle of length 5 of pairwise exchanges with $V^i = U^i = \{v_i\}$.

For Country 1 we add $2k - 1$ more vertices on the edge $(v_2, v_1)$ and $k - 1$ on the edge $(v_5, v_1)$. For country 2, we add $2k - 1$ more vertices on the edge $(v_3, v_2)$ and $k - 1$ on the edge $(v_2, v_1)$. For country 3, we add $2k - 1$ more vertices on the edge $(v_3, v_4)$ and $k - 1$ on the edge $(v_2, v_3)$. For country 4, we add $2k - 1$ on the edge $(v_4, v_5)$ and $k - 1$ on the edge $(v_4, v_3)$. Finally, for country 5 we add $2k - 1$ on the edge $(v_5, v_1)$ and $k - 1$ on the edge $(v_4, v_5)$. See Figure 2 for an illustration.

If neither of the two large cycles is chosen as the exchange, then there will be a country $i$ that has no vertices covered. By the construction, the country $i - 1$ (5, if $i = 1$) would block with him and both would improve by at least $k$. If the $v_1, v_2, v_3, v_4, v_5$ oriented large cycle is chosen, then in the small cycle between 1 and 2, both 1 and 2 improve by at least $k - 1$. If the $v_5, v_4, v_3, v_2, v_1$ oriented large cycle is chosen, then the small cycle between 4 and 5 blocks in a way such that both countries improve by at least $k - 1$.

Therefore, for any exchange $\mathcal{E}$, there is a blocking coalition, where all participants can improve by at least $k - 1$, so even if we add $k - 2$ additional donors, there will still be a blocking coalition. Hence, there is no $(k - 2)$-supplemented core here. As $k = \frac{|V|}{15} + \frac{1}{3}$ in this example, the statement follows. □

Theorems 3.1 and 5.2 show that even the trivial solution that assigns a distinct additional altruistic donor for each patient $v \in \cup_{i \in N} U^i$ is optimal up to a constant factor both for pairwise and cyclic exchanges. However, as we show next, if the number of types $t$ is also some constant, then a good upper bound exists on the required number of additional donors.

Our main result on the Supplemented Core for cyclic exchange is as follows.

Theorem 5.3. *Given a partitioned exchange economy with a compatibility graph that can be represented by* $t$ *types and a bound of* $\Delta$ *on the exchange cycle lengths, the* $(\Delta - 1)n(t + 1)$*-supplemented core is always nonempty. Furthermore, it also contains a Pareto-optimal solution.*

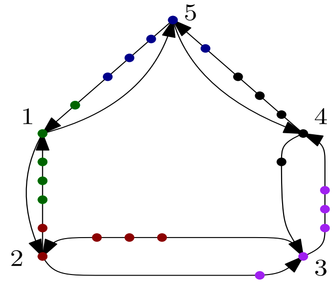


**Figure 2:** Illustration for Theorem 5.2 with $k = 2$. Different colored vertices denote the different countries.

We provide the proof in the appendix. The idea behind the proof follows from our description in Section 2.1. We begin by choosing an optimal type representation for $\mathcal{G}$ and finding a fractional domination solution obtained from Scarf's lemma and then we round it to an integral solution. Due to the complex combinatorics involved in cyclic exchanges, the rounding procedure may violate the capacity constraints on the number of donors of each type for each country. Nevertheless, our result guarantees that, for each country, the number of additional donors needed depends only on the number of types.

## 6 Simulations

In this section, we present our simulation results. We also examine a strengthened version of the core, described below.

Definition 6.1. *Given an exchange $\mathcal{E}$, and a coalition $P \subseteq N = \{1, .., n\}$, we say that $P$ is a* TU blocking coalition to $\mathcal{E}$, *if there exists an exchange $\mathcal{E}'$ in the graph $\mathcal{G}[\cup_{i \in P} V_i]$ such that $\sum_{i \in P} u_i(\mathcal{E}') > \sum_{i \in P} u_i(\mathcal{E})$. We say that $\mathcal{E}$ is in the TU core, if there is no TU blocking coalition to $\mathcal{E}$.*

Intuitively, $\mathcal{E}$ is in the TU core, if all coalitions $P \subseteq N$ receive at least as many transplants in the solution combined, as they could achieve alone.

Notice that TU core implies strong core, but not reversely. While in a weakly blocking coalition, the exchange $\mathcal{E}'$ must be a weak improvement for everyone, in a TU blocking coalition, some countries may be worse, only the total number of transplants of the countries must increase.

Next, we define the *supplemented TU-core.*

Definition 6.2. *Given additional altruistic donors $V^0$, an exchange $\mathcal{E}^*$ in the extended graph $\mathcal{G}^{+V^0}$ is a $V^0$-supplemented TU core if it is not TU blocked by any coalition $P \subseteq N = \{1, .., n\}$. If $|V^0| \leq d$, we also refer to $\mathcal{E}^*$ as a $d$-supplemented TU core.*

While we have seen that theoretically not much can be guaranteed in terms of even the strong core, our simulations showed that in practice, even this strongest version of the core is often nonempty and never needs many altruistic donors for existence.

**Setup.** We generated instances of partition exchange economies using a state-of-the-art generator [13, 24] (which is a refinement over the widely used Saidman-generator by Saidman et al. [26]), similarly to previous works [8, 9, 16]. Each instance contains 100-130 donor-recipient pairs, which is in line with most European kidney exchange programs, randomly assigned to between 4 and 9 countries. We experimented with both pairwise exchanges and exchanges involving cycles of length up to 3.

All simulations were run on an AMD Ryzen 7735HS processor with 16GB RAM. For each configuration (number of countries, cycle length), we ran simulations over 20 independent instances and tracked:

- the number of altruistic donors needed to obtain a core or TU core exchange,
- the number of steps taken by our heuristic algorithm to find a (supplemented) core exchange,
- the number of core and individual rationality (IR) violations in the initial maximum matching found in the first step of the heuristic.

The details of the computation are described in Appendix B.1. For computing solutions in the TU core and the supplemented TU core, we used an integer program IP: TU core. For finding a core, or supplemented core exchange, we used a heuristic that starts by finding an arbitrary maximum solution, and then if there exists a blocking coalition, it slightly increases the weight on the edges that go to the countries in the blocking coalition, and iterates with the new weighted objective. After 100 steps, if a core exchange was not found, it adds an additional altruistic donor.

**Discussion of results.** The results of our simulations are depicted on Tables 1, 2 and Tables 3-5 in Appendix B.2. Our simulations indicate that, in practical settings, the core of a partition exchange economy is predominantly nonempty, whereas the TU core may be empty in approximately $25\% - 50\%$ of cases. Notably, in over $90\%$ of instances, the initial maximum solution identified by our heuristic algorithm resided within the core. In scenarios where this was not the case, the heuristic successfully located a core solution within two or three iterations (after an additional donor was added when necessary). Interestingly, these instances often exhibited violations of individual rationality constraints, highlighting potential challenges in ensuring even the most basic form of incentive compatibility.

A particularly encouraging finding is that, across all instances of size 100–130, a single additional altruistic donor was always sufficient to achieve a core exchange. For the stronger notion of TU core stability, at most two additional donors were required. Thus, despite theoretical worst-case bounds suggesting the need for many supplementary donors, our experiments show that, in practice, the resource requirement is minimal. This demonstrates that our $d$-supplemented core concept is both practical and highly implementable, enabling strong incentive guarantees with negligible additional cost.

Finally, we observed that as the number of countries increased, the likelihood of an empty TU core also rose, and the heuristic more often required supplementary altruistic donors, likely because the core was indeed empty. This suggests a trade-off between market size and stability, and further motivates the use of supplemented cores in larger, decentralized kidney exchange settings.

**Table 1:** 6 countries, same size

| | $\Delta = 2$ | | $\Delta = 3$ | |
|---|---|---|---|---|
| | TU Core | Core | TU Core | Core |
| % of instances with no core found | 5% | 0% | 35% | 5% |
| Average number of donors needed | 0.05 | 0.00 | 0.35 | 0.05 |
| Max number of donors needed | 1 | 0 | 1 | 1 |
| Average number of steps in heuristic | – | 1.00 | – | 6.15 |
| Max number of steps in heuristic | – | 1 | – | 102 |
| Avg number of core violations in 1st step (IR, core) | – | (0.00,0.05) | – | (0.00,0.30) |
| Max number of core violations in 1st step (IR, core) | – | (0,1) | – | (0,4) |

## 7 Conclusions

This paper introduces the concept of the supplemented core in kidney exchange. Our main result shows that a small number of altruistic donors can stabilize the exchange market. This highlights a potentially new, yet important, role for altruistic donors in exchange systems. The applications extend beyond kidney exchange to other organ exchanges where organizational incentives play a critical role.

**Table 2:** 9 countries: 7 small, 2 large

| | $\Delta = 2$ | | $\Delta = 3$ | |
|---|---|---|---|---|
| | TU Core | Core | TU Core | Core |
| % of instances with no core found | 45% | 10% | 25% | 0% |
| Average number of donors needed | 0.50 | 0.10 | 0.25 | 0.00 |
| Max number of donors needed | 2 | 1 | 1 | 0 |
| Average number of steps in heuristic | – | 11.00 | – | 1.00 |
| Max number of steps in heuristic | – | 101 | – | 1 |
| Avg number of core violations in 1st step (IR, core) | – | (0.1,0.1) | – | (0.00,0.00) |
| Max number of core violations in 1st step (IR, core) | – | (1,1) | – | (0,0) |

Future work includes more empirical studies, exploration of computational challenges, and practical implementations.

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## A Omitted Proofs

### A.1 Proof of Lemma 2.2

*Proof.* Let us create an instance of Scarf's Lemma as follows. For the matrix $Q$, we create a row for each vertex $v \in V$. Furthermore, for each coalition $P \subseteq N$ and each utility vector $(u_1, \dots, u_{|P|})$ which is attainable by a feasible exchange $\mathcal{E} \subseteq E[P]$ (that is $u_i(\mathcal{E}) = u_i$ for $i \in P$), we create a column $(P, \mathcal{E})$ by choosing an arbitrary such exchange $\mathcal{E}$. The entry in the intersection of a row $v$ and a column $(P, \mathcal{E})$ is 1, if $v$ is covered by $\mathcal{E}$ (also implying that $v \in V^i$ for some $i \in P$ by $\mathcal{E} \subseteq E[P]$) and 0 otherwise. The bounding vector $q$ is chosen to be 1 for all rows. Both are nonnegative, hence they satisfy the conditions of Scarf's Lemma.

Then, we create the strict ranking required for the rows over their nonzero elements. For a vertex $v \in V^i$, this is created in a way such that $(P, \mathcal{E}) \succ_i (P', \mathcal{E}')$, whenever it holds that $u_i(\mathcal{E}) > u_i(\mathcal{E}')$. If $u_i(\mathcal{E}) = u_i(\mathcal{E}')$, then we break the ties in a way such that $(P, \mathcal{E}) \succ_i (P', \mathcal{E}')$, whenever $|P| < |P'|$.

By Scarf's Lemma, there exists an extreme point of this system that dominates every column in some row. Choose this solution $x^*$ in a way such that $|x^*|_1 = \sum_{P,\mathcal{E}} x^*_{P,\mathcal{E}}$ is maximal.

Define $y^*_c = \sum_{P,\mathcal{E}|c\in\mathcal{E}} x^*_{P,\mathcal{E}}$ for all cycles $c \in \mathcal{C}_\Delta$. Then, $\sum_{c|v\in c} y^*_c = \sum_{c|v\in c} \sum_{P,\mathcal{E}|c\in\mathcal{E}} x^*_{P,\mathcal{E}} = \sum_{P,\mathcal{E}|\mathcal{E} \text{ covers } v} x^*_{P,\mathcal{E}} = Q_v x^* \leq 1$, so $y^*$ is indeed a fractional exchange.

Let $\mathcal{E}^*$ be an exchange using some $d \geq 0$ altruistic donors, such that $u_i(\mathcal{E}^*) \geq \lfloor \sum_{c\in\mathcal{C}_\Delta} \gamma^i_c y^*_c \rfloor$ for all $i \in N$.

Suppose there is a blocking coalition $P$ with an exchange $\mathcal{E}_P$, where all $i \in P$ improve. By construction, there is an exchange $\mathcal{E}'_P$, such that $P, \mathcal{E}'_P$ has a column in $Q$ and $u_i(\mathcal{E}_P) = u_i(\mathcal{E}'_P)$ for all $i \in P$. Hence, as $x^*$ was a dominated solution, it dominated $P, \mathcal{E}'_P$ in some row $v$. Let $j$ be the country for which $v \in V^j$. For this $j \in N$, we have that $\sum_{P',\mathcal{E}'} Q_v x^*_{P',\mathcal{E}'} = 1$ and that for any $P', \mathcal{E}'$ with $j \in P'$ and $x^*_{P',\mathcal{E}'} > 0$ it holds that $u_j(\mathcal{E}') \geq u_j(\mathcal{E}'_P)$.

Hence, by the construction of $y^*$, we must have $\sum_{c\in\mathcal{C}_\Delta} \gamma^j_c y^*_c = \sum_{P',\mathcal{E}'} \sum_{c\in\mathcal{E}'} \gamma^j_c x^*_{P',\mathcal{E}'} = \sum_{P',\mathcal{E}'} u_j(\mathcal{E}') x^*_{P',\mathcal{E}'} \geq$

$$\sum_{P',\mathcal{E}'|\mathcal{E}' \text{ covers } v} u_j(\mathcal{E}')x^*_{P',\mathcal{E}'} \geq Q_v x^* \cdot \min_{P',\mathcal{E}'}\{u_j(\mathcal{E}') \mid Q_{v,(P',\mathcal{E}')}x^*_{P',\mathcal{E}'} > 0\} \geq u_j(\mathcal{E}'_P).$$

Therefore, we get that $u_j(\mathcal{E}^*) \geq \lfloor \sum_{c \in \mathcal{C}_\Delta} \gamma^j_c y^*_c \rfloor \geq u_j(\mathcal{E}'_P) = u_j(\mathcal{E}_P)$, contradicting that $j \in P$ improves in $\mathcal{E}_P$. □

### A.2 Proof of Theorem 3.1

*Proof.* We create an instance of a partition exchange economy with pairwise exchanges as follows. Let $K_l$ be the clique on $l$ vertices - i.e. the graph where any two vertices are connected. The graph $\mathcal{G}$ of the partition exchange economy consists of $3a$ vertex-disjoint $K_3$ graphs (i.e. triangles) $K_3^1, K_3^2, \ldots, K_3^{3a}$ and $9a$ vertex-disjoint $K_5$ graphs $K_5^1, K_5^2, \ldots, K_5^{9a}$, where $a \in \mathbb{N}$ is a parameter. We have that $|V| = 9a + 45a = 54a$, and the size of each country is $18a$.

There are three countries $1, 2, 3$. Each triangle $K_3^j$ contains one vertex from each country. Furthermore, $K_5^{(j-1)\cdot 3a+1}, \ldots, K_5^{(j-1)\cdot 3a+3a}$ contains 2 vertices from countries $j$ and $j+1$ and 1 vertex from $j+2$ for $j \in [3]$, where addition is taken modulo 3 (i.e. $2+1=3$, but $2+2=1$).

It is easy to see that for any pairwise exchange (matching), at least $12a$ vertices must remain unmatched, at least one for each disjoint clique in the graph.

Furthermore, if two countries join, then we claim that together they can create a matching where at most $6a$ of their vertices remain unmatched. Indeed, they can match all their vertices in the triangles as there is always an edge between them. Among $K_5^1, \ldots, K_5^{9a}$, there are $6a$ cliques, where one of them has an odd number of vertices. In such cliques, they must leave one of their vertices unmatched, but this can be an arbitrary one of their vertices, as any two vertex within a $K_5^j$ are connected. In the cliques $K_5^j$, where both of them have an even number of vertices, all can be matched.

Finally, a single country can also create a matching, leaving only $6a$ of its vertices unmatched. This holds because among the $9a$ cliques $K_5^1, \ldots, K_5^{9a}$, it has an even number of vertices in $6a$ of them, so all of them can be matched there. Hence, only $3a$ vertices in the $K_5^j$s and $3a$ in the triangles will be unmatched.

Let $d := a$. Suppose for the contrary that there exists a $3(d-1)$-supplemented core solution $\mathcal{E}$. By our above observations, the two worst-off countries together have at least $\frac{2}{3} \cdot (12a - (3d-3)) = 8a - 2d + 2 = 6a + 2$ unmatched vertices. Say one of them has $c \geq 0$ and the other at least $8a - 2d + 2 - c$. As each country can create a matching with at most $6a$ unmatched vertices, we must have that $c \leq 6a$ and $8a - 2d + 2 - c \leq 6a$, so $2a - 2d + 2 \leq c$. Hence, $2 \leq c \leq 6a = 8a - 2d + 2 - 2$ and so both countries must have at least 2 vertices unmatched. Hence, both of these two countries can improve their situation in a matching among themselves, where only $6a < 6a + 2$ vertices are unmatched, contradicting the fact that $M$ is in the $3(d-1)$-supplemented core.

As we have that $d = a$ and hence $d = \frac{|V|}{54}$, the statement follows. □

### A.3 Proof of Theorem 3.2

*Proof.* Take a labeling $f$ of $\mathcal{G}$ with set of types $T$ and let $|T| = t$. We refer to $f(v)$ as the *type* of $v$.

By Lemma 2.2, there exists a fractional exchange $y^*$, such that if $u_i(\mathcal{E}^*) \geq \lfloor \sum_{c \in \mathcal{C}_2} \gamma^i_c y^*_c \rfloor = \lfloor \sum_{e \in E} \gamma^i_e y^*_e \rfloor$ for all $i \in N$ for some exchange $\mathcal{E}^*$ using $d$ additional donors, then $\mathcal{E}^*$ is in the $d$-supplemented core. Let $y^*(E(U^i)) := \sum_{e \in E} \gamma^i_e y^*_e$.

We create a Polyhedron $\mathcal{P} = \{Az \leq a, Bz \geq b, z \geq 0\}$ as follows. The variables $z$ correspond to the edges of $\mathcal{G}$. Furthermore, the matrix $A$ is just the incidence matrix of $\mathcal{G}$ with $a \equiv 1$. In the matrix $B$, we have a row for each country, such that $B_{i,e} = \gamma_e^i = |e \cap U^i|$. It is easy to see that the row $B_i$ of a country $i$ is the sum of the rows $A_v$ for $v \in U^i$. Every entry of $B$ is from $\{0, 1, 2\}$. The vector $b$ is set such that $b_i = \lfloor y^*(E(U^i)) \rfloor$.

The polyhedron $\mathcal{P}$ is clearly nonempty, as $y^* \in \mathcal{P}$.

Let $z^*$ be an extreme point of $\mathcal{P}$ that maximizes $\sum\limits_{e \in E} (\sum\limits_{i \in N} \gamma_e^i) z_e^*$. If there is an integer coordinate $z_e^*$ in $z^*$, then we delete that column from $A$ and $B$ and set $a := a - z_e^* \cdot A_{\cdot,e}$ and $b := b - z_e^* \cdot B_{\cdot,e}$, where $Q_{\cdot,j}$ denotes columns $j$ for a matrix $Q$. Also, if a row becomes all zero, then we delete it. Furthermore, if $a_v$ becomes 0, then we can set all variables with a nonzero entry in the row $A_v$ to 0 and delete them. Hence, we can assume that $a$ is always 1 for every coordinate. If some row or column got deleted, then we update $z^*$ to an extreme point of the new polyhedron $\mathcal{P}$ that maximizes $\sum\limits_{e \in E} (\sum\limits_{i \in N} \gamma_e^i) z_e^*$.

If at some point, all variables become integral, then it gives an exchange $\mathcal{E}$ in the core by Lemma 2.2.

Otherwise, suppose that all variables of the extreme point $z^*$ are fractional.

Hence, we can use Lemma A.1 for the polyhedron $\mathcal{P}$ to show some properties that are satisfied by $z^*$.

Credit each variable $z_e^*$ of $z^*$ with 1 token. Then, we redistribute the tokens as follows. If $u \in V^i \setminus U^i$ for some $i \in N$, or $A_u z^* = 1$, then we give $\frac{1}{2}$ token to $A_u$. Else, $u \in U^i$, for some $i$, $A_u z^* < 1$ and we give $\frac{1}{2}$ token to $B_i$. Similarly, if $v \in V^j \setminus U^j$ for some $j \in N$, or $A_v z^* = 1$, then we give $\frac{1}{2}$ token to $A_v$. Otherwise, $v \in U^j$ for some $j$, $A_v z^* < 1$ and we give $\frac{1}{2}$ token to $B_j$. It is easy to see that each $z_e^*$ component distributes at most 1 token.

It is easy to see that each tight row $A_v$ gets at least one token, since each coordinate is $\{0, 1\}$ in $A_v$ and all coordinates are fractional, so there are at least two edges $e_1, e_2$ that give $\frac{1}{2}$ token to $A_v$.

Take a tight row $B_i$, such that not all $A_v$ rows with $v \in U^i$ are tight. Otherwise, the row $B_i$ would be generated by the tight rows $A_v$ for $v \in U^i$.

Then, take a vertex $v \in U^i$ with $A_v z^* < 1$. By the tightness of $B_i$, we have that there are at least two $v \in U^i$ with $A_v z^* < 1$, since $b_i = B_i z^* = \sum_{v \in U^i} A_v z^*$ and $b_i$ is integer. Therefore, $B_i$ gets at least one token too.

By Lemma A.1 we get that this is only possible, if each tight row that got at least one token got exactly one token, and if a row got less than one token, then it got 0.

From this, we conclude that there are at most two fractional edges of $z^*$ incident to any vertex $v$ with $A_v z^* = 1$. We claim that this is also true for a vertex $v$ with $A_v z^* < 1$. Indeed, if $v \in U^i$ for a country $i$ such that $B_i$ is tight, then all fractional edges incident to $v$ give a $\frac{1}{2}$ token to $B_i$'s row, so there are at most two. Finally, if $v \notin U^i$ for any $i \in N$, then $\frac{1}{2}$ of $z_e^*$'s token would have been given to $A_v$ for any fractional edge $e$ with $v \in e$, hence as only tight rows received tokens, there can be no fractional edges incident to $v$. Hence, the fractional edges of $z^*$ correspond to a disjoint set of cycles and paths in the graph $\mathcal{G}[\cup_{i \in N} U^i]$. Also, no paths can be odd-length, as otherwise it contradicts the fact that $z^*$ maximizes $\sum\limits_{e \in E} (\sum\limits_{i \in N} \gamma_e^i) z_e^*$. Furthermore, there can be no even length cycle or path $c$ either, since otherwise $z^*$ is not extreme: $z^* = \frac{1}{2}(z_1 + z_2)$, where $z_1 = z^* - \varepsilon \chi_{c_1} + \varepsilon \chi_{c_2}$ and $z_2 = z^* + \varepsilon \chi_{c_1} - \varepsilon \chi_{c_2}$ for some sufficiently small enough $\varepsilon > 0$ and $c = c_1 \cup c_2$ is chosen such that the edges of $c$ alternate between $c_1$ and $c_2$.

Hence, we conclude that the fractional edges of $z^*$ gives a vertex disjoint set of odd cycles $\mathcal{C}$. If $\mathcal{G}$ is bipartite, then this is a contradiction, which shows that the core is nonempty in this case, and $z^*$ corresponds to a Pareto-optimal exchange in the core.

Suppose that $\mathcal{G}$ is not bipartite.

We can suppose that $\mathcal{C}$ is an independent set of odd cycles. Otherwise, if there exists an edge $(u, v)$

with $u \in c_{j_1}, v\rho_{j_2}, j_1 \neq j_2$, then all vertices in $c_{j_1} \cup c_{j_2}$ can be covered with a matching. Hence, we may choose such a matching, and iterate this procedure until no such case exists.

From this, it follows that we can round $z^*$ to a matching $\mathcal{E}$ that satisfies that at most $\min\{\nu(\mathcal{G}), \frac{t}{3}\}$ vertices are unmatched from these cycles. Hence, with $\frac{t}{3}$ additional donors – at most one from each type – we can guarantee that $u_i(\mathcal{E}) \geq \lfloor y^*(E(U^i)) \rfloor$, so by Lemma 2.2 it gives a solution in the $\frac{t}{3}$-supplemented core.

If $\mathcal{E}$ is not Pareto-optimal, then take an exchange $\mathcal{E}'$ in $\mathcal{G}^{+V^0}$ that Pareto-dominates it and maximizes $\sum_{i \in N} u_i(\mathcal{E}')$. Then, $\mathcal{E}'$ is clearly Pareto-optimal. Also, $u_i(\mathcal{E}') \geq u_i(\mathcal{E}) \geq \lfloor \sum\limits_{c \in \mathcal{C}_\Delta} \gamma^i_c y^*_c \rfloor$ for all $i \in N$, so by Lemma 2.2 is in the the $(\Delta - 1)n(t+1)$-supplemented core too. □

### A.4 Proof of Remark 3.1

*Proof.* In the proof of Theorem 3.2, we chose an arbitrary matching that leaves only one vertex from each odd cycle $c \in \mathcal{C}$.

We will show that if we choose this matching properly, then it satisfies the claimed additional constraint.

Create a graph $\mathcal{H}$ as follows. The set of vertices of $\mathcal{H}$ is $N$ and there is an arc $ij$, if there exists an odd cycle $c \in \mathcal{C}$, where $i$ has a covered vertex and $j$ has an uncovered vertex. Let $A$ be the set of those countries such that they have strictly less than $\lceil |U^i \cap \mathcal{C}|/3 \rceil$ vertices uncovered and $B$ be the set of those countries such that they have strictly more than $\lceil |U^i \cap \mathcal{C}|/3 \rceil$ vertices uncovered. If $B$ is empty, we are done. If $B$ is nonempty, then by the maximality of the chosen matching, $A$ is also nonempty.

We claim that there is a path from $A$ to $B$ in $\mathcal{H}$. Suppose otherwise. Then, the set of vertices $\hat{B}$ not reachable from $A$ all satisfy that they have strictly more than $\lceil \sum_{i \in \hat{B}} |U^i \cap \mathcal{C}|/3 \rceil$ uncovered vertices. Furthermore, as there is no incoming edge to $\hat{B}$ in $\mathcal{H}$, it follows that any odd cycle that contains an uncovered vertex of a country from $\hat{B}$ satisfies that all other vertices also belongs to someone from $\hat{B}$. As every odd cycle has at least 3 vertices and each odd cycle satisfies $c \subset \mathcal{G}[\cup_{i \in N} U^i]$ as we observed in the proof of Theorem 3.2, we get that if we sum the number of uncovered vertices of each country in $\hat{B}$, then it is at most one-third of the total number of vertices of these countries in $\mathcal{C}$, contradicting the above observation about $\hat{B}$.

Hence, in this case we can find an $A - B$ path in $\mathcal{H}$ along which we can create a new matching, where someone from $A$ has one less vertex covered and someone from $B$ has one more vertex covered. Iterating this, we arrive at a point, where $B$ will be empty.

Hence, for each country, $\frac{|U^i|}{3}$ additional donors suffice. □

### A.5 Proof of Lemma 3.1

*Proof.* Let the number of groups be a constant $g$. Let $p$ be the smallest strictly positive probability between groups. We will not assume anything on the distribution of the vertices to the groups in $\Phi$.

We will prove a slightly stronger result, that is, we will show that the expected number of independent edges (i.e. vertex vertex-disjoint edges with no other edges between them) is also $\mathcal{O}(\log |V|)$. This statement is stronger, as any independent set of odd cycles also contains an independent set of edges of the same size. Let $\nu'(\mathcal{G})$ denote the maximum number of independent edges and let $X$ be the random variable s.t. for a graph $\mathcal{G}$, $X(\mathcal{G}) = \nu'(\mathcal{G})$.

For some fix distinct vertices $U = \{v_1, \dots, v_{2\ell}\}$, let $X_U$ be an indicator random variable that is equal to 1, if $\mathcal{G}[U]$ contains $\ell = \frac{|U|}{2}$ independent edges, and 0 otherwise.

$\mathbb{E}[X] = \sum_{k=1}^{|V|} k\mathbb{P}(X = k) = \sum_{k=1}^{|V|} \mathbb{P}(X \geq k)$.

In the following, we bound $\mathbb{P}(X \geq k)$. Clearly, $X \geq k$ can only happen if $\sum\limits_{U \subseteq V ||U|=2k} X_U \geq 1$.

Take a fix $U \subseteq V$ with $|U| = 2k$. For any set of $k$ independent edges, there must be at least $2k/g^2$ such that their starting vertices are in the same group and their ending vertices are also in the same group. We can assume that no edge exists between the same group. Hence, $k/g^2$ independent edges among these vertices is only possible, if for each vertex in group 1, it is connected to exactly one vertex in group 2. This can be happen in $(k/g^2)! \leq k^k$ many subgraphs. The probability that a fix subgraph happens is bounded by $(1-p)^{k^2/g^4 - k/g^2}$, because each edge from group 1 to group 2 have probability at least $p$ (since there can be edges between them), and $k/g^2(k/g^2 - 1)$ of these edges cannot be included.

Hence, using $g \geq 1$, we have that $\mathbb{P}(X_U \geq 1) \leq 2\binom{2k}{\lceil 2k/g^2 \rceil} k^k (1-p)^{k^2/g^4 - k/g^2} \leq 2(2^2 k^3 (1-p)^{k/g^4 - 1/g^2})^k$.

Then, $\mathbb{P}(X \geq k) \leq \sum\limits_{U \subseteq V ||U|=2k} \mathbb{P}(X_U \geq 1) \leq 2(|V| 4k^3 (1-p)^{k/g^4 - 1/g^2})^k \leq 2(4|V|^4 (1-p)^{k/g^4 - 1/g^2})^k$.

For any constant $\delta > 0$, this is bounded by $\frac{1}{|V|^\delta}$ for $k \geq D \log |V|$ for a constant $D = D(p, g, \delta)$, as $g$ is a constant and $1 - p < 1$ is also a constant. Hence, $\mathbb{P}(X \geq D \log |V|) \leq \frac{1}{|V|^\delta} \xrightarrow{|V| \to \infty} 0$. Therefore, $\sum_{k=1}^{|V|} \mathbb{P}(X \geq k) \leq \sum_{k=1}^{D \log |V|} \mathbb{P}(X \geq k) + 1 \leq D \log |V|$. □

### A.6 Proof of Theorem 5.3

*Proof.* Take a labeling $f$ of $\mathcal{G}$ with set of types $T$ and let $|T| = t$. We refer to $f(v)$ as the *type* of $v$.

By Lemma 2.2, we have that there exists a fractional exchange $y^*$, such that if $u_i(\mathcal{E}^*) \geq \lfloor y^*(E(V^i)) \rfloor$ for all $i \in N$ for an exchange $\mathcal{E}^*$ using $d$ additional donors, then $\mathcal{E}^*$ is in the $d$-supplemented core.

We create a polyhedron $\mathcal{P} = \{Az \leq a, Bz \geq b, z \geq 0\}$ as follows. The variables $z$ correspond to the cycles of $\mathcal{G}$ with length bounded by $\Delta$.

In the matrix $A$, we have a row $(t, i)$ for each type $t \in T$ and country $i \in N$. The entry in row $(t, i)$, column $c$ (corresponding the the cycle $c$) is $\beta_c^{t,i} = |\{v \in c \cap V^i \mid v \text{ has type } t\}|$ and the bound is $a_{t,i} = |\{v \in V^i \mid v \text{ has type } t\}|$. That is, we have inequalities of the form

$$\sum_{c \in \mathcal{C}_\Delta} \beta_c^{t,i} z_c \leq a_{t,i}.$$

In the matrix $B$, we have a row for each country $i \in N$. The entry in row $i$, column $c$ is $\gamma_c^i = |c \cap U^i|$ and the lower bound is $b_i = \lfloor \sum\limits_{c \in \mathcal{C}_\Delta} \gamma_c^i y_c^* \rfloor$. That is, we have inequalities of the form

$$\sum_{c \in \mathcal{C}_\Delta} \gamma_c^i z_c \geq \lfloor \sum_{c \in \mathcal{C}_\Delta} \gamma_c^i y_c^* \rfloor.$$

It is clear that $y^*$ satisfies these constraints, so $y^* \in \mathcal{P}$.

Consider the following rounding procedure. Initially, let $z^*$ be an extreme point that maximizes $\sum\limits_{c \in \mathcal{C}_\Delta} (\sum\limits_{i \in N} \gamma_c^i) z_c^*$.

While $z^*$ is not integral, we do the following

(1) For each integral component $c$ of $z^*$, we delete the variable $c$ and update the bounds accordingly - that is, if $z_c^* = 1$ for some $c \in \mathcal{C}_\Delta$, then we decrease $b_i$ by $\gamma_c^i$ and $a_{t,i}$ by $\beta_c^{t,i}$ for $t \in T$.

(2) If there are rows in $A$ or $B$ such that $A_{t,i}\mathbf{1} \leq a_{t,i} + \Delta - 1$ or $B_i z^* \leq \Delta - 1$, then we eliminate them from $\mathcal{P}$ (here $\mathbf{1}$ is the all 1 vector).

(3) Update $z^*$ to be an extreme point that maximizes $\sum\limits_{c \in \mathcal{C}_\Delta} (\sum\limits_{i \in N} \gamma_c^i) z_c^*$ of the updated polyhedron $\mathcal{P}$.

First of all, suppose that this procedure terminates in an integer valued $z^*$. Then, $z^*$ corresponds to an exchange $\mathcal{E}^*$ using cycles of length at most $\Delta$ that violates each constraint by at most $\Delta - 1$. This holds because $A_{t,i} z^*$ must remain below $A_{t,i}\lceil z^* \rceil \leq A_{t,i}\mathbf{1} \leq a_{t,i} + \Delta - 1$, if we eliminated row $A_{t,i}$ and $B_i z^*$ must remain above $0 \geq b_i - \Delta + 1$ if we eliminated row $B_i$.

For each country, we add $(b_i - B_i z^*)^+ + \sum_{t \in T}(A_{t,i} z^* - a_{t,i})^+ \leq (\Delta - 1)(t+1)$ additional donors, in total at most $(\Delta - 1)n(t+1)$ for all countries. With the help of these additional donors, we can create an exchange $\mathcal{E}$ that assigns at least $\sum\limits_{c \in \mathcal{C}_\Delta} \lfloor \gamma_c^i z_c^* \rfloor$ transplants for each country - as vertices with the same type $t$ in the same country $i$ are interchangable by the assumption that compatibility only depends on the type.

By Lemma 2.2, $\mathcal{E}$ is in the $(\Delta - 1)n(t+1)$-supplemented core.

If $\mathcal{E}$ is not Pareto-optimal, then take an exchange $\mathcal{E}'$ in $\mathcal{G}^{+V^0}$ that Pareto-dominates it and maximizes $\sum_{i \in N} u_i(\mathcal{E}')$. Then, $\mathcal{E}'$ is clearly Pareto-optimal. Also, $u_i(\mathcal{E}') \geq u_i(\mathcal{E}) \geq \lfloor \sum\limits_{c \in \mathcal{C}_\Delta} \gamma_c^i y_c^* \rfloor$ for all $i \in N$, so by Lemma 2.2 is in the the $(\Delta - 1)n(t+1)$-supplemented core too.

Therefore, to prove the Theorem, it only remains to show that the procedure must terminate. That is, if there are no more rows that can be eliminated, then $z^*$ must be integral.

We use the following well-known lemma from operations research (Schrijver [28]).

Lemma A.1. *Let $z^*$ be an everywhere strictly positive extreme point of $\{Qz \leq q, z \geq 0\}$. Then the number of variables equals the maximum number of linearly independent tight rows of $Q$.*

Suppose for the contrary that $z^*$ is not integral, but there are now rows that can be eliminated. As we eliminate integral variables in the procedure, all variables of $z^*$ can be assumed to be strictly positive and fractional. Hence, we can use Lemma A.1.

We give one token to each fractional component $z_c^*$. Then, we redistribute these tokens to the rows as follows. We give $\gamma_c^i \frac{z_c^*}{\Delta}$ from $z_c$'s token to the row $B_i$ and $\beta_c^{t,i} \frac{1 - z_c^*}{\Delta}$ to the row $A_{t,i}$ for $i \in N, t \in T$. Then, the total number of tokens that $z_c^*$ distributes is $\sum_{i \in N} \gamma_c^i \frac{z_c^*}{\Delta} + \sum_{i \in N} \sum_{t \in T} \beta_c^{t,i} \frac{1 - z_c^*}{\Delta} \leq z_c^* + 1 - z_c^* \leq 1$, since $|c| \leq \Delta$.

Take a tight row $B_i$. Then, $\Delta - 1 < B_i z^* = b_i \in \mathbb{Z}$, as the row cannot be eliminated, so $B_i z^* = \sum_{c \in \mathcal{C}_\Delta} \gamma_c^i z_c^* \geq \Delta$. This implies that any such row must obtain at least $\frac{1}{\Delta} \sum_{c \in \mathcal{C}_\Delta} \gamma_c^i z_c^* \geq 1$ tokens.

Next, take a tight row $A_{t,i}$. Then, $A_{t,i} z^* = a_{t,i}$ and $A_{t,i}\mathbf{1} \geq a_{t,i} + \Delta$, as $A_{t,i}\mathbf{1}$ is integer and the row cannot be eliminated. Putting these two together, we get that $A_{t,i}(1 - z^*) = \sum_{c \in \mathcal{C}_\Delta} \beta_c^{t,i}(1 - z_c^*) \geq \Delta$. Therefore, any such row must obtain at least $\frac{1}{\Delta} \sum_{c \in \mathcal{C}_\Delta} \beta_c^{t,i}(1 - z_c^*) \geq 1$ tokens.

Hence, we must have that every row with a nonzero entry must be tight and must receive exactly one token and they must be linearly independent, otherwise the number of linearly independent tight rows could not be equal to the number of variables as Lemma A.1 requires. Furthermore, any row that should receive some positive fraction of a token from any $z_c^*$ cannot have been eliminated by the same reasons. So if $\beta_c^{t,i} > 0$, then $A_{t,i}$ is not eliminated yet and if $\gamma_c^i > 0$ then $B_i$ is not eliminated yet. Finally, each component $z_c^*$ must distribute exactly one token, hence $\sum_{i \in N} \gamma_c^i = \Delta$ and $\sum_{i \in N} \sum_{t \in T} \beta_c^{t,i} = \Delta$ for any such cycle $c$.

Therefore, if we sum the vectors $A_{t,i}$ for all tight rows, we must get a vector $a'$ such that $a'_c = \Delta \; \forall c$

using that $\sum_{t \in T} \sum_{i \in N} \beta_c^{t,i} = \Delta$. Similarly, if we sum all tight $B_i$ rows, we must get the same vector $a'$, as $\sum_{i \in N} \gamma_c^i = \Delta$ too. This contradicts the fact that these tight rows are linearly independent.

□

# B Additional material for simulations

## B.1 Computation details

**Computation of coalition values.** To compute the maximum number of transplants $\nu(P)$ that a coalition $P \subseteq N$ can achieve on its own, we solved the following integer program (IP: (P,Δ)) on the subgraph $\mathcal{G}\left[\cup_{i \in P} V^i\right]$. Here, $\rho(v)$ and $\delta(v)$ denote the incoming and outgoing arcs of a vertex $v$, respectively.

$$\begin{aligned} \text{maximize} \quad & \sum_{\substack{e=(u,w) \in \mathcal{G}[\cup_{i \in P} V^i] \\ w \in \cup_{i \in P} U^i}} x_e \\ \text{subject to} \quad & \sum_{e \in \rho(v)} x_e = \sum_{e \in \delta(v)} x_e && \forall v \in V \\ & \sum_{e \in \rho(v)} x_e \leq 1 && \forall v \in V \\ & \sum_{e \in p} x_e \leq \Delta - 1 && \forall \text{path } p \subseteq E,\ |P| = \Delta \\ & x_e \in \{0, 1\} && \forall e \in E \end{aligned} \tag{IP: (P,Δ)}$$

**Finding core and supplemented core exchanges.** To find a core or a $d$-supplemented core exchange, we applied the following heuristic procedure. First, we checked whether the TU core was nonempty by solving the integer program (IP: TU core), which we obtain by adding to (IP: (P,Δ)) (with $P = N$) the following equations.

$$\sum_{\substack{e=(u,w) \\ w \in G[\cup_{i \in P} U^i]}} x_e \geq \nu(P) \quad \forall P \subseteq N \tag{IP: TU core}$$

If a TU core exchange was found, it was also a core exchange. Otherwise, we proceeded as follows: We solved (IP: (P,Δ)) for $P = N$ to find a maximum exchange. If the resulting exchange was not in the core, we searched for a blocking coalition $P$ and slightly increased the weights of arcs with endpoints in $\cup_{i \in P} U^i$ to favor this coalition. If a core exchange could not be found within 100 iterations, we added a new altruistic donor to $V^0$.

For the supplemented TU core, we used IP: TU core, and added donors one by one, until the integer program had a solution.

**Core stability verification.** To verify whether an exchange $\mathcal{E}$ belongs to the core, we iterated over all coalitions $P \subseteq N$ and checked whether $P$ could block $\mathcal{E}$ by solving the integer program we get by adding to (IP: (P,Δ)) the equations:

$$\sum_{\substack{e=(u,w) \\ w \in U^i}} x_e \geq u_i(\mathcal{E}) + 1 \quad \forall i \in P \tag{IP:core-check}$$

Our codes for the simulations are available in our GitHub repository Csáji [12].

### B.2 Omitted tables

In this section we list the additional results from the simulations that were omitted from the main body of the paper. In particular, we provide the simulation results for 4,5 and 8 countries. With few countries, the weak core was always nonempty, and even the TU core was much less often empty, than for many countries. Also, with few countries, the arbitrary maximum solutions found in the first step of our heuristic always turned out to be in the weak core already, without additional optimization.

**Table 3:** 4 countries, same size

| | $\Delta = 2$ | | $\Delta = 3$ | |
|---|---|---|---|---|
| | TU Core | Core | TU Core | Core |
| % of instances with no core found | 5% | 0% | 20% | 0% |
| Average number of donors needed | 0.05 | 0.00 | 0.20 | 0.00 |
| Max number of donors needed | 1 | 0 | 1 | 0 |
| Average number of steps in heuristic | – | 1.00 | – | 1.00 |
| Max number of steps in heuristic | – | 1.00 | – | 1.00 |
| Avg number of core violations in 1st step (IR, core) | – | (0.00,0.00) | – | (0.00,0.00) |
| Max number of core violations in 1st step (IR, core) | – | (0,0) | – | (0,0) |

**Table 4:** 5 countries, same size

| | $\Delta = 2$ | | $\Delta = 3$ | |
|---|---|---|---|---|
| | TU Core | Core | TU Core | Core |
| % of instances with no core found | 15% | 0% | 20% | 0% |
| Average number of donors needed | 0.15 | 0.00 | 0.20 | 0.00 |
| Max number of donors needed | 1 | 0 | 1 | 0 |
| Average number of steps in heuristic | – | 1.00 | – | 1.00 |
| Max number of steps in heuristic | – | 1.00 | – | 1.00 |
| Avg number of core violations in 1st step (IR, core) | – | (0.00,0.00) | – | (0.00,0.00) |
| Max number of core violations in 1st step (IR, core) | – | (0,0) | – | (0,0) |

**Table 5:** 8 countries: 5 small, 3 large

| | $\Delta = 2$ | | $\Delta = 3$ | |
|---|---|---|---|---|
| | TU Core | Core | TU Core | Core |
| % of instances with no core found | 45% | 5% | 50% | 0% |
| Average number of donors needed | 0.45 | 0.05 | 0.50 | 0.00 |
| Max number of donors needed | 1 | 1 | 1 | 0 |
| Average number of steps in heuristic | – | 6.00 | – | 1 |
| Max number of steps in heuristic | – | 101 | – | 1 |
| Avg number of core violations in 1st step (IR, core) | – | (0.05,0.05) | – | (0.00,0.00) |
| Max number of core violations in 1st step (IR, core) | – | (1,1) | – | (0,0) |

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