Lemma 1. *(Cicalese, Gargano, and Vaccaro 2013) The join $p \vee q$ of two probability vectors p and q saturates the triangular inequality for the distance defined as $d(p,q) := H(p) + H(q) - 2H(p \vee q)$.*

Proof.

$$\begin{aligned} d(p, p \vee q) + d(p \vee q, q) &= H(p) + H(p \vee q) - 2H(p \vee (p \vee q)) + H(p \vee q) + H(q) - 2H((p \vee q) \vee q) \\ &= H(p) + H(p \vee q) - 2H(p \vee q) + H(p \vee q) + H(q) - 2H(p \vee q) \\ &= H(p) + H(q) - 2H(p \vee q) \\ &= d(p,q). \end{aligned}$$

Result 1. *The measure of comparability of a vector p to the majorizing cone of a vector q defined as $E_q^-(p) := \min_{s \succ q} d(p,s)$ is equal to the distance $d(p, p \vee q)$.*

Proof.

$$\begin{aligned} E_q^-(p) &= \min_{s \succ q} d(p,s) \\ &\stackrel{\text{Lemma 1}}{=} \min_{s \succ q} d(p, p \vee s) + d(p \vee s, s) \end{aligned}$$

Let $s' = p \vee s$. The absorption law $a \vee a = a$, along with the associativity of the join operation, implies that choosing $s = s'$ (i.e. requiring that s majorizes p) makes the second term of the sum vanish, while the first remains invariant.

$$\begin{aligned} \implies E_q^-(p) &= \min_{s \succ q} d(p, p \vee s') + d(p \vee s', s') \\ &= \min_{s \succ q} d(p, p \vee (p \vee s)) + d(p \vee (p \vee s), p \vee s) \\ &= \min_{s \succ q} d(p, (p \vee p) \vee s) + d((p \vee p) \vee s, p \vee s) \\ &= \min_{s \succ q} d(p, p \vee s) + d(p \vee s, p \vee s) \\ &= \min_{s \succ q} d(p, p \vee s) \end{aligned}$$

Because s majorizes q, s' majorizes q too. Moreover, to be the join of p with another vector, s' must majorize p as well.

$$\begin{aligned} \implies E_q^-(p) &= \min_{s' \succ q,p} d(p, s') \\ &= \min_{s' \succ q,p} H(p) + H(s') - 2H(p \vee s') \\ &= \min_{s' \succ q,p} H(p) + H(s') - 2H(s') \\ &= \min_{s' \succ q,p} H(p) - H(s'). \end{aligned}$$

The Schur-concavity of the Shannon entropy implies that the maximum value of $H(s')$ in the subset of P^n $\{v \mid v \succ p, q\}$ is reached for a vector s' that is majorized by all other vectors in the subset. By definition, that vector is the join $p \vee q$. $\square$

Lemma 2. *$d'(p,q) = d'(p, p\wedge q) + d'(p\wedge q, q)$ for the quasi-distance defined as $d'(p,q) \coloneqq 2H(p\wedge q) - H(p) - H(q)$.*

Proof.

$$\begin{aligned} d'(p, p\wedge q) + d'(p\wedge q, q) &= 2H(p\wedge(p\wedge q)) - H(p) - H(p\wedge q) + 2H((p\wedge q)\wedge q) - H(p\wedge q) - H(q) \\ &= 2H(p\wedge q) - H(p\wedge q) - H(p) + 2H(p\wedge q) - H(p\wedge q) - H(q) \\ &= 2H(p\wedge q) - H(p) - H(q) \\ &= d'(p,q). \end{aligned}$$

Result 2. *The measure of preceding incomparability of a vector p to a vector q defined as $E_q^{prec}(p) \coloneqq \min_{s\prec q} d'(p,s)$ is equal to the quasi-distance $d'(p, p\wedge q)$.*

Proof.

$$\begin{aligned} E_q^{\text{prec}}(p) &= \min_{s\prec q} d'(p,s) \\ &\overset{\text{Lemma 2}}{=} \min_{s\prec q} d'(p, p\wedge s) + d'(p\wedge s, s) \\ &\implies \min_{s\prec q} d'(p,s) \geq \min_{s\prec q} d'(p, p\wedge s) \end{aligned} \tag{1}$$

Let $s' = p\wedge s$. Because s is majorized by q, s' is majorized by q as well. Moreover, to be the meet of p with another vector, s' must be majorized by p as well.

$$\begin{aligned} \implies \min_{s\prec q} d'(p, p\wedge s) &= \min_{s'\prec q,p} d'(p,s') \\ &= \min_{s'\prec q,p} 2H(p\wedge s') - H(p) - H(s') \\ &= \min_{s'\prec q,p} H(s') - H(p). \end{aligned}$$

The Schur-concavity of the Shannon entropy implies that the minimum value of $H(s')$ in the subset of P^n $\{v \mid v \prec p, q\}$ is reached for a vector s' that majorizes all other vectors in the subset. By definition, that vector is the meet $p\wedge q$, and so $\min_{s'\prec q,p} H(s') - H(p) = H(p\wedge q) - H(p)$. The vector $p\wedge q$ is also part of the original subset $\{v \mid v \prec q\}$, and so plugging $s = p\wedge q$ into (1) shows that the LHS realizes the value $H(p\wedge q) - H(p)$ as well, and the 2 minima must therefore be equal. $\square$

Lemma 3. *In any causal diamond $p, q, p\wedge q, p\vee q$ we have $d(p, p\vee q) \leq d(q, p\wedge q)$.*

Proof.

$$\begin{aligned} d(p, p\vee q) &= H(p) + H(p\vee q) - 2H(p\vee(p\vee q)) \\ &= H(p) - H(p\vee q) \\ &= H(p) + H(p\wedge q) - H(p\wedge q) - H(p\vee q) \\ &\overset{\text{supermod}}{\leq} H(p\wedge q) - H(q) \\ &= d(q, p\wedge q) \end{aligned}$$

Result 3. *There exists no bistochastic matrix D such that $E^-(Dp \parallel q) < E^-(p \parallel q)$.*

*Proof.*Let p' be a vector majorized by p, i.e. there exists a bistochastic matrix D such that $p' = Dp$. Let $q' = p' \vee q$, and let $p'' = p \wedge q'$. By hypothesis and by definition of q', p' is majorized by both p and q', which is equivalent to p' being majorized by $p \wedge q'$.

$$\implies d(p'', q') = H(p \wedge q') - H(q') \leq H(p') - H(q') = d(p', q') \tag{2}$$

$$\begin{aligned}
\implies \min_{p' \prec p} E^-(p' \parallel q) &= \min_{p' \prec p} d(p', p' \vee q) \\
&= \min_{p' \prec p} d(p', q') \\
&\overset{(2)}{=} \min_{p' \prec p} d(p'', q') \\
&= \min_{p' \prec p} d(p \wedge q', q') \\
&\overset{\text{Lemma 3}}{\geq} \min_{p' \prec p} d(p, p \vee q') \\
&= \min_{p' \prec p} d(p, p \vee (p' \vee q)) \\
&= \min_{p' \prec p} d(p, (p \vee p') \vee q) \\
&= d(p, p \vee q) \\
&= E^-(p \parallel q).
\end{aligned}$$

□

Corollary 3.1. *Let $|\psi\rangle$ and $|\phi\rangle$ be two quantum states, and let λ_ψ and λ_ϕ be the associated Schmidt vectors. If $|\psi\rangle \overset{LOCC}{\longrightarrow} |\phi\rangle$ with probability 1, then $E^-(\lambda_\psi \parallel \lambda_\alpha) \geq E^-(\lambda_\phi \parallel \lambda_\alpha)$ with λ_α some probability vector.*

Corollary 3.2. *Let ρ_{AB} be a bipartite quantum state, and let ρ_A and ρ_B be the reduced states. Let λ_{AB}, λ_A and λ_B be the vectors of eigenvalues of their density matrices. Then, if ρ_{AB} is separable, $E^-(\lambda_{AB} \parallel \lambda_B) \geq E^-(\lambda_A \parallel \lambda_B)$ and $E^-(\lambda_{AB} \parallel \lambda_A) \geq E^-(\lambda_B \parallel \lambda_A)$.*

One would hope that the analogue of result 3 would hold for the other measure of incomparability E^+, however it does not. The following counterexample shows that a bistochastic matrix D such that $E^+(Dp||q) < E^+(p||q)$ can exist, but also that a bistochastic matrix D' such that $E^+(D'p||q) > E^+(p||q)$ can exist too. Let $q = (0.6, 0.4)$, $p = (0.7, 0.29, 0.1)$, $p' = p \wedge q$ and $p'' = (0.7, 0.15, 0.15)$. One can verify that p majorizes both p' and p'', yet $E^+(p||q) = 0.0939$ bits, $E^+(p'||q) = 0$ bits and $E^+(p''||q) = 0.171$ bits.

Lemma 4. *$p \vee q \succ p \vee q' \quad \forall\, q' \prec q$.*

*Proof.*Let q' be any probability vector majorized by q and let p be some probability vector.

$$p \vee q = p \vee (q \vee q') = p \vee (q' \vee q) = (p \vee q') \vee q \succ p \vee q'.$$

Result 4. *There exists no bistochastic matrix D such that $E^-(p \parallel Dq) > E^-(p \parallel q)$.*

*Proof.*Let q' be a vector majorized by q, i.e. there exists a bistochastic matrix D such that $q' = Dq$.

$$\begin{aligned}\max_{q' \prec q} E^-(p \parallel q') &= \max_{q' \prec q} d(p, p \vee q') \\ &= \max_{q' \prec q} H(p) - H(p \vee q') \\ &\overset{\text{Lemma 4}}{=} H(p) - H(p \vee q) \\ &= d(p, p \vee q) \\ &= E^-(p \parallel q).\end{aligned}$$

□

Lemma 5. $p \wedge q \succ p \wedge q' \quad \forall\, q' \prec q$.

*Proof.*Let q' be any probability vector majorized by q and let p be some probability vector.

$$p \wedge q' = p \wedge (q \wedge q') = (p \wedge q) \wedge q' \prec p \wedge q.$$

Result 5. *There exists no bistochastic matrix D such that $E^+(p \parallel Dq) < E^+(p \parallel q)$.*

*Proof.*Let q' be a vector majorized by q, i.e. there exists a bistochastic matrix D such that $q' = Dq$.

$$\begin{aligned}\min_{q' \prec q} E^+(p \parallel q') &= \min_{q' \prec q} d(p, p \wedge q') \\ &= \min_{q' \prec q} H(p \wedge q') - H(p) \\ &\overset{\text{Lemma 5}}{=} H(p \wedge q) - H(p) \\ &= d(p, p \wedge q) \\ &= E^+(p \parallel q).\end{aligned}$$

□

Results 4 and 5 have corollaries analoguous to those of result 3.