Part 1 - Analytic Assignment

- Compute gradient vector for a plane in 3D space

$$z = f(x,y) = ax + by + c$$

$$\nabla f(x,y) = \frac{\partial(ax+by)}{\partial x} + \frac{\partial(ax+by)}{\partial y} = a + b$$

- Compute gradient vector for a hyperplane

$$z = f(x_1, x_2 \dots x_n) = \sum_{i=1}^{N} a_i (x_i - b_i) + S = a_1x_1 + a_2x_2 + \dots + a_Nx_N + d$$

(Hint: derivative of sum is the sum of derivatives)

$$\nabla f(x) = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \dots \frac{\partial f}{\partial x_n} = a_1 + a_2 + \dots + a_n$$

- compute partial derivative of paraboloid function

$$z = f(x,y) = A(x-x_0)^2 + B(y-y_0)^2 + c$$

$$f_x(x,y) = \left(\frac{\partial f(x,y)}{\partial x}\right)_y = \frac{\partial(A(x-x_0)^2 + B(y-y_0)^2 + Z)}{\partial x} = A\frac{\partial(x-x_0)^2}{\partial x} = A\frac{\partial(x^2 - 2x\cdot x_0 + x_0^2)}{\partial x} = A(2x - 2x_0)$$

$$f_y(x,y) = \left(\frac{\partial f(x,y)}{\partial y}\right)_x = B \cdot \frac{\partial(y-y_0)^2}{\partial y} = B(2y - 2y_0)$$

- compute following quantities and specify the shape of output.

$$x = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad y = (2\ 5\ 1) \quad A = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{pmatrix}$$

$x^T = [3\ 1\ 4]$ shape: [1×3]

$y^T = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ shape: [3×1]

$B^T = \begin{bmatrix} 3 & 5 & 1 \\ 5 & 2 & 4 \end{bmatrix}$ shape: [2×3]

- $x \cdot x = 3\times3 + 1\times1 + 4\times4 = 26$ shape: [1×1]

$x \cdot y^T = 3\times2 + 1\times5 + 4\times1 = 15$ shape: [1×1]

$x \times y = \cancel{3\times2+1\times5+4\times1=15}\ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} [2\ 5\ 1] = \begin{bmatrix} 3\times2 & 3\times5 & 3\times1 \\ 1\times2 & 1\times5 & 1\times1 \\ 4\times2 & 4\times5 & 4\times1 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{bmatrix}$ shape: [3×3]

$y \times x = 3\times2 + 1\times5 + 4\times1 = 15$ shape: [1×1]

$A \times x = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4\times3 + 5\times1 + 2\times4 \\ 3\times3 + 1\times1 + 5\times4 \\ 6\times3 + 4\times1 + 3\times4 \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \\ 34 \end{bmatrix}$ Shape: [3×1]

$A \times B = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4\times3 + 5\times1 + 2\times4 & 4\times5 + 5\times2 + 2\times4 \\ 3\times3 + 1\times1 + 5\times4 & 3\times5 + 1\times2 + 5\times4 \\ 6\times3 + 4\times1 + 3\times4 & 6\times5 + 4\times2 + 3\times4 \end{bmatrix} = \begin{bmatrix} 39 & 38 \\ 19 & 37 \\ 41 & 50 \end{bmatrix}$ shape: [3×2]

B.reshape(1,6) = [3 5 5 2 1 4]

Part 1: Linear Least square : single variable

$$y = m(x|p) = mx+b$$

$$p = (p_0, p_1) = (m, b)$$

$$SE_{line} = (y_1-(mx_1+b))^2 + (y_2-(mx_2+b))^2 + \cdots + (y_n-(mx_n+b))^2$$

→ find m, b by minimize SE_{line}

$$SE_{line} = y_1^2 - 2y_1(mx_1+b) + (mx_1+b)^2 + y_2^2 - 2y_2(mx_2+b) + (mx_2+b)^2 + \cdots + y_n^2 - 2y_n(mx_n+b) + (mx_n+b)^2$$

$$= (y_1^2+y_2^2+\cdots+y_n^2) - 2m(x_1y_1+x_2y_2+\cdots+x_ny_n) - 2b(y_1+y_2+\cdots+y_n)$$

$$+ m^2(x_1^2+x_2^2+\cdots+x_n^2) + 2mb(x_1+x_2+\cdots x_n) + nb^2$$

$$= n\overline{y^2} - 2m(n\overline{xy}) - 2b(n\overline{y}) + m^2(n\overline{x^2}) + 2mb(n\overline{x}) + nb^2$$

→ minimize SE_{line} ① $\frac{\partial SE}{\partial m} = 0$ ② $\frac{\partial SE}{\partial b} = 0$

$$-2n\overline{xy} + 2n\overline{x^2}m + 2bn\overline{x} = 0 \qquad -2n\overline{y} + 2mn\overline{x} + 2bn = 0$$

$$-\overline{xy} + m\overline{x^2} + b\overline{x} = 0 \qquad -\overline{y} + m\overline{x} + b = 0$$

$$m\overline{x^2} + b\overline{x} = \overline{xy} \qquad m\overline{x} + b = \overline{y}$$

since $y = mx+b$. from ① $\left(\frac{\overline{x^2}}{\overline{x}}, \frac{\overline{xy}}{\overline{x}}\right)$ is on the line

from ② $(\overline{x}, \overline{y})$ is on the line

therefore $m\left(\overline{x} - \frac{\overline{x^2}}{\overline{x}}\right) = \overline{y} - \frac{\overline{xy}}{\overline{x}}$

$$m = \frac{\overline{y} - \frac{\overline{xy}}{\overline{x}}}{\overline{x} - \frac{\overline{x^2}}{\overline{x}}} = \frac{\overline{x}\,\overline{y} - \overline{xy}}{(\overline{x})^2 - \overline{x^2}} = \frac{cov(x)}{var(x)}$$

$$b = \overline{y} - m\overline{x} = \overline{y} - \frac{cov(x)}{var(x)}\overline{x}$$

$$var(x) = \frac{1}{N}\sum_{i=1}^{N}(x_i-\overline{x})^2$$

$$= \frac{1}{N}\left[(x_1-\overline{x})^2 + (x_2-\overline{x})^2 + \cdots (x_n-\overline{x})^2\right]$$

$$= \frac{1}{N}\left[(x_1^2+x_2^2+\cdots x_n^2) - (2x_1\overline{x} + 2x_2\overline{x} + \cdots 2x_n\overline{x}) + (\overline{x}^2+\overline{x}^2+\cdots\overline{x}^2)\right]$$

$$= \frac{1}{N}\left[n\overline{x^2} - 2n\overline{x}\cdot\overline{x} + n\overline{x}^2\right]$$

$$= \overline{x}^2 - \overline{x^2}$$

$$cov(x) = \frac{1}{N}\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})$$

$$= \frac{1}{N}\left[(x_1-\overline{x})(y_1-\overline{y}) + (x_2-\overline{x})(y_2-\overline{y}) + \cdots + (x_n-\overline{x})(y_n-\overline{y})\right]$$

$$= \frac{1}{N}\left[(x_1y_1+x_2y_2+\cdots x_ny_n) - (x_1\overline{y}+x_2\overline{y}+\cdots+x_n\overline{y}) - (\overline{x}y_1+\overline{x}y_2+\cdots+\overline{x}y_n) + (\overline{x}\overline{y}+\overline{x}\overline{y}+\cdots\overline{x}\overline{y})\right]$$

$$= \frac{1}{N}\left[n\overline{xy} - n\overline{x}\overline{y} - \cancel{n\overline{x}}\overline{y} + \cancel{n\overline{x}}\overline{y}\right] = \frac{1}{n}\left[n\overline{xy} - n\overline{x}\overline{y}\right] = \overline{xy} - \overline{x}\overline{y}$$