ex 3.3

6. (1)、 3个实根 在 $(1,2)$, $(2,3)$, $(3,4)$

(2)、 设 n 个零点分别为 $x_1, x_2, \cdots, x_n$

由Rolle定理:

$P_n(x)$ 在 (x_1, x_2) 一定有实根, $(x_2, x_3), \cdots, (x_{n-1}, x_n)$ 同理

又 $P_n'(x)$ 有 $n-1$ 个根, 故上述每个区间有1个根, 为 $f'(x)$ 所有根

高阶导数同理

5. $\frac{|f(y)-f(x)|}{|y-x|} \le M|y-x|$

令 $y-x \to 0$ 得 $f'(x) \equiv 0$, $f(x) \equiv const$.

7. (1)、令 $f(x) = x^n$ $\quad f(a)-f(b) = a^n - b^n = n\xi^{n-1}(a-b)$, $a<\xi<b$

故 $nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$

(3). 令 $\frac{a}{b} = t$ $\quad$ 即证 $1-\frac{1}{t} \le \ln t \le t-1$

令 $f(t) = \ln t + \frac{1}{t} - 1$ $\quad f(1)=0$ $\quad f'(t) = \frac{t-1}{t^2} \ge 0$

$f(t)-f(1) = (t-1)f'(\xi) \ge 0$ $\quad f(t) \ge 0$

令 $t = \frac{1}{x}$ 可得: $\ln t \le t-1$

(5). 即证 $\frac{a+b}{2}\ln\frac{a+b}{2}-a\ln a<b\ln b-\frac{a+b}{2}\ln\frac{a+b}{2}$

令 $f(x)=x\ln x$

$f(\frac{a+b}{2})-f(a)=\frac{b-a}{2}f'(\xi)$ $\quad \xi\in(a,\frac{a+b}{2})$

$f(b)-f(\frac{a+b}{2})=\frac{b-a}{2}f'(\eta)$ $\quad \eta\in(\frac{a+b}{2},b)$

$f'(\xi)<f'(\eta)$ 故原不等式成立

9. 令 $h(x)=f(x)-g(x)$ $\quad h'(x)=f'(x)-g'(x)>0$

$h(x)-h(0)=h'(\xi)\cdot x>0$ $\quad h(x)>0$ $\quad f(x)>g(x)$

14、令 $g(x)=x^2$ 即证 $g'(c)[f(b)-f(a)]=[g(b)-g(a)]f'(c)$

令 $F(x)=\frac{f(b)-f(a)}{g(b)-g(a)}g(x)-f(x)$ $\quad F(a)=F(b)$

故 $\exists c\in(a,b)$, s.t. $F(c)=0$

15. 令 $g(x)=\ln x$ 即证 $g'(c)[f(b)-f(a)]=[g(b)-g(a)]f'(c)$

同上题

20、$2|f(x_1)-f(x_2)|=|f(x_1)-f(x_2)+f(x_1)-f(0)+f(1)-f(x_2)|$

$\leq|f(x_1)-f(x_2)|+|f(x_1)-f(0)|+|f(1)-f(x_2)|$

$<|x_1-x_2|+|x_1-0|+|1-x_2|=1$

$|f(x_1)-f(x_2)|<\frac{1}{2}$

23、令 $F(x)=f(x)e^{g(x)}$ $\quad F'(x)=e^{g(x)}(f'(x)+f(x)g'(x))$

$F(a)=F(b)=0 \Rightarrow \exists \xi\in(a,b)$ s.t. $F'(x)=0$

ex 3.4

1、(1)、$\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}}{\cos x}=2$

(3)、$\lim\limits_{x\to 0}\dfrac{\alpha e^{\alpha x}+\alpha\sin\alpha x}{\beta e^{\beta x}+\beta\sin\beta x}=\dfrac{\alpha}{\beta}$

(5)、$\lim\limits_{x\to 0}\dfrac{\sin x^2}{2\sin^2\frac{x}{2}}=\lim\limits_{x\to 0}\dfrac{x^2}{2(\frac{x}{2})^2}=2$

(7)、$\lim\limits_{x\to 0}\dfrac{(x-1)e^x+1}{3x}=\lim\limits_{x\to 0}\dfrac{e^x}{6}=\dfrac{1}{6}$

(9) $\lim\limits_{x\to 0^+}\dfrac{\tan 7x-1}{\tan 2x-1}=1$

(11)、$\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x}=\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}}{\cos x}=2$

(13)、$\lim\limits_{x\to 0}\dfrac{e^x(e^{\tan x-x}-1)}{\tan x-x}=\lim\limits_{x\to 0}e^x=1$

(15)、$\lim\limits_{x\to 0}\dfrac{e^x(\sin x+\cos x)-2x-1}{3x^2}$

$=\lim\limits_{x\to 0}\dfrac{2e^x\cos x-2}{6x}=\lim\limits_{x\to 0}\dfrac{e^x(\cos x-\sin x)}{3}=\dfrac{1}{3}$

(17)、$\lim\limits_{x\to\infty}x(e^{-\frac{1}{x}}-1)=\lim\limits_{x\to\infty}\dfrac{e^{-\frac{1}{x}}-1}{\frac{1}{x}}$

$=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x^2}e^{-\frac{1}{x}}}{-\frac{1}{x^2}}$

$=\lim\limits_{x\to\infty}-e^{-\frac{1}{x}}=-1$

(19). $\lim\limits_{x\to\frac{\pi}{2}^-}(\cos x)^{\frac{\pi}{2}-x} = \lim\limits_{x\to 0}(\sin x)^x$

$\lim\limits_{x\to 0} x\ln(\sin x) = \lim\limits_{x\to 0} x\ln x = 0$

$\lim\limits_{x\to\frac{\pi}{2}^-}(\cos x)^{\frac{\pi}{2}-x} = 1$

(21). $\ln(\cos\frac{a}{x})^x = x\ln(\cos\frac{a}{x}) = \dfrac{\ln(\cos\frac{a}{x})}{\frac{1}{x}}$

$\lim\limits_{x\to\infty}\dfrac{\ln(\cos\frac{a}{x})}{\frac{1}{x}} = \lim\limits_{x\to\infty}\dfrac{-\frac{a}{x^2}\sin\frac{a}{x}}{-\frac{1}{x^2}\cos\frac{a}{x}} = \lim\limits_{x\to\infty} a\tan\frac{a}{x} = 0$

$\lim\limits_{x\to\infty}(\cos\frac{a}{x})^x = 1$

(23). $\ln\left(\frac{\sin t}{t}\right)^{\frac{1}{t^2}} = \dfrac{\ln\frac{\sin t}{t}}{t^2}$ $\quad\lim\limits_{t\to 0}\dfrac{\ln\frac{\sin t}{t}}{t^2} = \lim\limits_{t\to 0}\dfrac{\frac{t}{\sin t}\cdot\frac{t\cos t-\sin t}{t^2}}{2t} = \infty$

$\lim\limits_{t\to 0}\left(\frac{\sin t}{t}\right)^{\frac{1}{t^2}} = \infty$

(25). $\lim\limits_{x\to\infty}(1+\frac{1}{x^2})^{x^2} = e$ $\quad\lim\limits_{x\to\infty}(1+\frac{1}{x^2})^x = \lim\limits_{x\to\infty}\sqrt[x]{e} = 1$

(27). $\lim\limits_{x\to 0}\dfrac{\ln(\frac{2}{\pi}\arccos x)}{x} = \lim\limits_{x\to 0}\dfrac{\frac{2}{\pi}\cdot(-\frac{1}{\sqrt{1-x^2}})}{\frac{2}{\pi}\arccos x} = -\frac{2}{\pi}$

$\lim\limits_{x\to 0}(\frac{2}{\pi}\arccos x)^{\frac{1}{x}} = e^{-\frac{2}{\pi}}$

(29). $\lim\limits_{x\to 1}\dfrac{\ln(1-x)}{\frac{1}{\ln x}} = \lim\limits_{x\to 1}\dfrac{-\frac{1}{1-x}}{-\frac{1}{\ln^2 x}\cdot\frac{1}{x}} = \lim\limits_{x\to 1}\dfrac{x\ln^2 x}{1-x}$

$= \lim\limits_{x\to 1} -(\ln^2 x+2\ln x) = 0$

(31). $\lim\limits_{x\to\infty}\dfrac{1-x\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim\limits_{x\to\infty} x^2(\ln(1+\frac{1}{x})-\frac{1}{x+1})$

$= \lim\limits_{x\to\infty}\left[x\ln(1+\frac{1}{x})^x - \frac{x^2}{x+1}\right] = +\infty$

4.

(1). $\lim\limits_{x\to 0} \dfrac{\sin\frac{k}{x}}{\frac{1}{x}} = 0$

(2). $\lim\limits_{x\to\infty} \dfrac{x-2\cos x}{x+2\cos x} = \lim\limits_{x\to\infty} \dfrac{1-2\cdot\frac{\cos x}{x}}{1+2\cdot\frac{\cos x}{x}} = 1$

~~若用~~ L'Hospital 法则:

$\lim\limits_{x\to+\infty} \dfrac{(x-2\cos x)'}{(x+2\cos x)'} = \lim\limits_{x\to+\infty} \dfrac{1+2\sin x}{1-2\sin x}$ 极限不存在

5. (1). $\lim\limits_{x\to+\infty} \dfrac{\ln^k x}{x^\alpha} = 0$ 故 $\ln^k x = o(x^\alpha)$

(3). $\ln\dfrac{x^x}{e^x} = x(\ln x-1)$ $\lim\limits_{x\to+\infty} x(\ln x-1) = +\infty$

故 $\lim\limits_{x\to+\infty} \dfrac{x^x}{e^x} = +\infty$ $e^x = o(x^x)$