ex 2.1

15. (1). $a_n > \frac{1}{n(n+1)} + \cdots + \frac{1}{(2n-1)2n} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \triangleq b_n$

$a_n < \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{2n(2n+1)} = \frac{1}{n+1} - \frac{1}{2n+1} = \frac{1}{2n+\frac{1}{n}+3} \triangleq C_n$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} C_n = 0 \quad \Rightarrow \lim\limits_{n\to\infty} a_n = 0$

(3). $0 < (n+1)^k - n^k <$

(5). $a_n = 2^{(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n})} = 2^{1-(\frac{1}{2})^n}$

$< 2^{1-(\frac{1}{2})^n} < 2$

(7). $\sqrt[n]{\cos 1} < a_n < \sqrt[n]{n}$

$0 < \cos 1 < 1 \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{\cos 1} = 1$

$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ 故 $\lim\limits_{n\to\infty} a_n = 1$

17. (2). $k \geq 1$: $0 < \frac{\ln n}{n^k} \leq \frac{\ln n}{n}$ $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$ 则 $\lim\limits_{n\to\infty} \frac{\ln n}{n^k} = 0$

$0 < k < 1$:

(4). $\lim\limits_{n\to\infty} \frac{n^k}{a^n} = \lim\limits_{n\to\infty} \frac{n^k - (n-1)^k}{a^n - a^{n-1}} = \lim\limits_{n\to\infty} \frac{P_{k-1}(n)}{a^{n-1}(a-1)}$

$= \cdots = \lim\limits_{n\to\infty} \frac{1}{a^{n-k}(a-1)^k} = 0$

18、(1)、 a_n单调递减且$a_n>0$ 故收敛

(3)、 a_n单调递减且$a_n>0$ 故收敛

20、(1)、 由于 $\sin x<x$， a_n单调递减 又$a_n>0$， 故a_n收敛

设$\lim\limits_{n\to\infty} a_n=a$ 则 $a=\sin a$ $a=0$

(3)、 a_n单调递增且$a_n<1$ 故收敛

设$\lim\limits_{n\to\infty} a_n=a$ $a=\frac{c}{2}+\frac{a^2}{2}$ $\Rightarrow$ $a=1-\sqrt{1-c}$

(5)、 a_n单调递减， $a_n>0$， 故a_n收敛

设$\lim\limits_{n\to\infty} a_n=a$ 则 $a=2-\frac{1}{a}$ $a=1$

23、(1)、 $\forall\varepsilon>0$， $\exists N=\log_q\frac{\varepsilon(1-q)}{M}>0$， 取 $n>N$， $p\in N^+$

$$a_{n+p}-a_n=\alpha_{n+1}q^{n+1}+\cdots+\alpha_{n+p}q^{n+p}$$
$$\leq M(q^{n+1}+\cdots+q^{n+p})=\frac{Mq^{n+1}(1-q^p)}{1-q}<\frac{Mq^{N+1}}{1-q}<\varepsilon$$

故a_n收敛

(3)、 $\forall\varepsilon>0$

$$a_{n+p}-a_n=\sin\frac{1}{\sqrt{n+1}}+\cdots+\sin\frac{1}{\sqrt{n+p}}$$

27. 若 $\lim\limits_{n\to\infty} a_n = a$. 则对 $\forall \varepsilon>0$, $\exists N$, s.t. $n>N$ 时 $|a_n-a|<\varepsilon$

此时 $\exists a_{n_i} > N$, 设为 N', $n_k > N'$ 时 $|a_{n_k}-a|<\varepsilon$

故 $\lim\limits_{k\to\infty} a_{n_k} = a$

28. (1). $b_n = \frac{n}{n+1}$ $\quad \lim\limits_{n\to\infty} b_n = 1$ $\qquad c_n = -\frac{n}{n+1}$ $\quad \lim\limits_{n\to\infty} c_n = -1$

b_n, c_n 均为 a_n 子列, 则 a_n 发散

(3).

29. (1). $\lim\limits_{n\to\infty} a_n = e$

(3). $\lim\limits_{n\to\infty} a_n = \dfrac{1}{\left(1+\frac{1}{2-n}\right)^{2-n}} = \dfrac{1}{e}$

(5). $\lim\limits_{n\to\infty} a_n = \left(\dfrac{1}{\left(1-\frac{1}{n^3}\right)^{-n^3}}\right)^2 = \dfrac{1}{e^2}$