ex 2.1

1. (1). $a_n=\frac{n}{n+1}$ (3). $a_n=\frac{1}{n}\cos\frac{n\pi}{2}$

2. (1).

ε	0.1	0.01	0.001	0.0001	0.00001
N	10	100	10^3	10^4	10^5

Pf. $\forall\varepsilon>0$, $\exists N=\lceil\frac{1}{\varepsilon}\rceil$, s.t. $n>N$ 时

$\left|\frac{n+1}{n}-1\right|=\frac{1}{n}<\varepsilon$ 故 $\lim\limits_{n\to\infty}\frac{n+1}{n}=1$

(3).

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Pf. $\forall\varepsilon>0$, $\exists N=\lceil\frac{1}{\varepsilon}\rceil$, s.t. $n>N$ 时

$\left|\frac{\sin n}{n}\right|<\frac{1}{n}<\varepsilon$ 故 $\lim\limits_{n\to\infty}\frac{\sin n}{n}=0$

(5). $\frac{n!}{n^n}<\frac{1}{n}$

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4. $\Rightarrow$: $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$, s.t. $n>N$时 $|a_n-a|<\varepsilon$

$$\big||a_n|-|a|\big| \le |a_n-a|<\varepsilon \quad 即 \lim_{n\to\infty}|a_n|=|a|$$

$\Leftarrow$: 反例: $a_n=\begin{cases}1, & n为奇数\\ -1, & n为偶数\end{cases}$

5. $\Rightarrow$: $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$, s.t. $n>N$时 $|a_n-a|<\varepsilon$

$$a-\varepsilon<a_n<a+\varepsilon \Rightarrow 2a-\varepsilon<a_n+a<2a+\varepsilon$$

$$|a_n^2-a^2|=|a_n-a||a_n+a|<\varepsilon(2a+\varepsilon)$$

令 $\varepsilon'=\varepsilon(2a+\varepsilon)$, $\varepsilon'\in(0,+\infty)$

即 $\forall \varepsilon'>0$, $\exists N\in\mathbb{N}^+$, s.t. $n>N$时 $|a_n^2-a^2|<\varepsilon'$

$\Leftarrow$: 反例: $a_n=-1$

10. $\forall \varepsilon>0$, $\exists N\in\mathbb{N}^+$, s.t. $n>N$时 $|a_n-a|<\varepsilon$

$$|10^{a_n}-10^a|=10^a|10^{a_n-a}-1|$$

令 $\varepsilon'=10^a|10^{\varepsilon}-1|$

$\forall \varepsilon'>0$, $\exists N\in\mathbb{N}^+$, s.t. $n>N$时 $|10^{a_n}-10^a|<\varepsilon'$

12. (1). $a_n=\dfrac{4+\frac{5}{n}+\frac{2}{n^2}}{3+\frac{2}{n}+\frac{1}{n^2}}$ $\quad \lim\limits_{n\to\infty}a_n=\dfrac{4}{3}$

(3). $a_n=\dfrac{\left(\sqrt{1+\frac{1}{n^2}}+1\right)^2}{\sqrt[3]{1+\frac{1}{n^6}}}$ $\quad \lim\limits_{n\to\infty}a_n=4$

(5). $\sqrt{n}-\sqrt{n+1} < \sin\sqrt{n+1}-\sin\sqrt{n} < \sqrt{n+1}-\sqrt{n}$

$\lim\limits_{n\to\infty}(\sqrt{n}-\sqrt{n+1}) = \lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0$

故 $\lim\limits_{n\to\infty}(\sin\sqrt{n+1}-\sin\sqrt{n}) = 0$

13. 令 $\varepsilon = \frac{a}{2}$ 则 $\exists N\in\mathbb{N}$, s.t. $n>N$ 时 $|a_n-a|<\frac{a}{2}$

即 $a_n > a-\frac{a}{2} = \varepsilon$

14. (1). $a_n = \frac{n-1}{2n} = \frac{1-\frac{1}{n}}{2}$ $\quad \lim\limits_{n\to\infty} a_n = \frac{1}{2}$

(3). $a_n = 1-\frac{1}{n}$ $\quad \lim\limits_{n\to\infty} a_n = 1$

(5). $1-\frac{2}{n(n+1)} = \frac{n^2+n-2}{n(n+1)} = \frac{(n+2)(n-1)}{n(n+1)}$

$a_n = \frac{4\times1}{2\times3}\cdots\frac{(n+2)(n-1)}{n(n+1)} = \frac{1}{3}\frac{n+2}{n}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{3}\cdot(1+\frac{2}{n}) = \frac{1}{3}$

(7). $a_n = \frac{1-q^{2n+1}}{1-q}$ $\quad \lim\limits_{n\to\infty} a_n = \frac{1}{1-q}$

(9). $a_n = \frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{1+\frac{1}{2n}}}{\sqrt{1+\frac{1}{n}}+1}$ $\quad \lim\limits_{n\to\infty} a_n = \frac{1}{2}$

(11).

24. 由Stolz定理，$\lim\limits_{n\to\infty}\dfrac{a_1+\cdots+a_n}{n}=\lim\limits_{n\to\infty}a_n=a$

$$\lim_{n\to\infty}\frac{a_1+2a_2+\cdots+na_n}{n^2}=\lim_{n\to\infty}\frac{na_n}{n^2-(n-1)^2}$$

$$=\lim_{n\to\infty}\frac{a_n}{2-\frac{1}{n}}=\frac{a}{2}$$

26. $\forall\varepsilon>0$，$\exists K_1, K_2, K_3$，s.t.

$k>K_1$ 时 $|a_{3k}-a|<\varepsilon$

$k>K_2$ 时 $|a_{3k+1}-a|<\varepsilon$

$k>K_3$ 时 $|a_{3k+2}-a|<\varepsilon$

取 $N=\max\{3K_1, 3K_2+1, 3K_3+2\}$，则有：

$\forall\varepsilon>0$，$\exists N\in\mathbb{N}$，s.t. $n>N$ 时 $|a_n-a|<\varepsilon$，故 $\lim\limits_{n\to\infty}a_n=a$