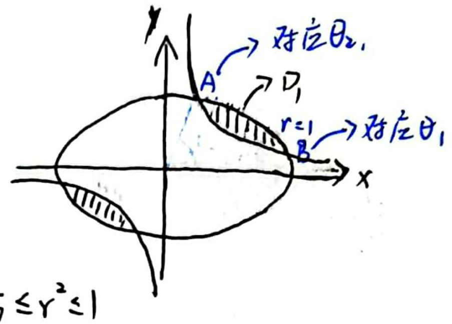
4.29 习题课讲义

(作业)

T_1 (习题10.2 3(1))

求下列曲线所围成的平面区域的面积

$x^2+2y^2=3$ 和 $xy=1$ (不含原点部分)



解: 用极坐标，只需求第一象限的面积 D_1

令 $x=\sqrt{3}r\cos\theta$, $y=\frac{\sqrt{6}}{2}r\sin\theta$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=\frac{3}{2}\sqrt{2}r$

由曲线围成的面积 D_1 知 (用θ型区域) $\begin{cases} x^2+2y^2\le 3 & ① \\ xy\ge 1 & ② \end{cases} \Rightarrow \frac{2\sqrt{2}}{3\sin 2\theta}\le r^2\le 1$

$\theta_1\le\theta\le\theta_2$ $\theta_1=\frac{1}{2}\arcsin\frac{2\sqrt{2}}{3}$, $\theta_2=\frac{\pi}{2}-\arcsin\frac{2\sqrt{2}}{3}$

(r,θ 范围的计算，将 x,y 极坐标形式带入 ① $\Rightarrow r^2\le 1$，代入②知 $\frac{3\sqrt{2}}{2}r^2\sin\theta\cos\theta\ge 1 \Rightarrow r^2\ge\frac{1}{\sin 2\theta}\cdot\frac{2\sqrt{2}}{3}$

交点A.B 即为 $\frac{1}{\sin 2\theta}\cdot\frac{2\sqrt{2}}{3}=1$ 时 $\Rightarrow \theta_1=\frac{1}{2}\arcsin\frac{2\sqrt{2}}{3}$, $\theta_2=\frac{\pi}{2}-\theta_1$)

$$\therefore \iint_{D_1} ds=\int_{\theta_1}^{\theta_2}\int_{\sqrt{\frac{2\sqrt{2}}{3\sin 2\theta}}}^{1}\frac{3}{2}\sqrt{2}r\,dr\,d\theta$$

$$=\int_{\theta_1}^{\theta_2}\frac{3\sqrt{2}}{4}\left(1-\frac{2\sqrt{2}}{3\sin 2\theta}\right)d\theta$$

$$=\frac{3}{4}\sqrt{2}(\theta_2-\theta_1)-\int_{\theta_1}^{\theta_2}\frac{1}{\sin 2\theta}d\theta=\frac{3\sqrt{2}}{4}(\theta_2-\theta_1)+\int_{\theta_1}^{\theta_2}\frac{d\cos\theta}{2\sin^2\theta\cos\theta}$$

令 $t=\cos\theta$

$$\int_{\theta_1}^{\theta_2}\frac{d\cos\theta}{2\sin^2\theta\cos\theta}=\int_{\frac{\sqrt{6}}{3}}^{\frac{\sqrt{3}}{3}}\frac{dt}{2(1-t^2)t}=\int_{\frac{\sqrt{3}}{3}}^{\frac{\sqrt{6}}{3}}\frac{1}{2(t^2-1)t}dt.$$

$$=\frac{1}{4}\int_{\frac{\sqrt{3}}{3}}^{\frac{\sqrt{6}}{3}}\left(\frac{1}{t+1}+\frac{1}{t-1}-\frac{2}{t}\right)dt=-\frac{1}{2}\ln 2$$

$$\therefore S=2\iint_{D_1}ds=\frac{3\sqrt{2}}{2}\left(\frac{\pi}{2}-\arcsin\frac{2\sqrt{2}}{3}\right)-\ln 2$$

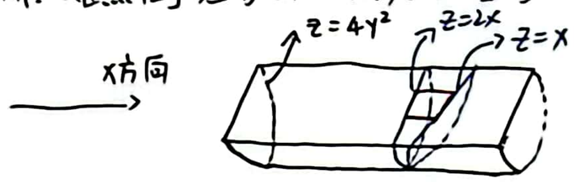
$$=\frac{3\sqrt{2}}{2}\arcsin\frac{1}{3}-\ln 2.$$

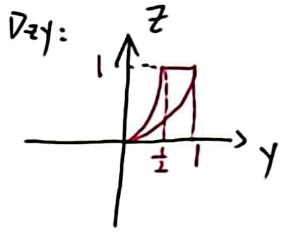
(其它题)

T_2 (习题10.3 3(5))

求 $\iiint_V x^2 dxdydz$，V是由曲面 $z=y^2$，$z=4y^2(y>0)$ 及平面 $z=x$，$z=2x$，$z=1$ 所围的区域.

解：难点在于想象出它的大概图象.





结合图形 $\iiint_V x^2 dxdydz = \iint_{D_{zy}} dzdy \int_{\frac{z}{2}}^{z} x^2 dx = \int_0^1 dz \int_{\frac{\sqrt{z}}{2}}^{\sqrt{z}} dy \int_{\frac{z}{2}}^{z} x^2 dx = \frac{7}{216}$

T_3 (习题10.3 3(8))

求 $\iiint_V (|x|+z) e^{-(x^2+y^2+z^2)} dxdydz$，V：$1 \le x^2+y^2+z^2 \le 4$

解：

对于 $\iiint_V z e^{-(x^2+y^2+z^2)} dxdydz$ 来说，积分区域关于xOy平面对称，函数关于z是奇的 $\Rightarrow = 0$

$\iiint_V |x| e^{-(x^2+y^2+z^2)} dxdydz$

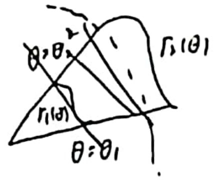
$= 2\iiint_{V\cap\{x>0\}} x e^{-(x^2+y^2+z^2)} dxdydz \xlongequal{\text{球面坐标变换}} 2\int_0^{\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_1^2 r\sin\theta\cos\varphi \cdot e^{-r^2} \cdot r^2 \sin\theta \, dr$

$= 2\int_0^{\pi} \sin^2\theta \, d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \, d\varphi \cdot \left(\frac{1}{2}\int_1^2 r^2 \cdot e^{-r^2} d(r^2)\right)$

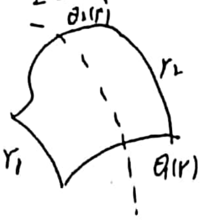
$= \pi\left(\frac{2}{e} - \frac{5}{e^4}\right)$

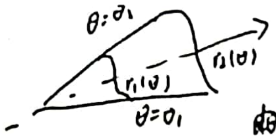
换极坐标的结果：分清是 θ-型区域，还是 r-型区域

θ型：



r型

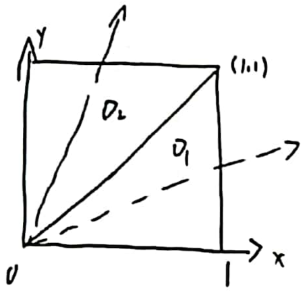




$dr \to d\theta$

$d\theta \to dr$

积分变换　1、将积分 $I=\iint\limits_{\substack{0\le x\le 1\\0\le y\le 1}} f(x,y)\,dxdy$ 化为极坐标形式 $\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}$



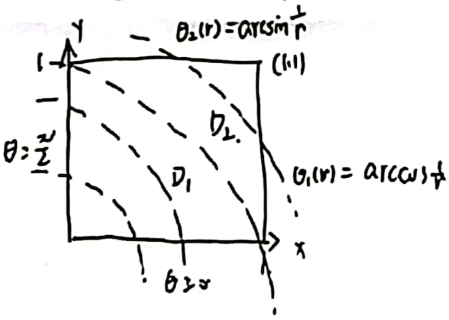
显然是 θ型区域比较简单 => 找r关于θ的函数

$D_1=\{(r,\theta)\mid 0\le\theta\le\frac{\pi}{4},\ 0\le r\le\frac{1}{\cos\theta}\}$

$D_2=\{(r,\theta)\mid \frac{\pi}{4}\le\theta\le\frac{\pi}{2},\ 0\le r\le\frac{1}{\sin\theta}\}$

$$I=\iint\limits_{\substack{0\le x\le 1\\0\le y\le 1}} f(x,y)\,dxdy=\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{1}{\cos\theta}} r\cdot f(r\cos\theta,r\sin\theta)\,dr$$
$$+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^{\frac{1}{\sin\theta}} r\cdot f(r\cos\theta,r\sin\theta)\,dr$$

若采用 r型区域 则：



$$I=\iint\limits_{\substack{0\le x\le 1\\0\le y\le 1}} f(x,y)\,dxdy=\int_0^1 dr\int_0^{\frac{\pi}{2}} f(r\cos\theta,r\sin\theta)\cdot r\,d\theta$$
$$+\int_1^{\sqrt{2}}dr\int_{\arccos\frac{1}{r}}^{\arcsin\frac{1}{r}} f(r\cos\theta,r\sin\theta)\,r\,d\theta.$$

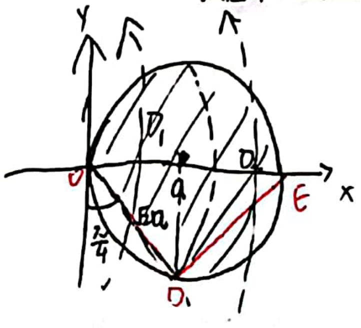
#

如何从
极坐标 => 直角坐标

$$K=\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta} f(r\cos\theta,r\sin\theta)\,r\,dr$$ 交换积分次序且化为直角坐标系.

明确此式为θ型积分区域.

抓住 $r=2a\cos\theta \Rightarrow r^2=2ar\cos\theta \Rightarrow x^2+y^2=2ax \Rightarrow (x-a)^2+y^2=a^2$. 边界

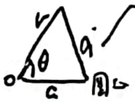


积分区域上 $r_{max}=2a$（即沿x轴）

利用ODE是等腰直角三角形

若采用r型区域，应利用曲线 $r=\sqrt{2}a$ 将区域划分为 $0\le r\le\sqrt{2}a$，$\sqrt{2}a\le r\le 2a$

$D_1=\{(r,\theta)\mid 0\le r\le\sqrt{2}a,\ -\frac{\pi}{4}\le\theta\le\arccos\frac{r}{2a}\}$



$D_2=\{(r,\theta)\mid \sqrt{2}a\le r\le 2a,\ -\arccos\frac{r}{2a}\le\theta\le\arccos\frac{r}{2a}\}$

$$\therefore k=\int_0^{\sqrt{2}a} r\,dr\int_{-\frac{\pi}{4}}^{\arccos\frac{r}{2a}} f(r\cos\theta,\ r\sin\theta)\,d\theta+\int_{\sqrt{2}a}^{2a} r\,dr\int_{-\arccos\frac{r}{2a}}^{\arccos\frac{r}{2a}} f(r\cos\theta,\ r\sin\theta)\,d\theta$$

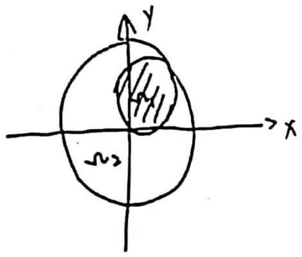
由圆方程 $x^2+y^2=2ax$ 得 $y=\pm\sqrt{2ax-x^2}$ $+$为上半圆，$-$为下半圆

$$\therefore k=\int_0^a dx\int_{-x}^{\sqrt{2ax-x^2}} f(x,y)\,dy+\int_a^{2a} dx\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y)\,dy$$ #

P187 T4 计算(综合题) $M=\iint\limits_{x^2+y^2\le 1}\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|dxdy$

解：$\because f(x,y)=\frac{x+y}{\sqrt{2}}-x^2-y^2=\frac{1}{4}-\left(x-\frac{1}{2\sqrt{2}}\right)^2-\left(y-\frac{1}{2\sqrt{2}}\right)^2$

故单位圆 $\Omega: x^2+y^2\le 1$，被分成两部分



$\Omega_1=\{(x,y)\mid f(x,y)\ge 0\}$.

$\Omega_2=\Omega-\Omega_1$

$$M=\iint\limits_{\Omega}|f|\,dxdy=\iint\limits_{\Omega_1} f\,dxdy-\iint\limits_{\Omega_2} f\,dxdy$$

$$=2\iint\limits_{\Omega_1} f\,dxdy-\iint\limits_{\Omega_1+\Omega_2} f\,dxdy=2\iint\limits_{\Omega_1} f\,dxdy-\iint\limits_{\Omega} f\,dxdy=I_1-I_2$$

$$I_1=2\iint\limits_{\Omega_1} f\,dxdy=2\iint\limits_{\Omega_1}\left[\frac{1}{4}-\left(x-\frac{1}{2\sqrt{2}}\right)^2-\left(y-\frac{1}{2\sqrt{2}}\right)^2\right]dxdy$$

令 $x-\frac{1}{2\sqrt{2}}=r\cos\varphi$, $y-\frac{1}{2\sqrt{2}}=r\sin\varphi$

$$\Rightarrow I_1=2\int_0^{2\pi}d\varphi\int_0^{\frac{1}{2}}\left(\frac{1}{4}-r^2\right)r\,dr=4\pi\left(\frac{1}{32}-\frac{1}{64}\right)$$

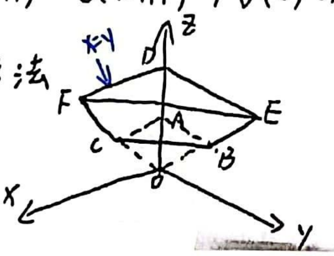
$$I_2=\iint\limits_{x^2+y^2\le 1}\Big(\underbrace{\frac{x+y}{\sqrt{2}}}_{\text{拆开看 对称性}=0}-x^2-y^2\Big)dxdy=-\iint\limits_{x^2+y^2\le 1}(x^2+y^2)\,dxdy=-\int_0^{2\pi}d\theta\int_0^1 r^3dr=-\frac{\pi}{2}$$

$$\therefore M=I_1-I_2=\frac{9}{16}\pi$$ #

三重积分：
- 截面法（化为1+2） $\int d\iint dd$
- 投影法（化为2+1） $\iint dd\int d$

计算积分 $I=\iiint\limits_V\frac{dv}{\rho^2}$，$\rho$是点$(x,y,z)$到$x$轴的距离，即$\rho^2=y^2+z^2$，$V$为棱台，其六个顶点为 $A(0,0,1)$, $B(0,1,1)$, $C(1,1,1)$, $D(0,0,2)$, $E(0,2,2)$, $F(2,2,2)$.

(1)：投影法

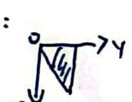


选择投影面为Oyz平面，投影出去为$ABED$梯形（记作W）

$\forall (y_0,z_0)\in W$，随着x增大，x从$0\longrightarrow y_0$.

$\therefore V=\{(x,y,z)\mid (y,z)\in W,\ 0\le x\le y\}$

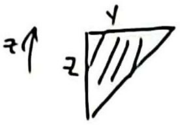
作俯视图：



$$\therefore I=\iint\limits_{W} dy\,dz\int_0^y \frac{dx}{y^2+z^2}=\iint\limits_{W}\frac{y}{y^2+z^2}\,dy\,dz=\int_1^2 dz\int_0^z\frac{y}{y^2+z^2}\,dy=\frac{1}{2}\ln 2.$$

② 截面法. 将V向z轴投影, 得到的区间是 [1, 2].

每一个截面为等腰直角三角形 D_z : $0\le y\le z$, $0\le x\le y$



$$\therefore I=\iiint\limits_{V}\frac{dv}{\rho^2}=\int_1^2 dz\iint\limits_{D_z}\frac{dx\,dy}{y^2+z^2}=\int_1^2 dz\int_0^z dy\int_0^y\frac{1}{y^2+z^2}\,dx=\frac{1}{2}\ln 2.$$

#

例（变量代换） $I=\iiint\limits_{\Omega}\left[\frac{1}{yz}\frac{\partial F}{\partial x}+\frac{1}{xz}\frac{\partial F}{\partial y}+\frac{1}{xy}\frac{\partial F}{\partial z}\right]dx\,dy\,dz$

其中Ω: $1\le yz\le 2$, $1\le xz\le 2$, $1\le xy\le 2$, 作积分变换 $u=yz$, $v=xz$, $w=xy$.

要求积分变换后积分出现u,v,w, 和F关于u,v,w的偏导数.(假设F有连续的一阶偏导数)

解: $u=yz$, $v=xz$, $w=xy$, 则 $J^{-1}=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}0&z&y\\z&0&x\\y&x&0\end{vmatrix}=2xyz=2\sqrt{uvw}$

$$uvw=x^2y^2z^2\Rightarrow x=\frac{\sqrt{uvw}}{u},\ y=\frac{\sqrt{uvw}}{v},\ z=\frac{\sqrt{uvw}}{w}$$

$$\therefore x'_u=-\frac{\sqrt{uvw}}{2u^2},\ y'_u=\frac{\sqrt{uvw}}{2uv},\ z'_u=\frac{\sqrt{uvw}}{2uw}$$

故有 $2uF'_u=2u(F'_x\cdot x'_u+F'_y\cdot y'_u+F'_z\cdot z'_u)$

$$=(-u^{-1}F'_x+v^{-1}F'_y+w^{-1}F'_z)\sqrt{uvw}\quad ①$$

由轮换对称性知: $2vF'_v=(u^{-1}F'_x-v^{-1}F'_y+w^{-1}F'_z)\sqrt{uvw}$ ②

$$2wF'_w=(u^{-1}F'_x+v^{-1}F'_y-w^{-1}F'_z)\sqrt{uvw}\quad ③$$

①+②+③ $2(uF'_u+vF'_v+wF'_w)=(u^{-1}F'_x+v^{-1}F'_y+w^{-1}F'_z)\sqrt{uvw}$

∴ 原积分号下的微分式

$$u^{-1}F'_x+v^{-1}F'_y+w^{-1}F'_z=\frac{2}{\sqrt{uvw}}(uF'_u+vF'_v+wF'_w)$$

$$\therefore I=\iiint\frac{2}{\sqrt{uvw}}(uF'_u+vF'_v+wF'_w)\cdot\underline{\frac{1}{2\sqrt{uvw}}}\,du\,dv\,dw$$

$$=\int_1^2 du\int_1^2 dv\int_1^2\left(\frac{1}{vw}F'_u+\frac{1}{uw}F'_v+\frac{1}{uv}F'_w\right)dw.$$

#

(10.4 T4)　f 连续

证明 $\int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_1) f(x_2) \cdots f(x_n)\, dx_n = \frac{1}{n!}\left[\int_0^a f(t)\,dt\right]^n$.

证明：记 $g_n(t) = \int_0^t dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_1) \cdots f(x_n)\, dx_n$.
(数归)

则 $g_1(t) = \int_0^t f(u)\,du$

假设 $g_{n-1}(t) = \frac{1}{(n-1)!}\left(\int_0^t f(u)\,du\right)^{n-1}$ $(n \geq 2)$

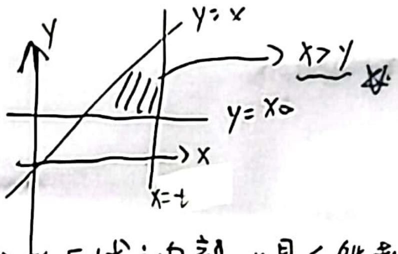
$g_n'(t) = \int_0^t dx_2 \cdots \int_0^{x_{n-1}} f(t) f(x_2) \cdots f(x_n)\, dx_n = f(t) \underline{\int_0^t dx_2 \int_0^{x_2} dx_3 \cdots \int_0^{x_{n-1}} f(x_2) \cdots f(x_n)\, dx_n}$

$\Rightarrow g_n'(t) = f(t) g_{n-1}(t) = \frac{1}{(n-1)!} f(t) \left(\int_0^t f(u)\,du\right)^{n-1}$ $\sim \int_0^t dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} f(x_1) \cdots f(x_{n-1})\, dx_{n-1}$

$\overset{☆}{=} \frac{1}{n!} \frac{d}{dt} \left(\int_0^t f(u)\,du\right)^n$.

$\Rightarrow g_n(t) = \frac{1}{n!}\left(\int_0^t f(u)\,du\right)^n$.　#.

例：求下列三重积分的极限，



$$\lim_{t \to x_0^+} \frac{1}{(t-x_0)^{n+4}} \iiint_\Omega (x-y)^n f(y)\, dx\, dy\, dz$$

Ω是由$y=x_0$ $(x_0>0)$，$y=x$，$x=t\,(>x_0)$，$z=x$ 及 $z=y$ 所围成的区域之内部，n是自然数.
$f(x)$在$[x_0, x_0+\delta]$ $(\delta>0)$上可微. $f(x_0)=0$.

解：由投影法画图判断 $\Omega = \{(x,y,z) : (x,y) \in D,\ y \leq z \leq x\}$.
(向xy平面投)

四面体在xy平面的投影区域为D，是xy平面上$y=x$，$x=t$，$y=x_0$三直线所围区域.

即 $D = \{(x,y) : x_0 \leq x \leq t,\ x_0 \leq y \leq x\}$

$= \{(x,y) : x_0 \leq y \leq t,\ y \leq x \leq t\}$.

由此可知 $\iiint_\Omega (x-y)^n f(y)\, dx\, dy\, dz$

$= \iint_D dx\, dy \int_y^x (x-y)^n f(y)\, dz = \int_{x_0}^t dy \int_y^t (x-y)^{n+1} f(y)\, dx = \frac{1}{n+2}\int_{x_0}^t (t-y)^{n+2} f(y)\, dy$

原式 $= \lim_{t \to x_0^+} \frac{1}{n+2} \frac{\int_{x_0}^t (t-y)^{n+2} f(y)\, dy}{(t-x_0)^{n+4}}$ (先二项展开).

$\left(\frac{1}{n+2}\right)$

$$\frac{\int_{x_0}^{t}(t-y)^{n+2} f(y)\,dy}{(t-x_0)^{n+4}}$$

(下面为方便省略积根号以及$\frac{1}{n+2}$)

$$=\frac{\int_{x_0}^{t} C_{n+2}^0 t^{n+2} f(y) - C_{n+2}^1 t^{n+1} y f(y) + C_{n+2}^2 t^n y^2 f(y) + \cdots C_{n+2}^{n+1} t (-1)^{n+1} y^{n+1} f(y) + C_{n+2}^{n+2}(-1)^{n+2} y^{n+2} f(y)\,dy}{(t-x_0)^{n+4}}$$

对t洛必达一项一项来

$$=\frac{C_{n+2}^0\left((n+2)t^{n+1}\int_{x_0}^t f(y) + t^{n+2} f(t)\right) - C_{n+2}^1\left((n+1)t^n\int_{x_0}^t y f(y) + t^{n+1} t\cdot f(t)\right)\cdots + C_{n+2}^{n+1}\left((-1)^{n+1}\int_{x_0}^t y^{n+1} f(y) + (-1)^{n+1} t\cdot t^{n+1} f(t)\right) + C_{n+2}^{n+2}(-1)^{n+2} t^{n+2} f(t)}{(n+4)(t-x_0)^{n+3}}$$

整理

$$=\frac{\underbrace{\left(C_{n+2}^0 - C_{n+2}^1 + C_{n+2}^2 + \cdots C_{n+2}^{n+1}(-1)^{n+1} + C_{n+2}^{n+2}(-1)^{n+2}\right)}_{=(1-1)^{n+2}=0} \boxed{t^{n+2} f(t)} + C_{n+2}^0 (n+2) t^{n+1}\int_{x_0}^t f(y) - C_{n+2}^1 (n+1) t^n \int_{x_0}^t y f(y) + \cdots C_{n+2}^{n+1}(-1)^{n+1}\int_{x_0}^t y^{n+1} f(y)}{(n+4)(t-x_0)^{n+3}}$$

← 无积分的项

← 有积分的项

对后面洛必达

$$=\frac{C_{n+2}^0(n+2)\left((n+1)t^n\int_{x_0}^t f(y) + t^{n+1} f(t)\right) - C_{n+2}^1(n+1)\left(n t^{n-1}\int_{x_0}^t y f(y) + t^n\cdot t f(t)\right) \cdots + \cdots + C_{n+2}^{n+1}(-1)^{n+1} t^{n+1} f(t)}{(n+4)(n+3)(t-x_0)^{n+2}}$$

$((t-1)^{n+2})' = (C_{n+2}^0 t^{n+2} + C_{n+2}^1(-1)t^{n+1} + \cdots C_{n+2}^{n+2}(-1)^{n+2})' = (n+2)C_{n+2}^0 t^{n+1} \cdots C_{n+2}^{n+1}(-1)^{n+1} = (n+2)(t-1)^{n+1}$

取 $t=1$ $\Rightarrow 0$

整理

$$=\frac{\underbrace{\left(C_{n+2}^0(n+2) - C_{n+2}^1(n+1) \cdots + C_{n+2}^{n+1}\cdot 1\cdot(-1)^{n+1}\right)}_{=0}\boxed{t^{n+1}f(t)} + C_{n+2}^0(n+2)(n+1)t^n\int_{x_0}^t f(y) - C_{n+2}^1(n+1)n t^{n-1}\int_{x_0}^t y f(y) \cdots + C_{n+2}^n(-1)^n\int_{x_0}^t y^n f(y)}{(n+4)(n+3)(t-x_0)^{n+2}}$$

$\vdots$

对后面重复此过程

$$=\frac{C_{n+2}^0(n+2)(n+1)\cdots 2t\int_{x_0}^t f(y) - C_{n+2}^1(n+1)n\cdots 1\cdot\int_{x_0}^t y f(y)}{(n+4)(n+3)\cdots 4\,(t-x_0)^3}$$

洛必达

$$=\frac{\overset{=1}{C_{n+2}^0}(n+2)!\int_{x_0}^t f(y) + \underbrace{C_{n+2}^0(n+2)!\,t\cdot f(t) - C_{n+2}^1(n+1)!\,t f(t)}_{=0}}{\frac{(n+4)!}{2}(t-x_0)^2}$$

$$= \frac{(n+2)!\int_{x_0}^t f(y)\,dy}{\frac{(n+4)!}{2}(t-x_0)^2} \overset{\text{洛}}{=} \frac{(n+2)!\, f(t)}{(n+4)!\,(t-x_0)} = \frac{(n+2)!\,(f(t) - \overset{=0}{f(x_0)})}{(n+4)!\,(t-x_0)} = \frac{1}{(n+4)(n+3)} f'(x_0)$$

$\therefore$ 原式 $= \frac{1}{(n+2)(n+3)(n+4)} f'(x_0^+)$ #