第6讲：多元函数微分法习题课(II)

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例1. 设 $f\in C^2(D)$，D是区域，$u=f(x+y+z,\ x^2+y^2+z^2)$，求 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$，$\frac{\partial^2 u}{\partial x^2}$，$\frac{\partial^2 u}{\partial x\partial y}$ 及 du.

解(1). 令 $v=x+y+z$，$w=x^2+y^2+z^2$，则 $u=f(v,w)$ 设 $f_1'=\frac{\partial f}{\partial v}$，$f_2'=\frac{\partial f}{\partial w}$，

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial f}{\partial v}\cdot 1+\frac{\partial f}{\partial w}\cdot 2x=f_1'+f_2'\cdot 2x;$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial f}{\partial v}\cdot 1+\frac{\partial f}{\partial w}\cdot 2y=f_1'+f_2'\cdot 2y;$$

(2). $\frac{\partial^2 u}{\partial x^2}=(f_1'+2xf_2')_x'=(f_1')_x'+(2xf_2')_x'=f_{11}''\cdot 1+f_{12}''\cdot 2x+(2x)_x'\cdot f_2'+2x(f_{21}''\cdot 1+f_{22}''\cdot 2x)=f_{11}''+4xf_{12}''+4x^2f_{22}''+2f_2'$.

$$\frac{\partial^2 u}{\partial x\partial y}=(f_1'+f_2'\cdot 2y)_x'=f_{11}''\cdot 1+f_{12}''\cdot 2x+2y(f_{21}''\cdot 1+f_{22}''\cdot 2x)$$
$$=f_{11}''+2f_{12}''\cdot(x+y)+4xyf_{22}''.$$

(3). $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

$$=(f_1'+2xf_2')dx+(f_1'\cdot 1+2yf_2')dy+(f_1'\cdot 1+2zf_2')dz.$$

注：$\because f\in C^2(D)$，$\therefore f_{12}''=f_{21}''$，即 $\frac{\partial^2 f}{\partial w\partial v}=\frac{\partial^2 f}{\partial v\partial w}$.

例2.(ex9.2/30, $u=u(x,y)\in C^2$)

试证方程：$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\cdot 2+\frac{\partial u}{\partial y}\cdot 6=0$ 在

线性变换 $\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$ 下，可化简为：$\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$.

证法(1)：从线性变换 $\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$ 可得线性逆变换：

$\begin{cases}x=\frac{1}{4}(\xi+\eta)\\ y=\frac{1}{4}(3\xi-\eta)\end{cases}\Rightarrow\frac{\partial x}{\partial\xi}=\frac{1}{4}=\frac{\partial x}{\partial\eta};\ \frac{\partial y}{\partial\xi}=\frac{3}{4},\ \frac{\partial y}{\partial\eta}=-\frac{1}{4}.$

从 $\frac{\partial u}{\partial\xi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\xi}=\frac{\partial u}{\partial x}\frac{1}{4}+\frac{\partial u}{\partial y}\cdot\frac{3}{4}\Rightarrow$

$\frac{\partial^2 u}{\partial\eta\partial\xi}=\left(\frac{\partial u}{\partial\xi}\right)'_\eta=\left(\frac{1}{4}\frac{\partial u}{\partial x}\right)'_\eta+\left(\frac{3}{4}\frac{\partial u}{\partial y}\right)'_\eta=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial\eta}\right)+$

$\frac{3}{4}\left(\frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\eta}\right)=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{1}{4}+\frac{\partial^2 u}{\partial x\partial y}\left(-\frac{1}{4}\right)\right)+\frac{3}{4}\left(\frac{\partial^2 u}{\partial y\partial x}\frac{1}{4}+\frac{\partial^2 u}{\partial y^2}\left(-\frac{1}{4}\right)\right)$

$=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}\right)$

即 $\begin{cases}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}=16\frac{\partial^2 u}{\partial\eta\partial\xi}\\ \frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=4\frac{\partial u}{\partial\xi}\end{cases}$

故原偏微分方程(PDE)化简为：$16\frac{\partial^2 u}{\partial\eta\partial\xi}+2\left(4\frac{\partial u}{\partial\xi}\right)=0$

即 $\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$.

2)

证法七)：从线性变换 $\begin{cases} \xi = x+y \\ \eta = 3x-y \end{cases} \Rightarrow \frac{\partial \xi}{\partial x}=1,\ \frac{\partial \xi}{\partial y}=1,$

$\frac{\partial \eta}{\partial x}=3,\ \frac{\partial \eta}{\partial y}=-1$. 而 $u(x,y)$ 通过中间变量可视为 ξ,η 的函数.

从而
$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}\cdot 1 + \frac{\partial u}{\partial \eta}\cdot 3 \\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi}\cdot 1 + \frac{\partial u}{\partial \eta}(-1) \end{cases}$$
，从而

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial u}{\partial \xi}\right)'_x + 3\left(\frac{\partial u}{\partial \eta}\right)'_x = \frac{\partial^2 u}{\partial \xi^2}\cdot 1 + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot 3 + 3\left(\frac{\partial^2 u}{\partial \eta \partial \xi}\cdot 1 + \frac{\partial^2 u}{\partial \eta^2}\cdot 3\right) \\ \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)'_x = \frac{\partial^2 u}{\partial \xi^2}\cdot 1 + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot 3 - \left(\frac{\partial^2 u}{\partial \eta \partial \xi}\cdot 1 + \frac{\partial^2 u}{\partial \eta^2}\cdot 3\right) \\ \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)'_y = \frac{\partial^2 u}{\partial \xi^2}\cdot 1 + \frac{\partial^2 u}{\partial \xi \partial \eta}(-1) - \left(\frac{\partial^2 u}{\partial \eta \partial \xi}\cdot 1 + \frac{\partial^2 u}{\partial \eta^2}(-1)\right) \end{cases}$$

即
$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 6\frac{\partial^2 u}{\partial \eta \partial \xi} + 9\frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \eta \partial \xi} - 3\frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \end{cases}$$

且 $2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 2\left(\frac{\partial u}{\partial \xi} + 3\frac{\partial u}{\partial \eta}\right) + 6\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 8\frac{\partial u}{\partial \xi}$，从而

$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 0$，化为

$$\frac{\partial^2 u}{\partial \xi^2}(1+2-3) + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot(6+4+6) + \frac{\partial^2 u}{\partial \eta^2}(9-6-3) + 8\frac{\partial u}{\partial \xi} = 0$$

即 $16\frac{\partial^2 u}{\partial \xi \partial \eta} + 8\frac{\partial u}{\partial \xi} = 0 \Rightarrow \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{1}{2}\frac{\partial u}{\partial \xi} = 0$. (3).

例3.(exq.3/13):

设$u=f(x,y,z)$，$g(x^2,e^y,z)=0$，$y=\sin x$，且$f,g\in C^1$，$\frac{\partial g}{\partial z}\neq 0$．求$\frac{du}{dx}$.

解：(1)．从$g(x^2,e^{\sin x},z)=0$及$g'_z=\frac{\partial g}{\partial z}\neq 0$，知．~~由隐函数~~

由方程$g(x^2,e^{\sin x},z)=0$可确定z是x,y的函数，从而

z是x的复合函数．故从$u=f(x,y,z)$知．u是x的一元函数．

$$\frac{du}{dx}=f'_1\cdot 1+f'_2\cdot y'_x+f'_3\cdot z'_x=f'_1+f'_2\cdot\cos x+f'_3\cdot z'_x.$$

令$F(x,y,z)=g(x^2,e^y,z)$，则$\begin{cases}F'_x(x,y,z)=g'_1\cdot 2x+g'_2 e^{\sin x}\cos x\\ F'_z(x,y,z)=g'_3\cdot 1=g'_3.\end{cases}$

$$z'_x=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}=-\frac{2xg'_1+e^{\sin x}\cos x\, g'_2}{g'_3}$$

故$\frac{du}{dx}=f'_1\cdot 1+f'_2\cdot y'_x+f'_3\cdot\left(-\frac{2xg'_1+e^{\sin x}\cos x\, g'_2}{g'_3}\right)$.

例4．说明：全微分也是有一阶微分形式不变性。

即，若$f(x,y)$可微，则不论x,y是自变量，还是中间变量，则$z=f(x,y)$总有：

$$dz = df(x,y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad (*_1)$$

证：(1) 当 x, y 是自变量且 $z=f(x,y)$ 可微时，有

$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$，即 $(*_1)$ 成立.

(2) 当 $z=f(x,y)$ 可微，$\begin{cases} x=g(s,t) \\ y=h(s,t) \end{cases}$ 可微，且 $f(g(s,t),h(s,t))$ 有意义时，z 通过中间变量 x, y 成了变量 s, t 的函数，此时.

有：$dx = \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt$，$dy = \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt$. 且

$$dz = \frac{\partial z}{\partial s}ds + \frac{\partial z}{\partial t}dt = \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)ds + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)dt$$

$$= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt\right)$$

$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$，即 x, y 是中间变量时，$(*)$ 仍成立。

利用全微分的一阶微分形式不变性，可导出多元可微函数的如下的微分四则运算法则：

(1). $d(u \pm v) = du \pm dv$；(2). $d(u \cdot v) = vdu + udv$

(3) $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$. 其中，$u, v$ 均可微，且 $v \neq 0$.

证(1). 令 $f(u,v)=u+v$，则 $f(u,v)\in C^1$，从而 $f(u,v)$ 可微.

无论 u,v 是自变量，还是中间变量，总有：

$$d(u+v)=df(u,v)=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=1du+1dv=du+dv$$

从而有：$d(u\pm v)=du\pm dv$. 这里 d 是全微分。

证(3). 令 $f(u,v)=\frac{u}{v}$，$(v\neq0)$，则 $f\in C^1 \Rightarrow d(\frac{u}{v})=df(u,v)$

$$=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=\frac{1}{v}du+(\frac{-u}{v^2})dv=\frac{vdu-udv}{v^2}.$$

设 $f(x,y)\in C^2$，则 $z=f(x,y)\Rightarrow dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$d(dz)\hat{=}d^2z=d(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy)=(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy)'_x dx+(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy)'_y dy$$

$\underline{\underline{dx,dy\text{视为常量}}}$ $(\frac{\partial^2 z}{\partial x^2}dx+\frac{\partial^2 z}{\partial x\partial y}dy)dx+(\frac{\partial^2 z}{\partial y\partial x}dx+\frac{\partial^2 z}{\partial y^2}dy)dy$

$$=\frac{\partial^2 z}{\partial x^2}dx^2+\frac{\partial^2 z}{\partial x\partial y}dydx+\frac{\partial^2 z}{\partial y\partial x}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2 \quad (\text{*2})$$

(*2)是 x,y 是自变量时的 $z=f(x,y)$ 的二阶微分. 二阶及以上的微分通常不再有形式不变性。

例5.(ex9.3/11(3)).设$u=u(x,y)$, $v=v(x,y)$是由方程组:

$$\begin{cases} u=f(ux, v+y) \\ v=g(u-x, v^2y) \end{cases}$$ 所确定的隐函数组,求 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

的Jacobi(雅可比)行列式 $\begin{vmatrix} u'_x, u'_y \\ v'_x & v'_y \end{vmatrix} \triangleq \frac{\partial(u,v)}{\partial(x,y)}$, $f,g \in C^1$.

解:令$A=ux$, $B=v+y$, $E=u-x$, $F=v^2y$,则 $\begin{cases} u=f(A,B) \\ v=g(E,F). \end{cases}$

方程组两边关于x求偏导: $\begin{cases} u'_x=f'_1\cdot(u+xu'_x)+f'_2\cdot(v'_x+0) \\ v'_x=g'_1\cdot(u'_x-1)+g'_2\cdot 2vv'_xy \end{cases}$

标准化: $\begin{cases} (xf'_1-1)u'_x+f'_2\cdot v'_x=-uf'_1 \\ g'_1u'_x+(2vyg'_2-1)v'_x=g'_1 \end{cases}$

令$D=\begin{vmatrix} xf'_1-1, & f'_2 \\ g'_1 & 2vyg'_2-1 \end{vmatrix}$. 则$D\neq 0$,且$D_1=\begin{vmatrix} -uf'_1 & f'_2 \\ g'_1 & 2vyg'_2-1 \end{vmatrix}$

$D_2=\begin{vmatrix} xf'_1-1 & -uf'_1 \\ g'_1 & g'_1 \end{vmatrix}$,由克莱姆(Cramer)法则

$u'_x=\frac{D_1}{D}$, $v'_x=\frac{D_2}{D}$.

方程组 $\begin{cases} u=f(A,B) \\ v=g(E,F) \end{cases}$ 两边对y求偏导.

$$\begin{cases} u'_y=f'_1\cdot xu'_y+f'_2\cdot(v'_y+1) \\ v'_y=g'_1\cdot u'_y+g'_2\cdot(2vv'_yy+v^2) \end{cases} \Leftrightarrow \begin{cases} (xf'_1-1)u'_y+f'_2v'_y=-f'_2 \\ g'_1u'_y+(2vg'_2y-1)v'_y=-g'_2v^2. \end{cases}$$

(D).

$D=\begin{vmatrix} xf_1'-1 & f_2' \\ g_1' & 2vyg_2'-1 \end{vmatrix}\neq 0$, $\tilde{D}_1=\begin{vmatrix} -f_2' & f_2' \\ g_2'v^2, & 2vyg_2'-1 \end{vmatrix}$, $\tilde{D}_2=\begin{vmatrix} xf_1'-1 & -f_2' \\ g_1' & g_2'v^2 \end{vmatrix}$

$u_y'=\frac{\tilde{D}_1}{D}$, $v_y'=\frac{\tilde{D}_2}{D}$, $\Rightarrow$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x' & u_y' \\ v_x' & v_y' \end{vmatrix}=\begin{vmatrix} D_1 & \tilde{D}_1 \\ D_2 & \tilde{D}_2 \end{vmatrix}/D.$$

例6 (ex9.3/15).

设 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$ 是由方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 确定的隐函数组，$F,G\in C^1$ 且 $\frac{\partial(F,G)}{\partial(u,v)}\neq 0$. 求 du, dv.

解：$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$; $dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$

在 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 两边关于 x 求偏导得：

$\begin{cases} F_x'\cdot 1+F_u'\cdot\frac{\partial u}{\partial x}+F_v'\frac{\partial v}{\partial x}=0 \\ G_x'\cdot 1+G_u'\frac{\partial u}{\partial x}+G_v'\frac{\partial v}{\partial x}=0 \end{cases}$，整理得：$\begin{cases} F_u'\frac{\partial u}{\partial x}+F_v'\frac{\partial v}{\partial x}=-F_x' \\ G_u'\frac{\partial u}{\partial x}+G_v'\frac{\partial v}{\partial x}=-G_x' \end{cases}$

令 $D=\begin{vmatrix} F_u' & F_v' \\ G_u' & G_v' \end{vmatrix}$. 则 $D=\frac{\partial(F,G)}{\partial(u,v)}\neq 0$.

令 $D_1=\begin{vmatrix} -F_x' & F_v' \\ -G_x' & G_v' \end{vmatrix}$, $D_2=\begin{vmatrix} F_u' & -F_x' \\ G_u' & -G_x' \end{vmatrix}$, 则 $D_1=\begin{vmatrix} F_v' & F_x' \\ G_v' & G_x' \end{vmatrix}=\frac{\partial(F,G)}{\partial(v,x)}$

$D_2=\begin{vmatrix} F_u' & -F_x' \\ G_u' & -G_x' \end{vmatrix}=\begin{vmatrix} F_x' & F_u' \\ G_x' & G_u' \end{vmatrix}=\frac{\partial(F,G)}{\partial(x,u)}$，依Cramer(克莱姆)

法则，$\frac{\partial u}{\partial x}=\frac{D_1}{D}=\frac{\partial(F,G)}{\partial(v,x)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$，$\frac{\partial v}{\partial x}=\frac{D_2}{D}=\frac{\partial(F,G)}{\partial(x,u)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

对方程组两边关于y求偏导，同理可得：

$\frac{\partial u}{\partial y}=\frac{\partial(F,G)}{\partial(v,y)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$，$\frac{\partial v}{\partial y}=\frac{\partial(F,G)}{\partial(y,u)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$，于是，

$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\left(\frac{\partial(F,G)}{\partial(v,x)}dx+\frac{\partial(F,G)}{\partial(v,y)}dy\right)\Big/\frac{\partial(F,G)}{\partial(u,v)}$；

$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=\left(\frac{\partial(F,G)}{\partial(x,u)}dx+\frac{\partial(F,G)}{\partial(y,u)}dy\right)\Big/\frac{\partial(F,G)}{\partial(u,v)}$.

(二) 作业：ex9.2/31；

ex9.3/6；7；8；10；11/(1)；14.

(三). 本下讲：方向导数与梯度 及 Jacobi行列式

(2023.3.20)