5.15

ex11.3: 1(6);

1.计算下列第二型曲线积分.

(6)$\int_L ydx+zdy+xdz$，L是$x+y=2$与$x^2+y^2+z^2=2(x+y)$的交线，从原点看去是顺时针方向.

这两个曲面的交线即为$x+y=2$与$x^2+y^2+z^2=4$的交线，这是一个封闭的曲线.

$Stokes$定理:

$\int_L ydx+zdy+xdz=\iint_S -1dydz-1dydz-1dxdy=\iint_S(-1,-1,-1)(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)dS=-2\sqrt{2}\pi$

ex11.4: 2;

2.求场$v=(x^3-yz)i-2x^2yj+zk$通过长方体$0\le x\le a,0\le y\le b,0\le z\le c$的外侧表面$S$的通量.

上周已做，$\frac{1}{3}a^3bc+abc$

ex11.5: 1(6),9(1)(2)(6)

1.计算下列曲面积分.

(6)$\iint_S(y^2+z^2)dydz+(z^2+x^2)dzdx+(x^2+y^2)dxdy$，$S$是上半球面$x^2+y^2+z^2=a^2(z\ge 0)$的上侧.

上周已做，$\frac{1}{2}\pi a^4$

9.计算下列曲线积分.

(1)$\oint_L ydx+zdy+xdz$，L是顶点为$A(1,0,0),B(0,1,0),C(0,0,1)$的三角形边界，从原点看去，$L$沿顺时针方向.

$Stokes$定理：

$LHS=\iint_S(-dydz-dzdx-dxdy)dS=\iint_S(-1,-1,-1)(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})dS=-\sqrt{3}\times\frac{\sqrt{3}}{4}\times 2=-\frac{3}{2}$

(2)$\oint_L(y-z)dx+(z-x)dy+(x-y)dz$，$L$是圆柱面$x^2+y^2=a^2$和平面$\frac{x}{a}+\frac{z}{h}=1(a>0,h>0)$的交线，从$x$轴正向看来，$L$沿逆时针方向.

$Stokes$定理：

$LHS = \iint_S(-2dydz - 2dzdx - 2dxdy) = \iint_S(-2,-2,-2)(\frac{h}{\sqrt{h^2+a^2}}, 0, \frac{a}{\sqrt{h^2+a^2}})dS$

$= \frac{-2(a+h)}{\sqrt{h^2+a^2}} \times \pi \times a \times \sqrt{h^2+a^2} = -2\pi a(a+h)$

(6)$\oint_L (y^2-z^2)dx + (2z^2-x^2)dy + (3x^2-y^2)dz$，其中$L$是平面$x+y+z=2$与柱面$|x|+|y|=1$的交线，从$z$轴正向看去，$L$为逆时针方向.

$Stokes$定理：

$LHS = \iint_S(-2y-4z)dydz + (-2z-6x)dzdx + (-2x-2y)dxdy$

$= \iint_S(-2y-4z, -2z-6x, -2x-2y)(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})dS = \frac{\sqrt{3}}{3}\iint_S(-8x-4y-6z)dS$

$= -\frac{\sqrt{3}}{3} \times 12 \times 2\sqrt{2} = -8\sqrt{6}$

5.17

ex11.7: 2,3(2),4,5(2),6(5)(6),10,13

2.求下列曲线积分.

(1)$\int_L (2x+y)dx + (x+4y+2z)dy + (2y-6z)dz$，其中$L$由点$P_1(a,0,0)$沿曲线$\begin{cases} x^2+y^2=a^2 \\ z=0 \end{cases}$到点$P_2(0,a,0)$，再由$P_2(0,a,0)$沿直线$\begin{cases} x+y=a \\ x=0 \end{cases}$到点$P_3(0,0,a)$.

$\nabla(x^2+xy+2y^2+2yz-3z^2) = (2x+y, x+4y+2z, 2y-6z)$

因此$LHS = x^2+xy+2y^2+2yz-3z^2|_{(a,0,0)}^{(0,0,a)} = -4a^2$

(2)$\int_{\overset{\frown}{AMB}} (x^2-yz)dx + (y^2-zx)dy + (z^2-xy)dz$，其中$\overset{\frown}{AMB}$是柱面螺线$x = a\cos\varphi, y = a\sin\varphi, z = \frac{h}{2\pi}\varphi$上点$A(a,0,0)$到$B(a,0,h)$这一段.

$\nabla(\frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 - xyz) = (x^2-yz, y^2-zx, z^2-xy)$

因此$LHS = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 - xyz|_{(a,0,0)}^{(a,0,h)} = \frac{1}{3}h^3$

3.证明下列向量场是有势场，并求出它们的势函数.

(2)$v = yz(2x+y+z)i + xz(2y+z+x)j + xy(2z+x+y)k$

$\nabla(x^2yz + xy^2z + xyz^2) = v$

因此$u = x^2yz + xy^2z + xyz^2 + C$

4.当a取何值时，向量场$F=(x^2+5ay+3yz)i+(5x+3axz-2)j+[(a+2)xy-4z]k$是有势场？并求出这时的势函数.

$\nabla\times F=((a+2)x-3ax,3y-(a+2)y,5+3az-3z-5a)=(0,0,0)$

所以$a=1$

$F=(x^2+5y+3yz)i+(5x+3xz-2)j+(3xy-4z)k$

$\nabla(\frac{1}{3}x^3+5xy+3xyz-2y-2z^2)=F$

所以势函数$\varphi=\frac{1}{3}x^3+5xy+3xyz-2y-2z^2+C$

5.求下列全微分的原函数u:

(2)$du=(x^2-2yz)dx+(y^2-2xz)dy+(z^2-2xy)dz$

$\nabla(\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{1}{3}z^3-2xyz)=(x^2-2yz,y^2-2xz,z^2-2xy)$

所以$u=\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{1}{3}z^3-2xyz+C$

6.验证下列积分与路径无关，并求出它们的值

(5)$\int_{(1,1,1)}^{(2,2,2)}(1-\frac{1}{y}+\frac{y}{z})dx+(\frac{x}{z}+\frac{x}{y^2})dy-\frac{xy}{z^2}dz$

$\nabla(x-\frac{x}{y}+\frac{xy}{z})=(1-\frac{1}{y}+\frac{y}{z},\frac{x}{z}+\frac{x}{y^2},-\frac{xy}{z^2})$

因此$LHS=x-\frac{x}{y}+\frac{xy}{z}|_{(1,1,1)}^{(2,2,2)}=3-1=2$

(6)$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}$，其中$(x_1,y_1,z_1),(x_2,y_2,z_2)$在球面$x^2+y^2+z^2=a^2$上

$\nabla(\sqrt{x^2+y^2+z^2})=(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}})$

所以$LHS=\sqrt{x^2+y^2+z^2}|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}=0$

10.已知$\alpha(0)=0,\alpha'(0)=2,\beta(0)=2$

(1)求$\alpha(x),\beta(x)$使曲线积分$\int_L Pdx+Qdy$与路线无关，其中
$P(x,y)=[2x\alpha'(x)+\beta(x)]y^2-2y\beta(x)\tan 2x$，$Q(x,y)=[\alpha'(x)+4x\alpha(x)]y+\beta(x)$

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$

$[\alpha''(x)+4\alpha(x)+4x\alpha'(x)]y+\beta'(x)-2[2x\alpha'(x)+\beta(x)]y+2\beta(x)\tan 2x=0$

$\alpha''(x)+4\alpha(x)-2\beta(x)=0,\beta'(x)+2\beta(x)\tan 2x=0$

$\beta(x)=2\cos 2x$

$\alpha(x)=x\sin 2x+a\cos 2x+b\sin 2x$，代入初值得$\alpha(x)=x\sin 2x+\sin 2x$

(2)求$\int_{(0,0)}^{(0,2)} Pdx+Qdy$

$[(2x(\sin 2x+2\cos 2x+2x\cos 2x)+2\cos 2x)y^2-4y\sin 2x,$

$((\sin 2x+2\cos 2x+2x\cos 2x)+4x(x\sin 2x+\sin 2x))y+2\cos 2x]$

$=\nabla(\frac{1}{2}\sin 2x+\cos 2x+x\cos 2x+2x^2\sin 2x+2x\sin 2x)y^2+2y\cos 2x$

所以$LHS=(\frac{1}{2}\sin 2x+\cos 2x+x\cos 2x+2x^2\sin 2x+2x\sin 2x)y^2+2y\cos 2x|_{(0,0)}^{(0,2)}=8$

13.设$f(x)$具有二阶连续导数，$f(0)=0,f'(0)=2$，且$[e^x\sin y+x^2y+f(x)y]dx+[f'(x)+e^x\cos y+2x]dy=0$为一全微分方程. 求$f(x)$及此全微分方程的通解.

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$

$f''(x)+e^y\cos y+2-e^x\cos y-x^2-f(x)=0,f''(x)-f(x)=x^2-2$

$f(x)=-x^2+Ae^x+Be^{-x}$，代入初始条件得$A=1,B=-1$，$f(x)=-x^2+e^x-e^{-x}$

此时$P=e^x\sin y+x^2y+(-x^2+e^x-e^{-x})y=e^x\sin y+(e^x-e^{-x})y$
$Q=-2x+e^x+e^{-x}+e^x\cos y+2x=e^x\cos y+e^x+e^{-x}$

$\nabla(e^x\sin y+e^xy+e^{-x}y)=(P,Q)$

通解为$e^x\sin y+e^xy+e^{-x}y+C=0$

5.19

ex7.1: 1(1),2(4)(5)(6)(9)(11),5,7

1.证明下列等式

(1)$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}$

$LHS=\sum_{n=1}^{\infty}\frac{1}{2}(\frac{1}{2n-1}-\frac{1}{2n+1})=\frac{1}{2}$

2.研究下列级数的敛散性

(4)$\sum_{n=1}^{\infty}\sin n$

发散, 因为$\lim_{n\to\infty}\sin n\neq 0$

(5)$\sum_{n=1}^{\infty}2^n\sin\frac{\pi}{3^n}$

收敛, 因为$LHS\leq\sum_{n=1}^{\infty}\pi(\frac{2}{3})^n$

(6)$\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$

发散, 因为$\sqrt[n]{n}\leq 2$, 所以有$\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}\geq\sum_{n=1}^{\infty}\frac{1}{2n}$

(9)$\sum_{n=1}^{\infty}\arctan\frac{\pi}{4n}$

发散, 记$a_n=\frac{\pi}{4n}, b_n=\arctan\frac{\pi}{4n}$,则$\lim_{n\to\infty}\frac{a_n}{b_n}=1$,两者同敛散

(11)$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$

收敛, $\frac{(n!)^2}{(2n)!}=\frac{1\times 2\times\cdots\times n}{(n+1)(n+2)\cdots(2n)}\leq\frac{1}{2^n}$

5.设正项级数$\sum_{n=1}^{\infty}a_n$收敛，证明：$\sum_{n=1}^{\infty}a_n^2$也收敛。试问反之是否成立?

$\sum_{n=1}^{\infty}a_n$收敛，因此n充分大时$a_n<1$，此时$a_n^2<a_n$，因此$\sum_{n=1}^{\infty}a_n^2$也收敛。

反之不成立，$a_n=\frac{1}{n}$

7.证明：若级数$\sum_{n=1}^{\infty}a_n^2$和$\sum_{n=1}^{\infty}b_n^2$收敛，则级数$\sum_{n=1}^{\infty}|a_nb_n|, \sum_{n=1}^{\infty}(a_n+b_n)^2$以及$\sum_{n=1}^{\infty}\frac{|a_n|}{n}$也收敛。

(1)

$|a_nb_n|\leq\frac{a_n^2+b_n^2}{2}$

(2)

$(a_n + b_n)^2 \leq 2(a_n^2 + b_n^2)$

(3)

取$b_n = \frac{1}{n}$，再利用(1)即可