第25讲：重积分复习与小结　2023.4.24

(一) 计算下列重积分 I

(1). $\iiint\limits_{\Omega} z\,dV$，$\Omega$ 由 $x^2+y^2+z^2=4$ 与 $x^2+y^2=3z$ 围成的区域.

(2). $\iint\limits_{D} \mathrm{sgn}(x^2-y^2+2)\,d\sigma$，$D$: $x^2+y^2\le 4$.　(ch10练/1)

(3). $\int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\,dx$　$(a>0, b>a)$

(4). $\int_0^1 \cos\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\,dx$，$(b>a>0)$　(ch10练/3)

(5). $\iint\limits_{|x|+|y|\le 1} \frac{af(x)+bf(y)}{f(x)+f(y)}\,dxdy$，$f>0$，$f\in C$，$a,b$ 为常数.

(6). $\iint\limits_{D} \frac{dxdy}{(1+x^2+y^2)^{\frac{3}{2}}}$，$D$: $\begin{cases} 0\le x\le 1 \\ 0\le y\le 1 \end{cases}$.

(7). $\iint\limits_{D} x(1+ye^{x^4+x^6})\,dxdy$，区域 D 由 $y=\sin x$，$x=-\frac{\pi}{2}$，$y=1$ 围成.

(8). $\iint\limits_{D} \left|\frac{x+y}{2}-x^2-y^2\right|dxdy$，$D$: $x^2+y^2\le 1$.　(ch10练/4)

(9). $\iiint\limits_{\Omega} (x^2+y^2+z^2)\,dV$，$\Omega$: $x^2+y^2+z^2\le x+y+z$.

(二) 证明题:

(1) 设 $f\in C$, 则 $\int_a^b e^{f(x)}dx\int_a^b e^{-f(y)}dy \ge (b-a)^2$. (ex10.2/5)

(2) $f\in C$ 且 $f(-x)=-f(x)$, 则 $\iint\limits_{|x|+|y|\le a} e^{f(x+y)}dxdy \ge 2a^2$. (ex10.2/6)

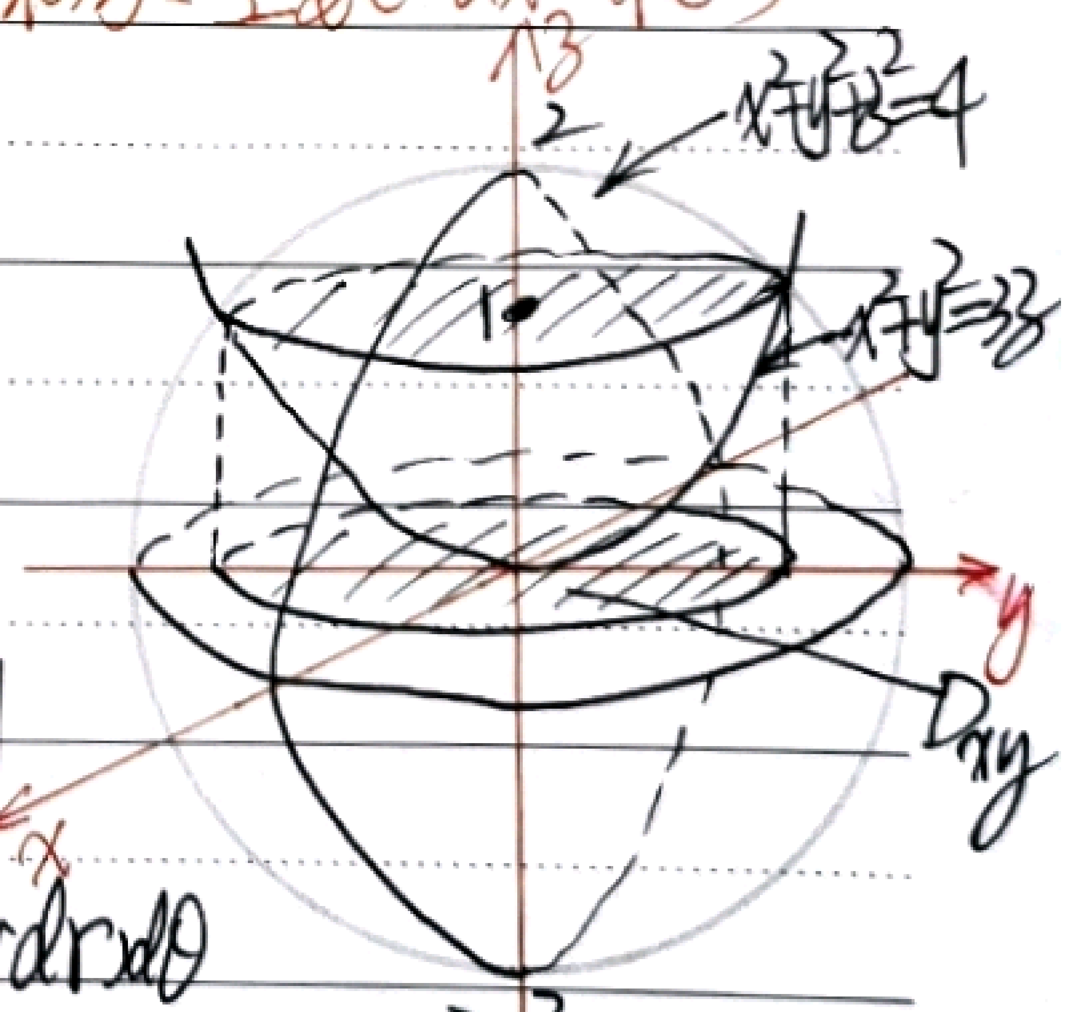
(3) 设 $a,b,\sigma_1,\sigma_2,\rho$ 为常数, $\sigma_1>0,\sigma_2>0$, $|\rho|<1$.

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-a}{\sigma_1}\right)^2-2\rho\left(\frac{x-a}{\sigma_1}\right)\left(\frac{y-b}{\sigma_2}\right)+\left(\frac{y-b}{\sigma_2}\right)^2\right]}$$

证明: $I=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)dxdy=1$.

(这说明, $f(x,y)$ 就是二维正态分布 $N(a,b,\sigma_1^2,\sigma_2^2,\rho)$ 的概率密度函数。(提示: 利用概率积分 $\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}dx=\sqrt{2\pi}$)

解 (一)/(1): "先一后二"法:



$$I=\iint\limits_{x^2+y^2\le 3}\left(\int_{\frac{1}{3}(x^2+y^2)}^{\sqrt{4-x^2-y^2}}z\,dz\right)dxdy$$

$$=\frac{1}{2}\iint\limits_{x^2+y^2\le 3}\left[(4-x^2-y^2)-\frac{1}{9}(x^2+y^2)^2\right]dxdy$$

$$\xlongequal[y=r\sin\theta]{x=r\cos\theta}\frac{1}{2}\int_0^{2\pi}\left(\int_0^{\sqrt{3}}\left[(4-r^2)-\frac{1}{9}r^4\right]rdr\right)d\theta$$

D_{xy}: $x^2+y^2\le 3$.

$$=\frac{1}{2}\int_0^{2\pi}d\theta\left(\int_0^{\sqrt{3}}(4r-r^3-\frac{1}{9}r^5)dr\right)$$

$$=\frac{13}{4}\pi.$$

(2).

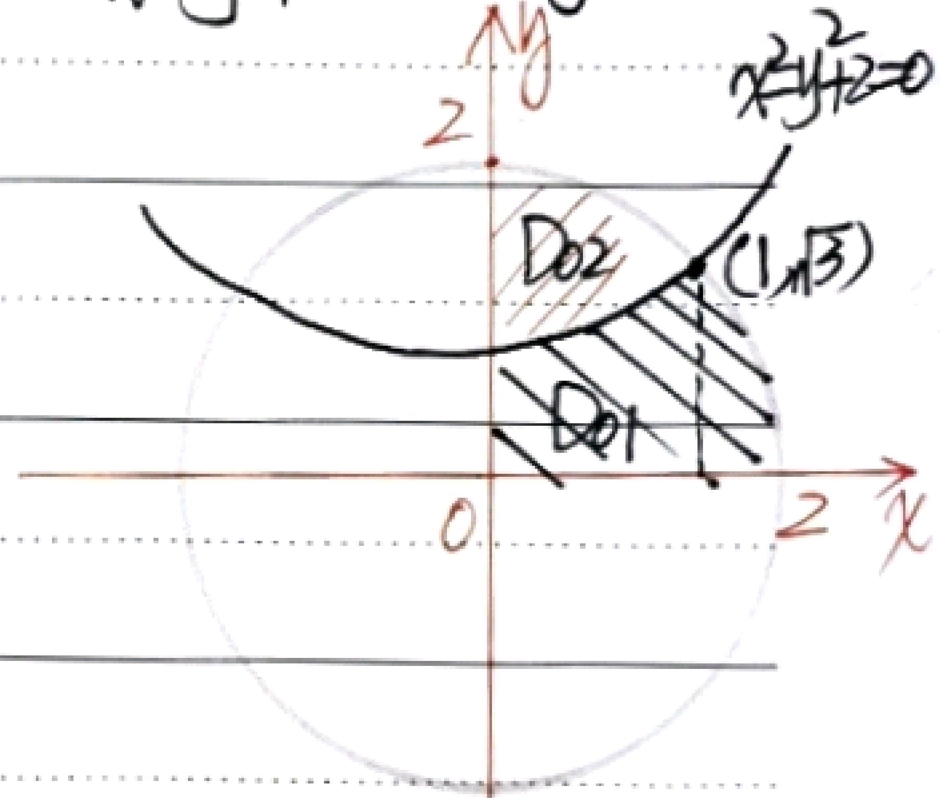
解(2)：先二后一法：

$$I=\iiint\limits_{\substack{x^2+y^2\le 3z\\0\le z\le 1}} z\,dV+\iiint\limits_{\substack{x^2+y^2\le 4-z^2\\1\le z\le 2}} z\,dV=\int_0^1 z\,dz\iint\limits_{x^2+y^2\le 3z} 1\,dxdy+\int_1^2 z\,dz\iint\limits_{x^2+y^2\le 4-z^2} 1\,dxdy$$

$$=\int_0^1 z\pi\cdot 3z\,dz+\int_1^2 z\pi(4-z^2)\,dz=\frac{13}{4}\pi.$$

解(一)/(2) (1°) $\operatorname{sgn}(x^2-y^2+2)=\begin{cases}1 & x^2-y^2+2>0\Leftrightarrow y^2<x^2+2\\ -1 & x^2-y^2+2<0\Leftrightarrow y^2>x^2+2\end{cases}$

(2°) 由奇偶对称性得知：



$$I=4\iint\limits_{D_0}\operatorname{sgn}(x^2-y^2+2)\,dxdy$$

$$=4\Big[\iint\limits_{D_{01}}(+1)\,dxdy+\iint\limits_{D_{02}}(-1)\,dxdy\Big]$$

$$=4\Big[\iint\limits_{D_{01}\cup D_{02}}(+1)\,dxdy-2\iint\limits_{D_{02}}dxdy\Big]$$

$$=4\Big[S(D_0)-2\int_0^1\Big(\int_{\sqrt{x^2+2}}^{\sqrt{4-x^2}}1\,dy\Big)dx\Big]$$

$$=4\Big[\frac{1}{4}\times\pi\cdot 2^2-2\int_0^1\big(\sqrt{4-x^2}-\sqrt{x^2+2}\big)dx\Big]$$

$$=4\pi-8\int_0^1\sqrt{4-x^2}\,dx+8\int_0^1\sqrt{x^2+2}\,dx=\frac{4}{3}\pi+4\ln(2+\sqrt{3}).$$

$D_0=D_{01}\cup D_{02}$

$D_0:\begin{cases}x^2+y^2\le 4\\ x\ge 0,\ y\ge 0.\end{cases}$

解(一)/(3). 利用 $\int_a^b x^u\,du=\frac{x^u}{\ln x}\Big|_a^b=\frac{x^b-x^a}{\ln x}\Rightarrow$

$$I=\int_0^1\Big(\sin\big(\ln\tfrac{1}{x}\big)\int_a^b x^u\,du\Big)dx=\int_0^1\int_a^b x^u\sin\big(\ln\tfrac{1}{x}\big)\,du\,dx$$

改变积分顺序 $\int_a^b\left(\int_0^1 x^u\sin(\ln\frac{1}{x})dx\right)du$

记 $I_0 \triangleq \int_0^1 x^u\sin(\ln\frac{1}{x})dx$ 中，令 $\ln\frac{1}{x}=v$，则 $x=e^{-v}$，$dx=-e^{-v}dv$，$I_0=\int_{+\infty}^{0}e^{-vu}\sin v(-e^{-v}dv)=\int_0^{+\infty}e^{-v(u+1)}\sin v\,dv$

~~通过分部积分~~ ~~再解方程~~ $\frac{1}{1+(1+u)^2}$，

故 $I=\int_a^b\frac{1}{1+(1+u)^2}du=\arctan(1+u)\Big|_a^b=\arctan(b+1)-\arctan(a+1)$.

例(一)/(4). 例写同(一)/(3). 答案：$\ln\sqrt{\frac{1+(b+1)^2}{1+(a+1)^2}}$.

例(一)/(5). 先证明当积分区域D的边界∂D满足：~~交换~~ 交换 x,y 的位置后，∂D的表达式不变时，有：$\iint\limits_D g(x,y)dxdy=\iint\limits_D g(y,x)dxdy$

证：作变换 $\begin{cases}x=y\\y=x\end{cases}$，则 $dxdy=\left|\frac{\partial(x,y)}{\partial(y,x)}\right|dydx=\left|\begin{vmatrix}0&1\\1&0\end{vmatrix}\right|dydx=1dydx=dxdy$. 从而 $\iint\limits_D g(x,y)dxdy\xlongequal[y=x]{\text{令}x=y}\iint\limits_D g(y,x)dxdy$.

$\because I=\iint\limits_{|x|+|y|\le1}\frac{af(x)+bf(y)}{f(x)+f(y)}dxdy=\iint\limits_{|x|+|y|\le1}\frac{af(y)+bf(x)}{f(y)+f(x)}dxdy$

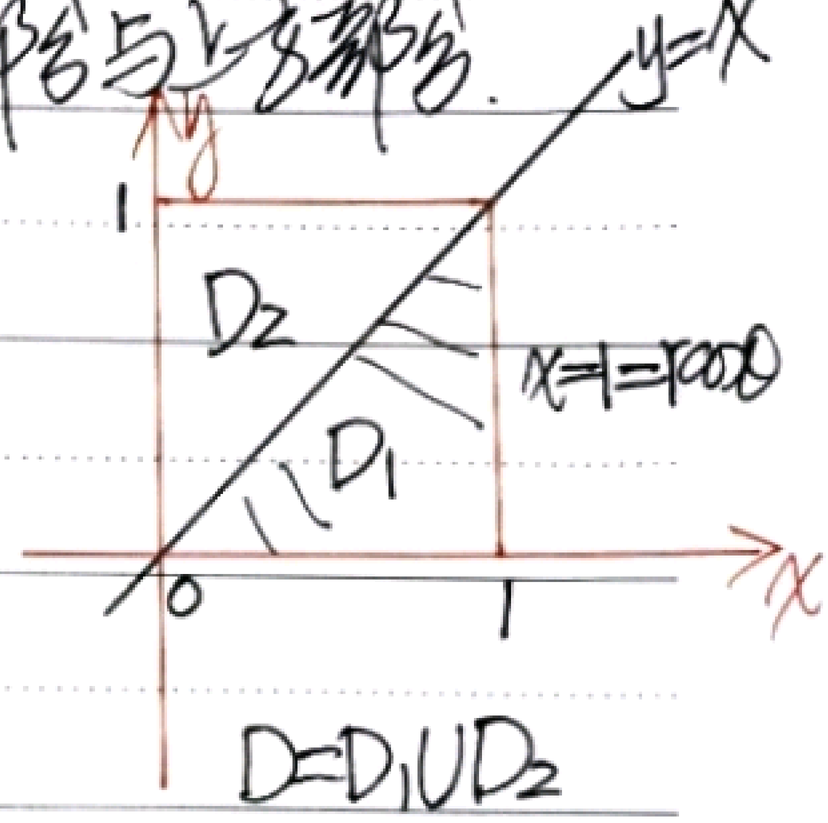
$\therefore 2I=\iint\limits_{|x|+|y|\le1}\frac{a\cancel{(f(x)+f(y))}+b\cancel{(f(x)+f(y))}}{\cancel{f(x)+f(y)}}dxdy=\iint\limits_{|x|+|y|\le1}(a+b)dxdy$

$=(a+b)S(D)=(a+b)(\sqrt{2})^2=2(a+b)$ $\therefore I=a+b$.

(4).

评(一)/(6)：先证明当积分区域关于 $y=x$ 对称且 $f(y,x)=f(x,y)$ 时，有 $\iint\limits_D f(x,y)dxdy=\iint\limits_D f(y,x)dxdy=2\iint\limits_{D_1} f(x,y)dxdy=2\iint\limits_{D_2} f(y,x)dxdy$

其中 D_1,D_2 是 D 在直线 $y=x$ 的下方部分与上方部分.



证：$\iint\limits_D f(x,y)dxdy=\iint\limits_{D_1\cup D_2} f(x,y)dxdy$

$=\iint\limits_{D_1} f(x,y)dxdy+\iint\limits_{D_2} f(x,y)dxdy$

而 $\iint\limits_{D_2} f(x,y)dxdy \xlongequal[y=x]{令x=y} \iint\limits_{D_1} f(y,x)dxdy$

$\xlongequal{f(y,x)=f(x,y)} \iint\limits_{D_1} f(x,y)dxdy$. 故 $\iint\limits_D f(x,y)d\sigma=2\iint\limits_{D_1} f(x,y)d\sigma$.

(2) $\because f(x,y)=\frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}$ 满足 $f(x,y)=f(y,x)$, $\forall(x,y)\in D$.

且 D 关于 $y=x$ 对称，$\therefore I=2\iint\limits_{D_1}\frac{dxdy}{(1+x^2+y^2)^{\frac{3}{2}}} \xlongequal[y=r\sin\theta]{令x=r\cos\theta}$

$2\cdot\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{1}{\cos\theta}}\frac{rdr}{(1+r^2)^{\frac{3}{2}}}=\frac{2}{2}\int_0^{\frac{\pi}{4}}\left(\int_0^{\sec\theta}(1+r^2)^{-\frac{3}{2}}d(1+r^2)\right)d\theta$

$=2\int_0^{\frac{\pi}{4}}\left(1-\frac{\cos\theta}{\sqrt{1+\cos^2\theta}}\right)d\theta=\pi/6$.

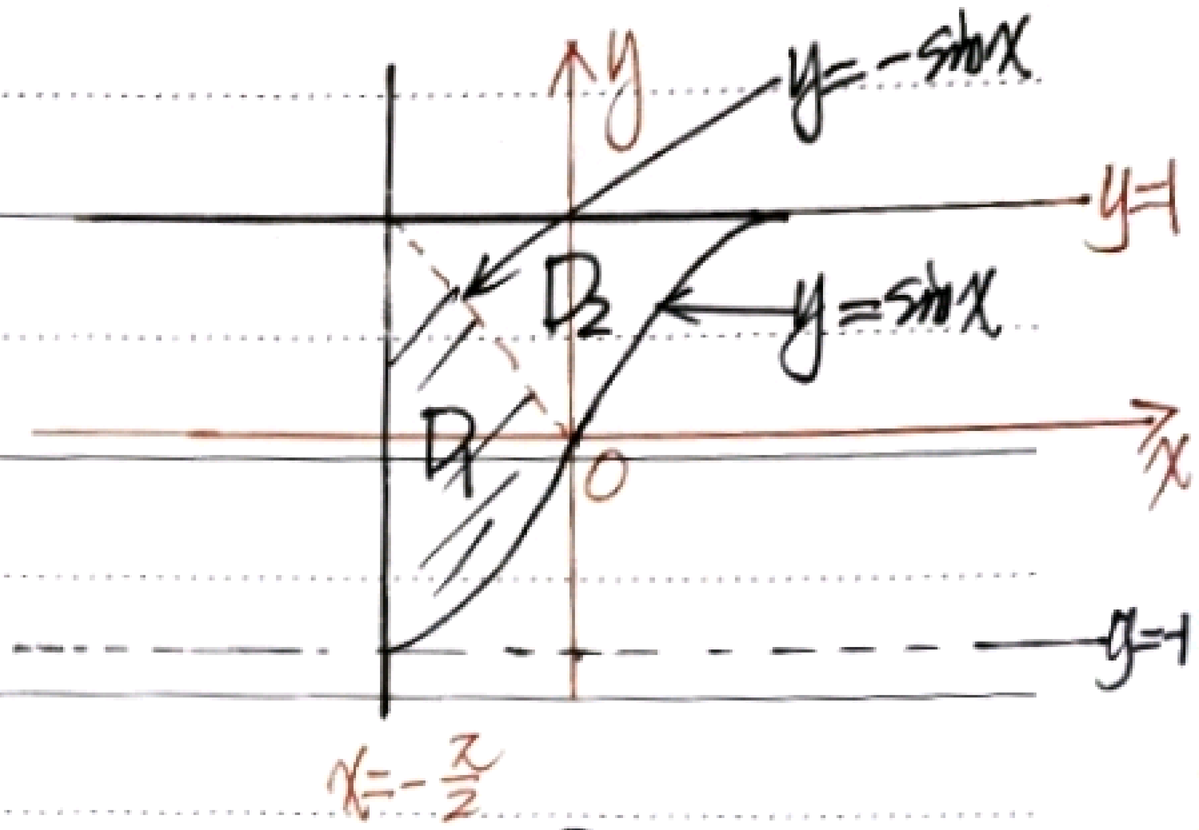
上述第(5)、第(6)题称为二重积分的轮换对称性，三重积分、n重积分中也具有轮换对称性. 如ch10综/2 可利用这种轮换对称性来解。

解(一)/(7)：作辅助线：

$y=-\sin x\,(-\frac{\pi}{2}\le x\le 0)$，则

$D=D_1\cup D_2$ 且 D_1, D_2 分别

关于 $x=0$，$y=0$ 坐标轴对称.



$D=D_1\cup D_2$.

而 $I=\iint\limits_{D}(x+xye^{x^4y^6})dxdy=\iint\limits_{D_1\cup D_2}(x+xye^{x^4y^6})dxdy$

$=\iint\limits_{D_1}xdxdy+\iint\limits_{D_2}xdxdy+\iint\limits_{D_1}xye^{x^4y^6}dxdy+\iint\limits_{D_2}xye^{x^4y^6}dxdy$

且由奇偶对称性，$\iint\limits_{D_2}xdxdy=0$，$\iint\limits_{D_1}xye^{x^4y^6}dxdy=0$.

$\iint\limits_{D_2}xye^{x^4y^6}dxdy=0$. 故

$I=\iint\limits_{D_1}xdxdy=\int_{-\frac{\pi}{2}}^{0}\left(\int_{\sin x}^{-\sin x}xdy\right)dx=\int_{-\frac{\pi}{2}}^{0}x(-2\sin x)dx$

$=2\int_{-\frac{\pi}{2}}^{0}xd\cos x=2x\cos x\Big|_{-\frac{\pi}{2}}^{0}-2\int_{-\frac{\pi}{2}}^{0}\cos xdx=0-2\sin x\Big|_{-\frac{\pi}{2}}^{0}=-2$.

解(一)/(8)：设 $f(x,y)=\frac{x+y}{2}-x^2-y^2$，则 $f(x,y)=\frac{1}{4}-(x-\frac{1}{\sqrt{8}})^2-(y-\frac{1}{\sqrt{8}})^2$

令 D_1：$(x-\frac{1}{\sqrt{8}})^2+(y-\frac{1}{\sqrt{8}})^2\le(\frac{1}{2})^2$，$D_2=D-D_1$. 则 $f(x,y)\begin{cases}\geq 0, & (x,y)\in D_1\\ \le 0, & (x,y)\in D_2\end{cases}$

$$I = \iint_{D_1+D_2} |f(x,y)|\,d\sigma = \iint_{D_1} f(x,y)\,d\sigma + \iint_{D_2} (-f(x,y))\,d\sigma$$

$$= 2\iint_{D_1} f(x,y)\,d\sigma - \iint_{D_1+D_2} f(x,y)\,d\sigma = 2\iint_{D_1} f(x,y)\,d\sigma - \iint_{D} f(x,y)\,d\sigma$$

而 $\iint_D f(x,y)\,d\sigma = \iint_{x^2+y^2\le 1} \left(\frac{x+y}{2} - x^2 - y^2\right) d\sigma = \frac12\iint_{x^2+y^2\le1} x\,d\sigma + \frac12\iint_{x^2+y^2\le1} y\,d\sigma$

$- \iint_{x^2+y^2\le1} (x^2+y^2)\,d\sigma = \frac12\times 0 + \frac12\times 0 - \int_0^{2\pi}\left(\int_0^1 r^2\, r\,dr\right)d\theta = -\frac{\pi}{2}.$

$\iint_{D_1} f(x,y)\,d\sigma = \iint_{D_1}\left[\frac14 - \left(x-\frac{1}{\sqrt8}\right)^2 - \left(y-\frac{1}{\sqrt8}\right)^2\right]dxdy \xrightarrow[y-\frac{1}{\sqrt8}=r\sin\theta]{x-\frac{1}{\sqrt8}=r\cos\theta}$

$\int_0^{2\pi}\left(\int_0^{\frac12}\left[\frac14 - r^2\right] r\,dr\right)d\theta = \int_0^{2\pi} d\theta \times \int_0^{\frac12}\left(\frac{r}{4} - r^3\right)dr = \frac{\pi}{32}.$

$\therefore I = 2\times\frac{\pi}{32} - \left(-\frac{\pi}{2}\right) = \frac{9}{16}\pi.$

例9(一)(9). $I = \iiint_\Omega x^2\,dV + \iiint_\Omega y^2\,dV + \iiint_\Omega z^2\,dV$ 轮换对称性!

而 $\iiint_\Omega z^2\,dV = \iiint_{x^2+y^2+z^2\le x+y+z} z^2\,dxdydz = \iiint_{z^2+y^2+x^2\le z+y+x} x^2\,dxdydz = \iiint_{x^2+z^2+y^2\le x+z+y} y^2\,dxdydz$

$\therefore I = 3\iiint_\Omega z^2\,dV = 3\int_{\frac{1-\sqrt3}{2}}^{\frac{1+\sqrt3}{2}} z^2\,dz \iint_{(x-\frac12)^2+(y-\frac12)^2\le\frac34-(z-\frac12)^2=\frac12+z-z^2} 1\,dxdy$

$= 3\int_{\frac{1-\sqrt3}{2}}^{\frac{1+\sqrt3}{2}} z^2\pi\left(\frac12+z-z^2\right)dz$

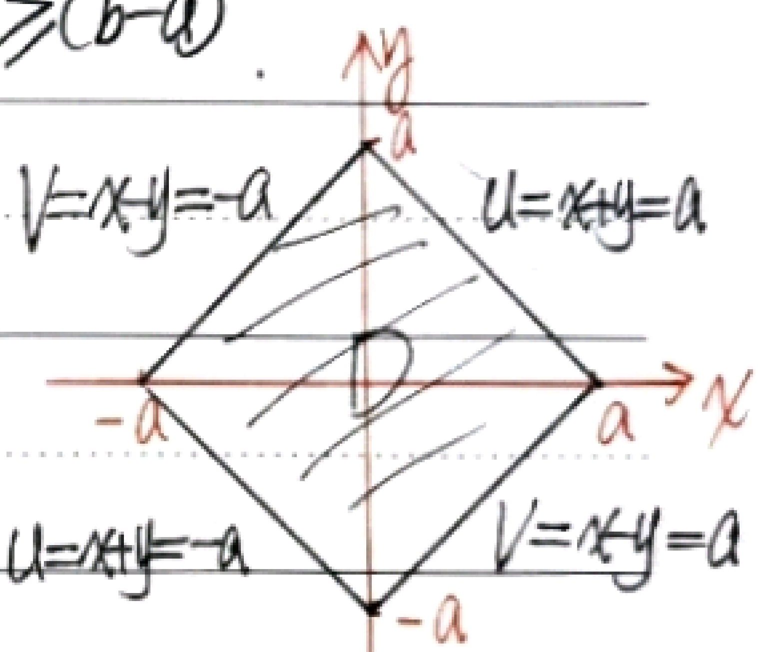
$=3\times\frac{\sqrt{3}}{5}\pi$.

证(三)/(1). ∵ $I=\int_a^b e^{f(x)}dx\times\left(\int_a^b e^{-f(y)}dy\right)=\int_a^b\int_a^b e^{f(x)-f(y)}dxdy$

又 $I=\int_a^b e^{f(y)}dy\times\left(\int_a^b e^{-f(x)}dx\right)=\int_a^b\int_a^b e^{f(y)-f(x)}dxdy$

∴ $2I=\int_a^b\int_a^b\left(e^{f(x)-f(y)}+e^{f(y)-f(x)}\right)dxdy\geqslant\int_a^b\int_a^b 2dxdy=2(b-a)^2$

故 $I\geqslant(b-a)^2$, 即 $\int_a^b e^{f(x)}dx\times\int_a^b e^{-f(y)}dy\geqslant(b-a)^2$.



证(三)/(2). 令 $\begin{cases}x+y=u\\x-y=V\end{cases}$ 则 $\begin{cases}-a\leq u\leq a\\-a\leq V\leq a\end{cases}$.

则 $dxdy=\left|\frac{\partial(x,y)}{\partial(u,V)}\right|dudV=\frac{1}{2}dudV$

$I=\iint\limits_{|x|+|y|\leq a}e^{f(x+y)}dxdy=\int_{-a}^{a}\left(\int_{-a}^{a}e^{f(u)}\frac{1}{2}du\right)dV=a\int_{-a}^{a}e^{f(u)}du$

又 $I=a\int_{-a}^{a}e^{f(u)}du\xlongequal{令u=-V}a\int_{a}^{-a}e^{f(-V)}(-dV)=a\int_{-a}^{a}e^{-f(u)}du$

∴ $2I=a\int_{-a}^{a}\left(e^{f(u)}+e^{-f(u)}\right)du\geqslant a\int_{-a}^{a}2du=(2a)^2\Rightarrow I\geqslant 2a^2$

即 $\iint\limits_{|x|+|y|\leq a}e^{f(x+y)}dxdy\geqslant 2a^2$.

证(三)/(3): ∵ $f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-a}{\sigma_1}-\rho\frac{y-b}{\sigma_2}\right)^2+(1-\rho^2)\left(\frac{y-b}{\sigma_2}\right)^2\right]}$

∴ $I=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\iint_{-\infty}^{+\infty}e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-a}{\sigma_1}-\rho\frac{y-b}{\sigma_2}\right)^2\right]}\cdot e^{-\frac{1}{2}\left(\frac{y-b}{\sigma_2}\right)^2}dxdy$

$$=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty} e^{-\frac{(y-b)^2}{2\sigma_2^2}}dy\int_{-\infty}^{+\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-a}{\sigma_1}-\rho\frac{y-b}{\sigma_2}\right)^2}dx$$

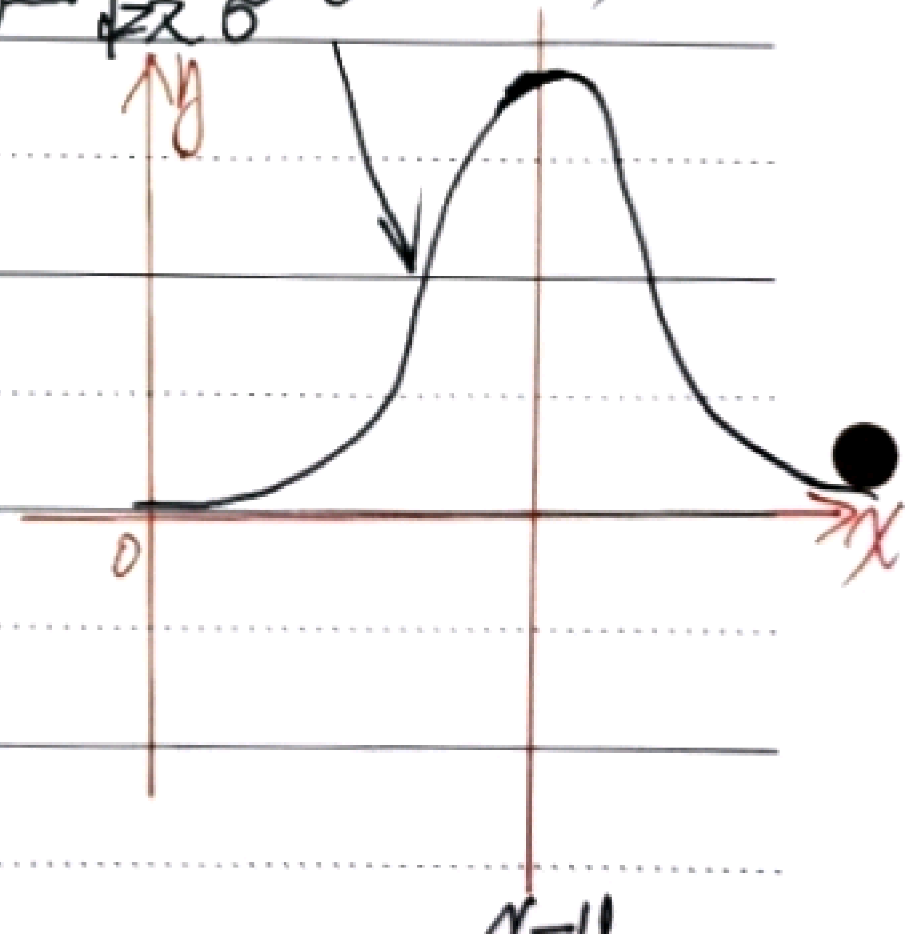
设 $\frac{1}{\sqrt{2}\sqrt{1-\rho^2}}\left(\frac{x-a}{\sigma_1}-\rho\frac{y-b}{\sigma_2}\right)=u$ 则 $dx=\sqrt{2}\sqrt{1-\rho^2}\,\sigma_1\,du$,

$$I=\frac{1}{2\pi\cancel{\sigma_1}\sigma_2\cancel{\sqrt{1-\rho^2}}}\int_{-\infty}^{+\infty} e^{-\frac{(y-b)^2}{2\sigma_2^2}}dy\int_{-\infty}^{+\infty} e^{-u^2}\sqrt{2}\cancel{\sqrt{1-\rho^2}}\cancel{\sigma_1}\,du$$

$$=\frac{1}{2\pi\sigma_2}\int_{-\infty}^{+\infty} e^{-\frac{(y-b)^2}{2\sigma_2^2}}dy\,(\sqrt{2}\sqrt{\pi})$$

令 $\frac{(y-b)}{\sqrt{2}\sigma_2}=v$ 则 $dy=\sqrt{2}\sigma_2\,dv$ $\quad \frac{\sqrt{2\pi}\times 1}{2\pi\cancel{\sigma_2}}\int_{-\infty}^{+\infty} e^{-v^2}\sqrt{2}\cancel{\sigma_2}\,dv=\frac{\sqrt{2\pi}}{2\pi}\sqrt{\pi}\sqrt{2}=1.$

注：一元正态分布的密度函数是 $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-u)^2}{2\sigma^2}}, x\in R$



三) 作业：

ex10.1/2(4),(7); ex10.2/3(2); 4; 6; 7.

ex10.3/3(7),(8); 4(4),(5).