(周一)

§10.1

T3 (1) 原式 $= 4\iint_{D_1}(x^2+y^2)\,dx\,dy$　　$D_1=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1\}$

$$= 4\int_0^1 dx\int_0^1 (x^2+y^2)\,dy$$
$$= \frac{8}{3}$$

(2) 原式 $= 0$　　(积分区域关于 y 对称, 被积函数关于 x 是奇的)

T5 (1) 记 $F(x)=\int_0^x f(t)\,dt \Rightarrow F'(x)=f(x),\ F(0)=0$

$$\therefore \int_0^a dx\int_0^x f(x)f(y)\,dy$$
$$= \int_0^a f(x)\,dx\int_0^x f(y)\,dy$$
$$= \int_0^a f(x)\cdot F(x)\,dx$$
$$= \int_0^a F(x)\,dF(x) = \frac{1}{2}F^2(x)\Big|_0^a = \frac{1}{2}F^2(a) = \frac{1}{2}\left(\int_0^a f(x)\,dx\right)^2$$

(2) $\int_0^a dx\int_0^x f(y)\,dy = \int_0^a dy\int_y^a f(y)\,dx$

$$= \int_0^a (a-y)f(y)\,dy = \int_0^a (a-x)f(x)\,dx$$

T6 $\iint_D \frac{\partial^2 f(x,y)}{\partial x\partial y}\,dx\,dy$

$$= \int_a^b dx\int_c^d \frac{\partial^2 f(x,y)}{\partial x\partial y}\,dy$$
$$= \int_a^b \frac{\partial f(x,d)}{\partial x} - \frac{\partial f(x,c)}{\partial x}\,dx$$
$$= f(x,d)\Big|_a^b - f(x,c)\Big|_a^b = f(b,d)-f(a,d)-f(b,c)+f(a,c)$$

T7 由积分中值定理: $\exists (x_r,y_r)\in B(0,r)$

s.t. $\frac{1}{\pi r^2}\iint_{x^2+y^2\le r^2} f(x,y)\,dx\,dy = f(x_r,y_r)$

由于 f 连续 $\Rightarrow \lim_{r\to 0} f(x_r,y_r) = f(0,0)$

即原命题成立.

§10.2

T1 (1) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \theta\in[0,\frac{\pi}{2}]\quad r\in[0,R]\qquad \frac{\partial(x,y)}{\partial(r,\theta)} = r$

原式 $= \int_0^{\frac{\pi}{2}} d\theta \int_0^R \ln(1+r^2)\cdot r\,dr \overset{t=1+r^2}{=} \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\int_1^{1+R^2} \ln t\,dt\right) d\theta = \frac{\pi}{4}\left((R^2+1)\ln(R^2+1) - R^2\right)$

T2 (1) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \theta\in[-\frac{\pi}{4},\frac{3}{4}\pi],\ r\in[0,\sqrt{2}\sin(\theta+\frac{\pi}{4})]\qquad \frac{\partial(x,y)}{\partial(r,\theta)} = r$

原式 $= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \int_0^{\sqrt{2}\sin(\theta+\frac{\pi}{4})} r\cdot r\,dr = \frac{8}{9}\sqrt{2}$

(2) $\begin{cases} x = ar\cos\theta \\ y = ar\sin\theta \end{cases} \Rightarrow \theta\in[0,\arctan\frac{a}{b}],\ r\in[0,2]\qquad \frac{\partial(x,y)}{\partial(r,\theta)} = abr$

原式 $= \int_0^{\arctan\frac{a}{b}} d\theta \int_0^2 r\cdot abr\,dr = \frac{8}{3}ab\cdot\arctan\frac{a}{b}$

T3 (1) 令 $x = \sqrt{3}r\cos\theta\qquad y = \frac{\sqrt{6}}{2}r\sin\theta\qquad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \frac{3}{2}\sqrt{2}r$

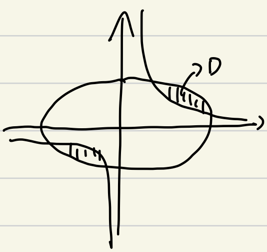
$\forall (x,y)\in D$ 都应有 (r,θ) 的关系式

$\frac{3}{2}\sqrt{2}r^2\sin\theta\cos\theta \geq 1$ 且 $r\leq 1$

记 $\theta_1 = \frac{1}{2}\arcsin\frac{2\sqrt{2}}{3}\qquad \theta_2 = \frac{\pi}{2} - \theta_1$

则 $\iint_D ds = \int_{\theta_1}^{\theta_2} d\theta \int_{\sqrt{\frac{2\sqrt{2}}{3\sin2\theta}}}^{1} \frac{3}{2}\sqrt{2}r\,dr = \int_{\theta_1}^{\theta_2} \frac{3}{4}\sqrt{2} - \frac{1}{\sin2\theta}\,d\theta$

$= \frac{3}{4}\sqrt{2}(\theta_2-\theta_1) + \underbrace{\frac{1}{2}\int_{\theta_1}^{\theta_2} \frac{d(\cos\theta)}{(1-\cos^2\theta)\cos\theta}}_{A}$



令 $t = \cos\theta$

$A = -\frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{\frac{\sqrt{6}}{3}} \frac{dt}{(1-t^2)t} = \frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{\frac{\sqrt{6}}{3}} \left(-\frac{1}{t} + \frac{1}{2}\cdot\frac{1}{1+t} - \frac{1}{2}\frac{1}{1-t}\right)dt$

$= -\frac{1}{2}\ln 2$

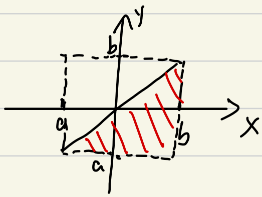
$\therefore S = 2\iint_D ds = \frac{3}{2}\sqrt{2}\left(\frac{\pi}{2} - \arcsin\frac{2\sqrt{2}}{3}\right) - \ln 2$

(形式不唯一)

(周三)

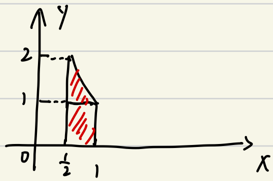
§10.1

T_1 (4)



$$\int_a^b dy \int_y^b f(x,y)\,dx = \int_a^b dx \int_a^x f(x,y)\,dy$$

(6)



$$\int_0^1 dy \int_{\frac{1}{2}}^1 f(x,y)\,dx + \int_1^2 dy \int_{\frac{1}{2}}^{\frac{1}{y}} f(x,y)\,dx$$

$$= \int_{\frac{1}{2}}^1 dx \int_0^{\frac{1}{x}} f(x,y)\,dy$$

T_2 (6)(8)已做 (6): $1-\sin 1$ (8): $\frac{\pi}{2}-1$

§10.2

T_2 (3) 令 $xy=u$ $\frac{y}{x}=v$ $\Rightarrow x=\sqrt{\frac{u}{v}}$, $y=\sqrt{uv}$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2v}\sqrt{\frac{v}{u}} & -\frac{u}{2v^2}\sqrt{\frac{v}{u}} \\ \frac{v}{2}\frac{1}{\sqrt{uv}} & \frac{u}{2}\frac{1}{\sqrt{uv}} \end{vmatrix} = \frac{1}{2v}$$

$$D' = \{(u,v) \mid 1 \le u \le 2,\ 1 \le v \le 2\}$$

$$原式 = \iint_{D'} \left(\frac{u}{v}+uv\right)\cdot\frac{1}{2v}\,du\,dv = \iint_{D'}\left(\frac{u}{2v^2}+\frac{u}{2}\right)du\,dv = \frac{9}{8}$$

(4) 令 $x^2=vy$, $y^2=ux$ $\Rightarrow x=(uv^2)^{\frac{1}{3}}$, $y=(u^2v)^{\frac{1}{3}}$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3} \qquad D' = \{(u,v) \mid b<u<a,\ n<v<m\}$$

$$原式 = \iint_{D'} \frac{1}{3}\,du\,dv = \frac{1}{3}(a-b)(m-n)$$

(7) 令 $x+y=u$, $x-y=v$ $\Rightarrow$ $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=\frac{1}{2} \qquad D'=\{(u,v)\mid -1\le u\le 1,\ -1\le v\le 1\}$$

$\therefore$ 原式 $=\iint_{D'}\frac{uv}{\sqrt{u+3}}\cdot\frac{1}{2}\,du\,dv=0$

(9) 令 $x=ar\cos\theta$, $y=ar\sin\theta$ $\quad \frac{\partial(x,y)}{\partial(u,v)}=-a^2r$

$D'=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le 2\pi\}$, $D''=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2}\}$

$\therefore$ 原式 $=\iint_{D'}|a^2r^2\sin\theta\cos\theta|\cdot a^2r\,dr\,d\theta \overset{\text{对称性}}{=} 4\iint_{D''}a^4r^3\sin\theta\cos\theta\,dr\,d\theta=\frac{1}{2}a^4$

T_3 (3) 令 $x+y=u$, $y=vx$ $\Rightarrow$ $a\le u\le b$, $k\le v\le m$

$x=\frac{u}{1+v}$, $y=\frac{uv}{1+v}$ $\Rightarrow$ $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{1+v} & -\frac{u}{(1+v)^2}\\ \frac{v}{1+v} & \frac{u}{(1+v)^2}\end{vmatrix}=\frac{u}{(1+v)^2}$

$\therefore$ $S=\iint_{D'}\frac{u}{(1+v)^2}\,du\,dv=\frac{1}{2}(b^2-a^2)\left(\frac{1}{1+k}-\frac{1}{1+m}\right)$

T_5 证明，记 $D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1\}$ $\qquad e^u\ge 1+u$

$$\int_0^1 e^{f(x)}dx\cdot\int_0^1 e^{-f(y)}dy=\iint_D e^{f(x)-f(y)}dx\,dy\ge\iint_D (1+f(x)-f(y))\,dx\,dy$$

$$=\iint_D dx\,dy+\iint_D f(x)\,dx\,dy-\iint_D f(y)\,dx\,dy=1$$

(周五)

§10.3

T_1 (1) 原式 $=\int_0^{\frac{1}{2}}dz\int_1^2 x\,dx\int_{-2}^{1}y\,dy=-\frac{9}{8}$

(2) 原式 $=\int_0^1\int_0^x\int_0^{xy}xy^2z^3\,dz\,dy\,dx=\frac{1}{364}$

T2 (1) 原式 $= \frac{1}{2}a^2 \int_0^2 dx \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy$

$= \frac{1}{2}a^2 \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 dr d\theta$

$= \frac{8}{9}a^2$

(2) 原式 $= \int_0^{2\pi} \int_0^R r^3 \sqrt{R^2-r^2}\, dr d\theta = \frac{4\pi}{15}R^5$

(3) 原式 $= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 r \cdot r^2 \sin\theta\, dr d\theta d\varphi = \frac{\pi}{8}$

T3 (1) 原式 $= \int_0^2 \int_0^{2\pi} \int_0^{\sqrt{2z}} r^3 dr\, d\theta\, dz = \frac{16}{3}\pi$

(3) 原式 $= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{\frac{r^2}{3}}^{\sqrt{4-r^2}} r \cdot z\, dz\, dr\, d\theta = \frac{13}{4}\pi$

(6) 原式 $= \iiint\limits_{x^2+y^2+z^2 \le 1} 1-x^2-y^2-z^2 dx dy dz + \iiint\limits_{1 \le x^2+y^2+z^2 \le 4} x^2+y^2+z^2-1\, dx dy dz$

$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (1-r^2) r^2 \sin\theta\, dr d\theta d\varphi + \int_0^{2\pi} \int_0^{\pi} \int_1^2 (r^2-1) r^2 \sin\theta\, dr d\theta\, d\varphi$

$= \frac{4}{15} \cdot 2\pi + \frac{116}{15} \cdot 2\pi$

$= 16\pi$

T7 换元知 $F(t) = \int_0^{2\pi} \int_0^{\pi} \int_0^t f(r^2) \cdot r^2 \sin\theta\, dr d\theta d\varphi$

$\therefore F(t) = 4\pi \int_0^t f(r^2) r^2 dr$

$\therefore F'(t) = 4\pi f(t^2) t^2$

T8 左 $= \int_{-1}^1 \left(\iint_{x^2+y^2 \le 1-z^2} f(z) dx dy \right) dz = \int_{-1}^1 f(z) \cdot \pi (1-z^2) dz$

$= \pi \int_{-1}^1 f(z) (1-z^2) dz =$ 右