Ex13.1 1(2)(3)(6)(7)(10)(14); 2(4)

1.判断下列反常积分的敛散性:

$$(2)\int_0^{+\infty}\sqrt{x}e^{-x}dx$$

$$LHS=\int_0^1\sqrt{x}e^{-x}dx(conv.)+\int_1^{+\infty}\sqrt{x}e^{-x}dx$$
 $\lim_{x\to+\infty}rac{\sqrt{x}e^{-x}}{x^{-2}}=0$ 而 $\int_1^{+\infty}x^{-2}dx=1$

因此conv

$$\begin{array}{l} (3) \int_{0}^{+\infty} \frac{x \arctan x}{\sqrt[3]{1+x^4}} dx \\ LHS = \int_{0}^{1} \frac{x \arctan x}{\sqrt[3]{1+x^4}} dx (conv.) + \int_{1}^{+\infty} \frac{x \arctan x}{\sqrt[3]{1+x^4}} dx \\ \lim_{x \to +\infty} \frac{x \arctan x}{\sqrt[3]{1+x^4}} / x^{-\frac{1}{3}} = \frac{\pi}{2} \\ \overline{\text{min}} \int_{1}^{+\infty} x^{-\frac{1}{3}} = +\infty \end{array}$$

因此div

$$(6)\int_0^1 rac{x^2}{\sqrt[3]{(1-x^2)^5}} dx$$
 $x=1$ 为瑕点
$$\lim_{x o 1} rac{x^2}{\sqrt[3]{(1-x^2)^5}} / rac{x}{\sqrt[3]{(1-x^2)^5}} = 1$$
, LHS 与 $\int_0^1 rac{x}{\sqrt[3]{(1-x^2)^5}} dx$ 同敛散.
$$\int_0^1 rac{x}{\sqrt[3]{(1-x^2)^5}} dx = rac{3}{4} (1-x^2)^{-rac{2}{3}} |_0^1 = +\infty$$

因此div

$$(7)\int_0^1 rac{dx}{e^{\sqrt{x}}-1}$$
 $x=0$ 为瑕点
$$\lim_{x o 0} rac{1}{e^{\sqrt{x}}-1}/rac{1}{\sqrt{x}}=1,\ LHS$$
与 $\int_0^1 rac{1}{\sqrt{x}}dx$ 同敛散.
$$\int_0^1 rac{1}{\sqrt{x}}dx=2\sqrt{x}|_0^1=2$$

因此conv

$$(10)\int_0^{rac{\pi}{2}}rac{\ln\sin x}{\sqrt{x}}dx$$
 $x=0$ 为瑕点

$$\lim_{x o 0} rac{\ln \sin x}{\sqrt{x}} / x^{-rac{3}{4}} = \lim_{x o 0} x^{rac{1}{4}} \ln \sin x = \lim_{x o 0} rac{rac{1}{ an x}}{-rac{1}{4} x^{-rac{5}{4}}} = 0$$

$$\int_0^{\frac{\pi}{2}} x^{-\frac{3}{4}} dx = 4x^{\frac{1}{4}} \Big|_0^{\frac{\pi}{2}} = 4(\frac{\pi}{2})^{\frac{1}{4}}$$

因此conv

(14)
$$\int_0^{+\infty} \frac{\arctan x}{x^{\mu}} dx$$

 $\mu \leq 0$ 时,显然**div**

$$\mu>0$$
时, $x=0$ 为瑕点, $LHS=\int_0^{rac{\pi}{4}}rac{\arctan x}{x^\mu}dx+\int_{rac{\pi}{4}}^{+\infty}rac{\arctan x}{x^\mu}dx$

这同敛散于
$$\int_0^{\frac{\pi}{4}} \frac{1}{x^{\mu-1}} dx + \frac{\pi}{2} \int_{\frac{\pi}{4}}^{+\infty} \frac{1}{x^{\mu}} dx$$

因此 $1<\mu<2$ 时, ${
m conv}$,否则 ${
m div}$

2.研究下列积分的条件收敛性与绝对收敛性:

(4)
$$\int_0^{+\infty} rac{\ln(1+x)}{x^2(1+x^p)} dx (p>0)$$

$$x=0$$
为瑕点, $LHS=\int_0^1rac{\ln(1+x)}{x^2(1+x^p)}dx+\int_1^{+\infty}rac{\ln(1+x)}{x^2(1+x^p)}dx$

这同敛散于
$$\int_0^1 \frac{1}{x} dx + M(M>0)$$

因此div

Ex13.3 1(1)(2); 2(1)(3); 3; 4(1)(2)

1.试用两种方法计算以下极限:

(1)
$$\lim_{\alpha \to 0} \int_{-1}^{1} \sqrt{x^2 + a^2} dx$$

方法一:

$$\int_{-1}^{1} \sqrt{x^2 + a^2} dx = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{1}{2} a^2 \ln(x + \sqrt{x^2 + a^2})|_{-1}^{1} = \sqrt{1 + a^2} + a^2 \ln \frac{1 + \sqrt{1 + a^2}}{-1 + \sqrt{1 + a^2}}$$
 因此 $\lim_{\alpha \to 0} \int_{-1}^{1} \sqrt{x^2 + a^2} dx = 1$

方法二:

$$\sqrt{x^2+a^2}$$
连续,因此 $\int_{-1}^1 \sqrt{x^2+a^2} dx$ 对 a 连续,所以 $\lim_{lpha o 0} \int_{-1}^1 \sqrt{x^2+a^2} dx = \int_{-1}^1 |x| dx = 1$

(2)
$$\lim_{\alpha \to 0} \int_{\alpha}^{1+\alpha} \frac{1}{1+x^2+\alpha^2} dx$$

方法一:

$$\int_{lpha}^{1+lpha} rac{1}{1+x^2+lpha^2} dx = rac{1}{\sqrt{1+lpha^2}} \mathrm{arctan} \, rac{x}{\sqrt{1+lpha^2}} |_{lpha}^{lpha+1}$$

因此
$$\lim_{lpha o 0}\int_lpha^{1+lpha}rac{1}{1+x^2+lpha^2}dx=rctan 1-rctan 0=rac{\pi}{4}$$

方法二:

$$f(lpha)=\int_{lpha}^{1+lpha}rac{1}{1+x^2+lpha^2}dx$$
连续,因此 $\lim_{lpha o 0}\int_{lpha}^{1+lpha}rac{1}{1+x^2+lpha^2}dx=f(0)=\int_{0}^{1}rac{1}{1+x^2}dx=rac{\pi}{4}$

$$2.求F'(\alpha)$$

(1)
$$F(\alpha) = \int_{\sin \alpha}^{\cos \alpha} e^{\alpha \sqrt{1-x^2}} dx$$

$$F'(\alpha) = \int_{\sin \alpha}^{\cos \alpha} \sqrt{1 - x^2} e^{\alpha \sqrt{1 - x^2}} dx - \sin \alpha e^{\alpha |\sin \alpha|} - \cos \alpha e^{\alpha |\cos \alpha|}$$

(3)
$$F(lpha)=\int_0^lpha rac{\ln(1+lpha x)}{x}dx$$

$$F'(lpha)=\int_0^lpharac{1}{1+lpha x}dx+rac{\ln(1+lpha^2)}{lpha}=rac{2\ln(1+lpha^2)}{lpha}$$

3.设f(x)在[a,b]上连续,证明:

$$y(x) = rac{1}{k} \int_{c}^{x} f(t) \sin k(x-t) dt$$
, $c, x \in [a,b)$

满足常微分方程 $y'' + k^2y = f(x)$, 其中c与k为常数.

$$y' = \frac{1}{k} \int_{c}^{x} kf(t) \cos k(x-t) dt = \int_{c}^{x} f(t) \cos k(x-t) dt$$

$$y'' = f(x) - \int_{c}^{x} kf(t) \sin k(x - t) dt$$

$$y''+k^2y=f(x)-\int_c^x kf(t)\sin k(x-t)dt+k^2rac{1}{k}\int_c^x f(t)\sin k(x-t)dt=f(x)$$

4.应用对参数进行微分或积分的方法, 计算下列积分:

(1)
$$\int_0^{rac{\pi}{2}} \ln(a^2 \sin^2 x + b^2 \cos^2 x) dx$$
 , $(a>0,b>0)$

$$\ln(a^2\sin^2 x + b^2\cos^2 x) = \ln a^2 + \ln(\sin^2 x + \frac{b^2}{a^2}\cos^2 x)$$

$$t = \frac{b}{a}$$

则
$$I(t)=\pi \ln a + \int_0^{rac{\pi}{2}} \ln(\sin^2 x + t^2 \cos^2 x) dx$$

$$I'(t) = \int_0^{\frac{\pi}{2}} \frac{2t \cos^2 x}{\sin^2 x + t^2 \cos^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{2t}{\tan^2 x + t^2} dx = \frac{\pi}{1+t}$$

$$I(t) = \pi \ln(t+1) + C$$

$$I(1) = \pi \ln a$$

$$C=\pi\lnrac{a}{2}$$
,于是 $I(rac{b}{a})=\pi\ln(rac{a}{2}(rac{b}{a}+1))=\pi\lnrac{a+b}{2}$

(2)
$$\int_0^\pi \ln(1-2a\cos x + a^2) dx$$
, $(0 \le a < 1)$

$$I(a)=\int_0^\pi \ln(1-2a\cos x+a^2)dx$$

$$I'(a) = \int_0^\pi \frac{2a - 2\cos x}{1 - 2a\cos x + a^2} dx = 0$$

 $LHS = I(0) = \int_0^\pi \ln 1 dx = 0$