第~~32~~28讲：Gauss公式及其应用　(2023,5,?)

(一) 复习:

设Ω是R^3中的区域，$\vec{A}(x,y,z)=(P,Q,R)\in C(\Omega)$.

光滑有向(即双侧)曲面$\Sigma\subset\Omega$，且$r(u,v)=(x(u,v),$ $y(u,v),z(u,v))\in C^1(\Omega)$，则:

$$\iint_{\Sigma}\vec{A}\cdot\vec{n}\,ds=\iint_{D_{uv}}(P,Q,R)\cdot\left(\pm\frac{r_u'\times r_v'}{|r_u'\times r_v'|}\right)|r_u'\times r_v'|\,du\,dv$$

$$=\pm\iint_{D_{uv}}\begin{vmatrix}P&Q&R\\x_u'&y_u'&z_u'\\x_v'&y_v'&z_v'\end{vmatrix}du\,dv=\iint_{\Sigma}P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy\quad(\text{式}1)$$

特别地，当Σ为显式曲面$z=z(x,y)\in C^1(D_{xy})$时.

有Σ: $r(x,y)=(x,y,z(x,y))\Rightarrow\begin{cases}r_x'=(1,0,z_x')\\r_y'=(0,1,z_y')\end{cases}$

$$\iint_{\Sigma}\vec{A}\cdot\vec{n}\,ds=\pm\iint_{D_{xy}}\begin{vmatrix}P&Q&R\\1&0&z_x'\\0&1&z_y'\end{vmatrix}dx\,dy$$

$$=\pm\iint_{D_{xy}}(R-z_x'P-z_y'Q)\,dx\,dy\xlongequal{\text{当}P=Q=0}\pm\iint_{D_{xy}}R(x,y,z(x,y))\,dx\,dy\quad(1)$$

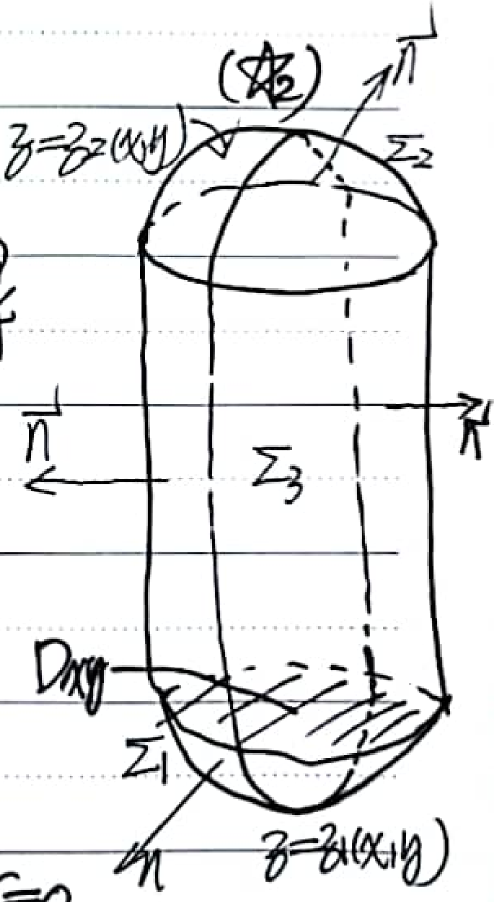
且$\Sigma: z=z(x,y)$取上侧时，取"+"号，Σ取下侧时取"−"号。

(三) Gauss公式：

设Ω为R^3中由分片光滑双侧曲面Σ围成的有界闭区域，且$\partial\Omega=\Sigma$取外侧. $\vec{A}(x,y,z)=(P,Q,R)\in C^1(\bar{\Omega})$，

则有：$\displaystyle\oiint_{\partial\Omega}\vec{A}\cdot\vec{n}\,dS=\oiint_{\partial\Omega}P\,dydz+Q\,dzdx+R\,dxdy=\iiint_{\Omega}(\nabla\cdot\vec{A})\,dV$

$$=\iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV$$



证(I)：设Ω是一个柱形曲面柱体

如右图所示，$\partial\Omega=\Sigma_1+\Sigma_2+\Sigma_3$

$\Sigma_1: z=z_1(x,y)\in C(D_{xy})$，取下侧；

$\Sigma_2: z=z_2(x,y)\in C(D_{xy})$，取上侧；

Σ_3为母线平行oz轴的柱面，$\vec{n}$中$\cos\gamma=0$

$\Rightarrow dxdy=\cos\gamma\,dS\equiv0$.

$$\oiint_{\partial\Omega} R(x,y,z)\,dxdy = \iint_{\Sigma_1} R(x,y,z)\,dxdy + \iint_{\Sigma_2} R(x,y,z)\,dxdy + \iint_{\Sigma_3} R(x,y,z)\,dxdy$$

$$= -\iint_{D_{xy}} R(x,y,z_1(x,y))\,dxdy + \iint_{D_{xy}} R(x,y,z_2(x,y))\,dxdy + 0$$

$$= \iint_{D_{xy}} [R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))]\,dxdy \quad \underline{\text{N-L公式}}$$

$$= \iint_{D_{xy}} \left(\int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R(x,y,z)}{\partial z}\,dz\right) dxdy = \iiint_{\Omega} \frac{\partial R}{\partial z}\,dV \qquad (*1)$$

同理，当Ω为前后型曲面柱体时，有

$$\oiint_{\partial\Omega} P(x,y,z)\,dydz = \iiint_{\Omega} \frac{\partial P}{\partial x}\,dV \qquad (*2)$$

当Ω为左右型曲面柱体时，有

$$\oiint_{\partial\Omega} Q(x,y,z)\,dzdx = \iiint_{\Omega} \frac{\partial Q}{\partial y}\,dV \qquad (*3)$$

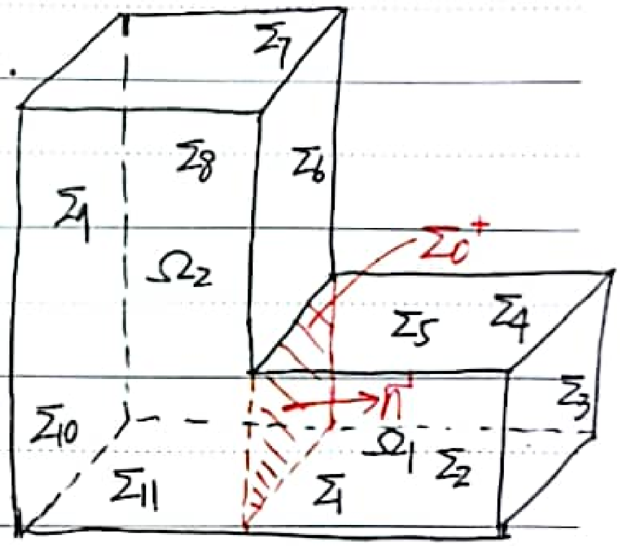
四. 当Ω同时为上下型、前后型、左右型曲面柱体，即Ω为"曲面长方体"时，(*1), (*2), (*3) 同时成立，相加得：

$$\oiint_{\partial\Omega} P\,dydz + Q\,dzdx + R\,dxdy = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV \qquad (3).$$

(四) 当Ω可分成有限个"曲面投影域"时. 在每个曲面投影域上使用Gauss公式. 最后相加时, 所有辅助曲面上的流量全部抵消. 仍有

$$\oiint_{\partial\Omega} Pdydz+Qdzdx+Rdxdy=\iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV.$$

- 例如. 右图Ω不是"曲面投影域". 但可分成两个"曲面投影域"Ω_1, Ω_2. 分别在Ω_1, Ω_2上应用Gauss公式:



$$\begin{cases}\oiint_{\partial\Omega_1}\vec{A}\cdot\vec{n}\,dS=\iiint_{\Omega_1}\nabla\cdot\vec{A}\,dV\\ \oiint_{\partial\Omega_2}\vec{A}\cdot\vec{n}\,dS=\iiint_{\Omega_2}\nabla\cdot\vec{A}\,dV\end{cases}$$

$$\Omega=\Omega_1+\Omega_2$$

$$\partial\Omega_1=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_0^-$$

$$\partial\Omega_2=\Sigma_6+\Sigma_7+\Sigma_8+\Sigma_9+\Sigma_{10}+\Sigma_{11}+\Sigma_0^+$$

相加时, 在辅助面Σ_0^+, Σ_0^-上的流量相抵. 最后有

$$\oiint_{\partial\Omega_1+\partial\Omega_2}\vec{A}\cdot\vec{n}\,dS=\oiint_{\partial\Omega}\vec{A}\cdot\vec{n}\,dS=\iiint_{\Omega_1+\Omega_2}\nabla\cdot\vec{A}\,dV=\iiint_{\Omega}\nabla\cdot\vec{A}\,dV.$$

- 例1. 设 $\vec{A}(x,y,z)=(x,y,z)$, 则 $\nabla\cdot\vec{A}=1+1+1=3$.

$$\oiint_{\partial\Omega} \vec{A}\cdot\vec{n}dS = \oiint_{\partial\Omega} xdydz+ydzdx+zdxdy = \iiint_{\Omega} 3dV = 3V(\Omega)$$

故 $V(\Omega) = \frac{1}{3}\oiint_{\partial\Omega} xdydz+ydzdx+zdxdy$ (※3)

例2. 求椭球面 Σ: $(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2=1$ 围成的椭球体 Ω 的体积 $V(\Omega)$.

解: $\Sigma=\partial\Omega$ 的参数方程为: $\begin{cases} x=a\sin\theta\cos\varphi \\ y=b\sin\theta\sin\varphi \\ z=c\cos\theta \end{cases}$, $\begin{cases} 0\le\theta\le\pi \\ 0\le\varphi\le2\pi \end{cases}$: $D_{\theta\varphi}$

$$V(\Omega)=+\frac{1}{3}\iint_{D_{\theta\varphi}}\begin{vmatrix} x & y & z \\ x'_\theta & y'_\theta & z'_\theta \\ x'_\varphi & y'_\varphi & z'_\varphi \end{vmatrix} d\theta d\varphi = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\begin{vmatrix} a\sin\theta\cos\varphi & b\sin\theta\sin\varphi & c\cos\theta \\ a\cos\theta\cos\varphi & b\cos\theta\sin\varphi & -c\sin\theta \\ -a\sin\theta\sin\varphi & b\sin\theta\cos\varphi & 0 \end{vmatrix}$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi}(abc\sin\theta\cos^2\theta+abc\sin^3\theta)d\theta d\varphi$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi}abc\sin\theta d\theta d\varphi=\frac{2\pi}{3}abc\times2=\frac{4}{3}\pi abc.$$

例3: 求 $\Phi=\iint_{\Sigma}(x^2,y^2,z^2)\cdot\vec{n}dS$, Σ 为上半球面 $x^2+y^2+z^2=a^2$ $(a>0)$ $(z\ge0)$ 的上侧.

解法(2): 补面 Σ_0: $z=0$, $x^2+y^2\le a^2$, $\vec{n}=-\vec{k}=(0,0,-1)$.

设 Σ_0 与 Σ 围成区域 Ω, 在 Ω 上应用Gauss公式:

$$\Phi=\oiint_{\Sigma+\Sigma_0}\vec{A}\cdot\vec{n}\,ds-\iint_{\Sigma_0}\vec{A}\cdot\vec{n}\,ds$$

而 $\oiint_{\Sigma+\Sigma_0}\vec{A}\cdot\vec{n}\,ds=\iiint_{\Omega}(\nabla\cdot\vec{A})\,dV=\iiint_{\Omega}(2x+2y+2z)\,dV$

$$\overset{\text{奇偶对称性}}{=}0+0+2\iiint_{\Omega}z\,dV\ \xlongequal[\substack{y=r\sin\theta\sin\varphi\\ z=r\cos\theta}]{x=r\sin\theta\cos\varphi}\ 2\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}d\theta$$

$$\int_0^a(r\cos\theta)r^2\sin\theta\,dr=2\times2\pi\left(\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta\times\int_0^a r^3dr\right)$$

$=4\pi\times\frac{1}{2}\times\frac{a^4}{4}=\frac{\pi}{2}a^4$. 且在 $\iint_{\Sigma_0}\vec{A}\cdot\vec{n}\,ds=$

$\iint_{\Sigma_0}x^2dydz+y^2dzdx+z^2dxdy$ 中, $\Sigma_0: z=0\Rightarrow dydz=0, dzdx=0,$

$z^2dxdy=0$. 故 $\iint_{\Sigma_0}\vec{A}\cdot\vec{n}\,ds=0$.

从而 $\Phi=\frac{\pi}{2}a^4-0=\frac{\pi}{2}a^4$.

解法(四). 第二型曲面积分的"偶零奇倍"法:

在 $\Phi=\iint_{\Sigma}x^2dydz+y^2dzdx+z^2dxdy$ 中.

(i) $\because \iint_{\Sigma}x^2dydz=\iint_{\Sigma_1+\Sigma_2}x^2dydz$, 其中 $\begin{cases}\Sigma_1: x=\sqrt{a^2-y^2-z^2}\geq0\\ \Sigma_2: x=-\sqrt{a^2-y^2-z^2}\leq0.\end{cases}$

且 Σ_1 取前侧, Σ_2 取后侧.

$\because \iint\limits_{\Sigma} x^2 dydz = \iint\limits_{\Sigma_1} x^2 dydz + \iint\limits_{\Sigma_2} x^2 dydz$

$= (+1)\iint\limits_{D_{yz}} (\sqrt{a^2-y^2-z^2})^2 dydz + (-1)\iint\limits_{D_{yz}} (-\sqrt{a^2-y^2-z^2})^2 dydz = 0$

$D_{yz}: y^2+z^2 \le a^2, x=0, z \ge 0.$

同理, $\iint\limits_{\Sigma} y^2 dzdx = \iint\limits_{\Sigma_3} y^2 dzdx + \iint\limits_{\Sigma_4} y^2 dzdx$, $\begin{cases} \Sigma_3 \text{ 取右侧} \\ \Sigma_4 \text{ 取左侧} \end{cases}$

$y=\sqrt{a^2-x^2-z^2} \quad y=-\sqrt{a^2-x^2-z^2}$

$= (+1)\iint\limits_{D_{xz}} (\sqrt{a^2-x^2-z^2})^2 dzdx + (-1)\iint\limits_{D_{xz}} (-\sqrt{a^2-x^2-z^2})^2 dzdx = 0$

(2°). $\iint\limits_{\Sigma} z^2 dxdy = (+1)\iint\limits_{x^2+y^2\le a^2} (a^2-x^2-y^2) dxdy \xlongequal[y=r\sin\theta]{x=r\cos\theta} \int_0^{2\pi}\left[\int_0^a (a^2-r^2) r dr\right] d\theta$

$= 2\pi \int_0^a (a^2 r - r^3) dr = 2\pi\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi}{2}a^4.$

故 $I = \iint\limits_{\Sigma} x^2 dydz + \iint\limits_{\Sigma} y^2 dzdx + \iint\limits_{\Sigma} z^2 dxdy$

$= 0 + 0 + \frac{\pi}{2}a^4 = \frac{\pi}{2}a^4.$

(三) 作业: P X11.5 / 1(1), (2), (3), (4), (5), (6); 4.

(四) 第39讲: Stokes 公式及其应用 (2023.5.15)

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(7)

关于Green第二公式及其证明：

(I) R^3中的Green第二公式：($\partial\Omega$分片光滑)

$$\oiint_{\partial\Omega}\left(a\frac{\partial b}{\partial n}-b\frac{\partial a}{\partial n}\right)ds=\iiint_{\Omega}(a\Delta b-b\Delta a)dxdydz \quad (\text{式}1)$$

其中，$a(x,y,z)$，$b(x,y,z)$在有界闭区域Ω中二阶偏导连续。$\partial\Omega$是Ω边界曲面，且$\partial\Omega$的单位法向量$\vec{n}$朝外，$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

(II) R^2中的Green第二公式(ex11.3/7/(3))：

$$\oint_{\partial D}\left(a\frac{\partial b}{\partial n}-b\frac{\partial a}{\partial n}\right)ds=\iint_{D}(a\Delta b-b\Delta a)dxdy \quad (\text{式}2).$$

其中，$a(x,y)$，$b(x,y)\in C^2(D)$，D是有界闭区域．∂D分段光滑且单位法向量朝外．$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

证(式1)：(1°)．$\because \frac{\partial b}{\partial n}=\nabla b\cdot\vec{n}=(b_x',b_y',b_z')\cdot\vec{n}$

$\therefore \oiint_{\partial\Omega}a\frac{\partial b}{\partial n}ds=\oiint_{\partial\Omega}(ab_x',ab_y',ab_z')\cdot\vec{n}\,ds\overset{\text{Gauss}}{=\!=}\iiint_{\Omega}[(ab_x')_x'+(ab_y')_y'+(ab_z')_z']dxdydz=\iiint_{\Omega}[a(b_{xx}''+b_{yy}''+b_{zz}'')+(a_x'b_x'+a_y'b_y'+a_z'b_z')]dxdydz$

$=\iiint_{\Omega}\left[a\left(\frac{\partial^2 b}{\partial x^2}+\frac{\partial^2 b}{\partial y^2}+\frac{\partial^2 b}{\partial z^2}\right)+(a_x',a_y',a_z')\cdot(b_x',b_y',b_z')\right]dxdydz$

$=\iiint_{\Omega}[a\Delta b+\nabla a\cdot\nabla b]dxdydz \quad (*1)$

(2°) 同理: $\oiint\limits_{\partial\Omega} b\frac{\partial a}{\partial n}dS = \iiint\limits_{\Omega}[b\Delta a + \nabla b\cdot\nabla a]dxdydz$ (*2).

(*1)−(*2) 得:

$$\oiint\limits_{\partial\Omega}\left(a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right)dS = \iiint\limits_{\Omega}(a\Delta b - b\Delta a)dxdydz.$$

证(*2): 设 ∂D 的单位切向量 $\vec{\tau}=(\cos\alpha,\cos\beta)$, 则 $\begin{cases}\cos\alpha\, ds = dx\\ \cos\beta\, ds = dy\end{cases}$

且此时, $\vec{n}=(\cos\beta,-\cos\alpha)$, 从而

$$\oint_{\partial D} a\frac{\partial b}{\partial n}ds = \oint_{\partial D} a(b'_x,b'_y)\cdot\vec{n}\,ds = \oint_{\partial D}(ab'_x,ab'_y)\cdot(\cos\beta\, ds,-\cos\alpha\, ds)$$

$$= \oint_{\partial D}(ab'_x,ab'_y)\cdot(dy,-dx) = \oint_{\partial D}(-ab'_y)dx+(ab'_x)dy \xlongequal[Q(x,y)=ab'_x,]{P(x,y)=-ab'_y,\ \text{Green}}$$

$$\iint\limits_{D}\left[\frac{\partial(ab'_x)}{\partial x} - \frac{\partial(-ab'_y)}{\partial y}\right]dxdy = \iint\limits_{D}[a(b''_{xx}+b''_{yy}) + (a'_x,a'_y)\cdot(b'_x,b'_y)]dxdy$$

$$= \iint\limits_{D}[a\Delta b + \nabla a\cdot\nabla b]dxdy,\quad \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right),\ \Delta=\nabla\cdot\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad (*3).$$

同理: $\oint_{\partial D} b\frac{\partial a}{\partial n}ds = \iint\limits_{D}[b\Delta a + \nabla b\cdot\nabla a]dxdy$ (*4)

(*3)−(*4) 得:

$$\oint_{\partial D}\left[a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right]ds = \iint\limits_{D}[a\Delta b - b\Delta a]dxdy$$

注：上述Green第二公式在数学物理方程课程中会多次遇到。