Ex13.4 1(1); 2(1)(3); 6; 7(1); 8(1)

1.确定下列含参变量反常积分的收敛域:

$$(1)\int_0^{+\infty} x^u dx$$

 $u \geq 0$ 时显然div

$$\int_0^{+\infty} x^u dx = \int_0^1 x^u dx + \int_1^{+\infty} x^u dx$$

$$u < -1$$
, $u > -1$ 时,均div

因此收敛域为∅

2.研究下列积分在指定区间上的一致收敛性:

$$(1) \int_{-\infty}^{+\infty} \frac{\cos ux}{1+x^2} dx \left(-\infty < u < +\infty\right)$$

考虑
$$+\infty$$
侧, $\int_{A'}^{A''} rac{\cos ux}{1+x^2} dx \leq \int_{A'}^{A''} rac{1}{1+x^2} dx = \arctan x|_{A'}^{A''}$

$$orall \epsilon > 0$$
,取 $X = an(rac{\pi}{2} - \epsilon)$,则 $\int_{A'}^{A''} rac{\cos ux}{1 + x^2} dx < \epsilon$

因此一致收敛。

$$(3)\int_0^{+\infty}\sqrt{\alpha}e^{-\alpha x^2}dx (0\leq \alpha<+\infty)$$

$$lpha>0$$
时, $\int_0^{+\infty}\sqrt{lpha}e^{-lpha x^2}dx=rac{\sqrt{\pi}}{2}$

$$\int_{0}^{A} \sqrt{\alpha} e^{-\alpha x^{2}} dx < A\sqrt{\alpha}$$

$$\int_{A}^{+\infty} \sqrt{\alpha} e^{-\alpha x^2} dx \ge \frac{\sqrt{\pi}}{2} - A\sqrt{a}$$

因此
$$\sup_{\alpha>0} \left| \int_{4}^{+\infty} \sqrt{\alpha} e^{-\alpha x^2} dx \right| = \frac{\pi}{2}$$

因此不一致收敛。

6.证明:
$$F(lpha)=\int_0^{+\infty}rac{\cos x}{1+(x+lpha)^2}dx$$
在 $0\leq lpha<+\infty$ 上是连续且可微的函数.

 $p(x)=rac{1}{1+x^2}$,则 $rac{\cos x}{1+(x+lpha)^2}\leq p(x)$,p(x)在 $[0,+\infty)$ 上可积,因此F(lpha)一致收敛,因此在 $0\leq lpha<+\infty$ 上连续。## 连续型证毕

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$$\exists f(x, lpha) = rac{\cos x}{1 + (x + lpha)^2}$$

$$\left|\frac{\partial f(x,\alpha)}{\partial \alpha}\right| = \left|\frac{-2(x+\alpha)\cos x}{(1+(x+\alpha)^2)^2}\right| \le \frac{\cos x}{1+(x+\alpha)^2}$$

因此 $\int_0^{+\infty} rac{\partial f(x,lpha)}{\partial lpha} dx$ 一致收敛,因此F(lpha)可微。

7.计算下列积分:

(1)
$$\int_0^1 rac{x^eta - x^lpha}{\ln x} dx (lpha, eta > -1)$$

被积函数可以表达为积分:
$$\frac{x^{\beta}-x^{\alpha}}{\ln x}=\int_{\alpha}^{\beta}x^{u}du$$

因此
$$LHS=\int_0^1 dx \int_{lpha}^{eta} x^u du=\int_{lpha}^{eta} du \int_0^1 x^u dx=\int_{lpha}^{eta} du rac{1}{u+1} x^{u+1} |_0^1=\int_{lpha}^{eta} rac{1}{u+1} du=\lnrac{eta+1}{lpha+1}$$

8.利用
$$\int_0^{+\infty}e^{-x^2}dx=rac{\sqrt{\pi}}{2}$$
及 $\int_0^{+\infty}rac{\sin x}{x}dx=rac{\pi}{2}$,计算:

(1)
$$\int_{-\infty}^{+\infty} rac{x}{\sigma\sqrt{2\pi}} e^{-rac{1}{2}(rac{x-a}{\sigma})^2} dx (\sigma>0)$$

$$u=rac{x-a}{\sqrt{2}\sigma}$$
,则 $dx=\sqrt{2}\sigma du$

$$LHS=\int_{-\infty}^{+\infty}rac{\sqrt{2}\sigma u+a}{\sigma\sqrt{2\pi}}e^{-u^2}\sqrt{2}\sigma du=rac{a}{\sigma\sqrt{2\pi}}\sqrt{2}\sigmarac{\sqrt{\pi}}{2}2=a$$

Ex13.2 1(2)(3); Ex13.4 6; 8(2)(4)(6)

1.计算反常积分:

(2)
$$\iint_{D} rac{dxdy}{(1+x+y)^{lpha}}$$
,其中 D 是第一象限, $lpha > 2$ 为常数

(3)
$$\iint_D \max\{x,y\}e^{-(x^2+y^2)}dxdy$$
,其中 D 是第一象限

6.证明:
$$F(lpha)=\int_0^{+\infty}rac{\cos x}{1+(x+lpha)^2}dx$$
在 $0\leq lpha<+\infty$ 上是连续且可微的函数.

8.利用
$$\int_0^{+\infty}e^{-x^2}dx=rac{\sqrt{\pi}}{2}$$
及 $\int_0^{+\infty}rac{\sin x}{x}dx=rac{\pi}{2}$,计算:

$$(2)\int_{-\infty}^{+\infty} \frac{(x-a)^2}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-a}{\sigma})^2} dx (\sigma > 0)$$

$$u=rac{x-a}{\sqrt{2}\sigma}$$
 , $\mathbb{N}dx=\sqrt{2}\sigma du$

$$LHS = \int_{-\infty}^{+\infty} u^2 \frac{\sigma}{\sqrt{\pi}} \sqrt{2} e^{-u^2} \sqrt{2} \sigma du = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} u^2 e^{-u^2} du = \frac{2\sigma^2}{\sqrt{\pi}} [-\frac{1}{2} u e^{-u^2}]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{1}{2} e^{-u^2} du] = \sigma^2$$

$$(4)\int_{0}^{+\infty} \frac{\sin^{2}x}{x^{2}} dx$$

$$LHS = -rac{\sin^2 x}{x}|_0^{+\infty} + \int_0^{+\infty} rac{\sin 2x}{x} dx = \int_0^{+\infty} rac{2\sin u}{u} rac{1}{2} du = rac{\pi}{2}$$

(6)
$$\int_0^{+\infty} \frac{\sin^4 x}{x^2} dx$$

$$LHS = -\tfrac{\sin^4 x}{x}|_0^{+\infty} + \int_0^{+\infty} \tfrac{4\sin^3 x \cos x}{x} dx = \int_0^{+\infty} \tfrac{\sin 2x (1-\cos 2x)}{x} dx = \int_0^{+\infty} \tfrac{\sin 2x}{x} dx - \int_0^{+\infty} \tfrac{\sin 4x}{2x} dx = \tfrac{\pi}{2} - \tfrac{\pi}{4} = \tfrac{\pi}{4}$$

Ex13.5 1; 2; 3(2)(4)(6); 4; 5

1.证明:

(1)
$$\Gamma(s) = 2 \int_0^{+\infty} x^{2s-1} e^{-x^2} dx (s>0)$$

$$\Gamma(s)=\int_0^{+\infty}t^{s-1}e^{-t}dt$$

$$t=x^2$$
, $\mathbb{J}dt=2xdx$

$$\Gamma(s) = \int_0^{+\infty} (x^2)^{s-1} e^{-x^2} 2x dx = 2 \int_0^{+\infty} x^{2s-1} e^{-x^2} dx$$

(2)
$$\Gamma(s)=a^s\int_0^{+\infty}x^{s-1}e^{-ax}dx(s>0,a>0)$$

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$$

$$t=ax$$
 , $\mathbf{y}dt=adx$

$$\Gamma(s) = \int_0^{+\infty} (ax)^{s-1} e^{-ax} a dx = a^s \int_0^{+\infty} x^{s-1} e^{-ax} dx$$

2.证明:

$$\mathrm{B}(p,q) = 2 \int_0^{rac{\pi}{2}} \sin^{2p-1} t \cos^{2q-1} t dt (p>0,q>0)$$

$$B(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$$

$$x = \sin^2 t$$
, $\mathbb{N} dx = 2\sin t \cos t dt$

$$B(p,q) = \int_0^{\frac{\pi}{2}} \sin^{2p-2}t \cos^{2q-2}t 2 \sin t \cos t dt = 2 \int_0^{\frac{\pi}{2}} \sin^{2p-1}t \cos^{2q-1}t dt$$

3.利用Euler积分计算:

$$(2)\int_0^a x^2 \sqrt{a^2 - x^2} dx$$

$$x=a\sqrt{t}$$
 , If $dx=rac{1}{2}at^{-rac{1}{2}}dt$

$$LHS = \int_0^1 a^2 t a \sqrt{1-t} \, frac{1}{2} a t^{-rac{1}{2}} dt = frac{1}{2} a^4 \mathrm{B}(frac{3}{2}, frac{3}{2}) = frac{1}{2} a^4 rac{\Gamma(frac{3}{2})\Gamma(frac{3}{2})}{\Gamma(3)} = frac{1}{2} a^4 frac{ frac{1}{4}\pi}{2!} = frac{1}{16} \pi a^4$$

$$(4)\int_0^1 x^{n-1}(1-x^m)^{q-1}dx(n,m,q>0)$$

$$x=t^{\frac{1}{m}},dx=\frac{1}{m}t^{\frac{1}{m}-1}dt$$

$$LHS = \int_0^1 t^{rac{n-1}{m}} (1-t)^{q-1} rac{1}{m} t^{rac{1}{m}-1} dt = rac{1}{m} \mathrm{B}(rac{n}{m},q)$$

$$(6) \int_0^{\frac{\pi}{2}} \tan^{\alpha} x dx (|\alpha| < 1)$$

$$\sin x = \sqrt{t}, \cos x = \sqrt{1-t}, x = \arcsin \sqrt{t}, dx = \frac{1}{\sqrt{1-t}} \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$LHS = \int_0^1 (rac{\sqrt{t}}{\sqrt{1-t}})^{lpha} rac{1}{2} t^{-rac{1}{2}} (1-t)^{-rac{1}{2}} dt = rac{1}{2} \int_0^1 t^{rac{1}{2}lpha - rac{1}{2}} (1-t)^{-rac{1}{2}lpha - rac{1}{2}} dt = rac{1}{2} \mathrm{B}(rac{1}{2}lpha + rac{1}{2}, -rac{1}{2}lpha + rac{1}{2})$$

4.计算极限:

$$\lim_{\alpha \to +\infty} \sqrt{\alpha} \int_0^1 x^{3/2} (1-x^5)^{\alpha} dx$$

$$x = t^{\frac{1}{5}}, dx = \frac{1}{5}t^{-\frac{4}{5}}dt$$

$$LHS = \lim \sqrt{lpha} \int_0^1 t^{rac{3}{10}} (1-t)^{lpha} rac{1}{5} t^{-rac{4}{5}} dt = \lim rac{1}{5} \sqrt{lpha} \mathrm{B}(rac{1}{2},lpha+1) = \lim rac{1}{5} \sqrt{lpha} rac{\sqrt{\pi} \Gamma(lpha+1)}{\Gamma(lpha+rac{3}{6})}$$

छि
$$\lim\sqrt{lpha}rac{\Gamma(lpha+1)}{\Gamma(lpha+rac{3}{2})}=\lim\sqrt{lpha}rac{\Gamma(lpha+rac{1}{2})}{\Gamma(lpha+1)}=k$$
्रण्री $\limlpharac{\Gamma(lpha+rac{1}{2})}{\Gamma(lpha+rac{3}{2})}=k^2=1, k=1$

于是
$$LHS = \frac{1}{5}\sqrt{\pi}$$

5.设a>0,试求由曲线 $x^n+y^n=a^n$ 和两坐标轴所围成的平面图形在第一象限的面积。

$$S=\int_0^a dx \int_0^{\sqrt[n]{a^n-x^n}} 1 dy = \int_0^a \sqrt[n]{a^n-x^n} dx$$

$$egin{aligned} x &= at^{rac{1}{n}}, dx = rac{a}{n}t^{rac{1}{n}-1}dt \ S &= \int_0^1 a\sqrt[n]{1-t}rac{a}{n}t^{rac{1}{n}-1}dt = rac{a^2}{n}\mathrm{B}(rac{1}{n},rac{1}{n}+1) \end{aligned}$$

证明本讲中的例1、例3、例5、例6

见49讲讲义