(周一)

T1 用五种方法计算 Ω: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ 的体积 $V(\Omega)$

法1 "先一后二"法

由 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow z^2=c^2(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})$ $\quad D=\{(x,y)\mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$

$$\therefore V(\Omega)=\iiint_\Omega dV=\iint_D dxdy\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dz$$

$$=\iint_D 2c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy \qquad D'=\{(r,\theta)\mid 0\le r\le 1, 0\le\theta\le 2\pi\}$$

$$\overset{\substack{x=ar\cos\theta\\ y=br\sin\theta}}{=}\iint_{D'}2c\sqrt{1-r^2}\,abr\,drd\theta$$

$$=2abc\int_0^{2\pi}d\theta\int_0^1(1-r^2)^{\frac{1}{2}}\cdot r\,dr=\frac{4}{3}\pi abc$$

法2 "先二后一"法

$$V(\Omega)=\iiint_\Omega dV=\int_{-c}^{c}dz\iint_D 1\,dxdy \qquad D=\{(x,y)\mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}\}$$

$$=\int_{-c}^{c}\pi\cdot a\sqrt{1-\frac{z^2}{c^2}}\cdot b\sqrt{1-\frac{z^2}{c^2}}\,dz$$ (注: 椭圆面积 $\pi\cdot a\cdot b$)

$$=\pi ab\int_{-c}^{c}1-\frac{z^2}{c^2}\,dz=\frac{4}{3}\pi abc$$

法三 "广义球坐标变换"

$$\begin{cases}x=ar\sin\theta\cos\varphi\\ y=br\sin\theta\sin\varphi\\ z=cr\cos\theta\end{cases}\qquad\begin{cases}0\le r\le 1\\ 0\le\theta\le\pi\\ 0\le\varphi\le 2\pi\end{cases}$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right|=abcr^2\sin\theta$$

$$\therefore V(\Omega)=\iiint_\Omega dV=\int_0^{2\pi}\int_0^{\pi}\int_0^1 abcr^2\sin\theta\,dr\,d\theta\,d\varphi=\frac{4}{3}\pi abc$$

法四 "累次积分法"

$$V(\Omega)=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dz\,dy\,dx$$

$$=2ab\pi\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$=\frac{4}{3}\pi abc$$

法五 "伸缩法"

$$\begin{cases}u=\frac{1}{a}x\\ v=\frac{1}{b}y\\ w=\frac{1}{c}z\end{cases}\Rightarrow\begin{cases}x=au\\ y=bv\\ z=cw\end{cases}\qquad\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=abc$$

$$\therefore V(\Omega)=\iiint_{\Omega}dV=\iiint_{\Omega'}abc\,dV\qquad \Omega'=\{(u,v,w)\mid u^2+v^2+w^2\le 1\}\ (球)$$

$$=abc\cdot\underbrace{\frac{4}{3}\pi\cdot 1^3}_{球体积}=\frac{4}{3}\pi abc$$

T2 计算 $I=\iiint_{x^2+y^2+z^2\le 1}\cos(ax+by+cz)\,dV$ 与 $\iiint_{x^2+y^2+z^2\le 1}(ax+by+cz)\,dV$

解：由18讲讲义例3知，若 $f\in C(\mathbb{R}^3)$

$$\iiint_{x^2+y^2+z^2\le 1}f(ax+by+cz)\,dV=\pi\int_{-1}^{1}f(\lambda u)(1-u^2)\,du\qquad \lambda=\sqrt{a^2+b^2+c^2}$$

$\therefore$ ①: $\iiint_{x^2+y^2+z^2\le 1}\cos(ax+by+cz)\,dV$

$$=\pi\int_{-1}^{1}\cos(\lambda u)(1-u^2)\,du$$

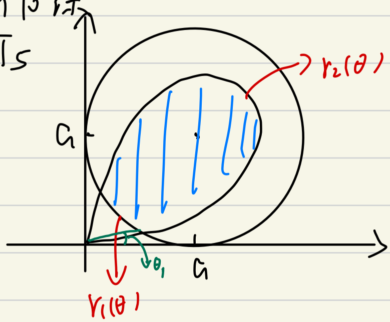
$$=-\frac{4\pi}{\lambda^2}\cos\lambda+\frac{4\pi}{\lambda^3}\sin\lambda\qquad\left(\lambda=\sqrt{a^2+b^2+c^2}\right)$$

②: $\iiint_{x^2+y^2+z^2\le 1}(ax+by+cz)^m\,dV=\pi\int_{-1}^{1}\lambda^m u^m(1-u^2)\,du$

$$=\begin{cases}0 & m\text{ 奇}\\ \dfrac{4\pi\lambda^m}{(m+1)(m+3)} & m\text{ 偶}\end{cases}\qquad\left(\lambda=\sqrt{a^2+b^2+c^2}\right)$$

Ch10 综

T5



图像关于 $y=x$ 对称

取极坐标 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

对于 $(x-a)^2+(y-a)^2 \le a^2$

则 $r^2-2ra(\cos\theta+\sin\theta)-a^2 \le 0$

$\therefore$ 取 $r_1(\theta) = a(\cos\theta+\sin\theta) - a\sqrt{\sin 2\theta}$

对于 $(x^2+y^2)^2 = 8a^2xy \Rightarrow r_2(\theta) = 2a\sqrt{\sin 2\theta}$

令 $r_1(\theta) = r_2(\theta) \Rightarrow 8\sin 2\theta = 1 \Rightarrow \theta_1 = \frac{1}{2}\arcsin\frac{1}{8}$

令 $D = \{(r,\theta) \mid r_1(\theta) \le r \le r_2(\theta),\ \theta_1 \le \theta \le \frac{\pi}{4}\}$

由对称性

$$S = 2\iint_D dx\,dy = 2\int_{\theta_1}^{\frac{\pi}{4}} d\theta \int_{r_1(\theta)}^{r_2(\theta)} r\,dr$$

$$= \int_{\theta_1}^{\frac{\pi}{4}} 4a^2\sin 2\theta - \left[a^2(\cos\theta+\sin\theta)^2 - 2a^2(\cos\theta+\sin\theta)\sqrt{\sin 2\theta} + a^2\sin 2\theta\right] d\theta$$

$$= a^2\int_{\theta_1}^{\frac{\pi}{4}} \left(2\sin 2\theta + 2(\cos\theta+\sin\theta)\sqrt{\sin 2\theta} - 1\right) d\theta$$

$$= a^2\Big[\underbrace{\cos\left(\arcsin\frac{1}{8}\right)}_{=\frac{3\sqrt{7}}{8}} \underbrace{- \frac{\pi}{4} + \frac{1}{2}\arcsin\frac{1}{8}}_{= -\frac{1}{2}\arccos\frac{1}{8}}\Big] + \underbrace{2a^2\int_{\theta_1}^{\frac{\pi}{4}} (\sin\theta+\cos\theta)\sqrt{\sin 2\theta}\, d\theta}_{I} \quad (*)$$

对于I：作 $\theta + \frac{\pi}{4} = t$，有

$$I = 2a^2\int_{\theta_1+\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}\sin t\sqrt{-\cos 2t}\, dt$$

$$= -2a^2\int_{\theta_1+\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1-2\cos^2 t}\ d(\sqrt{2}\cos t)$$

这里进行一个化简

$$\varphi=\left(\theta_1+\frac{\pi}{4}\right)=\frac{1}{2}\arcsin\frac{1}{8}+\frac{\pi}{4}$$

$$\cos 2\varphi=\cos\left(\arcsin\frac{1}{8}+\frac{\pi}{2}\right)=\frac{3\sqrt{7}}{8}\cdot 0-\frac{1}{8}\cdot 1=-\frac{1}{8}$$

$$\therefore\ \theta_1+\frac{\pi}{4}=\varphi=\frac{1}{2}\arccos\left(-\frac{1}{8}\right)$$

$$\therefore\ I=2a^2\int_{\frac{\pi}{2}}^{\frac{1}{2}\arccos(-\frac{1}{8})}\sqrt{1-(\sqrt{2}\cos t)^2}\,d\sqrt{2}\cos t$$

$\sqrt{2}\cos t=\sin u$

$$=2a^2\int_0^{\arcsin\frac{\sqrt{14}}{4}}\cos^2 u\,du=$$ ← 半角公式可求

$$=a^2\cdot\left(\frac{\sqrt{7}}{8}+\arcsin\frac{\sqrt{14}}{4}\right)$$

$$\therefore\ 由此式:\ S=a^2\left(\frac{3\sqrt{7}}{8}-\frac{1}{2}\arccos\frac{1}{8}\right)+a^2\left(\frac{\sqrt{7}}{8}+\arcsin\frac{\sqrt{14}}{4}\right)$$

$$=a^2\left(\frac{\sqrt{7}}{2}-\frac{1}{2}\arccos\frac{1}{8}+\arcsin\frac{\sqrt{14}}{4}\right)$$ (形式不唯一)

T6 显然 Ω 关于 xoy, yoz 平面对称

记 Ω_1: $(x^2+y^2)^2+z^4=y$, $x,y,z\geq 0$

用柱面坐标 $\Rightarrow$ Ω_1': $0\leq\theta\leq\pi/2$ $\quad 0\leq r\leq\sin^{1/3}\theta$ $\quad 0\leq z\leq(r\sin\theta-r^4)^{1/4}$

$$\therefore\ V=4\iiint_{\Omega'}dV=4\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sin^{\frac{1}{3}}\theta}dr\int_0^{(r\sin\theta-r^4)^{\frac{1}{4}}}r\cdot dz$$

$r=(x\sin\theta)^{\frac{1}{3}}$

$$=\frac{4}{3}\int_0^{\frac{\pi}{2}}d\theta\int_0^1 x^{-1/4}(1-x)^{1/4}\sin\theta\,dx$$

$$=\frac{4}{3}\int_0^1 x^{-1/4}(1-x)^{1/4}\,dx$$

$$=\frac{4}{3}B\left(\frac{3}{4},\frac{5}{4}\right)\overset{定理13.39}{=}\frac{4}{3}\frac{\Gamma(\frac{3}{4})\cdot\Gamma(\frac{5}{4})}{\Gamma(2)}$$

§13.5

$$\overset{B13-B14}{=}\frac{\sqrt{2}}{3}\pi$$

T8 分析：

从 $ax+by+c \to t\sqrt{a^2+b^2}+c \Rightarrow$ 考虑变量代换 $t=\frac{ax+by}{\sqrt{a^2+b^2}}$

从形式 $\sqrt{1-t^2} \Rightarrow$ 猜测另一个与之组成圆，即积分区域为 $s^2+t^2\le$ 一个常数

$\therefore$ 令 $s=\frac{-bx+ay}{\sqrt{a^2+b^2}}$

$\therefore$ 考虑坐标变换 $\begin{pmatrix} t \\ s \end{pmatrix}=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\cdot\frac{1}{\sqrt{a^2+b^2}}$ Jacobi $=1$

则 $s^2+t^2=x^2+y^2\le1$

$$\therefore \iint_{x^2+y^2\le1} f(ax+by+c)\,dx\,dy=\iint_{s^2+t^2\le1} f(t\sqrt{a^2+b^2}+c)\,ds\,dt$$

$$=\int_{-1}^{1} f(t\sqrt{a^2+b^2}+c)\,dt\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} ds$$

$$=2\int_{-1}^{1}\sqrt{1-t^2}\,f(t\sqrt{a^2+b^2}+c)\,dt$$

(周三)

§10.3

T5 (8)

(不妨 $a>0$)

$$V=4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a(\cos\varphi\sin\theta)^{\frac{1}{3}}} r^2\sin\theta\,dr\,d\varphi\,d\theta$$

$$=\frac{4}{3}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} a^3\sin^2\theta\cos\varphi\,d\varphi\,d\theta$$

$$=\frac{\pi}{3}a^3$$

T6 $D: (x-\frac{1}{2})^2+(y-\frac{1}{2})^2+(z-\frac{1}{2})^2\le(\frac{\sqrt{3}}{2})^2 \Rightarrow V(D)=\frac{4}{3}\pi(\frac{\sqrt{3}}{2})^3=\frac{\sqrt{3}}{2}\pi$

由 $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ 处的球面坐标可知

$$I=\iiint_D (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_0^{\frac{\sqrt{3}}{2}}\left[\frac{3}{4}+r(\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta)+r^2\right]\cdot r^2\sin\theta\,dr$$

$$=\frac{3}{5}\sqrt{3}\pi$$

$$\therefore \bar{f}=\frac{I}{V(D)}=\frac{6}{5}$$

T12 $\mu = \frac{k}{\rho}$，用球面坐标 (ρ,θ,φ)

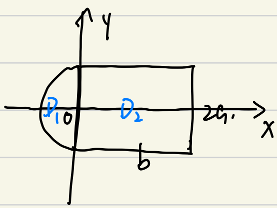
由对称性 $V=\{(\rho,\theta,\varphi) \mid r\le\rho\le R,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\varphi\le\frac{\pi}{2}\}$

$$m = 8\iiint_V \mu\, dV$$

$$= 8\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_r^R \frac{k}{\rho}\,\rho^2\sin\theta\, d\rho$$

$$= 2\pi k(R^2-r^2)$$

T14



建坐标如图，显然质心在x轴上，设密度为ρ为常数

$$\therefore 0 = x_G = \frac{\iint_D x\rho\,dxdy}{\iint_D \rho\,dxdy} = \frac{\iint_{D_1} x\,dxdy + \iint_{D_2} x\,dxdy}{\iint_{D_1}dxdy+\iint_{D_2}dxdy}$$

$$= \frac{-\frac{2}{3}a^3+ab^2}{\frac{\pi a^2}{2}+ab} \Rightarrow b=\frac{\sqrt{6}}{3}a$$

T16 $\rho = \frac{k}{\sqrt{x^2+y^2+z^2}}$，采用球面坐标系.

由对称性 $x_G=y_G=0$

$$z_G = \frac{\iiint_V z\rho\,dxdydz}{\iiint_V \rho\,dxdydz} = \frac{\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta} r\cos\theta\cdot\frac{k}{r}\cdot r^2\sin\theta\,dr}{\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\frac{k}{r}\cdot r^2\sin\theta\,dr} = \frac{4}{5}a$$

∴质心坐标为 $(0,0,\frac{4}{5}a)$

T19 由对称性知，引力在Ox轴和Oy轴上的射影为0 即 $F_x=F_y=0$

采用球面坐标系，引力在Oz轴上的射影为

$$F_z = \iiint_V \frac{G\rho z}{\sqrt{(x^2+y^2+z^2)^3}}\,dxdydz$$

$$= G\rho\int_0^{2\pi}d\varphi\int_0^R dr\int_{\pi-\alpha}^{\pi}\sin\theta\cos\theta\,d\theta$$

$$= -G\pi R\rho\sin^2\alpha$$（负表方向）

Ch10 综

T4 $f(x,y)=\frac{x+y}{\sqrt{2}}-x^2-y^2=\frac{1}{4}-(x-\frac{1}{2\sqrt{2}})^2-(y-\frac{1}{2\sqrt{2}})^2$

$\therefore$ 单位圆 $\Omega: x^2+y^2\le 1$，被分成两部

$$\Omega_1=\{(x,y)\mid f(x,y)\ge 0\}$$

$$\Omega_2=\Omega\setminus\Omega_1$$

$$I=\iint_\Omega |f(x,y)|\,dxdy=\iint_{\Omega_1} f\,dxdy-\iint_{\Omega_2} f\,dxdy$$

$$=2\iint_{\Omega_1} f\,dxdy-\iint_{\Omega} f\,dxdy=I_1-I_2$$

$$I_1=2\iint_{\Omega_1} f\,dxdy=2\iint_{\Omega_1}\left[\frac{1}{4}-(x-\frac{1}{2\sqrt{2}})^2-(y-\frac{1}{2\sqrt{2}})^2\right]dxdy$$

令 $x-\frac{1}{2\sqrt{2}}=r\cos\varphi,\quad y-\frac{1}{2\sqrt{2}}=r\sin\varphi$

$$\Rightarrow I_1=2\int_0^{2\pi}d\varphi\int_0^{\frac{1}{2}}(\frac{1}{4}-r^2)r\,dr=4\pi(\frac{1}{32}-\frac{1}{64})$$

$$I_2=\iint_{x^2+y^2\le 1}(\frac{x+y}{\sqrt{2}}-x^2-y^2)\,dxdy\overset{\text{对称性}}{=}-\iint_{x^2+y^2\le 1}(x^2+y^2)\,dxdy=-\int_0^{2\pi}d\theta\int_0^1 r^3\,dr$$

$$=-\frac{\pi}{2}$$

$$\therefore I=\frac{\pi}{16}+\frac{\pi}{2}=\frac{9}{16}\pi$$

(周五)

T1：见课本 P184-185

T2：作正交变换 $U=AX$，其中 $A=\begin{pmatrix}\frac{a_1}{\lambda} & \frac{a_2}{\lambda} & \cdots & \frac{a_6}{\lambda}\\ a_{21} & \ddots & & \vdots\\ \vdots & & & \\ \cdots & & & a_{66}\end{pmatrix}$，$\lambda=\sqrt{\sum_{i=1}^{6}a_i^2}$ $\det(A)=\pm 1$

此时 $a_1x_1+\cdots a_6x_6=\lambda u_1$

$$I=\iint\cdots\int_{u_1^2+\cdots u_6^2\le 1} f(\lambda u_1)\,du_1\cdots du_6=\int_{-1}^{1}f(\lambda u_1)\,du_1\int\cdots\int_{u_2^2+\cdots u_6^2\le 1-u_1^2}du_2\cdots du_6$$

$$=\frac{8}{15}\pi^2\int_{-1}^{1}(1-u_1^2)^{\frac{5}{2}}f(\lambda u_1)\,du_1$$

§10.4

T1 (1) $\frac{n}{3}$

(2) $I=\int\cdots\int_{[0,1]^n}\sum x_i^2+2\int\cdots\int_{[0,1]^n}\sum_{i<j}x_ix_j=\frac{n}{3}+\frac{n(n-1)}{4}$

$$=\frac{3n^2+n}{12}$$

(3) $\frac{1}{(2n)!!}=\frac{1}{2^n\cdot n!}$

Ch10 練

T3 $I_1=\int_0^1 \sin(\ln\frac{1}{x})\frac{x^b-x^a}{\ln x}dx$

$=\int_0^1 \sin(\ln\frac{1}{x})\left(\int_a^b x^t\,dt\right)dx$

$\overset{\ln x=y}{=}\int_{-\infty}^0\left(-\sin y\int_a^b e^{yt}\,dt\right)e^y\,dy$

$=-\int_a^b\underbrace{\int_{-\infty}^0 \sin y\, e^{(t+1)y}\,dy}_{I_3}\,dt$

$I_3=-\int_{-\infty}^0 e^{(t+1)y}\,d\cos y$

$=-e^{(t+1)y}\cos y\Big|_{-\infty}^0+\int_{-\infty}^0\cos y\,de^{(t+1)y}$

$=-1+\int_{-\infty}^0\cos y\cdot(t+1)e^{(t+1)y}\,dy$

$=-1+(t+1)\int_{-\infty}^0 e^{(t+1)y}\,d\sin y$

$=-1+(t+1)\left[e^{(t+1)y}\sin y\Big|_{-\infty}^0-\int_{-\infty}^0\sin y\cdot e^{(t+1)y}\cdot(t+1)dy\right]$

$=-1-(t+1)^2 I_3$

$\therefore I_3=\frac{-1}{1+(t+1)^2}\Rightarrow I_1=\int_a^b\frac{1}{1+(t+1)^2}d(t+1)$

$=\arctan(t+1)\Big|_a^b=\arctan(b+1)-\arctan(a+1)$

同理 $I_2=\int_a^b\frac{t+1}{1+(t+1)^2}dt$

$=\frac{1}{2}\int_a^b\frac{d(t^2+2t+2)}{t^2+2t+2}=\frac{1}{2}\ln(t^2+2t+2)\Big|_a^b=\frac{1}{2}\ln\frac{b^2+2b+2}{a^2+2a+2}$

$=\ln\sqrt{\frac{1+(b+1)^2}{1+(a+1)^2}}$