数学分析B2汪老师班第二周作业答案参考

Ex.9.2. 8.证明函数$\mathbf{u}=\frac{\mathbf{1}}{\sqrt{\mathbf{t}}}\mathrm{e}^{-\frac{\mathbf{x^2}}{\mathbf{4t}}}$ 满足热传导方程$\frac{\partial \mathbf{u}}{\partial \mathbf{t}}=\frac{\partial^2\mathbf{u}}{\partial \mathbf{x^2}}$.

$$\frac{\partial u}{\partial t}=-\frac{1}{2}\cdot t^{-\frac{3}{2}}e^{-\frac{x^2}{4t}}+\frac{1}{\sqrt{t}}\cdot e^{-\frac{x^2}{4t}}\frac{x^2}{4}\frac{1}{t^2}$$

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{t}}\cdot e^{-\frac{x^2}{4t}}\cdot\left(-\frac{1}{4t}\right)\cdot 2x$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{2}{4t\sqrt{t}}\cdot\left(e^{-\frac{x^2}{4t}}+x\cdot e^{-\frac{x^2}{4t}}\cdot\left(-\frac{2x}{4t}\right)\right)$$

11.设$\mathbf{r}=\sqrt{\mathbf{x^2}+\mathbf{y^2}+\mathbf{z^2}}$, 证明当 $\mathbf{r}\neq\mathbf{0}$ 时有

(1)$\frac{\partial^2\mathbf{r}}{\partial\mathbf{x^2}}+\frac{\partial^2\mathbf{r}}{\partial\mathbf{y^2}}+\frac{\partial^2\mathbf{r}}{\partial\mathbf{z^2}}=\frac{\mathbf{2}}{\mathbf{r}}$

(2)$\frac{\partial^2\ln\mathbf{r}}{\partial\mathbf{x^2}}+\frac{\partial^2\ln\mathbf{r}}{\partial\mathbf{y^2}}+\frac{\partial^2\ln\mathbf{r}}{\partial\mathbf{z^2}}=\frac{\mathbf{1}}{\mathbf{r^2}}$

(3)$\frac{\partial^2}{\partial\mathbf{x^2}}\frac{\mathbf{1}}{\mathbf{r}}+\frac{\partial^2}{\partial\mathbf{y^2}}\frac{\mathbf{1}}{\mathbf{r}}+\frac{\partial^2}{\partial\mathbf{z^2}}\frac{\mathbf{1}}{\mathbf{r}}=\mathbf{0}$

(1) $\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}\quad\frac{\partial^2 r}{\partial x^2}=\frac{\sqrt{x^2+y^2+z^2}-x\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$

$\sum_{x,y,z}\left(\frac{1}{r}-\frac{x^2}{r^3}\right)=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}$

(2) $\frac{\partial\ln r}{\partial x}=\frac{\partial\ln r}{\partial r}\cdot\frac{\partial r}{\partial x}=\frac{1}{r}\cdot\frac{x}{r}=\frac{x}{x^2+y^2+z^2}\quad\frac{\partial^2\ln r}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)=\frac{x^2+y^2+z^2-x\cdot 2x}{(x^2+y^2+z^2)^2}$

$\sum_{x,y,z}\frac{r^2-2x^2}{r^4}=\frac{3}{r^2}-\frac{2\left(x^2y^2+z^2\right)}{r^4}=\frac{1}{r^2}$

(3) $\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{-1}{r^2}\cdot\frac{x}{r}\quad\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\right)=\frac{-r^3+x\cdot 3r^2\cdot\frac{x}{r}}{r^6}$

$\sum_{x,y,z}\frac{r\left(-r^2+3x^2\right)}{r^6}=-\frac{3}{r^3}+\frac{3\left(x^2+y^2+z^2\right)}{r^5}=0$

19.求下列复合函数的偏导数或导数

(3)设$\mathbf{u}=\ln\left(\mathbf{x^2}+\mathbf{y^2}\right),\mathbf{x}=\mathrm{e}^{\mathbf{t+s+r}},\mathbf{y}=\mathbf{4}\left(\mathbf{s^2}+\mathbf{t^2}\right)$,求$\frac{\partial\mathbf{u}}{\partial\mathbf{r}},\frac{\partial\mathbf{u}}{\partial\mathbf{s}},\frac{\partial\mathbf{u}}{\partial\mathbf{t}}$

(4)设$\mathbf{u}=\frac{\mathrm{e}^{\mathbf{ax}}(\mathbf{y}-\mathbf{z})}{\mathbf{a^2}+\mathbf{1}},\mathbf{y}=\mathbf{a}\sin\mathbf{x},\mathbf{z}=\cos\mathbf{x}$, 求$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{x}}$

(3)$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{2x}{x^2+y^2}e^{t+s+r}+\frac{2y}{x^2+y^2}\cdot 0=\frac{2e^{2(t+s+r)}}{e^{2(t+s+r)}+16(s^2+t^2)^2}$

$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=\frac{2x}{x^2+y^2}e^{t+s+r}+\frac{2y}{x^2+y^2}8s=\frac{2e^{2(t+s+r)}+64s\left(s^2+t^2\right)}{e^{2(t+s+r)}+16(s^2+t^2)^2}$

$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=\frac{2x}{x^2+y^2}e^{t+s+r}+\frac{2y}{x^2+y^2}8t=\frac{2e^{2(t+s+r)}+64t\left(s^2+t^2\right)}{e^{2(t+s+r)}+16(s^2+t^2)^2}$

(4)$u=\frac{e^{ax}(a\sin x-\cos x)}{a^2+1}$

$$\begin{aligned}\frac{du}{dx}&=e^{ax}\cdot\frac{a}{a^2+1}(a\sin x-\cos x)+\frac{e^{ax}}{a^2+1}(a\cos x+\sin x)\\&=\frac{e^{ax}}{a^2+1}\left((a^2+1)\sin x\right)\\&=e^{ax}\sin x\end{aligned}$$

26.设$\mathbf{z=f(xy)}$,$\mathbf{f}$为可微函数,证明 $\mathbf{x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0}$.

$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=x\cdot f'(xy)\cdot y-yf'(xy)\cdot x=0$

27.设$\mathbf{z=f\left(\ln x+\frac{1}{y}\right)}$,$\mathbf{f}$ 为可微函数,证明 $\mathbf{x\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=0}$.

$x\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=x\cdot f'\left(\ln x+\frac{1}{y}\right)\cdot\frac{1}{x}+y^2f'\left(\ln x+\frac{1}{y}\right)\cdot\left(-\frac{1}{y^2}\right)=0$

29.若$\mathbf{u=F(x,y)}$,$\mathbf{F}$任意二阶偏导存在而$\mathbf{x=r\cos\varphi,y=r\sin\varphi}$.证明

$$\left(\frac{\partial \mathbf{u}}{\partial \mathbf{r}}\right)^2+\left(\frac{\mathbf{1}}{\mathbf{r}}\frac{\partial \mathbf{u}}{\partial \varphi}\right)^2=\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^2+\left(\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\right)^2$$

$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\varphi+\frac{\partial u}{\partial y}\sin\varphi$

$\frac{\partial u}{\partial \varphi}=\frac{\partial u}{\partial x}r\sin\varphi-\frac{\partial u}{\partial y}\cdot r\cos\varphi$ left=right

30.试证方程 $\mathbf{\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0}$ 经变化 $\xi=\mathbf{x+y},\eta=\mathbf{3x-y}$ 后变成 $\mathbf{\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0}$. 其中二阶偏导数均连续

$$\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$$

$$\begin{cases}x=\frac{1}{4}(\xi+\eta)\\ y=\frac{1}{4}(3\xi-\eta)\end{cases}$$

得到

$$\frac{\partial x}{\partial\xi}=\frac{\partial x}{\partial\eta}=\frac{1}{4},\frac{\partial y}{\partial\xi}=\frac{3}{4},\frac{\partial y}{\partial\eta}=-\frac{1}{4}$$

$$\frac{\partial u}{\partial\xi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\xi}=\frac{1}{4}\left(\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial^2 u}{\partial\eta\partial\xi}=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial\xi}+6\frac{\partial u}{\partial\xi}\right)=0$$

20.求下列复合函数的偏导数或导数,其中各题中的$\mathbf{f}$ 均有连续的二阶偏导

(2)设$\mathbf{u}=\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z}),\mathbf{x}=\sin\mathbf{t},\mathbf{y}=\cos\mathbf{t},\mathbf{z}=\mathrm{e}^{\mathbf{t}}$,求$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{t}}$;

(3)设$\mathbf{u}=\mathbf{f}\left(\mathbf{x}^{2}-\mathbf{y}^{2},\mathrm{e}^{\mathbf{xy}}\right)$,求$\frac{\partial\mathbf{u}}{\partial\mathbf{x}},\frac{\partial^{2}\mathbf{u}}{\partial\mathbf{x}\partial\mathbf{y}}$;

(4)设$\mathbf{u}=\mathbf{f}\left(\mathbf{x}+\mathbf{y}+\mathbf{z},\mathbf{x}^{2}+\mathbf{y}^{2}+\mathbf{z}^{2}\right)$, 求$\frac{\partial\mathbf{u}}{\partial\mathbf{x}},\frac{\partial^{2}\mathbf{u}}{\partial\mathbf{x}^{2}},\frac{\partial^{2}\mathbf{u}}{\partial\mathbf{x}\partial\mathbf{y}}$;

(2)

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\cos tf_1'-\sin tf_2'+\mathrm{e}^tf_3'$$

(3)

$$\frac{\partial u}{\partial x}=2xf_1'+y\mathrm{e}^{xy}f_2',$$
$$\frac{\partial^2u}{\partial x\partial y}=-4xyf_{11}''+\left(2x^2-2y^2\right)\mathrm{e}^{xy}f_{12}''+xy\mathrm{e}^{2xy}f_{22}''+(1+xy)\mathrm{e}^{xy}f_2'.$$

(4)

$$\frac{\partial u}{\partial x}=f_1'+2xf_2',\quad\frac{\partial^2u}{\partial x^2}=f_{11}''+4xf_{12}''+2f_2'+4x^2f_{22}'',$$
$$\frac{\partial^2u}{\partial x\partial y}=f_{11}''+(2x+2y)f_{12}''+4xyf_{22}''.$$

25.设$\mathbf{u}=\mathbf{f}(\mathbf{t}),\mathbf{t}=\varphi(\mathbf{xy},\mathbf{x}+\mathbf{y})$,其中$\mathbf{f},\varphi$分别具有连续的二阶导数及偏导数，求$\frac{\partial\mathbf{u}}{\partial\mathbf{x}},\frac{\partial\mathbf{u}}{\partial\mathbf{y}},\frac{\partial^{2}\mathbf{u}}{\partial\mathbf{x}\partial\mathbf{y}}$.

$\frac{\partial u}{\partial x}=f'\left(\varphi(xy,x+y)\right)\cdot\left(\varphi_1'y+\varphi_2'\right),\frac{\partial u}{\partial y}=f'\left(\varphi(xy,x+y)\right)\cdot\left(\varphi_1'x+\varphi_2'\right)$

$\frac{\partial^2u}{\partial x\partial y}=f''\cdot\left(\varphi_1'x+\varphi_2'\right)\cdot\left(\varphi_1'y+\varphi_2'\right)+f'\cdot\left(\varphi_{11}''xy+\varphi_{12}''x+\varphi_1'+\varphi_{21}''y+\varphi_{22}''\right)$

28.证明函数$\mathbf{u}=\varphi(\mathbf{x}-\mathbf{at})+\psi(\mathbf{x}+\mathbf{at})$满足波动方程

$$\frac{\partial^{2}\mathbf{u}}{\partial\mathbf{t}^{2}}=\mathbf{a}^{2}\frac{\partial^{2}\mathbf{u}}{\partial\mathbf{x}^{2}}$$

其中 φ,ψ 有连续的二阶微商

$$\frac{\partial^2u}{\partial t^2}=a^2\phi''(x-at)+a^2\psi''(x+at).$$
$$\frac{\partial^2u}{\partial x^2}=\phi''(x-at)+\psi''(x+at).$$

32.设变换$\begin{cases}u=x-2y,\\v=x+ay\end{cases}$ 可把方程$6\frac{\partial^{2}\mathbf{z}}{\partial\mathbf{x}^{2}}+\frac{\partial^{2}\mathbf{z}}{\partial\mathbf{x}\partial\mathbf{y}}-\frac{\partial^{2}\mathbf{z}}{\partial\mathbf{y}^{2}}=\mathbf{0}$简化为$\frac{\partial^{2}\mathbf{z}}{\partial\mathbf{u}\partial\mathbf{v}}=\mathbf{0}$. 求常数 **a.**其中二阶偏导数均连续

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\quad\frac{\partial z}{\partial y}=-2\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}$

$\frac{\partial^2z}{\partial x^2}=\frac{\partial^2z}{\partial u^2}+2\frac{\partial^2z}{\partial v\partial u}+\frac{\partial^2z}{\partial v^2}$

$\frac{\partial^2z}{\partial x\partial y}=-2\frac{\partial^2z}{\partial u^2}+(a-2)\frac{\partial^2z}{\partial u\partial v}+a\frac{\partial^2z}{\partial v^2}$

$\frac{\partial^2z}{\partial y^2}=4\frac{\partial^2z}{\partial u^2}-4a\frac{\partial^2z}{\partial u\partial v}+a^2\frac{\partial^2z}{\partial v^2}$

$6\frac{\partial^2}{\partial x^2}+\frac{\partial^2z}{\partial x\partial y}-\frac{\partial^2z}{\partial y^2}=(5a+10)\frac{\partial^2z}{\partial u\partial v}+\left(6+a-a^2\right)\frac{\partial^2z}{\partial v^2}\Rightarrow a=3$

Ex9.3 1.证明下列方程在指定点的附近对y有唯一解并求出y对x在该点处的一阶和二阶导数 **(1)** $\mathbf{x}^{2}+\mathbf{xy}+\mathbf{y}^{2}=\mathbf{7}$,在 $(\mathbf{2},\mathbf{1})$ 处

令 $F(x,y)=x^2+xy+y^2-7$

$F_x'(x,y)=2x+y \quad F_y'(x,y)=2y+x$

$F(2,1)=0 \quad F_x'(2,1)=5 \quad F_y'(2,1)=4$

故在$(2,1)$ 邻棫内隐函数存在且唯一 $\frac{\mathrm{d}y}{\mathrm{d}x}|_{(2,1)}=\frac{-F_x'(2,1)}{F_y'(2,1)}=-\frac{5}{4}$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}|_{(2,1)}=-\frac{F_y'\left(F_{xx}''+F_{xy}''\cdot\left(-\frac{F_x'}{F_y'}\right)\right)-F_x'\left(F_{yx}''+F_{yy}''\cdot\left(-\frac{F_x'}{F_y'}\right)\right)}{(F_y')^2}=-\frac{21}{32}$

2.求由下列方程所确定的隐函数的导数

(2)$\ln\sqrt{\mathbf{x^2+y^2}}=\arctan\frac{\mathbf{y}}{\mathbf{x}}$**,求** $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}}$**和**$\frac{\mathrm{d}^2\mathbf{y}}{\mathrm{d}\mathbf{x}^2}$;

(4) $\mathrm{e}^{-\mathbf{xy}}-\mathbf{2z}+\mathrm{e}^{\mathbf{z}}=\mathbf{0}$**,求** $\frac{\partial\mathbf{z}}{\partial\mathbf{x}},\frac{\partial\mathbf{z}}{\partial\mathbf{y}},\frac{\partial\mathbf{x}}{\partial\mathbf{y}},\frac{\partial^2\mathbf{z}}{\partial\mathbf{x}^2}$

(2) $\frac{dy}{dx}=\frac{x+y}{x-y},\frac{d^2y}{dx^2}=\frac{2\left(x^2+y^2\right)}{(x-y)^3}$

(4)

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=\frac{ye^{-xy}}{e^z-2} \quad \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=\frac{xe^{-xy}}{e^z-2} \quad \frac{\partial x}{\partial y}=-\frac{F_y'}{F_x'}=-\frac{x}{y}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{-y^2e^{-xy}\left(e^z-2\right)^2-y^2e^{z-2xy}}{\left(e^z-2\right)^3}$$

4.试求由下列方程所确定的隐函数的微分 (1) $\cos^2\mathbf{x}+\cos^2\mathbf{y}+\cos^2\mathbf{z}=\mathbf{1}$**,求**$\mathrm{d}\mathbf{z}$;

$-2\cos x\sin x\,\mathrm{d}x-2\cos y\sin y\,\mathrm{d}y-2\cos z\sin z\,\mathrm{d}z=0. \quad \mathrm{d}z=-\frac{\cos x\sin x\,\mathrm{d}x+\cos y\sin y\,\mathrm{d}y}{\cos z\sin z}.$

31.试证方程 $\frac{\partial^2\mathbf{u}}{\partial\mathbf{x}^2}+\mathbf{2}\frac{\partial^2\mathbf{u}}{\partial\mathbf{x}\partial\mathbf{y}}\cos\mathbf{x}-\frac{\partial^2\mathbf{u}}{\partial\mathbf{y}^2}\sin^2\mathbf{x}-\frac{\partial\mathbf{u}}{\partial\mathbf{y}}\sin\mathbf{x}=\mathbf{0}$**经过参数变换** $\xi=\mathbf{x}-\sin\mathbf{x}+\mathbf{y}$, $\eta=\mathbf{x}+\sin\mathbf{x}-\mathbf{y}$**后变成**$\frac{\partial^2\mathbf{u}}{\partial\xi\partial\eta}=\mathbf{0}$**. 其中二阶偏导数均连续.**

$\xi=x-\sin x+y,\eta=x+\sin x-y$

则 $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}(1-\cos x)+\frac{\partial u}{\partial\eta}(1+\cos x)$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}$

$\frac{\partial^2u}{\partial x^2}=\frac{\partial^2u}{\partial\xi^2}(1-\cos x)^2+\frac{\partial^2u}{\partial\eta\partial\xi}\left(1-\cos^2x\right)+\frac{\partial u}{\partial\xi}\sin x+\frac{\partial^2u}{\partial\eta^2}(1+\cos x)^2+\frac{\partial^2u}{\partial\xi\partial\eta}(1-$
$\cos^2x)+\frac{\partial u}{\partial\eta}(-\sin x)$

$\frac{\partial^2u}{\partial x\partial y}=\frac{\partial^2u}{\partial\xi^2}(1-\cos x)+\frac{\partial^2u}{\partial\eta\delta\xi}(1+\cos x)-\frac{\partial^2u}{\partial\eta^2}(1+\cos x)-\frac{\partial^2u}{\partial\xi\partial\eta}(1-\cos x)$

$\frac{\partial^2u}{\partial y^2}=\frac{\partial^2u}{\partial\xi^2}-\frac{\partial^2u}{\partial\eta\partial\xi}-\frac{\partial^2u}{\partial\xi\partial\eta}+\frac{\partial^2u}{\partial\eta^2}$

再由 $\frac{\partial^2u}{\partial\eta\partial\xi}=\frac{\partial^2u}{\partial\xi\partial\eta}$ 知 $\frac{\partial^2u}{\partial\eta\partial\xi}=0$

6.设 $\mathbf{z}=\mathbf{z}(\mathbf{x},\mathbf{y})$ **是由方程** $\mathbf{2}\sin(\mathbf{x}+\mathbf{2y}-\mathbf{3z})=\mathbf{x}+\mathbf{2y}-\mathbf{3z}$ **所确定的隐函数，试证：** $\frac{\partial\mathbf{z}}{\partial\mathbf{x}}+\frac{\partial\mathbf{z}}{\partial\mathbf{y}}=\mathbf{1}$.

令

$$F(x,y,z)=2\sin(x+2y-3z)-x-2y+3z$$

可得：

$$F_x'=2\cos(x+2y-3z)-1$$

$$F_y'=4\cos(x+2y-3z)-2$$

$$F_z'=-6\cos(x+2y-3z)+3$$

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=\frac{1}{3}\quad\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=\frac{2}{3}\quad\frac{\partial z}{\partial y}+\frac{\partial z}{\partial x}=1$$

7.设 $\mathbf{z}=\mathbf{z}(\mathbf{x},\mathbf{y})$ 是由方程 $\varphi(cx-az,cy-bz)=0$ 所确定的隐函数，试证：不论 φ 为怎样的可微函数，都有 $\mathbf{a}\frac{\partial\mathbf{z}}{\partial\mathbf{x}}+\mathbf{b}\frac{\partial\mathbf{z}}{\partial\mathbf{y}}=\mathbf{c}$.

$\frac{\partial z}{\partial x}=-\frac{\partial F}{\partial x}/\frac{\partial F}{\partial z}=\frac{c\varphi_1'}{a\varphi_1'+b\varphi_2'}$

$\frac{\partial z}{\partial y}=-\frac{\partial F}{\partial y}/\frac{\partial F}{\partial z}=\frac{c\varphi_2'}{a\varphi_1'+b\varphi_2'}$

从而 $a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c$

8.设 $\mathbf{z}=\mathbf{x}^2+\mathbf{y}^2$, 其中 $\mathbf{y}=\mathbf{y}(\mathbf{x})$ 为由方程 $\mathbf{x}^2-\mathbf{x}\mathbf{y}+\mathbf{y}^2=1$ 所定义的函数，求 $\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}\mathbf{x}}$ 及 $\frac{\mathrm{d}^2\mathbf{z}}{\mathrm{d}\mathbf{x}^2}$.

由 $\mathrm{d}\left(x^2-xy-y^2\right)=2x\,\mathrm{d}x-y\,\mathrm{d}x-x\,\mathrm{d}y+2y\,\mathrm{d}y=0$ 有 $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-y}{x-2y}(x\neq 2y)$,

于是

$$\frac{\mathrm{d}z}{\mathrm{d}x}=2x+2y\frac{\mathrm{d}y}{dx}=2x+y\frac{2x-y}{x-2y},\quad x\neq 2y$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=2+2\left(\frac{\mathrm{d}y}{dx}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2+2\left(\frac{2x-y}{x-2y}\right)^2+2y\left(\frac{-6(x^2-xy+y^2)}{(2y-x)^3}\right)$$

10.设 $\mathbf{x}=\mathbf{x}(\mathbf{z}),\mathbf{y}=\mathbf{y}(\mathbf{z})$ 是方程组 $\begin{cases}x+y+z=0,\\x^2+y^2+z^2=1\end{cases}$ 所确定的隐函数组，求 $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{z}},\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{z}}$

在两个方程两端对z求导得到 $\begin{cases}\frac{dx}{dz}+\frac{dy}{dz}+1=0\\2\frac{dx}{dz}x+2\frac{dy}{dz}y+2z=0\end{cases}$ 从而解得 $\begin{cases}\frac{dx}{dz}=\frac{y-z}{x-y}\\\frac{dy}{dz}=\frac{x-z}{y-x}\end{cases}$

11.设 $\mathbf{u}=\mathbf{u}(\mathbf{x},\mathbf{y}),\mathbf{v}=\mathbf{v}(\mathbf{x},\mathbf{y})$ 是由下列方程组所确定的隐函数组，求 $\frac{\partial(\mathbf{u},\mathbf{v})}{\partial(\mathbf{x},\mathbf{y})}$.

(1) $\begin{cases}u^2+v^2+x^2+y^2=1\\u+v+x+y=0\end{cases}$

$$\begin{cases}udu+vdv+xdx+ydy=0\\du+dv+dx+dy=0\\dv=\frac{(u-x)dx+(u-y)dy}{v-u}\\du=\frac{(v-x)dx+(v-y)dy}{u-v}\\\frac{\partial(u\cdot v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\\frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{v-x}{u-v}&\frac{v-y}{u-v}\\\frac{x-u}{u-v}&\frac{y-u}{u-v}\end{vmatrix}=\frac{vy-xy-uv+ux-(xv-uv-xy+uy)}{(u-v)^2}\\=\frac{v(y-x)+u(x-y)}{(u-v)^2}=\frac{x-y}{u-v}\end{cases}$$

14.设 $\mathbf{y}=\mathbf{y}(\mathbf{x}),\mathbf{z}=\mathbf{z}(\mathbf{x})$ 是由方程 $\mathbf{z}=\mathbf{x}\mathbf{f}(\mathbf{x}+\mathbf{y})$ 和 $\mathbf{F}(\mathbf{x},\mathbf{y},\mathbf{z})=\mathbf{0}$ 所确定的函数，其中 f 和 F 分别具有一阶连续导数和一阶连续偏导数．求 $\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}\mathbf{x}}$.

$$G(x,y,z)\triangleq xf(x+y)-z$$

则有：

$$G'_x = f(x+y) + xf'(x+y)$$

$$G'_y = xf'(x+y)$$

$$G'_z = -1$$

由隐函数定理可知：

代入得：

$$\frac{dz}{dx} = \frac{F'_x G'_y - F'_y G'_x}{F'_y G'_z - F'_z G'_y}$$

$$\frac{dz}{dx} = \frac{-F'_x f'(x+y)x + F'_y f(x+y) + F'_y f'(x+y)x}{F'_y + F'_z f'(x+y)x}$$