数学分析B2汪老师班第三周作业答案参考

9.2:21,22,23,24,36(2),(5),38

21.求函数 $u=xyz$ **在点** $(1,2,-1)$ **沿方向** $\boldsymbol{l}=(3,-1,1)$ **的方向微商．**

根据方向微商的计算公式

$$\frac{\partial u}{\partial \boldsymbol{l}}=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)\cdot\frac{\boldsymbol{l}}{|\boldsymbol{l}|}=(-2,-1,2)\cdot\frac{(3,-1,1)}{\sqrt{11}}=-\frac{3\sqrt{11}}{11}$$

22.试求函数 $z=\arctan\frac{y}{x}$ **在圆** $x^2+y^2-2x=0$ **上一点** $P\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ **处沿该圆周逆时针方向上的方向微商．**

$$\mathbf{l}=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)\quad \mathbf{grad}(z)=\left(-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$$

则在 $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ 点的方向微商为：

$$\mathbf{grad}(z)\cdot\vec{l}_{x=\frac{1}{2},y=\frac{\sqrt{3}}{2}}=\frac{1}{2}$$

23.求函数 $u=x^2+2y^2+3z^2+xy+3x-2y-6z$ **在点** $(1,1,-1)$ **的梯度和最大方向微商．**

$$u'_x=2x+y+3,u'_y=x+4y-2,u'_z=6z-2$$

$u'_x(1,1,-1)=6,u'_y(1,1,-1)=4,u'_z(1,1,-1)=-12$

$\mathbf{grad}(u)|_{(1,1,-1)}=(6,3,-12)$

$\left(\frac{\partial f}{\partial\vec{e}}\right)_{\max}=|\operatorname{gradu}|_{(1,1,-1)}|=3\sqrt{21}$

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24. 设 $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k},r=|\boldsymbol{r}|$**, 试求 (1)** $\operatorname{grad}\frac{1}{r^2}$**; (2)** $\operatorname{grad}\ln r$**.**

(1) 由 $\frac{1}{r^2}=\frac{1}{x^2+y^2+z^2}$ 有

$$
\begin{aligned}
\frac{\partial \frac{1}{r^2}}{\partial x} &= -\frac{2x}{\left(x^2+y^2+z^2\right)^2} = -\frac{2x}{r^4}, \\
\frac{\partial \frac{1}{r^2}}{\partial y} &= -\frac{2y}{\left(x^2+y^2+z^2\right)^2} = -\frac{2y}{r^4}, \\
\frac{\partial \frac{1}{r^2}}{\partial z} &= -\frac{2z}{\left(x^2+y^2+z^2\right)^2} = -\frac{2z}{r^4},
\end{aligned}
$$

所以 $\mathbf{grad}\frac{1}{r^2} = -\frac{2}{r^4}\mathbf{r}$

(2) 由 $\ln r = \frac{1}{2}\ln\left(x^2+y^2+z^2\right)$ 易有 $\mathbf{grad}\ \ln r = \frac{1}{r^2}\mathbf{r}$.

36.求下列复合函数的微分 $\mathrm{d}u$

(2) $u = f(\xi,\eta), \xi = xy, \eta = \frac{x}{y}$**;**

(5) $u = f(\xi,\eta,\zeta), \xi = x^2+y^2, \eta = x^2-y^2, \zeta = 2xy$**.**

(2) $\mathrm{d}u = \left(f_1'y + \frac{f_2'}{y}\right)\mathrm{d}x + \left(f_1'x - \frac{xf_2'}{y^2}\right)\mathrm{d}y.$

(5) $\mathrm{d}u = (2xf_1' + 2xf_2' + 2yf_3')\,\mathrm{d}x + (2yf_1' - 2yf_2' + 2xf_3')\,\mathrm{d}y.$

38.求直角坐标和极坐标的坐标变换 $x = x(r,\theta) = r\cos\theta, y = y(r,\theta) = r\sin\theta$ **的 Jacobi 行列式．**由题意得:

$$
\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}
$$

又知：

$$
\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta
$$

代入得：

$$
\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r
$$

9.4:3,4,8(1),(4),9,11,16(1),17(2)

3.证明曲线 $x = a\cos t, y = a\sin t, z = bt$ **的切线与** Oz **轴成定角．**

$\mathrm{r}'(t) = (-a\sin t, a\cos t, b)$ $\mathrm{k} = (0,0,1)$

$\cos\theta = \frac{r'\cdot k}{|r'|\cdot|k|} = \frac{b}{\sqrt{a^2+b^2}}$ 为常数

$\therefore$ 曲线的切线与 Oz 轴夹角为常值

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4. 设 $\boldsymbol{r} = \left(\frac{t}{1+t}, \frac{1+t}{t}, t^2\right)(t>0)$**，判断它是不是简单曲线，是不是光滑曲线，并求出它在** $t=1$ **时的切线方程和法平面方程．**

$x(t)$在$t>0$时单调，且$x'(t),y'(t),z'(t)$均连续,故$\boldsymbol{r}$是简单曲线也是光滑曲线.

$$\boldsymbol{r}'(t)=\left(\frac{1}{(1+t)^2},-\frac{1}{t^2},2t\right)$$

将 $t=1$ 代入得切线的方向向量 $\vec{v}=\left(\frac{1}{4},-1,2\right)$, 又 $\boldsymbol{r}(1)=\left(\frac{1}{2},2,1\right)$.
从而切线方程：$\frac{4x-2}{1}=\frac{y-2}{-1}=\frac{z-1}{2}$.
法平面方程：$\frac{1}{4}x-y+2z-\frac{1}{8}=0$.

8.求下列曲面在指定点的切平面和法线方程．
(1) $z=\sqrt{x^2+y^2}-xy$, 在点 $(3,4,-7)$;
(4) $4+\sqrt{x^2+y^2+z^2}=x+y+z$, 在点 $(2,3,6)$.
(1) $\boldsymbol{n}=(17,11,5)$, $\quad\boldsymbol{\pi}:17x+11y+5z-60=0$ $\frac{x-3}{17}=\frac{y-4}{11}=\frac{z+7}{5}$
(4) $\boldsymbol{n}=(5,4,1)$, $\quad\boldsymbol{\pi}:5x+4y+z-28=0$ $\frac{x-2}{5}=\frac{y-3}{4}=\frac{z-6}{1}$

9.求椭球面 $x^2+2y^2+z^2=1$ 上平行于平面 $x-y+2z=0$ 的切平面方程．
$x^2+2y^2+z^2=1$ 在点 (x_0,y_0z_0) 处切平面方程为 $x_0x+2y_0y+z_0z=1$ 平面 $x-y+2z=0$ 法向量 $(1.-1.2)$,故$(x_0,2y_0,z_0)=\lambda(1,-1,2)$

$$x_0^2+2y_0^2+z_0^2=1\Longrightarrow\quad\lambda=\pm\frac{\sqrt{22}}{11}$$

故切平面 $\left(x-\frac{\sqrt{22}}{11}\right)-\left(y+\frac{\sqrt{22}}{22}\right)+2\left(z-\frac{2\sqrt{22}}{11}\right)=0$
$\dot{x}'\left(x+\frac{\sqrt{2}}{11}\right)-\left(y-\frac{\sqrt{22}}{22}\right)+2\left(z+\frac{2\sqrt{22}}{11}\right)=0$

11.求椭球面 $x^2+2y^2+3z^2=21$ 上某点 M 处的切平面 π 的方程，使 π 过已知直线 $L:\frac{x-6}{2}=\frac{y-3}{1}=\frac{2z-1}{-2}$.
M_0. 处切平面方程 $x_0x+2y_0y+3z_0z=21$
π过直线(π过点$(6,3,\frac{1}{2})$且π的法向量垂直于$(2,1,-1)$) $\Longrightarrow\left\{\begin{array}{l}6x_0+6y_0+\frac{3}{2}z_0=21\\2x_0+2y_0-3z_0=0\end{array}\right.$

$$x_0^2+2y_0^2+3z_0^2=21\Longrightarrow\quad(x_0,y_0,z_0)=(3,0,2),(1,2,2)$$

$\pi:x+2z=7\qquad\pi:x+4y+6z=21$

16.求下列平面曲线在给定点的切线和法线方程
(1) $x^3y+xy^3=3-x^2y^2$, 在点 $(1,1)$;
$3x^2ydx+x^3dy+y^3dx+3xy^2dy=-2xy^2dx-2x^2ydy$

$$\frac{dy}{dx}=-\frac{y^3+2xy^2+3x^2y}{3xy^2+2x^2y+x^3}\Longrightarrow\left.\frac{dy}{dx}\right|_{(1,1)}=-1$$

切线:$y=-x+2$　法线:$y=x$

17(2)求下列曲线在给定点的切线和法平面方程

$\begin{cases} 2x^2+3y^2+z^2=47, \\ x^2+2y^2=z \end{cases}$ 在点 $(-2,1,6)$.

$F(x,y,z) \triangleq 2x^2+3y^2+z^2-47 \quad G \triangleq= x^2+2y^2-z$

$\nabla F(-2,1,6)=(-8,6,12)$

$\nabla G(-2,1,6)=(-4,4,-1)$

$\tau=\nabla F \times \nabla G=(-54,-56,-8)$ 切线:$\frac{x+2}{27}=\frac{y-1}{28}=\frac{z-6}{4}$　　法平面:$27(x+2)+28(y-1)+4(z-6)=0$

9.5:2(2),3,4(1),(3),(7),7(1),(3),(4)

2.求下列函数由点 (x_0,y_0) 变到 (x_0+h,y_0+k) 时函数的增量．

(2) $f(x,y)=x^2y+xy^2-2xy,(x_0,y_0)=(1,-1)$.

$h^2+hk^2+k^2-h^2-2hk+h-3k$

3.对于函数 $f(x,y)=\sin\pi x+\cos\pi y$, 用中值定理证明，存在一个数 $\theta, 0<\theta<1$ 使得

$$\frac{4}{\pi}=\cos\frac{\pi\theta}{2}+\sin\left[\frac{\pi}{2}(1-\theta)\right]$$

$f(x_0+h,y_0+k)-f(x_0,y_0)=f_x'(x_0+\theta h,y_0+\theta k)h+f_y'(y_0+\theta h,y_0+\theta k)k$

令 $x_0=0,y_0=-\frac{1}{2}\quad h=k=\frac{1}{2}$

4.求下列函数的Taylor公式，并指出展开式成立的区域．

(1) $f(x,y)=\mathrm{e}^x\ln(1+y)$ 在点 $(0,0)$, 直到三阶为止；

(3) $f(x,y)=\frac{1}{1-x-y+xy}$ 在点 $(0,0)$, 直到 n 阶为止；

(7) $f(x,y)=2x^2-xy-y^2-6x-3y+5$ 在点 $(1,-2)$ 的 Taylor 展开式．

(1) 成立区域：$\{(x,y)\mid y>-1\}$.

$$\begin{aligned} f(x,y)&=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o\left(x^3\right)\right)\left(y-\frac{1}{2}y^2+\frac{1}{3}y^3+o\left(y^3\right)\right) \\ &=y+xy-\frac{1}{2}y^2+\frac{1}{2}x^2y-\frac{1}{2}xy^2+\frac{1}{3}y^3+o\left(\rho^3\right) \end{aligned}$$

(3) 成立区域：$\{(x,y)\mid x<-1,y<-1\}$.

$$
\begin{aligned}
f(x,y) = \frac{1}{(1-x)(1-y)} &= \left(\sum_{i=0}^{n} x^i + o\left(x^n\right)\right)\left(\sum_{i=0}^{n} y^i + o\left(y^n\right)\right) \\
&= \sum_{k=0}^{n}\sum_{i=0}^{k} x^i y^{k-i} + o(\rho).
\end{aligned}
$$

(7) 成立区域：$\mathbb{R}^2$. 配方得：

$$
f(x,y) = 2(x-1)^2 - (y+2)^2 - (x-1)(y+2) + 5.
$$

7.求下列函数的极值．

(1) $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}(x>0, y>0)$;

(3) $f(x,y) = \mathrm{e}^{2x}\left(x+2y+y^2\right)$;

(4) $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$, 求隐函数 $y=y(x)$ 的极值．

(1) $\frac{\partial f}{\partial x} = y - \frac{50}{x^2}, \frac{\partial f}{\partial y} = x - \frac{20}{y^2}, \frac{\partial^2 f}{\partial x^2} = \frac{100}{x^3}, \frac{\partial^2 f}{\partial x \partial y} = 1, \frac{\partial^2 f}{\partial y^2} = \frac{40}{y^3}.$

令 $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0,$

可解得 $x=5, y=2$

当 $x>0, y>0$ 时，$Q(h,k) = 0.8h^2 + 2hk + 5k^2$ 是正定的，

因此 $(x,y) = (5,2)$ 是小极值点，极小值为 30．

(3) $\frac{\partial f}{\partial x} = 2e^{2x}\left(x+2y+y^2\right) + e^{2x}, \frac{\partial f}{\partial y} = e^{2x}(2+2y)$

$\frac{\partial^2 f}{\partial x^2} = e^{2x}\left(4x+8y+4y^2+\ 4\right), \frac{\partial^2 f}{\partial x \partial y} = e^{2x}(4+4y), \frac{\partial^2 f}{\partial y^2} = 2e^{2x}.$

令 $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0,$ 可解得 $x=0.5, y=-1.$

由于 $Q(h,k) = e\left(2h^2+2k^2\right)$ 是正定的，因此 $(x,y) = (0.5,-1)$ 是极小值点，极小值为 $-\frac{e}{2}$

(4) 记 $f(x,y) = \left(x^2+y^2\right)^2 - a^2\left(x^2-y^2\right)$

则 $\frac{\partial f}{\partial x} = 4x\left(x^2+y^2\right) - 2a^2x, \frac{\partial f}{\partial y} = 4y\left(x^2+y^2\right) + 2a^2y.$

因此 $\frac{dy}{dx} = -\frac{2x\left(x^2+y^2\right)-a^2x}{2y\left(x^2+y^2\right)+a^2y} = 0 \Leftrightarrow x=0,$ 或 $2\left(x^2+y^2\right) = a^2.$

若 $x=0,$ 那么 $f(x,y) = 0 \to y=0,$ 从而 $\frac{\partial f}{\partial y} = 0,$ 这说明 $y(x)$ 不存在．

若 $2\left(x^2+y^2\right) = a^2,$ 那么 $f(x,y) = 0 \to x^2 = \frac{3}{8}a^2, y^2 = \frac{1}{8}a^2, a \neq 0.$ 再通过计算 $\frac{d^2y}{dx^2}$

可知，$\left(\pm\sqrt{\frac{3}{8}}|a|, \pm\sqrt{\frac{1}{8}}|a|\right)$ 是极值点，y 极大值为 $\left.\sqrt{\frac{1}{8}}|a|\right)$, 极小值为 $\left.-\sqrt{\frac{1}{8}}|a|\right)$