

● 第18讲: 三重积分的计算与证明 (2023.4.17)

(一) 三重积分  $I = \iiint_{\Omega} f(x, y, z) dV$  的奇偶对称性 (设  $f \in C(\Omega)$ )

(1) 若  $f(x, y, z)$  关于  $z$  是奇(偶)函数, 且  $\Omega$  关于  $z=0$  的坐标面对称, 则  $\iiint_{\Omega} f(x, y, z) dV = 0$  ( $= \iiint_{\Omega_1} f(x, y, z) dV$ ), 其中,  $\Omega$  是

●  $\Omega$  在坐标面  $xOy$  之上或下部分。余类推。

(2) 若  $f(x, y, z)$  关于  $x, y, z$  分别都是偶函数, 且  $\Omega$  关于三个坐标面  $x=0; y=0; z=0$  都对称, 则必有

$\iiint_{\Omega} f(x, y, z) dV = 8 \iiint_{\Omega_0} f(x, y, z) dV$ ,  $\Omega_0$  是  $\Omega$  在第一卦限部分。

● 例1. 再次计算单位球体  $\Omega: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$  的体积  $V(\Omega)$ .

解:  $\because V(\Omega) = \iiint_{\Omega} 1 dx dy dz$  且  $f(x, y, z) \equiv 1$  关于  $x, y, z$  分别是偶函数,  $\Omega$  关于三个坐标面  $x=0, y=0, z=0$  都对称, 故

$V(\Omega) = 8 \iiint_{\Omega_0} 1 dx dy dz$ ,  $\Omega_0$  是  $\Omega$  在第一卦限部分. 作广义球

● 坐标变换  $\begin{cases} x = ar \sin \theta \cos \varphi \\ y = br \sin \theta \sin \varphi \\ z = cr \cos \theta \end{cases}$ , 则  $\begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$  (1).



- $$dV = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} \right| dr d\theta d\varphi = abc r^2 \sin \theta dr d\theta d\varphi$$

$$V(\Omega) = 8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 abc r^2 \sin \theta dr d\theta d\varphi$$

$$= 8abc \left( \int_0^{\frac{\pi}{2}} \sin \theta d\theta \right) \left( \int_0^{\frac{\pi}{2}} r^2 dr \right) \left( \int_0^{\frac{\pi}{2}} d\varphi \right) = \frac{4}{3} abc.$$

例2. 计算曲面  $\Sigma: (x^2+y^2)^2+z^4=y$  围成区域  $\Omega$  的

- $$\text{体积 } V(\Omega) = \iiint_{\Omega} 1 dV \quad (\text{choosing } 6)$$

解: (1) 从  $(x^2+y^2)^2+z^4=y \geq 0$  知, 域  $\Omega$  仅在半空间  $y \geq 0$

中存在, (2) 从  $-x$  到  $x$ ,  $-z$  到  $z$  的方程即  $\Sigma$

的方程不变知,  $\Omega$  关于  $x=0, z=0$  两坐标面对称, 且

- $$f(x, y, z) \equiv 1 \text{ 关于 } x, z \text{ 分别都是偶函数, 故有}$$

$$V(\Omega) = 4 \iiint_{\Omega_0} 1 dx dy dz, \quad \Omega_0 \text{ 是 } \Omega \text{ 在第一卦限部分.}$$

$$\text{作球坐标变换} \begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \text{ 且 } \begin{cases} (r^2 \sin^2 \theta)^2 + (r^2 \cos^2 \theta)^2 \leq y = r \sin \theta \sin \varphi \\ 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

$$\text{即 } 0 \leq r \leq \left( \frac{\sin \theta \sin \varphi}{\sin^4 \theta + \cos^4 \theta} \right)^{\frac{1}{3}}.$$



$$\begin{aligned}
 V(\Omega) &= 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta + \cos\theta}} \sin\theta r^2 dr d\theta d\phi \\
 &= \frac{4}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \sin\theta}{\sin\theta + \cos\theta} d\theta d\phi = \frac{4}{3} \left( \int_0^{\frac{\pi}{2}} \sin\theta d\theta \right) \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\sin\theta + \cos\theta} d\theta \\
 &= \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{\tan^2\theta \sec^2\theta}{1 + \tan\theta} d\theta = \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{\tan^2\theta d\tan\theta}{1 + \tan\theta} \quad \underline{\tan\theta = u} \\
 &= \frac{4}{3} \int_0^{+\infty} \frac{u^2 du}{1+u^4}, \quad \text{令 } I_0 = \int_0^{+\infty} \frac{u^2 du}{1+u^4} \quad \text{则}
 \end{aligned}$$

$$I_0 \stackrel{u=\frac{1}{v}}{=} \int_{+\infty}^0 \frac{\frac{1}{v^2} \cdot \frac{-1}{v^2} dv}{1 + \frac{1}{v^4}} = \int_0^{+\infty} \frac{dv}{1+v^4} = \int_0^{+\infty} \frac{du}{1+u^4} \Rightarrow$$

$$2I_0 = \int_0^{+\infty} \frac{u^2+1}{1+u^4} du = \int_0^{+\infty} \frac{1+u^2}{u^2+u^2} du = \int_0^{+\infty} \frac{d(u-\frac{1}{u})}{(u-\frac{1}{u})^2+2}$$

$$= \frac{1}{\sqrt{2}} \arctan \frac{u-\frac{1}{u}}{\sqrt{2}} \Big|_0^{+\infty} = \frac{1}{\sqrt{2}} \left( \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = \frac{\pi}{\sqrt{2}} \Rightarrow I_0 = \frac{\pi}{2\sqrt{2}}$$

$$V(\Omega) = \frac{4}{3} I_0 = \frac{4}{3} \times \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}}{3} \pi.$$

例3. 设  $(a, b, c) \neq 0 = (0, 0, 0)$  为常向量,  $f(x)$  连续函数.

$\Omega: x^2 + y^2 + z^2 \leq R^2$  ( $R > 0$  常数). 证明:

$$\iiint_{\Omega} f(ax+by+cz) dV = \pi \int_{-R}^R f(u) (R^2 - u^2) du, \quad \lambda = \sqrt{a^2+b^2+c^2} > 0.$$

证: 设  $A = \begin{pmatrix} \frac{a}{\lambda} & \frac{b}{\lambda} & \frac{c}{\lambda} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$  为正交矩阵, 即满足  $A^T = A^{-1}$  的三阶

$$\text{正交阵. 作正交变换: } \begin{pmatrix} u \\ v \\ w \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{a}{\lambda} & \frac{b}{\lambda} & \frac{c}{\lambda} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(3)



● 则  $u = \frac{a}{\lambda}x + \frac{b}{\lambda}y + \frac{c}{\lambda}z \Leftrightarrow ax + by + cz = \lambda u, \lambda = \sqrt{a^2 + b^2 + c^2} > 0$ .

且由  $A^T = A^{-1} \Rightarrow AA^T = E \Rightarrow |AA^T| = |A||A^T| = |A|^2 = |E| = 1 \Rightarrow$

$|A| = \pm 1$ . 即正交矩阵的行列式恒为  $\pm 1$ . *orthogonal transformation*

在几何中,  $|A| = +1$  的正交变换称为旋转变换;  $|A| = -1$  的

正交变换称为轴对称变换或镜面反射变换。

$$\text{而 } dV = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw = \frac{\frac{du dv dw}{\frac{\partial(u, v, w)}{\partial(x, y, z)}}}{|\pm 1|} = du dv dw. \quad \text{且}$$

$$R^2 \equiv x^2 + y^2 + z^2 = (x, y, z) \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad u^2 + v^2 + w^2 = (u, v, w) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (x, y, z) A^T A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ = (x, y, z) E \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x, y, z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2 \leq R^2$$

即正交变换一定是保距变换, 另外, 从  $\begin{cases} \vec{\alpha} = A\vec{x} \\ \vec{\beta} = A\vec{y} \end{cases}$ ,  $A$  是正交阵  $\Rightarrow$

$$\cos(\vec{\alpha}, \vec{\beta}) = \frac{\vec{\alpha} \cdot \vec{\beta}}{\|\vec{\alpha}\| \|\vec{\beta}\|} = \frac{\vec{\alpha}^T \vec{\beta}}{\|\vec{\alpha}\| \|\vec{\beta}\|} = \frac{\vec{x}^T A^T A \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \frac{\vec{x}^T E \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \frac{\vec{x}^T \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \cos(\vec{x}, \vec{y})$$

知, 正交变换是保持角度不变的变换——保角变换!

$$\text{从而 } \iiint_{x^2+y^2+z^2 \leq R^2} f(ax+by+cz) dV = \iiint_{u^2+v^2+w^2 \leq R^2} f(u) du dv dw \quad (4)$$



$$\bullet = \int_R^R f(u) du \iint_{u^2+v^2 \leq R^2-u^2} 1 dv dw = 2 \int_R^R f(u) (R^2-u^2) du$$

若  $f$  是奇函数时,  $\iiint_{\Omega} f(ax+by+c) dV = 0$ .

例4. ch10 总/8:  $(a,b) \neq 0 = (0,0)$ .  $f \in C$ ,  $\lambda = \sqrt{a^2+b^2} > 0$ .

$$\text{则: } \iint_{x^2+y^2 \leq 1} f(ax+by+c) dx dy = 2 \int_{-1}^1 \sqrt{1-t^2} f(\lambda t+c) dt.$$

证: 设  $A = \begin{pmatrix} a & b \\ a_2 & b_2 \end{pmatrix}$  为正交矩阵. 即  $A^T = A^{-1} \Rightarrow |A| = \pm 1$ .

作正交变换  $\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$  则  $\frac{a}{\lambda}x + \frac{b}{\lambda}y = u \Rightarrow ax+by = \lambda u$

$$\text{且 } dx dy = \frac{|\partial(x,y)/\partial(u,v)|}{|\partial(x,y)/\partial(u,v)|} du dv = \frac{du dv}{|A|} = \frac{du dv}{\pm 1} = du dv$$

$$u^2+v^2 = (u,v) \begin{pmatrix} u \\ v \end{pmatrix} = (x,y) A^T A \begin{pmatrix} x \\ y \end{pmatrix} = (x,y) E \begin{pmatrix} x \\ y \end{pmatrix} = x^2+y^2 \leq 1$$

$$\text{从而 } \iint_{x^2+y^2 \leq 1} f(ax+by+c) dx dy = \iint_{u^2+v^2 \leq 1} f(\lambda u+c) du dv$$

$$= \int_{-1}^1 f(\lambda u+c) \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 1 dv du = 2 \int_{-1}^1 \sqrt{1-u^2} f(\lambda u+c) du$$

$$= 2 \int_{-1}^1 \sqrt{1-t^2} f(\lambda t+c) dt, (\lambda = \sqrt{a^2+b^2} > 0)$$

(注在 ch11 总/4, 5, 6 三题中, ~~也可~~ 利用正交变换的题)

$$\bullet \text{ 例5. 证明: } \iint_{x^2+y^2 \leq 1} e^{x^2+y^2} d\sigma < \left( \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} e^{x^2} dx \right)^2 = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} e^{x^2+y^2} dx dy \quad (5).$$



证: 由奇偶对称性知:  $\iint_{x^2+y^2 \leq 1} e^{x^2+y^2} d\sigma = 4 \iint_{D_{II}} e^{x^2+y^2} d\sigma$ .

$$\iint_{D_2} e^{x^2+y^2} d\sigma = 4 \iint_{D_{2I}} e^{x^2+y^2} d\sigma, \quad D_1: x^2+y^2 \leq 1, \quad D_2: \begin{cases} |x| \leq \frac{\sqrt{2}}{2} \\ |y| \leq \frac{\sqrt{2}}{2} \end{cases}$$

$D_{1I}, D_{2I}$  分别是  $D_1, D_2$  在第一象限部分. 令  $D_0 = D_{1I} \cap D_{2I}$ .

$$\text{则 } S(D_1) = 2 = S(D_2) \Rightarrow S(D_{1I}) = S(D_{2I}) \Rightarrow S(D_{1I} - D_0) = S(D_{2I} - D_0)$$

$$\text{而 } \iint_{D_1 - D_0} e^{x^2+y^2} d\sigma < \iint_{D_{1I} - D_0} e^{x^2+y^2} d\sigma = e S(D_{1I} - D_0) = e S(D_{2I} - D_0) = \iint_{D_{2I} - D_0} e^{x^2+y^2} d\sigma$$

$$< \iint_{D_2 - D_0} e^{x^2+y^2} d\sigma \Leftrightarrow \iint_{D_{1I}} e^{x^2+y^2} d\sigma < \iint_{D_{2I}} e^{x^2+y^2} d\sigma \Leftrightarrow$$

$$\iint_{D_1} e^{x^2+y^2} d\sigma < \iint_{D_2} e^{x^2+y^2} d\sigma \Leftrightarrow \iint_{x^2+y^2 \leq 1} e^{x^2+y^2} d\sigma < \left( \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} e^{x^2} dx \right)^2.$$

(三) 作业:

(1) 用五种方法计算  $\Omega: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$  的体积  $V(\Omega)$ .

(2) 计算  $I = \iiint_{x^2+y^2+z^2 \leq 1} a x^2 + b y^2 + c z^2 dV$  与  $\iiint_{x^2+y^2+z^2 \leq 1} (a x^2 + b y^2 + c z^2)^m dV$

$(a, b, c) \neq 0$  为实数,  $m \in \mathbb{N}^+$

(3) ch10 例/5, 6, 8.

(6)