(周一)

§12.2

T_1 (1) $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx=\frac{2a}{\pi}$

$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{a}\cos nx\,dx=\frac{2\sin na}{n\pi}$

$b_n=0$

$\therefore f(x)\sim\frac{a}{\pi}+\sum_{n=1}^{\infty}\frac{2\sin na}{n\pi}\cos nx$

由parseval等式知 $\frac{2a^2}{\pi^2}+\sum_{n=1}^{\infty}\frac{4\sin^2 na}{n^2\pi^2}=\frac{1}{\pi}\int_{-\infty}^{+\infty}f^2(x)\,dx=\frac{2a}{\pi}$

$\therefore\sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}=\frac{\pi^2}{4}\left(\frac{2a}{\pi}-\frac{2a^2}{\pi^2}\right)=\frac{a}{2}(\pi-a)$

(2) $\sum_{n=1}^{\infty}\frac{\cos^2 na}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}=\frac{\pi^2}{6}-\frac{a}{2}(\pi-a)$

T_2 由parseval恒等式

$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx$

由于 f 平方可积

$\therefore\sum_{k=1}^{n}a_k^2\uparrow$ 且有界 $\Rightarrow\sum_{k=1}^{\infty}a_k^2$ 收敛，同理 $\sum_{k=1}^{\infty}b_k^2$ 收敛

由于 $\left|\frac{a_n}{n}\right|\le\frac{1}{2}\left(\frac{1}{n^2}+a_n^2\right)$ 且 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收敛

$\therefore\sum_{n=1}^{\infty}\frac{a_n}{n}$ 收敛，同理 $\sum_{n=1}^{\infty}\frac{b_n}{n}$ 收敛

T_3 $a_n=0$

$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx=\frac{2(1-(-1)^n)}{n\pi}$

$\therefore$ 由parseval等式知 $\frac{1}{\pi}\int_{-\pi}^{\pi}dx=\sum_{n=1}^{\infty}\frac{16}{(2n-1)^2\pi^2}$

$\therefore\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

注意到 $f(x)\sim\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin(2n-1)x$

上式~~积分~~的原函数可出现我们的目标

令 $g(x)=\begin{cases}1+x & 0\le x\le\pi\\ 1-x & -\pi\le x\le 0\end{cases}$

对 $g(x)$ 作展开

$\overline{a_0}=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\,dx=\pi+2$

$\overline{a_n}=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\cos nx\,dx=\frac{2}{\pi}\frac{(-1)^n-1}{n^2}$

$\overline{b_n}=0$

$\therefore x\in[0,\pi]$ 时，$1+x=\frac{\pi+2}{2}+\sum_{n=1}^{\infty}\overline{a_n}\cos nx=\frac{\pi+2}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi}\frac{-2}{(2n-1)^2}\cos(2n-1)x$

$\therefore\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}=\frac{\pi^2}{8}-\frac{\pi}{4}x$

T4 (1) $m=n=0$ 时 $\int_0^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} dx = \pi$

$m=n\neq 0$ 时 $\int_0^{\pi} \cos mx \cos nx \, dx = \frac{\pi}{2}$

$m\neq n$ 时 $\int_0^{\pi} \cos mx \cos nx \, dx = 0$

∴是正交函数系.

标准正交系为：$\{ \frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \cos x, \sqrt{\frac{2}{\pi}} \cos 2x \cdots\cdots \sqrt{\frac{2}{\pi}} \cos nx \cdots\cdots \}$

(2) $\langle \sin \frac{m\pi}{l} x, \sin \frac{n\pi}{l} x \rangle = \int_0^l \sin \frac{m\pi}{l} x \sin \frac{n\pi}{l} x \, dx = \frac{1}{2} \int_0^l \cos \frac{(m-n)\pi}{l} x - \cos \frac{(m+n)\pi}{l} x \, dx$

$m\neq n$ 时 上式 $=0$

$m=n$ 时 上式 $=\frac{l}{2}$

∴ 原函数系是正交系

且 $\{ \sqrt{\frac{2}{l}} \sin \frac{\pi}{l} x, \sqrt{\frac{2}{l}} \sin \frac{2\pi}{l} x \cdots\cdots \sqrt{\frac{2}{l}} \sin \frac{n\pi}{l} x \cdots\cdots \}$ 是标准正交系

§12.3

T8 对f作奇延拓，并延拓为以2π为周期的函数

∵ $f(0)=0$ $f(\pi)=0$

∴ f为连续函数

对f在$[-\pi,\pi]$上作Fourier展开，则

$a_n=0$

$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi} \left(\int_0^1 \frac{\pi-1}{2} x \sin nx \, dx + \int_1^{\pi} \frac{\pi-x}{2} \sin nx \, dx \right)$

$= \frac{\sin n}{n^2}$

∴ $f(x) = \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \sin nx$ $0\leq x\leq \pi$

T9 (1) $x\in[0,1]$时，两边求导 $\frac{\pi-1}{2} = \sum_{n=1}^{\infty} \frac{\sin n}{n} \cos nx$

取 $x=0$ $\Rightarrow \sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi-1}{2}$

在 $f(x) = \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \sin nx$ 中取 $x=1$

$\Rightarrow f(1) = \frac{\pi-1}{2} = \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2 = \frac{\pi-1}{2}$

(2) 由 parseval 等式

$$\frac{2}{\pi}\int_0^\pi f^2\,dx = \sum_{n=1}^{\infty}\frac{\sin^2 n}{n^4}$$

$$\int_0^\pi f^2\,dx = \int_0^1 \frac{(\pi-1)^2}{4}x^2\,dx + \int_1^\pi \frac{(\pi-x)^2}{4}\,dx = \frac{\pi}{12}(\pi-1)^2$$

$$\therefore \sum_{n=1}^{\infty}\frac{\sin^2 n}{n^4} = \frac{(\pi-1)^2}{6}$$

(周三)

§12.1

T2 (2)

$$a_0 = \frac{2}{T}\int_0^T \frac{x}{3}\,dx = \frac{T}{3}$$

$$a_n = \frac{2}{T}\int_0^T \frac{x}{3}\cos\frac{2n\pi}{T}x\,dx = 0$$

$$b_n = \frac{2}{T}\int_0^T \frac{x}{3}\sin\frac{2n\pi}{T}x\,dx = -\frac{T}{3n\pi}$$

$$\therefore f(x) \sim \frac{T}{6} + \sum_{n=1}^{\infty}\left(-\frac{T}{3n\pi}\sin\frac{2n\pi}{T}x\right) \qquad (0\le x\le T)$$

$$\text{收敛} = \begin{cases} \frac{T}{6} & x=0 \text{ or } T \\ \frac{x}{3} & 0<x<T \end{cases}$$

T5 (1) 将 $f(x)$ 偶延拓，再以 $T=2$ 为周期延拓到整个实轴

计算知 $S(x)$ 是 $f(x)$ 的 Fourier 级数.

$$\therefore S(x) = \begin{cases} f(x) & x\ne \pm\frac{1}{2} \\ \frac{3}{4} & x=\pm\frac{1}{2} \\ 0 & x=1 \end{cases} \qquad x\in(-1,1]$$

$$\therefore S\left(\frac{9}{4}\right) = S\left(\frac{1}{4}\right) = \frac{1}{4} \qquad S\left(-\frac{5}{2}\right) = S\left(-\frac{1}{2}\right) = \frac{3}{4}$$

(2) 同理

$$S(x) = \begin{cases} f(x) & x\in(-\pi,0)\cup(0,\pi) \\ 0 & x=0 \\ \frac{\pi^2}{2} & x=\pi \end{cases}$$

$$\therefore S(3\pi) = S(\pi) = \frac{\pi^2}{2}$$

$$S(-4\pi) = S(0) = 0$$

§12.2

T4 (3) $\langle \sin(2n+1)x, \sin(2m+1)x \rangle = \int_0^{\frac{\pi}{2}} \frac{\cos 2(n-m)x}{2} - \frac{\cos 2(m+n+1)x}{2} dx$

$m=n$ 时 上式 $=\frac{\pi}{4}$

$m\neq n$ 时 上式 $=0$

$\therefore$ 原函数是正交系.

正交标准系为 $\{\sqrt{\frac{4}{\pi}}\sin x, \sqrt{\frac{4}{\pi}}\sin 3x \ \cdots, \sqrt{\frac{4}{\pi}}\sin(2n+1)x \cdots\}$

(4) $\langle \cos\frac{(2n+1)\pi x}{2l}, \cos\frac{(2m+1)\pi x}{2l} \rangle = \int_0^l \frac{\cos\frac{\pi(m+n+1)}{l}x + \cos\frac{(n-m)\pi}{l}x}{2} dx$

$m=n$ 时 上式 $=\frac{l}{2}$

$m\neq n$ 时 上式 $=0$

$\therefore$ 原 ……

正 …… $\{\sqrt{\frac{2}{l}}\cos\frac{\pi}{2l}x, \sqrt{\frac{2}{l}}\cos\frac{3\pi}{2l}x, \cdots \sqrt{\frac{2}{l}}\cos\frac{(2n+1)\pi}{2l}x, \cdots\}$

T6 $a_n = \frac{2}{l}\int_0^l x\cos\frac{(2n+1)\pi}{2l}x\,dx$

$= \frac{4l}{(2n+1)\pi}\left((-1)^n - \frac{2}{(2n+1)\pi}\right)$

$\therefore f(x) \sim \frac{4l}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left((-1)^n - \frac{2}{(2n+1)\pi}\right)\cos\frac{(2n+1)\pi}{2l}x$

(周五)

§12.4

T1 (1) $a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\cos\lambda x\,dx = \frac{KT}{\lambda\pi}\sin\lambda T + \frac{K}{\lambda^2\pi}\cos\lambda T - \frac{K}{\lambda^2\pi}$

$b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\sin\lambda x\,dx = -\frac{KT}{\lambda\pi}\cos\lambda T + \frac{K}{\lambda^2\pi}\sin\lambda T$

$\therefore f(x) = \int_0^{+\infty} a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x\ d\lambda$

(2) $a(\lambda) = 0$ $\quad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\sin\lambda x\,dx = \frac{2}{\lambda\pi}(1-\cos\lambda)$

$\therefore f(x) = \int_0^{+\infty}\frac{2}{\lambda\pi}(1-\cos\lambda)\sin\lambda x\,d\lambda$

(3) $a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\cos\lambda x\,dx = \frac{2}{\pi}\int_{0}^{+\infty}\frac{1}{a^2+x^2}\cos\lambda x\,dx$

$\overset{例12.4.3}{=} \frac{1}{a}e^{-a\lambda}$

$b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\sin\lambda x\,dx = 0$

$\therefore f(x) = \int_{0}^{+\infty}\frac{1}{a}e^{-a\lambda}\cos\lambda x\,d\lambda$

T2 (1)

$F(\lambda) = \int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}\,dx \overset{分部}{=} \frac{-4ia\lambda}{(a^2+\lambda^2)^2}$

(3) $F(\lambda) = 2\int_{0}^{\frac{\pi}{2}}\cos x\cos\lambda x\,dx$

$= \int_{0}^{\frac{\pi}{2}}\cos(\lambda+1)x + \cos(\lambda-1)x\,dx$

$= \frac{2}{1-\lambda^2}\cos\left(\frac{\pi}{2}\lambda\right)$