3.6

习题 9.1

T12 令 $\begin{cases} x+y=2 \\ y/x=3 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{2} \\ y=\frac{3}{2} \end{cases} \Rightarrow f(2,3)=-2$

令 $\begin{cases} x+y=u \\ y/x=v \end{cases} \Rightarrow \begin{cases} x=\frac{u}{1+v} \\ y=\frac{uv}{1+v} \end{cases} \Rightarrow f(u,v)=\frac{u^2(1-v^2)}{(1+v)^2}$ 即 $f(x,y)=\frac{x^2(1-y^2)}{(1+y)^2}$

$= x^2\frac{1-y}{1+y}$

T13 $f[\varphi(x,y),\psi(x,y)]=[\varphi(x,y)]^{\psi(x,y)}=(x+y)^{x-y}$

$\varphi[f(x,y),\psi(x,y)]=f(x,y)+\psi(x,y)=x^y+x-y$

$\psi[\varphi(x,y),f(x,y)]=\varphi(x,y)-f(x,y)=x+y-x^y$

T14 (2) $\lim\limits_{\substack{x\to 0\\ y\to a}}\frac{\sin xy}{x}=\lim\limits_{\substack{x\to 0\\ y\to a}}\frac{\sin xy}{xy}\cdot y=a$

(7) $\because 0\le\frac{x^2+y^2}{e^{x+y}}\le\frac{(x+y)^2}{e^{x+y}}$ $\left(\lim\limits_{z\to+\infty}\frac{z^2}{e^z}=0\right)$

不妨 $x,y>0$

由夹逼准则 $\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}(x^2+y^2)e^{-(x+y)}=0$

(9) $\sqrt{xy+1}-1\sim\frac{1}{2}xy\quad(x,y\to 0)\Rightarrow\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{xy}{\sqrt{xy+1}-1}=2$

(10) 若取 $y=kx$，原式 $=\lim\limits_{x\to 0}\frac{\sqrt{kx^2+1}-1}{x+kx}=\lim\limits_{x\to 0}\frac{kx^2}{(1+k)x\left(\sqrt{kx^2+1}+1\right)}$

$=\lim\limits_{x\to 0}\frac{k}{1+k}\cdot\frac{1}{\sqrt{k+\frac{1}{x^2}}+\frac{1}{x}}=0$

若取 $y=x^2-x$，原式 $=\lim\limits_{x\to 0}\dfrac{x(x^2-x)}{x^2(\sqrt{x^3-x^2+1}+1)}=\lim\limits_{x\to 0}\dfrac{x-1}{\sqrt{x^3-x^2+1}+1}=-\dfrac{1}{2}$

∴ 极限不存在

T15 (1) $\lim\limits_{r\to 0^+} e^{\frac{1}{x^2-y^2}}=\lim\limits_{r\to 0^+} e^{\frac{1}{r^2\cos 2\theta}}$

观察知 极限存在 $\Leftrightarrow \cos 2\theta<0 \Leftrightarrow \theta\in(\frac{\pi}{4},\frac{3}{4}\pi)\cup(\frac{5}{4}\pi,\frac{7}{4}\pi)$

(2) $\lim\limits_{r\to +\infty} e^{x^2-y^2}\cdot \sin 2xy=\lim\limits_{r\to +\infty} e^{r^2\cos 2\theta}\cdot\sin(r^2\sin 2\theta)$

∵ $\sin(r^2\sin 2\theta)$ 为有界量，有极限即 $e^{r^2\cos 2\theta}\to 0$ or $\sin 2\theta=0$ 即可

∴ $\cos 2\theta<0$ or $\sin 2\theta=0$

$\theta\in(\frac{\pi}{4},\frac{3}{4}\pi)\cup(\frac{5}{4}\pi,\frac{7}{4}\pi)\cup\{0,\pi,2\pi\}$

T17 在 $y>x$ or $y<x$ 处连续，在 $y=x$ 上不连续

详见 课堂第三讲 讲义 P_2.

T18 沿着每一条射线，此时 α 固定

当 $\alpha\neq 0,\pi$，则 $\lim\limits_{t\to 0} f(t\cos\alpha, t\sin\alpha)=\lim\limits_{t\to 0}\dfrac{t^2\cos^2\alpha\cdot t\sin\alpha}{t^4\cos^4\alpha+t^2\sin^2\alpha}=\lim\limits_{t\to 0}\dfrac{t\sin\alpha\cos^2\alpha}{t^2\cos^4\alpha+\sin^2\alpha}=0=f(0,0)$

当 $\alpha=0$ or π 则 $\lim\limits_{t\to 0} f(t\cos\alpha,t\sin\alpha)=\lim\limits_{t\to 0}\dfrac{0}{t^4}=0=f(0,0)$

∴ 沿着每一条射线连续

但取 $y=kx^2$ 时 $\lim\limits_{x\to 0}\dfrac{x^2y}{x^4+y^2}=\lim\limits_{x\to 0}\dfrac{kx^4}{x^4+k^2x^4}=\dfrac{k}{1+k^2}$

∴ 极限不存在，即 $(0,0)$ 处不连续.

3.8

习题 9.2

T2 (2) $\frac{\partial z}{\partial x} = 3^{-\frac{y}{x}} \cdot \ln 3 \cdot \frac{y}{x^2}$

$\frac{\partial z}{\partial y} = 3^{-\frac{y}{x}} \cdot \ln 3 \cdot (-\frac{1}{x})$

(5) $\frac{\partial u}{\partial x} = \frac{1}{1+(\frac{x+y}{x-y})^2} \cdot \frac{-2y}{(x-y)^2} = \frac{-y}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{1+(\frac{x+y}{x-y})^2} \cdot \frac{2x}{(x-y)^2} = \frac{x}{x^2+y^2}$

(8) $\frac{\partial u}{\partial x} = e^{-z} + \frac{1}{x+\ln y}$ $\frac{\partial u}{\partial y} = \frac{1}{x+\ln y} \cdot \frac{1}{y}$ $\frac{\partial u}{\partial z} = -xe^{-z} + 1$

T3 $\frac{\partial f}{\partial x} = \frac{\sin x^2 y}{x^2 y} \cdot 2xy = \frac{2\sin x^2 y}{x}$ $\frac{\partial f}{\partial y} = \frac{\sin x^2 y}{x^2 y} \cdot x^2 = \frac{\sin x^2 y}{y}$

T4 $\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$

$\frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\cancel{\Delta y} \sin \frac{1}{(\Delta y)^2}}{\cancel{\Delta y}}$ 不存在

T6 $\begin{cases} z = \frac{x^2+y^2}{4} \\ y = 4 \end{cases} \Rightarrow z = \frac{1}{4}x^2 + 4$ (y=4截面上一条抛物线) $z' = \frac{1}{2}x$ $z'(2) = 1$

$\therefore \alpha = 45^\circ$

T13 (4) $\frac{\partial z}{\partial x} = \frac{1}{1+(\frac{y}{x})^2} \cdot (-\frac{y}{x^2}) = \frac{-y}{x^2+y^2}$ $\frac{\partial z}{\partial y} = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

$\therefore dz = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$

(6) $\frac{\partial z}{\partial x} = 4x^3 - 8xy^2$ $\frac{\partial z}{\partial y} = 4y^3 - 8yx^2$

$dz|_{(0,0)} = 0$

$dz|_{(1,1)} = -4dx - 4dy$

T16 显然 $\frac{|xy|}{x^2+y^2} \le \frac{1}{2}$, $(x,y) \neq (0,0)$

$\therefore$ 当 $(x,y) \neq (0,0)$ 时 $|f(x,y)| \le \frac{1}{2}|x|$

$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0)$ 连续.

$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0$ $\quad \frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0$

$\therefore$ 偏导数存在.

由于 $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(\Delta x,\Delta y)-f(0,0)}{\rho} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{(\Delta x)^2\Delta y}{((\Delta x)^2+(\Delta y)^2)^{\frac{3}{2}}}$ 不存在. $\therefore$ $(0,0)$ 处不可微

(因: 沿 $y=x$ 时 极限 $= \frac{1}{2^{\frac{3}{2}}} = \frac{\sqrt{2}}{4}$ $\Rightarrow$ 不存在
沿 x 轴方向, 即 $y=0$ 则 极限 $=0$)

3.10

习题 9.1

T17 (2) 当点 (x_0,y_0) 满足 $y_0 \neq 0$ 时. $\lim_{(x,y)\to(x_0,y_0)} x\sin\frac{1}{y} = x_0\sin\frac{1}{y_0} = f(x_0,y_0)$

当点 (x_0,y_0) 满足 $y_0=0$ 时且 $x_0\neq 0$ 时 $\lim_{(x,y)\to(x_0,0)} x\sin\frac{1}{y} = \lim_{y\to 0} x_0\sin\frac{1}{y}$ 不存在

当点 (x_0,y_0) 满足 $x_0=y_0=0$ 时 $\lim_{(x,y)\to(0,0)} x\sin\frac{1}{y} = 0 = f(x_0,y_0)$

综上 $f(x,y)$ 在 $x\in\mathbb{R}, y\neq 0$ 上和 $(0,0)$ 处是连续的, 在 $x\neq 0, y=0$ 处是不连续的

习题 9.2

T1 (3) $f'_x = \frac{1}{xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}} \cdot \left(y^2+2xy+\frac{(xy^2+yx^2)(y^2+2xy)}{\sqrt{1+(xy^2+yx^2)^2}}\right) = \frac{y^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$

$f'_y = \frac{1}{xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}} \cdot \left(2xy+x^2+\frac{(xy^2+yx^2)(2xy+x^2)}{\sqrt{1+(xy^2+yx^2)^2}}\right) = \frac{2xy+x^2}{\sqrt{1+(xy^2+yx^2)^2}}$

$\therefore f'_x(1,y) = \frac{y^2+2y}{\sqrt{1+(y^2+y)^2}}$

$f'_y(1,y) = \frac{2y+1}{\sqrt{1+(y^2+y)^2}}$

T2 (2) $\frac{\partial u}{\partial x} = y^z x^{y^z - 1}$ $\frac{\partial u}{\partial y} = x^{y^z} \cdot \ln x \cdot z y^{z-1}$

$\ln u = y^z \ln x \Rightarrow \frac{1}{u} \cdot \frac{\partial u}{\partial z} = y^z \cdot \ln y \cdot \ln x \Rightarrow \frac{\partial u}{\partial z} = x^{y^z} y^z \ln y \cdot \ln x$

T13 (2) $\frac{\partial z}{\partial x} = \frac{y(x^2+y^2) - xy \cdot 2x}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$ $\frac{\partial z}{\partial y} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$

$\therefore dz = \frac{y^3 - x^2 y}{(x^2+y^2)^2} dx + \frac{x^3 - xy^2}{(x^2+y^2)^2} dy$

T14 $\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$ (不妨 f, g 都是二元函数)

$\Delta g = g(x_0+\Delta x, y_0+\Delta y) - g(x_0, y_0)$

$\Delta(fg) = f(x_0+\Delta x, y_0+\Delta y) g(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \cdot g(x_0, y_0)$

$= f(x_0+\Delta x, y_0+\Delta y) g(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) g(x_0+\Delta x, y_0+\Delta y)$

$+ f(x_0, y_0) g(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) g(x_0, y_0)$

$= [f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)] g(x_0+\Delta x, y_0+\Delta y) + f(x_0, y_0) \cdot [g(x_0+\Delta x, y_0+\Delta y) - g(x_0, y_0)]$

在 $\Delta x, \Delta y \to 0$ 时. 由定理 9.14

上式 $= df \cdot g + f \cdot dg$ 即 $d(fg) = df \cdot g + f \cdot dg$

T15 $\frac{\partial f}{\partial x}(0,0) = 0$ $\frac{\partial f}{\partial y}(0,0) = 0$

$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\sqrt{|\Delta x \cdot \Delta y|} - 0 - 0 - 0}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \xlongequal{令 \Delta y = k\Delta x} \frac{\sqrt{|k|}}{\sqrt{1+k^2}}$ 极限不存在

$\therefore$ 在原点处不可微.

T17 $|f(x,y)| \le x^2+y^2$, 由两边夹 $f(x,y)$ 在 $(0,0)$ 处连续

$\lim_{\Delta x \to 0} \frac{f(\Delta x, 0)}{\Delta x} = \lim_{\Delta x \to 0} \Delta x \cdot \sin\frac{1}{|\Delta x|} = 0$

$\lim_{\Delta y \to 0} \frac{f(0, \Delta y)}{\Delta y} = \lim_{\Delta y \to 0} \Delta y \cdot \sin\frac{1}{|\Delta y|} = 0$

$\therefore f$ 在 $(0,0)$ 处偏导存在且都为0

当 $x^2+y^2 \neq 0$ 时 $\frac{\partial f}{\partial x} = 2x \sin\frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cos\frac{1}{\sqrt{x^2+y^2}}$

$\frac{\partial f}{\partial y} = 2y \sin\frac{1}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} \cos\frac{1}{\sqrt{x^2+y^2}}$

令 $y = kx$

$\lim\limits_{x\to 0} \frac{\partial f}{\partial x}(x, kx) = \lim\limits_{x\to 0} -\frac{x}{\sqrt{x^2+k^2x^2}} \cos\frac{1}{\sqrt{x^2+k^2x^2}}$ 不存在

$\lim\limits_{x\to 0} \frac{\partial f}{\partial y}(x, kx) = -\frac{kx}{\sqrt{x^2+k^2x^2}} \cos\frac{1}{\sqrt{x^2+k^2x^2}}$ 不存在

$\therefore$ 偏导数在 $(0,0)$ 处不连续

由于 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{((\Delta x)^2+(\Delta y)^2)\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}} - 0 - 0 - 0}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = 0$

$\therefore$ f 在 $(0,0)$ 处可微