专题讲座：重积分的应用举例 (2023.4.19)

例1. 设物体Ω：$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$的体密度为$\rho(x,y,z)=x^2+y^2+z^2$，求$\Omega$的总质量$M(\Omega)$

这也是$\rho_0=1$的物体Ω关于原点O的转动惯量

解：$M(\Omega)=\iiint\limits_{\Omega}(x^2+y^2+z^2)dV=\iiint\limits_{\Omega}x^2dV+\iiint\limits_{\Omega}y^2dV+\iiint\limits_{\Omega}z^2dV$

而$\iiint\limits_{\Omega}x^2dV=\int_{-a}^{a}x^2dx\iint\limits_{D_x}1dydz$，$D_x$：$\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1-\frac{x^2}{a^2}$

$=\int_{-a}^{a}x^2S(D_x)dx=\int_{-a}^{a}x^2\pi bc(1-\frac{x^2}{a^2})dx=\frac{4}{15}\pi a^3bc$.

同理，$\iiint\limits_{\Omega}y^2dV=\frac{4}{15}\pi b^3ac$，$\iiint\limits_{\Omega}z^2dV=\frac{4}{15}\pi c^3ab$.

$\therefore M(\Omega)=\frac{4}{15}\pi abc(a^2+b^2+c^2)$

例2. 求立体Ω：$(\frac{x}{a})^{\frac{2}{3}}+(\frac{y}{b})^{\frac{2}{3}}+(\frac{z}{c})^{\frac{2}{3}}\le 1$的体积$V(\Omega)$

解：$V(\Omega)=\iiint\limits_{\Omega}1dV=\iiint\limits_{\Omega}1dxdydz\ \underline{\underline{x=au^3,y=bv^3,z=cw^3}}$

$\iiint\limits_{u^2+v^2+w^2\le 1}1\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw=\iiint\limits_{u^2+v^2+w^2\le 1}27abcu^2v^2w^2dudvdw$

$\underline{\underline{u=r\sin\theta\cos\varphi,\ v=r\sin\theta\sin\varphi,\ w=r\cos\theta}}\ 27abc\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_0^1(r\sin\theta\cos\varphi)^2(r\sin\theta\sin\varphi)^2(r\cos\theta)^2r^2\sin\theta dr$

$=27abc\int_0^{2\pi}\cos^3\varphi\sin^2\varphi d\varphi\times\int_0^{\pi}\sin^5\theta\cos^2\theta d\theta\times\int_0^1 r^8 dr$

$=27abc\times 4\int_0^{\frac{\pi}{2}}(\cos^3\varphi-\cos^5\varphi)d\varphi\times 2\int_0^{\frac{\pi}{2}}(\sin^5\theta-\sin^7\theta)d\theta\times\frac{1}{9}$

$=12abc\left(\frac{1}{2}\times\frac{\pi}{2}-\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right)\times 2\left(\frac{4}{5}\times\frac{2}{3}\times 1-\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\right)=\frac{4}{35}\pi.$

例3. 求 $f(x,y,z)=x^2+y^2+z^2$ 在 $\Omega: x^2+y^2+z^2\le x+y+z$ 上的积分平均值 $\bar{f}$.

解: $\bar{f}=\iiint_\Omega f(x,y,z)dV \Big/ V(\Omega)$.

而 $\Omega: (x-\frac{1}{2})^2+(y-\frac{1}{2})^2+(z-\frac{1}{2})^2\le(\frac{\sqrt{3}}{2})^2$ 之体积 $V(\Omega)=$

$\frac{4}{3}\pi R^3=\frac{4}{3}\pi\left(\frac{\sqrt{3}}{2}\right)^3=\frac{\sqrt{3}}{2}\pi$. 且

$\iiint_\Omega f(x,y,z)dV \xlongequal[z=\frac{1}{2}+r\cos\theta]{x=\frac{1}{2}+r\sin\theta\cos\varphi,\ y=\frac{1}{2}+r\sin\theta\sin\varphi} \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{\frac{\sqrt{3}}{2}}\left[\left(\frac{1}{2}+r\sin\theta\cos\varphi\right)^2+\right.\right.\right.$

$\left.\left.\left.+\left(\frac{1}{2}+r\sin\theta\sin\varphi\right)^2+\left(\frac{1}{2}+r\cos\theta\right)^2\right]r^2\sin\theta\, dr\right)d\theta\right)d\varphi$

$=\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{\sqrt{3}}{2}}\left[\frac{3}{4}r^2\sin\theta+r^3\sin^2\theta(\cos\varphi+\sin\varphi)+r^3\sin\theta\cos\theta+r^4\sin\theta\right]dr\,d\theta\,d\varphi$

$=\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{\sqrt{3}}{2}}\left(\frac{3}{4}r^2\sin\theta+r^4\sin\theta\right)dr\,d\theta\,d\varphi$

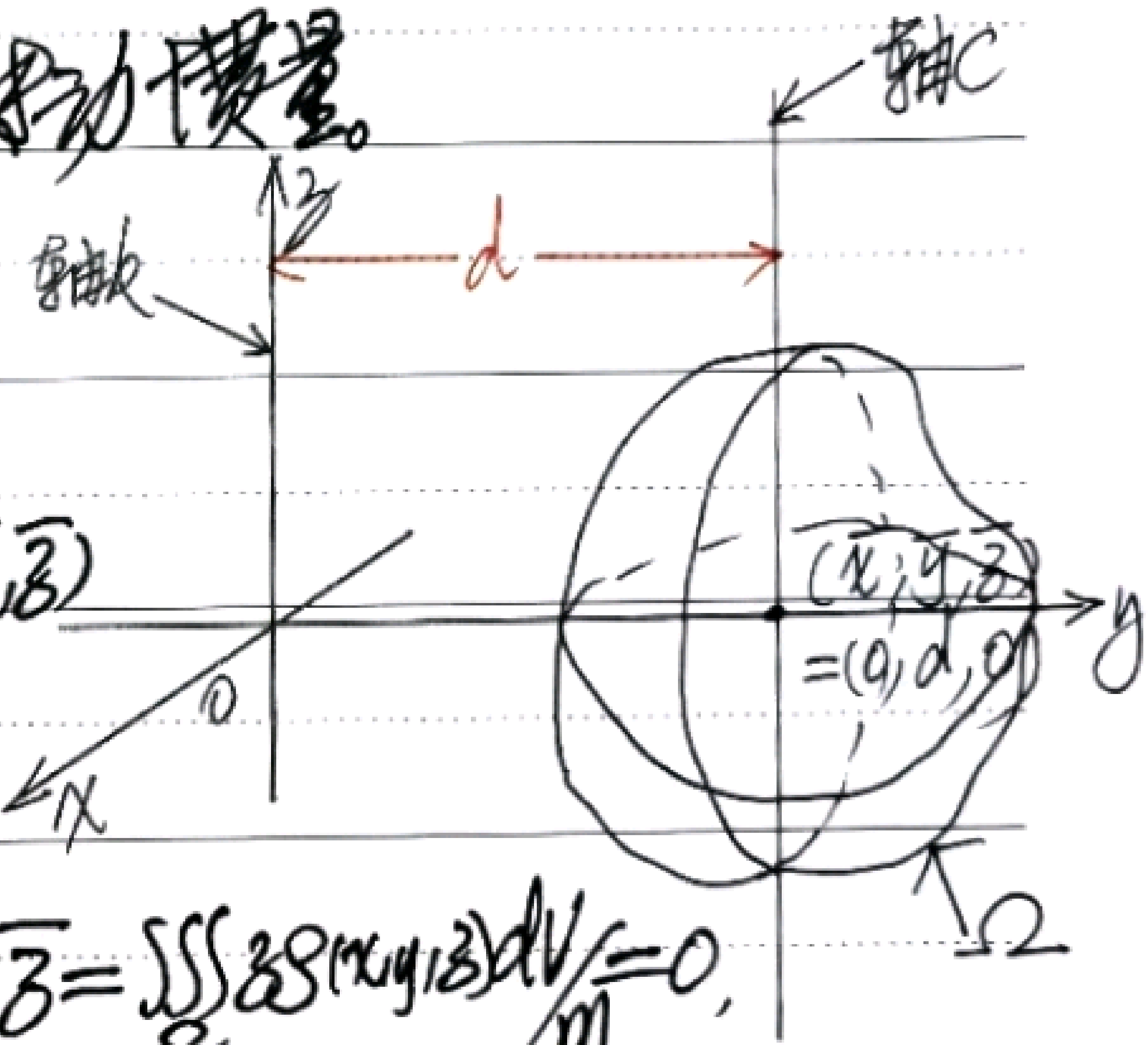
$=2\pi\int_0^{\pi}\sin\theta\, d\theta\times\int_0^{\frac{\sqrt{3}}{2}}\left(\frac{3}{4}r^2+r^4\right)dr=2\pi\times 2\left(\frac{1}{4}\times\frac{3\sqrt{3}}{8}+\frac{9\sqrt{3}}{5\times 32}\right)=\frac{3}{5}\sqrt{3}\pi.$

$\therefore \bar{f}=\frac{3}{5}\sqrt{3}\pi\Big/\frac{\sqrt{3}}{2}\pi=\frac{6}{5}.$

例4. 证明刚体转动惯量的平行轴定理:

$$J_k = J_c + m\cdot d^2$$ (d是两平行轴之间的距离)

m为物体Ω的质量: $m=\iiint\limits_{\Omega}\rho(x,y,z)\,dV$. J_c是Ω绕过物体重心$G(\bar{x},\bar{y},\bar{z})$之轴C的转动惯量, J_k是Ω绕与轴C平行的轴k的转动惯量。



证(1°)建立坐标系, 使轴k与Oz轴重合, 且重心$G(\bar{x},\bar{y},\bar{z})$位于Oy轴上: $G(0,d,0)$

即 $\bar{x}=\iiint\limits_{\Omega} x\rho(x,y,z)\,dV \Big/ m = \bar{z} = \iiint\limits_{\Omega} z\rho(x,y,z)\,dV \Big/ m = 0$,

$\bar{y}=\iiint\limits_{\Omega} y\rho(x,y,z)\,dV \Big/ m = d \Rightarrow \iiint\limits_{\Omega} y\rho(x,y,z)\,dV = md$.

(2°) $J_k = J_z = \iiint\limits_{\Omega}(x^2+y^2)\rho(x,y,z)\,dV$

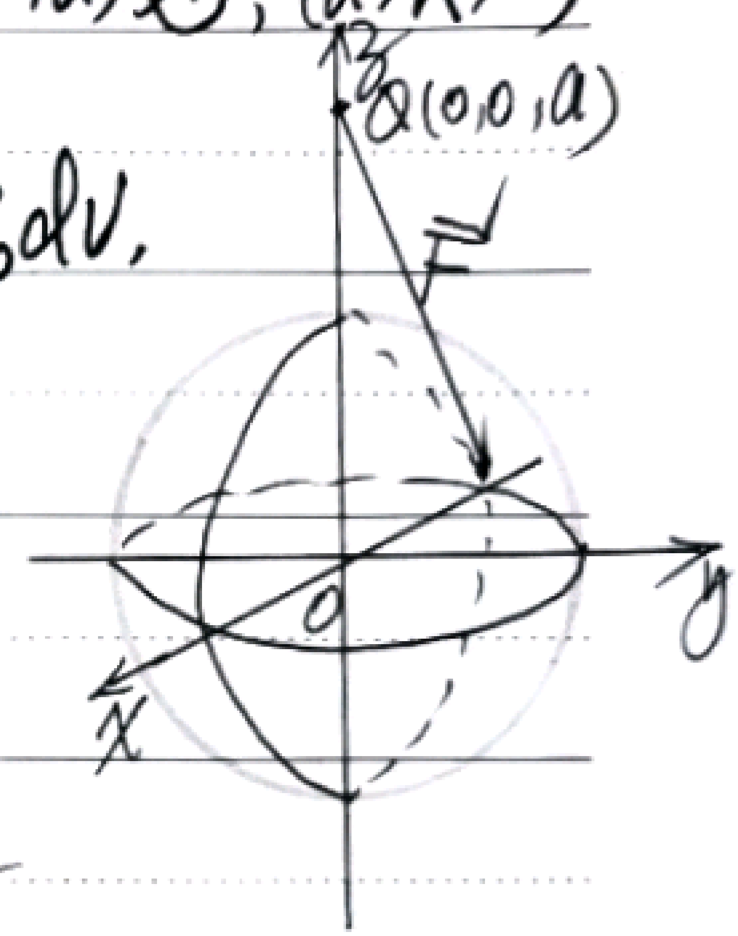
$J_c = \iiint\limits_{\Omega}(x^2+(y-d)^2+0^2)\rho(x,y,z)\,dV = J_k+\iiint\limits_{\Omega} d^2\rho\,dV - 2d\iiint\limits_{\Omega} y\rho\,dV$

$= J_k + d^2\cdot m - 2d(m\cdot d) = J_k - d^2\cdot m \Leftrightarrow J_k = J_c + m\cdot d^2$

例5. 求半径为R的均匀球体Ω对球外一质点Q的引力$\vec{F}=(F_x, F_y, F_z)$，其中质点Q的质量为m_0.

解：(1°) 设均匀球体$\Omega: x^2+y^2+z^2\le R^2$的体密度为$\rho_0>0$.

(2°) 建立坐标系，设Q点位于Oz轴上$(0,0,a)$处，$(a>R>0)$



(3°) 在Ω中任取一小区域Ω_i，质量$m(\Omega_i)=\rho_0 dV$.

$$d\vec{F}=k\frac{m_0(\rho_0 dV)}{x^2+y^2+(z-a)^2}\cdot\vec{F}^0$$

$$\vec{F}^0=\frac{(x,y,z-a)}{\sqrt{x^2+y^2+(z-a)^2}}$$

$$dF_x=\frac{km_0\rho_0\, x\,dxdydz}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}},\quad dF_y=\frac{km_0\rho_0\, y\,dxdydz}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}}$$

$$dF_z=\frac{km_0\rho_0(z-a)dxdydz}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}}$$，由奇偶对称性知：

$$\iiint_\Omega dF_x=F_x=0=F_y=\iiint_\Omega dF_y \quad 而\ F_z=\iiint_\Omega dF_z \xlongequal[z=z]{x=r\cos\theta,\ y=r\sin\theta}$$

$$km_0\rho_0\int_{-R}^{R}(z-a)dz\int_0^{2\pi}\int_0^{\sqrt{R^2-z^2}}\frac{r\,dr}{(r^2+(z-a)^2)^{\frac{3}{2}}}d\theta$$

$$=2\pi km_0\rho_0\int_{-R}^{R}\left((z-a)(-1)(r^2+(z-a)^2)^{-\frac{1}{2}}\Big|_0^{\sqrt{R^2-z^2}}\right)dz$$

$$=-2\pi km_0\rho_0\int_{-R}^{R}(z-a)\left(\frac{1}{\sqrt{R^2-2az+a^2}}-\frac{1}{a-z}\right)dz$$

$$= -2\pi k m_0 \rho_0 \left(\int_{-R}^{R} \frac{(z-a)dz}{\sqrt{R^2-2az+a^2}} + 2R \right)$$ 分部积分

$$= -2\pi k m_0 \rho_0 \left[\left(-\frac{1}{a}\right) \int_{-R}^{R} (z-a)\, d\sqrt{R^2-2az+a^2} + 2R \right]$$

$$= -2\pi k m_0 \rho_0 \left(\frac{2R^3}{3a^2} - 2R + 2R \right) = -\frac{\left(\frac{4}{3}\pi R^3 \rho_0\right) m_0 k}{a^2} = -k\frac{M_0 m_0}{a^2}$$

$$M_0 \triangleq \frac{4}{3}\pi R^3 \rho_0 \Rightarrow \vec{F} = \left(0, 0, -\frac{k m_0 M_0}{a^2}\right)$$

例16. 求以 $z=f(x,y)=\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|$ 为顶，以 $D: x^2+y^2\le 1$ 为底的曲顶柱体 Ω 的体积 $V(\Omega)=\iint_D \left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right| d\sigma$ (ch10练/4)

解：设 $D_1: \left(x-\frac{1}{\sqrt{8}}\right)^2+\left(y-\frac{1}{\sqrt{8}}\right)^2 \le \left(\frac{1}{2}\right)^2$，$D_2 = D - D_1$ 则 $D = D_1 + D_2$

$$V(\Omega)=\iint_D \left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right| d\sigma = \iint_{D_1} \left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right| d\sigma + \iint_{D_2} \left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right| d\sigma$$

令 $g(x,y)=\frac{x+y}{\sqrt{2}}-x^2-y^2$ 则 $V(\Omega)=\iint_{D_1} g(x,y)d\sigma - \iint_{D_2} g(x,y)d\sigma$

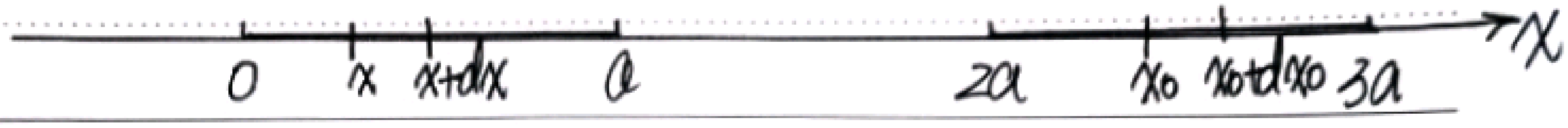
$$= 2\iint_{D_1} g(x,y)d\sigma - \iint_{D_1+D_2} g(x,y)d\sigma$$ 记P

$$2\iint_{D_1} g(x,y)d\sigma \xrightarrow[y-1/\sqrt{8}=r\sin\theta]{x-1/\sqrt{8}=r\cos\theta} 2\int_0^{2\pi}\int_0^{\frac{1}{2}} \left(\frac{1}{4}-r^2\right) r\,dr\,d\theta = \frac{\pi}{16}$$

$$\iint_{D_1+D_2} g(x,y)d\sigma = \iint_{x^2+y^2\le 1} \left(\frac{x+y}{\sqrt{2}}-(x^2+y^2)\right)d\sigma = 0 - \iint_{x^2+y^2\le 1}(x^2+y^2)d\sigma = -\frac{\pi}{2}$$

$$\therefore V(\Omega) = \frac{\pi}{16} - \left(-\frac{\pi}{2}\right) = \frac{9}{16}\pi$$

例7、设两等长为a，质量为m_0的均匀细杆位于同一直线上，其近端距离为a，求两杆间的引力F。



解：设两细杆所在直线为Ox轴，两杆的线密度均为$\rho_0=\frac{m_0}{a}$

$$F=\int_{2a}^{3a}\int_0^a \frac{k(\frac{m}{a}dx)(\frac{m}{a}dx_0)}{(x_0-x)^2}=\left(\frac{m}{a}\right)^2 k\int_{2a}^{3a}\left(\int_0^a \frac{dx}{(x-x_0)^2}\right)dx_0$$

$$=k\left(\frac{m}{a}\right)^2\int_{2a}^{3a}\left(\frac{1}{x_0-a}-\frac{1}{x_0}\right)dx_0=k\left(\frac{m}{a}\right)^2\ln\frac{x_0-a}{x_0}\Big|_{2a}^{3a}=k\left(\frac{m}{a}\right)^2\ln\frac{4}{3}.$$

作业：ex10.3：5(8)；6；12；14；16；19；ch10综/4.

第20讲：n重积分 $\int\int_{\Omega}\cdots\int f(x_1,x_2,\cdots,x_n)dx_1dx_2\cdots dx_n$

(2022.4.11)