第四讲：高阶偏导数与复合函数微分法 (2023.3.13)

(一) 高阶偏导数：

例1. 设 $z=x^2y^2+e^{xy^3}$，$(x,y)\in D=R^2$，

则 $\frac{\partial z}{\partial x}=2xy^2+e^{xy^3}\cdot y^3$，$\frac{\partial z}{\partial y}=2yx^2+e^{xy^3}\cdot 3xy^2$.

将 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ 继续求偏导数，就得到 z 高阶偏导数：

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\triangleq\frac{\partial^2 z}{\partial y\partial x}=(2xy^2+y^3e^{xy^3})'_y=4xy+3y^2e^{xy^3}+3y^5xe^{xy^3}$.

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\triangleq\frac{\partial^2 z}{\partial x\partial y}=(2yx^2+3xy^2e^{xy^3})'_x=4xy+3y^2e^{xy^3}+3xy^5e^{xy^3}=\frac{\partial^2 z}{\partial y\partial x}$

我们记作 $z''_{yx}=z''_{xy}$ 或 $f''_{yx}(x,y)=f''_{xy}(x,y)$，$\forall(x,y)\in R^2$ (1)

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=(2xy^2+e^{xy^3}y^3)'_x=2y^2+e^{xy^3}y^6\triangleq z''_{xx}=f''_{xx}(x,y)$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=(2yx^2+e^{xy^3}\cdot 3xy^2)'_y=2x^2+e^{xy^3}(3xy^2)^2+6xy\,e^{xy^3}=z''_{yy}=f''_{yy}$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}\Delta y=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$，$x,y$ 是自变量 (2).

令 $z=x$，则 $z=x+0\cdot y\Rightarrow dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=1\Delta x+0\Delta y=\Delta x$

而 $dz=dx=\Delta x$，令 $z=y$，则 $z=0x+y\Rightarrow dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y$

$=0\cdot\Delta x+1\cdot\Delta y=\Delta y$ 而 $dz=dy=\Delta y$.

即一切自变量的增量，都等于自变量的微分。

Th1：设 $z=f(x,y)\in C^2(D)$，D是区域，即 f 在D上二阶偏导函数连续，则必有 $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$，或 $f''_{xy}(x,y)=f''_{yx}(x,y)$.

证：对 $\forall M_0(x_0,y_0)$，$M_1(x_0+h,y_0)$，$M_2(x_0,y_0+k)$ 及 $M(x,y)\in D$，$M_3(x_0+h,y_0+k)\in D$

令 $\begin{cases} m(x)=f(x,y_0+k)-f(x,y_0) \\ n(y)=f(x_0+h,y)-f(x_0,y) \end{cases}$，则对 $\forall\ hk\neq 0$，有：

$m(x_0+h)-m(x_0)=f(x_0+h,y_0+k)-f(x_0+h,y_0)-f(x_0,y_0+k)+f(x_0,y_0)$

$n(y_0+k)-n(y_0)=f(x_0+h,y_0+k)-f(x_0,y_0+k)-f(x_0+h,y_0)+f(x_0,y_0)$，即有

$m(x_0+h)-m(x_0)\equiv n(y_0+k)-n(y_0)$ 即从微分中值定理有：

$m'_x(x_0+\theta_1 h)h\equiv n'_y(y_0+\theta_2 k)k$，$\theta_1,\theta_2\in(0,1)\Rightarrow$

$(f'_x(x_0+\theta_1 h,y_0+k)-f'_x(x_0+\theta_1 h,y_0))h\equiv(f'_y(x_0+h,y_0+\theta_2 k)-f'_y(x_0,y_0+\theta_2 k))k$

对 f'_x，f'_y 再次使用微分中值定理：

$f''_{xy}(x_0+\theta_1 h,y_0+\theta_3 k)\cancel{kh}\equiv f''_{yx}(x_0+\theta_4 h,y_0+\theta_2 k)\cancel{hk}$，$\theta_3,\theta_4\in(0,1)$

令 $h\to 0$，$k\to 0$，利用 f''_{xy}，f''_{yx} 的C性得：

$f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$. 由 $M_0(x_0,y_0)\in D$ 中的任意性

(2).

知：$f''_{xy}(x,y)=f''_{yx}(x,y)$ 即 $\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$，$\forall(x,y)\in D$.

同理可证，若 $z=f(x,y)\in C^3(D)$，D是区域，则

$$\frac{\partial^3 z}{\partial x\partial y^2}=\frac{\partial^3 z}{\partial y\partial x\partial y}=\frac{\partial^3 z}{\partial y^2\partial x},$$

若 $u=f(x,y,z)\in C^3(D)$，D是区域，则

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=\frac{\partial^3 u}{\partial x\partial z\partial y}=\frac{\partial^3 u}{\partial z\partial x\partial y}=\frac{\partial^3 u}{\partial z\partial y\partial x}=\frac{\partial^3 u}{\partial y\partial z\partial x}=\frac{\partial^3 u}{\partial y\partial x\partial z}\cdots$$

即在高阶偏导数连续的条件下，求偏导（高阶）时，与求偏导的顺序无关！

(三) 复合多元函数的微分法：

Th2：设 $z=f(u,v)\in D_1$ 中可微，而 $\begin{cases}u=g(x,y)\\v=h(x,y)\end{cases}\in D_2$ 中可微.

则有 $\begin{cases}\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}g'_x+\frac{\partial z}{\partial v}h'_x\\\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}g'_y+\frac{\partial z}{\partial v}h'_y.\end{cases}$ → 链式法则.

其中，复合 $f(g(x,y),h(x,y))\in D_2$ 中有意义。$z=f(g(x,y),h(x,y))$，$(x,y)\in D_2$

证：(1) 设 (x,y)，$(x+\Delta x,y)\in D_2$ 则 $\begin{cases}\Delta u_x=g(x+\Delta x,y)-g(x,y)\\\Delta v_x=h(x+\Delta x,y)-h(x,y)\end{cases}$

此时，$\Delta z_x=\frac{\partial z}{\partial u}\Delta u_x+\frac{\partial z}{\partial v}\Delta v_x+o(\rho)$. $\rho=\sqrt{(\Delta u_x)^2+(\Delta v_x)^2}\Rightarrow$

$\frac{\Delta z_x}{\Delta x}=\frac{\partial z}{\partial u}\frac{\Delta u_x}{\Delta x}+\frac{\partial z}{\partial v}\frac{\Delta v_x}{\Delta x}+\frac{o(\rho)}{\Delta x}$ 且 $\Delta x\to 0$ 时.

$\frac{\Delta u_x}{\Delta x}\to\frac{\partial u}{\partial x}$, $\frac{\Delta v_x}{\Delta x}\to\frac{\partial v}{\partial x}$, $\frac{o(\rho)}{\Delta x}=\frac{o(\rho)}{\rho}\frac{\rho}{\Delta x}=\frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u_x}{\Delta x}\right)^2+\left(\frac{\Delta v_x}{\Delta x}\right)^2}$

且 $\Delta x\to 0$ 时，$\sqrt{\left(\frac{\Delta u_x}{\Delta x}\right)^2+\left(\frac{\Delta v_x}{\Delta x}\right)^2}\longrightarrow\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$，从而 $\sqrt{\left(\frac{\Delta u_x}{\Delta x}\right)^2+\left(\frac{\Delta v_x}{\Delta x}\right)^2}$ 有界，

$\rho\longrightarrow 0\Rightarrow\frac{o(\rho)}{\rho}\longrightarrow 0$. 从而

$\frac{\partial z}{\partial x}=\lim\limits_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}\cdot g_x'+\frac{\partial z}{\partial v}\cdot h_x'$.

(三). 3个重要的例题： 三维调和方程!

例1. 证明函数 $u=\frac{1}{r}$, $r=\sqrt{x^2+y^2+z^2}>0$ 满足 Laplace 方程：$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$. $\forall(x,y,z)\in\Omega$: $x^2+y^2+z^2>0$.

例2. 证明二元函数：$u=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$ $(x>0, t>0, a>0$ 常$)$ 满足一维热传导方程：$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$.

例3. 证明：$u=\varphi(x-at)+\psi(x+at)$, $\forall\varphi,\psi\in C^2(I)$ 满足一维波动方程：$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ $(a>0$, 常数$)$

其中，$t>0$, $x\in(-\infty,+\infty)$.

(4).

证(1): $\frac{\partial u}{\partial x}=\frac{du}{dr}\frac{\partial r}{\partial x}=-\frac{1}{r^2}\cdot\frac{2x}{2r}=-\frac{x}{r^3}$, $\Rightarrow$

$\frac{\partial^2 u}{\partial x^2}=-\left(\frac{x}{r^3}\right)'_x=-\frac{1\cdot r^3-3r^2\frac{x}{r}x}{r^6}=-\frac{r^2-3x^2}{r^5}$, 由对称性,

$\frac{\partial^2 u}{\partial y^2}=-\frac{r^2-3y^2}{r^5}$, $\frac{\partial^2 u}{\partial z^2}=-\frac{r^2-3z^2}{r^5}$,

故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3r^2-3(x^2+y^2+z^2)}{r^5}=-\frac{3r^2-3r^2}{r^5}=0$. #

注(1) 在数学物理方程中,称 $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$ 是 Laplace方程的一个基本解。Laplace方程又称为调和方程。

注(2): $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$ 中,任意交换自变量的位置,仍与原来的函数相同,称这类函数为对称函数。

证(2): $\because \frac{\partial u}{\partial t}=\frac{(t^{-\frac{1}{2}})'_t}{2a\sqrt{\pi}}e^{-\frac{x^2}{4a^2t}}+\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\left(-\frac{x^2}{4a^2}t^{-1}\right)'_t$

$=\frac{-1}{4a\sqrt{\pi t}\,t}e^{-\frac{x^2}{4a^2t}}+\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\frac{x^2}{4a^2t^2}$

$=\frac{1}{4a\sqrt{\pi t}\,t}e^{-\frac{x^2}{4a^2t}}\left(-1+\frac{x^2}{2a^2t}\right)$, 且

$\frac{\partial u}{\partial x}=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\left(-\frac{x}{2a^2t}\right)$, $\Rightarrow \frac{\partial^2 u}{\partial x^2}=\frac{1}{2a\sqrt{\pi t}}\left(e^{-\frac{x^2}{4a^2t}}\left(-\frac{x}{2a^2t}\right)^2+\right.$

$\left.e^{-\frac{x^2}{4a^2t}}\left(\frac{-1}{2a^2t}\right)\right)=\frac{1}{4a\sqrt{\pi t}\,t}e^{-\frac{x^2}{4a^2t}}\left(\frac{x^2}{2a^4t}-\frac{1}{a^2}\right)\Rightarrow$

$a^2\frac{\partial^2 u}{\partial x^2}=\frac{1}{4a\sqrt{\pi t}\,t}e^{-\frac{x^2}{4a^2t}}\left(\frac{x^2}{2a^2t}-1\right)=\frac{\partial u}{\partial t}$ #.

注：在数理方程中，将二元函数 $u=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$ 称为

一维热传导方程：$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ 的基本解。

证(3)：令 $\begin{cases} v=x-at \\ w=x+at \end{cases}$，则 $u=\varphi(v)+\psi(w)$，由复合函数

求导法则：$\frac{\partial u}{\partial x}=\varphi'(v)\frac{\partial v}{\partial x}+\psi'(w)\frac{\partial w}{\partial x}=\varphi'(v)\cdot 1+\psi'(w)\cdot 1$

$\frac{\partial^2 u}{\partial x^2}=\varphi''(v)\cdot 1^2+\psi''(w)\cdot 1^2=\varphi''(v)+\psi''(w)$.

而 $\frac{\partial u}{\partial t}=\varphi'(v)\frac{\partial v}{\partial t}+\psi'(w)\frac{\partial w}{\partial t}=\varphi'(v)(-a)+\psi'(w)a\Rightarrow$

$\frac{\partial^2 u}{\partial t^2}=\varphi''(v)\frac{\partial v}{\partial t}(-a)+\psi''(w)\frac{\partial w}{\partial t}(a)=\varphi''(v)(-a)^2+\psi''(w)a^2$

$=a^2(\varphi''(v)+\psi''(w))=a^2\frac{\partial^2 u}{\partial x^2}$.

注：二维波动方程为：$\frac{\partial^2 u}{\partial t^2}=a^2(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$；三维波动方程为：

$\frac{\partial^2 u}{\partial t^2}=a^2(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$.　又称为哈密顿算子。

(四) 三维向量算子：$\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$；读作Nabla

(1) 设 $u=f(x,y,z)\in C^1$，则 $\nabla u=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})u=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})$.

(2). $\nabla\cdot\nabla\triangleq\Delta=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\cdot(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})=(\frac{\partial}{\partial x})^2+(\frac{\partial}{\partial y})^2+(\frac{\partial}{\partial z})^2\triangleq\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

设 $u=f(x,y,z)\in C^2$，则 $\Delta u=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$.

比如，三维Laplace方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$ 简记为

$\Delta u=0$；而三维波动方程：

$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$ 记为：$\frac{\partial^2 u}{\partial t^2}=a^2\Delta u$；

(五). 作业：

ex9.2：8；11；19(3),(4)；26；27；29；30.

(六). 多元函数微分学中的若干例函数：

(1). $f(x,y)=\sqrt{x^2+y^2}$ 在O(0,0)处C，不可偏导，从而不可微；

(2). $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & x^2+y^2>0\\ 0, & x^2+y^2=0\end{cases}$ 在O(0,0)处可偏导，不C，从而不可d；

(3). $f(x,y)=\sqrt{|xy|}$ 在O(0,0)处C，可偏导，但不可d；

(4). $f(x,y)=(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}$ 在O(0,0)处C，可偏导，可微，

但 $f_x'(x,y)$，$f_y'(x,y)$ 在O(0,0)处不C。

其中，$f_x'(x,y)=\begin{cases}2x\sin\frac{1}{\sqrt{x^2+y^2}}-\frac{x}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2>0.\\ 0, & x^2+y^2=0.\end{cases}$