1.

(a)

$$\mathbb{E}[X(t)] = \mathbb{E}(\sin Ut) = \int_0^{2\pi} \frac{1}{2\pi} \sin Ut \, \mathrm{d}U = 0 \qquad (t = 1, 2, \cdots)$$

$$\begin{aligned}
\mathrm{Cov}(X(t), X(s)) &= \mathbb{E}(\sin Ut \cdot \sin Us) \\
&= \frac{1}{2}\mathbb{E}\left[\cos(t-s)U - \cos(t+s)U\right] \\
&= \frac{1}{4\pi}\left\{ \frac{1}{t-s}\sin(t-s)U\Big|_0^{2\pi} - \frac{1}{t+s}\sin(t+s)U\Big|_0^{2\pi} \right\} \\
&= 0 \qquad (t \neq s)
\end{aligned}$$

当 $t = s$ 时 $\mathrm{Cov}(X(t), X(s)) = \mathbb{E}(\sin^2 Ut) = \frac{1}{2}$ $\quad \therefore$ 是宽平稳

考虑 $F_t(x) = \mathbb{P}(\sin Ut \leqslant x)$, 显然 $F_{t+h} = \mathbb{P}[\sin U(t+h) \leqslant x]$ 与其不一定相同 $\quad \therefore$ 不是严平稳

(b)

$$\mathbb{E}[X(t)] = \frac{1}{2\pi t}(1 - \cos 2\pi t)$$

$$\mathrm{Var}[X(t)] = \mathbb{E}\left(\sin Ut - \frac{1}{2\pi t}(1 - \cos 2\pi t)\right)^2 = \frac{1}{2} - \frac{\sin 4\pi t}{8\pi t} - \left(\frac{1 - \cos 2\pi t}{2\pi t}\right)^2$$

都与 t 相关 $\quad \therefore$ 不是宽平稳

若其严平稳, 则因二阶矩存在, 应为宽平稳, 矛盾. $\quad \therefore$ 不是严平稳.

3.

$$\begin{aligned}
\mathbb{E}(X_n) &= \mathbb{E}\left[\sum_{k=1}^N \sigma_k \sqrt{2}(\cos(a_k n)\cos U_k + \sin(a_k n)\sin U_k)\right] \\
&= \sum_{k=1}^N \sigma_k \sqrt{2}\left[\mathbb{E}(\cos U_k)\cos a_k n + \mathbb{E}(\sin U_k)\sin a_k n\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(X_n, X_m) &= \mathbb{E}(X_n X_m) - \mathbb{E}(X_n)\mathbb{E}(X_m) = \mathbb{E}(X_n X_m) \\
&= \mathbb{E}\left[\sum_{k=1}^N \sigma_k \sqrt{2}\cos(a_k n - U_k) \sum_{j=1}^N \sigma_j \sqrt{2}\cos(a_j m - U_j)\right] \\
&= \sum_{k=1}^N 2\sigma_k^2 \mathbb{E}\left[\cos(a_k n - U_k)\cos(a_k m - U_k)\right] + \sum_{k \neq j} 2\sigma_k \sigma_j \mathbb{E}[\cos(a_k n - U_k)]\mathbb{E}[\cos(a_j m - U_j)] \\
&= \sum_{k=1}^N \sigma_k^2 \mathbb{E}\left[\cos(a_k(n-m)) + \cos(a_k n + a_k m - 2U_k)\right] + 0 \\
&= \sum_{k=1}^N \sigma_k^2 \cos[a_k(n-m)]
\end{aligned}$$

只与 $n - m$ 有关 $\quad \therefore$ 宽平稳.

4.

要求

$$\mathbb{E}[Z(t)] = \sum_{k=1}^{n} \mathbb{E}(A_k \mathrm{e}^{j\omega_k t}) = const$$

$\therefore \mathbb{E}(A_k) = 0$, 要求

$$\mathrm{Cov}(Z(t), Z(s)) = \mathbb{E}[Z(t)\overline{Z(s)}] = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \mathbb{E}(A_k A_\ell) \cdot \mathrm{e}^{j\omega_k t - j\omega_\ell s}$$

只与 $t-s$ 有关

$\therefore \mathbb{E}(A_k A_l) = 0 \quad (k \neq \ell \text{ 且 } \omega_k \neq \omega_\ell)$

7.

(i)

$$\begin{aligned}
\mathbb{E}[Z(t)W(t)] &= \mathbb{E}[X(t+1)X(t-1)] = R(2) = 4\mathrm{e}^{-4} \\
\mathbb{E}[Z(t)W(t)]^2 &= \mathbb{E}\big[X^2(t+1) + 2X(t+1)X(t-1) + X^2(t-1)\big] \\
&= 2\mathbb{E}[X^2(t)] + 2R(2) \\
&= 2\big\{\mathrm{Var}[X(t)] - \mathbb{E}^2[X(t)]\big\} + 8\exp(-4) \\
&= 2R(0) + 8\exp(-4) \\
&= 8[1+\exp(-4)]
\end{aligned}$$

(ii) $Z(t) = X(t+1) \sim N(0, 2^2)$

$$\therefore f_Z(z) = \frac{1}{\sqrt{2\pi \cdot 2^2}} \mathrm{e}^{-\frac{z^2}{2\cdot 2^2}} = \frac{1}{\sqrt{8\pi}} \mathrm{e}^{-\frac{z^2}{8}}$$

$$\therefore \mathbb{P}[Z(t) < 1] = \int_{-\infty}^{1} f_Z(z)\,\mathrm{d}z = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{1} \mathrm{e}^{-\frac{z^2}{8}}\,\mathrm{d}z$$

(iii) 显然 $f_{Z,W}(z,w)$ 为二维正态分布概率密度函数, 协方差矩阵为

$$\boldsymbol{C} = \begin{pmatrix} 4 & 4\mathrm{e}^{-4} \\ 4\mathrm{e}^{-4} & 4 \end{pmatrix}$$

其逆矩阵

$$\boldsymbol{C}^{-1} = \begin{pmatrix} \frac{1}{4(1-\mathrm{e}^{-8})} & -\frac{\mathrm{e}^{-4}}{4(1-\mathrm{e}^{-8})} \\ -\frac{\mathrm{e}^{-4}}{4(1-\mathrm{e}^{-8})} & \frac{1}{4(1-\mathrm{e}^{-8})} \end{pmatrix}$$

其行列式 $|\boldsymbol{C}| = 16(1-\mathrm{e}^{-8})$, 期望向量 $\bar{\boldsymbol{\mu}} = (0,0)$

$$\begin{aligned}
\therefore f_{Z,W}(z,w) &= \frac{1}{2\pi|\boldsymbol{C}|} \exp\left\{ -\frac{1}{2}\Big((z,w) - \bar{\boldsymbol{\mu}}\Big)\boldsymbol{C}^{-1}\Big((z,w) - \bar{\boldsymbol{\mu}}\Big)^T \right\} \\
&= \frac{1}{8\pi\sqrt{1-\mathrm{e}^{-8}}} \exp\left\{ -\frac{z^2 + w^2 - 2\mathrm{e}^{-4}wz}{8(1-\mathrm{e}^{-8})} \right\}
\end{aligned}$$