

---

# **Calculus Revisited**

## **Part 1**

A Self-Study Course



---

### **Lecture Notes**

Center for Advanced  
Engineering Study

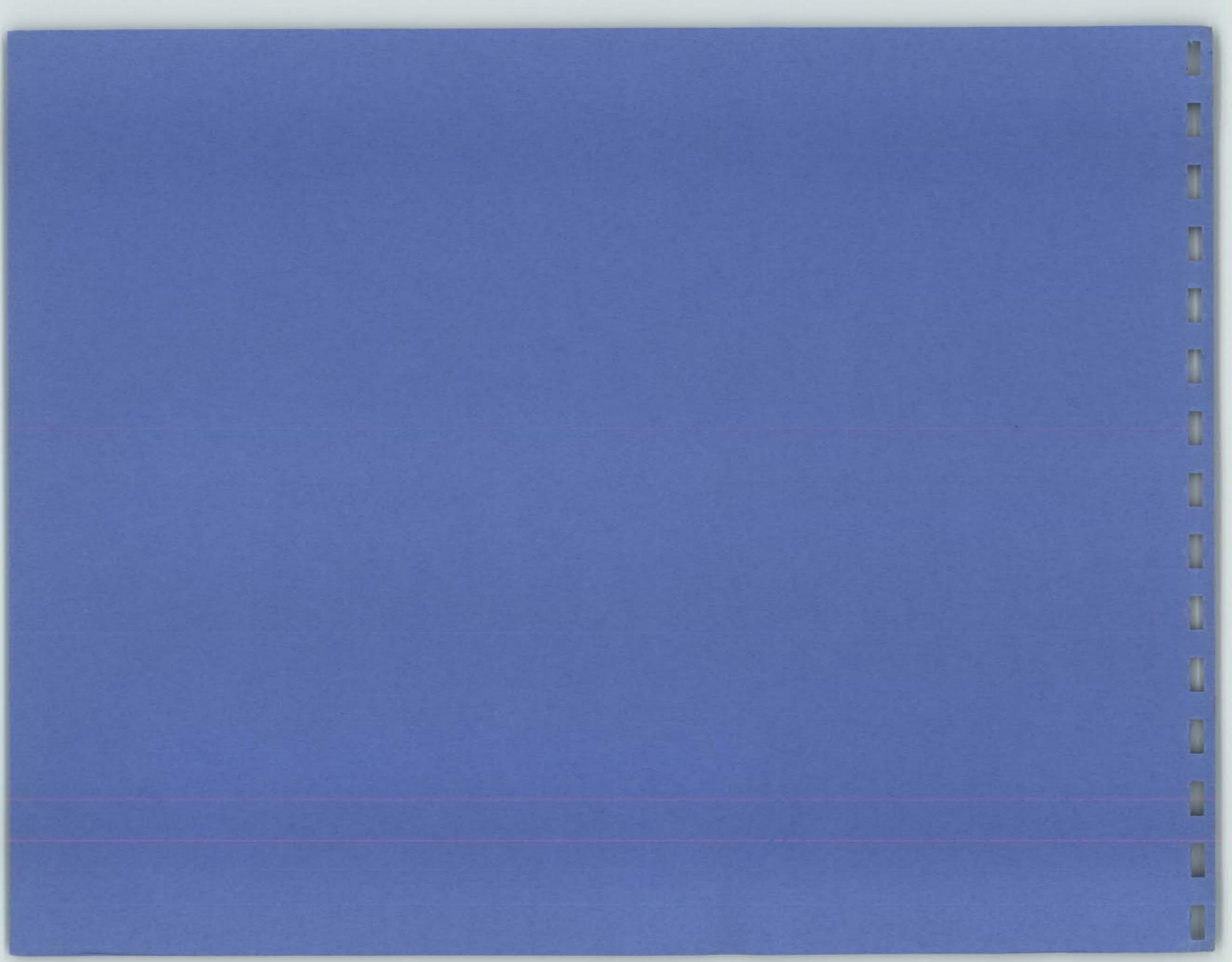
26-4002

---

Herbert I. Gross

---

Massachusetts  
Institute of Technology



---

CALCULUS REVISITED  
PART 1  
A Self-Study Course

---

LECTURE NOTES

---

Herbert I. Gross

---

Center for Advanced Engineering Study  
Massachusetts Institute of  
Technology

---



---

## Table of Contents

### Block I: Sets, Functions, and Limits

- 0.000 Preface
- 1.010 Analytic Geometry
- 1.020 Functions
- 1.025 Inverse Functions
- 1.030 Derivatives and Limits
- 1.040 Limits: A More Rigorous Approach
- 1.060 Mathematical Induction

### Block II: Differentiation

- 2.010 Derivatives of Some Simple Functions
  - 2.020 Approximations and Infinitesimals
  - 2.030 Composite Functions and the Chain Rule
  - 2.040 Differentiation of Inverse Functions
  - 2.045 Implicit Differentiation
  - 2.050 Continuity
  - 2.060 Curve Plotting
  - 2.070 Maxima-Minima
  - 2.080 Rolle's Theorem and its Consequences
  - 2.090 Inverse Differentiation
  - 2.100 The "Definite" Indefinite Integral
-

---

Block III: The Circular Functions

3.010 Circular Functions

3.020 Inverse Circular Functions

Block IV: The Definite Integral

4.010 2-dimensional Area

4.020 Marriage of Differential & Integral Calculus

4.030 3-dimensional Area (Volume)

4.040 1-dimensional Area (Arc Length)

Block V: Transcendental Functions

5.010 Logarithms without Exponents

5.020 Inverse Logarithms

5.030 What a Difference a Sign Makes

5.040 Inverse Hyperbolic Functions

Block VI: More Integration Techniques

6.010 Some Basic Recipes

6.020 Partial Fractions

6.030 Integration by Parts

6.040 Improper Integrals

---

---

Block VII: Infinite Series

7.010 Many Versus Infinite

7.020 Positive Series

7.030 Absolute Convergence

7.040 Polynomial Approximations

7.050 Uniform Convergence

7.060 Uniform Convergence of Series

---

## Block I: Sets, Functions, and Limits

0.000 Preface

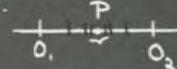
32 min.

Preface:

- Lectures
- Text
- Notes
- Study Guide
- Learning Exercises
- "Instantaneous Speed"

$\frac{0}{0} = \text{undefined}$   
 $\frac{10^{-6}}{10^{-12}} = 10^6; \frac{10^{-12}}{10^{-6}} = 10^{-6}$

$$b \times \left( \frac{a}{b} \right) = a$$

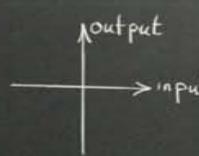
$$0 \times \left( \frac{a}{0} \right) = 0$$


How many pairs of observers do we need?

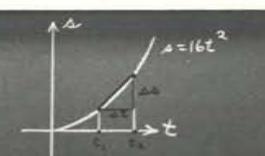
Functions

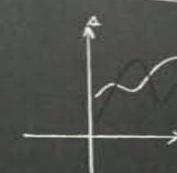
$$\Delta = 16t^2$$

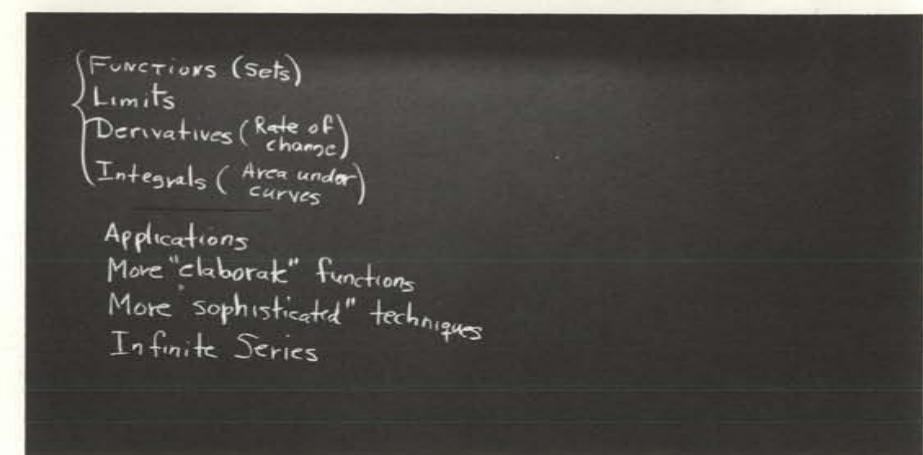
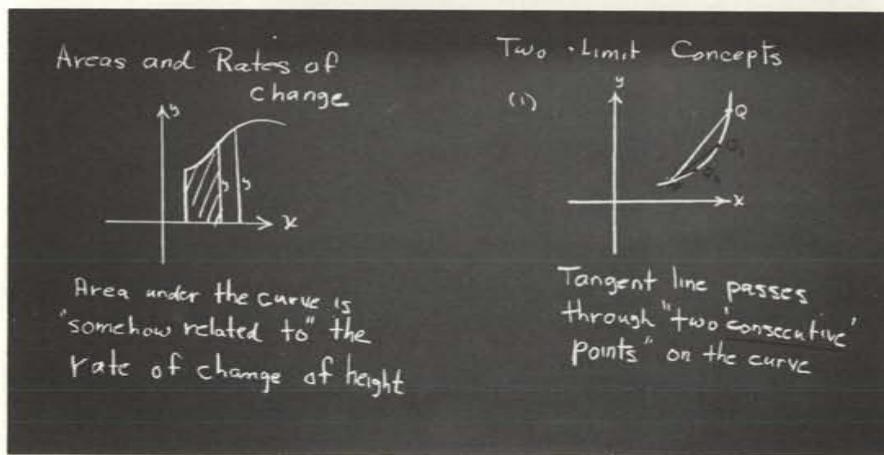
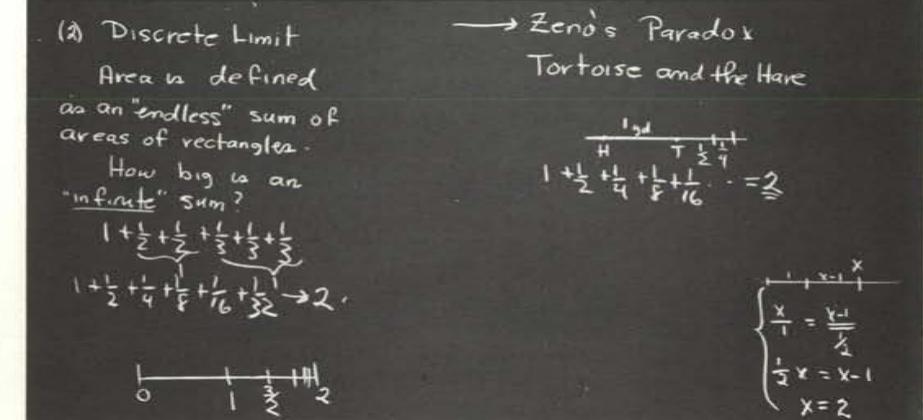
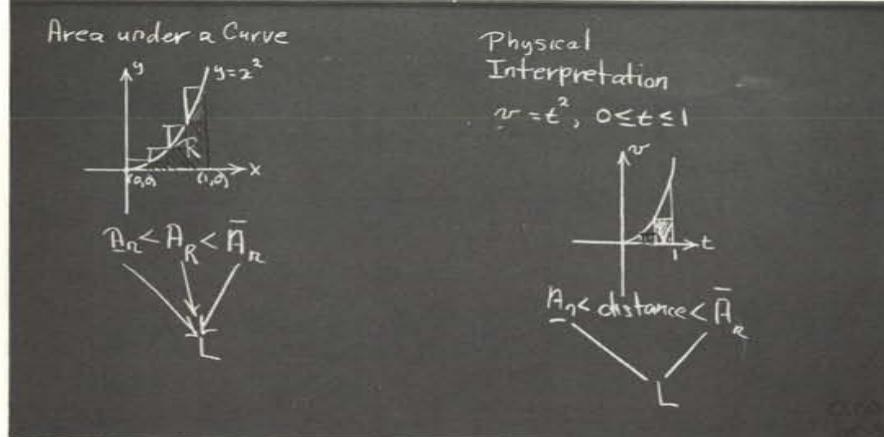
We can find  $\Delta$  for each  $t$



$\Delta = 16t^2$

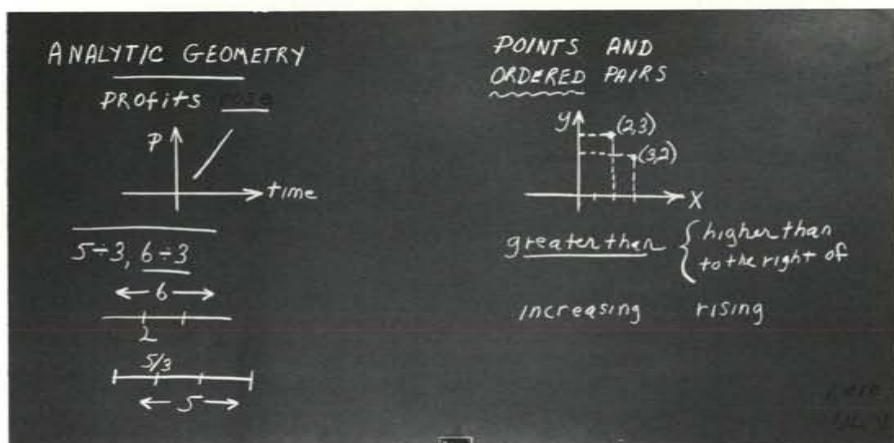


$$\Delta_{av} = \frac{\Delta s}{\Delta t}, [v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}]$$




## 1.010 Analytic Geometry

37 min.



$$s = 16t^2$$

$t \xrightarrow{\text{distance}}$   $\boxed{16t^2}$   $\xrightarrow{\text{machine}}$   $\boxed{16t^2}$   $\xrightarrow{\text{output}}$

$$V = \pi r^2 h$$

$(r,h) \boxed{V - \text{machine}}$   $\boxed{\pi r^2 h}$

$(2,3) \rightarrow 12\pi$   
 $(3,2) \rightarrow 18\pi$

$$\begin{aligned} & (a+b)^2 \\ & a^2 + 2ab + b^2 \end{aligned} \quad \left. \begin{array}{l} (a+b)^3 \\ a^3 + 3a^2b + 3ab^2 + b^3 \end{array} \right\} \quad \begin{array}{|c|c|} \hline a & b \\ \hline a^2 & ab \\ \hline ab & b^2 \\ \hline \end{array}$$

$$\begin{aligned} & (a+b)^4 \\ & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \end{aligned}$$

1.010  
12

$$S = \{(x, y) : x^2 + y^2 = 25\}$$

$(3, 4) \in S, (1, 2) \notin S$



$$x^2 + y^2 = 25$$

$(1, 2) \notin S$   
"inside" thickly

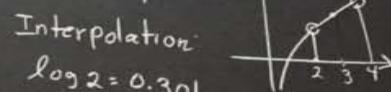
Why straight lines?



$$\log 2 = 0.301$$

$$\log 4 = 0.602$$

0.477



Equations of lines



line  $\parallel$  to y-axis  
 $\{(x, y) : x=a\} \rightarrow x=a$



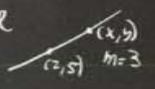
$$m = \frac{y - y_1}{x - x_1}$$

Example:

$$m_e = 3 \quad (2, 5) \in l$$

$$\frac{y-5}{x-2} = 3$$

$$y = 3x - 1$$



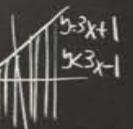
Is  $(8, 23)$  on  $l$ ?

$$y = 3x - 1$$

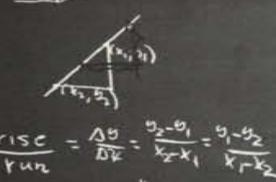
$$23 = 3(8) - 1$$

$$(8, 23)$$

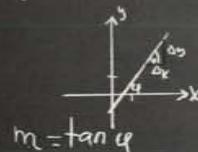
$$(8, 12)$$



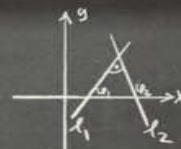
slope:



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$



$$m = \tan \alpha$$



$$m_1 = \tan u_1; m_2 = \tan u_2$$

$$\tan(u_1 - u_2) = \frac{\tan u_1 - \tan u_2}{1 + \tan u_1 \tan u_2}$$

$$\boxed{l_1 \perp l_2} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Simultaneous equations

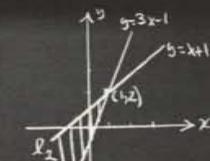
$$\begin{cases} y = 3x - 1 \\ y = x + 1 \end{cases}$$

$$3x - 1 = x + 1$$

$$2x = 2$$

$$x = 1$$

$$(1, 2)$$



$$\begin{cases} (1, 2) \in l_1 \cap l_2 \\ y < x + 1 \\ y > 3x - 1 \end{cases}$$

## 1.020 Functions

39 min.

Functions

$f: A \rightarrow B$  means that  
 $f$  is a rule which assigns  
 to each  $a \in A$  an element  $b \in B$

$A = \text{domain of } f$   
 $B = \text{range of } f$   
 $C = \text{Image of } f$   
 $= f(A)$

$f: A \rightarrow B$

Onto Functions

Range = Image

Example:  
 $\rightarrow A = \{1, 2, 3\}$   
 $f(1) = 4, 1 \in A$   
 $f(2) = 8$   
 $f(3) = 12$   
 $B = \{4, 8, 12\}$

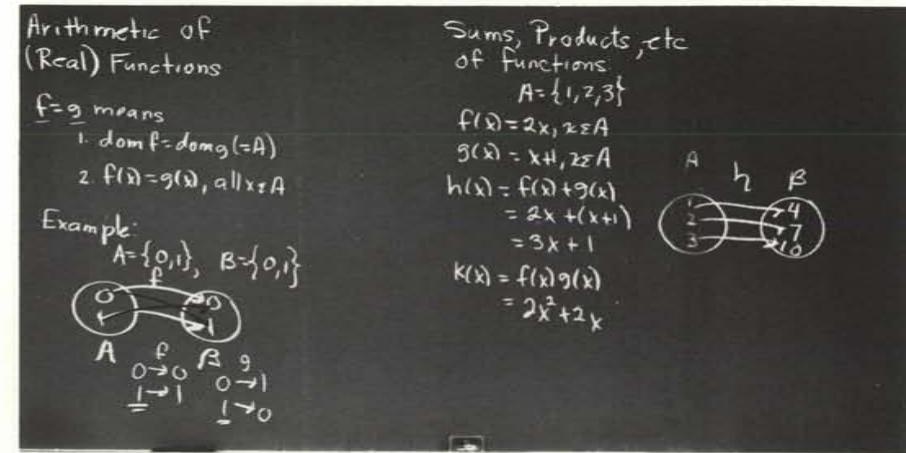
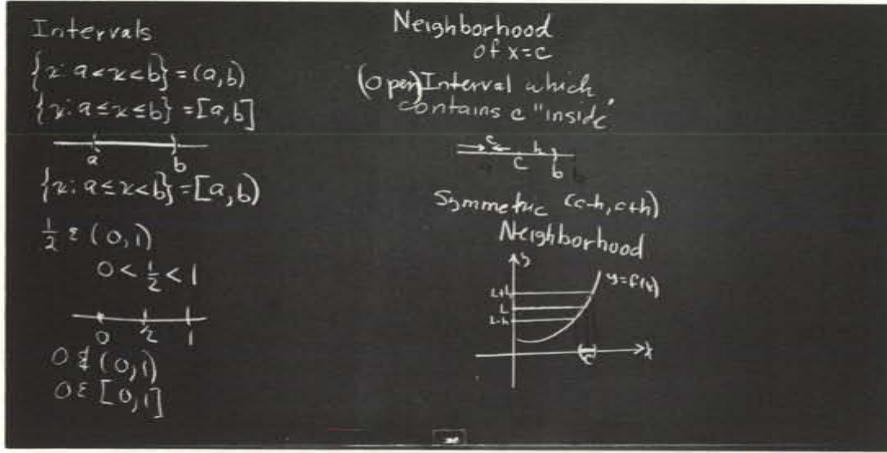
One-to-One Functions

$f(a_1) = f(a_2) \rightarrow a_1 = a_2$

Functions of a Real Variable

$s = 16t^2$

$s = \begin{cases} 0, t < 0 \\ 16t^2, 0 \leq t \leq t_G \\ 16t_G, t > t_G \end{cases}$



**Deleted Neighborhoods**

$f(x) = \frac{x^2 - 9}{x - 3}$

$f(3) = \frac{0}{0}$

$f(x) = \frac{(x+3)(x-3)}{(x-3)}$

$37$

"distance"  $(x_1, x_2) = \sqrt{x_2 - x_1}$

**Absolute Value**

$$|x_1 - x_2| = \sqrt{(x_1 - x_2)^2}$$

$$|x - 3| < 2$$

$$1 < x < 5$$

$$+\sqrt{(x-3)^2} < 2$$

$$(x-3)^2 < 4$$

$$\begin{cases} x^2 - 6x + 9 < 0 \\ (x-1)(x-5) < 0 \end{cases} \quad 1 < x < 5$$

$$> 0 < 0$$

**Composition of functions**

$$g = g \circ f \Rightarrow g(x) = \underline{g(f(x))}$$

$$f(x) = 2x, g(x) = x+1$$

$$g(x) = 2x+1$$

dom g = dom f

$$a+b = b+a$$

$$a \cdot b = b \cdot a$$

$$a/b \neq b/a$$

$p = f \circ g$

$p(x) = 2(x+1) = 2x+2$

$g(x) = 2x+1$

$\begin{cases} f(x)g(x) = 2x(x+1) \\ f \circ g(x) = 2x+2 \\ g \circ f(x) = 2x^2+1 \end{cases}$

1.025 Inverse Functions

40 min.

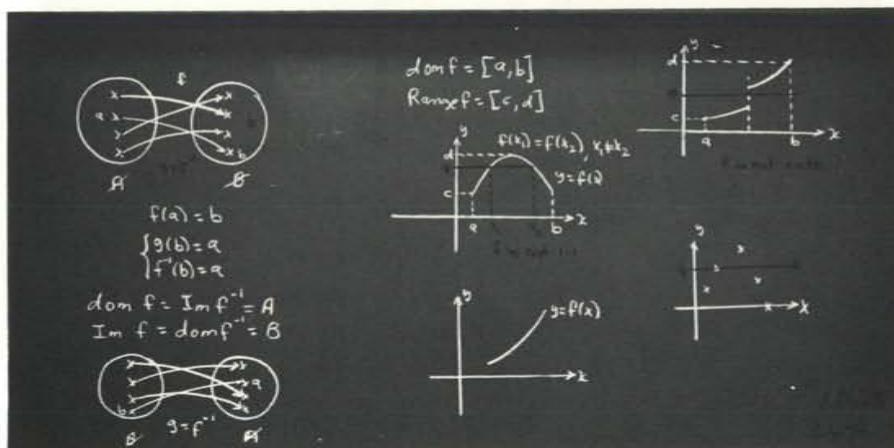
Inverse Functions  
 "Switch in Emphasis"

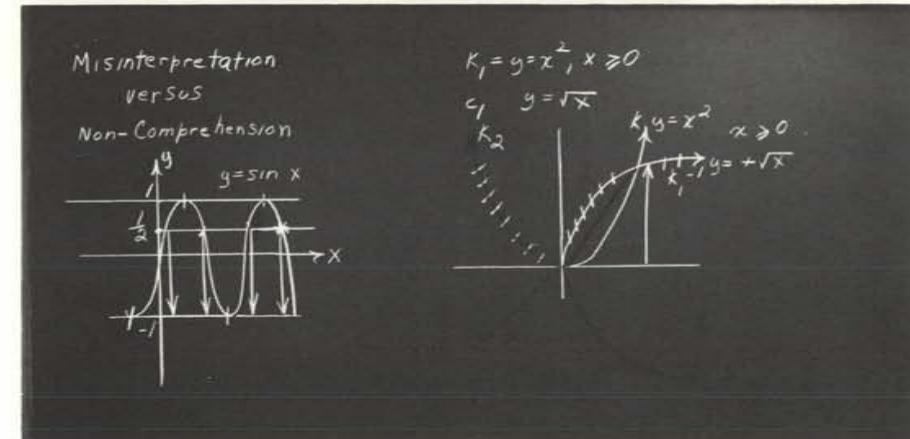
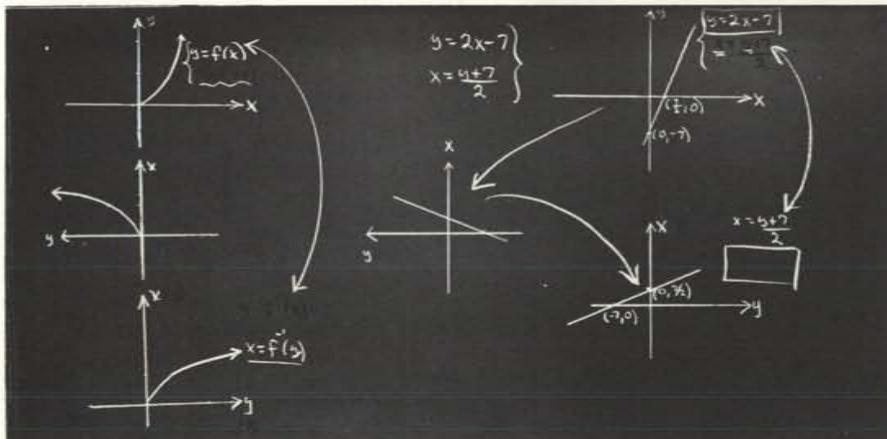
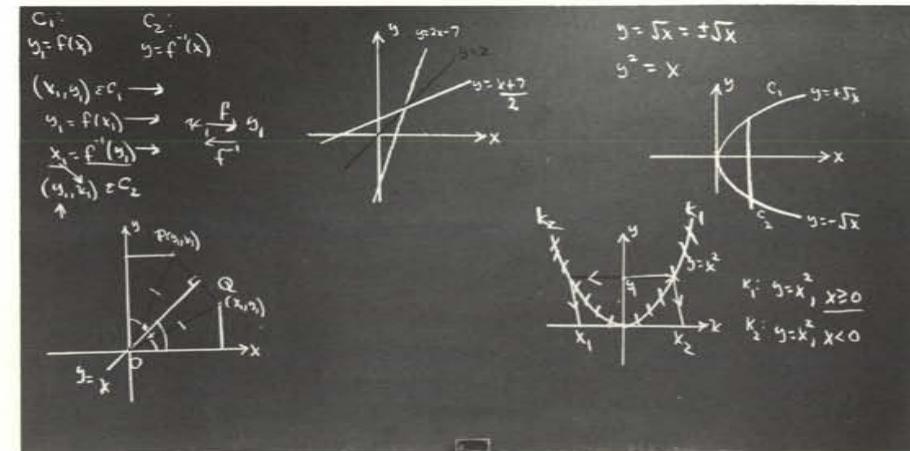
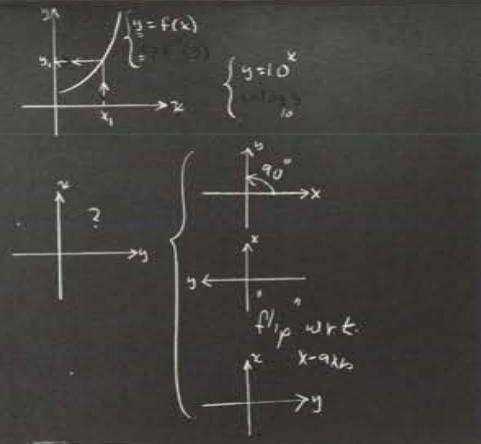
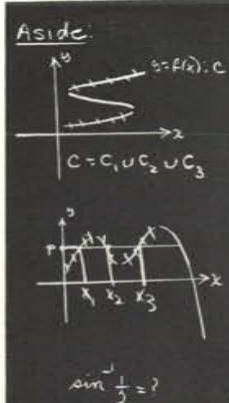
$$\begin{array}{rcl} 5-3=2 & \quad 2+3=5 \\ \uparrow & & \uparrow \\ 3+\underline{(5-3)}=5 & & * \\ \frac{3+(5-3)}{2}=5 & & \\ y=\log_b x ; b=x \\ y=\sin^{-1} x ; x=\sin y \end{array}$$

$y=f(x) ; x=f^{-1}(y)$

Example  
 $\rightarrow y=2x-7 ; y=f(x) = 2x-7$   
 $x=\frac{y+7}{2} ; x=f^{-1}(y) = \frac{y+7}{2}$

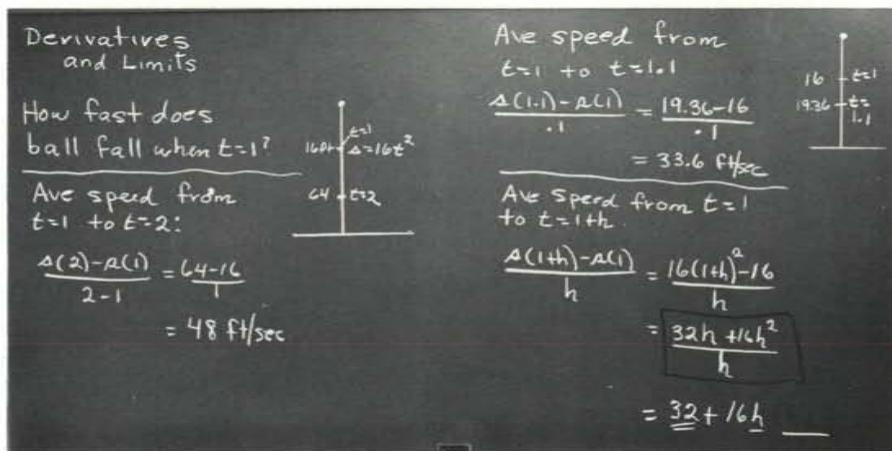
$\begin{array}{c} \text{input} \xrightarrow{\text{f-machine}} \boxed{\text{f-machine}} \xrightarrow{\text{output}} x \\ \downarrow \quad \uparrow \\ \text{input} \xrightarrow{\text{f'-machine}} \boxed{\text{f'-machine}} \xrightarrow{\text{output}} x \end{array}$

$$\begin{array}{l} f^{-1} \circ f = \text{Id}_{\text{int}} : c \xrightarrow{f} 2c-7 \xrightarrow{f^{-1}} \frac{(2c-7)+7}{2} = c \\ f(f^{-1}(d)) = d : d \xrightarrow{f^{-1}} \frac{d+7}{2} \xrightarrow{f} 2\left(\frac{d+7}{2}\right)-7 = d \end{array}$$




## 1.030 Derivatives and Limits

45 min.



"Appears" that speed is 32 ft/sec when  $t=1$   
I.e.,

$$\lim_{h \rightarrow 0} \left[ \frac{\Delta(1+h) - \Delta(1)}{h} \right] = \lim_{h \rightarrow 0} (32 + 16h) = 32 \text{ ft/sec}$$

$$\frac{\Delta(t_i+h) - \Delta(t_i)}{h} = \frac{16(t_i+h)^2 - 16t_i^2}{h} = 32t_i + 16h$$

By this approach,  
if  $s = 16t^2$  then at time  $t=t_i$ ,  $v = 32t_i$ , or:  $v = 32t$ ;  
where (instantaneous) speed has been defined as a limit of average speeds

$$y = f(x)$$

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

$$f'(x_1) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \right]$$

$$\lim_{\substack{x \rightarrow a \\ x \neq a}} f(x) = f(a)$$

$$\lim_{x \rightarrow 3} \left[ \frac{x^2 - 9}{x - 3} \right] = \frac{0}{0}$$

$$\frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{(x-3)}$$

$$= x+3$$

Given  $\epsilon > 0$  must find  $\delta > 0$  such that:

$$0 < |x-3| < \delta \rightarrow |x^2 - 2x - 3| < \epsilon$$

$$-\epsilon < x^2 - 2x - 3 < \epsilon$$

$$-\epsilon < x^2 - 2x + 1 - 4 < \epsilon$$

$$-\epsilon < (x-1)^2 - 4 < \epsilon$$

$$4 - \epsilon < (x-1)^2 < 4 + \epsilon$$

If  $x$  is "near" 3,  $x-1$  is positive

$$\sqrt{4 - \epsilon} < x-1 < \sqrt{4 + \epsilon}$$

$$\sqrt{4 - \epsilon} < x < 1 + \sqrt{4 + \epsilon}$$

$$\frac{x^2 - 9}{x - 3} = x + 3$$

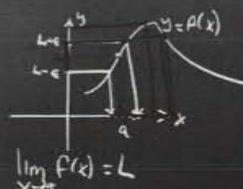
If  $x = 3$ ,  $x^2 - 9/x - 3$  is undefined

$$\lim_{x \rightarrow 3} \left[ \frac{x^2 - 9}{x - 3} \right] = \lim_{x \rightarrow 3} (x+3) = \underline{\underline{\infty}}$$

$$=$$

$$\text{if } x \neq 3.$$

$\lim_{x \rightarrow a} f(x) = L$  means  
Given  $\epsilon > 0$  we can find  
 $\delta > 0$  such that  
 $0 < |x-a| < \delta \rightarrow |f(x)-L| < \epsilon$



$$\sqrt{4 - \epsilon} - 2 < x - 3 < \sqrt{4 + \epsilon} - 2$$

Let  $\delta = \min \left\{ \frac{\sqrt{4 + \epsilon} - 2}{2}, \frac{\sqrt{4 - \epsilon} - 2}{2} \right\}$

Then:

$$0 < |x-3| < \delta \rightarrow |x^2 - 2x - 3| < \epsilon \rightarrow$$

$$\lim_{x \rightarrow 3} (x^2 - 2x) = 3$$

$$\begin{aligned} (1 + \sqrt{4 + \epsilon})^2 - 2(1 + \sqrt{4 + \epsilon}) &= \\ 1 + 2\sqrt{4 + \epsilon} + 4 + \epsilon - 2 - 2\sqrt{4 + \epsilon} &= 3 + \epsilon \end{aligned}$$

IF  $\epsilon < 1$  there exists  $\delta > 0$  such that

$$0 < |x-3| < \delta \rightarrow |x-3| < \frac{\epsilon}{5}$$

$$|x+1| < 5 \rightarrow |x-3||x+1| < \frac{\epsilon}{5} \cdot 5$$

1.040 Limits: A More Rigorous Approach

46 min.

Limits - A More Rigorous Approach

$\lim_{x \rightarrow a} f(x) = L$

For each  $\epsilon > 0$ , can find  $\delta > 0$  such that  $0 < |x-a| < \delta \rightarrow |f(x)-L| < \epsilon$

Theorem:

$\lim_{x \rightarrow a} c = c$   
Let  $f(x) = c$   
Then  $\lim_{x \rightarrow a} f(x) = c$

Given  $\epsilon > 0$  we must find  $\delta > 0$  such that  $0 < |x-a| < \delta \rightarrow |c - c| < \epsilon$

Thm:

$\lim_{x \rightarrow a} x = a$

Given  $\epsilon > 0$  choose  $\delta = \epsilon$   
must find  $\delta > 0$  such that  $0 < |x-a| < \delta \rightarrow |x-a| < \epsilon$

Thm:

$\lim_{x \rightarrow a} [f(x) + g(x)] =$   
 $\lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$

$\lim_{x \rightarrow a} f(x) = L_1, \lim_{x \rightarrow a} g(x) = L_2$  defn f

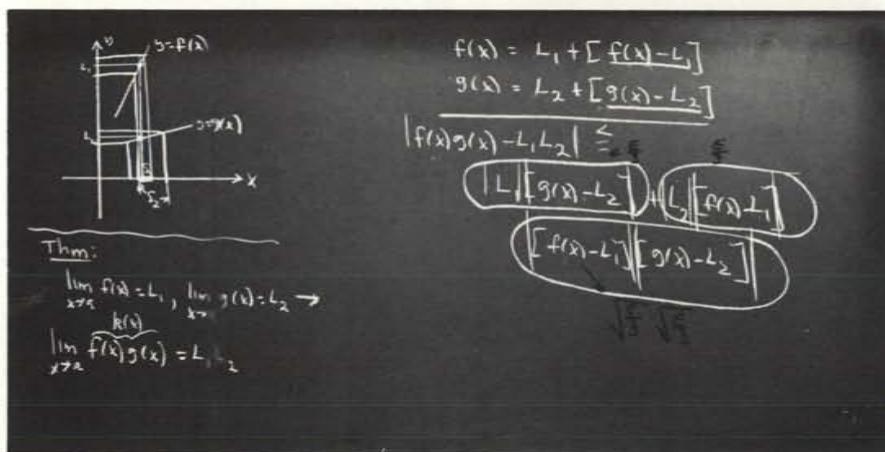
Let  $h(x) = f(x) + g(x)$  (HII)  
To prove:  $\lim_{x \rightarrow a} h(x) = L_1 + L_2$  defn g

Given  $\epsilon > 0$  must find  $\delta > 0$   
such that  $0 < |x - a| < \delta \rightarrow |(f(x) + g(x)) - (L_1 + L_2)| < \epsilon$

$$|(f(x) + g(x)) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)| \leq |f(x) - L_1| + |g(x) - L_2|$$

Given  $\epsilon > 0$ , let  
 $\epsilon_1 = \frac{\epsilon}{2}$   
 Can find  $\delta_1 > 0$  such that  
 $0 < |x - a| < \delta_1 \rightarrow |f(x) - L_1| < \epsilon_1 \quad (\lim_{x \rightarrow a} f(x) = L_1)$   
 Can find  $\delta_2 > 0$  such that  
 $0 < |x - a| < \delta_2 \rightarrow |g(x) - L_2| < \epsilon_2$   
 Let  $\delta = \min\{\delta_1, \delta_2\}$   
 $0 < |x - a| < \delta \rightarrow |f(x) - L_1| + |g(x) - L_2| < \epsilon$

Example  
 $\lim_{x \rightarrow 3} (x^2 + 7x) = ?$   
 $x \rightarrow 3 \quad (x - 3)k? \quad x^2 + 7x \approx 30.023$   
 $\rightarrow |\lim_{x \rightarrow 3} x^2 + |\lim_{x \rightarrow 3} 7x| =$   
 $(\lim_{x \rightarrow 3} x)(\lim_{x \rightarrow 3} x) + (\lim_{x \rightarrow 3} 7)(\lim_{x \rightarrow 3} x) =$   
 $(3)(3) + 7(3) = 30$



## 1.060 Mathematical Induction

29 min.

Mathematical Induction

$$\lim_{x \rightarrow a} [f_1(x) + f_2(x)] = \lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x)$$

How about

$$\lim_{x \rightarrow a} [f_1(x) + f_2(x) + f_3(x)] ?$$

$$\lim_{x \rightarrow a} [f_1(x) + f_2(x)] + \lim_{x \rightarrow a} f_3(x)$$

$$[\lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x)] +$$

$$\left\{ \begin{array}{l} \lim_{x \rightarrow a} [f_1(x) + f_2(x) + f_3(x) + f_4(x)] = \\ \lim_{x \rightarrow a} [f_1(x) + f_2(x) + f_3(x)] + \lim_{x \rightarrow a} f_4(x) = \\ \lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x) + \lim_{x \rightarrow a} f_3(x) + \lim_{x \rightarrow a} f_4(x) \end{array} \right.$$

$$1 + 2 + 3 + 4 = 10$$

$$12 \div 6 \div 2 =$$

$$12 \div 6 \div 2 = 4$$

odd + odd = ? even  
 Positive - positive = ?  
 $3 - 5$

Suppose

$$\lim_{x \rightarrow a} [f_1(x) + \dots + f_n(x)] =$$

$$\lim_{x \rightarrow a} f_1(x) + \dots + \lim_{x \rightarrow a} f_n(x)$$

$$\lim_{x \rightarrow a} [f_1(x) + \dots + f_n(x) + f_{n+1}(x)] =$$

MATHEMATICAL INDUCTION:

- (1) Show CONJECTURE TRUE FOR  $n=1$  ✓
- (2) Prove that truth for  $n=k$  implies the truth for  $n=k+1$  ✓

Assume:

$$\frac{1+2+\dots+n}{2} = \frac{n(n+1)}{2}$$

$$1+2+\dots+k = \frac{k(k+1)}{2}$$

$$1+2+\dots+k+k+1 =$$

$$\frac{k(k+1)}{2} + k+1 =$$

$$(k+1)\left(\frac{k}{2} + 1\right) =$$

$$\frac{(k+1)(k+2)}{2}$$

$P(n) = n^2 - n + 41$

$P(1) = 1^2 - 1 + 41 = 41$	$6 = 2 \times 3$
$P(2) = 2^2 - 2 + 41 = 43$	$7 = 7$
$P(3) = 3^2 - 3 + 41 = 47$	$8 = 2 \times 2 \times 2$
$\vdots$	$9 = 3 \times 3$
$P(40) = \text{prime}$	$10 = 2 \times 5$
$P(41) = 41^2 - 41 + 41$	$11 = 11$

$\underbrace{n+1 \text{ factors}}_{\text{different than } n}$

$\begin{cases} 59 \\ 60 \\ 61, 2, 3, 4, 5, 6, 7 \end{cases}$

UNIQUE FACTORIZATION THEOREM

True for  $n=7$   $\xrightarrow{x \rightarrow k+1}$

True for  $n \geq 7$

$1+2+3+\dots+98+99+100 =$

$$\frac{(100)(101)}{2}$$

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

"Next" one?

## Block II: Differentiation

### 2.010 Derivatives of Some Simple Functions

28 min.

#### Derivatives of Some Simple Functions

$$f'(x_i) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right]$$

Example 1:  $f(x) = x^3$

$$\begin{aligned} f(x_i + \Delta x) - f(x_i) &= (x_i + \Delta x)^3 - x_i^3 \\ &= 3x_i^2 \Delta x + 3x_i \Delta x^2 + \Delta x^3 \\ \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} &= 3x_i^2 + 3x_i \Delta x + \Delta x^2 \\ \therefore f'(x_i) &= 3x_i^2; f'(x) = 3x^2 \end{aligned}$$

#### Generalization

Let  $f(x) = x^n$   
n a positive whole number

$$\begin{aligned} f(x_i + \Delta x) - f(x_i) &= (x_i + \Delta x)^n - x_i^n \\ &= n x_i^{n-1} \Delta x + \Delta x^n \left( \frac{\text{number}}{\text{number}} \right) \end{aligned}$$

$$\begin{aligned} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} &= n x_i^{n-1} + \Delta x \left( \frac{\text{number}}{\text{number}} \right) \\ \therefore f'(x_i) &= n x_i^{n-1} \end{aligned}$$

$$f(x) = x^n \rightarrow f'(x) = n x^{n-1}$$

Example 2

f, g differentiable at  $x_i$   
(I.e.,  $f'(x_i)$  and  $g'(x_i)$  exist)

Define h by

$$h(x) = f(x) + g(x)$$

[ $x \in \text{dom } f \cap \text{dom } g$ ]

Then:

$h'$  exists

$$h'(x) = f'(x) + g'(x)$$

#### Proof

$$h(x_i + \Delta x) = f(x_i + \Delta x) + g(x_i + \Delta x)$$

$$\begin{aligned} h(x_i + \Delta x) - h(x_i) &= [f(x_i + \Delta x) + g(x_i + \Delta x)] \\ &\quad - [f(x_i) + g(x_i)] \end{aligned}$$

$$\begin{aligned} \frac{h(x_i + \Delta x) - h(x_i)}{\Delta x} &= \left[ \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right] \\ &\quad + \left[ \frac{g(x_i + \Delta x) - g(x_i)}{\Delta x} \right] \end{aligned}$$

$$\therefore h'(x_i) = f'(x_i) + g'(x_i)$$

Beware of "Self-Evident"

$$[f(x)g(x)]' \neq f'(x)g'(x)$$

$$f(x) = x+1 \rightarrow f'(x) = 1$$

$$g(x) = x-1 \rightarrow g'(x) = 1$$

$$f(x)g(x) = x^2 - 1 \rightarrow [f(x)g(x)]' = 2x$$

$$\therefore [f(x)g(x)]' \neq f'(x)g'(x)$$

In fact, if

$$h(x) = f(x)g(x)$$

then:

$$h(x_i + \Delta x) - h(x_i) = f(x_i + \Delta x)g(x_i + \Delta x)$$

$$- f(x_i)g(x_i)$$

$$= f(x_i + \Delta x)g(x_i + \Delta x) - f(x_i)g(x_i + \Delta x)$$

$$+ f(x_i)g(x_i + \Delta x) - f(x_i)g(x_i)$$

$$\therefore \frac{h(x_i + \Delta x) - h(x_i)}{\Delta x} = \left[ \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right] g(x_i + \Delta x)$$

$$h'(x_i) = f'(x_i)g(x_i) + f(x_i)\left[ \frac{g(x_i + \Delta x) - g(x_i)}{\Delta x} \right]$$

$$(1(x_i - 1) + (y_i + 1)) = 2x_i$$

Summary

The basic definition:

$$f'(x_i) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right]$$

never changes. But it can be manipulated to yield "convenient" "recipes".

\* It is not true always that  $\lim_{t \rightarrow a} g(t) = g(a)$

$$\text{For example, let } g(t) = \frac{t^2 - 1}{t - 1}$$

$$\lim_{t \rightarrow 1} g(t) = 2, \text{ } g(1) \text{ doesn't exist!}$$

( $\frac{0}{0}$ )

In our present case:

$$g(x_i + \Delta x) - g(x_i) = \left[ \frac{g(x_i + \Delta x) - g(x_i)}{\Delta x} \right] \Delta x$$

$$\lim_{\Delta x \rightarrow 0} [g(x_i + \Delta x) - g(x_i)] = \underbrace{g'(x_i)}_{\Delta x \rightarrow 0} \lim_{\Delta x \rightarrow 0} \Delta x = 0$$

$$\therefore \lim_{\Delta x \rightarrow 0} g(x_i + \Delta x) = g(x_i)$$

For a quotient

we can show that if  $h(x) = \frac{f(x)}{g(x)}$  then

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example.

$$f(x) = x^n \quad n, \text{ positive integer}$$

$$= \frac{(x^n)}{x^n}$$

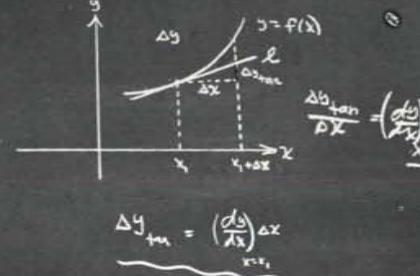
$$f'(x) = \frac{x^n(0) - 1(nx^{n-1})}{x^n} = -nx^{n-1}$$

$$= -n x^{n-1}$$

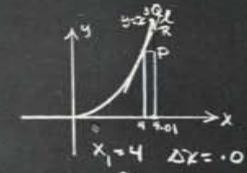
2.020 Approximations and Infinitesimals

34 min.

Approximations  
and  
Infinitesimals



$$(4.01)^3 = 64.481201$$



$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx}|_{x=4} = 3 \cdot 4^2$$

$$\frac{dy}{dx}|_{x=4} = 48$$

$$\Delta y_{\text{tan}} = 48(0.01)$$

$$\underline{64.48}$$

$$y = x^3$$

$$\left( \frac{dy}{dx} \right)_{x=x_1} = 3x_1^2$$

$$\Delta y_{\text{tan}} = 3x_1^2 \Delta x$$

$$\Delta y = (x_1 + \Delta x)^3 - x_1^3$$

$$= [3x_1^2 \Delta x] + [3x_1 \Delta x]^2 + \Delta x^3$$

$$\frac{\Delta y}{\Delta x} = \left( \frac{dy}{dx} \right)_{x=x_1} + k$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left[ \left( \frac{dy}{dx} \right)_{x=x_1} + k \right]$$

$$+ \lim_{\Delta x \rightarrow 0} k$$

$$\frac{dy}{dx} \Big|_{x=x_1} = \frac{dy}{dx} \Big|_{x=x_1} + \lim_{\Delta x \rightarrow 0} k$$

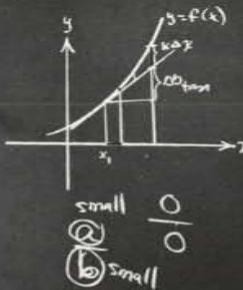
$$\boxed{\lim_{\Delta x \rightarrow 0} k = 0}$$

$$\Delta y = \left( \frac{dy}{dx} \right)_{x=x_1} \Delta x + k \Delta x$$

$\underbrace{\Delta y}_{\Delta b + \tan}$

$$\Delta y = \underbrace{\Delta y}_{\Delta x} + k \Delta x$$

where  $\lim_{\Delta x \rightarrow 0} k = 0$



Example

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$\begin{aligned} \rightarrow dy &= 3x^2 dx \\ \Delta y &= 3x^2 \Delta x \end{aligned}$$

$$\Delta y = \left( \frac{dy}{dx} \right)_{x=x_1} \Delta x + k \Delta x$$

where  $\lim_{\Delta x \rightarrow 0} k = 0$

$$\begin{aligned} f(x_1 + \Delta x) - f(x_1) &= \\ f'(x_1) \Delta x + k \Delta x & \\ \text{where } \lim_{\Delta x \rightarrow 0} k &= 0 \end{aligned}$$

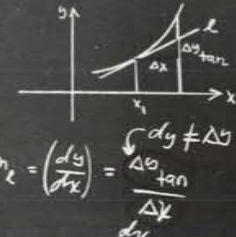
$$\frac{\Delta y}{\Delta t} = \left( \frac{dy}{dx} \right)_{x=x_1} \frac{\Delta x}{\Delta t} + k \frac{\Delta x}{\Delta t} \quad \left\{ \begin{array}{l} \lim_{\Delta t \rightarrow 0} k = 0 \\ \text{or} \end{array} \right.$$

$$\frac{16}{64} = \frac{1}{4} \quad \frac{19}{95} \quad \frac{24}{85} \quad \frac{49}{98}$$

$\left( \frac{dy}{dx} \right)$  is one symbol. IT  
is not  $dy \div dx$

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} &= \left[ \lim_{\Delta t \rightarrow 0} \left( \frac{dy}{dx} \right)_{x=x_1} \right] \left[ \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \right] \\ &\quad + \left[ \lim_{\Delta t \rightarrow 0} k \right] \left[ \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \right] \end{aligned}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} + \frac{dx}{dt}$$



$$m_k = \left( \frac{dy}{dx} \right) = \frac{\Delta y}{\Delta x}$$

## 2.030 Composite Functions and the Chain Rule

39 min.

Composite Functions  
and  
The Chain Rule

$$\left(\frac{dy}{dt}\right)_{t=t_1} = \left(\frac{dy}{dx}\right)_{x=x_1} \left(\frac{dx}{dt}\right)_{t=t_1}$$

$$\rightarrow \frac{\Delta y}{\Delta t} = \left(\frac{dy}{dx}\right)_{x=x_1} \frac{\Delta z}{\Delta t} + \cancel{(k)} \frac{\Delta z}{\Delta t}$$

$$\begin{cases} \left(\frac{dy}{dt}\right)_{t=t_1} = \left(\frac{dy}{dx}\right)_{x=x_1} \left(\frac{dx}{dt}\right)_{t=t_1} \\ (\lim_{\Delta z \rightarrow 0} k = 0) \end{cases}$$

$$\rightarrow \frac{\Delta y}{\Delta t} = \cancel{0} \frac{\Delta y}{\Delta t} = 0$$

Example

Find  $\frac{dy}{dx}$  if  $y = (x^2 + 1)^4$

$$y = (x^2 + 1)^4 \quad y = (u^2)^4$$

$$\frac{dy}{dx} = 2x \quad y = (u^2)^2$$

$$\frac{dy}{dx} = 2(u^2) \cdot u' \quad \frac{dy}{dx} = 2(u) \cdot u'$$

$$\frac{du}{dx} = 2x^2 \quad u = x^2 + 1$$

$$\frac{dy}{dx} = 2(x^2 + 1) \cdot 2x^2 \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = 4x^3 + 4x^2 \quad u' = 2x$$

$$\frac{dy}{dx} = 4x(x^2 + 1) \quad \frac{dy}{dx} = 4x(x^2 + 1)$$

$\rightarrow y = f(u)$

$\rightarrow u = g(x)$

$$\frac{dy}{dt} = f'(t)$$

$$\frac{dx}{dt} = g'(t)$$

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{f'(t)}{g'(t)}$$

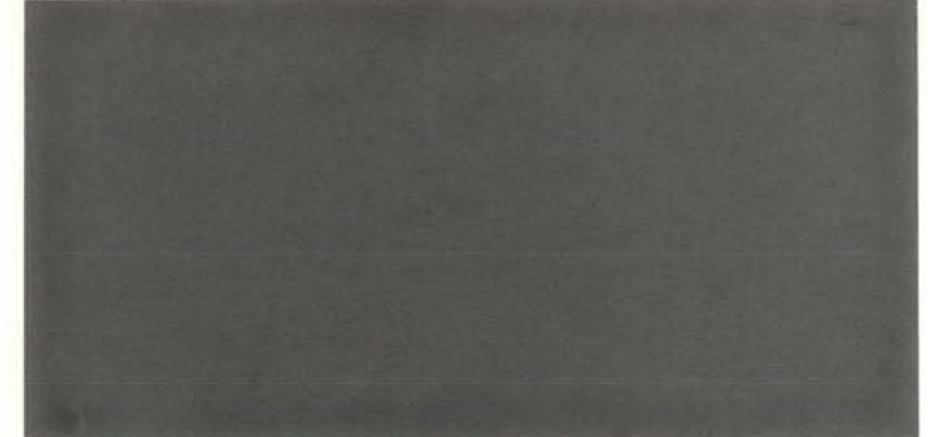
$$\begin{aligned}
 & x = f'(t) \Delta t + k_1 \Delta t \\
 & x = g'(t) \Delta t + k_2 \Delta t \\
 & \lim_{\Delta t \rightarrow 0} k_1 = \lim_{\Delta t \rightarrow 0} k_2 = 0 \\
 & \frac{\Delta x}{\Delta t} = \frac{[f'(t) + k_1] \Delta t}{[g'(t) + k_2] \Delta t} \\
 & \frac{\Delta x}{\Delta t} = \frac{f'(t)}{g'(t)}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{d^2y}{dx^2} + \frac{d^2y/dt^2}{d^2x/dt^2} = \frac{d^2y}{dt^2} \\
 & y = f(t) \rightarrow d^2y/dt^2 \\
 & \boxed{\frac{dy}{dt}} \quad y' = f'(t) \leftarrow k = g(t) \rightarrow d^2y/dt^2 \\
 & \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{f'(t)}{g'(t)} \right) \\
 & = \frac{d}{dt} \left[ \frac{f'(t)}{g'(t)} \right] \frac{dt}{dx} \\
 & = \frac{[g''(t)f'(t) - f''(t)g'(t)]}{[g'(t)]^2} \frac{dt}{dx}
 \end{aligned}$$

$$\begin{aligned}
 & y = f(u) \\
 & u = g(x) \\
 & \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \\
 & \text{Suppose } y = x: \\
 & \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \\
 & 1 = \left( \frac{dy}{du} \right) \left( \frac{du}{dx} \right)
 \end{aligned}$$

Example:

$y = t^4$   
 $x = t^2$   
 $\frac{dy}{dt} = 4t^3$   
 $\frac{dx}{dt} = 2t$   
 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2t^2$   
 check:  $y = x^2$   
 $\frac{dy}{dx} = 2x$



2.040 Differentiation of Inverse Functions

28 min.

Differentiation  
of  
Inverse Functions

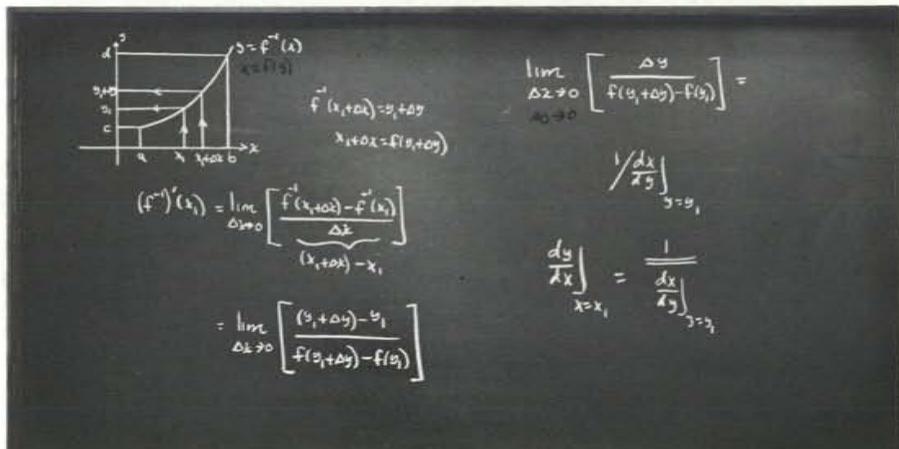
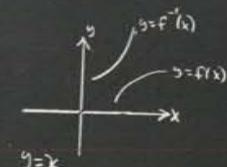
Find  $\frac{dy}{dx}$  if  $y = x^{\frac{1}{3}}$

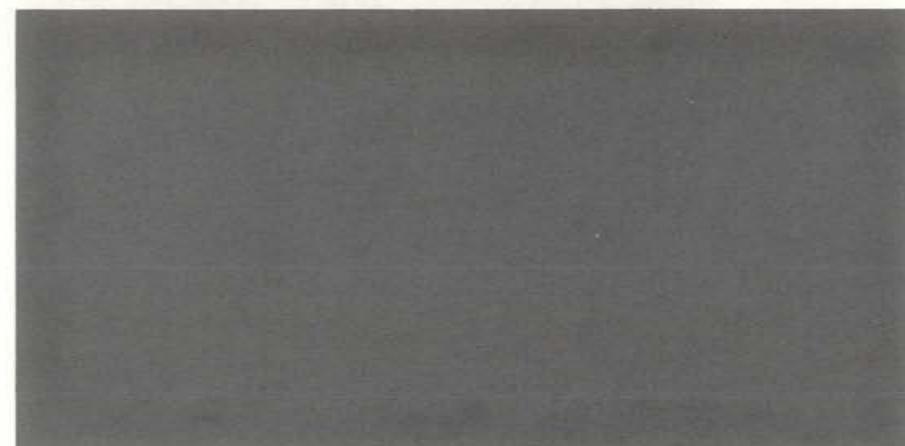
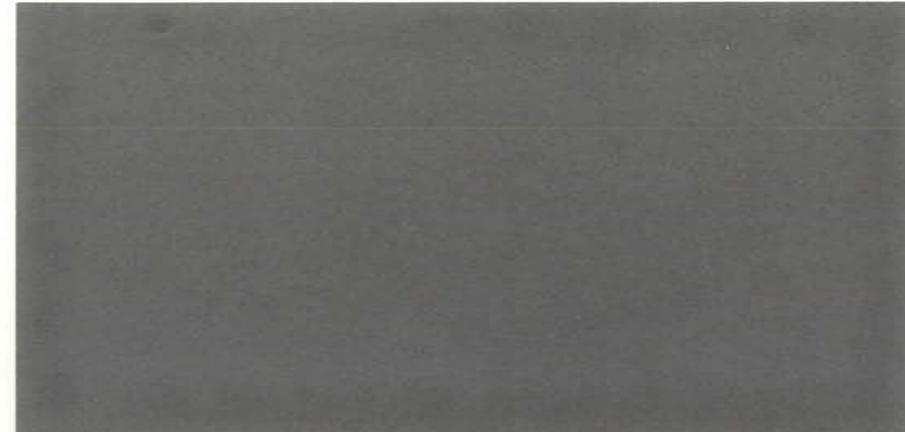
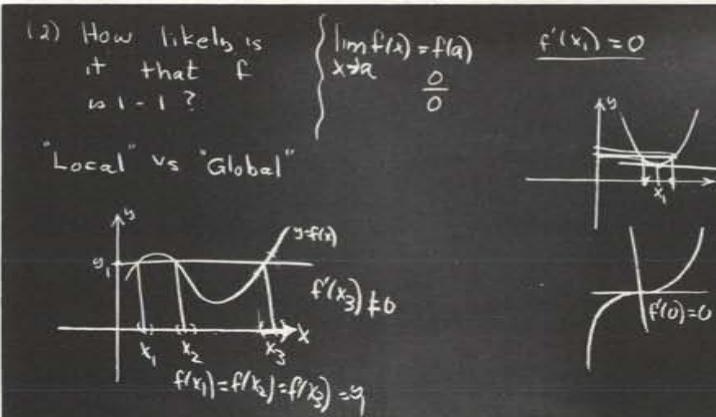
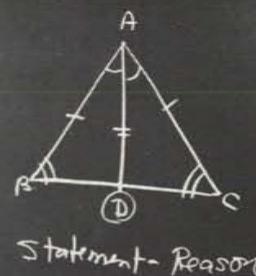
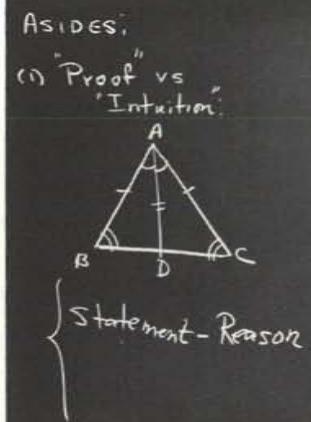
$$x = y^3$$

$$\frac{dx}{dy} = 3y^2$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2} = \frac{1}{3y^2} \cdot y = \frac{1}{3}y^{-2} \\ &= \frac{1}{3}(x^{\frac{1}{3}})^{-2} = \frac{1}{3}x^{-\frac{2}{3}}\end{aligned}$$

$$\begin{aligned}\frac{dy}{du} &= \frac{dy}{dx} \frac{dx}{du} \\ u &= y \\ 1 &= \left(\frac{dy}{dx}\right) \left(\frac{dx}{du}\right)\end{aligned}$$





2.045 Implicit Differentiation

40 min.

Implicit Differentiation

$y = f(x)$ ;  $\frac{dy}{dx} = ?$

$$x^8 + x^6 y^4 + y^6 = 3$$

Assume  $y = y(x)$   
such that

$$x^8 + x^6 y^4 + y^6 = 3$$

$x^2 = 4; x = -2, 2$   
 $2x = 0; x = 0$   
 $x^2 = 4 \text{ means } \{x \cdot x^2 = 4\} = \{-2, 2\}$

$$xy = 1 \rightarrow y = \frac{1}{x} (x \neq 0)$$

$$xy + y = 0$$

$$\frac{d}{dx}(xy) + \frac{dy}{dx} = 0$$

$$x \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$x = 2, y = \frac{1}{2}$$

$$x = -2, y = -\frac{1}{2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$x^2 + y^2 = 25$$

Assume  $y = y(x)$   
such that

$$x^2 + y^2 = 25$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{⑤}$$

$$y(x) = \pm \sqrt{25-x^2}$$

$$y_1(x) = +(25-x^2)^{\frac{1}{2}}$$

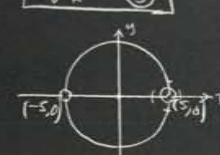
$$y_2(x) = -(25-x^2)^{\frac{1}{2}}$$

$$\frac{dy_1}{dx} = \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{(25-x^2)^{\frac{1}{2}}}$$

$$\frac{dy_2}{dx} = -\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{x}{(25-x^2)^{\frac{1}{2}}}$$

$$= \frac{x}{-y_2}$$


ASIDE:

$$y = x^{\frac{p}{q}}, p, q \text{ integers}$$

$$y^q = x^p$$

$$y = x^{\frac{p}{q}}$$

$$\begin{aligned} qy^{q-1} \frac{dy}{dx} &= px^{p-1} \\ \frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}} \\ &= \frac{px}{q(x^{\frac{p}{q}-1})} \\ &= \frac{p}{q} x^{\frac{p-1-p+1}{q}} = \frac{p}{q} x^{\frac{0}{q}-1} \end{aligned}$$

Find the equation of the line tangent to  $x^8 + x^6 y^4 + y^6 = 3$  at the point  $(1, 1)$ .

$$\begin{aligned} 8x^7 + 6x^5 y^4 + 4x^6 y^3 \frac{dy}{dx} + 6y^5 \frac{dy}{dx} &\equiv 0 \\ \left. \frac{dy}{dx} \right|_{(1,1)} &= -\frac{(8x^7 + 6x^5 y^4)}{4x^6 y^3 + 6y^5} \Big|_{(1,1)} \\ \left. \frac{dy}{dx} \right|_{(1,1)} &= -\frac{14}{10} = -\frac{7}{5} \end{aligned}$$

$$\frac{y-1}{x-1} = -\frac{7}{5}$$

$$7x + 5y = 12$$

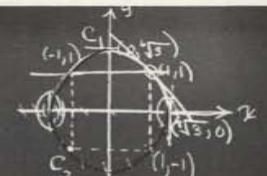
$$\begin{aligned} \frac{dy}{dx} &= 0 \iff x^5(8x^2 + 6y^4) = 0 \\ &\iff x = 0 \end{aligned}$$

$$\begin{aligned} \frac{dx}{dy} &= 0 \iff y^3(4x^6 + 6y^2) = 0 \\ &\iff y = 0 \end{aligned}$$

$$x = 0 \rightarrow y = \pm \sqrt[8]{3}$$

$$y = 0 \rightarrow x = \pm \sqrt[8]{3}$$

$$(x^8 + x^6 y^4 + y^6 \equiv 3)$$



$$\begin{aligned} C_1: \quad &x^8 + x^6 y^4 + y^6 = 3 \quad y \geq 0 \\ C_2: \quad &x^8 + x^6 y^4 + y^6 = 3 \quad y < 0 \end{aligned}$$

### Related Rates

A particle moves

$$\text{along the curve } x^2 + y^2 = 25$$

( $x, y$  in feet). At  $(3, 4)$ ,  $\frac{dx}{dt} = 8 \text{ ft/sec}$

Find  $\frac{dy}{dt}$ .

$$\begin{aligned} \frac{dy}{dt} &= -\frac{3}{4}(8) \\ &= -6 \text{ ft/sec} \end{aligned}$$

Assume "near"  $(3, 4)$   
 $x$  and  $y$  are differentiable  
 functions of  $t$ . Then:

$$\begin{aligned} x^2 + y^2 &\equiv 25 \rightarrow \\ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &\equiv 0 \\ \frac{dy}{dt} &= -\frac{x}{y} \frac{dx}{dt} \end{aligned}$$

2.050 Continuity

22 min.

Continuity

Does  $\lim_{x \rightarrow a} f(x) = f(a)$  ?

(1)  $f(a)$  must be defined. E.g.

$$\text{Let } f(x) = \frac{x^2 - 1}{x - 1}$$

Then:

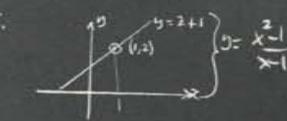
$$\lim_{x \rightarrow 1} f(x) = 2$$

But  $f(1) = \frac{0}{0}$  which

is undefined.

Pictorially:

$$\frac{x^2 - 1}{x - 1} = x + 1 \quad \text{except when } x = 1$$



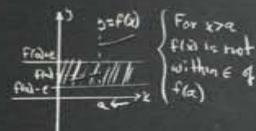
$$g(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

$$\therefore g(x) = x + 1$$

(2)  $f(x)$  is "near"  $f(a)$

when  $x$  is "near"  $a$ .

I.e., curve  $y = f(x)$  is "unbroken" in a neighborhood of  $x = a$ .



Definition:

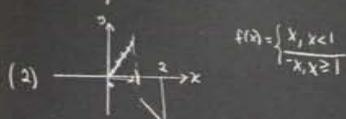
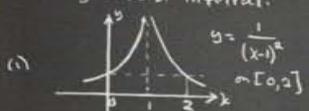
(1)  $f$  is called continuous at  $x = a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$

(2)  $f$  is called continuous on the interval  $I$  if  $\lim_{x \rightarrow a} f(x) = f(a)$  for each  $a \in I$ .



### Geometric Ideas

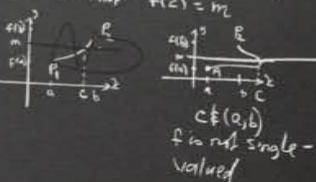
(1) Cont. functions assume their max. and min. values on any closed interval.



### (2) Intermediate Value Theorem

f cont. on  $[a, b]$ ,  $f(a) < f(b)$   
Let  $m$  be such that  
 $f(a) < m < f(b)$

Then we can find  $c \in (a, b)$  such that  $f(c) = m$



$f \text{ is not single-valued}$

### Analytic Ideas

(1)  $f, g$  cont. at  $x=a$ .

$$h(x) = f(x) + g(x) \quad \lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} [f(x) + g(x)]$$

$$= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \\ = f(a) + g(a)$$

$\therefore$  Sum of two continuous functions is a continuous function.

### (2) Differentiable $\rightarrow$ Continuous

$$f(x) - f(a) = \left[ \frac{f(x) - f(a)}{x-a} \right] (x-a)$$

$$\lim_{x \rightarrow a} [f(x) - f(a)] = \lim_{x \rightarrow a} \left[ \frac{f(x) - f(a)}{x-a} \right] \lim_{x \rightarrow a} (x-a) \\ = f'(a) \cdot 0 = 0$$

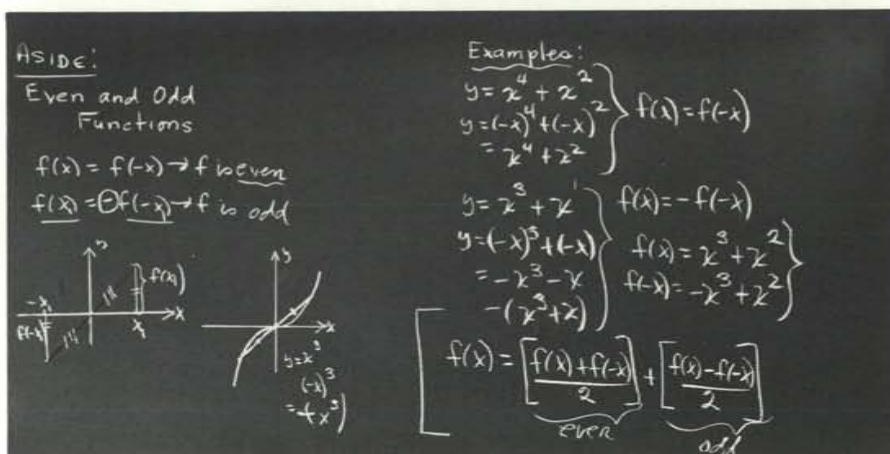
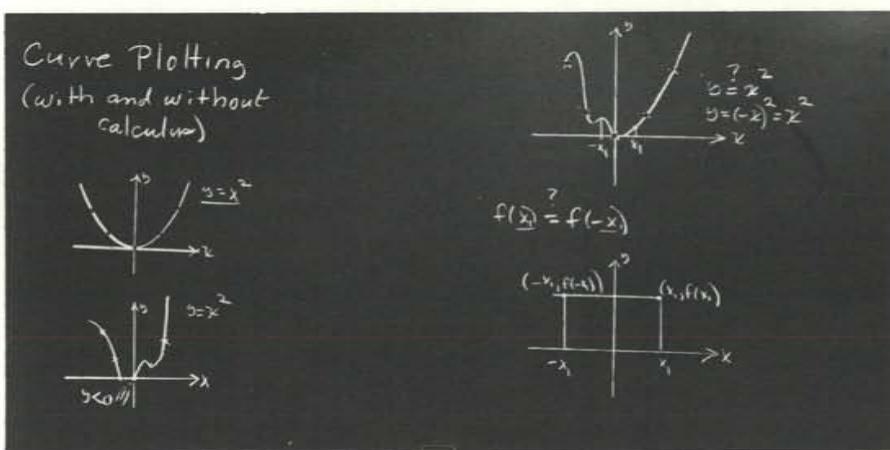
$$\therefore \lim_{x \rightarrow a} f(x) = f(a)$$

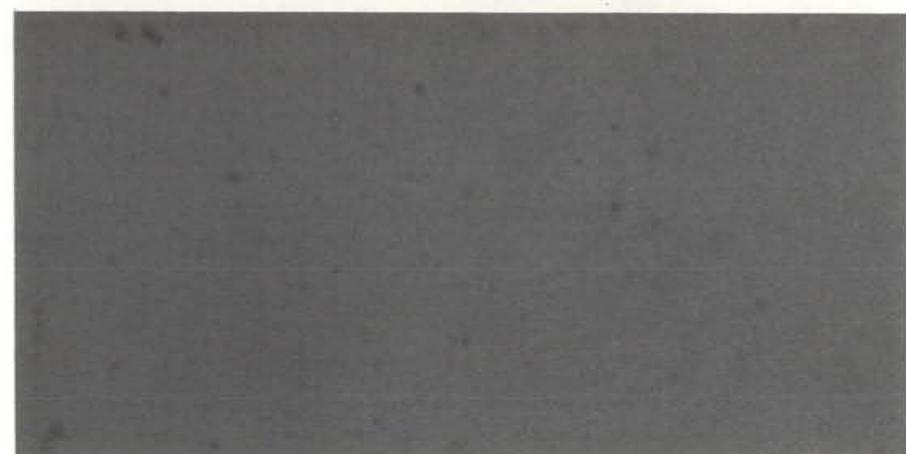
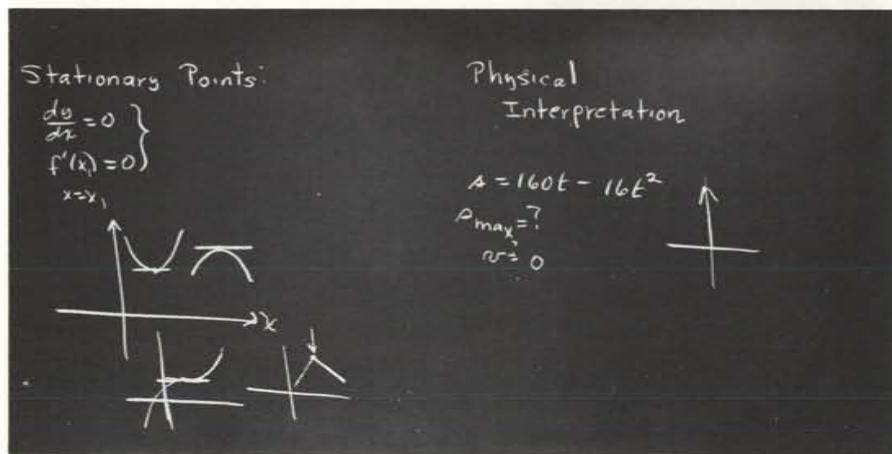
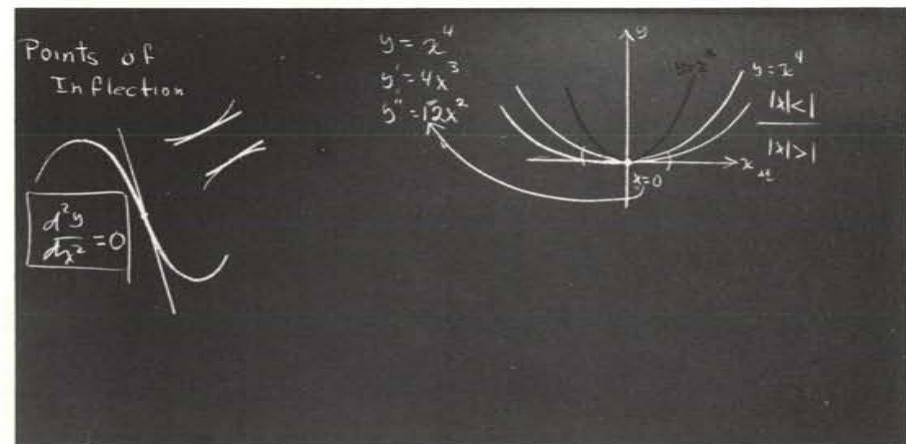
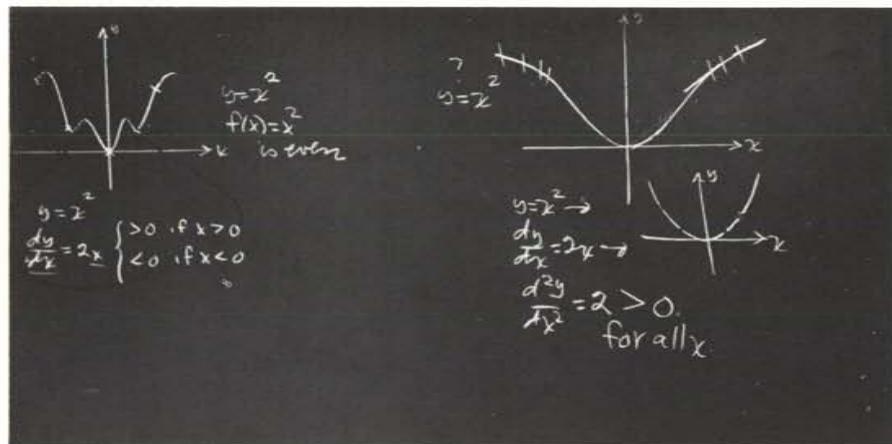
Pictorially:

$\left. \begin{matrix} \text{Smooth} \rightarrow \text{Unbroken} \\ \text{Unbroken} \rightarrow \text{Smooth} \end{matrix} \right\}$	
$\left. \begin{matrix} \text{Unbroken} \\ \text{but not smooth} \end{matrix} \right\}$	

2.060 Curve Plotting

31 min.





2.070 Maxima-Minima

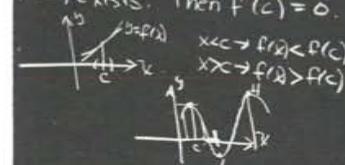
34 min.

Maxima-Minima

High Points-Low Points)

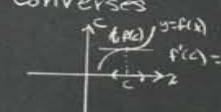
Fundamental Theorem:

Suppose  $f(c) \geq f(x)$   
for all  $x \in N_\delta(c)$  and suppose  
 $f'(c)$  exists. Then  $f'(c) = 0$ .

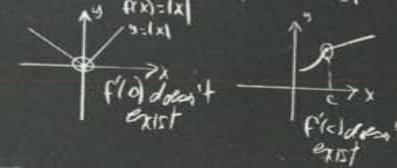


Cautions:

(1) Beware of false converses:



(2) Beware if  $f'(c)$  doesn't exist



(3) Beware of Endpoints

$$f(x) = x^2 \quad \text{dom } f = [2, 3]$$

$$2 \leq x \leq 3$$

$$f'(x) = 0 \quad \text{No!}$$

f' non-existent? No!

If  $f$  has a max or min  
it does not occur in  $(a, b)$

$$x \in N_\delta(c) \quad c - \delta < c < c + \delta$$





### SUMMARY:

Suppose  $f$  is continuous on  $[a, b]$ .  
Then to find candidates  $c$  such that  
 $f(c)$  is a minimum (maximum):

- (1) Find all  $c$  such that  
 $f'(c) = 0$
- (2) Find all  $c$  for which  
 $f'(c)$  fails to exist
- (3) Check  $x=a$  and  $x=b$

$$\begin{aligned} V &= 30\pi x^2 - \pi x^3 \\ 0 < x < 30 \\ \frac{dV}{dx} &= 60\pi x - 3\pi x^2 \\ \frac{d^2V}{dx^2} &= 60\pi - 6\pi x \\ \frac{dV}{dx} = 0 \iff &x = 0, 20 \\ \frac{d^2V}{dx^2} = 0 \iff &x = 10 \quad V = V(x) \\ &(20, 4000\pi) \quad 0 < x < 30 \end{aligned}$$

$\left\{ \begin{array}{l} V = (\pi x^2)y; \quad \begin{cases} x, y > 0 \\ xy = 30 \end{cases} \\ \frac{dV}{dx} = 2\pi xy + \pi x^2 \frac{dy}{dx}; \quad 1 + \frac{dy}{dx} = 0 \\ = 2\pi xy - \pi x^2 \quad (x \neq 0) \quad \frac{dy}{dx} = -1 \\ = 0 \iff x = 2y \\ \therefore x = 20 \quad y = 10 \end{array} \right.$

$$\begin{aligned} V &= \pi x^2 y \\ &= \pi x^2 (30-x) \\ &= 30\pi x^2 - \pi x^3, \quad 0 < x < 30 \\ V'(x) &= 60\pi x - 3\pi x^2 \\ &= 0 \iff x=0, x=20 \\ &\therefore y=10 \end{aligned}$$

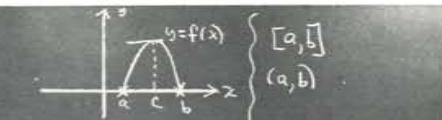
$$\begin{aligned} A_1 &= 400 & A_2 &= 450 \\ V_1 &= \pi(20)^2/10 & V_2 &= \pi(15)^2/15 \\ &= 4000\pi & &= 3375\pi \\ V &= \pi(\cancel{x})y \end{aligned}$$

2.080 Rolle's Theorem and its Consequences

30 min.

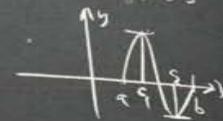
Rolle's Theorem  
and its  
Consequences

Let  $f$  be defined  
and continuous on  $[a, b]$   
and differentiable in  $(a, b)$ .  
Suppose also  $f(a) = f(b) = 0$ .  
Then  $f'(c) = 0$  for at least  
one  $c \in (a, b)$



Cautions:

- (1) There may be  
several  $c$ 's

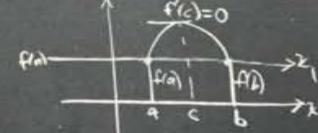


(2) (Aside)

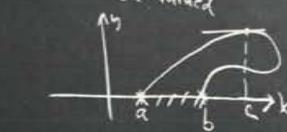
$f(a) = f(b) = 0$  is

too restrictive

$f(a) = f(b)$



- (3)  $f$  must be  
single-valued



### The Mean Value Theorem

Let  $f$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Then there exists a number  $c \in \varepsilon(a, b)$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



$$\frac{f(b) - f(a)}{b - a} = f'(c) \\ c \in \varepsilon(a, b)$$

(2)  $\underbrace{f'(x) = g'(x)}$   $\rightarrow$   $\underbrace{(a \leq x \leq b)}$

$f(x) - g(x)$  is constant



$$F(x) = f(x) - g(x)$$

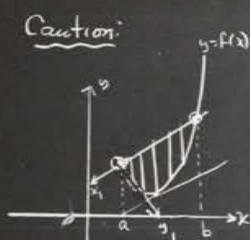
$$F'(x) = f'(x) - g'(x)$$

$$\equiv 0$$

$$F(x) = C$$

$$f(x) - g(x) = C$$

Caution:



Mean Value Theorem  
Supplies  
Rigor to Intuition

Example:

$$(1) F'(x) \equiv 0 \rightarrow F(x) = C \\ \rightarrow a \neq b \rightarrow F(a) = F(b)$$

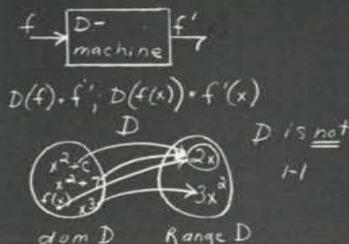
$$\boxed{\frac{F(b) - F(a)}{b - a} = F'(C)}$$

$$F'(a) - F'(b) = 0 \\ F'(a) = F'(b)$$

2.090 Inverse Differentiation

42 min.

Inverse Differentiation



$$\{x^2+c\} \xrightarrow[D^{-1}]{\text{machine}} 2x$$

$$\text{Suppose } D(f(x)) = 2x$$

$$\text{we know } D(x^2) = 2x$$

$$\therefore f(x) = x^2 \text{ constant}$$

$$\text{or: } f(x) \in \{x^2 + c\}$$

In general:

Suppose we are given  $f(x)$

$$\text{Let } E_f = \{f(x) + c\}$$

then

$$\begin{array}{c} f(x) \\ \xrightarrow[E_f]{\text{D-mach.}} f'(x) \end{array} \xrightarrow{\text{is 1-1}}$$

$$D^{-1}(f(x)) = \{G(x), G'(x) = f(x)\}$$

(Implicitly)

$$= \{F(x) + c, F'(x) = f(x)\}$$

(Explicitly)

$$\text{Example: Let } h(x) = x\sqrt{x^2+1}$$

$$h'(x) = \frac{d}{dx} x(x^2+1)^{\frac{1}{2}} = (2x) + (x^2+1)^{\frac{1}{2}}$$

$$= \frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}$$

$$= \frac{2x^2+1}{\sqrt{x^2+1}}$$

$$\therefore D^{-1}\left(\frac{2x^2+1}{\sqrt{x^2+1}}\right) = \{x\sqrt{x^2+1} + c\}$$

But even if we didn't  
know this, we would still  
have:

$$D^{-1}\left(\frac{2x^2+1}{\sqrt{x^2+1}}\right) = \left\{G(x), G'(x), \frac{2x^2+1}{\sqrt{x^2+1}}\right\}$$

Some "Recipos"

$$\text{D}^{-1}(x^n) = \frac{x^{n+1}}{n+1} + C$$

(since  $\text{D}\left(\frac{x^n}{n+1} + C\right) = x^n (n+1)\right)$

Example: Determine  $\text{D}^{-1}(x^7)$ .

$$\begin{cases} \text{D}(x^8) = x^7 \\ \frac{1}{8}\text{D}(x^8) = x^7 \\ \text{D}\left(\frac{1}{8}x^8\right) = \frac{1}{8}\text{D}(x^8) \\ \therefore \text{D}\left(\frac{1}{8}x^8\right) = x^7 \end{cases} \quad \begin{cases} \text{D}(cf) = \\ \text{D}(c\text{D}(f)) = \\ \text{D}\left(\frac{1}{8}(x^8+C)\right) = \frac{1}{8}\text{D}(x^8+C) \end{cases}$$

$$\text{D}^{-1}(x^7) = \left\{ \frac{1}{8}x^8 + C \right\}$$

Beware of non-constant factors

$$\text{D}'[(x^2+1)^7] = ?$$

$$\begin{aligned} \text{D}\left[(x^2+1)^7\right] &= 7(x^2+1)^6(2x) \\ &= 14x(x^2+1)^6 \\ &\text{D}\left[\frac{1}{16x}\text{D}\left[(x^2+1)^7\right]\right] = (x^2+1)^7 \\ &\text{D}\left[\frac{1}{16x}(14x(x^2+1)^6)\right] \neq (x^2+1)^7 \\ &\text{D}\left[\frac{1}{16}(x^2+1)^7\right] \neq (x^2+1)^7 \end{aligned}$$

Traditional Notation

$$g(x) \xrightarrow{\text{D-machine}} \frac{1}{[g(x)]^2}$$

$$g'(x) = \frac{1}{g^2(x)}$$

$$g^2(x)g'(x) = 1$$

$$\begin{cases} 3g^2(x)g'(x) = 3 \\ [g^3(x)]' = 3 = [3x]' \\ g^3(x) = 3x + C \\ g(x) = \sqrt[3]{3x+C} \end{cases}$$

$$\begin{aligned} \text{D}^{-1}(f(x)) &= \boxed{\int f(x) dx} \\ (1) \int x^n dx &= \frac{x^{n+1}}{n+1} + C, n \neq -1 \\ (2) \int [f(x)+g(x)]dx &= \int f(x)dx + \int g(x)dx \\ (3) \int kf(x)dx &= k \int f(x)dx \end{aligned}$$

Inverse Chain Rule

$$\text{D}[f(u)] = f'(u)$$

$$\text{D}\left[\frac{1}{5}(x^5+1)^5\right] = (x^5+1)^4$$

$$\text{D}\left[\frac{1}{5}(x^5+1)^5\right] =$$

$$\text{D}\left[\frac{1}{5}(x^5+1)^5\right] \text{D}(x^5+1) =$$

$$(x^5+1)^4 5x^4$$

$$\text{D}^{-1}\left[\frac{1}{5}(x^5+1)^5\right] =$$

$$\frac{1}{5}(x^5+1)^4$$

$$(1) \text{D}^{-1}[f(x)+g(x)] =$$

$$\text{D}^{-1}(f(x)+\text{D}^{-1}(g(x)))$$

$$\text{D}^{-1}(x^5+x^3) = ?$$

$$\begin{cases} \text{D}\left(\frac{1}{5}x^5\right) = x^5 \\ \text{D}\left(\frac{1}{3}x^3\right) = x^3 \end{cases} \quad \begin{cases} \text{D}(f+g) = \\ \text{D}(f)+\text{D}(g) \end{cases}$$

$$\text{D}\left(\frac{1}{5}x^5\right) + \text{D}\left(\frac{1}{3}x^3\right) = x^5+x^3$$

$$\text{D}\left(\frac{1}{5}x^5+\frac{1}{3}x^3\right) = x^5+x^3$$

$$\text{D}^{-1}(x^5+x^3) = \left\{ \frac{1}{5}x^5 + \frac{1}{3}x^3 + C \right\}$$

Why  $\int f(x) dx$ ?

Why not  $\int_x f(x) dx$ ?

$f(x) \xrightarrow{\text{D-machine}} \text{finds } y dx$

 $g'(x) = \frac{1}{[g(x)]^2}$ 

Let  $y = g(x)$

 $\frac{dy}{dx} = \frac{1}{y^2}$ 
 $y^2 dy = dx$ 
 $3y^2 dy = 3dx$ 
 $y^3 = 3x + C, y = \sqrt[3]{3x+C}$

Example:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \begin{cases} g(x) = -\frac{x}{y} \\ g(3) = 4 \end{cases}$$

and when  $x=3, y=4$

 $y dy = -x dx$ 
 $2y dy = -2x dx$ 
 $y^2 = -x^2 + C$ 
 $[g^2(x)]' = [-x^2]'$ 
 $x^2 + y^2 = C$ 
 $\frac{9+16}{9+16} = C$ 
 $x^2 + y^2 = 25$ 
 $g(x) = \pm \sqrt{C-x^2}$

2.100 The "Definite" Indefinite Integral

29 min.

The "Definite" Indefinite Integral

$$D^{-1}(f(x)) = \int f(x) dx$$

$$= \{G(x) + C\}$$

$$+ \{F(x) + C' - f\}$$

$$f(x) = x^2$$

$$\left[ \frac{1}{3}x^3 \right]_0^1 = x^2$$

$$f(x) = \frac{1}{3}x^3 + C$$

$$\text{Suppose } f(x_0) = y_0$$

$$\therefore C = y_0 - \frac{1}{3}x_0^3$$

$$f(x) = \frac{1}{3}x^3 + y_0 - \frac{1}{3}x_0^3$$

$$\left\{ \begin{array}{l} \frac{dy}{dx} = x^2 \\ y = \frac{1}{3}x^3 + C \\ y_0 = \frac{1}{3}x_0^3 + C \\ y = \frac{1}{3}x^3 + y_0 - \frac{1}{3}x_0^3 \end{array} \right.$$

Generalization

$$\frac{dy}{dx} = f(x) \quad a \leq x \leq b$$

$$\text{Assume } G'(x) = f(x)$$

$$y(x) = G(x) + C$$

$$y(a) = G(a) + C$$

$$C = y(a) - G(a)$$

$$y(x) = G(x) + y(a) - G(a)$$

$$y(b) = G(b) + y(a) - G(a)$$

$$y(b) - y(a) = G(b) - G(a)$$

$$\Delta y \Big|_{x=a}^{x=b} = G(x) \Big|_{x=a}^{x=b}, \quad G' = f$$

Notice that if  $H' = f$ , then

$$\Delta y \Big|_{x=a}^{x=b} = H(b) - H(a)$$

For in this case

$$H' = G'$$

$$H(a) = G(a) + C,$$

$$H(b) = G(b) + C,$$

$$H(b) - H(a) =$$

$$\underline{\underline{G(b) - G(a)}}$$

Geometric Interpretation

Physical Interpretation

$$v = t^2 \quad 0 \leq t \leq 1$$

$$v = \frac{dx}{dt}$$

$$x = \frac{1}{3}t^3 + C$$

when  $t=0$ , let  $x=x_0 (=x(0))$

$$\therefore C = x_0$$

$$\frac{x(t)}{t=0} = \frac{\frac{1}{3}t^3 + x_0}{t=0} \quad \frac{x(t)}{t=1} = \frac{x_0 + \frac{1}{3}}{t=1} \rightarrow x$$

$$x(1) = \frac{1}{3} + x_0$$

$$x(0) = x_0$$

$$\Delta x \Big|_{t=0}^{t=1} = \frac{1}{3}$$

Generalization

$$v = f(t), \quad a \leq t \leq b$$

$$G' = f$$

$$\therefore x(a) = G(a) + C$$

$$x(a) = G(a) + C$$

$$\therefore x(t) = G(t) + x(a) - G(a)$$

$$x(b) = G(b) + \cancel{x(a)} - G(a)$$

$$x(b) - x(a) = G(b) - G(a)$$

$$\Delta x \Big|_{t=a}^{t=b} = G(t) \Big|_{t=a}^{t=b}$$

$$G' = f$$

Once we invent  
 $\int f(x) dx$  to denote  
 $\{ G(x); G' = f \}$

why not invent  
 $\int_a^b f(x) dx$  to denote  
 $G(b) - G(a)$  ?

$$G(b) = \int_a^b f(x) dx$$

$$G(a) = \int_a^b f(x) dx$$

$$G(b) - G(a) = \int_a^b f(x) dx - \int_a^b f(x) dx$$

Summary

Suppose

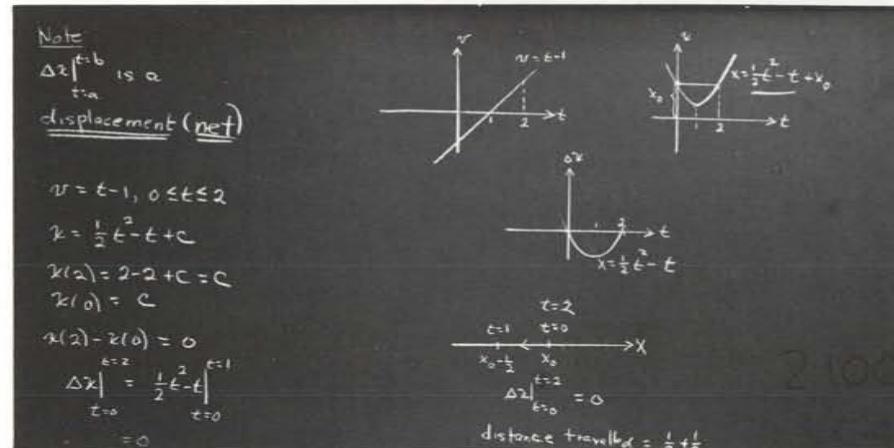
$$\frac{dy}{dx} = f(x), \quad a \leq x \leq b$$

Then  $\Delta y \Big|_{x=a}^{x=b} = \boxed{\int_a^b f(x) dx}$

$$= G(b) \Big|_{x=a}^{x=b}$$

$$= G(b) - G(a)$$

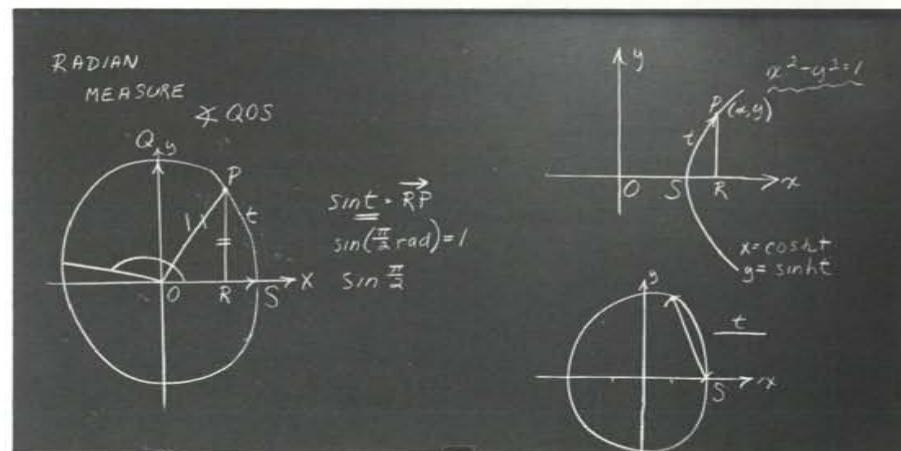
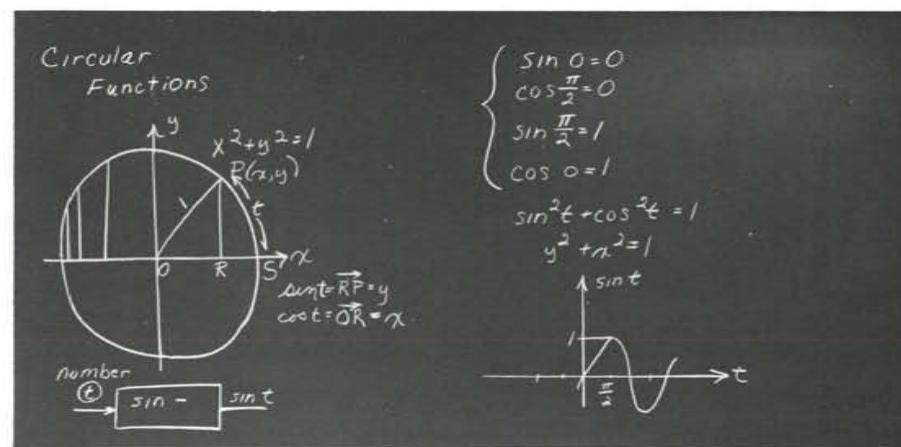
where  $G' = f$



## Block III: The Circular Functions

### 3.010 Circular Functions

35 min.



$$\begin{aligned}
 f(x) &= \sin x \\
 f'(x_0) &= \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \left[ \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \left[ \frac{\sin x_0 \cos \Delta x + \cos x_0 \sin \Delta x - \sin x_0}{\Delta x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \left[ \frac{\cos x_0 \left( \frac{\sin \Delta x}{\Delta x} \right) - \left( 1 - \cos x_0 \right) \sin \Delta x}{\Delta x} \right]
 \end{aligned}$$

$\lim_{t \rightarrow 0} \frac{\sin t}{t}$   
  
 $\frac{\sin t \cos t}{2} < \left(\frac{t}{2\pi}\right)\pi < \frac{\tan t}{2}$   
 $\cot t < \frac{t}{\sin t} < \frac{1}{\cot t}$   
 $\lim_{t \rightarrow 0} \frac{t}{\sin t} = 1$

$$\begin{aligned}
 x &= \sin kt \\
 \frac{dx}{dt} &= k \cos kt \\
 \frac{d^2x}{dt^2} &= -k^2 \sin kt \\
 \frac{d^2x}{t^2} &= -k^2 \frac{x}{t^2}
 \end{aligned}$$

$\int \cos u du = \sin u + C$   
 $\int \sin u du = -\cos u + C$   
 $\frac{d(\sin^n x)}{dx} = n \sin^{n-1} x \cos x$   
 $\int \sin^n x \cos x dx = \frac{1}{n+1} \sin^{n+1} x + C$

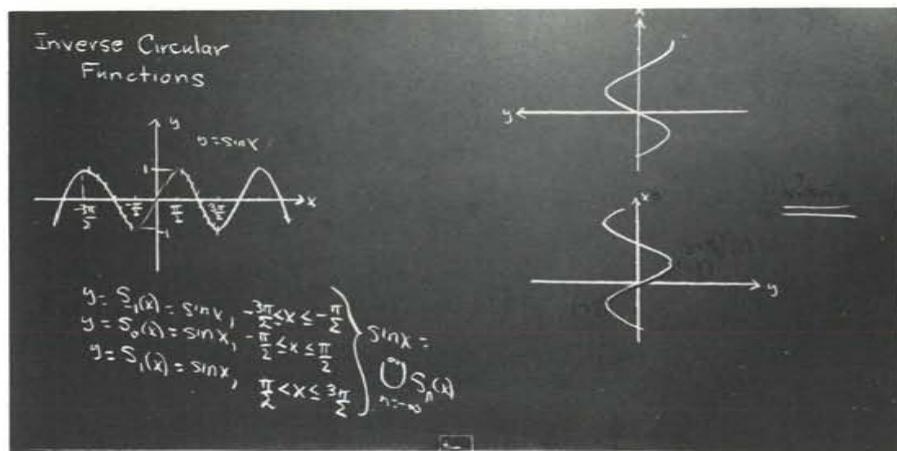
$$\begin{aligned}
 \dots \lim_{t \rightarrow 0} \left( \frac{\sin t}{t} \right) &= 1 \\
 \lim_{t \rightarrow 0} \frac{1 - \cos t}{t} &= \\
 \lim_{t \rightarrow 0} \frac{1 - \cos^2 t}{t(1 + \cos t)} &= \\
 \lim_{t \rightarrow 0} \left[ \left( \frac{\sin t}{t} \right) \left( \frac{0}{1 + \cos t} \right) \right] &= 1 \times 0 \\
 y = \arctan x \rightarrow \frac{dy}{dx} &= \cos x
 \end{aligned}$$

$y = \begin{cases} f(u) \\ \sin u \end{cases}$   
 $\frac{dy}{dx} = \frac{du}{dx} \frac{du}{dx}$   
 $= \cos x$   
 $\cos x = \sin \left( \frac{\pi}{2} - x \right)$   
 $\frac{d(\cos x)}{dx} = -\cos \left( \frac{\pi}{2} - x \right)$   
 $= -\sin x$

3010  
LC-4

3.020 Inverse Circular Functions

26 min.



$$\begin{aligned} y = \sin^{-1} x \text{ means } & \left. \begin{array}{l} x = \sin y, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\ y = \sin^{-1} x \text{ means } \end{array} \right\} \\ & y = S_o^{-1}(x) \\ & x = S_o(y) = \sin y, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\ \Rightarrow \frac{dx}{dy} &= \cos y \\ \frac{dy}{dx} &= \frac{1}{\cos y} \quad \left. \begin{array}{l} \frac{dy}{dx} = \frac{1}{\cos y} \\ y = \sin^{-1} x \\ x = \sin y, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\ x^2 = \sin^2 y \\ \cos^2 y = 1 - \sin^2 y \\ \cos y = \pm \sqrt{1 - x^2} \end{array} \right\} \end{aligned}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$


$$\sin \theta = x$$

$$\cos d\theta = dx$$

$$\sqrt{1-x^2} = \cos \theta$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos d\theta}{\cos \theta}$$

$$= \theta + C = \sin^{-1} x + C$$

$$y = \cos^{-1} x$$


$$\rightarrow y + (\frac{\pi}{2} - \theta) = \frac{\pi}{2}$$

$$\cos' x + \sin' x = \frac{\pi}{2}$$

$$\begin{aligned} \cos^{-1} x &\equiv \frac{\pi}{2} - \sin^{-1} x \\ \frac{d(\cos^{-1} x)}{dx} &= \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \\ &= -\frac{1}{\sqrt{1-x^2}} \end{aligned}$$

## Block IV: The Definite Integral

### 4.010 2-dimensional Area

36 min.

(2-dimensional) Area  
or  
"Calculus Revisited"  
Revisited

Area  
↓  
Integral Calculus | Diff'l Calculus  
Greece, 600 B.C. | 1680 A.D.  
Fixioms for Area

- (1)  $A = b h$ , for a rectangle
- (2)  $R \subset S$  implies  $A_R \leq A_S$
- (3)  $R = R_1 \cup \dots \cup R_n$  then

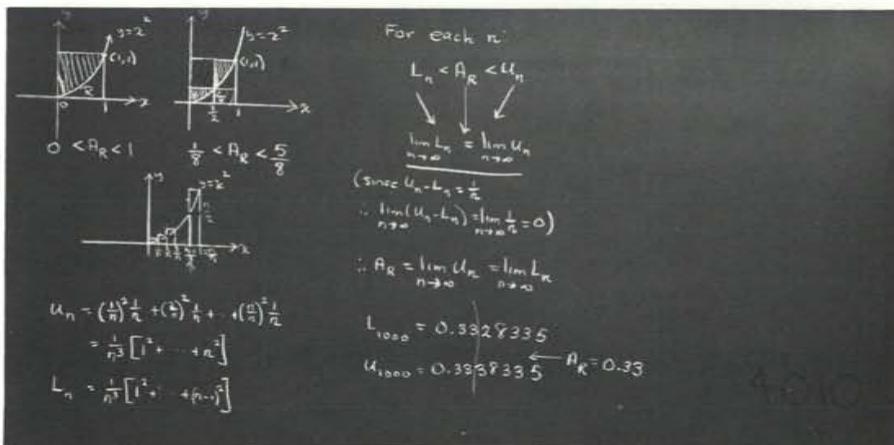
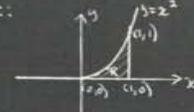
$$A_R = A_{R_1} + \dots + A_{R_n} = \sum_{k=1}^n A_{R_k}$$

#### Method of Exhaustion:

Given a region  $R$ , we "squeeze it" between two networks of rectangles.

#### Example:

We wish to determine the area,  $A_R$ , of the region  $R$  where:



$$\begin{aligned} U_n &= \frac{1}{n^2} \left[ \frac{n(n+1)}{6} (2n+1) \right] \\ &= \frac{1}{6} \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right) \\ &= \frac{1}{6} \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right) \\ &> \frac{1}{6} (1)(2) \end{aligned}$$

$L_n > \frac{1}{3} \leftarrow u_n$

$$\text{Similarly, } A_R$$

$$L_n = \frac{1}{6} \left( 1 - \frac{1}{n} \right) \left( 2 - \frac{1}{n} \right)$$

For each  $n$ ,  $L_n < \frac{1}{3}$ ,  $L_n < L_{n+1}$

$$\lim_{n \rightarrow \infty} L_n = \frac{1}{3}$$

$$\overrightarrow{L_n < \frac{1}{3} < L_{n+1}}$$

$$\lim_{n \rightarrow \infty} L_n = \frac{1}{3} \leftarrow l_n$$

### Generalization

Let  $f$  be continuous on  $[a, b]$  (and non-decreasing) and define  $R$  by



Partition  $[a, b]$  into  $n$  equal parts

$$a = x_0 < x_1 < \dots < x_n = b, \Delta x = b - a$$

and define

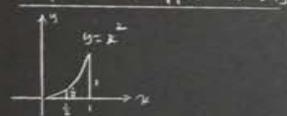
$$\begin{aligned} U_n &= f(x_k) \Delta x + \dots + f(x_n) \Delta x = \sum_{k=1}^n f(x_k) \Delta x \\ L_n &= f(x_0) \Delta x + \dots + f(x_{n-1}) \Delta x = \sum_{k=0}^{n-1} f(x_k) \Delta x \end{aligned}$$

$$\text{Then: (1) } \lim_{n \rightarrow \infty} U_n = \lim_{n \rightarrow \infty} L_n$$

$$\left( \text{since } U_n - L_n = [f(x_n) - f(x_0)] \Delta x \right)$$

$$(2) L_n < A_R < U_n \quad \left( f(x_n) - f(x_0) \right) \frac{1}{\Delta x} \quad \left\{ \begin{array}{l} A_R = \lim_{n \rightarrow \infty} U_n \\ = \lim_{n \rightarrow \infty} L_n \end{array} \right.$$

### Trapezoidal Approximations



$$T_3 = \frac{1}{6} + \frac{1}{2} \left( 1 + \frac{1}{3} \right) \frac{1}{3} = \frac{1}{6} + \frac{5}{18} = \frac{3}{8} = 0.375$$



$$\begin{aligned} T_4 &= \frac{1}{8} \left[ \frac{1}{16} + \frac{1}{16} \cdot \frac{1}{4} + \frac{1}{4} + \frac{9}{16} + \frac{9}{16} + 1 \right] \\ &= \frac{1}{128} \left[ 1 + 4 + 4 + 9 + 9 + 16 \right] \\ &= 0.344 \end{aligned}$$

### Definition

$f$  is called piecewise continuous on  $[a, b] \iff f$  is continuous except at a finite number of points where it has "jump" discontinuities



### Aside #1

Let  $c_k \in [x_{k-1}, x_k]$

$$\text{Form: } \sum_{k=1}^n f(c_k) \Delta x$$

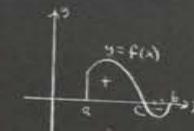
$$\text{Then: } L_n < \sum_{k=1}^n f(c_k) \Delta x < U_n$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x = \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} U_n \\ \lim_{n \rightarrow \infty} L_n \end{array} \right\} = A_R$$



### Aside #2

We can remove the restriction that  $f$  be non-negative if we replace area by net area

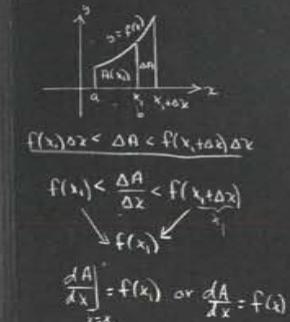


400

## 4.020 Marriage of Differential &amp; Integral Calculus

30 min.

The "Marriage" of  
Differential and Integral  
Calculus



First Fundamental  
Theorem of Integral Calc.

Suppose we know explicitly a function  $G$  such that  $G' = f$ . Then  $A(x) = G(x) + C$ .

Since  $A(a) = 0$ , we have

$$0 = A(a) = G(a) + C \Rightarrow C = -G(a)$$

Letting  $x = b$

$$A_R = A(b) = G(b) - G(a); G' = f$$

$$\text{But } A_R = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

That is, we can compute  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$  by use of "inverse derivatives."

Example:



$$A_R = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sin\left(\frac{k\pi}{n}\right) \frac{\pi}{n} (-1) \\ = G\left(\frac{\pi}{2}\right) - G(0), \text{ where } G'(x) = \sin x \\ = -\cos x \Big|_0^{\frac{\pi}{2}} = 0 - (-1) = 1$$

Aside

We earlier used  $\int_a^b f(x) dx$  to denote  $G(b) - G(a)$  where  $G' = f$ . Hence, we have shown that

$$A_R = \int_a^b f(x) dx = G(b) - G(a), G' = f$$

Historically,  $\int_a^b f(x) dx$  was invented to denote  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$ . Then by the First Fundamental Theorem,

$$\int_a^b f(x) dx = G(b) - G(a), G' = f$$

Second Fundamental Thm. of Integral Calc

Suppose we want  $A_R$  where

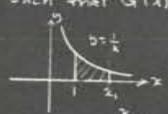


$$A_R = \int_1^2 \frac{dx}{x} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{1}{c_i}\right) \Delta x$$

$$= G(2) - G(1) \text{ where } G'(x) = \frac{1}{x}$$

But I don't know (explicitly) such a  $G$ !

In fact, because we can pinpoint  $A_R$ , we can "construct"  $G$  such that  $G'(x) = \frac{1}{x}$ . Namely,



$$G(x_i) = A(x_i) = \int_1^{x_i} \frac{dx}{x} \rightarrow$$

$$G'(x_i) = A'(x_i) = \frac{1}{x_i}$$

(Aside:  $G(x) = \ln x \therefore A_R = \ln 2$   
and we may, therefore, compute  $\ln 2$  as an infinite sum)

In general

Let  $f$  be continuous on  $[a, b]$ . Define  $G$  by:

$$G(x_i) = \int_a^x f(x) dx, \quad x_i \in [a, b]$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

Then:

$$G'(x_i) = f(x_i)$$

Summary.

(1) First Fund. Thm. allows us to compute  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$

Provided we can find  $G$  such that  $G' = f$ . In this case:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x = G(b) - G(a)$$

(2) Second Fund. Thm. allows us, given  $f$ , to construct  $G$  such that  $G' = f$ . Namely,

$$G(x) = \int_a^x f(x) dx \quad (= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x)$$

4.030 3-dimensional Area (Volume)

42 min.

3-dimensional Area



$$(1) V_{\text{cyl}} = Ah$$

$$(2) R \subseteq S \rightarrow V_R \leq V_S$$

$$(3) R = \bigcup_{k=1}^n R_k$$

$$\therefore V_R = \sum_{k=1}^n V_{R_k}$$

$$\frac{r(x)}{\Delta x} \approx \frac{r}{\Delta x}$$

$$\pi \left( \frac{r(x)}{\Delta x} \right)^2 \frac{\Delta x}{\Delta x} = \frac{\pi r^2 h}{\Delta x}$$

$$U_n = \frac{\pi r^2 h}{\Delta x} \sum_{k=1}^n k^2$$

$$= \frac{\pi r^2 h}{\Delta x} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6} \pi r^2 h (1 + \frac{1}{n})(2 + \frac{1}{n})$$

$$\lim_{n \rightarrow \infty} U_n = \frac{1}{3} \pi r^2 h$$

Solids of Revolution

$$V = \Delta V_1 + \dots + \Delta V_n = \sum_{k=1}^n \Delta V_k$$

$$A(M_k) \Delta x \leq \Delta V_k \leq A(M_k) \Delta x_K$$

$$\sum_{k=1}^n A(M_k) \Delta x_K \leq V \leq \sum_{k=1}^n A(M_k) \Delta x_K$$

$$V + \lim_{n \rightarrow \infty} U_n = \int_a^b f(x) dx$$

$$\frac{\Delta x_2 \rightarrow 0}{\Delta x_K}$$

$$A(m) \Delta x \leq \Delta V \leq A(M) \Delta x$$

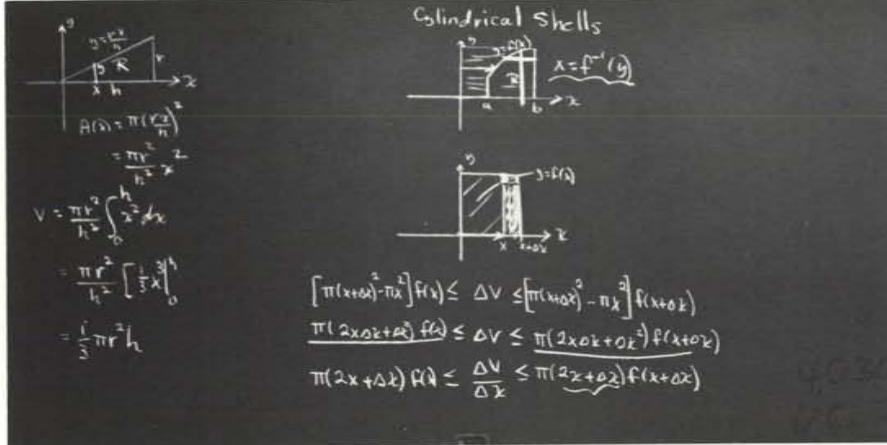
$$A(m) \leq \frac{\Delta V}{\Delta x} \leq A(M) \quad \frac{dV}{dx} = A(x)$$

$$\downarrow \quad \downarrow$$

$$A(x) \quad V = \int_a^b A(x) dx = G(x)|_a^b$$

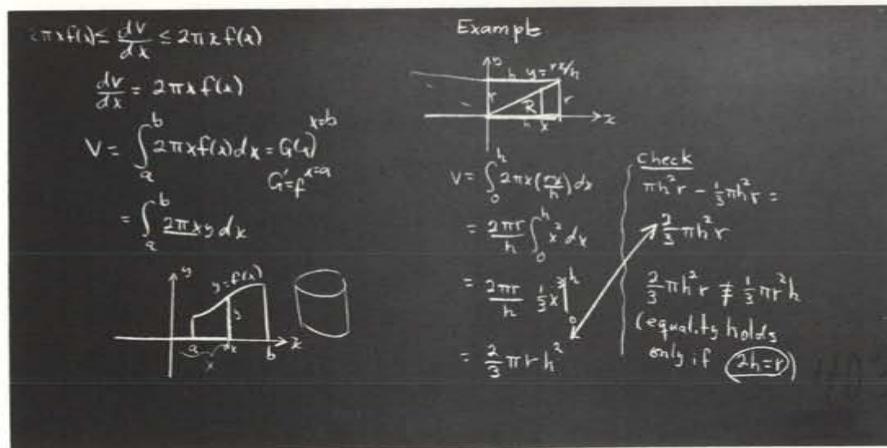
$$A(x) = \pi y^2 = \pi f^2(x)$$

$$V_R = \int_a^b \pi f^2(x) dy$$



$y = 2z - z^2, 0 \leq z \leq 2$   
 $x = \frac{2 \pm \sqrt{4-4y}}{2}$   
 $= \frac{1 \pm \sqrt{1-y}}{1-\sqrt{1-y}}$

$A(z) = \pi(1+\sqrt{1-y})^2 - \pi(1-\sqrt{1-y})^2$   
 $= 4\pi \sqrt{1-y}$   
 $V = 4\pi \int_0^1 (1-y)^{\frac{1}{2}} dy$   
 $= 4\pi \left[ -\frac{2}{3}(1-y)^{\frac{3}{2}} \right]_0^1$   
 $= \left( \frac{8\pi}{3} \right)$



$f \text{ cont on } [a, b]$   
 $\alpha = x_0 < x_1 < \dots < x_n = b$   
 $Q = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k$   
 $c_k \in [x_{k-1}, x_k]$   
 $\Delta x_k = x_k - x_{k-1}$   
 $Q = \int_a^b f(x) dx$   
 $= G(b) - G(a), G' = f$

## 4.040 1-dimensional Area (Arc Length)

36 min.

1-dimensional Area  
(arc length)Axiom #1

We can measure the length of any straight line segment.

Axiom #2

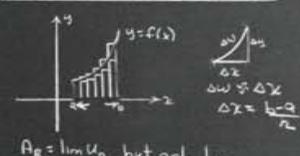
The length of the whole equals the sum of the lengths of the parts

However, it need not be true that  $R \subset S \rightarrow L_R \leq L_S$ .

For example:

Intuitive Approach

Lay off a "string" along  $\widehat{AB}$ . Then "straighten" the string and measure its length with a ruler.

Analytical Approach: Trial #1

$A_R = \lim_{n \rightarrow \infty} U_n$ , but only because

$$\left\{ \begin{array}{l} L_n < A_R < U_n \\ \text{and} \\ \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} U_n \end{array} \right.$$

However, if we let  $\Delta w \neq \Delta x$  and define  $f_a^b = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta x = \lim_{n \rightarrow \infty} \sum_{k=1}^n (b-a)$ , then  $f_a^b = b-a$  not  $w$ .

Analytical Approach: Trial #2

Now let  $\Delta w \neq \Delta x$ , and define  $L_a^b$  by

$$L_a^b = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta x$$

$$= \lim_{n \rightarrow \infty} \sqrt{\sum_{k=1}^n (\Delta x)^2}$$

"The" three questions

- (1) Does the limit,  $L_a^b$  exist?
- (2) If so, how do we compute it?
- (3) How does  $L_a^b$  compare with our intuitive ideas about arc length?

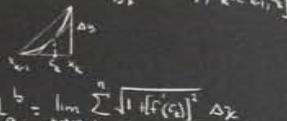
Answer to Question (2):

$$L_a^b = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{\Delta x + \alpha_k^2}$$

$$\sqrt{\Delta x + \alpha_k^2} = \sqrt{1 + \left(\frac{\Delta x}{\Delta x}\right)^2} \Delta x$$

If  $f'$  is differentiable on  $[a, b]$  we may invoke MVT to conclude

$$\frac{\Delta y}{\Delta x} = f'(c_k), c_k \in [x_{k-1}, x_k]$$



$$\therefore L_a^b = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x$$

Now if  $f'$  is continuous, so also are  $[f'(a)]^2$ ,  $[f'(b)]^2$  and  $\sqrt{1 + [f'(x)]^2}$ . Thus, in this case

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

(which may be hard to evaluate)

### Summary

If  $f$  is differentiable on  $[a, b]$  and  $f'$  is continuous on  $[a, b]$  then

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

In other words, we have let  $\Delta w_k$  and assumed that  $w = \sum_{k=1}^n \Delta w_k \approx \sum_{k=1}^n \Delta x$   
what we have shown is that  
 $\lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta x$  ( $= L_a^b$ ) exists. We have not shown that this limit is  $w$ !

In essence, how do we know if all the error has been squeezed out? This is precisely what Question (3) is all about.

### Generalization of Question (3)

Suppose  $w$  is any function defined on  $[a, b]$  and we assume that  $\Delta w \leq g(c_k) \Delta x$  where  $g$  is some "intuitive" function defined on  $[a, b]$

$$\text{Then: } w = \sum_{k=1}^n \Delta w_k \leq \sum_{k=1}^n g(c_k) \Delta x$$

and if  $g$  is continuous on  $[a, b]$ ,  $\lim_{n \rightarrow \infty} \sum_{k=1}^n g(c_k) \Delta x$  exists and is denoted by  $\int_a^b g(x) dx$ .

The question is:

$$\text{Does } w = \int_a^b g(x) dx?$$

Suppose  $\Delta w_k = g(c_k) \Delta x + \epsilon_k \Delta x$

$$\text{Then: } w = \sum_{k=1}^n \Delta w_k$$

$$= \sum_{k=1}^n g(c_k) \Delta x + \sum_{k=1}^n \epsilon_k \Delta x$$

$$\therefore w - \sum_{k=1}^n g(c_k) \Delta x = \sum_{k=1}^n \epsilon_k \Delta x$$

$$w - \int_a^b g(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \epsilon_k \Delta x \neq 0$$

Situation #1:  $\epsilon_k = C \neq 0$  for all  $k$

$$\text{Then } \lim_{n \rightarrow \infty} \sum_{k=1}^n \epsilon_k \Delta x = \lim_{n \rightarrow \infty} \sum_{k=1}^n C \Delta x$$

$$= \lim_{n \rightarrow \infty} C \left( \sum_{k=1}^n \Delta x \right)$$

$$= C(b-a) \neq 0$$

Situation #2:  $\epsilon_k = B \Delta x$

$$\text{Then } \sum_{k=1}^n \epsilon_k \Delta x = \sum_{k=1}^n B \Delta x^2$$

$$= \sum_{k=1}^n B \left[ \frac{(b-a)}{n} \right]^2$$

$$= n \left\{ B \left( \frac{b-a}{n} \right)^2 \right\}$$

$$\Rightarrow B(b-a)^2$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \epsilon_k \Delta x = 0$$

$$w = \int_a^b g(x) dx$$

In this situation

$$\Delta w \leq \Delta w \leq \overline{AB} + \overline{BC}$$

$$\overline{AB} = \sqrt{1 + [f'(x_{k-1})]^2} \Delta x$$

$$\overline{BC} = \epsilon \Delta x, \lim_{\Delta x \rightarrow 0} \epsilon = 0$$

$$\Delta w = \sqrt{1 + [f'(x_k)]^2} \Delta x$$

$$\geq \sqrt{1 + [f'(x_{k-1})]^2} \Delta x$$

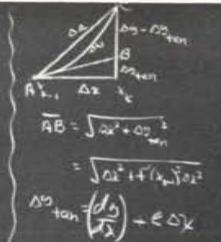
$$\sqrt{1 + [f'(x_{k-1})]^2} \Delta x \leq \Delta w \leq \sqrt{1 + [f'(x_k)]^2} \Delta x$$

$$+ \epsilon \Delta x$$

$$\therefore \Delta w = \sqrt{1 + [f'(x_{k-1})]^2} \Delta x + \epsilon \Delta x$$

$$\text{where } \epsilon \ll \Delta x$$

$$\lim_{\Delta x \rightarrow 0} \epsilon = 0$$



## Block V: Transcendental Functions

### 5.010 Logarithms without Exponents

34 min.

Logarithms Without Exponents

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$\underbrace{n+1}_{n=-1}$

$$\frac{dm}{dt} = km$$

$$\frac{dm}{m} = k dt$$

$$\int \frac{dm}{m} = kt + C$$

Differential Calculus Approach

$$y = L(x)$$

$$y' = \frac{1}{x}$$

$$y'' = -\frac{1}{x^2}$$

Integral Calculus Approach

$$y = L(x) = \int \frac{1}{x} dx$$

Integral Calculus Approach

Pick  $a > 0$  and once picked "fix" it.

Define  $L(x)$  by

$$L(x) = \int_a^x \frac{dt}{t}$$

$$\left[ \int_a^x f(t) dt \right]_a^x = f(x)$$

$y = \frac{1}{t}$

$y = L(x)$

### Logarithmic Functions

$f$  is called logarithmic if  $f(x_1x_2) = f(x_1) + f(x_2)$  for all  $x_1, x_2$  in dom  $f$ .

If  $f$  is logarithmic, then (1)  $f(1) = 0$ , since  $f(1) = f(x_1) = f(1) + f(1)$

(2) If  $f(\frac{1}{x})$  is defined then  $f(\frac{1}{x}) = -f(x)$ , since  $0 = f(1) = f(x \cdot \frac{1}{x}) = f(x) + f(\frac{1}{x})$

$$0 = f(x) + f(\frac{1}{x})$$

$$= f(x) + f(\frac{1}{x})$$

$$(3) f\left(\frac{x}{y}\right) = f(x) - f(y), \text{ since}$$

$$\underline{f\left(\frac{x}{y}\right)} = f\left(x \cdot \frac{1}{y}\right) = f(x) + f\left(\frac{1}{y}\right) \\ = f(x) - f(y)$$

$$(4) f(x^n) = nf(x) \text{ for any positive integer } n.$$

For:  $f(x^n) = f(\underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}})$

$$= \underbrace{f(x) + \dots + f(x)}_{n \text{ times}} = nf(x)$$

How is this related to  $L(x)$ ?

### Example

$$y = uv$$

$$\ln y = \ln(uv)$$

$$= \ln u + \ln v$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{u}{y} \frac{du}{dx} + \frac{v}{y} \frac{dv}{dx}$$

$$= uv \frac{du}{dx} + uv \frac{dv}{dx}$$

With "traditional" logarithms, if the base is  $b$  then

$$\log_b b = 1$$

Thus if  $e$  is to be the "base" for  $\ln x$ , then

$$\ln e = 1$$

Let's compare

$L(bx)$  with  $L(b) + L(x)$

$$\frac{dL(bx)}{dx} = \frac{dL(bx)}{d(bx)} \frac{d(bx)}{db} \\ = \frac{1}{bx} \cdot b = \frac{1}{x}$$

$$\frac{d(L(b) + L(x))}{dx} = \frac{1}{x} \quad ?=0$$

$$\therefore L(bx) = L(b) + L(x) + C$$

IF  $2 = 1$ :

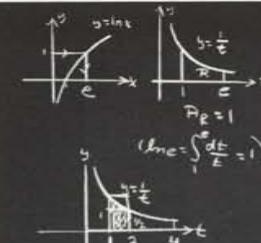
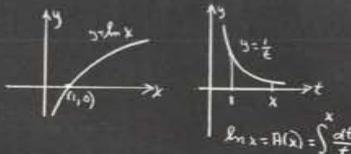
$$L(b) = L(b) + L(1) + C$$

$$\therefore C = -L(1) \quad \therefore L(1) = 0$$

$\therefore L(x)$  is logarithmic  $\leftrightarrow L(1) = 0$

Define  $\ln x$  to be this member of the family  $L(x)$  s.t.

That is:  $\frac{d(\ln x)}{dx} = \frac{1}{x}$  and  $\ln(1) = 0$



$$\frac{1}{x} < \ln x < 1$$

$$\ln e = 1$$

$$\ln 4 = \ln(2^2) = 2\ln 2 > 1$$

$$\therefore \ln 2 < \ln e < \ln 4 \quad \frac{d\ln x}{dx} = \frac{1}{x} > 0$$

$$2 < e < 4$$

$$\checkmark \frac{dm}{dt} = km$$

$$\frac{dm}{m} = k dt$$

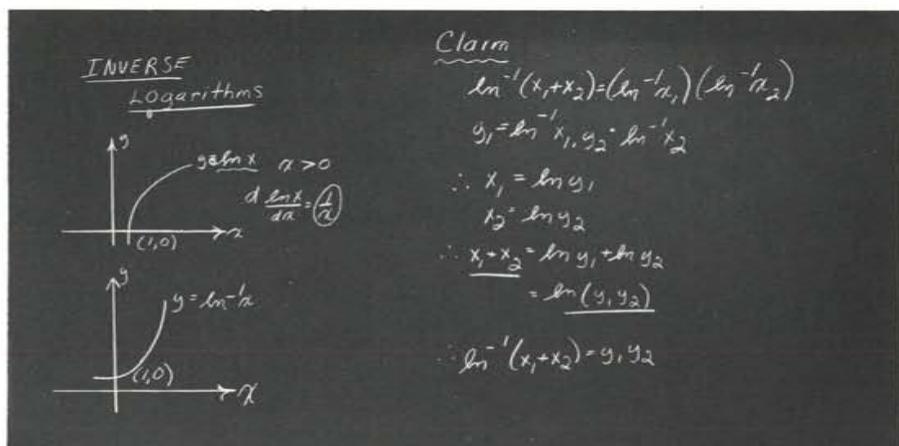
$$\int \frac{dm}{m} = kt + C$$

$$\boxed{\ln m = kt + C}$$

$$m = (\ln -)(kt + C)$$

## 5.020 Inverse Logarithms

21 min.



Find  $\frac{dy}{dx}$  if  $y = \ln^{-1}x$

$$y = \ln^{-1}x \rightarrow x = e^y$$

$$\frac{dx}{dy} = \frac{1}{y} \cdot \boxed{\frac{dy}{dx} = y}$$

$$\therefore \frac{d(\ln^{-1}x)}{dx} = \ln^{-1}x$$

$$\int \frac{dy}{y} \cdot \int dx$$

↑

Notation

$\ln^{-1}x$  is usually abbreviated by  $(e^x)$

This matches the identification of  $\ln x$  with  $\log_e x$

$$\therefore \frac{d(e^x)}{dx} = e^x$$

$$\frac{d e^u}{dx} = \frac{d e^u}{du} \cdot \frac{du}{dx}$$

$$= e^u \frac{du}{dx}$$

$$\int e^{-x^2} dx = ?$$

$$\int 2x e^{-x^2} dx = ?$$

$$u = -x^2$$

$$du = -2x dx$$

$$\int 2x e^{-x^2} dx =$$

$$\frac{\int -e^u du = -e^u + C = -e^{-x^2} + C}{\int e^{-x^2} d(-x^2) = e^{-x^2} + C}$$

$$\int -2x e^{-x^2} dx = e^{-x^2} + C$$

5020  
M-3

$$y'' - 5y' + 6y = 0$$

$$\text{Try } y_T = e^{rx}$$

$$y'_T = r e^{rx}$$

$$y''_T = r^2 e^{rx}$$

$$r^2 e^{rx} - 5r e^{rx} + 6e^{rx} = 0$$

$$e^{rx}(r^2 - 5r + 6) = 0$$

$$\therefore r = 2 \text{ or } r = 3$$

$$y = e^{2x}$$

$$y' = 2e^{2x}$$

$$y'' = 4e^{2x}$$

$$y'' - 5y' + 6y = 0$$

$$y = e^{3x}$$

$$y' = 3e^{3x}$$

$$y'' = 9e^{3x}$$

$$y'' - 5y' + 6y = 0$$

In general if  $a$  and  $b$  are constants  
 transforms into  $y'' + ay' + by = 0$

5020  
M-3

5.030 What a Difference a Sign Makes

27 min.

What a Difference a Sign Makes

$\left\{ \begin{array}{l} x^2 + y^2 = 1 \\ \text{Circular Functions} \end{array} \right.$

$\left\{ \begin{array}{l} x^2 - y^2 = 1 \\ \text{Hyperbolic Functions} \end{array} \right.$

**ASIDE**

(1)  $x^2 - y^2 = 1$  means  $x^2 + (iy)^2 = 1$  where  $i^2 = -1$

(2) Given  $a$  and  $b$  let  $x = \frac{a+b}{2}$ ,  $y = \frac{a-b}{2}$ , then  $x+y = a$ ,  $x-y = b$

$$\begin{aligned} D(e^t) &= e^t & C(t) &= e^{2t} + e^{-2t} \\ D(e^{-t}) &= -e^{-t} & S(t) &= e^{2t} - e^{-2t} \\ D(e^{t+i\theta}) &= e^t e^{i\theta} & C^2(t) - S^2(t) &= 4 \\ D(e^{t-i\theta}) &= e^t e^{-i\theta} & \left[ \frac{C(t)}{2} \right]^2 - \left[ \frac{S(t)}{2} \right]^2 &= 1 \\ \text{Let } \begin{cases} C(t) = e^t + e^{-t} \\ S(t) = e^t - e^{-t} \end{cases} & & \cosh t &= \frac{C(t)}{2} = \frac{e^t + e^{-t}}{2} \\ \therefore C'(t) &= S(t) & \sinh t &= \frac{S(t)}{2} = \frac{e^t - e^{-t}}{2} \\ S'(t) &= C(t) & x &= \cosh t \\ y &= \sinh t & \rightarrow & x^2 - y^2 = 1, x > 0 \end{aligned}$$

$$y = \tanh x$$

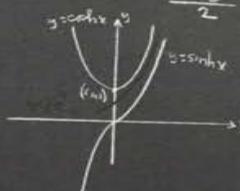
$$= \frac{\sinh x}{\cosh x}$$

$$\frac{d(\tanh x)}{dx} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\int \operatorname{sech}^2 x dx = \tanh x + C$$

$$\begin{cases} y = \cosh x = \frac{e^x + e^{-x}}{2} \\ y' = \sinh x = \frac{e^x - e^{-x}}{2} \\ y'' = \cosh x = \frac{e^x + e^{-x}}{2} \end{cases}$$



$$x = \sinh kt$$

$$\frac{dx}{dt} = k \cosh(kt)$$

$$\frac{d^2x}{dt^2} = k^2 \sinh kt$$

$$= k^2 x$$

Hyperbolic Functions  
are a solution to

$$\frac{d^2x}{dt^2} = k^2 x$$

Circular Functions  
solve  $\frac{d^2x}{dt^2} = -k^2 x$

5.040 Inverse Hyperbolic Functions

30 min.

Inverse Hyperbolic Functions

$y = \sinh x$

$y = \sinh^{-1} x$

$$\begin{cases} y = \sinh^{-1} x \\ x = \sinh y \\ \frac{dx}{dy} = \cosh y \\ \frac{dy}{dx} = \frac{1}{\cosh y} \end{cases}$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$\cosh y = \pm \sqrt{1 + \sinh^2 y}$$

$$= \sqrt{1+x^2}$$

$$\therefore \frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$$

$$A(t) = \boxed{\int_0^t \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} t}$$

$$\tan \theta = x \quad \rightarrow \quad \sinh \theta = x$$

$$\sec^2 \theta d\theta = dx \quad \cosh \theta d\theta = dx$$

$$\sqrt{1+x^2} = \sec \theta \quad \sqrt{1+\sinh^2 \theta} = \sqrt{1+\cosh^2 \theta}$$

$$\therefore \int \frac{dx}{\sqrt{1+x^2}} = \int \sec \theta d\theta \quad = \cosh \theta$$

$$x = \tan \theta \quad \therefore \int \frac{dx}{\sqrt{1+x^2}} = \int d\theta$$

$$1+x^2 = 1+\tan^2 \theta \quad = \theta + C$$

$$= \sec^2 \theta \quad = \sinh^{-1} x + C$$

$$\left\{ \begin{array}{l} \sec^2 \theta - \tan^2 \theta = 1 \\ \cosh^2 \theta - \sinh^2 \theta = 1 \end{array} \right.$$

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$x = \sinh y$$

$$= \frac{e^y - e^{-y}}{2}$$

$$\therefore 2x = e^y - \frac{1}{e^y}$$

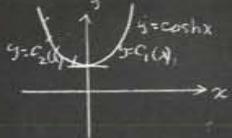
$$e^{2y} - 2xe^y - 1 = 0$$

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

$$e^{2y} = 2x \pm \sqrt{4x^2 + 4}$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

$$y = \ln(x + \sqrt{x^2 + 1})$$



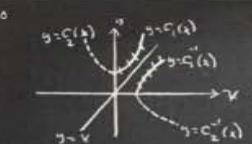
Define

$$C_1(x) = \cosh x, x \geq 0$$

$$C_2(x) = \cosh x, x \leq 0$$

Then  $C_1$  and  $C_2$  are each 1-1.

$\therefore C_1^{-1}, C_2^{-1}$  exist



$$\cosh^{-1} x = C_1^{-1}(x)$$

That is.

$$y = \cosh^{-1} x \text{ means}$$

$$x = \cosh y$$

$$\text{and } y \geq 0$$

$$(\text{dom } \cosh^{-1} x \text{ is } x \geq 1)$$

$$\text{Find } \frac{dy}{dx} \text{ if } y = \cosh^{-1} x$$

$$x = \cosh y, y \geq 0$$

$$\frac{dx}{dy} = \sinh y, y \geq 0$$

$$\frac{dy}{dx} = \frac{1}{\sinh y}$$

$$\cosh^2 y - \sinh^2 y \geq 1$$

$$\sinh y = \pm \sqrt{x^2 - 1}$$

$$y \geq 0 \rightarrow \sinh y \geq 0$$

$$\therefore \frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

$$\left\{ \begin{array}{l} \frac{d C_1^{-1}(x)}{dx} = \frac{1}{\sqrt{x^2 - 1}} \\ \frac{d C_2^{-1}(x)}{dx} = -\frac{1}{\sqrt{x^2 - 1}} \end{array} \right.$$

## Block VI: More Integration Techniques

### 6.010 Some Basic Recipes

30 min.

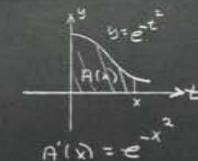
#### Some Basic Recipes

Pick a differentiable function  $G$ ; say  $G' = f$

$$\text{Then: } \int f(x) dx = G(x) + C$$

Given  $f$  the required  $G$  may not exist in "familiar" form

$$\text{For example: } f(x) = e^{-x^2}$$



Objective of this Block is to find "recipes" for finding  $G(x)$  for various types of  $f(x)$

$$\begin{aligned} \int u^n du &= \begin{cases} \frac{u^{n+1}}{n+1} + C, & n \neq -1 \\ \ln|u| + C, & n = -1 \end{cases} \\ \int (\sin x)^n d(\sin x) &= \begin{cases} \frac{1}{n+1} \sin^{n+1} x + C, & n \neq -1 \\ \ln|\sin x| + C, & n = -1 \end{cases} \\ \therefore \int \sin^n x \cos x dx &= \begin{cases} \frac{1}{n+1} \sin^{n+1} x + C, & n \neq -1 \\ \ln|\sin x| + C, & n = -1 \end{cases} \end{aligned}$$

$$\begin{aligned} \int \cos^n x dx &= \int \cos^6 x \cos x dx = \\ &\rightarrow \int (1 - \sin^2 x)^3 \cos x dx \\ \int \cos^6 \theta d\theta &= \int (\cos^2 \theta)^3 d\theta = \\ &\rightarrow \int \left(\frac{1 + \cos 2\theta}{2}\right)^3 d\theta = \\ &\frac{1}{8} \int (1 + 3\cos 2\theta + 3\cos^2 2\theta + \cos^3 2\theta) d\theta \end{aligned}$$

Sums and Differences  
of Squares

$$\int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$a \sin \theta = x, a \cos \theta d\theta = dx$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos \theta d\theta}{a \cos \theta} = \int d\theta$$

$$= \theta + C$$

$$= \sin \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$a \csc \theta = x, a \cot \theta d\theta = dx$$

$$\sqrt{x^2 - a^2} = a \csc \theta$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \cot \theta d\theta}{a \csc \theta}$$

$$= \int -\csc \theta d\theta$$

$$\left. \begin{array}{l} a \csc \theta = x \\ d\theta = a \sinh \theta d\theta \end{array} \right\} \int \frac{dx}{\sqrt{x^2 - a^2}} = \theta + C$$

$$\sqrt{x^2 - a^2} = \sqrt{a^2 (\cosh^2 \theta - 1)}$$

$$= a \sinh \theta$$

Completing the Square

$$ax^2 + bx + c =$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) =$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) =$$

$$a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

$$\int \frac{dx}{ax^2 + bx + c} =$$

$$\int \frac{dx}{a(x + \frac{b}{2a})^2 + c_1} =$$

$$\frac{1}{a} \int \frac{dx}{(x + \frac{b}{2a})^2 + c_1} =$$

$$\frac{1}{a} \int \frac{du}{(u^2 + c_1)} = \frac{1}{a} \int \frac{du}{u^2 + k^2}$$

$$\left. \begin{array}{l} \text{Let } u = x + \frac{b}{2a} \\ du = dx \end{array} \right\}$$

## 6.020 Partial Fractions

32 min.

Partial Fractions

Technique applies  
to  $\int \frac{P(x)}{Q(x)} dx$  where P and Q  
are polynomials in x ( $\deg P < \deg Q$ )

We can't factor  
 $x^2 + 1$  without using non-real numbers

$$\begin{aligned} x^2 + 1 &= x^2 - i^2 = (x+i)(x-i) \\ x^2 + 1 &= (x^2 + 1)^2 - 2x^2 \\ &= (x^2 + 1 + \sqrt{2}x)(x^2 + 1 - \sqrt{2}x) \end{aligned}$$

(1) We can handle linear  
and quadratic denominators

(2) Theoretically, every real  
polynomial can be factored  
into linear and quadratic terms

$$\begin{aligned} \int \frac{dx}{(x-1)(x^2+1)} &=? \\ \frac{1}{(x-1)(x^2+1)} &\equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \\ 1 &\equiv Ax^2 + A + Bx^2 - Bx + Cx - C \\ A+B &= 0 \\ C-B &= 0 \\ A-C &= 1 \\ \therefore A &= \frac{1}{2}, C = -\frac{1}{2} \\ B &= \frac{-1}{2} \end{aligned}$$

$$\begin{aligned} \frac{1}{(x-1)(x^2+1)} &\equiv \frac{1}{2} \left[ \frac{1}{x-1} - \frac{x+1}{x^2+1} \right] \\ &\equiv \frac{1}{2} \left( \frac{1}{x-1} \right) - \frac{1}{2} \left( \frac{x}{x^2+1} \right) - \frac{1}{2} \left( \frac{1}{x^2+1} \right) \\ \int \frac{dx}{(x-1)(x^2+1)} &= \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{x dx}{x^2+1} - \frac{1}{2} \int \frac{dx}{x^2+1} \\ &= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln(x^2+1) \\ &\quad - \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

### Polynomial Identities

$$a_2x^2 + a_1x + a_0 = b_2x^2 + b_1x + b_0$$

(1) Let  $x=0 \therefore a_0 = b_0$

$$\therefore x(a_2x+a_1) = x(b_2x+b_1)$$

$$\therefore a_2x+a_1 \equiv b_2x+b_1$$

(2) Let  $x=0 \therefore a_1=b_1$ , etc

$$\therefore 2a_2x+a_1 \equiv 2b_2x+b_1$$

$$2a_2 \equiv 2b_2$$

### Beware!

In general

$$a_1u_1 + a_2u_2 = b_1u_1 + b_2u_2$$

does not imply that

$$a_1=b_1 \text{ and } a_2=b_2$$

### Example

$$f(x) + g\left(\frac{x}{2}\right) \equiv f(x) + h\left(\frac{x}{2}\right)$$

But  $f \neq g$

$h \neq f$

"It has been discovered  
that..."

$$z = \tan \frac{x}{2}$$

$$\begin{aligned} dz &= \frac{1}{2} \sec^2 \frac{x}{2} dx \\ &= \frac{1+z^2}{2} dx \end{aligned}$$

$$\begin{aligned} \sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} \\ &= 2 \frac{z}{\sqrt{1+z^2}} \frac{1}{\sqrt{1+z^2}} \end{aligned}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1}{1+z^2} - \frac{z^2}{1+z^2} = \frac{1-z^2}{1+z^2}$$

$$\int \sec x dx =$$

$$\int \frac{dx}{\cos x} =$$

$$\int \frac{2dz}{1+z^2} \left( \frac{1+z^2}{1-z^2} \right) =$$

$$2 \int \frac{dz}{1-z^2}$$

$$\sin y = \frac{2z}{1+z^2} \leftarrow$$

## 6.030 Integration by Parts

26 min.

### Integration by Parts

$$d(uv) = u dv + v du$$

$$u dv = d(uv) - v du$$

$$\int u dv = uv - \int v du \quad (+C)$$

### Example:

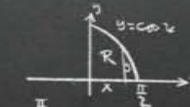
$$\text{Let } u = x, \frac{du}{dx} = 1, \quad v = \sin x, \frac{dv}{dx} = \cos x dx$$

$$\int u dv = \int x \cos x dx \quad <$$

$$\int v du = \int \sin x dx \quad <$$

$$\begin{aligned} \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C \end{aligned}$$

$$(\text{check: } x \cos x + \sin x - \sin x = x \cos x)$$



$$\begin{aligned} V_3 &= 2\pi \int_0^{\frac{\pi}{2}} x \cos x dx \\ &= 2\pi [G(\frac{\pi}{2}) - G(0)], G'(x) = \underline{x \cos x} \\ &= 2\pi (x \sin x + \cos x) \Big|_0^{\frac{\pi}{2}} = 2\pi (\frac{\pi}{2} - 1) \end{aligned}$$

$$\begin{aligned} \int \frac{x \cos x}{x} dx &\quad \left\{ u = x, dv = \cos x dx \right. \\ &\quad \left. du = dx, v = \sin x + C \right\} \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C \end{aligned}$$

$$\begin{aligned} \int \frac{x \cos x}{x} dx &\quad \left\{ u = \cos x, dv = x dx \right. \\ &\quad \left. du = -\sin x dx, v = \frac{1}{2}x^2 \right\} \\ &= \frac{1}{2}x^2 \cos x + \frac{1}{2} \int x^2 \sin x dx \end{aligned}$$

$$\left( \int x^2 \sin x dx = 2 \int x \cos x dx - x^2 \cos x \right)$$

$$\begin{aligned} \int x^2 \sin x dx &\quad \left\{ u = x^2, dv = \sin x dx \right. \\ &\quad \left. du = 2x dx, v = -\cos x \right\} \\ &= -x^2 \cos x + 2 \int x \cos x dx \end{aligned}$$

$$\int x \cos x = x \sin x + \cos x + C$$

$$\therefore \int x^2 \sin x dx = \boxed{-x^2 \cos x + 2x \sin x + 2 \cos x + C}$$

$$\int \tan^{-1} x \, dx$$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$u = \tan^{-1} x, dv = dx$$

$$du = \frac{dx}{1+x^2}, v = x$$

$$\int \tan^{-1} x \, dx =$$

$$x \tan^{-1} x - \underbrace{\int \frac{x \, dx}{1+x^2}}_{=}$$

$$x \tan^{-1} x - \underbrace{\frac{1}{2} \ln(1+x^2)}_{+C} + C$$

$$u = \tan^{-1} x \quad dv = dx$$

$$du = \frac{dx}{1+x^2} \quad v = x + C_1$$

$$\int \tan^{-1} x \, dx = (x + C_1) \tan^{-1} x$$

$$- \int \frac{(x+C_1) \, dx}{1+x^2}$$

$$= x \tan^{-1} x + C_1 \tan^{-1} x$$

$$- \int \frac{x \, dx}{1+x^2} - C_1 \tan^{-1} x$$

$$\int \ln x \, dx$$

$$u = \ln x, dv = dx$$

$$du = \frac{1}{x} dx, v = x + C$$

$$\int \ln x \, dx = x \ln x - \int dx$$

$$= (x \ln x - x) + C$$

$$A_R = \int_1^b \ln x \, dx = x \ln x - x \Big|_1^b$$

## 6.040 Improper Integrals

29 min.

Improper Integrals

Find the flaw!

$$\int_{-1}^1 \frac{dx}{x^2} = G(1) - G(-1)$$

$$G(x) = -\frac{1}{x}, G'(x) = \frac{1}{x^2}$$

$$G(1) = -1, G(-1) = 1$$

$$\therefore \int_{-1}^1 \frac{dx}{x^2} = -2$$

$$\left(\frac{1}{x^2} \geq 0 \therefore \int_{-1}^1 \frac{dx}{x^2} \geq 0\right)$$

Key Point

$$\int_a^b f(x) dx = G(b) - G(a)$$

$$G' = f$$

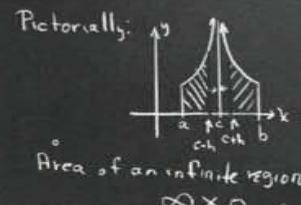
requires that  $f$  be  
(piecewise) continuous on  $[a, b]$   
 $\frac{1}{x^2} = \infty$  when  $x=0, 0 \in [-1, 1]$

Definition #1

$\int_a^b f(x) dx$  is called improper  
of the first kind  $\leftrightarrow f$  is  
infinite for at least one  $c \in [a, b]$

If  $c$  is the only point  
in  $[a, b]$  at which  $f$  is infinite,

we define  $\int_a^b f(x) dx =$   
 $\lim_{h \rightarrow 0^+} \left[ \int_a^{ch} f(x) dx + \int_{ch}^b f(x) dx \right]$



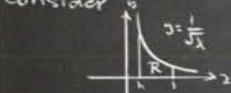
The question centers  
about whether  $h \rightarrow 0$   
"faster than"  $f(c+h) \rightarrow \infty$

In our example this  
didn't happen

$$A_R = \int_h^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_h^1 = \frac{1}{h} - 1$$

$$\lim_{h \rightarrow 0^+} A_R = \lim_{h \rightarrow 0^+} \frac{1}{h} - 1 = \infty - 1 = \infty$$

On the other hand,  
consider



$$A_R = \int_1^R \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_1^R = 2 - 2\sqrt{R}$$

$$\lim_{R \rightarrow \infty} A_R = 2 - 2\sqrt{\infty} = 2$$

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_0^1 = 2$$

### Definition #2

If  $f$  is infinite at  $x=c$ ,  
 $c \in [a, b]$  then the improper  
integral  $\int_a^b f(x) dx$  is convergent  
if:

$$\lim_{h \rightarrow 0^+} \left[ \int_a^{c+h} f(x) dx + \int_{c+h}^b f(x) dx \right] \text{ exists}$$

Otherwise it is called  
divergent

6040  
163

### Computational Aside

We do not have to be  
able to compute  $\int_a^\infty f(x) dx$   
to determine its convergence

For example, consider

$$\int_1^\infty e^{-x^2} dx. \quad \text{For } x \ge 1, x^2 \ge 1$$

$$\therefore x^2 \ge x$$

$$\therefore \int_1^\infty e^{-x^2} dx \le \int_1^\infty e^{-x} dx$$

$$= \left[ -e^{-x} \right]_1^\infty = e^{-1}$$

$$\lim_{b \rightarrow \infty} V_b = \pi$$

Infinite Area but finite volume!



$$A_R = \int_1^R \frac{dx}{\sqrt{x}} = \ln R$$

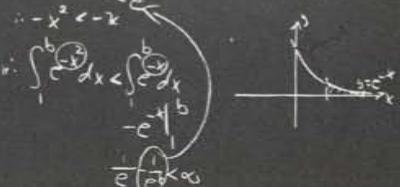
$$\lim_{R \rightarrow \infty} A_R = \infty = \int_1^\infty \frac{dx}{\sqrt{x}}$$

$$V_b = \pi \int_1^b \left( \frac{1}{x} \right)^2 dx$$

$$= \pi \left( -\frac{1}{x} \right)_1^b$$

$$= \pi \left( 1 - \frac{1}{b} \right)$$

$$\lim_{b \rightarrow \infty} V_b = \pi$$



### Examples:

(1) Note that  $\frac{1}{x^2}$  and  $\frac{1}{\sqrt{x}}$   
( $x \ge 0$ ) are inverses



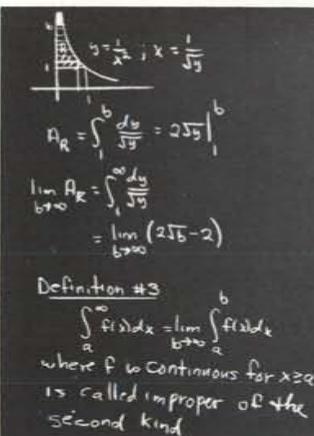
$$\begin{aligned} \int_1^\infty \frac{dx}{x^2} &= \lim_{b \rightarrow \infty} \left[ \int_1^b \frac{dx}{x^2} \right] \\ &= \lim_{b \rightarrow \infty} \left( -\frac{1}{x} \Big|_1^b \right) \\ &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{b} + 1 \right] \\ &= 1 \end{aligned}$$

### Summary:

The limits of  
 $\int_a^\infty$ ,  $\int_{-\infty}^b$ ,  $\int_{-\infty}^\infty$

give us warning to  
beware.

However always  
examine  $\int_a^b f(x) dx$   
for "infinities" of  $f$



$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

where  $f$  is continuous for  $x \ge a$   
is called improper of the  
second kind

## Block VII: Infinite Series

### 7.010 Many Versus Infinite

26 min.

#### Many Versus Infinite

$$\begin{aligned} N &= 10^{10} \\ &= 10,000,000,000 \end{aligned}$$

$$N+1, N+2, N+3, \dots$$

N is no nearer the end of the number system than n.

#### Additional Examples

- ① 1, 3, 2, 5, 7, 4, 9, 11, 6, ...  
 No matter where we stop (even at  $10^{10}$ ) there are twice as many "odds" as "evens"

$$\begin{aligned} ② \quad 1 + (-1) + 1 + (-1) + \dots \\ [1 + (-1)] + [1 + (-1)] + \dots = 0 \} \\ 1 + [(-1) + 1] + [(-1) + 1] + \dots = 1 \} \end{aligned}$$

Notice the need for order as well as the terms

Note:

Our intuition is defied because it doesn't apply!

why deal with infinite sums? Because we need them!

$$\rightarrow A_E = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

How shall we add infinitely many terms?

Consider  $q_1 = \frac{1}{2}, q_2 = \frac{1}{4}$

$$\begin{aligned} q_3 &= \frac{1}{8} \text{ and we want} \\ q_1 + q_2 + q_3 \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \end{aligned}$$

(Do not confuse the  $a$ 's and  $A$ 's. The sequence of numbers being added to  $a_1, a_2, a_3$  ( $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ ) while the partial sums in the sequence  $a_1, a_2, A_3$  ( $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}$ ). The sum is the number  $A_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ )

(Do not confuse the  $a_i$ 's and  $\frac{1}{a_i}$ . The sequence of numbers being added is  $a_1, a_2, a_3, \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$  while the partial sums are the sequence  $a_1, a_2, a_3, \left(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}\right)$ . The sum is the number  $a = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ )

Generalized, if  $a_n = \frac{1}{2^n}$   
then  $a_n = a_1 + \dots + a_n = \frac{1}{2} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$

For example

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{1024} = 1 - \frac{1}{1024}$$

So, how about

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots$$

$$\lim_{n \rightarrow \infty} a_n = 1 \quad \text{Define } \sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} \left( \frac{1}{2} + \frac{1}{4} + \dots \right) = 1$$

### Pictorially

$$-\frac{1}{2} \leq \frac{1}{\sqrt{1-e^{\frac{1}{2}x}}} \leq -\frac{1}{e^{\frac{1}{2}x}}$$

After "a while" all  $s_n$   
are within  $\epsilon$  of  $L$ .

### Basic Definitions

The infinite sequence  $b_1, b_2, b_3, \dots = \{b_n\}$  is said to converge to the limit L (written  $\lim_{n \rightarrow \infty} b_n = L$ )  $\Leftrightarrow \forall \epsilon > 0$ , we can find  $N(\epsilon)$  such that  $n > N \rightarrow |b_n - L| < \epsilon$

### Again Pictorially

$$\lim_{n \rightarrow \infty} a_n = L \quad \frac{Q(1)C_1^{\frac{1}{n}}}{a_n} \underset{n \rightarrow \infty}{\xrightarrow{\text{注意到}} 0} 0$$



limit: infinite sequence  
as  
last term: finite sequence

Notice same limit theorems as before apply.

### Example

$$\lim_{n \rightarrow \infty} \left( \frac{2n+3}{5n+7} \right) = \lim_{n \rightarrow \infty} \left[ \frac{2 + \frac{3}{n}}{5 + \frac{7}{n}} \right] = \frac{2}{5}$$

$$\sum_{n=1}^{\infty} \left( \frac{2n+3}{5n+7} \right) = \infty \quad \text{since}$$

"after a while" each term  
behaves like  $\frac{2}{x}$ . That is

$$\frac{5}{12} + \frac{5}{17} + \frac{9}{22} + \frac{11}{25} +$$

That is

$$\sum_{n=1}^{\infty} a_n \text{ converges} \rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

(For if  $\lim_{n \rightarrow \infty} a_n = L \neq 0$ , then

$$\sum_{n=k}^{\infty} a_n \text{ "behaves like" } \sum_{n=k}^{\infty} L$$

On the other hand  $\lim_{n \rightarrow \infty} a_n = 0$   
 is not enough to guarantee  
 the convergence of  $\sum a_n$

### Example

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \dots$$

## 7.020 Positive Series

34 min.

Positive SeriesOrdering:

$$\begin{aligned} S &= \{11, 8, 9, 7, 10\} \\ &= \{7, 8, 9, 10, 11\} \end{aligned}$$

$7 \leq x, 11 \geq x$  where  $x \in S$   
 $\therefore 7$  is a lower bound for  $S$   
 $11$  is an upper bound for  $S$

$$\begin{array}{c} \oplus \quad \oplus \quad \oplus \\ 7 \quad 8 \quad 9 \quad 10 \quad 11 \end{array}$$

$7$  is the greatest lower bound for  $S$   
 $11$  is the least upper bound for  $S$

These results are more subtle for infinite sets:

Examples:

(1) Let  $S = \{a_n : a_n = \frac{n}{n+1}\}$   
 $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$

$1$  is lub for  $S$ , but  $1 \notin S$

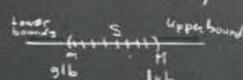
(2) Let  $S = \{a_n : a_n = \frac{1}{n}\}$   
 $S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$0$  is glb for  $S$ , but  $0 \notin S$

Formal Definitions:

(#1)  $M$  is called an upper bound for  $S \iff M \geq x$ , for all  $x \in S$

- (#2)  $M$  is called a least upper bound for  $S$  if (1)  $M$  is an upper bound for  $S$   
and (2)  $L \leq M \rightarrow L$  is not u.b. for  $S$   
( $M$  need not belong to  $S$ )



- (#3) A set is bounded if it has both an upper and a lower bound

Key Property:

Every bounded set has a glb and lub

- (#4) A sequence  $\{a_n\}$  is called monotonic non-decreasing if  $a_n \leq a_{n+1}$  for all  $n$   
(That is:  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq a_{n+1} \leq \dots$ )

For such sequences, two possibilities exist:

- (i)  $\{a_n\}$  has no upper bound  
In this case we write

$$\lim_{n \rightarrow \infty} a_n = \infty$$

(An example is:  
 $1, 2, 3, \dots, n, \dots$  where  $a_n = n$ )

(2)  $\{a_n\}$  has an upper bound

Then  $\limsup_{n \rightarrow \infty} a_n = L$  where

$$L = \text{lub}\{a_n\}$$

"Proof"

$$\begin{aligned} & \text{at least one } \\ & a_n \geq q_n \forall n \in \mathbb{N}, n \in (L, L+\epsilon) \\ & \text{Therefore } \\ & L \leq \text{lub}\{a_n\} \\ & \text{No } a_n \in (L, L+\epsilon) \\ & n > N \rightarrow a_n \geq q_N \\ & \therefore n \geq N \rightarrow L - \epsilon < a_n \leq q_N \leq L \\ & \rightarrow |a_n - L| < \epsilon \\ & \rightarrow \limsup_{n \rightarrow \infty} a_n = L \end{aligned}$$

### Positive Series

If  $a_n \geq 0$  for each  $n$ , then  $\sum a_n$  is called a positive series.

In this case the sequence of partial sums is monotonic non-decreasing.

Therefore

If  $\sum a_n$  is a positive series, it either diverges to  $\infty$  or it converges to the limit  $L$  where  $L$  is the lub for the sequence of partial sums.

### Comparison Test

Suppose  $\sum c_n$  is a convergent positive series and  $0 \leq u_n \leq c_n$  for each  $n$ .

Then  $\sum u_n$  also converges.

"Proof"

$$\begin{aligned} & \text{Let } T_n = c_1 + \dots + c_n \\ & S_n = u_1 + \dots + u_n \text{ for each } n \end{aligned}$$

$$\limsup_{n \rightarrow \infty} T_n = T = \text{lub}\{T_n\}$$

$$\therefore S_n \leq T \text{ for each } n$$

$$\therefore S_n \text{ is bounded (and monotone non-decreasing)} \therefore \lim_{n \rightarrow \infty} S_n \in \mathbb{R}$$

### Notes:

(1) The condition  $0 \leq u_n \leq c_n$  for all  $n$  can be weakened to  $0 \leq u_n \leq c_n$  for all  $n \geq N$  (i.e., for  $n$  "sufficiently large") since convergence depends on the "end" of the sequence

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \\ & \rightarrow 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots \end{aligned}$$

(2) If  $u_n \geq d_n$  where  $\sum d_n$  is a positive divergent series then  $\sum u_n$  also diverges (since its convergence would imply  $\sum d_n$  converges).

### Ratio Test

Given  $\sum a_n$ , form the sequence  $\left\{\frac{a_{n+1}}{a_n}\right\} = \{u_n\}$

(For example, given  $\sum \frac{10^n}{n!} a_n$  to  $\limsup_{n \rightarrow \infty} \frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!} = \frac{10}{n+1}$ )

Now, assuming  $\lim_{n \rightarrow \infty} u_n$  exists, call it  $p$ . Then

$\sum a_n$  converges if  $p < 1$   
diverges if  $p > 1$

If  $p = 1$ , test fails

Note: Even if  $u_n < 1$  for every  $n$ ,  $p$  may still equal 1.

For example,  $\lim_{n \rightarrow \infty} \frac{n}{n(n+1)} = 1$ , but  $\frac{n}{n+1} < 1$

### Geometric "Proof" for p<1

$$\frac{a_{n+1}}{a_n} < p$$

There exists  $N$  such that  $n > N \rightarrow \frac{a_{n+1}}{a_n} < r < 1$

$$\frac{a_{n+1}}{a_n} < r \therefore a_{n+1} < r a_n < r^2 a_n$$

$$\sum a_n < a_1 \left( \sum_{n=1}^N r^n \right) \text{ converges geometric series}$$

### Integral Test

#### Series and Improper Integrals

Suppose there is a decreasing continuous function  $f(x)$  such that  $f(n) = u_n$  is the  $n^{\text{th}}$  term of the positive series  $u_1 + u_2 + \dots + \dots$

Then:  $\sum u_n$  converges

if and only if  $\int_1^\infty f(x) dx$  converges

### "Proof"



$$u_1 + u_2 + \dots + u_n > \int_1^n f(x) dx$$

$$\int_1^\infty f(x) dx \text{ diverges} \rightarrow \sum u_n \text{ diverges}$$

$$u_1 + u_2 + \dots + u_n < \int_1^n f(x) dx$$

$$\begin{aligned} & \sum_{n=1}^\infty u_n < u_1 + u_2 + \dots + u_n < u_1 + \int_1^\infty f(x) dx \rightarrow \int_1^\infty f(x) dx \text{ converges} \\ & \therefore \int_1^\infty f(x) dx \text{ converges} \rightarrow \sum u_n \text{ converges} \end{aligned}$$

## 7.030 Absolute Convergence

21 min.

Absolute Convergence

Consider:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

we see,

- (1) Terms alternate in sign
- (2)  $n^{\text{th}}$  term  $\rightarrow 0$
- (3) Terms decrease in magnitude

Claim:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges}$$

Geometric Proof

(It turns out that  
 $L = \ln 2$   
but who'd have guessed that!)

So WHAT? . . .

Definition

1.  $\sum_{n=1}^{\infty} a_n$  is said to converge absolutely if  $\sum_{n=1}^{\infty} |a_n|$  converges

2. A series which converges but not absolutely is called conditionally convergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is conditionally convergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges, but

because +'s cancel -'s

not because the terms get small fast enough!

That is, if we replace each term by its magnitude, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges

### The Subtlety of Conditional Convergence

The sum of a cond. conv. series depends on the order of the terms

Example

Divide the terms of  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$  into 2 "teams" (sets)

$$\rightarrow P = \left\{ 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots \right\}$$

$$\rightarrow N = \left\{ -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \dots \right\}$$

Both  $\sum_{n=1}^{\infty} \frac{1}{2n-1}$  and  $\sum_{n=1}^{\infty} \frac{1}{2n}$  diverge to  $\infty$

Let us, for example, make the sum  $\frac{3}{2}$ . We start with  $P$  and write  $1 + \frac{1}{3} + \frac{1}{5} + \dots$  until the sum first exceeds (or equals)  $\frac{3}{2}$ . (This must happen since  $1 + \frac{1}{3} + \frac{1}{5} + \dots$  diverges to  $\infty$ )

In particular,

$$1 + \frac{1}{3} + \frac{1}{5} = \frac{23}{15} > \frac{3}{2}$$

We next "annex" members of  $N$  until the sum falls below  $\frac{3}{2}$ .

### Summary - So Far:

If  $\sum_{n=1}^{\infty} a_n$  is cond. conv., its

limit exists, but the limit changes as the order of the terms is changed. That is, rearranging the terms changes the series.

This is not true in "finite" arithmetic!

The moral:

DON'T "MONKEY" WITH CONDITIONAL CONVERGENCE

The beauty of absolute convergence is that the sum is the same for every rearrangement of the terms!

Details are left to the supplementary notes!

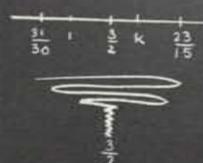
$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} = \frac{31}{30} < \frac{3}{2}$$

Continue with  $P$ ,

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} = \frac{247}{210} < \frac{3}{2}$$

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \dots + \frac{1}{13} = \frac{131,093}{90,090} < \frac{3}{2}$$

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \dots + \frac{1}{15} = \frac{2,056,485}{1,351,350} = k > \frac{3}{2}$$

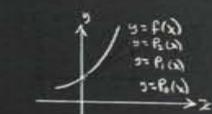


( $\frac{3}{2}$  was not important - though the arithmetic gets "messier" if we choose a larger number)

7.040 Polynomial Approximations

32 min.

Polynomial Approximations



$$P_0(x) = f(x)$$

$$P_1(x) = f(x) + f'(x)x \\ (y = mx + b)$$

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

$$P_2(0) = a_0 = f(0)$$

$$P_2'(0) = a_1 = f'(0)$$

$$P_2''(0) = 2a_2 = f''(0) \rightarrow a_2 = \frac{f''(0)}{2!}$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\text{If } P_n(x) = a_0 + a_1 x + \dots + a_n x^n,$$

$$P_n^{(n)}(x) = n! a_n$$

$$P_n^{(n)}(0) = n! a_n = f^{(n)}(0)$$

$$\therefore a_n = \frac{f^{(n)}(0)}{n!}$$

$$\therefore P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

$$= a_0 + a_1 x + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

and we let

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \text{ denote } \lim_{n \rightarrow \infty} P_n(x) = P(x)$$

Example:

$$\text{Let } f(x) = e^x$$

$$\text{Then } f^{(n)}(x) = e^x$$

$$\therefore f^{(n)}(0) = 1 \quad \therefore \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

$$\therefore P_n(x) = \sum_{k=0}^n \frac{x^k}{k!} \\ = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$P_0(x) = 1$$

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

We would like to be able to compare  $f(x)$  with  $\lim_{n \rightarrow \infty} P_n(x)$

This involves 3 questions:

① Does  $\lim_{n \rightarrow \infty} P_n(x)$  exist?  $P(x) = ?$

② If so, does it equal  $f(x)$ ?

③ Letting  $P(x) = \lim_{n \rightarrow \infty} P_n(x)$ , does  $P$  possess the "polynomial-properties" possessed by each  $P_n$ ?

### Counter-Examples

① Let  $f(x) = \frac{1}{1-x}$

$$f'(x) = (1-x)^{-2}$$

$$f''(x) = 2(1-x)^{-3}$$

$$f'''(x) = 3!(1-x)^{-4}$$

$$f^{(n)}(x) = n!(1-x)^{-n-1}$$

$$\frac{f^{(n)}(0)}{n!} = 1$$

$$\therefore f(x) = \frac{1}{1-x} \rightarrow P_n(x) = 1+x+\dots+x^n$$

Now:

$$f(2) = -1$$

$$P(2) = \lim_{n \rightarrow \infty} P_n(2) = \infty$$

② Let  $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 4 & \text{if } x > 2 \end{cases}$

$$f(0) = 0, f'(0) = 0, f''(0) = 2$$

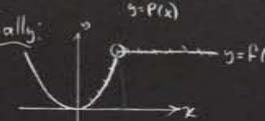
$$f^{(n)}(0) = 0, n > 2$$

$$\therefore P_0(x) = 0, P_1(x) = 0, P_2(x) = x^2$$

$$P_n(x) = x^2, n > 2$$

$$\therefore P(x) = \lim_{n \rightarrow \infty} P_n(x) = x^2 \neq f(x)$$

Pictorially:



②  $\sum \frac{|x|^n}{n!}$  yields

$$u_n = \frac{|x|^n}{n!}, \quad u_{n+1} = \frac{|x|^{n+1}}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{|x|}{n+1}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1$$

for all  $x$

$\therefore \sum \frac{x^n}{n!}$  converges for  $|x| < \infty$

In general, given  $\sum a_n x^n$

there exists  $M$  such that

$\sum a_n x^n$  convs. abs. if  $|x| < M$   
diverges if  $|x| > M$

Pictorially:

$$\sum a_n x^n$$

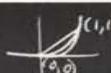
$\xrightarrow{\text{If } \sum a_n x^n \text{ converges}}$

(If  $\sum a_n x^n$  converges then  
 $\sum a_n x^n$  converges abs if  $|x| < |x_1|$ )

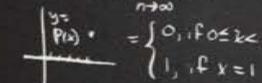
$\xleftarrow{\text{If } \sum a_n x^n \text{ diverges}}$

(If  $\sum a_n x^n$  diverges then  
 $\sum a_n x^n$  diverges if  $|x| > |x_1|$ )

③ Let  $P_n(x) = \frac{x^n}{n!}$   
 $\text{dom } P_n = [0, 1]$



$$\text{Then: } P(x) = \lim_{n \rightarrow \infty} P_n(x)$$



i.e. Each  $P_n$  is continuous when  $x=1$   
but  $P$  is not!

The Ratio Test and  
Absolute Convergence:

$\sum a_n x^n$  converges if  $\sum_{n=0}^{\infty} |a_n x^n|$  does.

But we may test  $\sum |a_n x^n|$  for  
convergence by the ratio test etc.

Examples:

(1)  $\sum_{n=0}^{\infty} |n x^n|$  converges  $\Leftrightarrow |x| < 1$

$$\left\{ \begin{array}{l} (\text{Let } u_n = |n x^n|, u_{n+1} = |(n+1)x^{n+1}|) \\ \therefore \frac{u_{n+1}}{u_n} = |x| < 1 \\ \therefore \rho = \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} |x| < 1 \end{array} \right.$$

$\therefore \sum_{n=0}^{\infty} x^n$  converges (absolutely) if  $|x| < 1$

$\sum a_n x^n$

$\xrightarrow{\text{Suppose } \sum a_n x_1^n \text{ conv}}$

$$\sum a_n x_2^n \text{ div}$$

"Taylor's Theorem with  
Remainder"

Using integration by parts  
repeatedly (as shown in our text)  
it follows that if  $f$  and its first  
 $n+1$  derivatives exist at  $x=0$ , then

$$f(x) = f(0) + \underbrace{\int_0^x f'(t) dt}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(t)}{n!} x^{n+1}}_{R_n(x)}$$

$$\text{where } P_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt$$

$$\boxed{f(x) = P(x) \Leftrightarrow \lim_{n \rightarrow \infty} R_n(x) = 0}$$

7.050 Uniform Convergence

28 min.

Uniform Convergence

Review: If  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in [a, b]$  we say that  $\{f_n\}$  converges to  $f(x)$  on  $[a, b]$ .

Example: Suppose  $f_n(x) = \frac{n}{2n+1}x^2$ . Then  $f(x) = \lim_{n \rightarrow \infty} f_n(x) = \frac{1}{2}x^2$ . Hence  $\{\frac{n^2}{2n+1}\}$  converges to  $\frac{1}{2}x^2$ . Now  $\lim_{n \rightarrow \infty} f_n(2) = 2$ ,  $\lim_{n \rightarrow \infty} f_n(4) = 8$ .  
 $n > N_1 \rightarrow |f_n(2) - 2| < \epsilon$ ,  $n > N_2 \rightarrow |f_n(4) - 8| < \epsilon$ . Of course,  $N_1$  and  $N_2$  need not be the same.

Two Basic Definitions

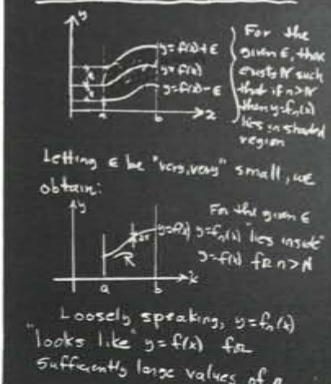
(1) Let  $\text{dom } f_n = [a, b]$ . We say  $\{f_n\}$  converges pointwise to  $f$  on  $[a, b] \Leftrightarrow \lim_{n \rightarrow \infty} f_n(x) = f(x)$  for each  $x \in [a, b]$ .

That is, given  $\epsilon > 0$  we can find  $N_1$  such that  $n > N_1$  implies  $|f_n(x) - f(x)| < \epsilon$  for  $\forall$  given  $x \in [a, b]$ .

In general, the choice of  $N_1$  depends on the choice of  $x$ , and there are infinitely many such choices in  $[a, b]$ .

(2) If we can find one  $N$  such that  $n > N \rightarrow |f_n(x) - f(x)| < \epsilon$  for every  $x \in [a, b]$  we say that the convergence is uniform.

A Pictorial Interpretation



From our picture it should seem clear that if  $\{f_n\}$  converges uniformly to  $f$  on  $[a, b]$ , and if each  $f_n$  is continuous on  $[a, b]$ , then

(2)  $f$  is also continuous on  $[a, b]$ .

(Note: This result, as we have already seen, need not be true for pointwise convergence. Recall our example in which  $f_n(x) = x^n$ ,  $\text{dom } f_n = [0, 1]$ .)

Since  $f$  continuous at  $x = x_i$  means  $\lim_{x \rightarrow x_i} f(x) = f(x_i)$  and since  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$

we may rewrite (1) as:

$$\lim_{x \rightarrow x_i} \left[ \lim_{n \rightarrow \infty} f_n(x) \right] = \lim_{n \rightarrow \infty} f_n(x_i) = \lim_{n \rightarrow \infty} \left[ \lim_{x \rightarrow x_i} f_n(x) \right]$$

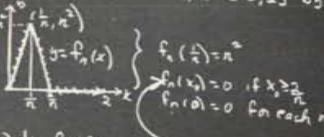
$$(2) \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

$$+ \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx$$

These two results are proven as theorems in the supplementary notes. A theorem about differentiation is also proved.  
(Differentiation is in a way more subtle than integration since it requires "smoothness" as well as "unbrokenness")

Note: (2) need not be true if the convergence is not uniform

Example: Define  $f_n$  on  $[0, 2]$  by:



- (a)  $\lim_{n \rightarrow \infty} f_n(0) = 0$
- (b) If  $0 < x_0 \leq 2$ , we may find  $N$  such that  $n > N \Rightarrow \frac{1}{n} < x_0$   
 $\therefore n > N \Rightarrow f_n(x_0) = 0$   
 $\therefore \lim_{n \rightarrow \infty} f_n(x_0) = 0$

Hence, if  $\left\{ \sum_{k=0}^{\infty} a_k x^k \right\}$  converges

uniformly to  $\sum_{k=0}^{\infty} a_k x^k$

then, for example,

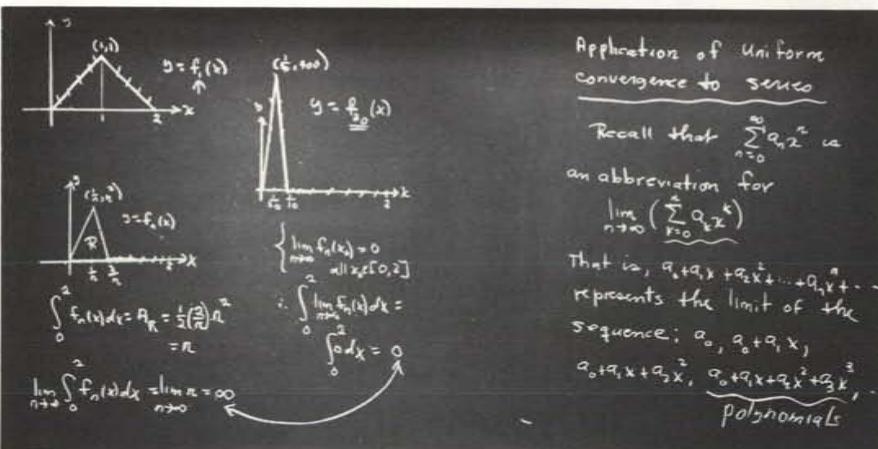
$\sum_{k=0}^{\infty} a_k x^k$  must be continuous

Since each  $\sum_{k=0}^{\infty} a_k x^k$  is.

Also, if  $[a, b]$  is included in the interval of uniform convergence then  $\int_a^b \left( \sum_{k=0}^{\infty} a_k x^k \right) dx = \sum_{k=0}^{\infty} \left( \int_a^b a_k x^k dx \right)$

Next time we shall show that within the interval of absolute convergence  $\left\{ \sum_{k=0}^{\infty} a_k x^k \right\}$  converges uniformly to  $\sum_{k=0}^{\infty} a_k x^k$

In other words within the radius of convergence  $\sum_{k=0}^{\infty} a_k x^k$  "enjoys" the "usual" polynomial properties associated with  $\sum_{k=0}^{\infty} a_k x^k$



7.060 Uniform Convergence of Series

27 min.

Uniform Convergence  
of Series

Weierstrass M-Test

Suppose  $\sum_{n=1}^{\infty} M_n$  is a positive convergent series and that  $|f_n(x)| \leq M_n$  for each  $n$  and each  $x \in [a, b]$ .

Then  $\left\{ \sum_{k=1}^n f_k(x) \right\}$  converges uniformly to  $\sum_{k=1}^{\infty} f_k(x)$  ( $= f(x)$ ) on  $[a, b]$ .

Proof

$$\begin{aligned} |f(x) - \sum_{k=1}^n f_k(x)| &= \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\ &= \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\ &\leq \sum_{k=n+1}^{\infty} |f_k(x)| \\ &\leq \sum_{k=n+1}^{\infty} M_k \\ &\leq \epsilon \text{ for } n \text{ sufficiently large} \end{aligned}$$

independently of  $x$

An Example:

$$\cos x + \cos \frac{4x}{4} + \dots + \cos \left( \frac{n^2 x}{n^2} \right) + \dots = \sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^2}$$

$$\left| \frac{\cos(n^2 x)}{n^2} \right| \leq \frac{1}{n^2} \text{ for each } n \text{ and all } x$$

$\sum \frac{1}{n^2}$  is a pos. conv. series

$$\therefore \left\{ \sum_{k=1}^n \frac{\cos k^2 x}{k^2} \right\} \text{ conv. unif. to } \sum_{k=1}^{\infty} \frac{\cos k^2 x}{k^2}$$

$\therefore$  let us compute, say,

$$\int_0^t (\cos x + \cos \frac{4x}{4} + \dots + \cos \frac{n^2 x}{n^2}) dx \\ (= \int_0^t \left( \sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^2} \right) dx)$$

By uniform convergence, we have:

$$\begin{aligned} \sin x + \sin \frac{4x}{4} + \sin \frac{9x}{9} + \dots &= \int_0^t \sin x dx \\ &= \sin t + \sin \frac{4t}{4} + \dots \\ &= \left( \sum_{n=1}^{\infty} \frac{\sin n^2 t}{n^2} \right) \end{aligned}$$

### Application to Power Series

Let  $|x_0| < |x_1| < R$  where

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges for } |x| < R$$

$$\lim_{n \rightarrow \infty} a_n x_1^n = 0 \quad (\text{Since } \sum a_n x^n \text{ converges})$$

$\therefore$  Given  $M > 0$ , there exists  $N$  such that  $n > N \rightarrow |a_n x_1^n| < M$

Key idea is:

$$\begin{aligned} |a_n x_0^n| &= |a_n \left(\frac{x_0}{x_1}\right)^n x_1^n| \\ &\leq |a_n x_1^n| \left|\frac{x_0}{x_1}\right|^n \\ &\leq M \left|\frac{x_0}{x_1}\right|^n \end{aligned}$$

Now  $\left|\frac{x_0}{x_1}\right|$  is a positive constant  $< 1$

$\therefore \sum_{n=0}^{\infty} M \left|\frac{x_0}{x_1}\right|^n$  is a positive convergent (geometric) series

$\therefore$  By Weierstrass M-test

$\sum_{n=0}^{\infty} a_n x^n$  is uniformly convergent

if  $|x| < R$  where  $R$  is the radius of convergence for  $\sum_{n=0}^{\infty} a_n x^n$

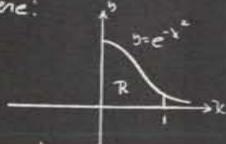
In other words

$$A_R = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{210} + \dots$$

$\approx 0.748$

### Example

Find the area of  $R$  where:



$$A_R = \int_0^1 e^{-x^2} dx, \text{ but we do not know (explicitly) } g(x)$$

$$\text{such that } g'(x) = e^{-x^2}$$

We do know:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

for all  $x$

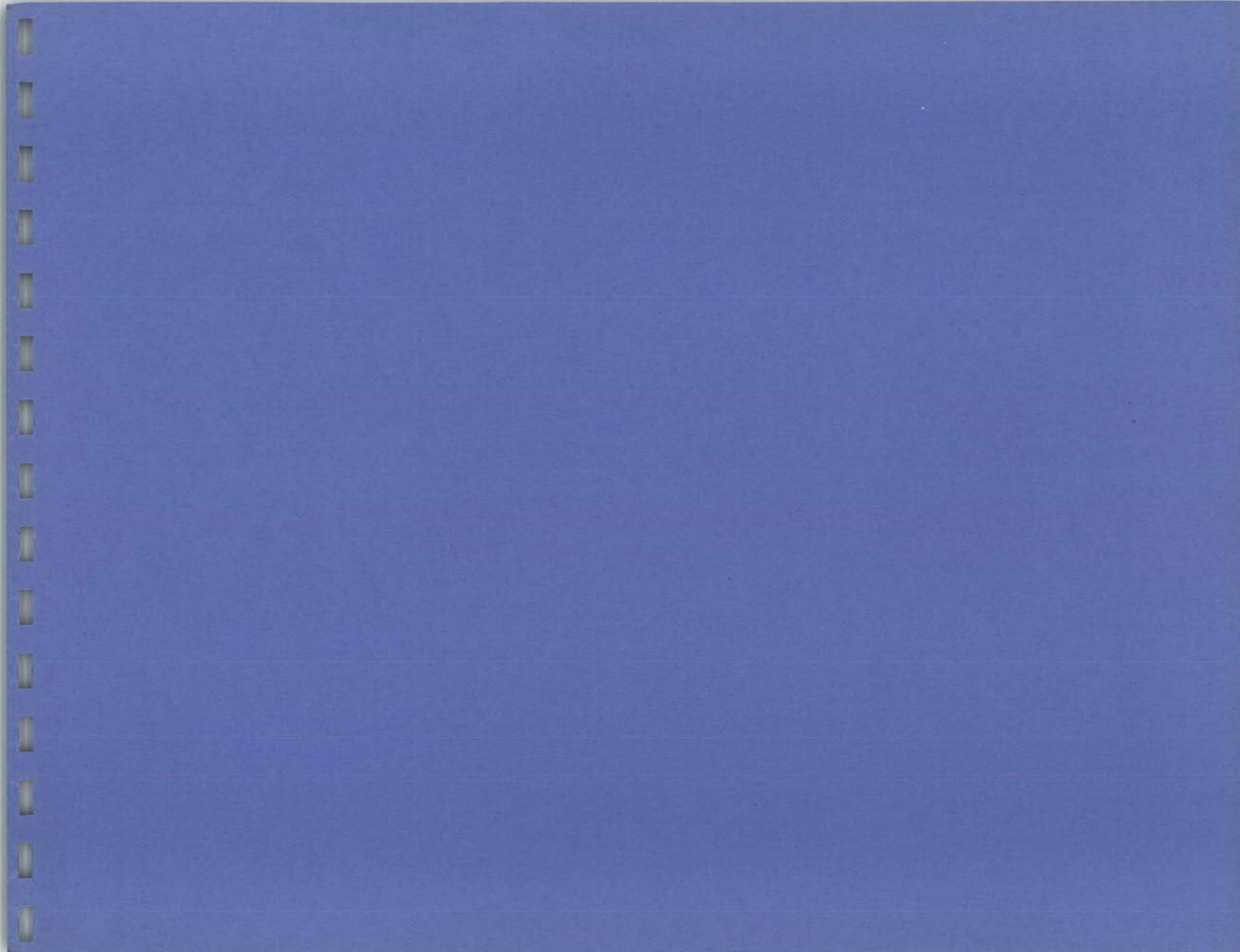
$$\therefore e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \dots + \frac{(-1)^n x^{2n}}{n!}$$

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \int_0^1 \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \right) dx \\ &= \sum_{n=0}^{\infty} \left( \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx \right) \\ &\quad (\text{by uniform conv}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} \end{aligned}$$









8  
30

MIT OpenCourseWare  
<http://ocw.mit.edu>

**Resource: Calculus Revisited**  
Herbert Gross

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.