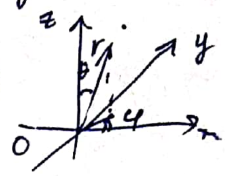
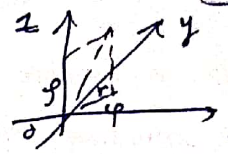
$$f(x_0+dx) = f(x_0) + f'(x_0)\,dx + \dots + \frac{f^{(n)}(x_0)}{n!}dx^n + o(dx^n)$$

$$dx\,dy\,dz = r^2\sin\theta \cdot dr\,d\theta\,d\varphi$$

MA





$$dx\,dy\,dz = r\,dr\,d\varphi\,dz$$

$$dx\,dy = r\,dr\,d\varphi$$

**Chain Rule**

$$\frac{\partial(u_1,\dots u_m)}{\partial(x_1,\dots x_n)} = \frac{\partial(u_1,\dots u_m)}{\partial(t_1,\dots t_k)} \cdot \frac{\partial(t_1,\dots t_k)}{\partial(x_1,\dots x_n)}$$

Jacobian matrix

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

**Gradient vector**

$$\nabla f = f'_x\vec{i} + f'_y\vec{j} + f'_z\vec{k}$$

$$\frac{\partial u(M_0)}{\partial \vec{l}} = \lim_{\rho\to 0}\frac{u(M)-u(M_0)}{\rho} \quad (\vec{M M_0} = \rho\vec{l})$$

$$= \vec{grad}\,u(M_0)\cdot\vec{l}$$

$$\vec{grad}\,u(M_0) = \frac{\partial u}{\partial x}(M_0)\vec{i} + \frac{\partial u}{\partial y}(M_0)\vec{j} + \frac{\partial u}{\partial z}(M_0)\vec{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\,\vec{c}) - \vec{c}(\vec{a}\,\vec{b})$$

$$(\vec{a},\vec{b},\vec{c}) = \vec{a}(\vec{b}\times\vec{c}) = \vec{b}(\vec{c}\times\vec{a})$$

$f$ continuous $f_{xy} = f_{yx}$

$$\frac{d}{d\alpha}\int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx = f(\psi(\alpha),\alpha)\,\psi'(\alpha) - f(\varphi(\alpha),\alpha)\,\varphi'(\alpha) + \int_{\varphi(\alpha)}^{\psi(\alpha)}\frac{\partial}{\partial\alpha}f(x,\alpha)\,dx$$

**Mean value theory:** $f(b)-f(a) = f'(c)(b-a)$, $c\in[a,b]$

$$f(x_0+dx, y_0+dy) = f(x_0,y_0) + df + \frac{1}{2}d^2f$$

$d^2f > 0$: min $\quad d^2f$: indefinite $\to$ saddle

$d^2f < 0$: max $\quad d^2f$: semidefinite ?

$$f(x_1,x_2,x_3) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{33} & a_{32} & a_{33}\end{pmatrix}$$

$\Delta_1 = |a_{11}|$

$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}$

$\Delta_3 = |A|$

$f$ is positive definite: $\Delta_1 > 0$ +, $\Delta_2 > 0$ +, $\Delta_3 > 0$ +

negative definite: $\Delta_1 < 0$ −, $\Delta_2 > 0$ +, $\Delta_3 < 0$ −

$f(x,y)$ in constrain $g(x,y) = 0$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases} \to \begin{cases} (f-\lambda g)_x = 0 \\ (f-\lambda g)_y = 0 \end{cases}$$

$L(x,y,\lambda) = f - \lambda g$ : Lagrange

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\operatorname{div}F = \nabla\cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$F = f(x,y,z)\,i + g(x,y,z)\,j + h(x,y,z)\,k$$

$$\operatorname{curl}F = \operatorname{rot}F = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h\end{vmatrix}$$

$$\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\cdot dt$$

F: conservative field: $F(x,y) = \nabla\varphi(x,y)$

$$\Rightarrow \int_C F(x,y)\,dr = \int_C \nabla\varphi\,dr = \varphi(x_1,y_1) - \varphi(x_0,y_0)$$

$$\left(\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}\right)$$

F: conservative field (3D): $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}; \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}; \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$

$$\to \int_C F(x,y,z)\,dr = \varphi(x_1,y_1,z_1) - \varphi(x_0,y_0,z_0)$$

**Green's theorem**

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA$$

**Surface integrals**

$$\iint_G f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

$$r = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k$$

$$\iint_G f(x,y,z)\,dS = \iint f(x(u,v),y(u,v),z(u,v))\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dA$$

$$n(u,v) = \frac{\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}}{\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|}$$

$$\Phi = \iint_\sigma F \cdot n \, dS = \iint_R F \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) dA = \iint_R F \cdot \nabla G \, dA$$

($\sigma$ smooth parametric surface represent by vector equation $r = r(u,v)$ in which $(u,v)$ varies over a region $R$ in $uv$-plane

Divergence theorem

$G$: solid whose surface $\sigma$ is oriented outward. $n$ is outward unit normal on $\sigma$

$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$

$$\iint_\sigma F \cdot n \, dS = \iiint_G \operatorname{div} F \, dV$$

Stoke's theorem

$\sigma$ piecewise smooth oriented surface that bounded by simple, closed, piecewise curve $C$ with positive orientation

$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$

$T$. tangent vector to $C$

$$\oint_C F \cdot T \, ds = \iint_\sigma (\operatorname{curl} F) \cdot n \, dS$$

Pointwise convergence: $\forall x \in D, \forall \varepsilon > 0, \exists N_{\varepsilon,x}$

$\{f_n(x): n = 0,1,2,\dots\}$ sequence of functions defined on $I$, limit function

$f_n(x) \to f(x)$ as $n \to \infty$, each $x \in I$

Uniform convergence $\to$ pointwise convergence

$\sup |f_n(x) - f(x)| \leq \varepsilon$

$\forall \varepsilon > 0 \ \exists N_\varepsilon, \forall n > N_\varepsilon \ \forall x \in D$

$\to |f_n(x) - f(x)| < \varepsilon$

$\sup X$ : smallest upper boundary

$\inf X$ : greatest lower boundary

$f_n(x) \rightrightarrows f(x)$

$\to f(x)$ continous on $D$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim f_n(x)\,dx = \int_a^b f(x)\,dx$$

$f_n(x)$ continously differentable on $D$

$\exists x_0 \in D$: $\exists \lim_{n\to\infty} f_n(x_0)$ $\quad f_n'(x) \rightrightarrows d(x)$ on $D$

$$\to \left(\lim_{n\to\infty} f_n(x)\right)' = \lim_{n\to\infty} f_n'(x)$$

$(a^x)' = a^x \ln a$ ; $(\log_a x)' = \frac{1}{x \ln a}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ ; $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{x^2+1}$ ; $(\operatorname{arccot} x)' = \frac{-1}{1+x^2}$

$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$ $\quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$\sinh x = \frac{e^x - e^{-x}}{2}$ ; $\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x}$ ; $\operatorname{sech} x = \frac{1}{\cosh x}$

$$\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|\sqrt{x^2+a} + x\right| + C$$

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2} x\sqrt{1+x^2} + \frac{1}{2}\ln\left|x + \sqrt{1+x^2}\right| + C$$

$\rho = n\left(\frac{S_{\infty y_n}}{S_{\infty y_{n+1}}} - 1\right)$ $< \begin{matrix} \rho > 1: \text{converg} \\ < 0: \text{diverg} \end{matrix}$

Converge: $\left[\begin{matrix} \left|\frac{a_{n+1}}{a_n}\right|_{n\to\infty} < 1 \\ \sqrt[n]{|a_n|}_{n\to\infty} < 1 \end{matrix}\right.$

Taylor formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \theta(x - x_0)) \quad (0<\theta<1)$$

$0 \leq u_n \leq v_n$ ...

Radius of convergence: $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

$\sum_n a_n x^n$ $\left(R = \frac{1}{\rho}; \rho = \lim_{n\to\infty} \sqrt[n]{a_n}\right)$

Fourier series: $[-\pi, \pi]$: of $f(x)$

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\,dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\,dx$$

Complex: $f(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{inx}$ ; $\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$

Fourier transform of $f$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\,dx \qquad (\xi \in \mathbb{R})$$

Inverse transform : $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\,d\xi$ $\quad (x \in \mathbb{R})$

Fourier integral: $A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\xi) \cos \lambda\xi \, d\xi$; ...

$$f(x) = \int_0^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda$$

$\cos \cdot \cos = \frac{1}{2}(\cos + \cos)$ $\quad \sin \cdot \cos = \frac{1}{2}(\sin + \sin)$

$\sin \cdot \sin = -\frac{1}{2}(\cos - \cos)$