Rolle Th.

$$(3)\begin{cases} \cdot f(x) \text{ is cont in } [a,b] \\ \cdot f(x) \text{ is diff on } (a,b) \\ \cdot f(a) = f(b) \end{cases}$$

$\Rightarrow \exists \xi \in (a,b): f'(\xi) = 0$

~~Lagrang Th.~~ Mean Value Th.

$$\begin{cases} f(x) \text{ is conts } [a,b] \\ f(x) \text{ is diff on } (a,b) \end{cases}$$

$\Rightarrow \exists \xi \in (a,b): f(b) - f(a) = f'(\xi)(b-a)$

Cauchy Th.

$$\begin{cases} f(x), g(x) - \text{Cont on } [a,b] \\ f(x), g(x) - \text{diff on } (a,b) \end{cases}$$

$$\Rightarrow \exists \xi \in (a,b): \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

Taylor Th.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$f(x) = P_0(x) + r_0(x)$$

Larange reminder (Th 1)

$\forall x \in U_\delta(x_0), \exists \xi \in (x_0, x)$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Th2.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o\left((x-x_0)^n\right) \quad (x \to x_0)$$

Th 3

$$\begin{cases} \cdot f^{(n)}(x_0) \text{ exists} \\ \cdot f(x) = \sum_{k=0}^{n} a_k (x-x_0)^k + o\left((x-x_0)^n\right) \end{cases}$$

$(x \to x_0)$

$$\Rightarrow a_k = \frac{f^{(k)}(x_0)}{k!} \quad k = 0, 1, 2, 3, \dots$$

Limit estimate using Taylor formula

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|\sqrt{x^2+a} + x\right| + c$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arctan x)' = \frac{1}{1+x^2}$$

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty}\left[\int_a^b c_k (x-x_0)^k dx\right]$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \qquad -1 < x \le 1$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots \quad -\infty < x < +\infty$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad -\infty < x < +\infty$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad -\infty < x < +\infty$$

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \quad p > 1: \text{converge} \quad 0 < p \le 1: \text{diverge}$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad |r| < 1$$

$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ holds at x_0 iff $\lim_{n \to +\infty} R_n(x) = 0$

$$R_n(x) \le \frac{M}{(n+1)!}|x-x_0|^{n+1}$$

M: upper bound for $|f^{(n+1)}(x)|$

Calculus I ②

$$P(x) = (x-\alpha)\, Q(x) + P(\alpha)$$

Limit definition: $\lim_{x \to a} f(x) = L$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0:$

$\forall x \in \dot{U}_\delta(a) \Rightarrow f(x) \in \dot{U}_\varepsilon(L)$

Continuity at a point

$\Leftrightarrow$ (3) $\begin{cases} \cdot\ f \text{ is defined in } U_\delta(a) \\ \cdot\ \lim_{x \to a} f(x) = A \\ \cdot\ A = f(a) \end{cases}$

Composition $f \circ g = f(g(x))$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1$$

$f(x), g(x)$ is defined in $\dot{U}_\delta(a)$

$f(x), g(x)$ non-zero in $\dot{U}_\delta(a)$

$\Rightarrow f(x) \sim g(x)$ equivalent

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \Rightarrow f(x) \underset{x \to x_0}{=} O(g(x))$$

$f(x)$ is defined in $U_\delta(x_0)$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\varphi(y) = f^{-1}(x)$$

$$\Rightarrow \varphi'(y) = \frac{1}{f'(x)}$$

Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\boxed{\frac{x^2 - 3}{(x^2+x+5)^3 (x-1)^2} = \frac{\alpha_1 x + \beta_1}{(x^2+x+5)^3} + \frac{\gamma}{(x-1)^2} + \frac{\alpha_2 x + \beta_2}{(x^2+x+5)^2} + \frac{\tilde{\gamma}}{(x-1)} + \frac{\alpha_3 x + \beta_3}{(x^2+x+5)}}$$

Hyperbolic function

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x}$$

$$\lim_{x \to 0} \frac{\operatorname{sh} x}{x} = 1 \qquad \lim_{x \to 0} \frac{\operatorname{th} x}{x} = 1$$

Trigonometric function

$\sin x \qquad \cos x \qquad \tan x$

$$\sec x = \frac{1}{\cos x} \qquad \csc = \frac{1}{\sin x}$$

Leibniz formula

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} C_n^k\, u^{(k)} \cdot v^{(n-k)}$$

$$C_n^k + C_n^{k+1} = C_{n+1}^k \qquad C_n^0 = C_n^n = 1$$

Fermat Th.

$\begin{cases} f(x) \text{ has local } \cancel{\text{theo}} \text{ extremum at } x_0 \\ f(x) \text{ is diff at } x_0 \end{cases}$

$\Rightarrow f'(x_0) = 0 \rightarrow$ critical point

Second derivative test (single variable)

$f''(x_0) < 0 \Rightarrow$ local maximum

$f''(x_0) > 0 \Rightarrow$ local minimum

$f''(x_0) = 0 \Rightarrow$ inconclusive

Higher-derivative test

$$f'(c) = \dots = f^{(n)}(c) = 0, \quad f^{(n+1)}(c) \neq 0$$

1) n – odd, $f^{(n+1)}(c) < 0 \Rightarrow$ local maximum
2) n – odd, $f^{(n+1)}(c) > 0 \Rightarrow$ local minimum
3) n – even, $f^{(n+1)}(c) < 0 \Rightarrow$ strictly decreasing point of reflection
4) n – even, $f^{(n+1)}(c) > 0 \Rightarrow$ — increasing —