$$\|a\|_p = \left(\Sigma |a_i|^p\right)^{1/p}$$

$(AB)^T = B^T A^T$ $(A^{-1})^T = (A^T)^{-1}$

$A = LDU$

$A = A^T \rightarrow A = LDL^T$

Symmetric: $A^T = A$ Skew-symmetric $A^T = -A$

$$S = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ a_{ij} & & 1 \end{bmatrix} \rightarrow S^{-1} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ -a_{ij} & & 1 \end{bmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Four subspace

- $C(A^T)$ $A^T y$ row space : dimension $= r$
- ⊥ $C(A)$ $= Ax$ column space : dimension $= r$
- $N(A)$ $Ax = 0$ Null space : $n - r$
- ⊥ $N(A^T)$ $A^T y = 0$ Nullspace of A^T (left null space of A) $m - r$

$x^T(A^T y) = (xA)^T y = 0^T y = 0$

$R^T R$ always symmetric

Projection of b on a : $p = \hat{x} a = \frac{a^T b}{a^T a} a$

Least square approximation

Ax = b has no sol → solve $A^T A \hat{x} = A^T b$

$\hat{x} = (A^T A)^{-1} A^T b$

projection of b onto subspace is p:

$p = A\hat{x} = Pb = A(A^T A)^{-1} A^T b$

$E = \|Ax - b\|^2$ as small as possible

Projection matrix: P $(\rightarrow p = Pb)$

$P = \frac{aa^T}{a^T a}$ $P^T = P$, $P^2 = P$

Orthonormal matrix: $Q = [q_1 \cdots q_n]$ $Q^T Q = I$

$Q^{-1} = Q^T$

LA

Gram-schmidt: $A = QR$

$A = (a, b, c)$

$Q = (q_1, q_2, q_3)$

$q_1 = \frac{\alpha}{\|a\|}$ $q_2 = \frac{\beta}{\|\beta\|}$ $q_3 = \frac{\gamma}{\|\gamma\|}$

$\beta = b - (q_1^T b) q_1$

$\gamma = c - (q_1^T c) q_1 - (q_2^T c) q_2$

$R = Q^T A$

$z = r\cos\theta + ir\sin\theta = re^{i\theta}$

$\bar{z} = r\cos\theta - ir\sin\theta$ conjugate of z

$\|z\|^2 = \bar{z}^T z = z^H . z = |z_1|^2 + \cdots + |z_n|^2$

$A^H = \bar{A}^T =$ "A Hermitian" $(A^H)^H = A$

conjugate transpose of AB: $(AB)^H = B^H . A^H$

Inner product: $u^H v$

Hermitian matrix: $A^H = A$ $(a_{ij} = \bar{a}_{ji})$

$\det A^H = \overline{\det A}$ $z^H A z$ is real.

Unitary matrix : square matrix with orthonormal columns: $U^H U = I \rightarrow U^H = U^{-1}$

Fourier matrix :

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdot & 1 \\ 1 & w & w^2 & \cdot & w^{n-1} \\ 1 & w^2 & w^4 & \cdot & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdot & w^{(n-1)^2} \end{bmatrix}$$

$F_{jk} = w_n^{jk}$, $w_n = e^{\frac{2\pi i}{n}}$

$Fc = y$ $F^{-1} = \frac{\bar{F}}{n}$

~~Dia~~ $A x = \overset{\swarrow \text{ eigenvalue}}{\lambda} x \leftarrow$ eigenvector

$A: \lambda_1, \lambda_2$

$\rightarrow A^{-1}: \frac{1}{\lambda_1}, \frac{1}{\lambda_2}$

$\rightarrow (A - \lambda I) x = 0$

$\rightarrow \det(A - \lambda I) = 0 \rightarrow \lambda \rightarrow x$

$S = [x_1 \ \ x_2 \ \cdots \ x_n]$

Diagonalizing a matrix : $S^{-1} A S = \Lambda$

$$AS = S\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & & \\ 0 & \cdots & & \lambda_n \end{bmatrix} = S\Lambda$$

$A^k = S\Lambda^k S^{-1}$

$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$ $\lambda_1 \lambda_2 \ldots \lambda_n = \det(A)$

Orthogonal $PP^T = I$

Projection $P^2 = P$

Permutation: every row & column contains a single 1 with 0's everywhere else

Hermitian $P^H = P$

Diagonalizable: sufficient Eigenvectors make the matrix diagonalizable

Markov matrix: every element of matrix is positive & sum of column elements equals to 1

FFT $m = \frac{n}{2}$ $\quad y = F_n c$

$$c' = \begin{pmatrix} c_0 \\ c_2 \\ \vdots \\ c_{n-2} \end{pmatrix} \qquad c'' = \begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{n-1} \end{pmatrix} \qquad \omega = \frac{2\pi i}{n}$$

$$y' = F_m c' \qquad y'' = F_m c''$$

$$y = \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} y_0' + y_0'' \\ y_1' + \omega y_1'' \\ \vdots \\ y_{m-1}' + \omega^{m-1} y_{m-1}'' \\ y_0' - y_0'' \\ \vdots \\ y_{m-1}' - \omega^{m-1} y_{m-1}'' \end{pmatrix}$$

Symmetric matrix is positive definite $\rightarrow$ sum of squares $f = x^T A x$.
if all pivots are positive
semidefinite if no pivots are negative
...

A_1 + −
A_2 + +
A_3 + −

$A = A^T \rightarrow A = Q \Lambda Q^T$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{~~e^{i\theta}~~ \, e^{i\theta} - e^{-i\theta}}{2i}$$

$$x_1 x_2 = \frac{c}{a}$$

$$x_1 + x_2 = -\frac{b}{a}$$

Differential equation: $u' = Au$

$$u = Sv \rightarrow S\frac{dv}{dt} = ASv \Rightarrow \frac{dv}{dt} = \Lambda v$$

$$v(t) = e^{\Lambda t} v(0)$$

$$u(t) = S e^{\Lambda t} S^{-1} v(0) = e^{At} u(0)$$

$$e^{At} = 1 + At + \frac{(At)^2}{2!} + \cdots$$

stable: all $\lambda < 0$ $\quad (\mathrm{Re}\, \lambda_i < 0)$

Neutrally stable: $\lambda \leq 0$ $\quad (\mathrm{Re}\, \lambda_i \leq 0)$

Unstable: $\exists \lambda > 0$ $\quad (\mathrm{Re}\, \lambda > 0)$

$u = u_n + u_p \qquad u_p = A^{-1} \cdot b$

$u_n = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

Singular value decomposition SVD

$A = U \Sigma V^T$; U: orthogonal, Σ: diagonal (non-negative), V: orthogonal

$v_1, v_2, \ldots v_r$: orthonormal basis for row space
$u_1, u_2, \ldots u_r$: ... column space
$v_{r+1}, \ldots v_n$: ... nullspace ...

A symmetric positive definite its Eigenvectors are orthogonal : $A = Q \Lambda Q^T$

- $C^T C = V \Sigma^T \Sigma V^T \rightarrow$ normal eigenvector $\rightarrow V \Sigma$
- $CV = U\Sigma$ make colums normal $\rightarrow U$

$\rightarrow C = U \Sigma V^T$

Similar matrices A and $B = M^{-1} A M$

$MB = AM$

let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \ldots$

Linear transformations: $\begin{cases} T(v+w) = T(v) + T(w) \\ T(cv) = cT(v) \end{cases}$

$T(v) = Av$

$T: R^n \rightarrow R^m$

$(v_1, v_2, \ldots v_n) \quad (w_1, w_2, \ldots w_m) \quad$ (basis)

$T(v_i) = a_{1i} w_1 + a_{2i} w_2 + \ldots + a_{mi} w_m$

$T(v_i) = A v_i$

Two sided inverse: $AA^{-1} = A^{-1}A = I$

Left inverse: $A^{-1}_{left} = (A^T A)^{-1} A^T$ (full column rank)

Right inverse: $AA^{-1}_{right} = I \qquad A^T (AA^T)^{-1}$ (full row rank)

Pseudoinverse A^+ of A: $x = A^+ A x \ \forall x \in C(A^T)$

$A = U\Sigma V^T \rightarrow A^+ = V \Sigma^+ U^T \quad (r = m = n \rightarrow \Sigma^+ = \Sigma^{-1})$