INNOPOLIS UNIVERSITY

Differential Equations
Computational Practicum - Task 14
$y' = (1 + \frac{y}{x}) \ln \frac{x+y}{x} + \frac{y}{x}$

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Consider the initial value problem with the ODE of the first order and some interval:

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \\ x \in [x_0, X] \end{cases}$$

In this document, I will analyze a specific problem (task 14) where $f(x, y) = (1+\frac{y}{x})\ln\frac{x+y}{x} + \frac{y}{x}$, $x_0 = 1$, $y_0 = 2$ and $X = 6$. The exact solution and it's corresponding approximations in three numerical methods, i.e. Euler's method, improved Euler method and Runge-Kutta method will be implemented in a Java application. This application also provides data visualization; and allows user to change initial value x_0, y_0, X and grid size hence ability to compare methods.

The overview of the problem and methods will be presented in the first part. Following is about implementation in Java. Finally, analysis using implemented application will be provided.

1 Solution

Consider the initial value problem with the ODE of the first order and some interval:

$$\begin{cases} y' = f(x, y) = (1+\frac{y}{x})\ln\frac{x+y}{x} + \frac{y}{x} \\ y(x_0) = y_0 \\ x \in [x_0, X] \end{cases}$$

Obviously, $f(x, y)$ is defined when $x \neq 0$ and $\frac{x+y}{x} > 0$

1.1 Exact solution

Let $r = \frac{y}{x}$, $y' = (1+\frac{y}{x})\ln\frac{x+y}{x} + \frac{y}{x}$ becomes: $r'x + r = (1+r)\ln(1+r) + r$

$$\Rightarrow \frac{dr}{(1+r)\ln(1+r)} = \frac{dr}{x}$$

$$\Rightarrow \int_{\frac{y_0}{x_0}}^{r} \frac{dr}{(1+r)\ln(1+r)} = \int_{x_0}^{x} \frac{dx}{x}$$

$$\Rightarrow \ln\frac{\ln(1+r)}{\ln(1+\frac{y_0}{x_0})} = \ln\frac{x}{x_0}$$

$$\Rightarrow y = x(1+\frac{y_0}{x_0})^{\frac{x}{x_0}} - x$$

Therefore, $y = x(1+\frac{y_0}{x_0})^{\frac{x}{x_0}} - x$ where $x \neq 0$ and $\frac{x+y}{x} > 0$ is the exact solution of $y' = (1+\frac{y}{x})\ln\frac{x+y}{x} + \frac{y}{x}$

1.2 Numerical Methods

We are interested in computing approximate values of solutions of $y' = f(x,y),\ y(x_0) = y_0$ at equally spaced points $x_0, x_1, .., x_n = X$ in an interval $[x_0, X]$.

$$x_i = x_0 + ih, i = 0, 1, ..., n \text{ where } h = \frac{X - x_0}{n}$$

Local error (error at the ith step): $e_i = y(x_i) - y_i$

Global error: $e = e_n = y(X) - y_n$

1.2.1 Euler Method

Euler method is the simplest numerical method which is based on the assumption that the tangent line at $(x_i, y(x_i)$ approximate the integral curve over $[x_i, x_{i+1}]$. It starts with the known value $y(x_0) = x_0$ and computes $y_1, y_2, ..., y_n$ by the formula:

$$y_{i+1} = y_i + hf(x_i, y_i)\ , 0 \leq i \leq n-1$$

1.2.2 Improved Euler Method

The improved Euler method starts with the known value $y(x_0) = x_0$ and computes $y_1, y_2, ..., y_n$ by the formula:

$$k_{1i} = f(x_i, y_i)$$
$$k_{2i} = f(x_i + h, y_i + hk_{1i})$$
$$y_{i+1} = y_i + \frac{h}{2}(k_{1i} + k_{2i})$$

1.2.3 Runge-Kutta Method

Runge-Kutta is the most widely method. It starts with the known value $y(x_0) = x_0$ and computes $y_1, y_2, ..., y_n$ by the formula:

$$k_{1i} = f(x_i, y_i)$$
$$k_{2i} = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i})$$
$$k_{3i} = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{2i})$$
$$k_{4i} = f(x_i + h, y_i + hk_{3i}$$
$$y_{i+1} = y_i + \frac{h}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i})$$

2 JAVA APPLICATION

Above solutions are implemented in Java using Netbeans and SceneBuilder. This application provides GUI which allows user to alter x_0, y_0, X and grid size n in constraint, i.e. $x_0, X \neq 0$, $\frac{x_0+y_0}{x_0} > 0$ (to ensure that $f(x,y)$ is defined) and $X > x_0$, $n \in N$, $n > 0$. Since $f(x,y)$ is not defined at $x = 0$, for the ease of implementation, I applied one more constraint to user input: x_0 and X must have the same sign (both positive or both negative).

2.1 USER GUIDE

Highlight features

- Allow changing initial values x_0, y_0, X and grid size N.
- Allow choosing methods: exact solution, Euler method, improved Euler method and Runge-Kutta method.
- Provide visualization for solutions, local errors and global errors for chosen methods.

Figure 2.1: Application screenshot with default values

How to use

1. Enter valid initial values x_0, y_0, and X, grid size N and grid range. Default values: $x_0 = 1.0$, $y_0 = 2.0$, $X = 6.0$, $N = 10$ and grid range from 5 to 30 with step of 2. Any invalid input will be informed.

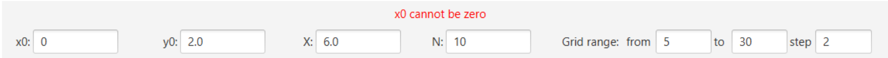


Figure 2.2: Enter values

2. Choose methods to process. Both Exact, Euler, Improved Euler and Runge-Kutta are chosen by default.

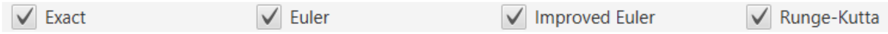
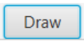


Figure 2.3: Choose methods

3. Click 

button to start analyzing.
Global errors with specified N will be presented right below chosen methods.

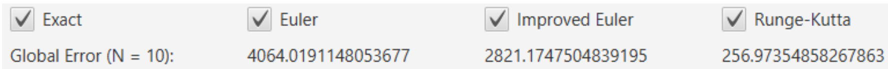


Figure 2.4: Global Errors at N

Also, plots of solutions, local errors, global errors in grid range will be updated in drawing space.

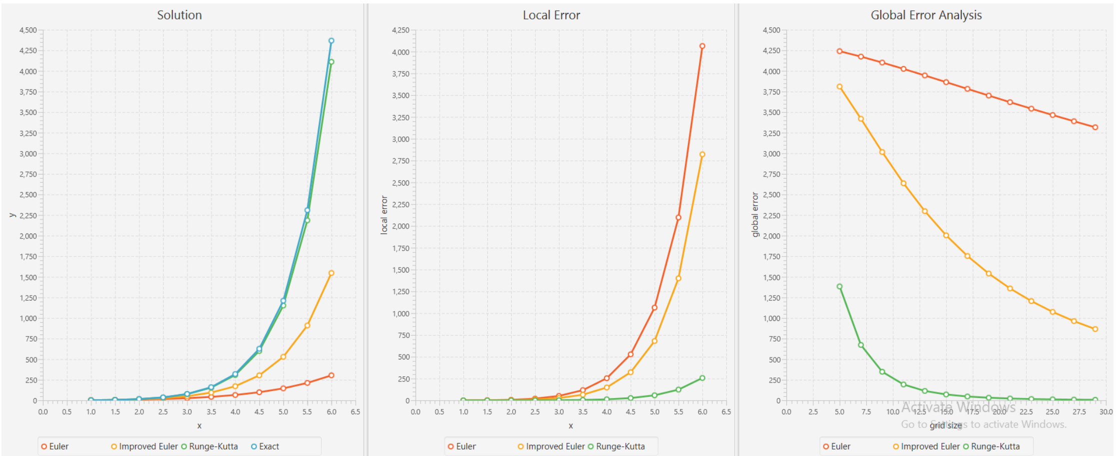
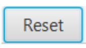


Figure 2.5: Plots of solutions, local errors at N; and global errors in grid range

Click 

button to reset all elements to default values.

2.2 Implementation

The application is written in Java using Netbeans and SceneBuilder. Since for each initial value x_0, and y_0; and X, exact solution and it's approximation should be provided, I decided to used only one class named Fode14 for our ODE. In GUI, there is only two event handlers (for clicking "Draw" and "Reset" buttons) which are in FXMLDocumentController class.

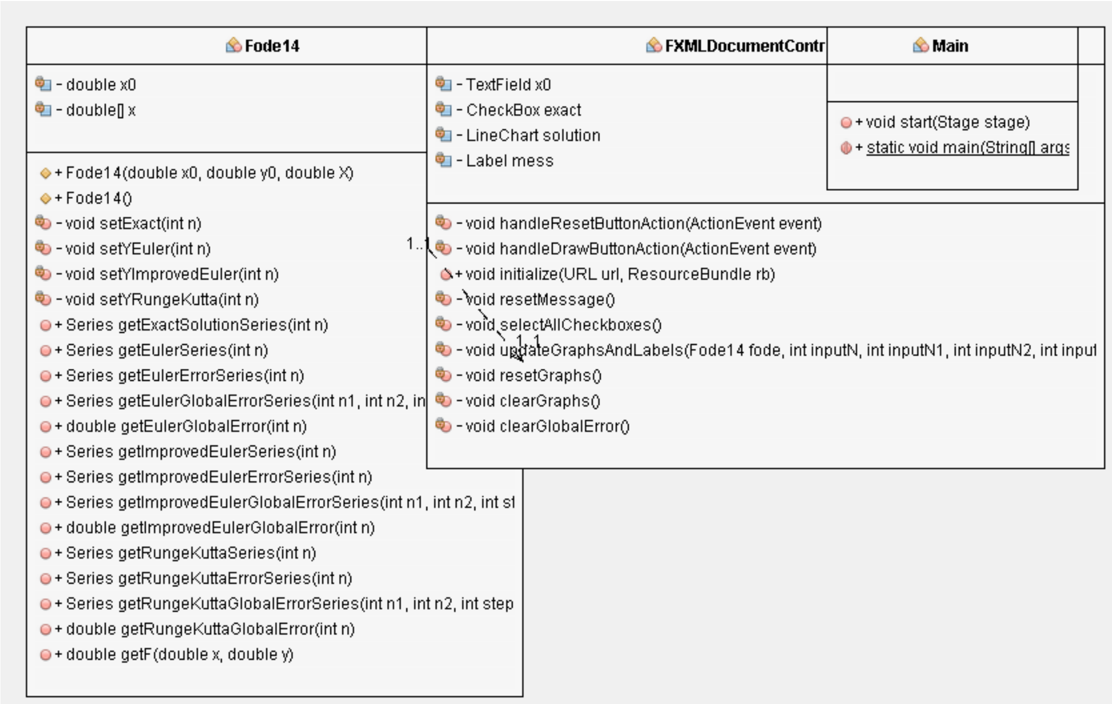


Figure 2.6: UML Diagram

Below is implementation of Runge-Kutta method (as functions in Fode14). It's structure is applicable for others.

```
private void setYRungeKutta(int n) throws Exception {
    setExact(n);

    if (yRungeKutta == null || yRungeKutta.length != n+1) {
        yRungeKutta = new double[n+1];
        yRungeKutta[0] = y0;

        double k1, k2, k3, k4;
        double h = (X-x0)/n;
        for (int i = 0; i < n; ++i) {
            k1 = getF(x[i], yRungeKutta[i]);
            k2 = getF(x[i] + h/2.0, yRungeKutta[i] + h*k1/2.0);
            k3 = getF(x[i] + h/2.0, yRungeKutta[i] + h*k2/2.0);
            k4 = getF(x[i] + h, yRungeKutta[i] + h*k3);
            yRungeKutta[i+1] = yRungeKutta[i] + h*(k1+2*k2+2*k3+k4)/6.0;
        }
    }
}
```

```java
public Series getRungeKuttaSeries(int n) throws Exception {
    Series rungeKutta = new Series();
    rungeKutta.setName("Runge-Kutta");

    setYRungeKutta(n);
    for (int i = 0; i <= n; ++i) {
        rungeKutta.getData().add(new XYChart.Data(x[i], yRungeKutta[i]));
    }
    return rungeKutta;
}

public Series getRungeKuttaErrorSeries(int n) throws Exception {
    Series rungeKutta = new Series();
    rungeKutta.setName("Runge-Kutta");

    setYRungeKutta(n);
    for (int i = 0; i <= n; ++i) {
        rungeKutta.getData().add(new XYChart.Data(x[i], yExact[i] -
            yRungeKutta[i]));
    }
    return rungeKutta;
}

public Series getRungeKuttaGlobalErrorSeries(int n1, int n2, int step)
    throws Exception {
    if (n1 <= 0 || n2 <= 0 || n1 >= n2 || step <= 0) {
        throw new Exception("Invalid Grid range input");
    }
    Series rungeKutta = new Series();
    rungeKutta.setName("Runge-Kutta");

    for (int i = n1; i <= n2; i += step) {
        rungeKutta.getData().add(new XYChart.Data(i,
            getRungeKuttaGlobalError(i)));
    }

    return rungeKutta;
}

public double getRungeKuttaGlobalError(int n) throws Exception {
    setYRungeKutta(n);
    return yExact[n] - yRungeKutta[n];
}
```

3 ANALYSIS

Figure 3.1: Analysis with $x_0 = 0.1$, $y_0 = 2$, $X = 6.0$, $N = 10$, grid range = $(5, 30, 2)$

Analyzing using the software, we can infer that:

- Runge-Kutta method has the highest performance (closest to exact solution, lowest local and global errors). Following is Improved Euler, and Euler method.
- For $x_0 = 0.1$, $y_0 = 2$, $X = 6.0$, from $N = 20$, increasing N does not significantly affect accuracy of Runge-Kutta method. This number for improved Euler method is around 150, and for Euler method is around 2000.