Computational Methods

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Readings: Shreve Chapter 6

Risk neutral pricing

- Dynamics of the asset price under the EMM

$$dS_t = (r_t - q_t)S_t dt + \sigma_t S_t dB_t^*$$

where B_t^* is a standard BM in a probability space $(\Omega, \mathbb{Q}, \mathcal{F})$ with natural filtration $\mathcal{F}_t = \sigma(B_s, s \in [0, t])$

- **Risk neutral pricing** of a derivative with payoff $V_T \in \mathcal{F}_T$

$$V_t = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(- \int_t^T r_s ds \right) V_T \middle| \mathcal{F}_t \right]$$

- Analytical solutions often not available

Numerical methods

- Numerical integration if the density is known
- Numerical solutions of PDE's
  - Theoretical: connection between risk neutral expectation and PDE via **Feynman-Kac Theorem**
  - Numerical: **finite difference** method, **finite element** method
- **Fourier transform** method: very powerful if the characteristic function of the underlying process is known
- **Monte carlo simulation**: competitive for multi-dimensional applications

Fourier transform methods

- Fourier transform methods in financial engineering
  - Pricing European options (Carr and Madan 1999) **http://www.math.nyu.edu/research/carrp/**
  - Pricing exotic derivatives (e.g., barrier, lookback, Asian options)
  - Pricing options with early exercise features (Bermudan, American options)
- Topics to be covered
  - Fourier transform and inverse Fourier transform
  - Discrete Fourier transform (DFT) and Fast Fourier transform (FFT)
  - Pricing European options in jump models

Fourier transform

- Fourier transform of $f \in L^1(\mathbb{R})$:

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)dx$$

- Characteristic function of a continuous r.v. X with pdf $f(x)$

$$\phi(\xi) = \mathbb{E}[e^{i\xi X}] = \int_{\mathbb{R}} e^{i\xi x} f(x)dx = \mathcal{F}f(\xi)$$

- In financial engineering, often knows c.f., but not pdf: affine models, Lévy models
- **Caution**: Fourier transform may be defined slightly differently

- Convolution of $f, g \in L^1(\mathbb{R})$

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy = \int_{\mathbb{R}} g(y)f(x-y)dy$$

Moreover, $f * g \in L^1(\mathbb{R})$

$$\begin{aligned}\int_{\mathbb{R}} |(f * g)(x)|dx &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)g(x-y)|dydx \\ &= \int_{\mathbb{R}} |f(y)| \int_{\mathbb{R}} |g(x-y)|dxdy \\ &= \int_{\mathbb{R}} |f(y)|dy \cdot \int_{\mathbb{R}} |g(z)|dz\end{aligned}$$

Convolution theorem

- **Convolution Theorem**: if $f, g \in L^1(\mathbb{R})$, then

$$\mathcal{F}(f * g)(\xi) = \mathcal{F}f(\xi) \cdot \mathcal{F}g(\xi)$$

$$\begin{aligned}
\int_{\mathbb{R}} e^{i\xi x}(f * g)(x)dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\xi x} f(y)g(x-y)dydx \\
&= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} e^{i\xi x} g(x-y)dxdy \\
&= \int_{\mathbb{R}} e^{i\xi y} f(y)dy \cdot \int_{\mathbb{R}} e^{i\xi z} g(z)dz
\end{aligned}$$

Fourier inversion

- **Fourier Inversion**: if $f, \hat{f} \in L^1(\mathbb{R})$

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}} e^{i\xi x} f(x)dx \\ f(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \hat{f}(\xi)d\xi \end{aligned}$$

- Recover pdf of a r.v. from its c.f.

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \phi(\xi)d\xi$$

Discrete Fourier transform

- **Discrete Fourier transform (DFT)** of $\{f_0, \cdots, f_{M-1}\}$

$$\hat{f}_n = \sum_{m=0}^{M-1} e^{-2\pi imn/M} f_m, \quad n = 0, 1, \cdots, M-1$$

  E.g., discretization of the Fourier transform

- **Inverse discrete Fourier transform (IDFT)**

$$f_m = \frac{1}{M} \sum_{n=0}^{M-1} e^{2\pi imn/M} \hat{f}_n, \quad m = 0, 1, \cdots, M-1$$

Fast Fourier transform

- Direct computation of DFT/IDFT: $O(M^2)$ operations
- Fast Fourier transform reduces computational cost to $O(M\ln(M))$
- Many free FFT packages available (e.g., **www.fftw.org**)
- **Caution**: slightly different definitions possible, make sure what is the package computing

Fourier transform method for European options

- **Carr-Madan method** for pricing European options in models with known c.f. (Carr and Madan 1999)
- Asset price follows (under EMM $\mathbb{Q}$)

$$S_t = S_0 e^{X_t}$$

- Risk neutral pricing for European call

$$c = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_0 e^{X_T} - K)^+]$$

- Assume the c.f. of X_T is given by $\phi(\xi)$

- In the BSM model (under EMM),

$$X_T = (r - q - \frac{1}{2}\sigma^2)T + \sigma B_T^*$$

$$\phi(\xi) = \exp\left(i\xi(r - q - \frac{1}{2}\sigma^2)T - \frac{1}{2}\sigma^2 T\xi^2\right)$$

- In many alternative models, ϕ is known, and the pdf of X_T is very complicated or not known analytically

- Consider call price as a function of $k = \ln(K/S_0)$

$$c(k) = S_0 e^{-rT} \mathbb{E}^{\mathbb{Q}}[(e^{X_T} - e^k)^+] = S_0 e^{-rT} \int_k^\infty (e^x - e^k) p(x) dx$$

  where $p(x)$ is the pdf of X_T
- Fourier transform of $c(k)$ not well defined: $c(k) \to S_0 e^{-qT}$ as $k \to -\infty$
- **Dampening** necessary: for $\alpha > 0$

$$c^\alpha(k) = e^{\alpha k} c(k) = S_0 e^{-rT} e^{\alpha k} \int_k^\infty (e^x - e^k) p(x) dx$$

- Fourier transform of $e^{\alpha k}\int_k^\infty (e^x - e^k)p(x)dx$

$$
\begin{aligned}
\psi(\xi) &= \int_{\mathbb{R}} e^{i\xi k + \alpha k} \int_k^\infty (e^x - e^k)p(x)dxdk \\
&= \int_{\mathbb{R}} p(x) \int_{-\infty}^x e^{i\xi k + \alpha k}(e^x - e^k)dkdx \\
&= \int_{\mathbb{R}} p(x) \frac{e^{(i\xi+\alpha+1)x}}{(i\xi+\alpha)(i\xi+\alpha+1)}dx \\
&= -\frac{\phi(\xi - i(\alpha+1))}{(\xi - i\alpha)(\xi - i(\alpha+1))}
\end{aligned}
$$

- **Fourier inversion representation** for call option price

$$c(k) = -\frac{1}{2\pi} S_0 e^{-rT-\alpha k} \int_{\mathbb{R}} e^{-i\xi k} \frac{\phi(\xi - i(\alpha+1))}{(\xi - i\alpha)(\xi - i(\alpha+1))} d\xi$$

where $k = \ln(K/S_0), \alpha > 0$

- For put option with strike K and maturity T:

$$p(k) = -\frac{1}{2\pi} S_0 e^{-rT-\alpha k} \int_{\mathbb{R}} e^{-i\xi k} \frac{\phi(\xi - i(\alpha+1))}{(\xi - i\alpha)(\xi - i(\alpha+1))} d\xi$$

where $k = \ln(K/S_0), \alpha < -1$

Discrete approximation

- Trapezoidal scheme with step size h

$$\int_{\mathbb{R}} f(x)dx \approx (\cdots + f(-h) + f(0) + f(h) + \cdots)\, h$$

- Trapezoidal scheme for the inverse Fourier transform integral with step size h

$$c(k) \approx \frac{1}{2\pi} S_0 e^{-rT-\alpha k} \sum_{m=-\infty}^{\infty} e^{-imhk} \psi(mh) h$$

- Surprisingly, trapezoidal scheme has exponentially decaying errors in the above case: error $\sim O(e^{-C/h})$ for some $C > 0$

- Truncate the infinite series for a large enough integer M

$$c(k) \approx \frac{1}{2\pi} S_0 e^{-rT-\alpha k} \sum_{m=-M}^{M-1} e^{-imhk} \psi(mh) h$$

- Computational cost for pricing one call option: $O(M)$
- If interested in pricing $2M$ call options with different strike prices, computational cost is $O(M \ln(M))$ instead of $O(M^2)$
- Make the sum a DFT so that the FFT can be applied

- Price options with $k = \ln(K/S_0) \in \{k_0, k_1, \cdots, k_{2M-1}\}$ where $k_n = k_0 + n\delta, n = 0, \cdots, 2M-1$
- For $n = 0, \cdots, 2M-1$, compute

$$
\begin{aligned}
\sum_{m=-M}^{M-1} e^{-imhk_n}\psi(mh)h &= \sum_{m=-M}^{M-1} e^{-imh(k_0+n\delta)}\psi(mh)h \\
&= \sum_{m=-M}^{M-1} e^{-imnh\delta} e^{-imhk_0}\psi(mh)h \\
&= \sum_{m=0}^{2M-1} e^{-i(m-M)nh\delta} e^{-i(m-M)hk_0}\psi((m-M)h)h \\
&= e^{iMnh\delta+iMhk_0} h \sum_{m=0}^{2M-1} e^{-imnh\delta} e^{-imhk_0}\psi((m-M)h)
\end{aligned}
$$

- FFT can be used if $h\delta = 2\pi/(2M)$, i.e., $\delta = \pi/(Mh)$

- Pricing European options in **Kou's double exponential jump diffusion model**

$$X_t = \mu t + \sigma B_t^* + \sum_{n=1}^{N_t} Z_n$$

N_t - Poisson process with intensity λ, $\{Z_n\}$ - i.i.d. double exponentially distributed jump sizes with pdf

$$p\eta_1 e^{-\eta_1 x}\mathbf{1}_{\{x>0\}} + (1-p)\eta_2 e^{\eta_2 x}\mathbf{1}_{\{x<0\}}$$

- μ determined so that the *discounted* gain process is a martingale

$$\mu = r - q - \frac{1}{2}\sigma^2 + \lambda(1 - \frac{p\eta_1}{\eta_1 - 1} - \frac{(1-p)\eta_2}{\eta_2 + 1})$$

- pdf of X_T very complicated, in terms of special functions. c.f. of X_T

$$\phi(\xi) = \exp\left(i\xi\mu T - \frac{1}{2}\sigma^2 T\xi^2 - \lambda T\left(1 - \frac{p\eta_1}{\eta_1 - i\xi} - \frac{(1-p)\eta_2}{\eta_2 + i\xi}\right)\right)$$

- Price 1-year European call with $S_0 = K = 100$, $r = 0.05$, $q = 0.02$, $\sigma = 0.1$, $\lambda = 3$, $p = 0.3$, $\eta_1 = 40$, $\eta_2 = 12$, $0 < \alpha < \eta_1 - 1$
- Implementation (Matlab, Mathematica, C++)
  - Define ϕ and ψ
  - For fixed α, h, M
  - Compute a single call option price

$$c(k) \approx \frac{1}{2\pi} S_0 e^{-rT-\alpha k} \sum_{m=-M}^{M-1} e^{-imhk} \psi(mh) h$$

  - or compute multiple option prices using the FFT

- Exact price $= 8.877004872829$. For $\alpha = 5$ and $M = 800$

h	Price	Error
5	9.00981014	0.13
4	8.90739428	0.03
3	8.87944905	2E-03
2	8.87701903	1E-05
1	8.87700487	2E-12

Partial differential equation approach

- BSM equation for a path-independent European derivative in the BSM model
- Establish the connection between risk neutral pricing and the PDE via **Feynman-Kac theorem**
- In general, the PDE needs to be numerically solved
- Finite difference methods for solving the PDE

Stochastic differential equations

- A **stochastic differential equation (SDE)** is of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

with **drift** $\mu(t, X_t)$, **diffusion** $\sigma(t, X_t)$, initial condition (suppose the current time is t)

$$X_t = x, \quad x \in \mathbb{R}$$

- Seek a solution $\{X_u, u \geq t\}$ such that $X_t = x$ and

$$X_u = x + \int_t^u \mu(s, X_s)ds + \int_t^u \sigma(s, X_s)dB_s$$

- **Example 1: GBM** in the Black-Scholes-Merton model

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t^*$$

with solution $S_T = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T + \sigma B_T^*\right)$

- **Example 2: Vasicek** interest rate model

$$dR_t = (a - bR_t)dt + \sigma dB_t$$

with solution $R_t = \frac{a}{b} + e^{-bt}(R_0 - \frac{a}{b}) + \sigma e^{-bt} \int_0^t e^{bs} dB_s$

- **Example 3: Hull-White** interest rate model

$$dR_t = (a(t) - b(t)R_t)dt + \sigma(t)dB_t$$

where $a, b, \sigma > 0$ are deterministic functions of time

$$R_t = R_0 e^{-\int_0^t b(u)du} + \int_0^t e^{-\int_u^t b(s)ds} a(u)du + \int_0^t e^{-\int_u^t b(s)ds} \sigma(u)dB_u$$

- **Example 4: CIR** interest rate model

$$dR_t = (a - bR_t)dt + \sigma\sqrt{R_t}dB_t$$

- Generally difficult to solve a SDE. Under mild smoothness and boundedness conditions for the drift and diffusion, there exists a unique solution to the SDE

Simulate a SDE

- Expectation of $f(X_T)$ when X_T is the solution of the SDE with initial condition $X_t = x$

$$g(t,x) = \mathbb{E}_{t,x}[f(X_T)]$$

- Monte carlo simulation: divide $[t, T]$ into subintervals with length δ. Start with $X_t = x$,

$$X_{t+\delta} = x + \mu(t,x)\delta + \sigma(t,x)\sqrt{\delta}Z_1,$$

$$X_{t+2\delta} = X_{t+\delta} + \mu(t+\delta, X_{t+\delta})\delta + \sigma(t+\delta, X_{t+\delta})\sqrt{\delta}Z_2,$$

Continue until one gets a realization of $f(X_T)$. Then repeat.

Markov property

- The solution X_t to the SDE is a **Markov process**: for $t < T$,

$$\mathbb{E}[f(X_T)|\mathcal{F}_t] = g(t, X_t)$$

where

$$g(t, x) = \mathbb{E}_{t,x}[f(X_T)]$$

The future of the process only depends on the current value of the process, not on its history

Feynman-Kac theorem

- **Feynman-Kac Theorem.** *suppose* X_t *solves*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

and

$$g(t, x) = \mathbb{E}_{t,x}[f(X_T)].$$

Then $g(t, x)$ *solves the following PDE*

$$\frac{\partial g}{\partial t} + \mu(t, x)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma(t, x)^2\frac{\partial^2 g}{\partial x^2} = 0$$

with terminal condition $g(T, x) = f(x)$.

- Compared to monte carlo simulation, numerical method for solving the PDE
  - converges faster
  - produces multiple values of $g(t,x)$ simultaneously
- Proof of the Feynman-Kac Theorem
  - show that $g(t,X_t)$ is a martingale
  - $g(t,X_t)$ has zero drift
- $g(t,X_t)$ is a martingale: for $s \leq t$,

$$g(s,X_s) = \mathbb{E}[f(X_T)|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[f(X_T)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[g(t,X_t)|\mathcal{F}_s]$$

- $g(t, X_t)$ satisfies

$$
\begin{aligned}
dg &= \dot{g}\,dt + g' dX_t + \frac{1}{2} g'' dX_t \cdot dX_t \\
&= \dot{g}\,dt + g'(\mu dt + \sigma dB_t) + \frac{1}{2} g'' \sigma^2 dt \\
&= \left[\dot{g} + \mu g' + \frac{1}{2}\sigma^2 g''\right] dt + \sigma g' dB_t
\end{aligned}
$$

- zero drift implies that

$$
\frac{\partial g}{\partial t} + \mu(t,x)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 g}{\partial x^2} = 0
$$

- **Feynman-Kac Theorem for Derivatives Pricing.** *suppose X_t solves*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

and

$$g(t, x) = \mathbb{E}_{t,x}[e^{-r(T-t)}f(X_T)].$$

Then $g(t, x)$ solves the following PDE

$$\frac{\partial g}{\partial t} + \mu(t, x)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma(t, x)^2\frac{\partial^2 g}{\partial x^2} - rg(t, x) = 0$$

with terminal condition $g(T, x) = f(x)$.

- $e^{-rt}g(t, X_t)$ is martingale: for $s \leq t$,

$$\begin{aligned} e^{-rs}g(s, X_s) &= \mathbb{E}[e^{-rT}f(X_T)|\mathcal{F}_s] \\ &= \mathbb{E}[\mathbb{E}[e^{-rT}f(X_T)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[e^{-rt}g(t, X_t)|\mathcal{F}_s] \end{aligned}$$

- $e^{-rt}g(t, X_t)$ satisfies

$$\begin{aligned} d(e^{-rt}g) &= e^{-rt}dg + gde^{-rt} + 0 \\ &= e^{-rt}[dg - rgdt] \end{aligned}$$

zero drift implies

$$\frac{\partial g}{\partial t} + \mu(t, x)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 g}{\partial x^2} - rg = 0$$

BSM equation

- Path-independent European derivatives with payoff $V(S_T)$ in the BSM model

$$V(t, S_t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} V(S_T) | \mathcal{F}_t]$$

where

$$dS_t = (r - q) S_t dt + \sigma S_t dB_t^*$$

- BSM equation by the Feynman-Kac theorem

$$\frac{\partial V}{\partial t} + (r - q) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

CEV model

- **Example:** the (**Constant elasticity of variance (CEV)**) model

$$dS_t = (r-q)S_t dt + \sigma S_t^{\beta+1} dB_t^*,$$

- Allows stochastic volatility σS_t^{β}
- Under the CEV model, the value function of a European path-independent derivative solves

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^{2(\beta+1)}\frac{\partial^2 V}{\partial S^2} - rV = 0$$

- Analytical solution available for vanilla options in terms of noncentral χ^2 distributions

2-d stochastic differential equations

- 2-dimensional SDEs

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} dB_{1t} \\ dB_{2t} \end{pmatrix}$$

  where (B_{1t}, B_{2t}) is a 2-d BM, and $\mu_1, \mu_2, \sigma_{ij}$ are functions of t, X_t, Y_t

- The solution (X_t, Y_t) of the 2-d SDE is a Markov process: for $t < T$,

$$\mathbb{E}[f(X_T, Y_T)|\mathcal{F}_t] = g(t, X_t, Y_t)$$

  where

$$g(t, x, y) = \mathbb{E}_{t,x,y}[f(X_T, Y_T)]$$

2-d Feynman-Kac theorem

- **2-d Feynman-Kac Theorem for Derivatives Pricing:** suppose (X_t, Y_t) solves the previous 2-d SDE and

$$g(t,x,y) = \mathbb{E}_{t,x,y}[e^{-r(T-t)} f(X_T, Y_T)].$$

Then $g(t,x,y)$ solves

$$\frac{\partial g}{\partial t} + \mu_1 \frac{\partial g}{\partial x} + \mu_2 \frac{\partial g}{\partial y} + \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)\frac{\partial^2 g}{\partial x^2}$$

$$+(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})\frac{\partial^2 g}{\partial x \partial y} + \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2)\frac{\partial^2 g}{\partial y^2} - rg = 0$$

Example: pricing and hedging Asian options

- Consider an Asian call in the BSM model with payoff

$$V_T = (\bar{S}_T - K)^+, \quad \text{where } \bar{S}_T = \frac{1}{T}\int_0^T S_t dt$$

- Introduce new *variable*

$$Y_t = \int_0^t S_u du, \quad \text{or equivalently,} \quad dY_t = S_t dt$$

- The solution (S_t, Y_t) of the following 2-d SDE is Markov

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sigma S_t dB_t^*, \\ dY_t &= S_t dt \end{aligned}$$

- The value of the Asian option $V(t, S_t, Y_t)$

$$V(t, S_t, Y_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \left(\frac{1}{T} Y_T - K \right)^+ \middle| \mathcal{F}_t \right]$$

- By 2-d Feynman-Kac theorem $(\mu_1 = (r-q)S, \mu_2 = S, \sigma_{11} = \sigma S)$, V solves

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial Y} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

subject to terminal condition

$$V(T, S, Y) = (Y/T - K)^+$$

- For hedging: the dynamics of $V(t, S_t, Y_t)$

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{\partial V}{\partial Y}dY_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS_t \cdot dS_t \\
&\quad + \frac{\partial^2 V}{\partial S \partial Y}dS_t \cdot dY_t + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}dY_t \cdot dY_t \\
&= \left(\frac{\partial V}{\partial t} + (r-q)S_t\frac{\partial V}{\partial S} + S_t\frac{\partial V}{\partial Y} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right)dt \\
&\quad + \sigma S_t \frac{\partial V}{\partial S}dB_t^* \\
&= rVdt + \sigma S_t \frac{\partial V}{\partial S}dB_t^*
\end{aligned}
$$

- The dynamics of the discounted value $e^{-rt}V(t, S_t, Y_t)$

$$\begin{aligned} d(e^{-rt}V) &= e^{-rt}dV + Vde^{-rt} \\ &= e^{-rt}(dV - rVdt) \\ &= e^{-rt}\sigma S_t \frac{\partial V}{\partial S} dB_t^* \end{aligned}$$

- To hedge a short position on an Asian call, a replicating portfolio satisfies

$$d(e^{-rt}X_t) = e^{-rt}\Delta_t \sigma S_t dB_t^*$$

so at time t, one should hold the following amount of shares

$$\Delta_t = \frac{\partial V}{\partial S}(t, S_t, Y_t)$$

Finite difference method

- Popular numerical methods: **finite difference method (FDM)**, **finite element method (FEM)**
- Consider the BSM equation

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

with terminal condition

$$V(T,S) = V_T(S), \quad S \in (0,\infty)$$

- Change of variable:

$$x = \ln(S), \quad f(t,x) = V(T-t, e^x)$$

Time variable t in $f(t,x)$ is option **time to maturity**

$$\frac{\partial f}{\partial t} = -\frac{\partial V}{\partial t}, \quad \frac{\partial f}{\partial x} = e^x \frac{\partial V}{\partial S} = S\frac{\partial V}{\partial S},$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \frac{\partial V}{\partial S} + e^{2x}\frac{\partial V^2}{\partial S^2} = \frac{\partial f}{\partial x} + S^2 \frac{\partial V^2}{\partial S^2}$$

- f solves the following PDE

$$-\frac{\partial f}{\partial t} + (r - q - \frac{1}{2}\sigma^2)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} - rf = 0$$

  with initial condition

$$f(0, x) = V_T(e^x), \quad x \in \mathbb{R}$$

  E.g., for a vanilla call, $f(0, x) = (e^x - K)^+$
- *The above PDE can further be reduced to the form of a heat equation:* $f_t = f_{xx}$.

- **Localization** when necessary: solve the PDE on a large enough computational domain $[x_{min}, x_{max}]$
- Imposing **boundary conditions** for $f(t, x_0)$ and $f(t, x_{M+1})$ when necessary (usually can be determined from the properties of the derivative)
- **Spatial discretization**: divide $[x_{min}, x_{max}]$ into $M+1$ equal intervals

$$\Delta x = \frac{x_{max} - x_{min}}{M+1}, \quad x_i = i\Delta x, \quad i = 0, 1, \cdots, M+1$$

- Denote the following vector ($\mathbf{f}_i(t) = f(t, x_i)$, $0 \le i \le M+1$)

$$\mathbf{f}(t) = \begin{pmatrix} f(t, x_1) \\ \vdots \\ f(t, x_M) \end{pmatrix}$$

- **Central difference** scheme for spatial discretization:

$$\frac{\partial f}{\partial x}(t, x_i) \leftarrow \frac{f(t, x_{i+1}) - f(t, x_{i-1})}{2\Delta x} = \frac{\mathbf{f}_{i+1}(t) - \mathbf{f}_{i-1}(t)}{2\Delta x}$$

$$\frac{\partial^2 f}{\partial x^2}(t, x_i) \leftarrow \frac{f(t, x_{i+1}) - 2f(t, x_i) + f(t, x_{i-1})}{\Delta x^2}$$

$$= \frac{\mathbf{f}_{i+1}(t) - 2\mathbf{f}_i(t) + \mathbf{f}_{i-1}(t)}{\Delta x^2}$$

- This scheme is second order accurate in space with error $O(\Delta x^2)$.
- Substitute the above into the PDE,

$$-\frac{\partial \mathbf{f}_i}{\partial t}+\frac{\mu}{2\Delta x}(\mathbf{f}_{i+1}-\mathbf{f}_{i-1})+\frac{\sigma^2}{2\Delta x^2}(\mathbf{f}_{i+1}-2\mathbf{f}_i+\mathbf{f}_{i-1})-r\mathbf{f}_i=0$$

where $\mu = r - q - \frac{1}{2}\sigma^2$. That is,

$$-\frac{\partial \mathbf{f}_i}{\partial t}+\underbrace{\left(\frac{\sigma^2}{2\Delta x^2}-\frac{\mu}{2\Delta x}\right)}_{a}\mathbf{f}_{i-1}-\underbrace{\left(r+\frac{\sigma^2}{\Delta x^2}\right)}_{b}\mathbf{f}_i+\underbrace{\left(\frac{\sigma^2}{2\Delta x^2}+\frac{\mu}{2\Delta x}\right)}_{c}\mathbf{f}_{i+1}=0$$

- Obtain

$$-\frac{\partial \mathbf{f}_i}{\partial t} + a\mathbf{f}_{i-1} - b\mathbf{f}_i + c\mathbf{f}_{i+1} = 0$$

- Spatial discretization in matrix form

$$\frac{\partial \mathbf{f}}{\partial t} + \mathbb{A}\mathbf{f} + \mathbf{F} = 0$$

where

$$\mathbb{A} = \begin{pmatrix} b & -c & & \\ -a & b & \ddots & \\ & \ddots & \ddots & -c \\ & & -a & b \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} -af_0 \\ 0 \\ \vdots \\ 0 \\ -cf_{M+1} \end{pmatrix}$$

- We obtain the following system of ODEs

$$\frac{\partial \mathbf{f}}{\partial t} + \mathbb{A}\mathbf{f} + \mathbf{F} = 0$$

with known initial condition $\mathbf{f}(0)$

- **Time discretization**: divide time interval $[0, T]$ into N equal intervals, each with length $\Delta t = T/N$. Let $t_n = n\Delta t$, $n = 0, \cdots, N$. Denote

$$\mathbf{f}^n = \mathbf{f}(t_n) = \begin{pmatrix} f(t_n, x_1) \\ \vdots \\ f(t_n, x_m) \end{pmatrix}, \quad n = 0, \cdots, N$$

$\mathbf{f}^0$ is known from the initial condition

- **Explicit (forward) Euler scheme**

$$\frac{\partial \mathbf{f}}{\partial t}(t_n) = \frac{\mathbf{f}(t_{n+1}) - \mathbf{f}(t_n)}{\Delta t} = \frac{\mathbf{f}^{n+1} - \mathbf{f}^n}{\Delta t}$$

  substitute into the ODE ($n = 0, \cdots, N-1$)

$$\frac{\mathbf{f}^{n+1} - \mathbf{f}^n}{\Delta t} + \mathbb{A}\mathbf{f}^n + \mathbf{F} = 0 \Rightarrow \mathbf{f}^{n+1} = (I - \Delta t \mathbb{A})\mathbf{f}^n - \Delta t \mathbf{F},$$

  where I is the $M \times M$ identity matrix

- Involves matrix vector multiplication only; first order accurate in time: $O(\Delta t)$; **conditionally stable**: requires $\Delta t < c\Delta x^2$

- **Implicit (backward) Euler scheme**

$$\frac{\partial \mathbf{f}}{\partial t}(t_n) = \frac{\mathbf{f}(t_n) - \mathbf{f}(t_{n-1})}{\Delta t} = \frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\Delta t}$$

substitute into the ODE ($n = 1, \cdots, N$)

$$\frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\Delta t} + \mathbb{A}\mathbf{f}^n + \mathbf{F} = 0 \Rightarrow (I + \Delta t \mathbb{A})\mathbf{f}^n = \mathbf{f}^{n-1} - \Delta t \mathbf{F},$$

- At each time step, a linear system of equations needs to be solved (tri-diagonal systems can be solved easily); **unconditionally stable**; first order accurate in time: $O(\Delta t)$

- **Crank-Nicolson scheme**

$$\text{Explicit}: \quad \mathbf{f}^{n+1} = (I - \Delta t \mathbb{A})\mathbf{f}^n - \Delta t \mathbf{F}$$

$$\text{Implicit}: \quad (I + \Delta t \mathbb{A})\mathbf{f}^{n+1} = \mathbf{f}^n - \Delta t \mathbf{F}$$

$$\text{Crank-Nicolson}: \quad (I + \frac{1}{2}\Delta t \mathbb{A})\mathbf{f}^{n+1} = (I - \frac{1}{2}\Delta t \mathbb{A})\mathbf{f}^n - \Delta t \mathbf{F}$$

- Matrix vector multiplication and solving a linear system of equations needed at each time step; **second order accurate in time**: $O(\Delta t^2)$; **unconditionally stable**

- **Extrapolation** based on implicit Euler scheme: error expansion of the implicit Euler scheme ($\mathbf{f}(T)$*: exact solution to the ODE,* $\mathbf{f}^N$*: numerical solution with* N *time steps*)

$$\mathbf{f}(T) - \mathbf{f}^N = c_1 \Delta t + c_2 \Delta t^2 + \cdots$$

From two approximations $\mathbf{f}^{N_1}$ and $\mathbf{f}^{N_2}$, eliminate the first order terms

$$\mathbf{f}(T) - \mathbf{f}^{N_1} = c_1 \Delta t_1 + c_2 \Delta t_1^2 + \cdots$$

$$\mathbf{f}(T) - \mathbf{f}^{N_2} = c_1 \Delta t_2 + c_2 \Delta t_2^2 + \cdots$$

where $t_1 = T/N_1$, $t_2 = T/N_2$. Then

$$\mathbf{f}(T) - \frac{N_2 \mathbf{f}^{N_2} - N_1 \mathbf{f}^{N_1}}{N_2 - N_1} = -c_2 \Delta t_1 \Delta t_2 + \cdots$$

- In general, if we obtain approximations $\mathbf{f}_{11}, \mathbf{f}_{21}, \cdots, \mathbf{f}_{k1}$ using number of time steps $N_1, N_2, \cdots, N_k$, respectively, we can obtain an approximation with error $O(\Delta t_1 \cdots \Delta t_k)$

time steps	implicit Euler approximations	*extrapolation*		
$N_1 = 1$	$\mathbf{f}_{11}$			
$N_2 = 2$	$\mathbf{f}_{21}$	$\mathbf{f}_{22}$		
$N_3 = 3$	$\mathbf{f}_{31}$	$\mathbf{f}_{32}$	$\mathbf{f}_{33}$	
$N_4 = 4$	$\mathbf{f}_{41}$	$\mathbf{f}_{42}$	$\mathbf{f}_{43}$	$\mathbf{f}_{44}$
$Total = 10$				

- **Example**: pricing double barrier put in Kou's jump diffusion model (Implicit Euler: 873 sec, Extrapolation: 0.05 sec)