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Lecture Note 8.1: Structural Models of Credit Risk

We now turn our attention to debt markets. We will spend much of the remainder of the term exploring products whose underlying source of risk is the survival/default of risky borrowers.

This lecture explores the insight we can gain through applying continuous-time no-arbitrage methodology to the problem of valuing different parts of a firm's capital structure. As financial engineers, we need models like this in order to design securities for firms, as well as to trade and hedge them.

The premise of the models is that all a firm's liabilities – bonds, stock, warrants, pension liabilities, or anything else –can be viewed as derivatives whose underlying is the assets to which they are claims. This gives us a powerful method for approaching credit risk.

Even though the models we will see are simplifications, they are still very useful in practice. Moreover, this is an active area of research, in which better models are still being developed.

Outline:

I. Corporate Debt Markets.
II. Structural Models of Credit Risk
III. Equity-based Models
IV. Improving the Models
V. Summary

I. Corporate Debt Markets.

- This is supposed to be a course about financial derivatives, but it is also a course about how real world markets work.
  - ▶ So let's just spend a minute talking about corporate bonds.

- There are several trillion dollars in publicly traded bonds in the world's major economies. Mostly they are quite *illiquid*.
  - ▶ When bonds are issued they are usually sold to passive funds and insurance companies, some of whom literally never sell them.
  - ▶ The average corporate bond in the U.S. trades something like once a week!
  - ▶ Not much scope for high-frequency trading here.

- The market for corporate debt is mostly over the counter (OTC) even though bonds are sometimes also listed on exchanges for regulatory reasons.
  - ▶ Like all OTC markets, they are not very transparent.
  - ▶ In the U.S., there is a requirement to report trades (within 15 minutes) to a central database called TRACE which the public can see.
    - ∗ As far as I know, no other country has anything like this.
  - ▶ There are also a few electronic platforms; not formally SEFs, but the same idea.

- ▶ Bonds are electronically cleared through the Depository Trust Corporation (DTC) or Euroclear.

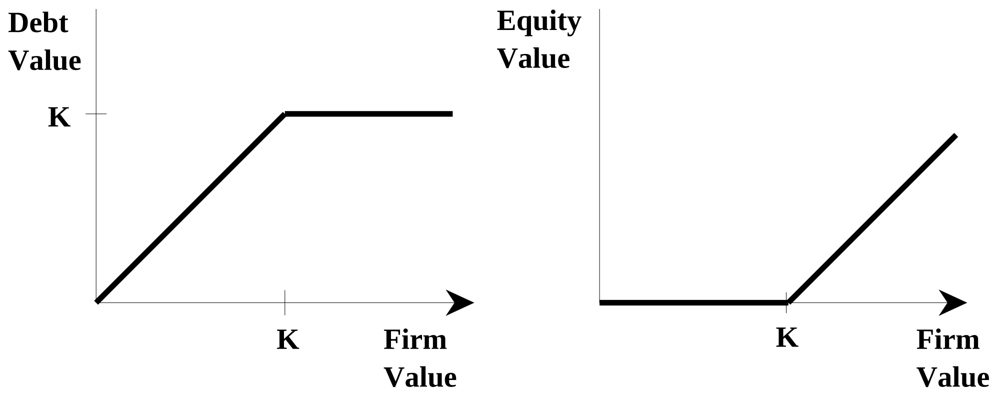
- Bonds prices are almost always quoted as a percentage of their face amount (or par).
  - ▶ The amount of money you actually pay, though, if you buy a bond is the trade amount *plus* the amount of coupon interest accrued since the last payment date.
    - ∗ In the U.S. most bonds pay fixed coupons semi-annually. However, because of this market convention, investors treat the coupon obligation as legally accruing continuously.
  - ▶ If, however, market participant think the next coupon is unlikely to be paid (i.e., for a company about to default), then they drop this convention and prices are quoted "flat".

- The rules for short-selling bonds are similar to those for stocks.
  - ▶ You have to find someone to lend it to you first, and agree on a fee.
  - ▶ You typically have to pay them the accrued interest too, like a dividend.
  - ▶ And you have to return it whenever they want it back.

- Most corporate debt is actually not in the form of bonds; it is just bank lending.
- However, a lot of debt that starts out as a bank loan *does* get later traded on a secondary market.
  - ▶ This is almost entirely an institutional market, and there is no public reporting of anything.
  - ▶ Holders of bank loans have access to inside information on the borrowers health. And they are not supposed to be allowed to trade based on this information....
- The contractual terms of every bond (and loan) are described in a *prospectus* or *indenture*.
  - ▶ If the bonds are sold to the public, then the same information must be disclosed to the SEC prior to the sale.
  - ▶ The prospectus can often be 50 pages long or more. You have to read the whole thing to really know what you are trading. Every one is different.
  - ▶ Besides the coupon and maturity, the most important thing is the stated seniority.
    - ∗ Bonds typically call themselves either *senior* or *junior*.
    - ∗ In addition, they may be *secured* – meaning that they have an explicit right to a certain physical assets of the company if it defaults – or *unsecured*.

II. Structural Models of Credit Risk

- I'm sure you have all learned in corporate finance that one can think of a firm's equity as a call option on the value of its assets. This picture probably looks familiar:



- The idea is: the firm has debts of face value K, and at their maturity, T, stock holders get $\max[V - K, 0]$ where V is the value of the underlying assets. Bond holders get the rest.

- This is called the **Merton model** of risky debt.

- As simple as the set-up is, it gives us the power to actually give specific, quantitative answers to questions about the co-determination of bond and stock prices for such an idealized firm.

- For example: it is well known that increasing V's volatility transfers value from bond holders to stock holders.
  - ▶ The model tells us <u>how much</u>.

- Of course, we wouldn't expect the predictions to apply exactly to any real firm. But the model suggests how such relationships work across firms generally.
  - ▶ It is also the starting point for learning how to handle more complex capital structures.

- I want to focus now on the specific topic of credit spreads, i.e. what determines the relative value of corporate bonds. The credit spread is just defined as the difference between a risky bond's yield-to-maturity and that of an equivalent riskless one.

- Let's set the basic assumptions we are going to use throughout the lecture. We are in a world with **perfect markets** and **continuous trading.** But some of the details now need to be clarified.

  1. It is now the total value of the firm's assets that evolves according to the standard geometric Brownian motion

  $$\frac{dV}{V} = (\mu - \Pi)\, dt \ + \ \sigma\, dW.$$

  We are taking this specification as given. In particular:

  - ▶ σ is the volatility of the underlying asset value.

  - ▶ Also we assume we know the total payout rate to all holders of claims to V – denoted Π. This is the sum of all coupons, dividends, and net new financings that the company will engage in – modeled as a constant percentage of asset value.

2. There are no taxes or reorganization costs. We don't actually need this, but when it holds we have **The Modigliani-Miller Theorem.**

Recap: *The market value of the firm is independent of its capital structure and depends only on the cash-flows from the underlying assets.*

The Argument: If there were any particular best capital structure, and the firm didn't have it, somebody would just buy up all its securities and finance it with the better structure – thus making arbitrage profits. Since anybody could do this, the market prices of the existing claims will be bid up to the point where arbitrage is impossible, which means the value of the firm with the original structure is just the same as with the best one.

As a consequence, V equals the sum of the prices of all the firm's claims.

3. The firm won't do any unanticipated financing between now and T. We need this because if they issued more claims it could change how the firm's value is divided up at T.

▶ Also note that this assumption means we know the firm's payout policy won't/can't change. The Merton model assumes no dividends.

4. All contracts are enforcable and cannot be renegotiated. This rules out, for example, strategic bankruptcy filings.

- Before going on, we should note an interesting fact: we do not have to assume the firm's underlying assets are somehow themselves tradable in the market.

- We have seen that we can value derivatives on non-traded risk factors.

- In fact, if we want to go a level deeper, we can *solve for* V!
  - To illustrate, suppose the earnings stream from a firm's assets obey
$$\frac{dE}{E} = \mu_E \, dt + \sigma_E \, dW.$$
  - What is the value of the rights to receive E forever?
  - We know that as long as markets are complete with respect to E risk, that $V(E)$ obeys our canonical PDE. Hence we know from Feynman-Kac that the solution is
$$\mathrm{E}_t^Q[\int_t^\infty DF_{t,s} \cdot E_s \, ds].$$
  - We know the risk-neutralized law of E is
$$\frac{dE}{E} = [\mu_E - \lambda^E \sigma_E] \, dt + \sigma_E \, dW.$$
  - Let's make our life simple and let the riskless rate be constant. Then
$$V(E) = \mathrm{E}_t^Q[\int_t^\infty e^{-r(s-t)} \cdot E_s \, ds].$$

▶ If the integral converges and we can apply Fubini, then $V(E) =$

$$\int_t^{\infty} e^{-r(s-t)} \mathrm{E}_t^Q[E_s]\, ds = \int_t^{\infty} e^{-r(s-t)} E_t e^{[\mu_E - \lambda^E \sigma_E](s-t)}\, ds.$$

Or

$$V(E) = \frac{E_t}{r - [\mu_E - \lambda^E \sigma_E]}$$

- Pretty cool.
  - ▶ Our methodology lets us value arbitrary cash-flow streams!

- Notice that E_t is also the cash-flow available to pay all the firms claims, and it is a constant fraction of V as we assumed above:

$$\Pi \equiv \frac{E}{V} = r - [\mu_E - \lambda^E \sigma_E].$$

- Also, what is the risk-neutral drift of V?
  - ▶ Well $dV/V = dE/E$ so both processes have risk neutral drift of $(\mu_E - \lambda^E \sigma_E)$.
  - ▶ But note that this is also equal to $(r - \Pi)$. As for a traded asset, it's just the riskless rate minus the payout yield!

- Hence it is without loss of generality to assume that we start with V whether or not the assets are literally traded.

- Returning to our development, if we assume V is the sole source of risk, then, with our assumptions, suppose we are given an arbitrary claim, F, whose cash-flows depend on V Let's let F have a payout per unit time (a coupon rate) of Γ.

- As we know, the continuous-time no-arbitrage argument then implies that F obeys:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} + \frac{\partial F}{\partial t} - rF + (r - \Pi)V \frac{\partial F}{\partial V} + \Gamma = 0. \tag{1}$$

- Now all we have to do is specify the boundary conditions that characterize any specific security, and then solve (1) subject to them.

- Let's look at some specific examples.

(A) Credit spreads in the Merton model.

- Formally, in terms of the framework we just set up, Π and Γ are both zero, because the bonds are zero-coupon and the firm doesn't pay dividends. The debt has face value $F^\star$ and maturity T and the boundary conditions are

$$\begin{aligned} F_T(V) &= \min[V, F^\star] \\ F_t(V = 0) &= 0 \quad \text{for all } t < T \\ F_t(V = \infty) &= F^\star \cdot e^{-r(T-t)} \quad \text{for all } t < T \end{aligned}$$

- For this model, we don't need to re-solve the PDE because we already know what the solution is.

- If the equity value is $c(V)$, then $F = V - c$. Since c is just given by the Black-Scholes formula with V being the underlying and $F^\star$ being the strike.

- Rerranging the terms slightly shows that $V - c$ is

$$F^\star e^{-r(T-t)} \left\{ \mathcal{N}(d_2) + \frac{1}{d}\mathcal{N}(-d_1) \right\}$$

where $d \equiv F^\star e^{-r(T-t)}/V$ is the "leverage" of the firm (and has nothing to do with d_1, d_2).

- The continuously compounded **yield-to-maturity**, y_τ, (where $\tau \equiv T - t$) is always defined for a zero coupon bond by the equation

$$e^{-y_\tau\ \tau} = \frac{F}{F^\star} \quad \text{hence} \quad y_\tau = -\frac{1}{\tau}\log(\frac{F}{F^\star}).$$

Example: A two year discount bond whose price is 87 (per 100 face value) has y-t-m

$$-(1/2)\log(0.87) = -0.5 \times -.139 = 0.0696 \approx 7\% \text{ per year.}$$

- With our formulas, we can calculate the yield spread in this model

$$y_\tau - r = -\frac{1}{\tau}\log[\mathcal{N}(d_2) + \frac{1}{d}\mathcal{N}(-d_1)].$$

Example: Find the credit spread on a five-year bond on a firm whose assets have 20% volatility, when the risk-free rate is 6% and the firm's assets are worth twice the face value of the debt.

- First, notice that we don't care what V and $F^\star$ are, We only need their ratio.

$$d = \frac{F^\star}{V} \cdot e^{-0.06 \cdot 5} = 0.5 \cdot e^{-0.06 \cdot 5} = 0.370$$

because we are given that $V = 2F^\star$.

- Let's suppose $F^\star = 100$. This will give us the answer as a percent of the bond's face value, which is how bonds are really quoted.

- We just use the regular formulas for d_1 and d_2.

$$d_1 = \frac{\log\left(\frac{V}{F^\star e^{-r(T-t)}}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$

$$= \frac{\log\frac{1}{0.37} + 0.5\,(0.2^2)\,5}{0.2\,\sqrt{5}} = 2.44$$

Likewise

$$d_2 = \frac{\log\frac{1}{0.37} - 0.5\,(0.2^2)\,5}{0.2\,\sqrt{5}} = 2.00$$

- So the bond is worth

$$100e^{-0.06 \cdot 5}(\mathcal{N}(2.00) + (1/.37)\mathcal{N}(-2.44)) = 73.84\%$$

- This gives a credit spread of

$-(1/5) \cdot \log(.7384) - 0.06 = .0607 - .06 = 0.0007 =$ 7 basis points

- These are from Merton's 1974 paper

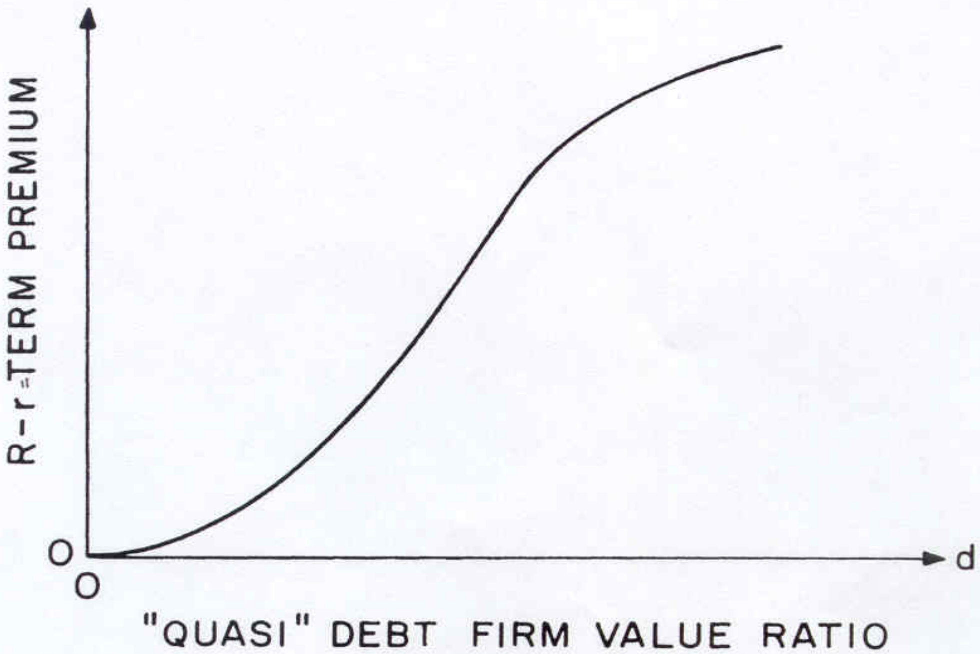


FIGURE 1

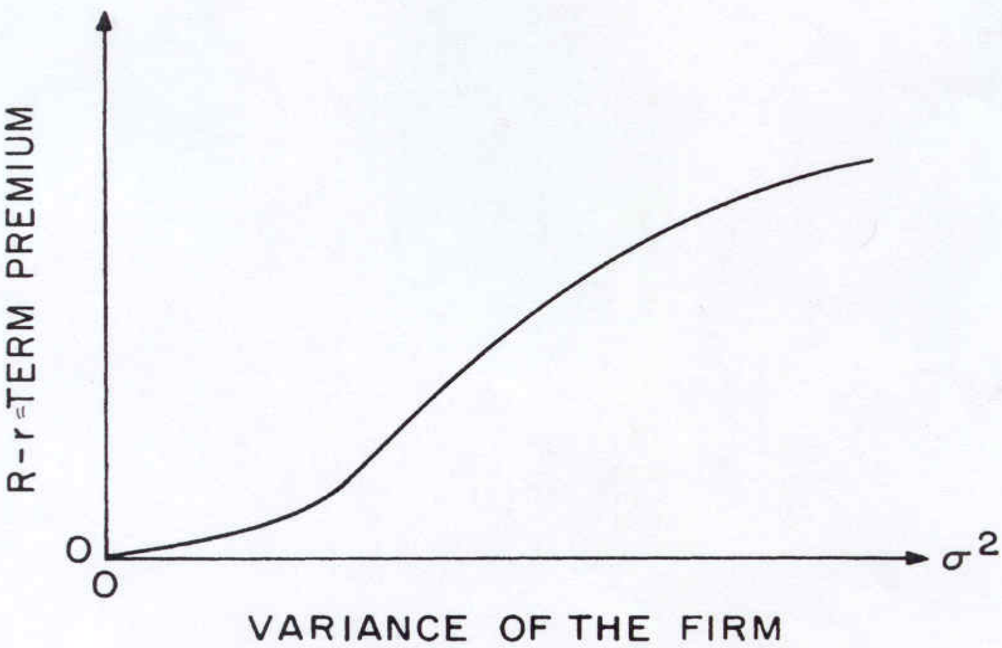


FIGURE 2

The Pricing of Corporate Debt 131

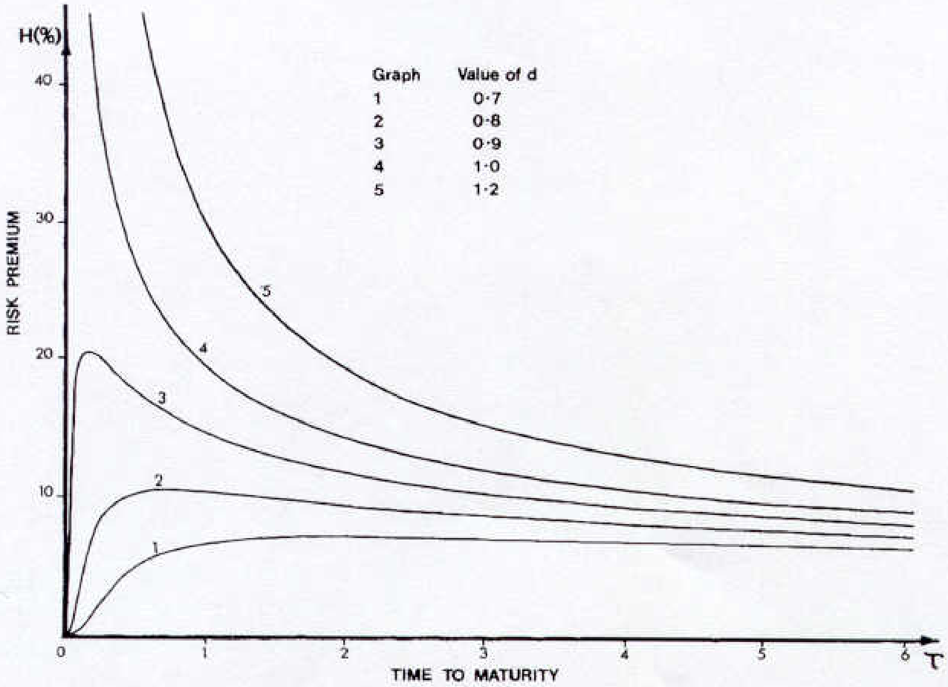


Figure 1. Risk premium as a function of time to maturity.

- This graph, which shows the <u>term structure of credit spreads</u>, is the most interesting one. It tells us that there are two very distinct cases in this model.
  - ▶ For highly levered firms ($d \geq 1$), the costs of short-term debt are enormous, but decline rapidly with maturity.
  - ▶ For firms with not much debt ($d < 1$), long-term debt is *more* expensive.
- Here firm volatility was set to 40%.

- The Merton model introduces a very fundamental idea that other researchers have built on:

  We can model credit risk with no-arbitrage methods.

- This approach to credit is called **structural modeling.**
  - ▶ It is "structural" in the sense that attempts to model default in terms of the structure of each firm's assets and liabilities.
  - ▶ It is often contrasted with "reduced form" models, which extract market odds on default from similarly rated securities, and use these to value a firm's bonds.
    - ∗ This approach can be very useful too. But it doesn't explicitly link valuation to arbitrage.

- Notice that one can use the Merton model in practice despite the fact that the two most important inputs to the model, V_t and σ_V, are unobservable. **How?**
  - ▶ From the firm's stock price, we can immediately compute the market value of equity, S.
  - ▶ Moreover <u>its</u> volatility, σ_S, can be estimated from historical stock returns or option implied volatilities.
  - ▶ We also have formulas for what both of these quantities are, in terms of the model's other inputs.

$$S(V, \sigma_V) = V\, \mathcal{N}(d_1) - e^{-r(T-t)}\, F^\star\, \mathcal{N}(d_2) \qquad (2)$$

$$\sigma_S(V, \sigma_V) = \frac{\partial S(V)}{\partial V}\, \frac{V}{S(V)}\, \sigma_V \qquad (3)$$

- ▶ The second equation just follows from applying Ito's lemma to the first equation. (And the first term on the right in the second equation is just the Black-Scholes delta, of course.)
- ▶ Given values for the quantities on the left, we have two equations in two unknowns.
- ▶ Although they are highly nonlinear, (remember V appears inside d_1 and d_2!) we can always solve them for V and σ_V with the help of a computer.

(B) Other Bankruptcy Assumptions.

- A wide range of structural models have been developed with the goal of trying to relax as many as possible of the unrealistic assumptions in the Merton model.
- It turns out the most important part of a structural specification is how it models default events.

  How is default triggered?

  How much is recovered upon default?

- In the Merton model, creditors recover all of V, and default can only happen at the one time horizon T.

- This is not a very realistic depiction of bankruptcy.
  - ▶ It means the firm can become massively insolvent $(V < F^{\star})$ and the creditors can't do anything. Management gets to stay in charge, and the equity holders get to keep their call option on V.

  - ▶ Conversely, if there were any coupon interest on the debt, owners might *want* to declare bankruptcy if asset values deteriorated a lot.

- Improving this picture was an early priority in the structural modeling game.

- An alternative assumption would be that at some low level $V = V_B$ the firm is put out of its misery and the creditors (bond holders) take control and divide the remaining assets.
  - ▶ Black and Cox (1976) modified the basic Merton model to include such a *bankruptcy barrier*.

  - ▶ A similar, more flexible version was developed by **Longstaff and Schwartz** in 1995.

- Now, the idea is bond holders enforce a <u>protective covenant</u>, and liquidate the firm before T if assets lose too much value.

- Notice that now the bankruptcy event is not defined by reference to any one specific debt issue.
  - ▶ This makes it possible to incorporate more complex capital structures than allowed by Merton, including coupon debt.

- Formally, we solve our PDE subject to the new boundary conditions.

$$
\begin{aligned}
F(T, V) &= F^{\star}, \quad V > V_B(T) \\
F(t, V_B(t)) &= (1 - W)F^{\star}, \quad t \leq T
\end{aligned}
$$

- Here W is the "writedown" due to the costs of bankruptcy.
  - ▶ Different bonds of the same issuer can have different W, of course, reflecting their seniority.
  - ▶ Even though the law stipulates "absolute priority" of creditor classes, in practice it rarely happens that way.

- A simple choice is just to have V_B be a constant.

- Unlike Merton, assets will never be liquidated to pay off maturing bonds. So the implicit assumption is that the firm can re-finance maturing issues as long as $V > V_B$.

- Here's what credit spreads look like as a function of

  $X = V_t/V_B =$ assets as a fraction of the lower bound (i.e. how much they can still lose before defaulting), and

  $W =$ the percent of the face value that will be lost in bankruptcy.

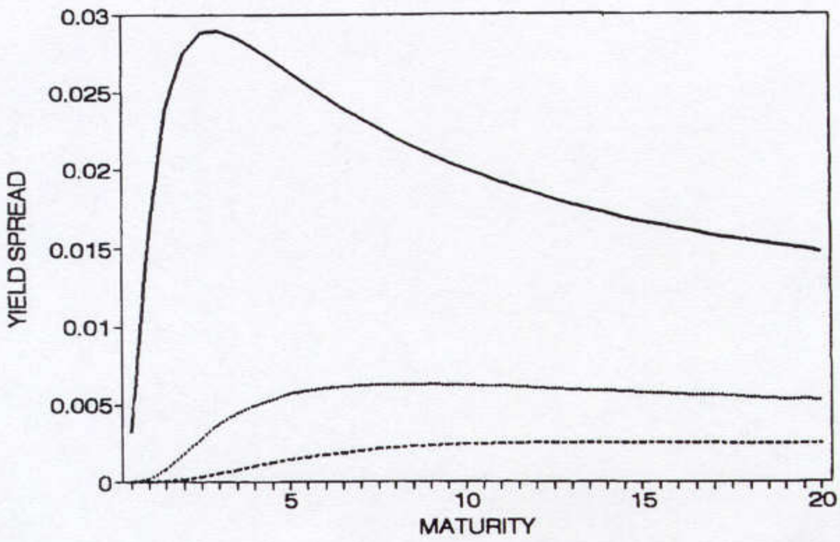


Figure 2. Credit spreads for an 8 percent bond for different values of *X*. The parameter values used are $r = 0.04$, $w = 0.5$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

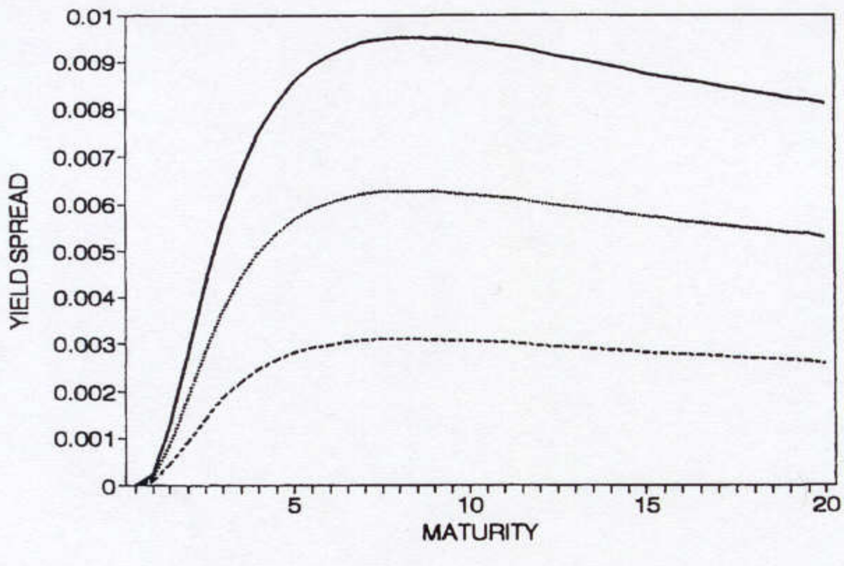


Figure 3. Credit spreads for an 8 percent bond for different values of *w*. The parameter values used are $X = 2.0$, $r = 0.04$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

- The credit spreads the model predicts actually do agree pretty well with observed market levels.
  - As a typical practice, one might calibrate V_B to match the true default probability to a particular horizon, as proxied by the firm's credit rating
  - The asset value, its volatility, and its expected growth rate can then be extracted from observed values for the same quantities for the firm's equity, as with the Merton model.
- In fact, any statistical model of V_t and any specification of V_B immediately implies two, related, very useful metrics:

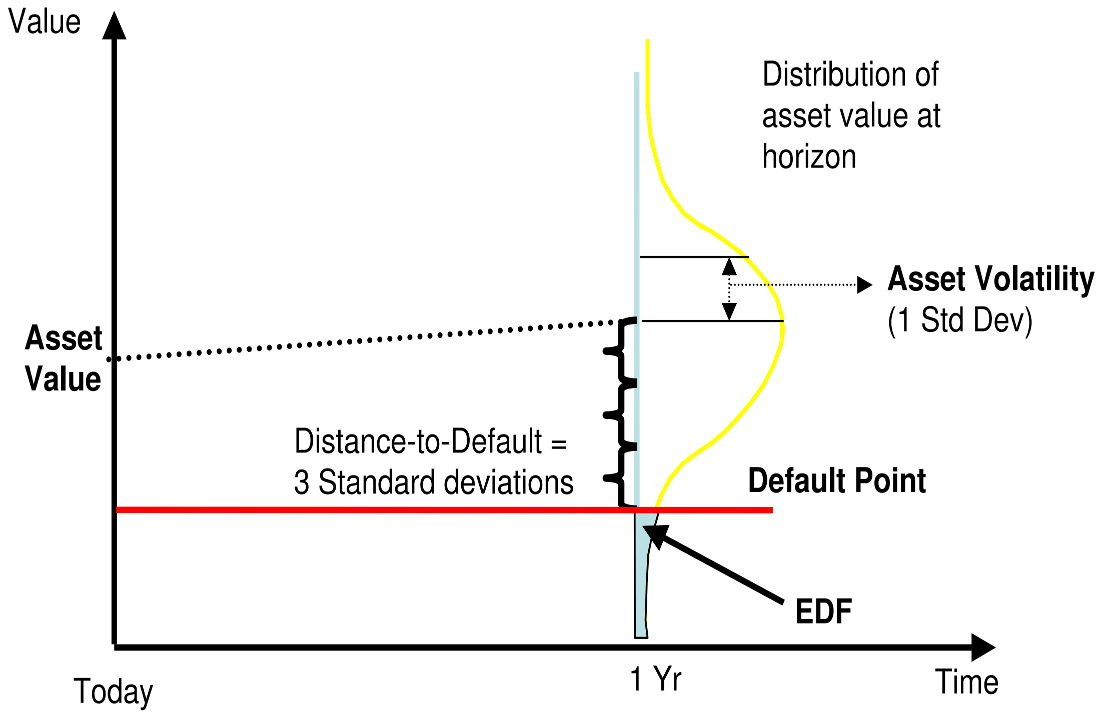
  Distance to default (DD) which is how far away V is from V_B in units of volatility looking ahead to some time T.

  Default probability within that time also called estimated default frequency (EDF).
- If V_B is constant and V_t has constant μ and σ then there are nice formulas for both these quanities:

$$DD = \frac{\log(V_t/V_B) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad EDF = 1 - \mathcal{N}(DD)$$

- Notice that these are both measures of TRUE probabilities of default, not "risk adjusted" ones.

- This picture is from one firm, Moodys/KMV, that sells forecasts of bankruptcy risk based on a structural model.



- These types of model <u>are</u> really used by big banks and hedge funds. You <u>can</u> actually model real companies' capital structure unlike the Merton model.
  - Bonds can have coupons.
  - The firm can have many different outstanding debt issues.
  - It's not hard to add random interest rates. And r can be correlated with V.

- Structural models for the most part cannot be solved in closed form. But we know we can always solve the PDE that characterizes each claim by our probabilistic technique.
  - ▶ Longstaff & Schwartz actually solved their model in an economy with stochastic r_t.
  - ▶ Notice that under the adjusted model, bankruptcy will typically occur <u>more</u> frequently than its true probability.
    - $*$ Why? Think about the simulation paths....
  - ▶ Can you see why no-arbitrage credit spreads are predicted to be lower when r goes up? There is some evidence that this is actually true empirically.
- Notice what the technique is doing: simulating forward V_t from $t = 0$ to $t = T$ – but *stopping when V_t hits V_B* – if that happens before T.
- So we can represent the value of an ordinary coupon bond (with principal$= 1$, coupon$= \Gamma$) in the LS model as:

$$\begin{aligned} F(V_0) &= B_{0,T}\ \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau>T\}}] \\ &\quad +\Gamma \sum_{T_n} B_{0,T_n}\ \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau>T_n\}}] \\ &\quad +(1-W) \int_0^T B_{0,t}\ \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau\in dt\}}]dt. \end{aligned}$$

  where τ denotes the random time at which V first hits (crosses) V_B. (The sum on the second line runs over all the coupon payment dates.)

- So, we could actually evaluate this in closed form if we knew the *first-passage-time density* for our geometric Brownian motion.
  - Note that the probability of hitting the barrier anytime before T is not the same as the probability of being below the barrier at T (which is the EDF).
  - And, of course, the Q in the notation is to remind you that you need to compute the probabilities using the altered model for dV (unlike the EDF).
- For the case of constant volatility and fixed barrier, the density of the first passage time is:

$$\text{Prob}^Q[\tau \in dt] = \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau \in dt\}}]$$

$$= \frac{\log\left(\frac{V}{V_B}\right)}{\sqrt{2\pi\sigma^2 t^3}} \exp\left\{-\frac{1}{2}\frac{(\log[V/V_B] + (r - \frac{1}{2}\sigma^2)t)^2}{\sigma^2 t}\right\} \; dt.$$

  (Replace r with $r - \Pi$ if the payouts to V's claims are non-zero.)
- You can use this density to do the integrals above analytically. But it's not clear this is any easier than just doing the simulations!

III. Equity-Based Models

- Now I want to show you another way of approaching risky bonds that can be much more practical for hedging.

- I mentioned that the Longstaff-Schwartz type models tell you how to hedge risky debt, but a problem is they tell you how to do it *in terms of units of V* – the whole firm.
  - To actually do that you'd have to be trading in little baskets of the firm's entire capital structure, which isn't very realistic.

- A clever trader could get around that by reasoning that, if a firm has two bonds, and we know the sensitivity of each to V then we can just use one to hedge the other. <u>True.</u>

- An even better way to do it would be to substitute the stock for one of the bonds, since stocks are more widely traded.

- So we would need to know what the model said the sensitivity of S to V was.

- But to price stocks using that model you have to first model every single other claim in the firm's capital structure, and then view equity as the residual
  - This is way too cumbersome to be practical.
  - And it might not be a very good model of stock prices. (They could turn out to be negative!)

- Why not try a more direct approach?

- If we know F (a given bond's price) is a function of V, and S is a function of V, why not just cut V out of the model all together and model F as a function of S?

- In short, go back to

$$\frac{dS}{S} = \mu_S \, dt \; + \; \sigma_S(S,t) \, dW$$

- Before we go this route, we had better think carefully about one issue though:

  Does it make sense to write down a model of the share price <u>before</u> we look at the firm's capital structure?

- In the Merton model, for example, the stock's volatility is an *endogenous* function of the firm's capital structure.

- What are we assuming if, instead, we specify an (exogenous) volatility for dS in advance?

- The answer is simple: we are now treating dV as endogenous.
  - That is, once we solve for the values of all claims, and add them all up, their sum V will obey some complicated process that might not look at all like geometric Brownian motion.
  - That's fine. It simply means we think we know more about the behavior of stock prices than we do about asset values.
- The point to recognize, however, is that we do now have to put some effort into writing down our specification of dS that realistically describes how a (levered!) firm's share price will behave.
- We do run in to one problem right away if, for example, we want to just re-use the Black-Scholes stock model.

  How do we capture bankruptcy?
- A natural answer is: bankruptcy is when the stock hits zero.
- But the log-normal process can't hit zero. **Why?**
- So what to do?

- One solution: specify some small positive stock price (e.g. 0.01) which corresponds to "bankruptcy".

  Problem: Might get very different answers depending on if you choose 0.01 or 0.02, etc.

- Better solution: **use a different model**.
  - I already argued that the **constant elasticity of variance** (CEV) model, for example, is a more realistic depiction of volatility dynamics.
  - Recall it says:

$$dS = \mu\, S\, dt \; + \; \omega \sqrt{S}\, dW \quad \text{or}$$

$$\frac{dS}{S} = \mu\, dt \; + \; \frac{\omega}{\sqrt{S}}\, dW.$$

  - For our purposes today, an even more important thing that's good about the CEV model is that it allows S to hit zero. This takes some math to prove. But, as you would expect (and want), it turns out that the chances of hitting zero are higher when μ is lower and/or σ is higher.
  - One thing you *can* see from $dS = \mu S\, dt + \omega\sqrt{S}\, dW$ is that, once S does hit zero, it stops. It really does die.
  - I simulated 1000 paths of the model for 10 years using daily steps and a starting volatility of 30% and an expected return of 6%.
    - About 3% went bankrupt – about what you might find for A or Baa companies.

- With this process, it's easy to go through our PDE argument again (I'll let you do this for yourself), and wind up with

$$\frac{1}{2}\omega^2 S\frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t} - rF + S(r-d)\frac{\partial F}{\partial S} + \Gamma = 0. \qquad (4)$$

- Now another advantage of thinking of S as the underlying asset is that some bonds have boundary conditions that are explicitly given in terms of the stock price. The most common ones are **convertible bonds.**

- These allow the holder the right to exchange them (at any time, usually) for a certain fixed number of shares.
  - ▶ Usually the indenture specifies a *conversion price*, S_C, which is related to the number of shares per bond, N, by

$$N = F^{\star}/S_C$$

    where $F^{\star}$, as before, is the face value of the bond.

- This would be a nightmare to try to implement via Longstaff & Schwartz. But it's simple now. It just requires that we use a terminal payoff of $\max[F^{\star}, NS_T]$ and impose an early-exercise constraint.

- So, all together, the set of boundary conditions is now

$$\begin{aligned} F(S_T, T) &= \max[F^{\star}, NS_T] \text{ for all } S_T > 0 \\ F(S_t, t) &\geq NS_t \text{ for all } t \leq T \\ F(0, t) &= 0 \text{ for all } t \leq T \end{aligned}$$

- Here, the second condition is the early-exercise condition; the third is bankruptcy.

- By the way, if you thought there was some reason to believe you would recover more than zero, just change the third one to something like

  $$F(S = 0) = (1 - W)F^{\star} \text{ for all } t < T.$$

  In practice most convertibles are very junior, unsecured debts. So it's rare that they get anything in bankruptcy.

- Another feature of real-world bonds is that the issuers usually have the right to redeem them early for a specific **call price** F^C. In fact, there are usually different call prices over the life of the bond.

- To translate that into boundary conditions, we need to know when redemption will actually occur.

- When should manager's exercise their right to call bonds?

- Answer: they should do it whenever the bonds would be worth more than F^C if they didn't call them. *Why?*

  - Because their job is to maximize the stock holder's value, which, in this case, is the same as minimizing bond holder's value.

  - To see this, imagine managers found their bonds were worth $F^C + \$1$. Then they could always make their share holders \$1 by calling the bonds for F^C and issuing new bonds with a lower coupon which was worth only F^C.

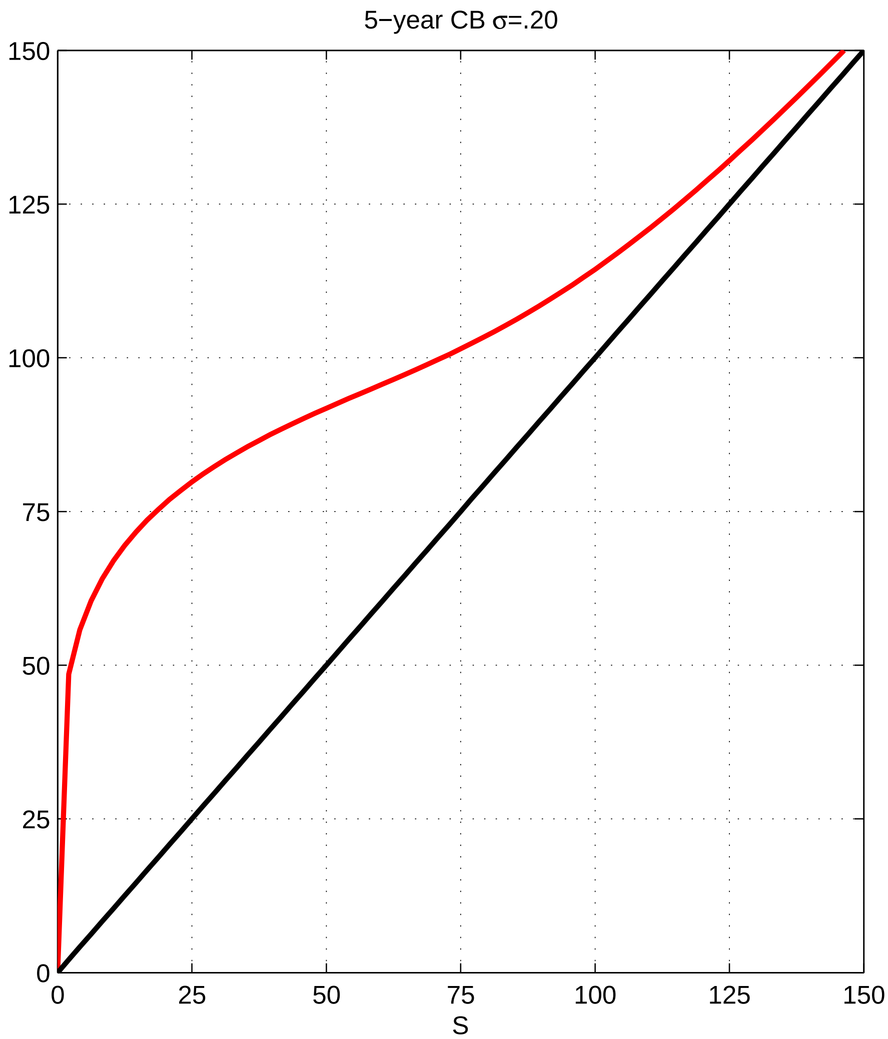
- So, for our valuation problem, handling call features is simple: we just assume the firm's managers are rational and hence *treat the call price as the ceiling on the bond's value.*

- This means we can add a fourth boundary condition:

$$F_t(S_t) \leq F^C(t) \text{ for all } t < T.$$

- Now we just solve the PDE (numerically).

- Here's what the solutions look like .



- Based on this plot, do CB holders prefer more or less σ?
- This is for a non-callable bond. How do you think a callability alters the picture?

- Notice that a convertible bond is **not at all** like just a (riskless) bond plus a call.
  - We are explicitly taking into account the credit risk of the principal re-payment.
- These models work extremely well in practice, both for valuation and hedging.
- It's also easy to play "what if" games, and figure out how changing or adding different features changes the price function. For example:
  - We could make the bonds *puttable* by enforcing a different lower boundary condition.
  - We could add sinking-funds/ammortizing principal.
  - We could see how much we would need to raise the coupon if we wanted to lower the number of shares and keep price constant.
  - And so on.

  This could be very useful to a potential issuer.
- Finally, we can return to the problem we started with and just value an ordinary risky bond by letting the number of shares you get go to zero.

IV. Better Models

- Firm credit risk models have been enhanced in recent years to try to capture progressively more realistic features of debt markets.

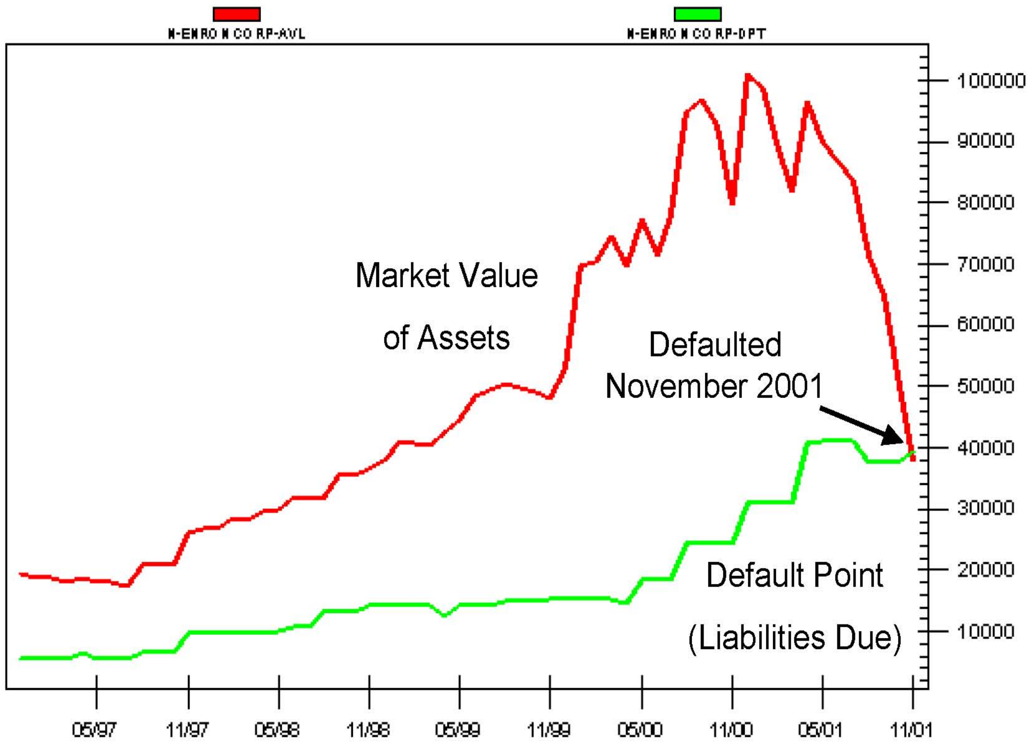
- You already know how we could incorporate more sources of randomness. Some obvious dimensions here include:

(A) Stochastic recovery rates.

(B) Stochastic asset volatility.

(C) Stochastic default thresholds.

- Here is KMV's description of Enron's demise:



- Going beyond these possibilities, recent research has started to explore further generalizations.

  Surprises. The models we looked at are based on continuous state variables. In particular, this means *bankruptcy is never a surprise.* This is unrealistic. Indeed, traditional structural models have a hard time explaining non-zero credit spreads for highly rated companies and short term debt for the this reason. The natural fix is to incorporate some form of jumps into the models. Models that include discontinuous shocks to asset values perform significantly better.

  Future financing policies. It is very important to try to capture not just the value of a firm's claims today, but what *changes* are likely to occur. Predicting what claims will the firm sell (or buy) in the future allows us to more accurately assess the ways in which assets will be divided in the future, and how risky those assets are likely to be. Lots of money has been lost by people using static models which failed to account for the shock to credit that can occur when, for example, a firm suddenly increases leverage a lot (e.g., for an LBO or a recapitalization.) Models of *dynamic capital structure* attempt to capture the optimization process of firm managers as they adjust their debt to capture tax benefits and hedge against bankruptcy risk.

Restructurings Several interesting models explicitly include the the potential for *strategic renegotiation* between debtors and equity holders outside of bankruptcy. Interestingly, this possibility does not always make debt riskier.

Financial constraints. Most models assume that firms can raise new capital to meet *cash-flow* needs – either from new equity or new debt – as long as it is solvent. More realistic models take into account that external finance may be difficult or impossible to raise in bad climates (i.e., when the firm has shortfalls of cash). Thus *liquidity*, as well as solvency, can be a trigger for credit events.

- This is a very lively field – and one where you could make real contributions!

V. Summary

- We looked at how no-arbitrage methodology could be applied to the hard problem of valuing securities subject to default risk.

- Even the simplified models we have seen today are enjoying a lot of attention right now from traders, risk managers, rating agencies, and regulators.

- We examined two classes of models.

(A) The first took the firm's *asset value process* as given and tried to quantify how risky all of the securities were, relative to each other.

(B) The second took the firm's *stock price process* as given and gave us a useful way to price and hedge real-world bonds.

- In these models there is the common implication that *bankruptcy risk is hedgable* and as a consequence **the price of credit risk should not depend on the true probability of default, but on the risk-neutral probability.**

- There generally aren't closed-form solutions to these models.
  - ▶ But we can always solve the PDE by taking discounted risk-neutral expectations of the payoffs to each claim.

Lecture Note 8.1: Summary of Notation

Symbol	Page	Meaning
V	p2	*value of a firm's assets*
μ, σ	p3	*expected growth rate and volatility of* V
Π	p3	*total yield per unit time to all firm's claim holders*
E	p5	*firm's earning stream (net of investment)*
λ^E	p5	*market price of earnings risk*
F	p7	*value of some particular claim on the firm*
Γ	p7	*payout per unit time to holders of claim* F
$F^\star$	p7	*face value of zero coupon bond in Merton model*
c	p8	*value of equity in Merton model*
d	p8	$PV(F^\star)/V$ *in Merton model*
y_τ	p8	*yield to maturity of 0-coupon bond maturing in* τ *years*
$S(V,t), \sigma_S(V,t)$	p13	*equity value and volatility in Merton model*
V_B	p14	*asset value which triggers bankruptcy*
X	p15	V/V_B *in the Longstaff-Schwartz model*
W	p15	*percent of face value lost in bankruptcy*
DD, EDF	p17	*distance to default,estimated default frequency*
τ	p18	*first (random) time* V_t *falls to level* V_B
$E^Q[\ \ ]$	p18	*expectation computed using RN probabilities*
$\mu_S, \sigma_S(S,t)$	p22	*expected growth rate and volatility of share price* S
ω	p24	*volatility when* $S=1$ *in CEV model*
N,S_C	p25	$N =$ *number of shares received on conversion of bond with face value* $F^\star$ *and conversion price* S_C
F_C	p26	*call price in effect at time* t