FIN516: Term-structure Models

Lecture 2: Black's formula

Dr. Martin Widdicks

UIUC

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Overview

- One of the simplest models to use to value interest rate derivatives is Black's model, based on the derivations in Black (1976)
- Unlike many of the other early interest rate models, in Black's model it is the *forward rate* that is modeled rather than the *short rate*.
- Analyzing the Black model is instructive for many reasons. First, it shows how to derive basic, analytic formulas for the value of caplets, caps and European swaptions.
- It demonstrates the advantages of modeling the forward rate directly, rather than always having to model the short rate.
- It also shows us what results from stochastic calculus we will require - most notably the ability to change *numeraire* and an understanding of Girsanov's theorem.
- Finally, Black's formula is the industry standard for valuing caps and will be a key input to the LIBOR Market Model.

Fischer Black

Fischer Black (1938 - 1995) is most famous for the Black-Scholes-Merton option pricing equations but he also developed the Black, Black-Derman-Toy and Black-Karasinski interest rate models.

Quiz

What is the forward price F for delivery at time t a stock with price S at $t = 0$?

- $F = Se^{rt}$
- $F = Se^{-rt}$
- $F = Se^{\mu t}$ ($\mu =$ expected return over t).

Quiz

What is the forward price F for delivery at time t a stock with price S at $t = 0$?

- $F = Se^{rt}$ CORRECT
- $F = Se^{-rt}$
- $F = Se^{\mu t}$ ($\mu =$ expected return over t).

Black's model for stock futures

- The Black model was originally designed for the valuations of options on commodities but it is most relevant for pricing interest rate derivatives.
- To see how it will work, first consider options on stock futures prices. Consider a non-dividend paying stock price, S_t, that follows geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where, as usual, μ is the drift, σ is the volatility, and W is a Brownian motion term. From simple no arbitrage relations, the futures (or forward) price on this stock, on a contract expiring at time T is given by $F_t = S_t e^{r(T-t)}$. So, by Îto's lemma:

$$dF_t = (\mu - r) F_t dt + \sigma F_t dW_t$$

- According to the fundamental theorem of finance, under the risk-neutral measure $\mathbb{Q}$ the stock follows

$$dS_t = r S_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{1}$$

Futures price process

- Thus, by Îto's lemma,

$$dF_t = \sigma F_t dW_t^{\mathbb{Q}}$$

  or in other word, F follows a martingale (or has no drift) under the risk-neutral measure.

- From here, it is possible to value options on the futures price using an adaptation to the Black-Scholes formula (see Problem Set 1).
- With interest rate derivatives we will end up in a similar scenario, where the forward rate will follow a martingale process, the only difference is that it it will take a bit more work to determine which *measure* the forward rate will be driftless under.

Martingales: a quick reminder

- Above we stated that under the risk-neutral measure, $\mathbb{Q}$, the stock price follows equation 1 above. This result can be obtained formally by considering the fundamental theorem of finance. This states that if we have complete markets then the price of any asset scaled by a positive asset (called the **numeraire** asset) is a martingale under the measure associated with that numeraire. Thus for an Asset with price A_t at time t:
$$\frac{A_t}{N_t} = E_t^{\mathbb{Q}}\left[\frac{A_T}{N_T}\right]$$
where N_t is the value of the numeraire asset at time t.
- Above we were considering the case where $A_t = S_t$ and $N_t = B(t)$, the money market account from the previous lecture. In this scenario, if we assume $B(0) = 1$ we have ...

Martingales: a quick reminder

- ...

$$\frac{S_0}{B(0)} = S_0 = E_0^{\mathbb{Q}}\left[\frac{S_T}{B(T)}\right]$$

$$S_0 = E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right) S_T\right]$$

- This can then be used to determine the drift of S_t under the risk-neutral measure and then for deriving the Black-Scholes option pricing formula.
- As a result of Girsanov's theorem the volatility term is unchanged under different measures. We will see more of this in lecture 4.

Quiz

How do we write S_t in terms of S_T and r_s?

- $S_t = E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right) S_T\right]$
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- $S_t = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) S_0\right]$

Quiz

How do we write S_t in terms of S_T and r_s?

- $S_t = E_0^{\mathbb{Q}} \left[\exp\left(-\int_0^T r_s ds \right) S_T \right]$
- $S_t = E_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_s ds \right) S_T \right]$ CORRECT
- $S_t = E_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_s ds \right) S_0 \right]$

Application of Black's formula for interest rates

- From the previous lecture we have

$$F(t;T,T+\tau) = \frac{1}{\tau}\left(\frac{P(t,T)}{P(t,T+\tau)} - 1\right) \quad (2)$$

$$= \frac{(P(t,T) - P(t,T+\tau))/\tau}{P(t,T+\tau)} \quad (3)$$

  and this is a portfolio of two zero coupon bonds (the *asset*) scaled by the zero coupon bond $P(t,T+\tau)$, thus by the fundamental theorem above the forward rate is a martingale under the $P(t,T+\tau)$ measure.

- Also we have that

$$S_{T_0,T_n}(t) = \frac{P(t,T_0) - P(t,T_n)}{\sum_{i=0}^{n-1} \tau_i P(t,T_{i+1})} == \frac{P(t,T_0) - P(t,T_n)}{Fixed_t(T,\ldots,T_n)} \quad (4)$$

  which is a portfolio of two zero coupon bonds (the *asset*) scaled by a portfolio of zero coupon bonds, thus by the fundamental theorem above the swap rate rate is a martingale under the measure associated with a portfolio of zero coupon bonds.

Assumptions about the forward and swap rates

- If the forward rate $F(t; T + T + \tau)$ is lognormal with constant volatility then

$$dF_t = \mu(F_t, t)F_t dt + \sigma_{Black} F_t dW_t$$

  (time notation supressed) but under the appropriate measure, with $P(t, T + \tau)$ as numeraire, then F is a martingale and so

$$dF_t = \sigma_{Black} F_t dW_t$$

- In a while we will relax the assumption that the volatility must be constant but at the moment we are treating this analagously to the Futures option above.

Caplet formula

- Now consider a caplet, the payoff is given by

$$
\begin{aligned}
\mathrm{Payoff}(Caplet)_{T+\tau} &= N(\max(L(T, T+\tau) - K, 0))\tau \\
&= N(\max(F(T; T, T+\tau) - K, 0))\tau
\end{aligned}
$$

- Under the $P(t, T+\tau)$ measure then the discounted caplet is also a martingale and so if $Cpl(0, T, T+\tau, K)$ denotes the value of the caplet at time t with a cap of K resetting at time T and paying at $T+\tau$ then

$$\frac{Cpl(0, T, T+\tau, K)}{P(0, T+\tau)} = E_0^{\mathbb{F}_i}\left[\frac{\max(F(T; T, T+\tau) - K, 0)N\tau}{P(T+\tau, T+\tau)}\right]$$

or

$$Cpl(0, T, T+\tau, K) = P(0, T+\tau)NE_0^{\mathbb{F}_i}\left[\max(F(T; T, T+\tau) - K, 0)\right]\tau$$

Caplet formula

- Where $\mathbb{F}_i$ denoted expectations under the particular forward rate measure, so all that is left to calculate is $E_0^{\mathbb{F}_i}[\max(F(T;T,T+\tau)-K,0)]$.
- From solving the SDE above (as for stocks under GBM)

$$F(T;T,T+\tau)=F(0;T,T+\tau)\exp\left(-\frac{1}{2}\sigma_{Black}^2 T+\sigma_{Black}W(T)\right)$$

Quiz

Why is the discounted caplet price also a martingale under the measure with $P(t, T+\tau)$ as numeraire?

- Because the forward rate is a martingale.
- All traded assets scaled by the numeraire asset are martingales in the appropriate measure.
- Because its price can also be expressed as a fraction with $P(t, T+\tau)$ as the denominator.

Quiz

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- Because the forward rate is a martingale.
- All traded assets scaled by the numeraire asset are martingales in the appropriate measure. CORRECT
- Because its price can also be expressed as a fraction with $P(t, T + \tau)$ as the denominator.

Black's Caplet formula

Thus, we have

$$
\begin{aligned}
E_0^{\mathbb{F}_i} &= \int_K^\infty (F(T;T,T+\tau)-K)dP(F(T;T,T+\tau)) \\
&= \frac{1}{\sqrt{2\pi}}\int_{K^*}^\infty \left(F(0;T,T+\tau)\exp\left(-\frac{1}{2}\sigma_{Black}^2 T+\sigma_{Black}x\sqrt{T}\right)-K\right)\exp\left(-x^2/2\right)dx
\end{aligned}
$$

where x has been described as a normally distributed variable and K^* is the value of x that ensures $F(T;T,T+\tau)>K$, namely

$$K^* = \frac{-\ln\frac{F(0;T,T+\tau)}{K}+\frac{1}{2}\sigma_{Black}^2 T}{\sigma_{Black}\sqrt{T}}.$$

This (eventually - you should probably do it once to satisfy your curiosity) simplifies to

$$F(0;T+\tau)\frac{1}{\sqrt{2\pi}}\int_{K^*-\sigma_{Black}\sqrt{T}}^\infty \exp(-y^2/2)dy - K\frac{1}{\sqrt{2\pi}}\int_{K^*}^\infty \exp(-y^2/2)dy$$

which gives the usual formula ...

Caplet formula

- . . .

$$Caplet(0, T, T+\tau, K) = NP(0, T+\tau)\tau\left(F(0; T, T+\tau)N(d_1) - KN(d_2)\right) \quad (5)$$

where

$$d_1 = \frac{\ln\left(\frac{F(0;T,T+\tau)}{K}\right) + \frac{1}{2}\sigma^2_{black}T}{\sigma_{Black}\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{F(0;T,T+\tau)}{K}\right) - \frac{1}{2}\sigma^2_{black}T}{\sigma_{Black}\sqrt{T}}$$

- This formula can easily be adapted for time varying volatilities by replacing the volatility σ_{Black} with the root mean square volatility. Thus,

$$\sigma^2_{Black}T = \int_0^T \sigma^2(t)dt$$

From caplets to caps

- The market typically reports the values of caps and not caplets. We will be more interesting in caplets as they can provide information for calibrating our models. However, we can move between caps and caplets as follows.
- The (discounted) payoff of a cap

$$\begin{aligned} Cap(0, T_0, \ldots, T_n, K) &= \sum_{i=0}^{n-1} N\tau_i \max(L(T_i, T_{i+1}) - K, 0) P(0, T_{i+1}) \\ &= \sum_{i=0}^{n-1} N\tau_i \max(F(T_i; T_i, T_{i+1}) - K, 0) P(0, T_{i+1}) \end{aligned}$$

can naturally be broken down into a series of caplets

$$\begin{aligned} Cap(0, T_0, \ldots, T_n, K) = \; & N\tau_1 \max(F(T_1; T_1, T_2) - K, 0) P(0, T_2) \\ & + N\tau_2 \max(F(T_2; T_2, T_3) - K, 0) P(0, T_3) \\ & + \ldots \end{aligned}$$

From caplets to caps: Cap volatility?

- Now, we can use our caplet formula to deduce the value of the first caplet, using the $P(t, T_2)$ as a numeraire asset. However, for the second caplet we would like to use a different numeraire asset - the $P(t, T_3)$ in order to use the Black formula. Are we able to do this?
- In this scenario, as the rate $F(t; T_1, T_2)$ has no impact on the valuation of the second caplet then there is no difficulty in using the Black formula for each of the caplets, with a different numeraire asset for both.
- This is not always going to be the case as we will see with a later example of a LIBOR-in-arrears FRA.
- Another issue is what volatility, σ_{Black} to use for each of the caplets. As with the Black-Scholes model, we developed the Black model assuming that the forward rate volatilities were either constant or time dependent. In practice we will find that forward rate volatilities are certainly not constant.

From caplets to caps: Cap volatility?

- As a result, as with Black-Scholes we can use the Black model to back out an *implied volatility* for the particular forward rate used to price each of the caplets. From here we can attempt to model the term structure of volatilities. We will return to this in great detail in the second half of the course.
- Often the cap prices are quoted simply as a cap implied volatility, assuming that the volatility is constant for the duration of the cap. This is usually called a **flat volatility**. When each of the caplet volatilities are quoted then this is known as **spot volatility**.
- Figure 2.1 shows the typical trends of cap volatilities as the time to maturity of the cap increases. There is usually a marked hump at maturities of 2-3 years with the volatility decreasing for longer maturities. There are many explanations for this and just a shape is very common.

Quiz

The implied volatility I calculate for each caplet above is from Black formulas with different numeraire assets. Is there a problem of comparability?

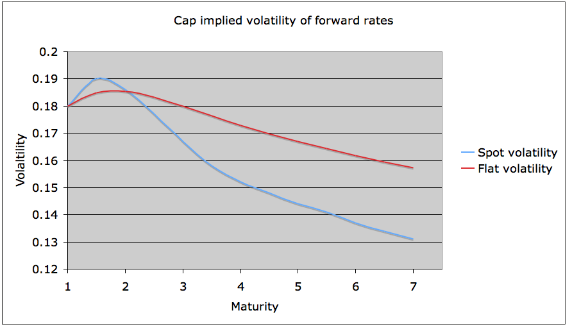
- Yes, the volatility is only valid for that numeraire asset
- No, it is fine as long as volatility is constant
- No, volatility is unchanged when you change measure

Quiz

The implied volatility I calculate for each caplet above is from Black formulas with different numeraire assets. Is there a problem of comparability?

- Yes, the volatility is only valid for that numeraire asset
- No, it is fine as long as volatility is constant
- No, volatility is unchanged when you change measure CORRECT

Figure 2.1


Cap implied volatility of forward rates
Volatility
0.2
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
1
2
3
4
5
6
7
Maturity
Spot volatility
Flat volatility

Pricing swaptions

- From above the swap rate scaled by an appropriate combination of zero coupon bonds is also a martingale, in this case

$$S_{T,T_n}(0) = E_0^{\mathbb{S}_i}\left[S_{T,T_n}(T)\right]$$

Where $\mathbb{S}_i$ denotes the appropriate swap rate measure and $Fixed_t(T, \ldots, T_n)$ is the numeraire asset, where:

$$Fixed_t(T, \ldots, T_n) = \sum_{i=0}^{n-1} \tau_i P(t, T_{i+1})$$

here the fixed stream of payments, a coupon paying bond starting at time T. From this formulation the swap rate is a martingale and so is any other asset scaled by the same numeraire, such as a **swaption**.

$$\frac{PS(0, T, \mathcal{T}, N, K)}{Fixed_0(T, \ldots, T_n)} = E_0^{\mathbb{S}_i}\left[\frac{PS(T, T, \mathcal{T}, N, K)}{Fixed_T(T, \ldots, T_n)}\right] \tag{6}$$

Pricing swaptions

- Where, recall

$$PS(T,T,\mathcal{T},N,K) = N\max\left(S_{T,T_n}(T)-K,0\right) Fixed_T(T,\ldots,T_n).$$

- Thus, we have that $S(t)$(subscripts removed for simplicity) follows a martingale,

$$dS(t) = \sigma_{S,Black} S(t) dW$$

(time subscripts dropped) where $\sigma_{S,Black}$ is the volatility of the swap rate, thus we can solve to derive

$$S(T) = S(0)\exp\left(-\frac{1}{2}\sigma^2_{S,Black}T + \sigma_{S,Black}W(T)\right)$$

Pricing swaptions

- Equation 6 shows that

$$PS(0,T,\mathcal{T},N,K) = NFixed_0(T,\dots,T_n)E_0^{\mathbb{S}_i}\left[\max\left(S_{T,T_n}(T)-K,0\right)\right]$$

and so this is analagous to the caplet formula above and we can write down the formula for a swaption as ...

$$PS(0,T,\mathcal{T},N,K) = Fixed_0(T,\dots,T_n)\left(S_{T,T_n}(0)N(d_1)-KN(d_2)\right) \tag{7}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_{T,T_n}(0)}{K}\right)+\frac{1}{2}\sigma^2_{S,black}T}{\sigma_{S,Black}\sqrt{T}} \\ d_2 &= \frac{\ln\left(\frac{S_{T,T_n}(0)}{K}\right)-\frac{1}{2}\sigma^2_{S,black}T}{\sigma_{S,Black}\sqrt{T}} \end{aligned}$$

Pricing swaptions

- And

$$Fixed_0(T, \ldots, T_n) = \sum_{i=0}^{n-1} \tau_i P(0, T_{i+1})$$

- As before, values of swaptions are used to back out the implied swap volatility $\sigma_{S,Black}$, suggesting that traders do not necessarily believe that swap rates have constant volatility, but that the market implied volatility can be useful in calibrating models.

Quiz

To consistently price caps and swaptions what must we assume?

- Forward rates are lognormally distributed
- Swap rates are lognormally distributed
- Forward rates and swap rates are lognormally distributed

Quiz

To consistently price caps and swaptions what must we assume?

- Forward rates are lognormally distributed
- Swap rates are lognormally distributed
- Forward rates and swap rates are lognormally distributed CORRECT

So far

- By a particular choice of numeraire and the assumption that forward and swap rates are lognormally distributed gives rise to the Black formulas that allow us to value caplets (and thus caps) and European swaptions. This relies upon an estimate of the volatility of the forward rates and the swap rates, and this volatility may be time dependent.
- If we assume that these formulas are consistent than we require both forward rates and swap rates to both be lognormal. However, we know that there is a relationship between forward rates and swap rates

$$S_{T_0,T_n}(t) = \sum_{i=0}^{n-1} \omega_i F(t; T_i, T_{i+1})$$

where

$$\omega_i = \frac{\tau_i P(t, T_{i+1})}{\sum_{j=1}^{n-1} \tau_j P(t, T_{j+1})}.$$

Are both rates lognormal?

- Now if we assume that forward rates are lognormal then it is possible to show that the swap rates cannot be lognormal (you can show this by applying Îto's lemma and noting that the Brownian motion term is not of the form volatility multiplied by swap rate).
- Thus it is not theoretically sound to use both Black's cap formula and Black's swaption formula as if one holds then the other will not. In practice however, Rebonato (1999) shows that the errors from assuming that both rates are lognormal are small and generally the market ignores this.
- A more striking problem will be always selecting numeraire assets so that all traded assets follow a martingale process. The next example illustrates why this can be a problem.

LIBOR-in-arrears

- Consider a slight adaptation to the caplet, the LIBOR-in-arrears caplet. The only difference is that instead of receiving the right to receive $\tau(F(T;T,T+\tau)-K)$ at time $T+\tau$ then you receive the money at time T instead (assume $N=1$).
- The payoff in present value terms has now changed from

$$\tau\max(F(T;T,T+\tau)-K,0)P(0,T+\tau)$$

to

$$\tau\max(F(T;T,T+\tau)-K,0)P(0,T)$$

- If, as above, we take the numeraire to be $P(t,T+\tau)$ then the forward rate is still a martingale as before but now the expectation has changed. If $CapletA(t,T,T+\tau,K)$ is the value of the contract at time t then

$$\frac{CapletA(0,T,T+\tau,K)}{P(0,T+\tau)}=E_0^{\mathbb{F}}\left[\frac{\tau\max(F(T;T,T+\tau)-K,0)}{P(T,T+\tau)}\right]$$

LIBOR-in-arrears

- But no longer does $P(T+\tau, T+\tau)$ cancel but instead we need to calculate the expectation of

$$\max(F(T; T, T+\tau) - K, 0)(1 + \tau F(T; T, T+\tau))$$

assuming that $F(T; T, T+\tau)$ is a martingale.

- This is far harder work that what we had before as now we have to calculate the expectation of the square of a lognormally distributed variable.
- The alternative is to use $P(t, T)$ as the numeraire asset. However, under the $P(t, T)$ measure we do not necessarily know that $F(T; T, T+\tau)$ is a martingale and so we would need to calculate its drift under the new measure. This is also not a straightforward task!

Quiz

Why is $F(T; T, T+\tau)$ not necessarily a martingale when $P(t, T)$ is the numeraire asset?

- Because this is not the denominator in its definition
- $P(t, T)$ is not a traded asset
- $P(t, T)$ is stochastic

Quiz

Why is $F(T; T, T+\tau)$ not necessarily a martingale when $P(t, T)$ is the numeraire asset?

- Because this is not the denominator in its definition CORRECT
- $P(t, T)$ is not a traded asset
- $P(t, T)$ is stochastic

A possible contract from Joshi

- The following nice example illustrates some of the interesting issues that can arise from our interest rate models.
- Consider a contract consisting of two forward rate agreements which span the time periods ($t_1 \to t_2$ and $t_2 \to t_3$) of the same length. Assume that the *reversing pair* contract consists of a long position (pay fixed) in the first contract and a short position (receive fixed) in the second.
- The current price is trivial - the sum of the two FRAs.
- Now as the structure of interest rates changes then interesting things can happen. If the yield curve shifts up or down than the value of the *reversing pair* changes by very little as any gains or losses in the first contract will be compensated by losses or gains from the second contract.
- However, if rates change so that the first rate goes down and the second rate goes up (the shape of the yield curve changes) then we lose on both contracts. So this contract reflects changes in the shape of the yield curve.

Options on these contracts

- Of course, we could overcome this risk by having *two* reversing contracts in adjacent time periods, this is now insensitive to changes in the slope but will be subject to the curvature of the yield curve . . .
- The situation becomes yet more interesting when we consider options on the reversing pair (or even two reversing pairs) called the *reverse option*. Now it is not so easy as we cannot split this into two options on FRAs as it is an option to enter into two FRAs simultaneously.
- Thus we cannot simply write down the value of this option as it will be dependent upon two forward rates and their joint distribution.
- In particular, all of the variation in option value will come from the divergence or convergence of the two rates making up the reversing pair.

Quiz

What will be the most important determinant of the price of the reversing option?

- The time 0 forward rates
- The volatility of the two forward rates
- The correlation between the two forward rates

Quiz

What will be the most important determinant of the price of the reversing option?

- The time 0 forward rates
- The volatility of the two forward rates
- The correlation between the two forward rates CORRECT

Volatility

- Of course, we have a lot of information about the process followed by the forward rates from the vanilla products. The FRAs tell us the initial value of the rates, the caps and floors give us the volatilities of each of the forward rates (assuming Black's model is correct).
- However, the constant volatility assumption of the Black model is clearly incorrect as there are some clear time dependent volatility features. This doesn't *really* matter for caps as we are considering the total volatility from now until the start of the appropriate forward rate and so the implied volatility is the root-mean-square volatility from now until the start date of the FRA.
- However, in the reverse option, the cap prices will give us the volatility from now until the start of the first rate (t_1) and also the start of the second rate (t_2) but **not** the expiry of the option.
- So, to know the volatility of a forward rate for any period not equal to the time until its expiry we need to assume a functional form for this *instantaneous* volatility retaining the property that the r.m.s volatility over the entire lifetime still matches the market value.

Drift and correlation

- Now, let's assume that we have chosen a formula for the instantaneous volatility, then we next pick an appropriate numeraire asset. Now we have two forward rates with different finishing times and so we cannot pick a numeraire that makes both of the rates driftless. So for at least one of the rates we will need to determine its drift.
- Now lets assume we have the drifts then we still need to jointly simulate (or calculate) the joint paths (or distributions) taken by the forward rates - which requires us to know the correlation between the rates.
- As with the volatility, we need to be careful when modeling the correlation. In particular we need to be careful when considering the interaction between volatility and correlation.

Quiz

If we have a contract whose payoff is dependent upon more than one forward rate, are all of these forward rates driftless?

- Yes
- No
- Perhaps

Quiz

If we have a contract whose payoff is dependent upon more than one forward rate, are all of these forward rates driftless?

- Yes
- No CORRECT
- Perhaps

Building a market model

- This discussion of the Black formula lead directly into the idea of Market Models, specifically the LIBOR market model that we will study in detail in the second half of the course.
- To generate a market model, we match the Black formulas for either caps or swaptions to the market prices for these instruments. When choosing forward rates, the cap prices will be automatically matched by assigning appropriate volatilities of the forward rates. When choosing swap rates to be martingales then the swaption prices will be matched.
- However, if we match cap prices exactly then we may not be able to match swaption prices exactly and vice versa. So should we base our model on forward rates or swap rates? In general, we will choose forward rates, for reasons we will see later.

Valuing swaptions when forward rates are martingales

- It is possible to show (see problem set 1) that the payoff of a swaption can be written in terms of forward rates starting at different times. This will mean that the joint distribution of all the forward rates will be important when valuing swaptions using forward rates.
- As a result, not only will we need to model the volatilities of the forward rates carefully but we will also need to model the correlation between each of the forward rates carefully in order to calibrate our models to both cap prices and swaption prices. See the reversing pair example.
- This is going to be a difficult task and there will be a variety of ways of doing this, as we will see.

Modeling large numbers of forward rates

- The final problem is when building our market model we will be valuing derivatives whose payoffs depend upon a large number of different forward rates. In order to do this we will need to simulate (or model) all of the forward rates assuming one bond as numeraire.
- Of course, by choosing a particular numeraire, only one of the forward rates will be guaranteed to be a martingale, thus we will need to calculate the drifts of the remaining forward rates. This will require knowledge of stochastic calculus and Girsanov's theorem.
- As we will eventually see these drifts may well be functions of the path taken by the forward rate. As a result it will be difficult to value derivatives in Market Models using backward numerical methods such as trees and finite difference methods, instead Monte Carlo techniques will need to be used.
- We would also like to value derivatives with early exercise features, but Monte Carlo methods do not naturally deal well with early exercise decisions, we will need to develop techniques to deal with this problem.

Modeling large numbers of forward rates

- We would also like to value derivatives with early exercise features, but Monte Carlo methods do not naturally deal well with early exercise decisions, we will need to develop techniques to deal with this problem.
- One way to make valuing American options using Monte Carlo is to reduce the overall number of stochastic factors thus another important tool will be Principal Component Analysis.

Quiz

If forward rates depend upon the path taken until time t why is it difficult to use trees?

- There will be too many steps in the tree
- The trees won't necessarily recombine
- The tree uses backward induction

Quiz

If forward rates depend upon the path taken until time t why is it difficult to use trees?

- There will be too many steps in the tree
- The trees won't necessarily recombine CORRECT
- The tree uses backward induction

Quiz

Why is it difficult to use Monte Carlo methods to value American options??

- It is a forward looking method
- It is inherently random
- There are too many paths to calculate

Quiz

Why is it difficult to use Monte Carlo methods to value American options??

- It is a forward looking method CORRECT
- It is inherently random
- There are too many paths to calculate

Seven steps to the LMM

1. Write the derivative's payoff in terms of the behavior of forward rates.
2. Observe the current forward rates and their market implied volatility.
3. Choose a numeraire that makes the rates as driftless as possible.
4. Compute the drifts of the forward rates in the appropriate measure
5. Choose or infer the instantaneous volatility curves for the forward rates
6. Choose or infer instantaneous correlations between different forward rates
7. Run a Monte Carlo simulation

Going back to the beginning, short rates

- To understand how to value derivatives using Market Models it makes sense to start with a simple term structure model and then move on to more complex models.
- We will start by seeing how we can value derivatives using a generic tree approach, then we will look at the tools required to build the tree. In particular, understanding how we can being to calibrate models to the yield (or LIBOR) curve using simple models
- These 'simple' models are short rate models, and we will look in detail at three short rate models: Black-Derman-Toy, Hull-White, Black-Karasinski and see how these models are useful but also the limitations that they have in pricing all derivatives accurately.
- We will then look at multi-factor models and end up spending most of the course creating, calibrating and implementing Market Models.

Conclusion

- We have seen the most basic of models - the Black model, and seen how it can be used to come up with analytic values for caps and swaptions.
- The Black model is very simple and makes incorrect assumptions such as a constant volatility for forward or swap rates. However, it is going to be the building block of the LMM model later.
- The derivation of the Black model enabled us to see some key techniques that we will return to later - the importance of the numeraire asset, the interrelations between forward rates, swap rates and zero coupon bonds.
- We also saw that using the Black model to price both caps and swaptions is inconsistent which is something that we will bear in mind when considering market models later.