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Lecture Note 5.1: Continuous-Time Arbitrage

As everybody knows, in 1973, Black and Scholes and Merton derived a closed-form solution for the price of a European call on an asset whose price is a geometric random walk in continuous time.

Why should we care? After all, we already know we can get the same answer – and can price much more interesting derivatives – just by using a binomial model with a large N.

The main reason to study the Black-Scholes-Merton model is to really understand how the no-arbitrage argument works in continuous time. This reveals an incredibly powerful ways to approach *all* derivatives.

Our goals for today are to make sure you really understand the economics of continuous time replication/hedging and also begin to think about the effect of relaxing its assumptions.

Outline:

I. The Black-Scholes-Merton argument
II. Recognizing all the assumptions
III. The solution for a European call
IV. Risk-neutral expectations
V. Continuous-time arbitrage in the real world
VI. Summary

I. The Black-Scholes-Merton Argument

- For much of the rest of the course, we will analyze derivatives in the context of continuous time economies.
  - ▶ This is not because they are more realistic.
  - ▶ It is because they are easier for mathematicians.
- In delving in to continuous time models, it can be easy to loose track of the no-arbitrage mechanism that underpins everything.
- You have seen derivations of the Black-Scholes equations already and I assume you are familiar with Itô's lemma and the basic properties of stochastic differential equations.
- I am going to walk you through my version of the logic once more in order to emphasize precisely why it all works.
- Last week, we discussed the geometric Brownian motion model as the continuous-time limit of binomial processes. We describe such processes via the notation:

$$
\begin{aligned}
\frac{dS}{S} &= \mu\, dt + \sigma\, dW \quad \text{or} \\
dS &= \mu S dt + \sigma S dW \quad \text{or, in general,} \\
dS &= a(S,t)dt + b(S,t)dW.
\end{aligned}
$$

  Here a is the *drift* and b is the *diffusion coefficient*.
  - ▶ Then a/S is the expected return and b/S is the volatility.

- For such a process, Itô just says that, if C is a smooth enough function to have partial derivatives, then

$$dC(S,t) = \frac{\partial C}{\partial S}\,dS \;+\; \frac{\partial C}{\partial t}\,dt \;+\; \frac{1}{2}b^2\frac{\partial^2 C}{\partial S^2}\,dt \qquad (1)$$

- Sometimes we will use subscripts to denote partial derivatives, so that equation (1) becomes

$$dC = C_S\,dS \;+\; C_t\,dt \;+\; \frac{1}{2}b^2 C_{SS}\,dt$$

or

$$dC = (C_S\,a(S,t) + C_t + \frac{1}{2}\,b(S,t)^2\,C_{SS})\,dt + C_S\,b(S,t)\,dW$$

▶ This is the basic notation we will use.

- Unlike the binomial case, the continuous-time no arbitrage argument isn't going to give us the value of an option directly as the value of a replicating portfolio. Instead, it's going to give us a *relationship* that characterizes the option's value.

Historical Note: The argument Black & Scholes actually used is much more convoluted than the one we are going to see, which was developed by Robert Merton.

Merton didn't get his name on the formula, but he did share the Nobel Prize for it.

- Here's the argument.

Step 1. Assume the price of the underlying asset follows

$$\frac{dS}{S} = \mu\, dt \; + \; \sigma\, dW. \tag{2}$$

Assume it pays no dividends. Let r be the continuously compounded, instantaneous risk-free interest rate. You can borrow or lend at this rate, so interest accrues at rate $r\, dt$. (All units are annualized.)

Step 2. Suppose that the price of a call is ONLY a function of S and t. That is, **there are no other sources of randomness affecting the value of the option.** Also assume the function $C(S, t)$ has two continuous partial derivatives with respect to its arguments. (We can verify this after we solve the problem.)

Step 3. Construct a portfolio that is long the call and short $\partial C/\partial S$ shares of stock. Invest an amount I in the risk-free asset. (We will determine what these quantities are later in the argument.) Write the position value as

$$\Pi \equiv C - C_S \cdot S + I.$$

Step 4. At all times, increase/decrease your risk-free investment by the cost of any changes in the stock position. With this self-financing policy, there is no cash in-flow or out-flow to your position. So its changes in value over dt come only from the positions you start the period with:

$$d\Pi = dC - C_S dS + rI dt.$$

This is a restriction on dI. Without it, in general, $d\Pi$ *also* would reflect changes due to $I(S)$ and $C_S(S)$ changing.

Step 5. Use Itô's lemma to rewrite the dC term:

$$\begin{aligned} dC - C_S dS + rI dt &= C_s\, dS \; + \; C_t\, dt + \; \frac{1}{2}\sigma^2 S^2 C_{SS}\, dt - C_s\, dS + rI dt \\ &= C_t\, dt \; + \; \frac{1}{2}\sigma^2 S^2 C_{SS}\, dt + rI dt \qquad (3) \end{aligned}$$

This is the instantaneous profit from our position.

Step 6. Deduce that, since this gain is riskless and has no cash-flows, **if it were not also equal to $\mathbf{r\,\Pi\,dt}$ there would be an arbitrage opportunity.** This is the key step.

Step 7. Conclude,

$$r(C - C_S \cdot S + I)\, dt = C_t\, dt \; + \; \frac{1}{2}\sigma^2 S^2 C_{SS}\, dt + rI dt$$

Hence:

$$\mathbf{\frac{1}{2}\sigma^2 S^2 C_{SS} + C_t - rC + rSC_S = 0}.$$

- That's it. That's seven steps to a Nobel Prize.

A General Derivatives Equation.

- Perhaps you are wondering why

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - rC + rS\frac{\partial C}{\partial S} = 0 \tag{4}$$

is worth so much excitement. It doesn't tell us the price of a call, it doesn't look like it tells us the price of anything.

- The reason is that the argument is incredibly general.

- If you re-do the steps assuming the derivative has payouts $\Gamma(t)$ per unit time and the underlying has yield d, the derivation is the same and the equation just says

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - rC + (r-d)S\frac{\partial C}{\partial S} + \Gamma = 0 \tag{5}$$

- *Any* derivative on S must satisfy such an equation. We said C was supposed to be the price of a call. But we never referred to this. We never used any special features of calls in the argument.

- This means that we can price all possible derivatives under this model by solving (4) or (5).
  - ▶ And, in fact, it is a very familiar (to mathematicians) type of partial differential equation (PDE), whose solution techniques are well understood.

- In other words, the PDE gives us everything but the final answer.

- Different securities impose different boundary conditions on the solution to (4), which means they impose different payoff functions at maturity, or along barriers etc.

- **Example of boundary conditions:** Consider an "up and out" knock-out call. If the strike is K and the knock-out happens at price X, then to get the value of the call while it is alive, we would solve the equation with the additional conditions:

$$\begin{aligned} C(T, S_T) &= \max[S_T - K, 0] \text{ for } S_T < X \\ C(t, S_t) &= 0 \text{ for all } t \text{ and all } S_t \geq X \\ C(t, 0) &= 0 \text{ for all } t \end{aligned}$$

- Once we have boundary conditions, the theory of partial differential equations also guarantees that
  1. **a solution exists,** which is lucky, and moreover,
  2. **only one solution exists.**

- That tells us that, if we find a particular solution to the equation, it must be **the** price of our derivative.
  - ▶ Otherwise, we wouldn't know that there wasn't some other one out there, and we wouldn't know which one ought to be the price.

- Our PDE also gives us some interesting insights about what the different terms mean. Consider.

Interpretation in terms of hedging profit & loss.

Imagine we are hedging a position. We buy the derivative, sell $\delta = C_S$ shares, and finance the whole thing by lending. How does our profit and loss behave over time?

- Assuming our theory holds, we know that we will make zero profit *over each instant.*
- But now we can say something about the components of our gains/losses.
- Since we are hedged, our random gains and losses from the stock and the call exactly offset.
- Our interest costs (per unit of time) are going to be: $r(SC_S - C)$.
- The option has time-decay of $C_t \equiv \frac{\partial C}{\partial t}$ (regardless of what the stock price does).
- But now look at our PDE again:

$$\text{hedge book profit} = 0 = C_t + r(SC_S - C) + \frac{1}{2}\sigma^2 S^2 C_{SS}.$$

- That third term must also be part of our profit/losses or else things won't add up. But what is it?
- Where else can P&L come from?

- It must be that the gains or losses per unit time from our hedge adjustment are

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \equiv \frac{1}{2}\sigma^2 S^2 \ \gamma.$$

- In particular, this will be positive or negative depending on the sign of "gamma", the <u>curvature</u> of the derivative price with respect to the stock.

<u>Interpretation of absence of μ.</u>

- Perhaps the most important point to notice about the Black-Scholes-Merton PDE: the underlying stock's expected rate of return μ appears *nowhere*.

  ∗ *An option on a stock expected to go <u>up</u> 20% a year is worth the same as one on a stock expected to go <u>down</u> 20% a year.*

    - all other things (σ, S, K, T, r) being equal.

  ∗ We noticed this in the binomial model's solution. But here we see it *even before we ever solve the equation.*

  ∗ And it applies to <u>all</u> derivatives, not just options.

II. Recognizing the Assumptions.

The Black-Scholes model makes a lot of assumptions. In discussing these, I want to distinguish between three different types of assumption.

(A) Old Assumptions.

- Just like the binomial model, and, indeed, *all* our no-arbitrage models, the Black-Scholes model uses the assumption of perfect-markets

  - ▶ No transations costs.
  - ▶ Riskless borrowing and lending and short-selling are possible.
  - ▶ No taxes.
  - ▶ No counterparty/legal risk.

- We don't want to brush these aside. (In fact, we will talk a lot about them later.) But first, I want to focus your attention on the **new** assumptions we had to make.

(B) New Assumptions

- There are really only two. But they matter.

  1. Continuous trading is possible.

  2. Prices cannot jump.

- Obviously these two are related. If either is not satisfied, we cannot construct instantaneously riskless portfolios.

- In practice, in the largest currency and fixed-income markets, continuity is not a bad approximation.

- Still, that doesn't tell us anything. **We don't know that a little violation means a little change to the model's results.**

- We will have to investigate that.

- The last category of assumptions are those concerning the interest rate and volatility processes. We need to clarify exactly what is old here and what is new.

(C) Interest Rate and Volatility Assumptions.

- When Black,Scholes and Merton solved the PDE (4), they did so under the <u>additional</u> assumptions that

  1. The volatility σ is a constant.

  2. The risk-free rate r is constant.

  3. d is zero.

- But what did we actually have to assume about about r, d, and σ when deriving the PDE?

- Really **they can be any functions of S and t.**

- So, for example, you could even have random interest rates in a currency model as long as the exchange rate determined the interest rates.

- The key point is that **there is only allowed to be one source of randomness in the economy** which affects the derivative. We express this by saying that the Black-Scholes model is a <u>one state variable model.</u>

- Actually, we had precisely the same level of generality in the binomial model. We were allowed to vary r across different nodes of the tree, and could even have varied the tree spacing (u and d) at different times and levels. It increased computational complexity, but it didn't affect the no-arbitrage argument.

- Now let's have a look at what happens when we <u>do</u> make the additional assumptions

  1. The volatility σ is a constant.
  2. The risk-free rate r is constant.
  3. d is zero.

III. The Black-Scholes Solution

(A) Theorem (Black and Scholes (1973), Merton (1973)): The solution to (4) when r and σ are constant, and subject to the boundary condition

$$c(S_T, T) \; = \; \max[S_T - K, 0]$$

is given by

$$c(S, K, T, r, \sigma) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \qquad (6)$$

where $\mathcal{N}(\cdot)$ is the **cumulative normal distribution function**, and where

$$d_1 \; \equiv \; \frac{\ln\left(\frac{S}{Ke^{-r(T-t)}}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and} \qquad (7)$$

$$d_2 \; \equiv \; \frac{\ln\left(\frac{S}{Ke^{-r(T-t)}}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)}. \qquad (8)$$

(B) Proof?

- A sufficient proof of the theorem, is just to plug the answer in to the PDE and verify that it works (and satisfies the boundary conditions). Then, because we know PDE solutions are unique, there can't be any other one.

- There is absolutely nothing interesting about all the calculus involved in solving the PDE "from scratch".

- We'll show later that you can also prove it from our general risk-neutral expectations formula.

Properties of the solution.

(A) Interpretation of the $\mathcal{N}$s.

- We can look at the Black-Scholes formula as another example of a formula we saw last time:

$$c = \delta S - L$$

- With a little calculus, we can differentiate the formula and prove that

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1).$$

(Use Leibnitz' Rule and the fact that $\mathcal{N}(x) = 1 - \mathcal{N}(-x)$.)

- This gives us a nice interpretation of $\mathcal{N}(d_1)$. It is just **the stock hedge ratio under the Black-Scholes model.**

- Next, from the Black-Scholes formula, we see that

$$c - \mathcal{N}(d_1)S = -Ke^{-r(T-t)}\mathcal{N}(d_2)$$

- But also

$$c - \mathcal{N}(d_1)S = c - \frac{\partial C}{\partial S}S = c - \delta S$$

- And we know that $c - \delta S = -L$ is the quantity of risk-free bonds in the replicating portfolio.

- Hence

$$L = Ke^{-r(T-t)}\mathcal{N}(d_2).$$

- This gives us an interpretation of $\mathcal{N}(d_2)$. It is **the percentage of the present value of the strike price we need to borrow to form the replicating portfolio.**

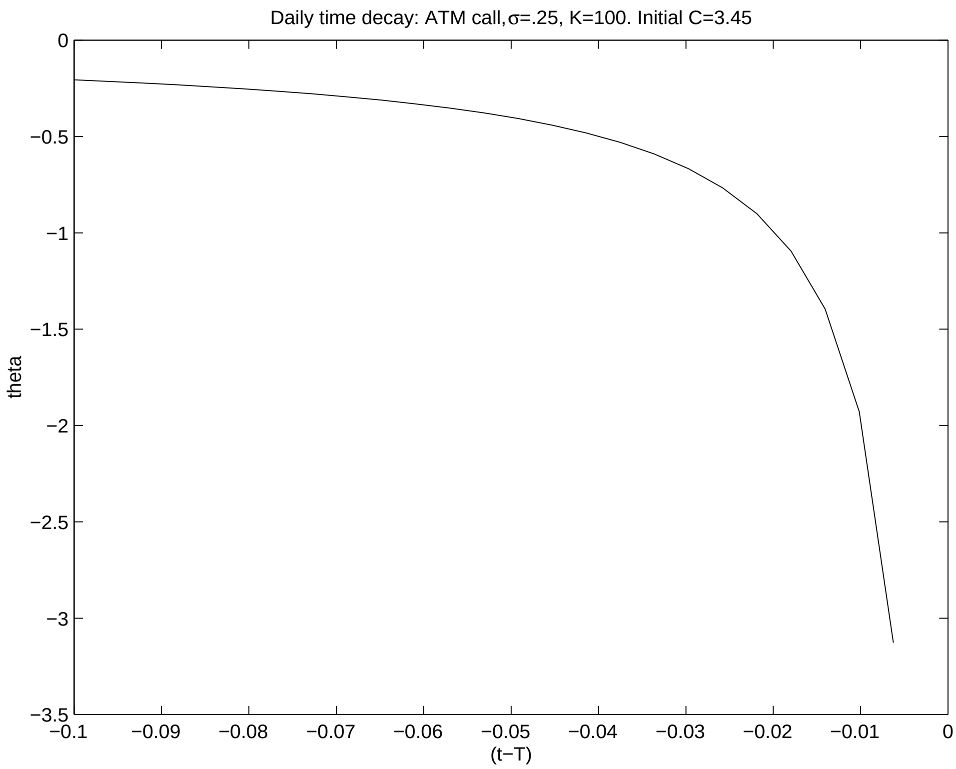
(B) Linearity of hedge ratios.

- Having a closed-form solution makes it easy to compute the Black-Scholes delta and gamma of a call.
  - ▶ I won't reproduce the formulas here. You can look them up in any text book.
- Using put-call-parity you can easily derive the analogous expressions for delta and gamma of European puts on a non-dividend paying stock.
- Having these formulas makes risk management of large books of many options easy.
  - ▶ The sensitivities just add linearly.
- This is a basic property from calculus. But it would not be true if the options values themselves did not add linearly.
  - ▶ Of course, if $c(K_1, T_1)$ and $c(K_2, T_2)$ both solve the Black-Scholes PDE, then $\Pi \equiv w_1 \cdot c(K_1, T_1) + w_2 \cdot c(K_2, T_2)$ solves it as well, and gives the value of a portfolio of w_1 units of option 1 and w_2 units of options 2.
- Market makers can have open positions in hundreds of strikes and expirations for each single stock or currency pair. Risk managers can monitor the entire trading book exposure in real-time because the hedge ratios can be computed so quickly.

(C) Time decay and expected returns.

- Here's one of the partial derivatives: $\frac{\partial c}{\partial t}$, sometimes called "theta".

  ▶ If you don't remember the analytical formula, you can always just compute these sensitivities by varying the input (in this case, time to expiration) by a little and then making your computer evaluate the rate of change of the solution.



- Surprisingly, nearly all of the time-decay in the option's premium is concentrated at the very end of its life.

- **Q:** Does this graph show that calls have negative expected return?

- In fact, let's consider what we know about the expected return of a derivative.
- Suppose we have an arbitrary derivative $Q(S)$ on a non-dividend paying asset S that obeys $dS = \mu S\ dt + \sigma S\ dW$.
- The instantaneous expected (excess) return to Q is the predictable part of dQ/Q minus the riskless rate. (I'm assuming there are no payouts to holding Q itself.)
- Using Ito's lemma, we can write
$$dQ = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Q}{\partial S^2} + \mu S \frac{\partial Q}{\partial S} + \frac{\partial Q}{\partial t}\right) dt + \text{something}\ \ dW.$$
- But the Black-Scholes PDE says$Q$ satisfies:
$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Q}{\partial S^2} + \frac{\partial Q}{\partial t} = rQ - rS\frac{\partial Q}{\partial S}.$$
- So the dt terms of dQ are
$$\mu S \frac{\partial Q}{\partial S} - rS\frac{\partial Q}{\partial S} + rQ.$$
- Dividing by Q to get expected *percentage* change, and subtracting r we get the expression
$$\frac{\partial Q}{\partial S}\ \frac{S}{Q}\ (\mu - r).$$
- This shows that the expected excess return is the product of the delta, the *gearing*, and the stock's risk premium.
  - ▶ Assuming the latter is positive, Q's risk premium is positive or negative depending upon the sign of delta.

(D) Implied Volatility versus Actual Volatility

- Another feature you have already noticed about the Black-Scholes solution is that it contains only one input that is not directly observable: the volatility σ.

- If we plot the call value against σ for any particular option we notice three things:

  1. The curve is strictly increasing.

  2. For $\sigma = 0$ it just gives the intrinsic value of the option.

  3. For $\sigma = \infty$ it just gives S.

- Mathematically, these facts imply that the pricing function has a unique inverse. For every possible call price (within the bounds allowed by static arbitrage) there is one, and only one σ which, plugged into the formula, would give that price.

- That particular σ is called the option's Black-Scholes **implied volatility** (BSIV) for the observed price.

- As a practical matter, options prices are quoted by traders *in units of BSIV.*

  ▶ Two counterparties can negotiate in "vol" units even as the underlying price S_t moves around a lot.

  ▶ Then, when they agree to trade, they both just plug the agreed vol into the same formula and both obtain the same dollar price.

  Trades are also commonly done "delta neutral", so that the parties are also agreeing to trade the stock at S_t when they agree on the option trade.

- **Don't be confused:**
  - ▶ Talking about an option's price in terms of these BSIV doesn't imply that either party *believes* the model that produced the formula.
  - ▶ You might come across statements like "The market uses Black-Scholes to price options," which can be very misleading.
- A close analogy is with the yield-to-maturity of a bond.
  - ▶ It's just a handy way of normalizing prices – in a highly nonlinear way – across different maturities and coupons.
    - ∗ In actual money units, bond prices and options prices are difficult to compare.
  - ▶ The BSIV, like the YTM, gives an intuitive, economic sense of relative valuation.
- We also need to bear in mind that, unless the model it exactly correct, we cannot equate BSIVs to the true standard deviation of the underlying returns – or even view them as forecasts of this quantity.

IV. Risk-neutral expectations.

- **Question:** How would the Black-Scholes PDE we derived have been different if the stock price model had been:

$$\frac{dS}{S} = r\,dt \;+\; \sigma\,dW \tag{9}$$

instead of

$$\frac{dS}{S} = \mu\,dt \;+\; \sigma\,dW? \tag{10}$$

- **Answer:** not at all.

- We already noticed that it doesn't matter what μ is. So it might as well have been r as anything else, for all we care.

- But that first stock model would be peculiar.

  It would say the stock's expected return over dt is the same as that of a perfectly riskless deposit.

- That would mean investor's were getting no compensation for the fact that the stock is risky.

- Such a model could <u>only</u> hold **if everybody were risk-neutral.**
  - ▶ Or, at least, neutral with respect to S-risk.

- They aren't. And that model <u>doesn't</u> hold.

- But if it did, then how would people value options?
  - ▶ Well, if they are neutral with respect to the risks in S, they must be neutral with respect to the risks in *derivatives that depend only on S*.

- ▶ Then, **risk-neutral people value assets as the expected value of their future cash-flows**, discounted at the risk free rate.

- So we know how they would compute the value of any European derivative on S. They'd:

  (1) Use their instantaneous stock model to figure out the probability distribution of all the possible time-T stock prices.

  (2) From that, deduce the distribution of all possible final call values.

  (3) From that, calculate their expected payoff from holding the call.

  (4) Then discount that number by multiplying by $B_{0,T}$.

- Of course, if they preferred, they could get it the same way we have to: by solving an ugly partial differential equation, subject to particular boundary conditions.
  - ▶ But both methods would have to give them the same answer.

- Actually, if they priced the option the second way, as we already observed, they would have to solve the exact same PDE, with the same boundary conditions that we would.

- Which means....

- ...THEY MUST GET THE SAME ANSWER THAT WE DO.
- So what?
- This tells us that we must also be allowed to use their other method.
  - ▶ We could have gotten our solution, by changing *our* stock model to *theirs*,
  - ▶ Just **switch** μ **to** $\mathbf{r}$ and then do steps (1)-(4) above.
  - ▶ That doesn't mean we believe their model. It's just a computational trick.
- Is it **always true** – with ANY model of stock price behavior – that you can value ANY derivative with this method?
- No.
  - ▶ Look back at the logic we used, and you will see that it is specialized to this case.
    - ∗ We needed to know that our derivative solved a particular PDE which didn't involve the true μ.
  - ▶ Other models might lead to a different PDE or to no PDE at all.
- So we can't apply the simple rule above in every situation. But when we can, it is often the best way to value derivatives.

V. Continuous-time arbitrage in the real world.

- In most real markets, continuous trading is obviously not feasible.
  - In the fastest electronic markets, algorithms can update hedge positions as continuously as you might want.
  - But even then, (a) you do not want to constantly be paying transactions costs; and (b) markets close and/or lose liquidity.
- If we recognize this plain truth, what happens to the model's conclusions?
- In effect, any discrete hedging/replication policy *contains some risk*.
  - Textbook perfect arbitrage becomes impossible.
  - In attempting to do the hedging, we face a trade-off between the replication risk and the transactions costs as we adjust the rehedging frequency.
- To illustrate, let's imagine that *all the Black-Scholes assumptions hold* except for continuous trading.
  - Specifically, let's say that the market closes for a period of time Δt (e.g., overnight) but that the underlying asset value evolution dS continues during this interval.

- We can quantify some of the issues by thinking about a Taylor series expansion of the P&L over Δt of a delta-hedged position in a security C.

$$\Delta C = \quad \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\Delta S)^2$$

$$+\frac{\partial^2 C}{\partial S \partial t}(\Delta S)(\Delta t) + \frac{1}{2}\frac{\partial^2 C}{\partial t^2}(\Delta t)^2 + \ldots$$

N.B. We <u>don't</u> have Ito's lemma now, since Δt is not infinitesimal. That's why we have all the extra terms.

- Each term involving ΔS represents a risk. Suppose we are trying to hedge.

- One step we can take to eliminate extra terms here is to modify the delta:

$$\delta^\star = \frac{\partial C}{\partial S} + \frac{\partial^2 C}{\partial S \partial t}(\Delta t).$$

(This adjustment will usually be very small.)

- More important, the remaining replication error $\Delta C - \delta^\star \Delta S$

$$= \frac{\partial C}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 C}{\partial t^2}(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\Delta S)^2 + o(\Delta t)^2 \text{ terms}$$

- The first two terms aren't random, and the higher order terms are *stochastically* small in the sense that their expected values will all contain terms like Δt raised to a power higher then 2.
  - ▶ Remember Δt is a number like .002, i.e. half a day.

- That leaves one term that captures the bulk of our re-hedging risk.

- Looking at the term, we can immediately deduce the most important feature of that risk: *It is proportional to the curvature of the position.*

- Hence, we can see that **if we have a zero-gamma position, we are essentially hedged against the risk of large changes over finite time intervals.**

- For this reason, maintaining "gamma-neutral" portfolios is sound risk management practice.

- This opens the door to a host of important questions.

- Most importantly, **How much will that cost?**
  - ▶ The diffusion-based analysis did not envision any need to gamma hedge and thus does not take into account the costs of doing so. So the model's solution is incomplete.

- As a practical matter, being delta and gamma neutral will have to mean that the replicating/hedging portfolio for a given option *will contain other options* as well as the underlying and riskless assets.
  - ▶ We can view the error in the Black-Scholes valuation as the sum of the cost of these extra options over all periods for which we cannot adjust our hedge.

- ▶ Notice that if we have a long gamma position then we *make money* on large moves *of either sign*.
  - ∗ This means that to gamma hedge we would be selling other options (e.g. short-term straddles) which would *lower* our cost.
  - ∗ So the <u>bias</u> in diffusion models is in the direction of undervaluing positive curvature positions.

- However, to precisely compute the exact value of a gamma-neutral strategy is a complex problem.
- It requires specifying a model of exactly when you and how you can trade.
  - ▶ If your delta hedging is limited by jumps, then we will need a model of the timing and amount of jumps.
    - ∗ Simplest example: a trinomial world!
  - ▶ If the limiting factor is transactions costs, then we will need to solve an additional problem of determining *how often you should rehedge.*
    - ∗ Of course the answer depends on the exact cost structure.
    - ∗ It will also depend on some preference parameters: how much are you willing to pay in transactions costs to lower your risk.

- An important observation is that your total valuation for a derivative may depend on positions that you already have.
  - For example, if you are short a European put, and you buy a European call of the same strike and expiration, you no longer have to do any re-hedging at all. Your expected transactions costs become zero.
  - *So you clearly can pay more for the call than someone who wasn't short the put.*
- But this brings up a puzzle. If everybody values a given option differently , whose valuation matters? Who sets the price?
- In particular: *whose transactions costs and risk aversion matter?*
- This is a classic question from micro-economics.
- We are essentially asking about the marginal production cost of a good, where "the good" is now the option in question and the production process is the replication strategy.
- In the major liquid capital markets in the big economies it is pretty reasonable to view the situation as a competetive one among producers:
  - There are lots of big financial firms capable of selling you any one product.
  - If they get uncompetetive, nothing stops new firms from entering.

- So in this situation the classical conclusion is that *the price will be set by the dealer with the lowest marginal costs.*

- It is reasonable to expect that, in a competetive market, the marginal bid and offer prices will be determined by the agent or institution with the lowest required compensation – the least risk-aversion, and the smallest transaction costs.

- To summarize, exact replication and no arbitrage pricing can still work in markets with discrete hedging, jumps, and transactions cost.
  - But computing them will require quantifying the costs of strategies that hedge multiple sources of risk.

- Under *incomplete markets* the theory can not, in general, deliver s unique price.

VI. Summary

- We derived a general partial differential equation for any derivative in a continuous time economy whose uncertainty is generated by a single Brownian motion price process for the underlying asset.
  - ▶ If it is violated then there is an *instantaneous* arbitrage opportunity.

- The main <u>new</u> assumptions to get the PDE were:

  1. Continuous trading is possible.

  2. Prices cannot jump.

- When volatility and interest rates are constant, the equation happens to have a closed-form solution for the case of a European call. That solution was discovered by Merton and Black and Scholes.

- The formula itself plays a rather limited role in financial engineering.

- The continuous-time framework opens the door to very powerful generalizations that we will see going forward.

Lecture Note 5.1: Summary of Notation

Symbol	Page	Meaning
dS	*p2*	*change in price of asset whose price is* S *over time interval* dt
dW	*p2*	*random normal variable of mean zero and variance* dt
μ, σ	*p2*	*expected percent change (continuously compounded) and standard deviation of percent changes in* S
$a(\,), b(\,)$	*p2*	*generic drift and diffusion coefficients for* dS
dC	*p3*	*change in price of derivative whose price* C *is known to depend only on* S *and* t
$\alpha(\,), \beta(\,)$	*p3*	*functions to be determined decribing expected change and std. dev. of changes in* C
C_t	*p3*	$\frac{\partial C}{\partial t}$ *the rate of change of* C *with respect to* t
C_S	*p3*	$\frac{\partial C}{\partial S}$ *the rate of change of* C *with respect to* S
C_{SS}	*p3*	$\frac{\partial^2 C}{\partial S^2}$ *the rate of change of* C_S *with respect to* S
Π	*p4*	*value of portfolio long one unit of derivative, short* C_S *units of the underlying, and with* I *invested in the risk-free asset*
r	*p4*	*instantaneous (e.g. overnight) riskless rate, assumed constant*
d	*p6*	*continuous payout percent on underlying asset*
Γ	*p6*	*continuous payout of derivative*
γ	*p9*	*"gamma"* $= \frac{\partial^2 C}{\partial S^2}$*, the convexity of* C
$\mathcal{N}(x)$	*p13*	*cumulative normal distribution function evaluated at* x
d_1, d_2	*p13*	*arguments of* $\mathcal{N}$ *appearing in the Black-Scholes formula*
L	*p14*	*amount of borrowing in replicating portfolio (minus* I *above)*
BSIV	*p16*	*Black-Scholes implied volatility*
$Q(S)$	*p17*	*Value of an arbitrary derivative on* S
Δt	*p23*	*Discrete re-hedging frequency in continuous time model*