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Lecture Note 4: Dynamic Arbitrage

Introduction:

Last week we examined robust static relationships that options prices must obey. Unfortunately, we can only take this so far. Option payoffs cannot be *exactly* replicated by static positions in the underlying securities. The really brillant part of derivatives theory is the recognition that *sometimes* dynamic replication can work instead.

In this note we re-visit the familiar binomial model as a way of illuminating the important issues with dynamic arbitrage. When does it work? What exactly is required? What types of derivatives can it price?

Besides laying the conceptual groundwork for more complex models, our discussion will show you that this model is not just an academic toy.

Outline:

I. The Stochastic Setting.

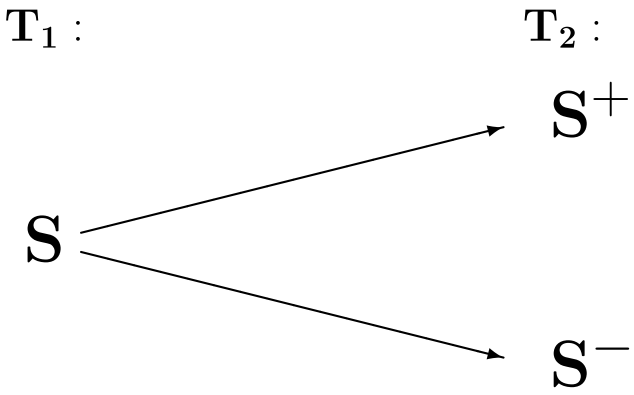
II. European Claims

III. Non-European Claims

IV. Summary

I. The Setting.

- The binomial model postulates that time is divided up into **discrete** periods, and, at each instant in time, the price of the underlying asset (call it a stock) can only go up to a specified higher price or down to a specified lower one. It can't stay the same. It can't do anything else.



- This sounds absurd, until you realize that the length of the time step could be interpreted as half a second. And the stock step size could be one "tick". Then, looking one minute ahead, there are 120 possible outcomes.

- What determines whether the price goes up or down?

- We don't know. It's random.
  - ▶ At each instant in time, Nature flips a coin.
  - ▶ The coin has probability π of landing "up" and $1 - \pi$ of landing "down".

⋆ It's worth stopping for a moment and considering how much is really being assumed here. We've all used probabalistic models before. We're used to making statistical statements about markets.

⋆ But now we are saying that there really is a "true" model and we know what it is.

⋆ And we know its parameters, and that the market obeys the same laws through out time.

⋆ Recognize that these types of assumptions are very different from simple assumptions about market mechanics, like predictable carry costs, etc. Now we are making assumptions about the nature of our uncertainty.

⋆ Now we have to ask ourselves a new type of question about our pricing conclusions: *what if the model is wrong?*

- The reason the binomial model is really so flexible is that you can make the grid (or **tree**) that the stock traverses have differing step sizes, and you can allow the probability π to be different at different points (or **nodes**) on the grid.

- A special case is a grid with constant <u>percentage</u> step sizes.
  - ▶ We fix numbers $u > 1$ (say 1.10) and $d < 1$ (say 0.90), and then allow the stock to move from S to either $u \cdot S$ or $d \cdot S$.
  - ▶ So, for example, after four "ups" and two "downs" the stock will have gotten to $u^4 d^2 S$.
  - ▶ Notice that you get to the same place regardless of the order of ups and downs. This is called a **recombining tree.**
  - ▶ In terms of continuously compounded returns (log differences), our grid has fixed absolute step sizes $\log u$ and $\log d$.

$$\begin{tikzcd}[column sep=huge] \mathbf{T_1}: & \mathbf{T_2}: \\ & \textbf{log S + log u} \\ \textbf{log S} \arrow[ur] \arrow[dr] & \\ & \textbf{log S + log d} \end{tikzcd}$$

- The path of the log price follows a random walk with constant mean (= *expected return.*)

- Once we fix the tree, the choice of π determines the mean and variance of the stock price (and/or its returns) at each step.

$$\mathrm{E}[S_{t+1}] = \pi S_t u + (1-\pi) S_t d \quad \text{so}$$
$$\frac{\mathrm{E}[S_{t+1}]}{S_t} \equiv \bar{\mu} = \pi u + (1-\pi) d$$

$$\mathrm{E}\left[\log(S_{t+1}/S_t)\right] = \pi \log u + (1-\pi) \log d$$
$$\mathrm{Var}[\log(S_{t+1}/S_t)] = \pi(1-\pi)[\log(u/d)]^2$$

- The last formula follows from the law:

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right]^2$$

and a little algebra.

- Suppose π is fixed for a tree of N time steps, each of length $\Delta t \equiv T/N$. There will be $N+1$ possible final prices S_N.
  - Pick one of the terminal nodes. What is the probability that S ends up there?
  - We can compute the probability of ending at the point $u^j d^{N-j} S_0$ by multiplying the probability of any one path that ends there (which is just $\pi^j (1-\pi)^{N-j}$) times the number of such paths.
  - There turn out to be

$$\frac{N!}{j!(N-j)!}$$

of them, where

$$N! \equiv N \cdot (N-1) \cdot (N-2) \cdots 2 \cdot 1.$$

- So the formula for the distribution (with $j = 0, 1, \ldots N$) is:

$$\text{Prob}(S_N = u^j d^{N-j} S_0) = \frac{N!}{j!(N-j)!} \pi^j (1-\pi)^{N-j}.$$

- This is the N-step binomial distribution.

- On a log scale, the individual returns are independent and identically distributed. So it's easy to calculate the mean and variance of $\log S_N/S_0$:

$$\mu_N \equiv \mathrm{E}\left[\log(S_N/S_0)\right] = \mathrm{E}\left[\sum_{k=0}^{N-1} \log(S_{k+1}/S_k)\right]$$
$$= N \cdot (\pi \log u + (1-\pi) \log d).$$
$$\sigma_N^2 \equiv \mathrm{Var}[\sum_{k=0}^{N-1} \log(S_{k+1}/S_k)] = N \cdot (\pi(1-\pi)[\log(u/d)]^2).$$

- In other words, **the mean and variance of returns are just their one-period values times the number of periods.**

• Frequently, in practice, we will want to set up a grid to model an asset with a given mean and volatility. So, with N/T fixed, we will want to come up with u, d and π to match μ and σ at horizon T.

  - Since we have three things to play with, there is an extra degree of freedom. We can use it to simplify calculations.

- $*$ Set $d = 1/u$.

- Now have two equations in two unknowns:

$$N \log(u)(2\pi - 1) = \mu\, T, \quad N[\log(u^2)]^2 \pi(1-\pi) = \sigma^2\, T$$

- Solution (you don't need to remember this):

$$\begin{aligned}
\ln(u) &= \sigma\sqrt{\frac{T}{N}}\left(1 + \frac{\mu^2 T}{\sigma^2 N}\right)^{1/2} \\
\ln(d) &= -\ln(u) = -\sigma\sqrt{\frac{T}{N}}\left(1 + \frac{\mu^2 T}{\sigma^2 N}\right)^{1/2} \\
\pi &= \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\frac{T}{N}}\left(1 + \frac{\mu^2 T}{\sigma^2 N}\right)^{-1/2}
\end{aligned}$$

- For large N/T this is approximately:

$$u = e^{\sigma\sqrt{\Delta t}} \approx 1 + \sigma\sqrt{\Delta t}$$
$$\pi = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

where $\Delta t \equiv T/N$ is the length of each time step.

- Now suppose that we divide time up more finely, i.e. increasing N while keeping T fixed.
  - ▶ If we keep adjusting u and π as above we keep the mean and variance per unit time fixed.
  - ▶ The Central-Limit Theorem guarantees that the time-T distribution of the return (i.e. $\log(S_T/S_0)$) converges to a normal distribution:

$$\log(S_T/S_0) = r_T \sim \mathcal{N}\left(\mu T, \sigma^2 T\right).$$

  - ▶ This holds for any time horizon, not just the end, since there are an infinite number of steps in any small interval.
  - ▶ In fact, the percentage change in the stock from any t to any $s > t$ is $\mathcal{N}\left(\mu(s-t), \sigma^2(s-t)\right)$. And, of course, each instant's change is still independent.
  - ▶ In the limit, the process looks the same no matter how small the time interval. It is infinitely wiggly and continuous.
  - ▶ We describe this process by the stochastic differential equation:

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \mu\, dt \;+\; \sigma\, dW.$$

    - ∗ the dt is an infinitely small Δt.
    - ∗ dW is the random part of the change:$\sim \mathcal{N}(0, dt)$.
    - ∗ The cumulatives sum, W, is called a **Wiener process**.
  - ▶ This S process is a **geometric Brownian motion**. we will come back to it (often).

- When we use a binomial model to price derivatives, we will find that it gives different values with different choices of the number of steps, N, even when you pick u, d and π to keep the mean and variance of the underlying constant.

- On the other hand, if you plot the values you get for different N, you see that they are always converging as N goes up.

- This is because the underlying binomial processes are converging to a unique Brownian motion process. So the derivative prices are converging to their value in a continuous-time world.
  - ▶ For many types of problems, only a handful of steps are required to get close to the limiting value.

- In practice, people naturally think of their binomial scheme not as a serious representation of the real world, but instead as a computational device for getting at the "true" continuous-time solutions.

- It is worth remembering, however, that sometimes time and prices really *do* move discretely and continuous trading is a very bad assumption.
  - ▶ Markets close
  - ▶ Liquidity can dry up
  - ▶ Prices can gap up or down

- In some situations the binomial model may actually be more realistic.

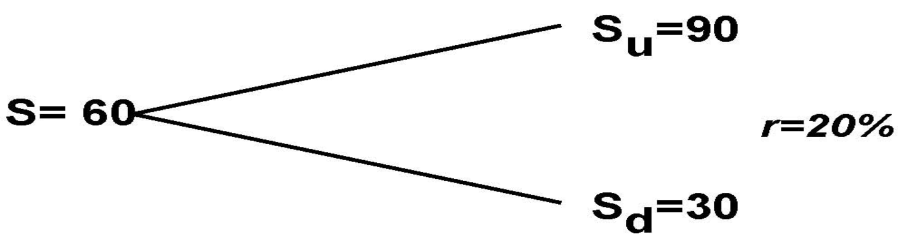
II. No-Arbitrage Pricing of European Derivatives.

(A) The one-step argument.

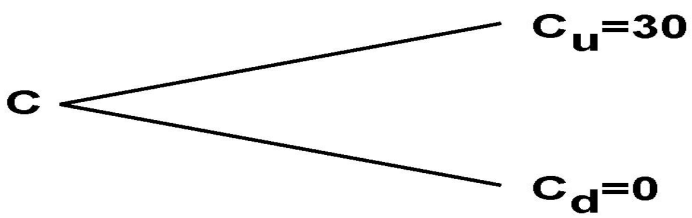
Consider the problem of valuing a call in a binomial world when there is only one time step remaining before expiration.

Assume the underlying asset is a non-dividend paying stock and there are no transactions costs.

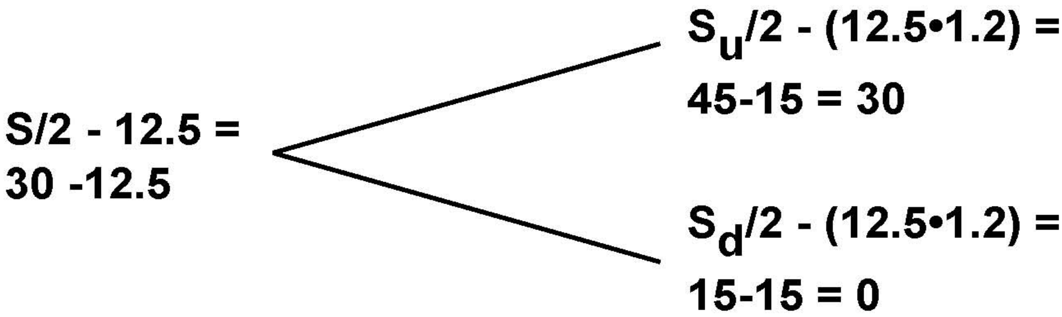
- Say the current per-period interest rate is 20%, the current stock price is 60 and the stock price can either fall to 30 or rise to 90.



- At the end of the period a call with a strike price of 60 will be worth either 0 or 30.



- Suppose we buy one-half share and borrow \$12.50.



- Since this portfolio represents the payoff to the call, the call must have the same value as the portfolio.

$$C = \frac{S}{2} - 12.50 = 30 - 12.50 = 17.50$$

- Therefore, this call is worth **17.50.**

- **What if the call option is trading at \$18.50?**
  - ▶ We then have an arbitrage opportunity, since we can replicate the payoff of the call option by spending only \$17.50.
    - ∗ Sell one call option;
    - ∗ Buy the replicating portfolio: buy one half share of the stock and borrow \$12.50.
  - ▶ The cashflow at time 0 is \$1.00.
  - ▶ The payoff from the replicating portfolio, by design, will exactly cover the short option obligation.

- While this little example does not seem very deep, we should consider some important questions about this simple set-up.

1. How did we guess the replicating portfolio?

- We need to solve for the replicating portfolio by using the conditions that the portfolio generate the same payoffs as the options.

- Suppose a portfolio of δ shares of stocks plus L in borrowing can replicate the call payoff.
  If the stock goes up to \$90,
  $$90\delta - 1.2L = 30; \tag{1}$$
  If the stock goes down,
  $$30\delta - 1.2L = 0. \tag{2}$$
  Solving (1) and (2) will give us $\delta = 0.5$ and $L = 12.5$

- Interpreting δ.
  - The portfolio should have the same sensitivity to the stock price as that of the call.
  - This is the hedge ratio or the "delta" of the call price with respect to the stock price:
    $$\delta \equiv \frac{\Delta C}{\Delta S} = \frac{30 - 0}{90 - 30} = \frac{1}{2}.$$
  - (It's just a coincidence that it happened to be 1/2 with the numbers here.)

2. Would the argument work in a trinomial world?

3. Why doesn't the option price $C(S_t)$ depend on the probability π?

- Mechanical Answer: To figure out a replication, you have to match cash flows in every possible state. But the probabilities of those states don't enter into the construction.

- But shouldn't the call price depend on the stock's expected return?

  ▶ It does, in so far as the stock's **current** price is determined by the returns people demand to hold it. But **given the current stock price**, we don't need any more information.

- Does this mean that a news event that leads to an increase in the assessed probability for an up-move in the stock price does not affect the value of the call option?

4. What else did we have to assume?

1. Underlying is a freely tradeable (and shortable) asset.

2. No possibility of cash-flows between grid nodes.

(B) Multi-period case.

- We now know how to value a call at any stock price (node) on our grid with one-step left in its life.
  - ▶ But there's nothing special about the last time-step.
    - ∗ To get the price one period earlier, we just repeat the replication strategy, this time choosing δ and L at each node to match the next period call values, whether the stock moves up or down.
  - ▶ Then, for the next time step......

- In other words, we're done! There really is no more to the pricing argument than what we saw in the one-step example.
- The key is that at each node there are two outcomes to hedge against and we have two securities to hedge with.
- As we work backward in time, we get *both* the option price and the replicating portfolio (hedge ratio plus amount of borrowing).
- We not only determine what the no-arbitrage price *should be*, but also the exact strategy we will follow if it *isn't*.

(C) Generalizing the calculation.

- As before, let:

$$\begin{aligned} u &= 1 + \text{rate of return if stock goes up} \\ d &= 1 + \text{rate of return if stock goes down} \\ \bar{r} &= 1 + \text{interest rate for borrowing and lending} \end{aligned}$$

  ▶ Note that the rates here are <u>simple, per-period</u> rates. So if the continuously-compounded annualized rate is r, $\bar{r} \approx e^{r\,\Delta t}$.

- We assume we <u>know</u> the two possible values the derivative could have next period. Call them C_u and C_d.

- We want to replicate these, so we require that:

$$\begin{aligned} \delta uS - \bar{r}L &= C_u \\ \delta dS - \bar{r}L &= C_d \end{aligned} \tag{3}$$

- Computing the hedge ratio yields:

$$\begin{aligned} \delta(uS - dS) &= C_u - C_d \\ \delta &= \frac{C_u - C_d}{S(u-d)} \end{aligned} \tag{4}$$

- Computing the required amount of the debt (using equations (3) and (4)):

$$
\begin{aligned}
& \frac{C_u - C_d}{S(u-d)} \cdot uS - \bar{r}L = C_u \\
\Longrightarrow \quad & (C_u - C_d) \cdot u - \bar{r}L \cdot (u-d) = C_u \cdot (u-d) \\
\Longrightarrow \quad & \bar{r}L \cdot (u-d) = dC_u - uC_d \\
\Longrightarrow \quad & L = (dC_u - uC_d)/(\bar{r}(u-d)) \qquad (5)
\end{aligned}
$$

- So the value of the derivative is the value of the replicating portfolio:

$$
\begin{aligned}
C &= \delta S - L \\
&= \frac{C_u - C_d}{u-d} - \frac{dC_u - uC_d}{\bar{r}(u-d)} \\
&= \left(\frac{1 - d/\bar{r}}{u-d}\right) C_u + \left(\frac{u/\bar{r} - 1}{u-d}\right) C_d \\
&= \frac{1}{\bar{r}}\left(\frac{\bar{r} - d}{u-d}\right) C_u + \frac{1}{\bar{r}}\left(\frac{u - \bar{r}}{u-d}\right) C_d \\
&= \frac{\mathbf{q}\,\mathbf{C_u} + (\mathbf{1}-\mathbf{q})\,\mathbf{C_d}}{\bar{\mathbf{r}}} \qquad (6)
\end{aligned}
$$

where we have defined

$$
q = \frac{\bar{r} - d}{u-d} \quad \text{and} \quad (1-q) = \frac{u - \bar{r}}{u-d}.
$$

- **Notice:** q does not depend on where we are in the tree or what any of the payoffs are.

- Equation (6) tells you how to move backwards along the grid. All you need to know is the values one step ahead!

- What changes here if there are payouts to the underlying?
  - ▶ Consider a one-period call on the U.S. Dollar from the perspective of a Japanese investor.
  - ▶ Let:

$$
\begin{aligned}
u &= 1 + \text{rate of return if Yen price of Dollar goes up} \\
d &= 1 + \text{rate of return if ¥/\$ goes down} \\
\bar{r}_d &= 1 + \text{interest rate for borrowing and lending Yen} \\
\bar{r}_f &= 1 + \text{interest rate for borrowing and lending Dollars}
\end{aligned}
$$

  - ▶ As before, we are somewhere on our grid and we know the two possible values, C_u and C_d, that the call could have next period.
  - ▶ Now if we hold δ Dollars, we put these in U.S. risk-free deposits and they each become $\delta\bar{r}_f$ Dollars (worth $\delta uS\bar{r}_f$ or $\delta dS\bar{r}_f$) next period. So the two replication conditions become:

$$
\begin{aligned}
\delta u\bar{r}_f S - \bar{r}_d L &= C_u &\quad (7) \\
\delta d\bar{r}_f S - \bar{r}_d L &= C_d &\quad (8)
\end{aligned}
$$

  - ▶ We do the exact same algebra as before.

    1. Solve these two equations for δ and L.

    2. Plug these into the no-arbitrage equation

$$C = \delta S - L$$

    and re-arrange.

- I'll let you fill in the details. In the end we get the same relation:

$$\mathbf{C} \;=\; \frac{\bar{\mathbf{q}}\,\mathbf{C_u} + (\mathbf{1}-\bar{\mathbf{q}})\,\mathbf{C_d}}{\bar{\mathbf{r}}_\mathbf{d}}$$

- Only now we have

$$\bar{q} \equiv \frac{(\bar{r}_d/\bar{r}_f) - d}{u-d} \quad \text{and} \quad (1-\bar{q}) \equiv \frac{u - (\bar{r}_d/\bar{r}_f)}{u-d}.$$

  * Note that if we have continuously compounded domestic and foreign interest rates r_d and r_f (instead of simple one-period rates) then

$$(\bar{r}_d/\bar{r}_f) \approx e^{(r_d - r_f)\Delta t}.$$

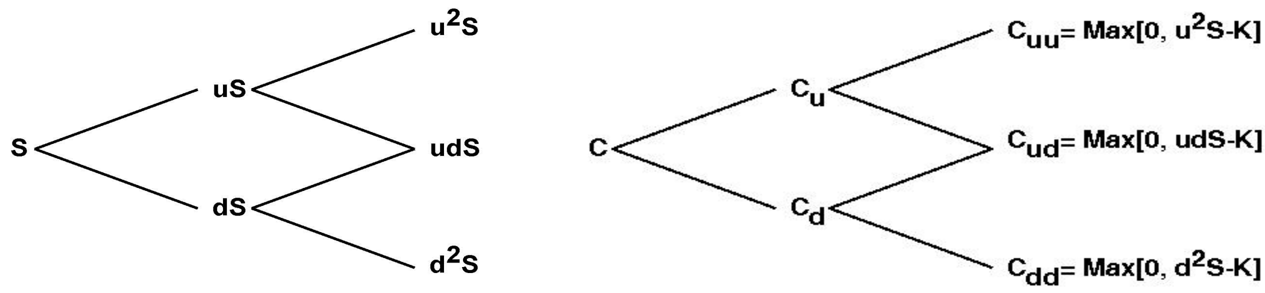
  * Note also, the denominator in the discounting formula is still just $\bar{r}_d$ NOT $(\bar{r}_d/\bar{r}_f)$.

- This analysis applies equally well to models with convenience yields, borrowing fees, storage costs etc, as long as *we know these rates in advance.*

  * But the method **doesn't** require that they are constant over all of the grid!

  * In the more general case, you could just compute a different $\bar{q}$ at each node.

- With two periods to go the picture is:

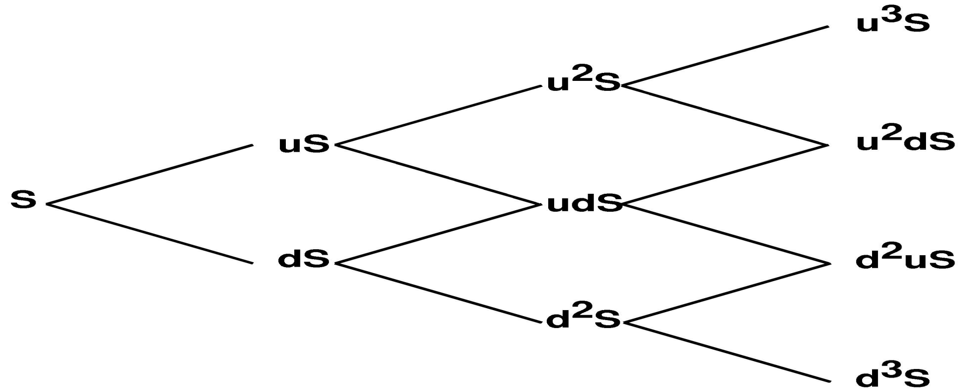


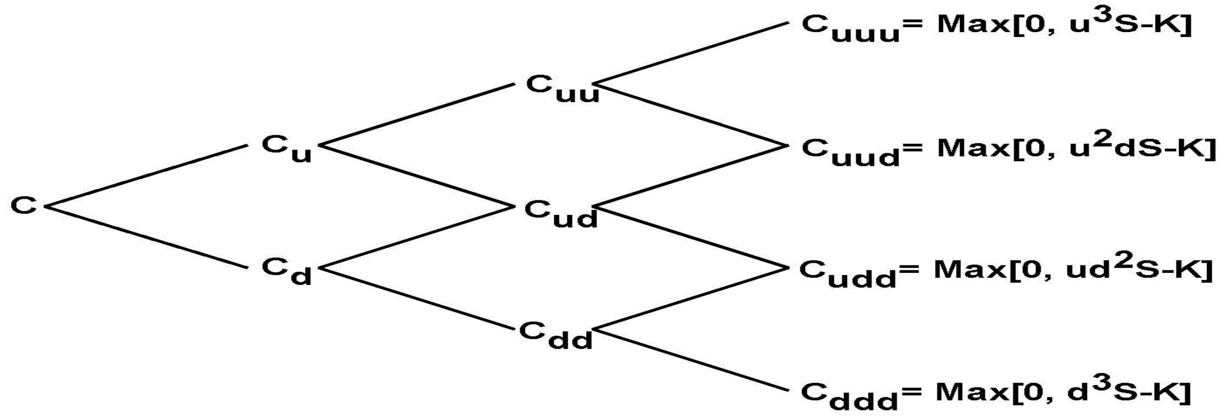
  - From the one-period case
$$
\begin{aligned}
C_u &= \frac{qC_{uu} + (1-q)C_{ud}}{\bar{r}} \\
C_d &= \frac{qC_{ud} + (1-q)C_{dd}}{\bar{r}}
\end{aligned}
$$

  - Once we have computed C_u and C_d, we have a one-period case.
$$
\begin{aligned}
C &= \frac{qC_u + (1-q)C_d}{\bar{r}} \\
&= \frac{q^2 C_{uu} + q(1-q)C_{ud} + (1-q)qC_{du} + (1-q)^2 C_{dd}}{\bar{r}^2} \\
&= \frac{q^2 C_{uu} + 2q(1-q)C_{ud} + (1-q)^2 C_{dd}}{\bar{r}^2}
\end{aligned}
$$
using the fact that $C_{ud} = C_{du}$.

  - We're using the fact that the one-step formula only involved q, which doesn't change. So we didn't have to solve for δ and L all over again.

- The three period case:





$$C = \frac{1}{\bar{r}^3}\left[q^3 C_{uuu} + 3q^2(1-q)C_{uud} + 3q(1-q)^2 C_{udd} + (1-q)^3 C_{ddd}\right]$$

▶ We could keep going but it would be tedious.

▶ What are we accomplishing?

(D) A General Pricing Formula for European Contingent Claims

- What we just did in the two and three period cases is combine the one-period formulas to eliminate the intermediate values of the derivative.

- This shows something remarkable: *we can write the value of the derivative today solely as a function of its payoffs at expiration.*

  - ▶ All we are doing is multiplying the different possible payoffs by weighting factors, and then adding them up.

  - ▶ This will work for any **European** derivative, because all the cash-flows they generate are at expiration.

- The formulas are getting messy. But we can simplify them for the general case of N periods to go – *if we can see the pattern. in those weighting factors.*

- The key is to recognize the ones we just derived for $N = 2$ and 3. **They are the same as the binomial probabilities of the underlying ending up at each different node.**

- We met those probabilities earlier in the lecture.

- Pick a particular node $u^j d^{N-j} S_0$ defined by

$$\begin{aligned} N &= \text{number of total steps} \\ j &= \text{number of } \textbf{up} \text{ steps needed to get to that point} \\ N-j &= \text{number of } \textbf{down} \text{ steps needed} \end{aligned}$$

- There are $\frac{N!}{j!(N-j)!}$ different paths that can get us there. The probability of each one was $\pi^j(1-\pi)^{N-j}$. So the total probability is

$$\left(\frac{N!}{j!(N-j)!}\right)\pi^j(1-\pi)^{N-j}$$

- Now the general expression for those weights in our multi-period pricing formula turns out to be

$$\left(\frac{N!}{j!(N-j)!}\right)q^j(1-q)^{N-j}$$

- In other words, **the weighting factors applied to each terminal payoff are just the probabilities of those payoffs, when π has been replaced by q**.

- So we can now jump straight to the general formula that must hold for an ARBITRARY derivative that pays off the amount $C_T(S_T(j))$ in N periods if the underlying ends up at

$$S_T(j) = S_t\, u^j d^{N-j}.$$

- It must be:

$$C_t \;=\; \left(\frac{1}{\bar{r}^N}\right) \sum_{j=0}^{N} \left(\frac{N!}{j!(N-j)!}\right) q^j (1-q)^{N-j} C_T(S_T(j))$$

where I have written in the general expression for the coefficients.

- This formula applies to ANY payoff function $C_T(\ )$.

- It tells us that a European claim's value can be interpreted as **the mathematical expectation of its payoff** discounted at the risk free rate

  ▶ ...using the wrong probabilities!

(E) The State Price Density.

- If you think back to last week's lecture, you may recall seeing a formula a lot like that before. **It is a special case of our general derivatives pricing equation:**

$$C_t = \sum_{x=0}^{\infty} C_T(x)\ b_T(x) = B_{t,T}\ \sum_{x=0}^{\infty} C_T(x)\ q_T(x).$$

  where $C_T(x)$ is the payoff function if the underlying asset ends at $S_T = x$, and $b_T(x)$ are the butterflies – that is – the prices of claims paying off 1 if $S_T = x$ and 0 otherwise.

- Now the formula on the previous page was also a sum over all the final stock prices, which were indexed by j the number of up-steps.

- So for this model, we can immediately see

$$b_T(j) = \left(\frac{1}{\bar{r}^N}\right)\ \left(\frac{N!}{j!(N-j)!}\right) q^j(1-q)^{N-j}.$$

- But $\left(\frac{1}{\bar{r}^N}\right)$ is just $B_{t,T}$, the price of a riskless bond maturing at T. (Recall $\bar{r}$ was one plus the per-period simple rate.)

- And we defined $b_T(x)/B_{t,T}$ to be $q_T(x)$ the state-price (or risk-neutral) probability that $S_T = x$.

- So for this model

$$\begin{aligned} \text{Risk-Neutral Probability that } S_T = S_t\, u^j d^{N-j} &= \\ \text{Risk-Neutral Probability of } j \text{ up moves out of } N &= \left(\tfrac{N!}{j!(N-j)!}\right) q^j (1-q)^{N-j}. \end{aligned}$$

- And, as we saw earlier, this is just another case of the N-step binomial distribution with parameter q.

This is our first example of an explicit recipe for the state price density.

- For this model, the SPD has the same form as the "true" stock price distribution, but with π replaced by q.
- How is one related to the other?
  - ▶ Consider the true one-period expected simple return on the stock:

$$\mathrm{E}\left[\frac{S_{t+1}}{S_t}\right] = \pi u + (1-\pi) d \equiv \bar{\mu}.$$

$$\text{So } \; \pi = \frac{\bar{\mu} - d}{u - d}.$$

  - ▶ Compare that to

$$q \equiv \frac{\bar{r} - d}{u - d}.$$

  - ▶ We see that q *would be* the same as π <u>if</u> the stock's expected return were $\bar{r}$ instead of $\bar{\mu}$.

- We will find that this relationship holds for many models: **To get the RND from the true distribution, just change the true expected return of the stock to the risk-free rate.**

- If we lived in a world where q actually *was* the up-move probability, then it would look like the stock *or any other asset* had expected return $\bar{r}$, i.e. the riskless rate.

- This is why q and $1 - q$ are sometimes called risk-neutral probabilities of up and down moves on the tree.

- As we emphasized last lecture, these aren't really the probabilities of anything. For our purposes, they are just terms in a formula: <u>a calculating device.</u>

III. Non-European Claims

The binomial set-up is really transparent and quite flexible. These features make it easy to adapt to handle some valuation problems with derivative whose cash-flows may occur before expiration.

(A) American Options

- American style options are substantially harder to price in most models.
  - ▶ The no-arbitrage value of the option is the value it has under the smartest possible exercise strategy.
  - ▶ The holder of the option essentially has a *second* option at each instant to exchange his original option for the underlying asset.
  - ▶ We saw that early exercise of a call is never optimal *if there are no payouts to the underlying.*
  - ▶ But if there *are* such payouts, holding a call that is deep in-the-money may be foregoing more in yield than the time-value of the option is worth.
  - ▶ So computing the optimal exercise point involves weighing the interest lost against the time decay of the option premium.
  - ▶ The solution varies throughout the life of the option, and is very difficult to describe analytically.

- In the binomial model, though, it's trivial:
  - ▶ Just work backwards as usual, but replace the model's value with the option's intrinsic value at every node where the latter is higher.
- Consider a currency option example.
  - ▶ Parameters: $S = 100¥/\$,\ K = 100¥/\$, r_d = 1\%,\ r_f = 6\%$ (quarterly compounded annual rates). $T = 1$ year.
  - ▶ The yen is the domestic currency. The underlying asset (the dollar) is assumed to have a high yield.
  - ▶ Assume a 4-step tree.
  - ▶ To construct the tree
    - ∗ Choose u, d to model a 12% annual volatility for the spot rate.
    - ∗ As we have seen, one way of doing this is $u = 1/d = \exp(\sigma\sqrt{\Delta t}) = 1.0618.$
    - ∗ Using our formula for the psuedo-probability:

$$\bar{q} = \frac{(\bar{r}_d/\bar{r}_f) - d}{u - d} = 0.382.$$

▶ Here's what the tree looks like for a European call:

Spot ¥/$				
				127.12
			119.72	
		112.75		112.75
	106.18		106.16	
100.00		100.00		100.00
	94.18		94.18	
		88.69		88.69
			83.53	
				78.66

European call				
				27.12
			18.20	
		9.94		12.75
	4.93		4.86	
2.32		1.86		0.00
	0.71		0.00	
		0.00		0.00
			0.00	
				0.00

▶ For an American call, proceed as before to time step $T-1$. Then notice that we should exercise at both in-the-money nodes.

	27.12
$\mathbf{18.20 < 19.72}$	
	12.75
$\mathbf{4.86 < 6.18}$	
	0.00
0.00	
	0.00
0.00	
	0.00

		27.12
	19.72	
$\mathbf{11.32 < 12.75}$		12.75
	6.18	
2.36		0.00
	0.00	
		0.00
	0.00	
		0.00

▶ Then do the same for the next step. Here only the most in-the-money node should be exercised.

▶ Then at the last two steps, the time-value of the option is big enough that we shouldn't exercise early. Final tree:

American	**call**			27.12
			19.72	
		12.75		12.75
	6.32		6.18	
2.96		2.36		0.00
	0.90		0.00	
		0.00		0.00
			0.00	
				0.00

- Notes about the solution:
  - ▶ The American call is quite a lot more valuable here.
    - ∗ The difference is greatest for high S/K or low r_d/r_f.
  - ▶ Tree gives us *optimal* exercise strategy, as well as the corrected hedge ratios.

- With discrete payouts, what do we do?
  - ▶ Now, instead of adjusting the q, we adjust the grid.
  - ▶ Simplest procedure is to assume (a) that the dividend ex-date coincides with one of the time steps; and (b) at each node, the company pays a dividend that is a *known percentage of the stock price at that time.*

* Notice this is different from assuming a known fixed amount (regardless) of the stock price.

▶ Then at the ex-dividend node, we *divide it into two* virtual nodes, i.e., right before and right after the dividend.

* Then we check whether we want to exercise at each one.

• **Example:** Consider this stock tree:

$$
\begin{array}{ccccccc}
 & & & & & & 86.4 \\
 & & & & & 72 & \\
 & & 72 & \searrow & 60 & & 57.6 \\
 & 60 & & & & 48 & \\
50 & & 48 & \searrow & 40 & & 38.4 \\
 & 40 & & & & 32 & \\
 & & 32 & \searrow & 26.7 & & 25.6 \\
 & & & & & 21.3 & \\
 & & & & & & 17.1
\end{array}
$$

▶ Here the company will pay a 16.66% dividend to people that own the stock by the third date.

* Do you see why a fixed dividend percentage is a convenient assumption?

▶ If you exercise a call on that date you get the money. So, for a call, use the cum-div prices (to the left of the arrows) to figure out exercise value.

* For a put you would always wait until the stock goes ex. So use those prices to evaluate the put exercise strategy.

- As above, we compare the "dead" value to the "live" value – what the option will be worth if we don't exercise.
- Suppose $q = 0.55, \bar{r} = 1.02, K = 40$. Then unexercised options will be worth

$$
\begin{array}{ccc}
 & & 46.4 \\
 & 32.8 & \\
21.9 & & 17.6 \\
 & 9.5 & \\
5.2 & & 0 \\
 & 0 & \\
0 & & 0 \\
 & 0 & \\
 & & 0
\end{array}
$$

- So, comparing these to what we could get by exercising <u>prior to</u> the dividend, we learn that it *is optimal* to exercise at the top two nodes.
- Replace the values there with intrinsic, and finish the tree:

$$
\begin{array}{ccccccc}
 & & & & & & 46.4 \\
 & & & & & 32.8 & \\
 & & 32 & \nwarrow & 21.9 & & 17.6 \\
 & 20.8 & & & & 9.5 & \\
13.05 & & 8 & \nwarrow & 5.2 & & 0 \\
 & 4.2 & & & & 0 & \\
 & & 0 & \nwarrow & 0 & & 0 \\
 & & & & & 0 & \\
 & & & & & & 0
\end{array}
$$

- Notice I skipped the step of checking for exercise at all the steps other than the third. Why?

(B) Other Non-European Derivatives.

- One often encounters other derivatives that have cash-flows before T – either contingent, as in American options, or not. The binomial model is often very helpful for these.

- Let's mention a few.

1. Convertible bonds give the holder the right to exchange for stock, which makes them like American options.

- But they <u>also</u> make coupon interest payments. So when moving along the grid, we simply add these amounts to the bond value on the dates for which they are paid.

- In fact, we could use the same procedure to value non-convertible bonds too!

  ▶ We'll return to this later in the course.

2. "Bermudan" options are a type of exotic that allows early exercise *at some times but not at others*.

- Examples:

  ▶ Executive stock option that can't be exercised for first 5 years.

  ▶ Cash-settled index options that can only be exercised at each day's close.

- Simple to handle with binomial tree:
  - ▶ Just enforce the early-exercise inequality at time steps for which it is permitted.

3. Binary or "touch" options pay off a fixed amount (e.g. 1.0) at any date upon which the underlying crosses or hits some pre-specified boundary.

- Actually these are not "options" at all.
- To value them, we consider any node that is across the boundary as a terminal point and assign the value 1.0 at it.
- Closely related are *knock-out* options that die (become instantly worthless) if another barrier is crossed.
- Notice that there is some implicit *path dependency* in the payoffs of these securities.

IV. Summary.

- In this note we reviewed how dynamic arbitrage can work in conjunction with specific *stochastic assumptions.*
  - ▶ In the case of the the binomial model, it works because, by assumption, we can hedge perfectly over each time step using two securities.

- It's not as silly as it first looks, because we can build complicated trees from simple ones.
  - ▶ One does not have to use $i.i.d.$ simple grids.
    - $*$ We can allow the volatility of the underlying to change out the grid by choosing different u and d.
  - ▶ In fact, there is also no assumption that payouts and interest rates are constant, as long as you know which rates to use at which nodes.

- By iterating backwards in time, we get the options price **and** the replicating portfolio at every point on a grid.
  - ▶ To do the arbitrage (or replicate the payoff), we just follow whatever path nature takes, changing our hedge as we go forward. This is a <u>self-financing, dynamic replication strategy</u>.
  - ▶ The strategy generates interests costs and hedge-adjustment (trading P&L) cost. These are certain in advance, and hence **an option is worth what it costs to hedge it.**