Change of Numéraire

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Readings: Shreve Chapter 9

A currency option example

A currency option example

Example

The exchange rate between USD and Euro is Q_t (USD/Euro). The current exchange rate is $Q_0 = 1.44$ USD/Euro. Risk free interest rates for USD and Euro are $r = 0.27\%$ and $r_f = 1.28\%$ per year with continuous compounding. Q_t follows a geometric Brownian motion with volatility $\sigma = 20\%$. What is the value (in USD) of a European call to buy 1.42 USDs for 1 Euro in 3 months?

Treat it as a put

- A call to buy 1.42 USDs for 1 Euro = a put to sell 1 Euro for 1.42 USDs with $K = 1.42$ USD/Euro

  `If Euro is worth less than` $K = 1.42$: `deliver Euros and get 1.42 USD per Euro`

  `If Euro is worth more than` $K = 1.42$: `do not exercise`

- Payoff $V_T = (K - Q_T)^+$ (in USD); Black-Scholes formula

$$V_0 = Ke^{-rT}N(-d_-) - Q_0 e^{-r_f T} N(-d_+) = 0.049 \text{ USD},$$

$$d_\pm = \frac{\ln(Q_0/K) + (r - r_f \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Treat it as a call

- A call to buy 1.42 USDs at $1/K = 1/1.42$ Euro/USD
- Call payoff for each USD traded: $(1/Q_T - 1/K)^+$, where $1/Q_t$ is the price of USD (in Euro)

$$1.42 \times \text{call price (in Euro)} \times Q_0$$

- Black-Scholes formula for the call price? Suppose

$$dQ_t = (r - r_f)Q_t dt + \sigma Q_t dB_t^*$$

where B_t^* is a Brownian motion under **"the risk neutral measure"** $\mathbb{Q}$

- Stochastic differential equation for $1/Q_t$

$$d\frac{1}{Q_t} \overset{???}{=} (r_f - r)\frac{1}{Q_t}dt + \sigma \frac{1}{Q_t} dB_t^*$$

A paradox

- Itô's formula: $f(x) = 1/x, f'(x) = -1/x^2, f''(x) = 2/x^3$

$$
\begin{aligned}
d\frac{1}{Q_t} &= df(Q_t) \\
&= f'(Q_t)dQ_t + \frac{1}{2}f''(Q_t)(dQ_t)^2 \\
&= -\frac{1}{Q_t^2}\Big((r - r_f)Q_t dt + \sigma Q_t dB_t^*\Big) + \frac{1}{Q_t^3}\sigma^2 Q_t^2 dt \\
&= (r_f - r + \sigma^2)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}dB_t^*
\end{aligned}
$$

- Why isn't the drift $r_f - r$ under measure $\mathbb{Q}$?
- Need a drift $r_f - r$ for $1/Q_t$ under a risk neutral measure to apply the Black-Scholes formula

How to discount

- **Discounting**: the risk neutral measure $\mathbb{Q}$ under which

$$dQ_t = (r - r_f)Q_t dt + \sigma Q_t dB_t^*$$

  is such that the discounted gain process is martingale; the **USD risk free rate is used for discounting**

- $1/Q_t$ (price of USD in Euro): want the discounted gain process to be a martingale, should use **Euro risk free rate for discounting**

Review: finding risk neutral measure

- In the physical world,

$$
\begin{aligned}
dQ_t &= (\mu - r_f)Q_t dt + \sigma Q_t dB_t \\
&= (r - r_f + \mu - r)Q_t dt + \sigma Q_t dB_t \\
&= (r - r_f)Q_t dt + \sigma Q_t(\theta dt + dB_t), \quad \theta = \frac{\mu - r}{\sigma} \\
&= (r - r_f)Q_t dt + \sigma Q_t dB_t^*
\end{aligned}
$$

- Apply the Girsanov theorem with Radon-Nikodým derivative

$$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\theta^2 T - \theta B_T\right)$$

Under measure $\mathbb{Q}$, B_t^* is a standard Brownian motion

Review: risk neutrality

- Under risk neutral measure, discounted gain process is a martingale
- **Domestic money market account**: if you deposit 1 USD, there will be $M_t = e^{rt}$ USDs at time t; r is the USD risk free rate; domestic discount factor: $D_t = 1/M_t = e^{-rt}$
- At time 0, invest Q_0 and obtain one Euro (and deposit). At time t, have $M_t^f = e^{r_f t}$ Euros that are worth Q_t USDs per Euro. **Gain process**: $Q_t M_t^f$

$$
\begin{aligned}
d(D_t Q_t M_t^f) &= Q_t d(D_t M_t^f) + D_t M_t^f dQ_t \\
&= D_t M_t^f (dQ_t - (r - r_f) Q_t dt) \\
&= \sigma D_t Q_t M_t^f dB_t^*
\end{aligned}
$$

Review: risk neutral pricing

- Under risk neutral measure, discounted value process of the derivative is a martingale: let V_t be the value of the derivative at t

$$D_t V_t = \mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t]$$

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} V_T | \mathcal{F}_t]$$

- With $V_T = (K - Q_T)^+$, the Black-Scholes formula can be used to compute the above

Numéraire

- Both V_t/M_t and $Q_t M_t^f/M_t$ ("Values" of the derivative and the foreign money market account denominated in units of the domestic money market account) are martingales under measure $\mathbb{Q}$
- **Numéraire** is the unit of account in which the values of other assets are measured
- Stock A worths \$100, stock B worths \$50. If stock B is used as numéraire: stock A is worth 2 units of stock B
- **Domestic money market account used as numéraire**: at time t, the derivative is worth V_t/M_t units of the domestic money market account
- **Foreign money market account**: if you deposit 1 Euro, there will be M_t^f Euros at time t, worth of $Q_t M_t^f$ USDs, or $Q_t M_t^f/M_t$ units of the domestic money market account

Foreign money market account as numéraire

- Use **foreign money market account as the numéraire**: "values" in terms of $Q_t M_t^f$
- "Value" of the domestic money market account in units of the foreign money market account:

$$\frac{M_t}{Q_t M_t^f}$$

- Find the measure under which the above is a martingale

- Recall that

$$d\frac{M_t}{M_t^f} = \frac{M_t}{M_t^f}(r - r_f)dt$$

$$d\frac{1}{Q_t} = (r_f - r + \sigma^2)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}dB_t^*$$

- Using Itô product rule,

$$\begin{aligned} d(\frac{M_t}{Q_t M_t^f}) &= \frac{M_t}{M_t^f}d\frac{1}{Q_t} + \frac{1}{Q_t}d\frac{M_t}{M_t^f} \\ &= -\sigma\frac{M_t}{Q_t M_t^f}(-\sigma dt + dB_t^*) \\ &= -\sigma\frac{M_t}{Q_t M_t^f}dB_t^f, \quad dB_t^f = -\sigma dt + dB_t^* \end{aligned}$$

Foreign risk neutral measure

- Apply the Girsanov theorem with Radon-Nikodým derivative

$$Z_T^f = \frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\sigma^2 T + \sigma B_T^*\right) = \frac{D_T Q_T M_T^f}{Q_0}$$

  Under measure $\mathbb{Q}^f$, B_t^f is a standard Brownian motion
- "Values" of the derivative and the domestic money market account denominated in units of the foreign money market account (i.e., $\frac{V_t}{Q_t M_t^f}$ and $\frac{M_t}{Q_t M_t^f}$) are martingales under measure $\mathbb{Q}^f$

Foreign risk neutral pricing

- Foreign risk neutral pricing

$$D_t V_t = \mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t] = Z_t^f \mathbb{E}^{\mathbb{Q}^f}\left[\frac{D_T V_T}{Z_T^f} | \mathcal{F}_t\right]$$

$$\frac{V_t}{Q_t M_t^f} = \mathbb{E}^{\mathbb{Q}^f}\left[\frac{V_T}{Q_T M_T^f} | \mathcal{F}_t\right]$$

$$V_t = Q_t \mathbb{E}^{\mathbb{Q}^f}\left[e^{-r_f(T-t)} \frac{V_T}{Q_T} \middle| \mathcal{F}_t\right]$$

- V_T/Q_T: derivative payoff in foreign currency; $\mathbb{E}^{\mathbb{Q}^f}\left[e^{-r_f(T-t)} V_T/Q_T \middle| \mathcal{F}_t\right]$: derivative price in foreign currency with correct discounting; Q_t price of foreign currency in USD

The currency example revisited

- The value (in USD) of the contract

$$V_0 = KQ_0\mathbb{E}^{\mathbb{Q}^f}[e^{-r_f T}(\frac{1}{Q_T} - \frac{1}{K})^+]$$

- But under measure $\mathbb{Q}^f$

$$\begin{aligned} d\frac{1}{Q_t} &= (r_f - r + \sigma^2)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}dB_t^* \\ &= (r_f - r)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}(-\sigma dt + dB_t^*) \\ &= (r_f - r)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}dB_t^f \end{aligned}$$

with the desired format!!!

- Apply the Black-Scholes formula with $1/Q_0 = 1/1.44$, $1/K = 1/1.42$, $\sigma = 20\%$, $T = 0.25$, risk free rate $r_f = 1.28\%$, continuous yield $r = 0.27\%$

$$\begin{aligned} V_0 &= 1.42 \times (\frac{1}{Q_0} e^{-rT} N(d_+) - \frac{1}{K} e^{-r_f T} N(d_-)) \text{ Euros} \\ &\quad \times 1.44 \text{ USD/Euro} = 0.049 USD \end{aligned}$$

$$d_\pm = \frac{\ln(\frac{1/Q_0}{1/K}) + (r_f - r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Stock as numeraire

Using stock as numeraire

- Carr, P., and D. Madan, 2009, Saddlepoint methods for option pricing, Journal of Computational Finance, 13(1), 49-61.

  ... saddlepoint methods are used to compute the probability that the stock is in-the-money for the risk neutral probability and the reweighted probability **when the stock is itself taken as a numeraire**. *Thus two saddlepoint approximations are involved in constructing one call option price*

- Black-Scholes formula for European vanilla puts

$$V_0 = Ke^{-rT} N(-d_-) - S_0 e^{-qT} N(-d_+), d_\pm = \frac{\ln(\frac{S_0}{K}) + (r - q \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

- Underlying asset price: $S_t = S_0 e^{X_t}$; put price = **cash-or-nothing** put price − **asset-or-nothing** put price:

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(K - S_T)^+] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(K - S_T)\mathbf{1}_{\{S_T \le K\}}] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[K\mathbf{1}_{\{S_T \le K\}}] - e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{\{S_T \le K\}}] \end{aligned}$$

- In BSM, $X_t = (r - q - \frac{1}{2}\sigma^2)t + \sigma B_t^*$; the cash-or-nothing put price is

$$Ke^{-rT}\mathbb{Q}(X_T \le \ln(\frac{K}{S_0})) = Ke^{-rT}N(-d_-)$$

- Where does the second probability come from?

- Use the **gain process** $e^{qt}S_t$ of the stock **as numeraire**
- Value of the domestic money market account $M_t = e^{rt}$ in units of the numeraire

$$\frac{M_t}{e^{qt}S_t} = \frac{e^{(r-q)t}}{S_t}$$

- Find equivalent measure so that the above is a martingale

- Recall that $dS_t = (r-q)S_t dt + \sigma S_t dB_t^*$

$$d\frac{1}{S_t} = (q - r + \sigma^2)\frac{1}{S_t}dt - \sigma\frac{1}{S_t}dB_t^*$$

$$d(e^{(r-q)t}) = e^{(r-q)t}(r-q)dt$$

- By Itô product rule

$$\begin{aligned} d(\frac{e^{(r-q)t}}{S_t}) &= e^{(r-q)t}d\frac{1}{S_t} + \frac{1}{S_t}d(e^{(r-q)t}) \\ &= -\frac{\sigma e^{(r-q)t}}{S_t}(-\sigma dt + dB_t^*) \\ &= -\frac{\sigma e^{(r-q)t}}{S_t}dB_t^S, \quad dB_t^S = -\sigma dt + dB_t^* \end{aligned}$$

Measure $\mathbb{Q}^S$

- Define measure $\mathbb{Q}^S$ via Radon-Nikodým derivative

$$Z_T^S = \frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\sigma^2 T + \sigma B_T^*\right) = e^{(q-r)T + X_T}$$

  Under measure $\mathbb{Q}^S$, B_t^S is a standard Brownian motion
- 'Value of the domestic money market account in units of the numeraire, i.e., $\frac{M_t}{e^{qt} S_t}$, is a martingale under measure $\mathbb{Q}^S$

- $\{D_t V_t, 0 \le t \le T\}$ is a martingale under measure $\mathbb{Q}$:

$$D_t V_t = \mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t]$$

- $\{D_t V_t / Z_t^S, 0 \le t \le T\}$ is a martingale under measure $\mathbb{Q}^S$ (easy to prove using Property II in the proof of the Girsanov theorem):

$$\frac{D_t V_t}{Z_t^S} = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{D_T V_T}{Z_T^S} | \mathcal{F}_t\right]$$

Risk neutral pricing under measure $\mathbb{Q}^S$

$$V_t = Z_t^S \mathbb{E}^{\mathbb{Q}^S}\left[\frac{D_T V_T}{D_t Z_T^S} | \mathcal{F}_t\right], \quad V_0 = \mathbb{E}^{\mathbb{Q}^S}\left[e^{-rT} \frac{V_T}{Z_T^S}\right]$$

Asset-or-nothing put

- For the asset-or-nothing put with payoff $S_T \mathbf{1}_{\{S_T \le K\}}$, the price is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^S}\left[e^{-rT}\frac{S_T \mathbf{1}_{\{S_T \le K\}}}{Z_T^S}\right] &= S_0 e^{-qT}\mathbb{E}^{\mathbb{Q}^S}[\mathbf{1}_{\{S_T \le K\}}] \\ &= S_0 e^{-qT}\mathbb{Q}^S(X_T \le \ln\frac{K}{S_0}) \end{aligned}$$

- In BSM, $X_T = (r - q - \frac{1}{2}\sigma^2)T + \sigma B_T^* = (r - q + \frac{1}{2}\sigma^2)T$ $+\sigma B_T^S$, we obtain

$$S_0 e^{-qT} N(-d_+)$$

General models

- In more general models (Lévy, stochastic volatility), European vanilla put price is

$$V_0 = Ke^{-rT}\mathbb{Q}(X_T \le \ln \frac{K}{S_0}) - S_0 e^{-qT}\mathbb{Q}^S(X_T \le \ln \frac{K}{S_0})$$

- Denote the characteristic function of X_T under measure $\mathbb{Q}$ by ϕ_T

$$\phi_T(\xi) = \mathbb{E}^{\mathbb{Q}}[e^{i\xi X_T}]$$

Since discounted gain process is martingale

$$\mathbb{E}^{\mathbb{Q}}[e^{-(r-q)T}S_T] = S_0 \rightarrow \mathbb{E}^{\mathbb{Q}}[e^{X_T}] = \phi_T(-i) = e^{(r-q)T}$$

Characteristic function under measure change

- Characteristic function of X_T under measure $\mathbb{Q}^S$

$$\begin{aligned}
\phi_T^S(\xi) &= \mathbb{E}^{\mathbb{Q}^S}[e^{i\xi X_T}] \\
&= \mathbb{E}^{\mathbb{Q}}[Z_T^S e^{i\xi X_T}] \\
&= \mathbb{E}^{\mathbb{Q}}[e^{(q-r)T} e^{i\xi X_T + X_T}] \\
&= \frac{\phi_T(\xi - i)}{\phi_T(-i)}
\end{aligned}$$

- For pricing European vanilla options, it suffices to invert ϕ_T and ϕ_T^S to compute the probabilities

From characteristic function to cdf

- **Gil-Pelaez formula** for inverting a characteristic function ϕ to compute the cdf F

$$F(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{i\xi x}\phi(-\xi) - e^{-i\xi x}\phi(\xi)}{i\xi} d\xi$$

- A **Hilbert transform implementation** that exhibits exponential convergence (see my papers available online)

$$F(x) \approx \frac{1}{2} + \frac{i}{2} \sum_{m=-M}^{M} e^{-ix(m-1/2)h} \frac{\phi((m-1/2)h)}{(m-1/2)\pi}, \quad h > 0, M \geq 1$$

- No dampening needed when using this approach

Quanto options

Quantos

- **Quanto derivatives**: settlement of the contract is in one currency while the underlying asset is in another currency
- CME Nikkei 225 (dollar) futures: the underlying stock index is in Japanese yen, while the futures payoff is in USD

  If a US bank expects a bull Japanese stock market, it might long CME Nikkei 225 (dollar) futures. June Nikkei 225 (dollar) futures price is currently 10075. If at maturity, the actual value of Nikkei 225 is 10080, the bank's profit is $5. It is not necessary to exchange USD for yen, buy Japanese stocks, and sell them later to get yen, and exchange yen for USD

- International diversification and speculation on foreign markets made easier for domestic investors

Multidimensional Girsanov Theorem

- Let $T > 0$ be fixed, $B_t = (B_{1t}, \cdots, B_{dt})$ be a $d-$dimensional standard Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and $\theta_t = (\theta_{1t}, \cdots, \theta_{dt})$ be a $d-$dimensional adapted process. Define

$$Z_t = \exp\left(-\int_0^t \theta_u \cdot dB_u - \frac{1}{2}\int_0^t ||\theta_u||^2 du\right),$$

$$B_t^* = B_t + \int_0^t \theta_u du \quad (\text{that is, } dB_t^* = \theta_t dt + dB_t).$$

Define probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$. Then B_t^* is a standard $d-$dimensional Brownian motion under measure $\mathbb{Q}$

Risk neutral measure in multiasset models

- Suppose there are two assets, governed by a $2-$dimensional standard Brownian motion

$$dS_{it} = (\mu_i - q_i)S_{it}dt + S_{it}\sum_{j=1}^{2}\sigma_{ij}dB_{jt}, \quad i = 1, 2$$

- Discount factor $D_t = e^{-rt}$, discounted gain process $e^{(q_i - r)t}S_{it}$ (the following derivation still goes through when q_i and r are adapted processes)
- Find measure under which the discounted gain processes are martingales

- Note that

$$de^{(q_i-r)t} = e^{(q_i-r)t}(q_i - r)dt$$

- By Ito product rule

$$d\left(e^{(q_i-r)t}S_{it}\right) = e^{(q_i-r)t}S_{it}\left((\mu_i - r)dt + \sum_{j=1}^{2}\sigma_{ij}dB_{jt}\right)$$

- Make the drift term zero for some $\theta = (\theta_1, \theta_2)$ with $\theta_i dt + dB_{it} = dB^*_{it}, i = 1, 2$

Market price of risk equations

- θ must satisfy the **market price of risk equations** below

$$\sigma_{11}\theta_1 + \sigma_{12}\theta_2 = \mu_1 - r$$

$$\sigma_{21}\theta_1 + \sigma_{22}\theta_2 = \mu_2 - r$$

- Assuming $\sigma_{11}\sigma_{22} \neq \sigma_{12}\sigma_{21}$, there is a solution $\theta = (\theta_1, \theta_2)$
- Construct measure $\mathbb{Q}$ using the Girsanov theorem with the above θ

- Under measure $\mathbb{Q}$, $B_t^* = (B_{1t}^*, B_{2t}^*)$ is a 2-dimensional standard Brownian motion, discounted gain processes are martingales

$$d\left(e^{(q_i - r)t} S_{it}\right) = e^{(q_i - r)t} S_{it} \sum_{j=1}^{2} \sigma_{ij} dB_{jt}^*$$

- Risk neutrality

$$dS_{it} = (r - q_i) S_{it} dt + S_{it} \sum_{j=1}^{2} \sigma_{ij} dB_{jt}^*, \quad i = 1, 2$$

Quanto options

- Consider a **quanto option** with payoff $(S_T - K)^+$ US dollars, where S_T is the price of a European stock denominated in Euro
- Domestic risk free interest rate r; foreign risk free interest rate r_f; exchange rate Q_t (USD/Euro) governed by

$$dQ_t = (\mu_1 - r_f)Q_t dt + \sigma_1 Q_t dB_{1t}$$

where B_{1t} is a standard Brownian motion under the **physical measure**

Foreign stock

- Foreign stock with price S_t (in foreign currency) and dividend yield q

$$dS_t = (\mu_2 - q)S_t dt + \sigma_2 S_t dB_{3t}$$

  where $dB_{3t} = \rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}$, $-1 < \rho < 1$, B_{2t} is a standard Brownian motion under the physical measure that is independent of B_{1t}

- B_{1t}, B_{3t} are correlated with correlation coefficient ρ

$$dB_{1t}dB_{3t} = \rho dt$$

- Seek an equivalent martingale measure with the **domestic money market account as numeraire**

Value of the stock in domestic currency

- **Value of the stock in domestic currency**: $S_t^d = S_t Q_t$

$$
\begin{aligned}
dS_t^d &= d(S_t Q_t) \\
&= S_t dQ_t + Q_t dS_t + dS_t dQ_t \\
&= S_t Q_t \Big((\mu_1 - r_f)dt + \sigma_1 dB_{1t} \\
&\quad + (\mu_2 - q)dt + \sigma_2 dB_{3t} + \rho\sigma_1\sigma_2 dt \Big) \\
&= (\mu_1 + \mu_2 - r_f + \rho\sigma_1\sigma_2 - q) S_t^d dt + S_t^d (\sigma_1 dB_{1t} + \sigma_2 dB_{3t}) \\
&= (\mu^d - q) S_t^d dt + S_t^d \left((\sigma_1 + \rho\sigma_2) dB_{1t} + \sigma_2 \sqrt{1-\rho^2} dB_{2t} \right)
\end{aligned}
$$

where $\mu^d = \mu_1 + \mu_2 - r_f + \rho\sigma_1\sigma_2$

- Find θ such that

$$\sigma_1\theta_1 = \mu_1 - r$$
$$(\sigma_1 + \rho\sigma_2)\theta_1 + \sigma_2\sqrt{1-\rho^2}\theta_2 = \mu^d - r$$

- Construct measure $\mathbb{Q}$ according to the 2−dimensional Girsanov theorem. Under measure $\mathbb{Q}$,

$$dQ_t = (r - r_f)Q_t dt + \sigma_1 Q_t dB^*_{1t}$$

$$\begin{aligned} dS^d_t &= (r-q)S^d_t dt + S^d_t\left((\sigma_1 + \rho\sigma_2)dB^*_{1t} + \sigma_2\sqrt{1-\rho^2}dB^*_{2t}\right) \\ &= (r-q)S^d_t dt + S^d_t\left(\sigma_1 dB^*_{1t} + \sigma_2 dB^*_{3t}\right) \end{aligned}$$

where $B^*_{3t} = \rho B^*_{1t} + \sqrt{1-\rho^2}B^*_{2t}$

Risk neutral valuation

- The value of the quanto option is given by

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+]$$

- Need $dS_t = d(S_t^d/Q_t)$ under measure $\mathbb{Q}$

$$d\frac{1}{Q_t} = (r_f - r + \sigma_1^2)\frac{1}{Q_t}dt - \sigma_1\frac{1}{Q_t}dB_{1t}^*$$

$$dS_t^d = (r-q)S_t^d dt + S_t^d\left(\sigma_1 dB_{1t}^* + \sigma_2 dB_{3t}^*\right)$$

- By Itô product rule,

$$\begin{aligned} dS_t &= d\frac{S_t^d}{Q_t} \\ &= S_t^d d\frac{1}{Q_t} + \frac{1}{Q_t} dS_t^d + dS_t^d d\frac{1}{Q_t} \\ &= S_t\Big((r_f - r + \sigma_1^2)dt - \sigma_1 dB_{1t}^* \\ &\quad +(r-q)dt + \sigma_1 dB_{1t}^* + \sigma_2 dB_{3t}^* - (\sigma_1^2 + \rho\sigma_1\sigma_2)dt\Big) \\ &= (r_f - q - \rho\sigma_1\sigma_2)S_t dt + \sigma_2 S_t dB_{3t}^* \end{aligned}$$

- For the **convenience** of applying Black-Scholes formula, write

$$dS_t = (r - \hat{q})S_t dt + \sigma_2 S_t dB_{3t}^*, \quad \hat{q} = r - r_f + q + \rho\sigma_1\sigma_2$$

- By Black-Scholes call price formula, the value of the quanto option is

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+] \\ &= S_0 e^{-\hat{q}T} N(d_+) - Ke^{-rT} N(d_-) \end{aligned}$$

where

$$d_\pm = \frac{\ln(S_0/K) + (r - \hat{q} \pm \frac{1}{2}\sigma_2^2)T}{\sigma_2\sqrt{T}}$$