FIN516: Term-structure Models

Lecture 6: Numerical methods for one-factor short rate models: BDT

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Introduction

- Having introduced the traditional one factor models (Vasicek, CIR, HW) we will now construct numerical methods for valuing interest rate derivatives.
- As we have seen, some of these models have analytic formulas for some derivatives whereas other models do not even lead to analytic formulas for the zero coupon bond price.
- Here we will develop binomial and trinomial trees for the instantaneous short rate, sometimes matching the bond prices to the market term structure, other times also fitting them to the term structure of volatility or to market cap or swaption prices.
- We will start with the BDT model, and show how to fit it to bond prices, then volatilities and then move on to the HW model, and then finally the BK model.

Black Derman and Toy in continuous time

- Another popular one-factor model is the Black-Derman and Toy (1990) model (paper on COMPASS). It is usually known as BDT.
- This was developed in discrete time using the binomial model as we will see next. It can, however, be considered as a continuous time model too.
- As a continuous time model the BDT can be described as

$$d \ln r(t) = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r(t)\right] dt + \sigma(t) dW$$

- This looks similar to the HW model as there is a long term time varying mean $\theta(t)$ and it generally exhibits mean reversion.
- The main advantage of the BDT method is that spot rates are lognormally distributed rather than normally distributed, so they cannot go negative.

Quiz

What is $\sigma(t)$ the BDT model?

- The volatility of the instantaneous spot rate.
- The volatility of the forward rate.
- The volatility of the forward swap rate.
- All of the above.

Quiz

What is $\sigma(t)$ the BDT model?

- The volatility of the instantaneous spot rate. CORRECT
- The volatility of the forward rate.
- The volatility of the forward swap rate.
- All of the above.

Black, Derman and Toy

- However, this feature comes at a cost. There are no analytic formulas even for the zero coupon bond price and so numerical methods must be used.
- This may not seem to be a major drawback as to value many derivatives numerical must be used even for the Vasicek model. However, when we discussed numerical methods we saw that having analytic zero coupon bond prices was very useful and simplifies the valuation process considerably.
- As with the HW method, σ is often modeled as a constant rather than a time varying parameter, again because of the difficulty in matching the volatility term structure, even with time varying parameters. In this case

$$d \ln r(t) = \theta(t)dt + \sigma dW(t)$$

  as $\sigma'(t) = 0$ if $\sigma(t)$ is constant.
- Note that the rate of mean reversion is tied up with the volatility. This makes calibration methods easier but leads to a restriction of parameters, making it harder to fit the model to market data.

Black Derman and Toy: Binomial tree

- Divide the time period T into $N\Delta t$ years, as we have a binomial tree then there are i time steps and $i+1$ states $0, 1, \ldots, N$.
- $r(i,j)$ is the annualized one-period short rate at period i and state j. The short rate evolves to either $r(i+1,j)$ or $r(i+1,j+1)$ when there are down or up moves respectively.
- As in Lecture 3, here we assume that the probability of up and down movements are both equal to $\frac{1}{2}$.
- Let the current market yield be $\hat{Y}(i)$ on an i-period zero-coupon bond, and $\hat{\sigma}_Y(i)$ be the corresponding **yield** volatility. The price of the zero coupon bond $\hat{P}(0, i\Delta t) = \hat{P}(i)$ is given as

$$\hat{P}(i) = \frac{1}{(1+\hat{Y}(i)\Delta t)^i}$$

Black Derman and Toy: Binomial tree

- Let $P(i)$ and $Y(i)$ be the price and yield from the BDT binomial tree, and $\sigma_Y(i)$ be the volatility corresponding to the n-period yield.
- The model is calibrated so that $P(i) = \hat{P}(i)$ and $\sigma_Y(i) = \hat{\sigma}_Y(i)$ for all i.
- Note that $Y(i)$ could be simple interest, continuously compounded, or any other convention as it is the zero coupon bond prices, $\hat{P}(i)$ that we will actually calibrate to.

Quiz

Do we know the volatility of the yield?

- Yes, we can get it from market data.
- It can be estimated but is not easily determined from market data.
- No, it is an unknown quantity.

Quiz

Do we know the volatility of the yield?

- Yes, we can get it from market data. Correct
- It can be estimated but is not easily determined from market data.
- No, it is an unknown quantity.

Building the tree from time zero

- Consider the first zero coupon bond $P(1)$ which can be written as

$$P(1) = e^{-r(0,0)\Delta t}$$

assuming that $r(0,0)$ is a continuously compounded rate.

- And so, if we rearrange then

$$r(0,0) = -\frac{\ln P(1)}{\Delta t}$$

- To begin with assume that we know the volatility of the short rate process at each point in time. To recover the conditional variance we require that

$$vol = \frac{\ln yield_{up} - \ln yield_{down}}{2}$$

where $yield_{up}/yield_{down}$ is the yield on a bond maturing at the next time step, seen from the up/down state. The way we accomplish this is to introduce a time varying volatility term $\sigma(i)$ where

$$r(i,j+1) = r(i,j)e^{2\sigma(i)\sqrt{\Delta t}}$$

this determines the spacing between the steps.

Building the tree from time zero

- Note that the zero coupon bond at time $i = 2$ can be expressed as

$$P(2) = e^{-r(0,0)\Delta t}\left[\frac{1}{2}e^{-r(1,1)\Delta t} + \frac{1}{2}e^{-r(1,0)\Delta t}\right]$$

- There appears to be three unknowns in this equation but recall that we know $r(0,0)$ from the price of the first zero coupon bond, and, assuming that we know $\sigma(1)$ then we can determine $r(1,1)$ from $r(1,0)$ or vice versa.
- Thus, there is only one unknown parameter and so the market price of the bond is simply a function of one unknown.
- Let's introduce a new parameter $U(i)$ which will be our free parameter, so that

$$r(i,j) = U(i)e^{(2j-i)\sigma(i)\sqrt{\Delta t}}$$

where $0 \leq j \leq i$.

Calibrating to bond prices

- As in lecture 3 we introduce a new security $\hat{\pi}(i,j)$ which pays out 1 at time i but only in state j.
- Strictly speaking we should specify from which nodes the security is issued but at the moment it is issued at time 0 (and therefore state 0).
- Then the current value of a zero coupon bond of maturity $n\Delta t$ is

$$P(i) = \sum_{j=0}^{i} \hat{\pi}(i,j)$$

and recall that we can also write

$$P(i+1) = \sum_{j=0}^{i} \hat{\pi}(i,j) e^{-r(i,j)\Delta t}$$

Calibrating to bond prices

- We have seen this in lecture 3 that we can use the result that

$$
\begin{aligned}
\hat{\pi}(i,0) &= \frac{1}{2}\hat{\pi}(i-1,0)e^{-r(i-1,0)\Delta t} \\
\hat{\pi}(i,j) &= \frac{1}{2}\hat{\pi}(i-1,j)e^{-r(i-1,j)\Delta t} + \frac{1}{2}\hat{\pi}(i-1,j-1)e^{-r(i-1,j-1)\Delta t} \\
\hat{\pi}(i,i) &= \frac{1}{2}\hat{\pi}(i-1,i-1)e^{-r(i-1,i-1)\Delta t}
\end{aligned}
$$

- We now have all of the tools required to generate the BDT that is fitted to yield curve data.

BDT algorithm

1. First determine $\hat{\pi}(0,0)$ from the market price of zero coupon bonds:
$$\hat{P}(0) = \hat{\pi}(0,0) = 1$$
2. The solve to find $r(0,0)$ by solving
$$\hat{P}(1) = \hat{\pi}(0,0)e^{-r(0,0)\Delta t}$$
3. Next, calculate the Arrow-Debreu prices at time 1 as follows:
$$\begin{aligned}\hat{\pi}(1,0) &= \frac{1}{2}\hat{\pi}(0,0)e^{-r(0,0)\Delta t}\\ \hat{\pi}(1,1) &= \frac{1}{2}\hat{\pi}(0,0)e^{-r(0,0)\Delta t}\end{aligned}$$
4. Then calibrate $U(1)$ to the market price $\hat{P}(2)$
$$\hat{P}(2) = \sum_{j=0}^{1} \hat{\pi}(1,j)\exp\left(-U(1)e^{(2j-1)\sigma(1)\sqrt{\Delta t}}\Delta t\right)$$
5. Then determine $r(1,j)$ and then use to calculate the new $\hat{\pi}(2,j)$ and then solve to find $U(2)$ and then proceed until $i = N$.

Finding $U(i)$

- We need to use some kind of iterative procedure to determine the values of $U(i)$.
- The simplest would be to use Newton iteration, the variable is $U(i)$, the function that we are solving is

$$f(U(i)) = \hat{P}(i+1) - \sum_{j=0}^{i} \hat{\pi}(i,j) \exp\left(-U(i) e^{(2j-i)\sigma(i)\sqrt{\Delta t}} \Delta t\right) = 0$$

- We need an initial guess for $U^0(i)$, perhaps $U(n-1)$, and then a second guess $U^1(i) = U(n-1) - 0.1\%$. Then we can estimate the derivative of the function and then complete the Newton iteration:

$$U^{k+1}(i) = U^k(i) - \frac{f(U^k(i))}{f'(U^k(i))}$$

Example

- Let's start with a simple example. Let's assume a constant continuously compounded 5% yield, $\Delta t = 1$ and try to calibrate the model to yearly zero coupon bond prices up to four years.
- To see how the model works we are using MS Excel, and this would normally have to be coupled with a VBA code for the numerical iteration or repeated use of MS Excel tools such as Solver or Goal Seek. This becomes more onerous as we have more and more steps.
- We will form three trees, the spot rate tree, the discount rate tree d and the Arrow-Debreu price tree, $\hat{\pi}$. At time 0

$$
\begin{aligned}
U(0) &= r(0,0) = 0.05 \\
d(0,0) &= e^{-r(0,0)} = 0.951229 \\
\hat{\pi}(1,0) &= \hat{\pi}(0,0)\frac{1}{2}d(0,0) = 0.4756 \\
\hat{\pi}(1,1) &= \hat{\pi}(0,0)\frac{1}{2}d(0,0) = 0.4756
\end{aligned}
$$

Example

- Then we need to find $U(1)$ by solving the following equation:

$$\begin{aligned} 0.9048 &= 0.4756 \exp\left(-U(1)e^{-\sigma\sqrt{\Delta t}}\Delta t\right) \\ &\quad +0.4756 \exp\left(-U(1)e^{\sigma\sqrt{\Delta t}}\Delta t\right) \end{aligned}$$

  The solution is, $U(1) = 0.04976$ or 4.976%.

- Then we start again by determining the $\hat{\pi}$ values:

$$\begin{aligned} \hat{\pi}(2,0) &= \hat{\pi}(1,0)\frac{1}{2}d(1,0) = 0.2273 \\ \hat{\pi}(2,1) &= \hat{\pi}(1,0)\frac{1}{2}d(1,0) + \hat{\pi}(1,1)\frac{1}{2}d(1,1) = 0.4524 \\ \hat{\pi}(2,2) &= \hat{\pi}(1,1)\frac{1}{2}d(1,1) = 0.2251 \end{aligned}$$

  and then repeat the above process to fill in the rest of the values in the spreadsheet.

Quiz

If we want more time steps but there are no traded bonds, what do we do?

- It is not possible to have time steps where there are no bond prices.
- Use the yield curve to interpolate the zero coupon bond price.
- Only calibrate when there are prices, and keep $U(i)$ fixed in between.

Quiz

If we want more time steps but there are no traded bonds, what do we do?

- It is not possible to have time steps where there are no bond prices.
- Use the yield curve to interpolate the zero coupon bond price.
  Correct
- Only calibrate when there are prices, and keep $U(i)$ fixed in between.

Calibrating to cap data

- The above calibration relied on a mysterious calculation of the volatility of zero coupon bond yields.
- Typically when refer to calibrations we refer to calibrating our model to the market price of either caps or swaptions. We will see if we can adapt the BDT to calibrate it to cap (or caplet) prices.
- This will be surprisingly easy, and partly explains the strange structure of the process chosen in the BDT model.
- Afterwards, we will detail how to do a calibration to known yield volatilities but the most usual calibration to caps is detailed below.

BDT implications for volatility of spot rate

- Kazziha and Rebonato (1998) show that for a general stochastic process of the form:

$$d\ln r(t) = [a(t)\,(b(t) - \ln r(t))] + \sigma(t)dW(t)$$

where $a(t), b(t)$ and $\sigma(t)$ are deterministic functions of time then

$$\mathrm{var}[\ln r(T)] = \exp\left[-2\int_0^T a(s)ds\right]\int_0^T \sigma^2(t)\exp\left[2\int_0^t a(s)ds\right]dt$$

BDT implications for volatility of spot rate

- The BDT model is a special model where $a(t) = -f'(t)$ and $f(t) = \ln \sigma(t)$ and so this equation simplifies to

$$\begin{aligned}
\mathrm{var}[\ln r(T)] &= \exp\left[2f(T) - f(0)\right] \int_0^T \sigma^2(t) \exp\left[-2f(t) - f(0)\right] dt \\
&= \exp\left[2f(T)\right] \int_0^T \sigma^2(t) \exp\left[-2f(t)\right] dt \\
&= \sigma(T)^2 T
\end{aligned}$$

- This is remarkable result that despite the process being mean reverting the unconditional variance of the short rate is simply the instantaneous variance at time T. It does not depend upon the previous short rate values or the previous instantaneous volatility parameters.

Quiz

In problem set 3, we actually used the BDT tree set up with $\sigma(i) = 0.1$. What Black implied volatilities did you find for caps and swaptions?

- I have no idea!
- ≈ 0.1
- $<< 0.1$
- $>> 0.1$

Quiz

In problem set 3, we actually used the BDT tree set up with $\sigma(i) = 0.1$. What Black implied volatilities did you find for caps and swaptions?

- I have no idea!
- ≈ 0.1 Correct
- $<< 0.1$
- $>> 0.1$

Lessons for calibration to cap prices

- The Black model, uses the unconditional variance of the forward rates (or the spot rate at the expiry of the caplet) then the quoted Black caplet volatilities can essentially be used as inputs for the instantaneous volatilities in the BDT model.
- They do not match exactly (see Rebonato (1998)) because the spot rate for the cap is a three-month rate, whereas in the BDT model the rate is typically much shorter. However, the approximation is very close and can be used as an excellent first approximation.
- So the Black model assumes a constant volatility structure for the lifetime of the caplet. When calibrating the BDT model we will adapt a similar procedure, using the implied Black volatility as the volatility for each caplet. So the volatility $\sigma(i)$ will only change on the maturity of each caplet.
- Note that although the rate $L(T_k, T_{k+1})$ is clearly known at the reset T_k this is not true of the *short rates* from T_k to T_{k+1} and so we have to continue to evolve those until T_{k+1}.

Algorithm for cap calibration

- First strip the caplet volatilities from the market cap prices. Assume that we have caplets on three month LIBOR, and these caps mature at time T_{k+1} and pay $N \max[L(T_k, T_{k+1}) - K, 0]\tau_k$.
- Recall that the Black implied caplet volatility is for the period up to the reset T_k but is extracted from the caplet with maturity T_{k+1}. We will denote these volatilities as $\sigma_{\text{Black}}(T_k)$
- Then we have a series of reset dates T_k and associated volatilities $\sigma_{\text{Black}}(T_k)$, and so in a tree with N time steps, we wish to specify the volatility of the short rate at each time step, $\sigma(i)$. We also have time steps i_k that correspond to the reset dates of the caplets, such that $i_k \Delta t = T_k$.
- Note that to fully calibrate the model, even with piecewise constant volatility, then you would have to undergo a complex process that we will describe in "BDT: volatility". However, with the nice feature of the BDT instantaneous volatility the process is much simplified.

Algorithm for cap calibration

- The algorithm proceeds as follows:
- Set the initial volatilities, $\sigma(i)$, based on the caplets with resets at T_k.
- Starting with the first caplet, with a reset date of T_1, maturing at T_2. This gives us volatilities for time steps up to $(i_2 - 1)\Delta t$:

$$\sigma(0), \ldots, \sigma(i_2 - 1) = \sigma_{\mathrm{Black}}(T_1)$$

and then, in general

$$\sigma(i_k), \ldots, \sigma(i_{k+1} - 1) = \sigma_{\mathrm{Black}}(T_k)$$

Algorithm for cap calibration

- Based on these volatilities, we can then calibrate the model to the yield curve, by finding $U(i)$ in the standard calibration above.
- Then we can iterate the piecewise constant volatilities to exactly match the caplet prices.
- Afterthe volatility iteration the $P(i)$ values will have changed again but should be very close to $\hat{P}(i)$. If necessary you can repeat the iteration, finding the new $U(i)$ values (using the previous $\sigma(i)$ value) and then generate new volatility estimates, but typically one iteration is enough.

Example

- Imagine we have $T = 1.25$ and $N = 15$, so $\Delta t = 1/12$. We have four caplets with resets $T_1 = 0.25$, $T_2 = 0.5$, $T_3 = 0.75$, $T_4 = 1$ and the following Black caplet vols: $\sigma_{\text{Black}}(T_1) = 0.75, \sigma_{\text{Black}}(T_2) = 0.70, \sigma_{\text{Black}}(T_1) = 0.65, \sigma_{\text{Black}}(T_1) = 0.60$
- Then set up the vols:

$$
\begin{aligned}
\sigma(1) = \quad & \ldots \quad = \sigma(5) = 0.75 \\
\sigma(6) = \quad & \ldots \quad = \sigma(8) = 0.70 \\
\sigma(9) = \quad & \ldots \quad = \sigma(11) = 0.65 \\
\sigma(12) = \quad & \ldots \quad = \sigma(14) = 0.60
\end{aligned}
$$

  (Note, $\sigma(15)$ is unimportant for valuation).
- Now, with these vols, find $U(i)$ for $i = 1, \ldots, 15$ as in Section "BDT: tree" above.

Example

- Now price the caplets using the tree and see how closely they match the Black prices. If they are not close you can change the volatilities so that they match precisely.
- We have four volatilities to vary:

$$\begin{aligned}
\sigma(1) = &\quad \dots \quad &= \sigma(5) = \sigma_1 \\
\sigma(6) = &\quad \dots \quad &= \sigma(8) = \sigma_2 \\
\sigma(9) = &\quad \dots \quad &= \sigma(11) = \sigma_3 \\
\sigma(12) = &\quad \dots \quad &= \sigma(14) = \sigma_4
\end{aligned}$$

  to match the caplets exactly.
- Now the $P(i)$ values will have changed slightly (although the errors are usually small) and so you can repeat the process again to find the $U(i)$ values.

Setting $\sigma(i)$

- In the first example we assumed that $\sigma(i)$ was constant. This led to no mean reversion in the spot rate.
- For cap calibration we can assume that the vols are piecewise constant and use the pleasant feature of the BDT underlying process to easily calibrate to caplets.
- In the original exposition, it is assumed that the volatility of the yield is known and so I will also show you this case, although the standard cap calibration will nearly always be preferred.

$P_U(i)$ and $P_D(i)$

- Let's introduce new notation, at node (i, j) then the short rate is still given by
$$r(i, i) = U(i)e^{(2j-i)\sigma(i)\sqrt{\Delta t}}$$

- Starting at $(0, 0)$, let the up node $(1, 1)$ be U and the down node $(1, 0)$ be D and define $P_U(i)$ and $P_D(i)$ as the discount function evaluated at these two nodes for $i > 1$. Also, let $R_U(i)$ and $R_D(i)$ be the continuously compounded rates computed from the zero coupon bond prices.
- Specifying the spot rate yield and the volatility curve at time 0 is equivalent to specifying $P_U(i)$ and $P_D(i)$ at $i = 1$.
- This is the first step in setting up the tree is to determine $P_U(i)$ and $P_D(i)$ for all $i \geq 2$. The known zero coupon bond prices and the new P functions are related by the following:
$$e^{-r(0,0)\Delta t}\left[0.5P_U(i) + 0.5P_D(i)\right] = P(i)$$

$P_U(i)$ and $P_D(i)$

- The initial volatilities can be recovered from the standard BDT equation:

$$\sigma_R(i)\sqrt{\Delta t} = \frac{1}{2}\ln\frac{\ln P_U(i)}{\ln P_D(i)}$$

- Combining the two above equations gives us:

$$P_D(i) = P_U(i)^{\exp(-2\sigma_R(i)\sqrt{\Delta t})}$$

where $P_U(i)$ is the solution of

$$P_U(i) + P_U(i)^{\exp(-2\sigma_R(i)\sqrt{\Delta t})} = 2P(i)e^{r(0,0)\Delta t}$$

$\hat{\pi}_U(i,j)$ and $\hat{\pi}_D(i,j)$

- As before we use forward induction with the Arrow-Debreu prices. There is a small adjustment now, as we need a set of AD prices for U and D. Denote the following:
  - $\hat{\pi}_U(i,j)$: The value, seen from node U of a security that pays off 1 if node (i,j) is reached and zero otherwise.
  - $\hat{\pi}_D(i,j)$: The value, seen from node D of a security that pays off 1 if node (i,j) is reached and zero otherwise.
- The tree is now constructed using a similar method to previously, we have similar equations for the zero coupon bond prices:

$$P_U(i+1) = \sum_{j=0}^{i} \hat{\pi}_U(i,j) e^{-r(i,j)\Delta t}$$

$$P_D(i+1) = \sum_{j=0}^{i} \hat{\pi}_D(i,j) e^{-r(i,j)\Delta t}$$

Quiz

What is $\hat{\pi}_U(1,1)$?

- 1
- $e^{-r(0,0)\Delta t}$
- 0

Quiz

What is $\hat{\pi}_U(1,1)$?

- 1 Correct
- $e^{-r(0,0)\Delta t}$
- 0

Constructing the tree

- Where, of course

$$r(i,j) = U(i)e^{(2j-i)\sigma(i)\sqrt{\Delta t}}$$

- These equations have two unknowns $U(i)$ and $\sigma(i)$ that can solved by a two-dimensional Newton iteration.
- The forward step is again analogous to the simpler case:

$$\begin{aligned}
\hat{\pi}_U(i,1) &= \frac{1}{2}\hat{\pi}_U(i-1,1)e^{-r(i-1,1)\Delta t} \\
\hat{\pi}_U(i,j) &= \frac{1}{2}\hat{\pi}_U(i-1,j)e^{-r(i-1,j)\Delta t} + \frac{1}{2}\hat{\pi}_U(i-1,i-1)e^{-r(i-1,i-1)\Delta t} \\
\hat{\pi}_U(i,i) &= \frac{1}{2}\hat{\pi}_U(i-1,i-1)e^{-r(i-1,i-1)\Delta t}
\end{aligned}$$

with the $\hat{\pi}_D$ equations being analogous.

The algorithm

1. Recored initial values: $U(0) = r(0,0) = R(1)$, $\hat{\pi}_U(1,1) = \hat{\pi}_D(1,0) = 1$, $\sigma(0) = \sigma_R(1)$, discount factor $d(0,0) = e^{-r(0,0)\Delta t}$.
2. Derive $P_U(i)$ and $P_D(i)$ for $i = 2, \ldots, N$ by solving the simultaneous equations above.
3. Find $U(1)$ and $\sigma(1)$ by iteration and thus determine $r(1,j)$ and $d(1,j)$.
4. Calculate the values of $\hat{\pi}_D(2,j)$ and $\hat{\pi}_U(2,j)$, then again solve to find $U(2)$ and $\sigma(2)$ by iteration and thus determine $r(2,j)$ and $d(2,j)(= e^{-r(2,j)\Delta t})$
5. Continue to find all rates $r(i,j)$.

Aside: Newton in 2-D

- We now need to use some kind of iterative procedure to determine the values of $U(i)$ and $\sigma(i)$. This is now two non-linear equations in two unknowns.
- The simplest would be to use Newton iteration, the variables are $U(i)$ and, the functions we are solving are

$$
\begin{aligned}
f(U(i),\sigma(i)) &= P_U^M(i+1) - \sum_{j=1}^{i} \hat{\pi}_U(i,j) \exp\left(-U(i)e^{(2j-i)\sigma(i)\sqrt{\Delta t}}\Delta t\right) = 0 \\
g(U(i),\sigma(i)) &= P_D^M(i+1) - \sum_{j=0}^{i-1} \hat{\pi}_D(i,j) \exp\left(-U(i)e^{(2j-i)\sigma(i)\sqrt{\Delta t}}\Delta t\right) = 0
\end{aligned}
$$

- Let $\mathbf{x} = (U(i),\sigma(i))$, $\mathbf{f} = \begin{pmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{pmatrix}$. We need an initial guess for $\mathbf{x}^0 = (U^0(i),\sigma^0(i)$, perhaps $(U(i-1),\sigma(i-1)$, and then a second guess $\mathbf{x}^1 = (U^1(i),\sigma^1(i)) = (U(i-1) - 0.1\%, \sigma(i-1) - 0.1\%)$.

Aside: Newton in 2-D

- We then get the next guess as

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta\mathbf{x}$$

where

$$\Delta\mathbf{x} = -(J(\mathbf{x}^k))^{-1}\mathbf{f}(\mathbf{x}^k)$$

and J is the Jacobian

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial U(i)} & \frac{\partial f}{\partial \sigma(i)} \\ \frac{\partial g}{\partial U(i)} & \frac{\partial g}{\partial \sigma(i)} \end{pmatrix}$$

Example

- Let's start with a simple example. Let's assume a constant continuously compounded 5% yield, $\Delta t = 1$ and try to calibrate the model to yearly zero coupon bond prices up to four years.
- The addition is that we now know the yearly yield volatilities which we have extracted from market data, these are decreasing over time, from 0.1 in year 1 to 0.07 in year 4.
- We now construct trees for $\hat{\pi}_U$ and $\hat{\pi}_D$ rather than just $\hat{\pi}$ but note that these trees now start from either state $(1,0)$ or $(1,1)$ and so are smaller than the tree for $\hat{\pi}$ in the previous example.
- The initial information gives us:

$$\begin{aligned} U(0) &= r(0,0) = R(0,1) = 0.05 \\ \sigma(0) &= \sigma_R(0,1) = 0.1 \\ d(0,0) &= e^{-r(0,0)\Delta t} = 0.9512 \\ \hat{\pi}_U(1,1) &= 1 \\ \hat{\pi}_D(1,0) &= 1 \end{aligned}$$

Example

- Then we solve to find the P_U and P_D values for each n all the way up until year 4 - not 3 that we construct the tree for.
- At $t = 2$ to do this we use solver or other iteration to solve the following equation

$$P_U(2) + P_U(2)^{\exp(-(0.09)2\sqrt{1})} = 2(0.9048)e^{0.05(1)}$$

which solves to give $P_U(2) = 0.9470$ and then derive P_D using the following equation:

$$P_D(2) = 0.9470^{\exp(-2(0.09)\sqrt{1})}$$

which gives $P_D = 0.9554$. This is continued to find the P_U and P_D values at $i = 3$ and $i = 4$.

Example

- Then we proceed to find the value of $U(1)$ and $\sigma(1)$ that matches the bond prices and the volatility of the yield. There are two equations and two unknowns as follows:

$$
\begin{aligned}
P_U(2) &= \hat{\pi}_U(1,1)\exp\left(-U(1)e^{\sigma(1)(1)\sqrt{1}}(1)\right) \\
0.9470 &= \exp\left(-U(1)e^{\sigma(1)(1)\sqrt{1}}(1)\right) \\
P_D(2) &= \hat{\pi}_D(1,0)\exp\left(-U(1)e^{\sigma(1)(-1)\sqrt{1}}(1)\right) \\
0.9554 &= \exp\left(-U(1)e^{\sigma(1)(-1)\sqrt{1}}(1)\right).
\end{aligned}
$$

The solution is $U(1) = 0.0498, \sigma(1) = 0.9$.

- Then we can extract the key values at $i = 1$ as follows:

$$
\begin{aligned}
r(1,1) &= 0.0498\exp(0.9(1)\sqrt{1})(1) \\
&= 0.5450 \\
r(1,0) &= 0.0498\exp(0.9(-1)\sqrt{1})(1) \\
&= 0.4552
\end{aligned}
$$

Example

- The final tricky part is to make sure that you construct the AD securities correctly at the next step.

$$
\begin{aligned}
\hat{\pi}_U(2,2) &= 0.5\hat{\pi}_U(1,1)d(1,1) = 0.4735 \\
\hat{\pi}_U(2,1) &= 0.5\hat{\pi}_U(1,1)d(1,1) = 0.4735 \\
\hat{\pi}_D(2,1) &= 0.5\hat{\pi}_D(1,0)d(1,0) = 0.4777 \\
\hat{\pi}_D(2,0) &= 0.5\hat{\pi}_D(1,0)d(1,0) = 0.4777
\end{aligned}
$$

- Then we can proceed as above to find the values of $U(2)$ and $\sigma(2)$.
- See the spreadsheet BDTyieldvol.xls for more details.

Conclusion

- Her we have seen how to build a standard binomial tree for short rate models - in particular for the BDT model.
- It is possible also to adjust the tree so that it matches the current term structure of interest rates and also, if desired, a particular volatility structure.
- The main appeal of the BDT model is that its particular form leads to the short rate volatility being simply the instantaneous volatility at time t, this makes it very useful for calibrating to market cap data.
- As we will see, the BDT has drawbacks which result from this structure.