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Lecture Note 2.1: Swaps

In the next few lectures we will look at swap contracts in our no-arbitrage framework.

Swaps are an amazingly broad class of derivatives. We will describe the major types of swaps and indicate some of the different financial engineering problems that swaps are designed to solve.

Our main goal today is to try to understand the valuation principles behind all swaps. We begin by analyzing swaps that are a lot like forwards contracts. As with forwards, we will figure out the fair *rate* for a new swap (so that its price is zero) and the fair *price* of an old swap. We start by solving these problems in a world of perfect markets.

Outline:

I. What are swaps?

II. Commodity swaps.

III. Currency swaps.

IV. A note about the assumptions.

V. Summary

I. Definition of a swap.

- A swap contract is an agreement between two parties to exchange two *streams* of some good or cash flows, over several dates until some terminal horizon T.

- The simplest swap can be viewed as the exchange of

  (**A**) A fixed quantity of a good, for

  (**B**) A fixed amount of money
  (or fixed monetary amount of the good)

  at a <u>series</u> of pre-specified dates.

- It is a somewhat surprising fact that all the swaps we will consider can be built up from simple swaps like this.

- Besides the good, the swap contract also specifies:
  - ▶ the dates of exchange.
  - ▶ delivery instructions for both parties (including netting).
  - ▶ conditions for assignability or cancellation;
  - ▶ remedies in the case of default.

- Swaps have a lot of similarities to forwards.
  - ▶ Swaps are traded exclusively in OTC markets (for now).
  - ▶ The terms are customizable.
  - ▶ Neither side pays the other to enter a swap.

- Most contracts follow the legal template of the ISDA Master Agreement, and are enforceable under U.S. or English law.

II. Swap pricing: Commodity swaps.

- We now examine the no-arbitrage valuation of several types of swaps. For this part of the lecture, we assume **markets are perfect**. The most important aspects of this assumption are

  Riskless borrowing and lending. In particular, we imagine that there is a term structure of riskless zero coupon bonds of price $B_{t,T}$ for any maturity you want *and* you can buy them (lend) or costlessly short sell them (borrow).

  Reliable counterparties. Unless otherwise indicated, we will assume that both sides to any swap contract are certain to perform their obligations over the life of the swap.

- A commodity swap is identical to a long-term fixed-price contract to deliver the good.

- The reason for their existence is obvious: both producers and consumers may wish to lock in the revenue or cost of some physical product that they know they will need repeatedly.
  - ▶ Energy products (heating oil, jet fuel, etc) are prominent examples.

Example: What is this contract worth?

- Consider the two sides of a two-year, semi-annual swap on 100 Oz of gold for \$1500. Value each side separately.

  Side 1 Delivery of 100 Oz of gold in 6 months, 100 Oz of gold in 12 months, 100 Oz of gold in 18 months, and 100 Oz of gold in 24 months.

  - ▶ A gold "annuity".
  - ▶ Can convert to a \$ payment stream via gold forward sales.

  Side 2 Delivery of \$150,000 at same dates.

  - ▶ Equivalently \$150,000 worth of gold at the then-prevailing spot price.
    - ∗ Possibly simpler to implement, since then just exchange the net amount of gold.
  - ▶ Either way, value as just a certain payout stream of cash.

Example (continued) So suppose interest rates and forward prices are as shown below:

Date	Bond	Fwd	Side 1 delivers		Side 2 delivers	
$t - t_0$	$e^{-r_t(t-t_0)}$	$F_{t_0,t}$	Gold or	t_0 dollars	Gold or	t_0 dollars
0	1.0	1400 (spt)	-	-	-	-
0.5	0.970	1435	100	139196	$1500/S_{t_1}$	145500
1.0	0.941	1485	100	139740	$1500/S_{t_2}$	141150
1.5	0.912	1540	100	140450	$1500/S_{t_3}$	136800
2.0	0.883	1565	100	138190	$1500/S_{t_4}$	132450
Total present value				-557575		-555900
Net value to Side 2						**+1675**

- No-arbitrage pricing at its simplest, once you take as given:
  - ▶ forward prices of gold to each delivery date; and
  - ▶ prices of $ to each date.

- Replicating portfolios:

  Side 1: buy forwards to each date (price=0) and buy enough T-bills to fund the forward purchases.

  Side 2: buy T-bills.

- **Conclusion:** Commodity swaps can be viewed as derivatives whose underlyings are riskless bonds and forwards on the commodity.

- Let's express the swap value in a formula. Say today is time $= 0$.

- Value of the contract to gold receiver (side 2) at $t = 0$:

$$V_0 = \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} e^{-r_t t}\ (100\ F_{0,t}) - \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} e^{-r_t t}\ (\$150,000)$$

$$= (100) \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} B_{0,t}\ F_{0,t} - (100\ X) \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} B_{0,t}$$

where X is the swap price in the contract (here 1500).

- What \$/oz. swap price must prevail today ($t = 0$) for a fair swap? Just set $V = 0$, and solve for X.

$$X_0 = \frac{\sum_t^T B_{0,t}\ F_{0,t}}{\sum_t^T B_{0,t}}.$$

- This swap "rate" tells you the fair price to use for a T-year fixed-price contract.
  - ▶ Not today's spot price.
  - ▶ A weighted average of forward prices.
    - ∗ What do the weights look like as a function of time-horizon?

Example (continued)

- What is the fair price today for a *new* gold swap like the one we valued above?

$$\frac{1391.96 + 1397.40 + 1404.50 + 1381.90}{0.970 + 0.941 + 0.912 + 0.883} = 1504.50$$

- We also know (from last week) that we can view forward contracts as themselves derivatives.
  - So we could also view *swaps* as derivatives of the underlying spot market.
  - But, recall, to price forwards by no-arbitrage we had to assume **known carry costs**:
  - If this condition is met, then we can replace all the Fs in the above formula by the function of S that we derived last week.

$$X_0 = \frac{\sum_t^T B_{0,t}\ S_0\ e^{(r-y+u)t}}{\sum_t^T B_{0,t}} = S_0\ \frac{\sum_t^T e^{(-y+u)t}}{\sum_t^T e^{-rt}}.$$

- So, we can still price swaps by no-arbitrage, even if a forward market *doesn't* exist (if the underlying is a tradeable asset with predictable carry cost).

- But it is important to understand that we can value the swap by no-arbitrage even if we *can't* value the forwards by no-arbitrage – as long as there is a forward market.
  - In that case, the valuation formula *does* still apply, even for things that cannot be transported into the future.
    - For example, the swap formula works for a swap contract that promises delivery of fresh (unfrozen!) fish to a sushi restaurant at the start of every week of the year... as long as there are traded forwards on fish.
    - Another example might be electricity, which is not storable.
  - **Q:** What if we have a *cash-settled swap* based on the realization of a non-physical statistic X_t – like inflation or the weather – at a series of future dates? Does the formula still apply?

- A simple variation on a standard swap is a **forward swap**. This just means both parties agree to begin the exchanges at a date sometime in the future.

Example (continued)

Q: What is the fair price today for our gold swap if we agree to start it in one year (i.e. skip the first two exchanges)?

A: This one-year-forward one-year swap must again have a price that makes the net value today of both sides zero.

$$0 = \sum_{t=1.5}^{2} B_{0,t}\ (100\ F_{0,t}) - \sum_{t=1.5}^{2} B_{0,t}\ (100\ X_{0,1,1}).$$

Or, with our numbers,

$$X_{0,1,1} = \frac{1404.50 + 1381.90}{0.912 + 0.883} = 1552.30$$

III. Currency swaps.

- Currency swaps are identical to commodity swaps where the commodity is the foreign currency (euros, say).

- Same formulas for value and fair rate (with appropriate units). (Replace $y_t - u_t$ with $r_t^{foreign}$.)

- Actually, those are the formulas for just the most basic currency swap – an **interest only** swap.

- In practice, many currency swaps originate as the exchange of cash-flows from two bonds. So, *unlike* commodity swaps, there is then an additional exchange of the principal amounts of the bonds at the start and end dates.
  - ▶ Still simple to value (assuming riskless payments) once forward prices are given.
  - ▶ Introduces another free parameter in the contract: the relative principal amounts.
    - ∗ The periodic payments, being coupons, are then expressed as a % of these amounts.

Example

- 7 year maturity, 1.25% coupon bond on ¥1b to be "swapped into dollars." (Assume annual coupons.) *How do we set the other side so as to make the swap fair?*

- ▶ Could either fix dollar principal amount and solve for fair dollar interest payment, *or*
- ▶ Fix dollar interest payment (e.g. 7-year treasury rate) and solve for principal amount.

- What would be the fair dollar interest rate today if dollar principal is \$10m?
- Just equate values of both currency flows and solve for X:

$$\yen 1,000 \cdot (1 - 0.0125 \sum_{t=1}^{7} e^{-r_t^{\yen} t} - e^{-r_7^{\yen} \cdot 7})$$
$$= S_0 \cdot \$10 \cdot (1 - \frac{X}{100} \sum_{t=1}^{7} e^{-r_t^{\$} t} - e^{-r_7^{\$} \cdot 7}).$$

where S_0 is today's spot yen-per-dollar rate.

- The first line is the value of the yen cash-flows promised by the side who issued the yen bond.

(A) He gets the principal from the issue at the start (hence the $+1$ in parantheses).

(B) Then he pays the interest payments (middle) and final principal (last term).

The formula just equates the value of this side to the value of the dollar flows to be received in the swap.

- Currency swaps are really easy to value and price (as long as we ignore counterparty default risk).

- Let's at least start to think about the effect of counterparty risk by means of a little example.

- **Example:** Suppose the risk-free term-structure is flat at 1.5% in the U.S. and flat at 0.15% in Japan. (These are continuously compounded.) The spot exchange rate is 120 ¥/$.

(A) Suppose you have the opportunity to trade in 3-year ¥/$ semi-annual *interest-only* swaps for 118 ¥/$. Assume markets are perfect. Construct an "arbitrage" (specify the trades, and show the cash-flows at each date).

Answer:

From the interest-rate term-structures we can immediately compute the no-arbitrage forward prices of FX, and the Japanese zero-coupon bond prices (remember yen is the numeraire if that is how the currency is quoted). Plugging into the no-arbitrage swap formula, we find a fair price today of

$$s = 120 * \frac{\sum_{t=0.5}^{3} e^{-0.015t}}{\sum_{t=0.5}^{3} e^{-0.0015t}} = 117.21.$$

So if we can trade at $118.00 > 117.21$, that means someone is overpaying for forward dollars. We can enter into the swap on the side that delivers dollars and receives yen.

If we do this for a notional amount of ¥118 billion, that means we will pay $1 billion every six months for the next three years. We finance each payment by buying (today) $ 1 billion face amount of Treasury bills (or riskless deposits) maturing at each date. This costs us $\sum_{t=0.5}^{3} e^{-0.015\ t} = 5.845$ billion dollars.

At the same time, we sell ¥118b face amount of riskless zero-coupon yen bonds maturing at each payment date which we will re-pay with our receipts from the swap. This sale nets us $118 \cdot \sum_{t=0.5}^{3} e^{-0.0015\ t} = 706.145$ billion yen today, which we convert into dollars in the spot market into $5.885b. This gives us enough to pay for our dollar bonds with $40 million profit left over.

However, I put arbitrage in quotation marks above for a reason...

(B) Assume you did the trades in (A), and after one and a half years your swap counterparty goes bankrupt. At that time the spot FX rate is 125, the dollar term structure is unchanged, and yen rates have risen to 0.50% to all maturities. What is the replacement value to you of the swap, i.e., the value of the remaining payments that are owed to you minus the value of the remaining payments that you owe?

Note that this is one definition of the mark-to-market value of the swap on this date.

Answer:

The replacement value of the original swap to you is the net cost of synthesizing the remaining payments at today's rates. This is

$$125 \sum_{t=0.5}^{1.5} e^{-0.015\ t} - 118 \sum_{t=0.5}^{1.5} e^{-0.005\ t}$$

which is 17.19 billion yen or 143.2 million dollars. (The number is 24.19 billion yen if you include the payment due on the current date.) The counterparty is a winner and you are a loser.

This means that you owe them this amount. It is a liability that the bankruptcy administrator will collect from you.

(C) How does the default event affect the "arbitrage" profit you thought you had?

Answer:

Actually, your arbitrage is unaffected!

If you liquidate all the zero-coupon bond positions you took in (A), they are worth

$$125 \sum_{t=0.5}^{1.5} e^{-0.015\ t} - 118 \sum_{t=0.5}^{1.5} e^{-0.005\ t}$$

which is exactly the same number as the replacement value that you owe..

So you will settle the claim against you at no net loss and you get to keep the profit you made at time zero.

- **The Point:** Counterparty risk is *asymmetric*, in that it only adversely affect positions that have evolved over time in your favor.
  - ▶ In this example, you are fortunate that you were a LOSER on your swap when the counterparty went bankrupt!
  - ▶ The real risk is that the counterparty defaults when you have a net positive value position, i.e., he effectively owes you.
    - ∗ In that case you may receive only a small fraction of the replacement value.
- The other thing to notice is that it would have been impossible to predict at the start of the swap how big your economic exposure to default risk really was.
  - ▶ It depends on how interest rates and the currency price move and when the default event occurs.
- We will continue our discussion of counterparty risk through out the term!

IV. A Note about one assumption.

- We will talk a lot about the assumptions behind the no-arbitrage valuation formula in the next lectures.

- I just want to call your attention to one respect in which we already know the analysis will be robust: transactions costs.

- Here the key is to notice that, as with forwards, the replicating positions for swaps are **static**.
  - ▶ As soon as we initiate the position, we can take all the replicating transactions instantly, and then never change them.

- So, if those initial transactions are costly (due to fees, bid/ask spreads, etc.) we can just build those costs into the value of the derivative.
  - ▶ In other words, our theory will again give us upper and lower no-arbitrage bounds for the swap rate, instead of a single price.

- **Remember:** our theory has absolutely nothing to say about what price should prevail for a derivative inside its no-arbitrage bounds. In particular, there's nothing special about the perfect-markets price: no tendency to revert towards it; nothing intrinsically "fair" about it.

V. Summary.

- The swaps we looked at today are just like baskets of forward contracts (all having the same forward price).

- So it is not surprising that the no-arbitrage swap price should be a weighted average of the underlying forward prices.
  - ▶ You **do** need to know that formula.

- The replicating/hedging positions for each side of a swap are just positions in forwards and zero coupon bonds whose quantities are equal to the quantity of the promised payments.

- If forward markets don't exist, the formula can still be enforced by arbitrage as long as the assumption of known carry costs holds, so that synthetic forwards can be created.

Lecture Note 2.1: Summary of Notation

Symbol	Page	Meaning
$S_{t_0}, F_{t_0,t}$	*p5*	*spot and forward-to-t price of an asset at time t_0*
$B_{t_0,t}$	*p5*	*riskless zero coupon bond maturing at t*
r_t	*p5*	*continuously-compounded riskless rate until time t*
$V_0(S_0)$	*p6*	*value at $t=0$ of an existing swap when spot is at S_0*
$\sum_{t=\frac{1}{2}:\frac{1}{2}}^{2}$	*p6*	*indicates that the expression following is to be*
		evaluated with $t=\frac{1}{2}, 1, \frac{3}{2}, 2$ and the terms added up
y, u	*p7*	*borrowing rate and storage cost (c.c.) for a commodity*
X_t	*p8*	*arbitrary statistic for cash-settled forward or swap*
S_0	*p11*	*spot yen/dollar exchange rate*
X_{0,t_1,t_2}	*p11*	*fair rate at $t=0$ for a forward swap running from t_1 to t_2*