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FIN 512 - Spring 2015
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Lecture Note 3.2: No-Arbitrage Restrictions and State Price Densities

Introduction:

- In this note we will pursue our inquiry into the restrictions that absence of arbitrage places on allowable options prices. Put-Call Parity constrained a put in terms of a call of the same strike and expiration. We now look at restrictions that calls (or puts) place on other calls of the same expiration.

- This leads us to an interesting insight into how derivatives are priced, and a representation for those prices that is entirely model-free. We will see the power of this representation, and discuss its (mis-)interpretation.

Outline:

I. Options Prices as a Function of Strike Price

II. More About Butterflies

III. An Abstract Pricing Result

IV. Extracting State Price Densities: An Example

V. Interpretation of Risk-Neutral Probabilities

VI. Conclusion

I. Options Prices as a Function of Strike Price

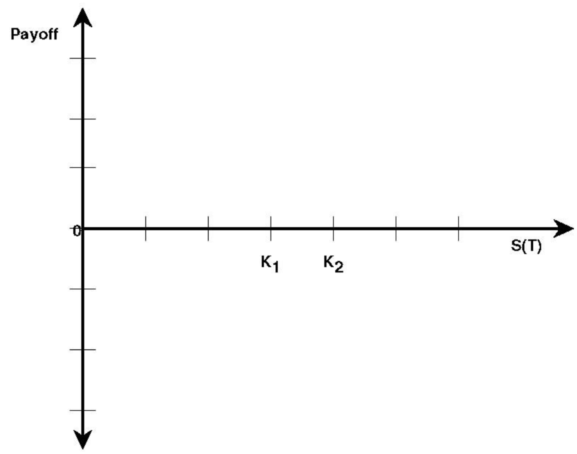
The situation is the following: markets are perfect; there are **no payouts** on the underlying; and all options are European and expire at T. I will drop all the arguments to the pricing functions except K, the strike.

(A) Spreads:

Consider the following combinations of options of the same type but different exercise prices ("vertical spreads") Here $K_2 > K_1$.

1. Bear Put Spread

- Cost today: $p(K_2) - p(K_1)$ (≥ 0)

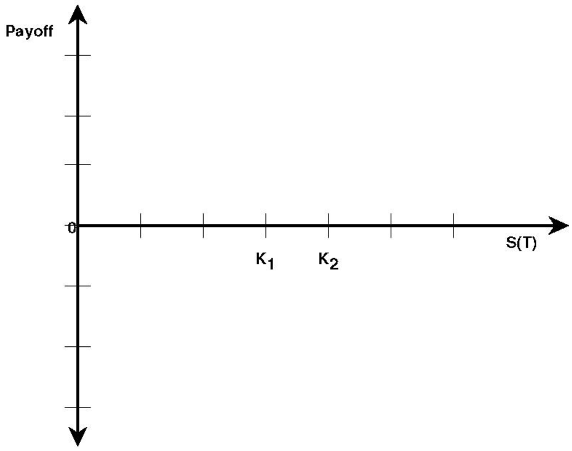


- Since the payoff is positive for all final stock prices, the position value (or price, or cost) must also be positive at all points in time. That is:

$$\text{At } t: \quad p(K_2) - p(K_1) \geq 0$$

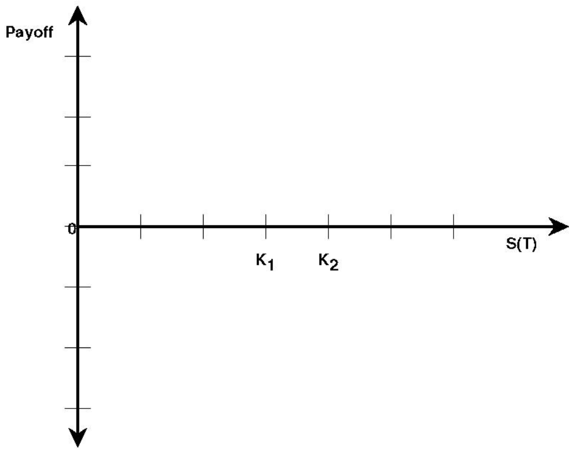
2. Bull Put Spread

- $p(K_1) - p(K_2)$ (≤ 0)



3. Bull Put Spread + Bond (with face value $K_2 - K_1$)

- $p(K_1) - p(K_2) + (K_2 - K_1) \cdot B$ (≥ 0)

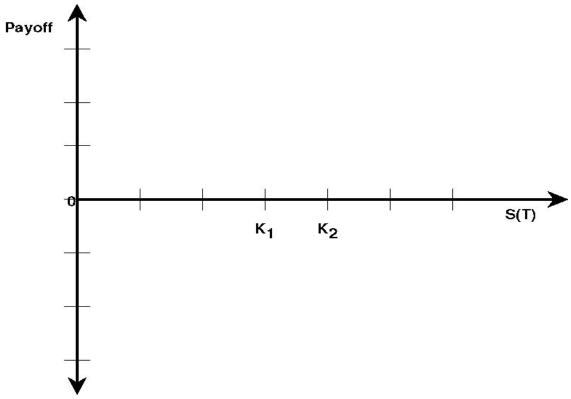


- This position always has positive value, which implies that, for a \$ 1 increase in the strike, the price of the put can increase by no more than the present value of \$ 1.

$$0 \leq \frac{p(K_2) - p(K_1)}{K_2 - K_1} \leq B \quad \Rightarrow \quad 0 \leq \frac{\partial p}{\partial K} \leq B$$

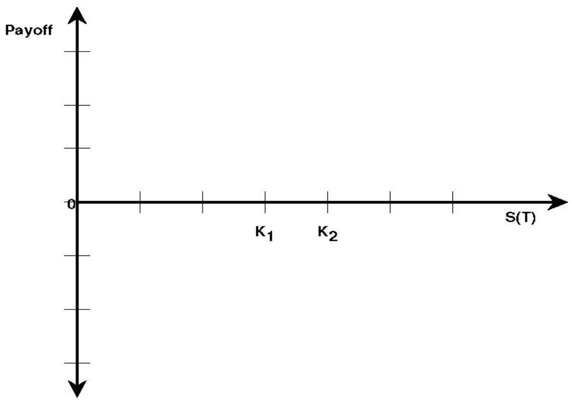
4. Bear Call Spread

- $c(K_2) - c(K_1)$ (≤ 0)



5. Bear Call Spread + Bond (with face value $K_2 - K_1$)

- $c(K_2) - c(K_1) + (K_2 - K_1) \cdot B$ (≥ 0)



- This position always has positive value, which implies that, for a \$ 1 increase in the strike, the price of the call can decrease by no more than the present value of \$ 1.

$$-B \leq \frac{c(K_2) - c(K_1)}{K_2 - K_1} \leq 0 \quad \Rightarrow \quad -B \leq \frac{\partial c}{\partial K} \leq 0$$

6. Slope restrictions for American options

- For American calls and puts, the arbitrage arguments are basically the same, except the written options may be exercised against you at any time.

- If this happens you will have to close out the position, and therefore, on vertical spreads, you would not gain the interest on the exercise price.

- For American options, as the strike changes from K_1 to K_2, the option price needs to change by more than $|K_2 - K_1|$ for an arbitrage opportunity to exist.

- Taking this into consideration, the arbitrage (slope) restriction for American options are:

$$-(K_2 - K_1) \leq C(K_2) - C(K_1) \leq 0$$
$$\Rightarrow -1 \leq \frac{\partial C}{\partial K} \leq 0$$
$$0 \leq P(K_2) - P(K_1) \leq (K_2 - K_1)$$
$$\Rightarrow \ 0 \leq \frac{\partial P}{\partial K} \leq 1.$$

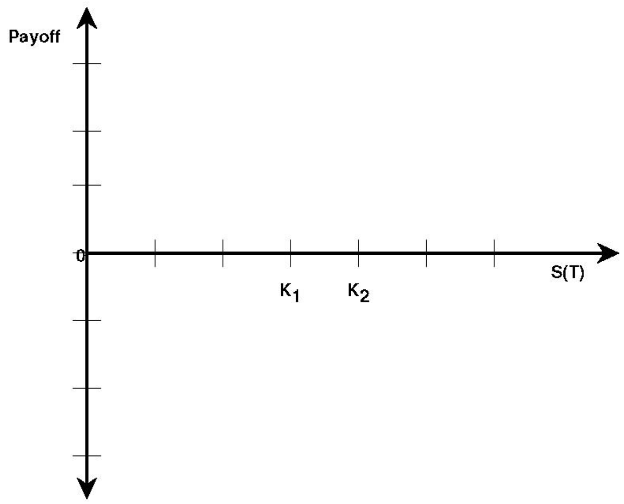
7. Questions:

- Is there an arbitrage here?

  IBM is currently trading at $100. One-year 100 puts are at $\$2\frac{1}{2}$, and One-year 110 puts are at $\$12\frac{1}{4}$. Your cost of money (simple rate) is 5% between now and expiration. (Options are European.)

Transaction	Cashflow at t	Payoff at T		
		$S_T < 100$	$100 < S_T < 110$	$S_T > 110$
	$-\$2.5$ $\$12.25$	$100 - S_T$ $-[110 - S_T]$	0 $-[110 - S_T]$	0 0

- In the diagram, what is the future payoff to the spread position you have taken on?



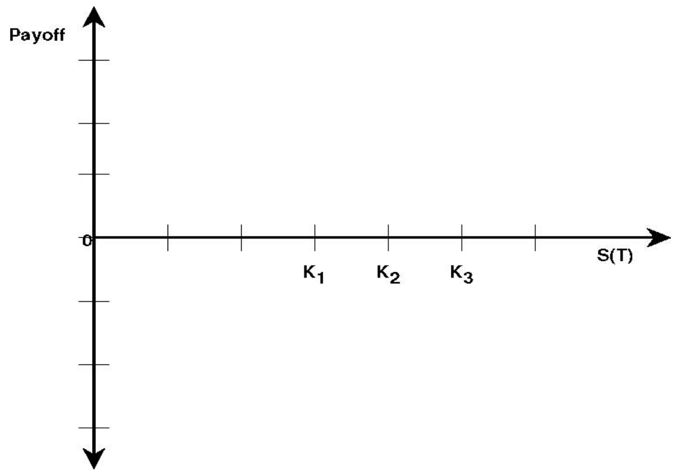
- What happens if there are payouts to the underlying?

(B) Butterfly Spreads and Convexity Restrictions:

Now consider a position involving three strikes, with $K_1 < K_2 < K_3$ and $K_2 = (K_1 + K_3)/2$. That is, the middle one midway between the other two. For some reason this is called a "butterfly".

1. Butterfly Spread (with calls)

- $c(K_1) - 2c(K_2) + c(K_3)$ (≥ 0)

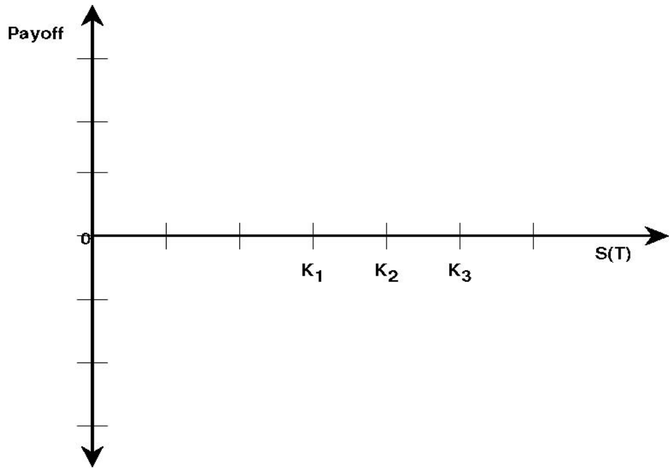


- This spread can also be viewed as a combination of bullish and bearish call spreads at different exercise prices:

$$c(K_1) - 2c(K_2) + c(K_3) = [c(K_3) - c(K_2)] - [c(K_2) - c(K_1)]$$

2. Butterfly Spread (with puts)

- $p(K_1) - 2p(K_2) + p(K_3)$ (≥ 0)



- And again, this spread can also be viewed as a combination of bullish and bearish vertical spreads at different exercise prices:

$$p(K_1) - 2p(K_2) + p(K_3) = [p(K_3) - p(K_2)] - [p(K_2) - p(K_1)]$$

- Also, note that the European put and call butterfly spreads must have the same price.

Question: How are these payoffs (puts or calls) affected if there is a yield or dividend on the underlying?

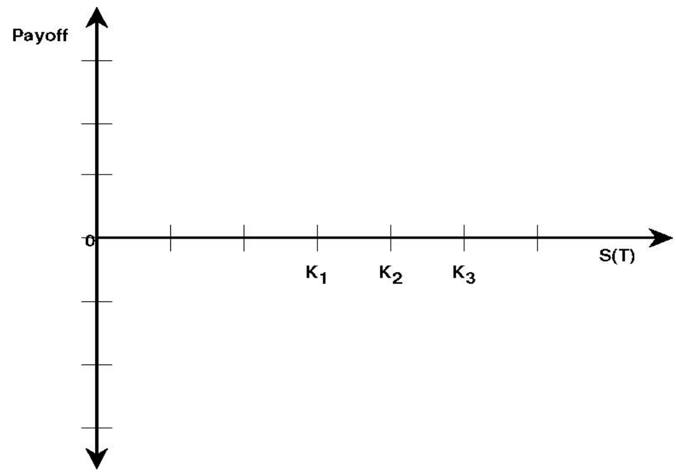
Question: Can butterfly payoffs for American options ever be negative?

3. Example (Butterfly Spreads):

MSFT is currently trading at $\$48\frac{1}{8}$. June 65 puts are at 17, June 70 puts are at 22, and June 75 puts are at 26. Your cost of money is 3.25%(simple) until June. What is the arbitrage opportunity and how can it be exploited?

Transaction	Payoff at t	Payoff at T			
		$S_T < 65$	$65 < S_T < 70$	$70 < S_T < 75$	$S_T > 75$
	$-\$17$	$65 - S_T$	0	0	0
	$-\$26$	$75 - S_T$	$75 - S_T$	$75 - S_T$	0
	$\$44$	$-2[70 - S_T]$	$-2[70 - S_T]$	0	0
	$\$1$	0	$S_T - 65$	$75 - S_T$	0

What is the payoff diagram to the arbitrage position?



4. Convexity

- Since butterfly spreads always have positive payoffs, the position value is always positive:

$$[c(K_3) - c(K_2)] - [c(K_2) - c(K_1)] \geq 0$$

and since $K_3 - K_2 = K_2 - K_1 > 0$, we have

$$\frac{c(K_3) - c(K_2)}{K_3 - K_2} - \frac{c(K_2) - c(K_1)}{K_2 - K_1} \geq 0$$

and similarly

$$\frac{p(K_3) - p(K_2)}{K_3 - K_2} - \frac{p(K_2) - p(K_1)}{K_2 - K_1} \geq 0$$

- This says the change in the slope of $c(K)$ and $p(K)$ is positive as we move to the right. Or *the slopes are increasing as K increases.*

- This is the same as saying that $c(S, K, t, T)$ and $p(S, K, t, T)$ are <u>convex</u> in K.

- For strike prices that are very close together ($\Delta K \to 0$) we would write:

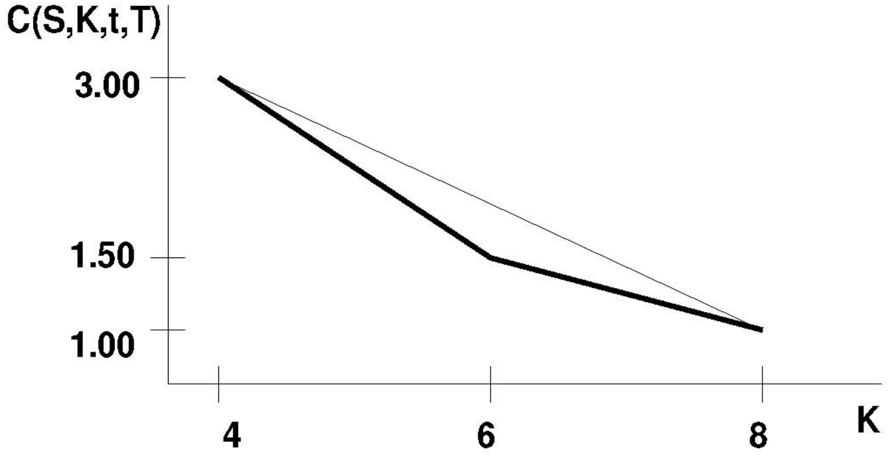
$$\frac{\partial}{\partial K}\left(\frac{\partial c(S, K, t, T)}{\partial K}\right) = \frac{\partial^2 c(S, K, t, T)}{\partial K^2} \geq 0$$

and

$$\frac{\partial}{\partial K}\left(\frac{\partial p(S, K, t, T)}{\partial K}\right) = \frac{\partial^2 p(S, K, t, T)}{\partial K^2} \geq 0$$

- This says the slope of the slope is positive.

- Graphically, for calls, the plot of price versus exercise price must look like this:



  - ▶ What does the plot of price versus exercise price for puts look like?

- Also, by absence of arbitrage, the price of butterfly spreads for European puts and calls must be the same:

$$c(K_3) - 2c(K_2) + c(K_1) = p(K_3) - 2p(K_2) + p(K_1)$$

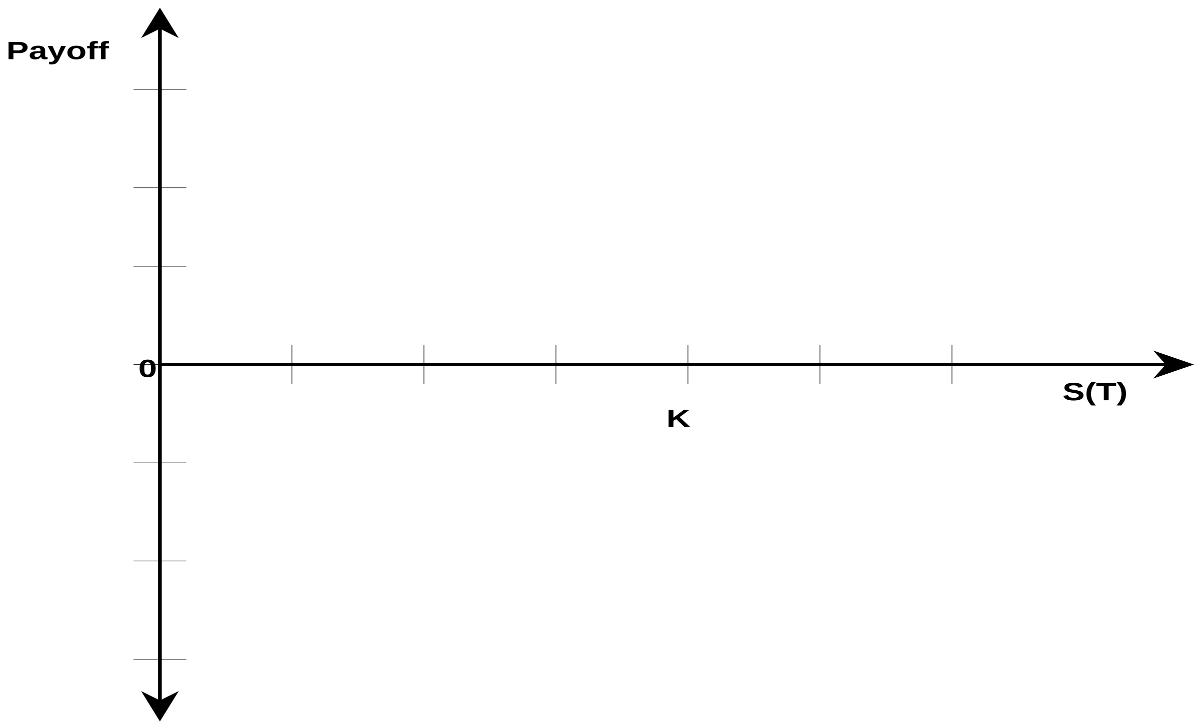
$$\Longrightarrow \quad \frac{\partial^2 c(S,K,t,T)}{\partial K^2} = \frac{\partial^2 p(S,K,t,T)}{\partial K^2}$$

II. More About Butterflies

- The convexity result we just saw involved the thought experiment of supposing we had strike prices that were arbitrarily close together. But suppose instead there are only as many strikes as there are possible final stock prices (e.g. one every $ 0.01).

- Now what does the payoff look like if we buy 100 (or $1/\Delta$) butterflies?

  ▶ $100 \cdot (c(K - \Delta) - 2c(K) + c(K + \Delta))$



- Treat this package as a security. Call its price b_K.

  ▶ Obviously, $b_K \geq 0,$ for all K .

- What if you have a portfolio of one for every possible K?
  - What's the payoff?
  - How much does it cost?
- Of course, since we manufactured the b_k out of calls, we must be able to re-synthesize the calls by an appropriate portfolio of <u>them</u>.
- Here's how: buy none of the b_k for $k \leq K$, then buy an increasing quantity of each one as you increase k

$$
\begin{array}{rclllll}
c(K) & = & +1\Delta & (c(K) & -2\,c(K+\Delta) & +\,c(K+2\Delta) &)/\Delta \\
 & & +2\Delta & & (c(K+\Delta) & -2\,c(K+2\Delta) & +\,c(K+3\Delta))\ \ )/\Delta \\
 & & +3\Delta & & & (c(K+2\Delta) & -2\,c(K+3\Delta)\ \ +\cdots \\
 & & \vdots & & & \vdots & \\
\hline
 & = & & 1\cdot c(K) & +0\cdot c(K+\Delta) & +0\cdot c(K+2\Delta) & \cdots
\end{array}
$$

- So, if we start with these butterflies, we can *synthesize* other payoffs.
- To put it another way, we can view them as <u>primary</u> securities, whose prices can tell us the value of other derivatives!

III. A General Formula for Derivatives Prices

(A) *If we can price butterflies, we can price any European claim.*

- We now know that starting with enough call prices we can extract this set of prices $\{b_k\}$.
- We already saw that we could reconstruct both a bond and a call from the $\{b_k\}$ by combining them in portfolios.
- In fact, we can likewise build up *any* time T payoff profile. **How?**
- Easy: if we want to have payoff g_1 when S ends up at S_1 at time T (and pays NOTHING for all other outcomes) just buy that many (g_1) of the S_1-butterflies. Total price: $g_1 \cdot b_1$.
- Then, if we want a payoff which ALSO pays us g_2 when S ends up at S_2, add these in the same way. The value of our derivative now is

$$g_1 \cdot b_1 + g_2 \cdot b_2$$

- So the price of any payoff *function*, g_k, must be

$$\sum_{k=0}^{\infty} g_k \; b_k$$

For any European claim, if we can replicate its payoff function we're done! There are no other cash-flows.

- Notation: Here the subscript k runs over all the possible terminal (time T) values of S.
  - For instance, if the possible prices are: $0.00, 0.01, 0.02, \ldots$, then $S_0 = 0, S_1 = 0.01$, etc.
  - And g_1, for example, signifies how much we want our derivative to pay off if S ends up at S_1.
- Sometimes it's easier to model prices as being continuous.
  - In that case, we would specify a continuous payoff function, $g(S)$, for our derivative, and a continuous function, $b(S)$, for our time-T butterflies.
- For this case, we would write the valuation formula above with an *integral* sign replacing the *summation* sign.

$$\int_{x=0}^{\infty} g(x)\ b(x)\ dx$$

- For a call, this would give the pricing formula:

$$\int_{x=0}^{\infty} \max[x - K, 0]\, b(x)\, dx = \int_{x=K}^{\infty} (x - K)\, b(x)\, dx$$

- One more example: The price of the stock itself must be

$$\int_{x=0}^{\infty} x\ b(x)\ dx$$

(assuming no payouts).

- If these formulas don't hold, there is an arbitrage somewhere.

(B) So what?

- It's not particularly surprising that, if you *start* with the prices of an infinite number of calls you can value other payoff profiles by no-arbitrage.
- The interesting thing is that we now know some things that must be true for *any* pricing model, if it rules out arbitrage.
- It must specify a family of functions $b(x;T)$ (one for each horizon T) having these properties:

  1. $b(x;T) \geq 0, \ \forall\, x, \ \forall\, T$

  2. $\int_{x=0}^{\infty} \ b(x;T) \ dx = B_{t,T}$

  3. $\int_{x=0}^{\infty} x \ b(x;T) \ dx = S_t, \ \forall\, T.$

  ($\forall$ means "for all values of".)

- Moreover, if we start with any function like this, and price derivatives according to our formula, then we know there is no arbitrage:
  - ▶ All positive payoffs will be given positive prices.
- So, in a sense, all of derivatives pricing now boils down to asking: which such $b(\ )$ should we use?
- Or, if we can get reliable prices of butterflies, we can deduce the market's model.

- **Terminology:** our unit butterflies are more formally called "state-contingent claims" or "Arrow-Debreu securities".

- The function $b(\ )$ is sometimes called the **pricing kernel**.

- We can divide $b(\ )$ by $B_{t,T}$ to make it sum to one. With this convention, the kernel $q = b/B_{t,T}$ is referred to as the **state-price density** or the **risk-neutral density**.

- What we saw above is that butterfly prices are the same as risk-neutral probabilities (multiplied by $B_{t,T}$) when the strike prices are very close together.

- We also saw that these butterfly prices, scaled by ΔK, are the same as the *curvature* or *convexity* of the function $c(K)$.

  - Putting the last two points together, the **Breeden and Litzenberger Theorem** says that this curvature is one way of identifying the risk-neutral probabilities.

(C) Options restrictions revisited

To show the power of the abstract pricing representation we developed, let's re-derive some of those static no-arbitrage relationships.

- *Result:* Put price upper bound.

- *New Proof:*

$$\begin{aligned} B \cdot K - p &= \int_{x=0}^{\infty} K\, b(x)\, dx \; - \; \int_{x=0}^{K} (K - x)\, b(x)\, dx \\ &= \int_{x=K}^{\infty} K\, b(x)\, dx \; + \; \int_{x=0}^{K} x\, b(x)\, dx \\ &> 0. \end{aligned}$$

- *Result:* European put-call parity.

- *New Proof:*

$$\begin{aligned} p - c &= \int_{x=0}^{\infty} \max[K - x, 0]\, b(x)\, dx \; - \; \int_{x=0}^{\infty} \max[x - K, 0]\, b(x)\, dx \\ &= \int_{x=0}^{K} (K - x)\, b(x)\, dx \; - \; \int_{x=K}^{\infty} (x - K)\, b(x)\, dx \\ &= \int_{x=0}^{\infty} K\, b(x)\, dx \; - \; \int_{x=0}^{\infty} x\, b(x)\, dx \\ &= K \cdot B - S. \end{aligned}$$

- *Result:* Convexity of calls with respect to strike.
- *New Proof:*

$$\begin{aligned}
\frac{\partial c}{\partial K} &= \frac{\partial}{\partial K}\int_{x=K}^{\infty}(x-K,0)\,b(x)\,dx \\
&= \frac{\partial}{\partial K}(\int_{x=K}^{\infty} x\,b(x)\,dx - K\int_{x=K}^{\infty} b(x)\,dx) \\
&= (-Kb(K)) - K(-b(K)) - \int_{x=K}^{\infty} b(x)\,dx \\
&= -\int_{x=K}^{\infty} b(x)\,dx \;<0 \\
\frac{\partial^2 c}{\partial K^2} &= -(0-b(K)) \;>0.
\end{aligned}$$

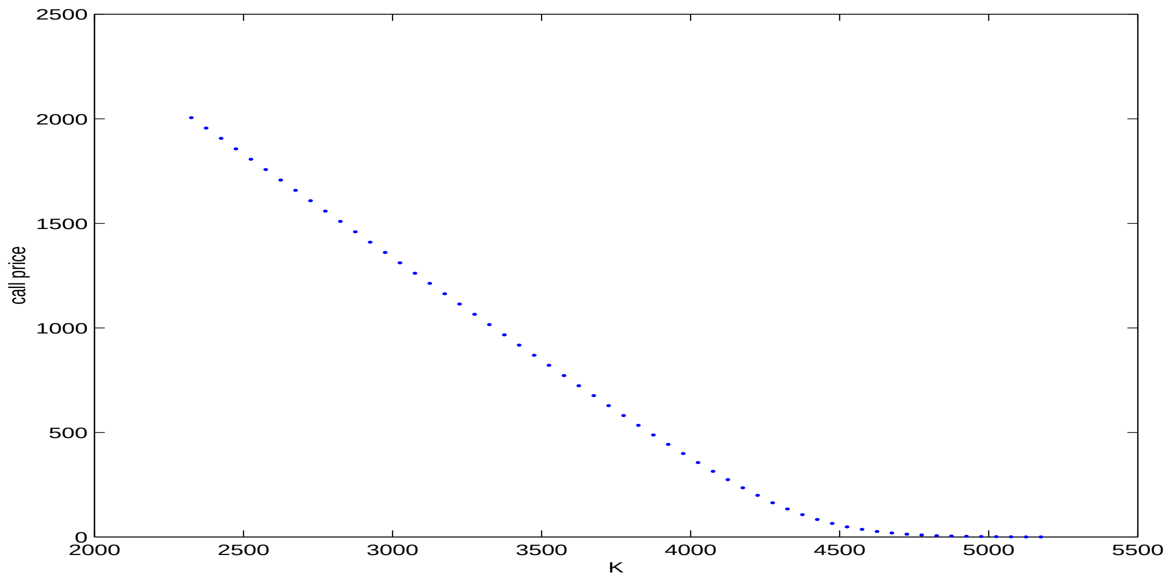
(D) The state prices (butterflies) <u>can't</u> tell you:

- What the price of your derivative will be tomorrow.
  - ▶ The bs can change from minute to minute.
- What the price of a derivative with a different expiration should be.
  - ▶ Every instant, there are a set of bs for each horizon.
- How to hedge your derivative with the underlying.

Still, these things really do exist, and we really can exploit them.

IV. Extracting State Price Densities: An Example

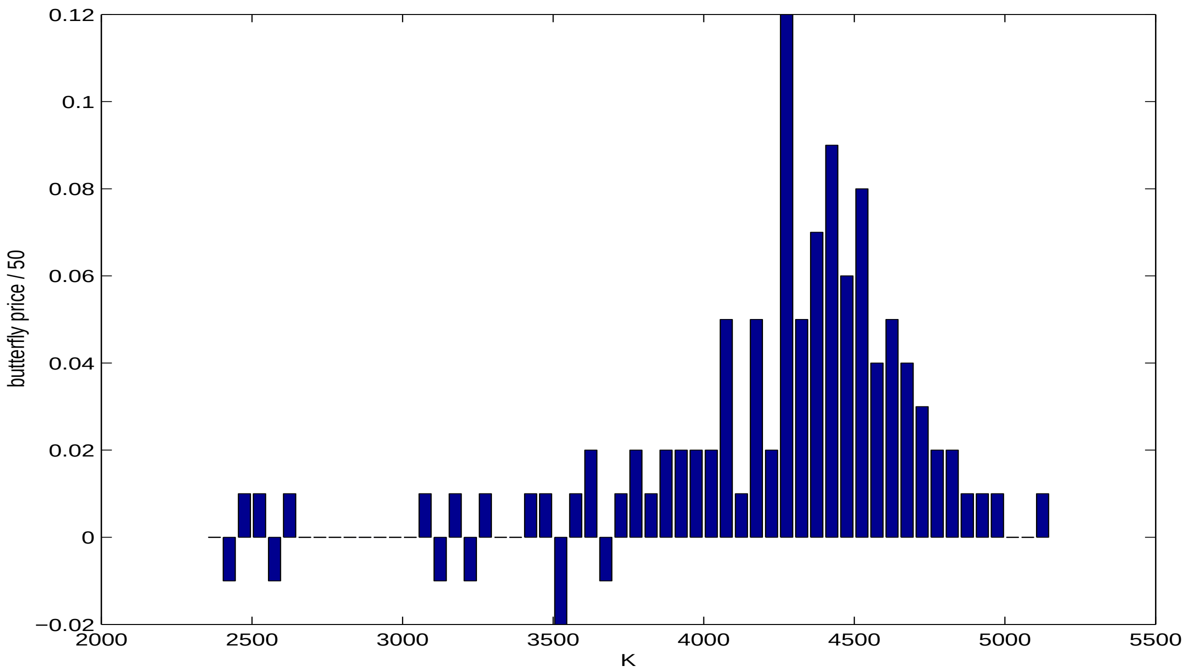
- Can we actually get at the risk neutral density in reality?
- Could be very practical – allowing us to price arbitrary derivatives without having to develop complicated models.
- Try it!
- LIFFE-traded FTSE calls and puts (European style) closing prices supplied by the exchange for every strike from 2400 to 5200 (with $\Delta K = 50$).
- Data (December call prices, $S_t \approx 4350$):



- Now just extract the butterflies for each strike, e.g.

$$b_{4000} = \frac{c_{3950} - 2\ c_{4000} + c_{4050}}{50} \text{ etc.}$$

- Result:

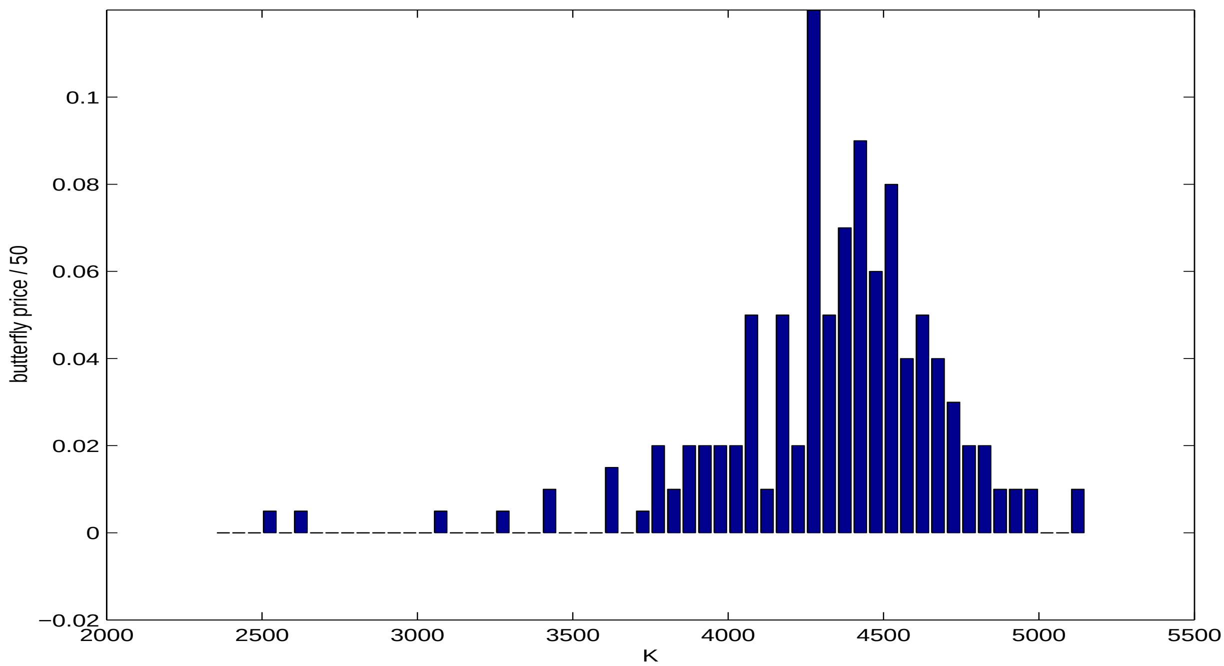


- What's the matter? Are there arbitrage opportunities ?

- Unlikely. Data is just bad.
  - Bid-ask spreads for calls$\approx$ 5pts.
  - Quotes are mixed actual trades and exchange estimates.
  - Other factors:
    - Some errors
    - Problems from non-simultaneity
    - Rounding (discreteness) problems
    - Market-maker "mistakes".

- We need some way to estimate the most likely true prices before we can trust butterflies.

- One solution: Fit the nearest convex function to the data.
  - This method "corrects" the prices by the least amount.
  - Other possibilities:
    - ∗ Non-parametric smoothing.
    - ∗ Splines, orthogonal functions, wavelettes, etc.
  - ...Or get better data direct from market makers!
- Resulting $\{b_i\}$:



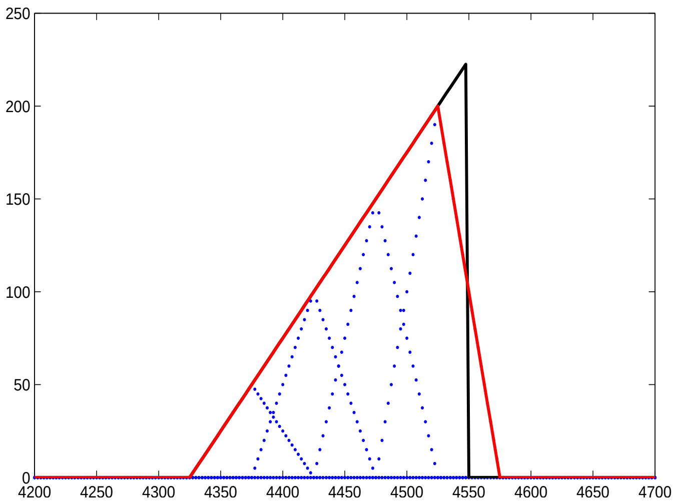
- Can we use this density to price derivatives?
- Yes!
  - ...If these really <u>are</u> tradeable prices.
  - Let's do an example.

- **Example:** On this day the price of a 4325 call was 134p. (This was close to at-the-money, and there were 2 months to expiration.) What should the price of a 4550-knock-out call be with the same strike?
  - The call pays zero if the FTSE is above 4550 at expiration.
  - Follow our recipe: find the price of a portfolio that holds $\max[j - 4325]$ butterflies at each strike j less than 4550.
  - Answer:

Strike	Q	b_j	Cost
4375	50	0.07	3.5
4425	100	0.09	9
4475	150	0.06	9
4525	200	0.08	16
Knock-out price:			37.50

  - Neat! A no-arbitrage price without any models!

- Here's the claim we actually priced:



- Limitations:
  - ▶ Of course, we could not price a derivative that had payoffs outside the range of available strike prices.
  - ▶ And attempting to actually TRADE in all these butterflies (hence actually to <u>do</u> the replication) could be costly.
- Still, notice how little we used here: No assumptions about continuous trading or constant interest rates. We didn't even need to know where spot was!
- We were simply regarding our knock-out as a derivative *of all the butterflies* or the calls – not as a derivative of the FTSE index.
  - ▶ The more prices you take as given the more other things you can price by no-arbitrage.
- People really do use techniques like this when they want to avoid all the assumptions that go with more "sophisticated" pricing methodologies.

V. Interpretation of State Price Densities

- As you will have noticed, the normalized butterlfies look a lot like a probability density.
- Indeed, if we define $q_k \equiv b_k/B_{t,T}$ then we have

$$
\begin{array}{lcl}
q(x) \geq 0, \text{ for all } x & \text{or} & q_k \geq 0, \text{ for all } k \\
q(x) < 1, \text{ for all } x & \text{or} & q_k < 1, \text{ for all } k \\
\displaystyle\int_0^\infty q(x)\, dx = 1 & \text{or} & \displaystyle\sum_{k=0}^{\infty} q_k = 1
\end{array}
$$

  which are the defining properties of a probability measure.

- This can cause a lot of confusion.
- Suppose our data above had found that the all the butterflies below 3725 added up to a risk-neutral probability of 0.01.
- This might tempt you to make statements like:

  "The market thinks there is a 1% chance the FTSE will crash below 3725 by expiration."

  Or

  "Our best estimate is that there is a 1% chance the FTSE will fall below 3725."

- These are wrong.

  Risk-neutral probabilities aren't necessarily the <u>true</u> (or statistical) probabilities of anything.

- Just because some numbers have the mathematical properties of probabilities, doesn't mean that they ARE (true) probabilities.

- Likewise, our pricing formula for derivatives has the mathematical form of a discounted expected value of the payoff function with respect to price.

$$\int_{x=0}^{\infty} g(x)\ b(x)\ dx = B_{t,T}\ \int_{x=0}^{\infty} g(x)\ q(x)\ dx$$

  But that doesn't mean the price is the market's (or anyone's) "expectation" of the payoff.

- Analogy: Do the (parimutuel) odds on eight horses in a given race exactly equal their statistical likelihood of winning? **No!**
  - ▶ On average, people prefer to bet on long-shots. (It's more fun when you win.)
  - ▶ Statistically, those horses are "over-priced" relative to the favorites.
  - ▶ A risk-neutral/fun-neutral person would always buy (bet on) favorites (and short long-shots).
    - ∗ But that's not the same thing as an arbitrage opportunity!
  - ▶ There are just more people playing for amusement than to beat treasury bonds.

⋆ So is there any connection between the SPD and "true" probabilities?

- First, a deeper question: What are "true" probabilities?
  - ▶ The concept only has meaning in the context of a "true" stochastic model.
  - ▶ And, even then, has meaning only with respect to a specific set of conditioning information.
  - ▶ We do not have, nor are we ever likely to have, a model of asset prices that *anyone* thinks is THE true one.
- So, the question ($\star$) pre-supposes that you have already accepted a specific statistical set of probabilites as "true".
- We can still ask two related questions:
  1. If a specified statistical model is true, does that restrict the SPD?

  2. Does the observed SPD restrict the set of statistical models that might be true?
- We are going to see that the answer to both is yes.
- For now, think about horse races ...

VI. Conclusions

- We have tried to learn as much as we can about option prices based purely on static arbitrage arguments.

- We took this approach to its extreme and deduced that, if we take the prices of enough *other* options as given, we can price anything.
  - ▶ First extract the state price density.
  - ▶ Then take mean payoff with respect to it, and discount by risk-free rate.
  - ▶ If the price is anything else, there is arbitrage.

- To be able to price options "from scratch" we need to introduce statistical models of the ways in which the underlying asset's price changes through time.

Lecture Notes 3.1 and 3.2: Summary of Notation

SYMBOL	WHERE	MEANING
p, c	*p2, L3.1*	*price of European put and European call*
P, C	*p2, L3.1*	*price of American put and American call*
K	*p2, L3.1*	*exercise price of an option*
t', t''	*p9, L3.1*	*the time right before and right after a stock goes ex-dividend*
$PV(D)$	*p14, L3.1*	*present value of dividends remaining before* T
$\partial p/\partial K$	*p3, L3.2*	*rate of change in the of value of a put as its exercise price changes*
$\partial^2 p/\partial K^2$	*p10, L3.2*	*rate of change of rate of change (or curvature, or convexity) in put value as its exercise price changes*
Δ	*p12, L3.2*	*minimum increment between stock prices*
b_k	*p12, L3.2*	*value of a butterfly centered at strike* $k \cdot \Delta$
g_k	*p14, L3.2*	*payoff of a European derivative if* $S_T = k \cdot \Delta$
$\int_{x=0}^{\infty} g(x)\ b(x)\ dx$	*p15, L3.2*	*sum of all payoffs times the corresponding butterflies, when stock price is allowed to be continuous*
Q	*p23, L3.2*	*quantity of butterflies purchased at each strike*
$q_k, q(x)$	*p25, L3.2*	*state-price (or "risk neutral") density for* S_T