# Stochastic Processes

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Readings: Shreve Chapter 2

# Stochastic processes

- A **stochastic process** $X = \{X_t, t \in \mathbb{T}\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of r.v.s on $\Omega$ indexed by time $t \in \mathbb{T}$
  - Continuous time: $\mathbb{T} = [0, \infty)$, discrete time $\mathbb{T} = \{t_0, t_1, \cdots\}$
  - The process starts at a constant $X_0$ at time 0
  - For any $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ is a **sample path** or **realization** of the process $X$
- A continuous time stochastic process is
  - continuous if sample paths are continuous *a.s.*
  - **cadlag** if sample paths are right continuous with left limits *a.s.*

# Discrete time example

**Example (Example 3)**

In a 3-day period, stock price either doubles (with probability $p = \frac{1}{2}$) or halves (with probability $1 - p = \frac{1}{2}$) in one day. Current stock price is 8.

- Flipping a fair coin for three times, **sample space**

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

- $\sigma-$**algebra** $\mathcal{F} = 2^{\Omega}$: collection of all possible events (subsets of $\Omega$)
- **Probability measure**

$$\mathbb{P}(HHT) = \cdots = \mathbb{P}(TTT) = \frac{1}{8}$$

# Filtration

- Stock price process $\{S_t, t = 0, 1, 2, 3\}$
  - Discrete time stochastic process
  - Sample path corresponding to $\omega = \{HTT\}$

$$S_0(\omega) = 8, \quad S_1(\omega) = 16, \quad S_2(\omega) = 8, \quad S_3(\omega) = 4$$

- **Filtration**: a sequence of increasing $\sigma-$algebras indexed by time $\mathbb{F} = \{\mathcal{F}_t\}$, $\mathcal{F}_t \subset \mathcal{F}$ contains information learned by observing the system up to time $t$

# Filtration example

- In **Example 3**: by time 1, we know whether the outcome of the first flipping is $H$ (stock price goes up) or $T$ (stock price goes down)

$$A_H = \{HHH, HHT, HTH, HTT\}, \ A_T = \{THH, THT, TTH, TTT\}$$

- Define $\sigma-$algebra $\mathcal{F}_1$ such that it contains

$$A_H, \quad A_T = A_H^c, \quad A_H \cup A_T = \Omega, \quad A_H \cap A_T = \emptyset$$

- $\mathcal{F}_1 = \{A_H, A_T, \emptyset, \Omega\}$ contains information learned by observing the outcome of the first coin flipping
- By time 1, we know whether $\omega \in A_H$ (hence $S_1(\omega) = 16$) or $\omega \in A_T$ (hence $S_1(\omega) = 4$); doesn't allow you to determine $S_2(\omega), S_3(\omega)$
- $A_H$ and $A_T$ are **atoms** of $\mathcal{F}_1$
- Events like $\{HHH, HHT\}$, $\{HHH, HTH, THH, TTH\}$ are not in $\mathcal{F}_1$

- $\mathcal{F}_2 = \sigma(\{A_{HH}, A_{HT}, A_{TH}, A_{TT}\})$ contains information learned by observing the first two coin flippings

$$A_{HH} = \{HHH, HHT\}, A_{HT} = \{HTH, HTT\},$$

$$A_{TH} = \{THH, THT\}, A_{TT} = \{TTH, TTT\}$$

- It contains events like $A_H = A_{HH} \cup A_{HT}$, $A_T = A_{TH} \cup A_{TT}$,

$$A_{HH} \cup A_{TH}, A_{HH} \cup A_{TT}, A_H \cup A_{TH}, etc.$$

but not

$$\{HHH\}, \{HHH, HTH, THH, TTH\}, etc.$$

- By time 2, we know whether $\omega \in A_{HH}$ or $\omega \in A_{HT}$ or $\omega \in A_{TH}$ or $\omega \in A_{TT}$ and the corresponding $S_1(\omega), S_2(\omega)$, but not $S_3(\omega)$
- $A_{HH}, A_{HT}, A_{TH}, A_{TT}$ are **atoms** of $\mathcal{F}_2$
- $\mathcal{F}_0 = \{\emptyset, \Omega\}$
- $\mathcal{F}_3 = 2^{\Omega}$: the eight atoms are $\{HHH\}, \{HHT\}, \cdots, \{TTT\}$
- **Filtration** $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$: $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \mathcal{F}$
- **Filtered probability space** $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$

# Conditional expectation

- Given $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. **Conditional expectation** $\mathbb{E}[X|\mathcal{F}_t]$ : expectation of $X$ conditional on information at time $t$
- In **Example 3**, consider expectation of $S_3$ conditional on information available by observing the first two coin flipping

$$\mathbb{E}[S_3|\mathcal{F}_2]$$

- Recall the filtration

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset, \Omega\} \\ \mathcal{F}_1 &= \sigma(\{A_H, A_T\}) \\ \mathcal{F}_2 &= \sigma(\{A_{HH}, A_{HT}, A_{TH}, A_{TT}\}) \\ \mathcal{F}_3 &= 2^\Omega \end{aligned}$$

- For $\omega = HH* \in A_{HH}$,

$$\mathbb{E}[S_3|\mathcal{F}_2](\omega) = pS_3(HHH) + (1-p)S_3(HHT) = \frac{1}{2}(64+16) = 40$$

For $\omega = HT* \in A_{HT}$,

$$\mathbb{E}[S_3|\mathcal{F}_2](\omega) = pS_3(HTH) + (1-p)S_3(HTT) = \frac{1}{2}(16+4) = 10$$

- Similarly,

$$\omega = TH* \in A_{TH}, \quad \mathbb{E}[S_3|\mathcal{F}_2](\omega) = 10$$

$$\omega = TT* \in A_{TT}, \quad \mathbb{E}[S_3|\mathcal{F}_2](\omega) = 5/2$$

- $\mathbb{E}[S_3|\mathcal{F}_2]$ can be considered as an **estimate of $S_3$ based on information available** up to time 2 (similar to unconditional expectation)
- Seen at time 0, $\mathbb{E}[S_3|\mathcal{F}_2]$ is a random variable (in contrast to unconditional expectation); its value will be revealed by time 2
- Recall that $\mathcal{F}_2 = \sigma(\{A_{HH}, A_{HT}, A_{TH}, A_{TT}\})$. $A_{HH}, A_{HT}, A_{TH}, A_{TT}$ are **atoms** of $\mathcal{F}_2$
- $\mathbb{E}[S_3|\mathcal{F}_2](\omega)$ is constant on each atom
- $\mathbb{E}[S_3|\mathcal{F}_2]$ is **$\mathcal{F}_2-$measurable** (knowing info up to time 2, one knows $\mathbb{E}[S_3|\mathcal{F}_2]$)

$$\{\mathbb{E}[S_3|\mathcal{F}_2] \in B\} \in \mathcal{F}_2, \quad \forall B \in \mathcal{B}(\mathbb{R})$$

- Expectation of $S_3$ conditional on information available by observing the first coin flipping

$$\mathbb{E}[S_3|\mathcal{F}_1](\omega) = p^2 S_3(HHH) + p(1-p)S_3(HHT) + (1-p)pS_3(HTH)$$
$$+(1-p)^2 S_3(HTT) = 25, \quad \omega = H** \in A_H$$
$$\mathbb{E}[S_3|\mathcal{F}_1](\omega) = p^2 S_3(THH) + p(1-p)S_3(THT) + (1-p)pS_3(TTH)$$
$$+(1-p)^2 S_3(TTT) = \frac{25}{4}, \quad \omega = T** \in A_T$$

- $\mathbb{E}[S_3|\mathcal{F}_1]$ is a $\mathcal{F}_1-$measurable r.v.
- It is constant on atoms $A_H, A_T$ of $\mathcal{F}_1 = \sigma(\{A_H, A_T\})$

- In general, would like to define $\mathbb{E}[X|\mathcal{G}]$ for a $\sigma-$algebra $\mathcal{G} \subset \mathcal{F}$
- Recall $\mathbb{E}[S_3|\mathcal{F}_2]$, for $\omega = HH* \in A_{HH}$,

$$\begin{aligned}\mathbb{E}[S_3|\mathcal{F}_2](\omega) &= pS_3(HHH) + (1-p)S_3(HHT) \\ &= S_3(HHH)\frac{\mathbb{P}(\{HHH\})}{\mathbb{P}(A_{HH})} + S_3(HHT)\frac{\mathbb{P}(\{HHT\})}{\mathbb{P}(A_{HH})}\end{aligned}$$

That is,

$$\mathbb{E}[S_3|\mathcal{F}_2](\omega)\mathbb{P}(A_{HH}) = \sum_{\omega' \in A_{HH}} S_3(\omega')\mathbb{P}(\{\omega'\})$$

$$\int_{A_{HH}} \mathbb{E}[S_3|\mathcal{F}_2](\omega)d\mathbb{P}(\omega) = \int_{A_{HH}} S_3(\omega)d\mathbb{P}(\omega)$$

- The above holds for all atoms of $\mathcal{F}_2$: $A_{HH}, A_{HT}, A_{TH}, A_{TT}$, and hence holds for any $A \in \mathcal{F}_2$:

$$\int_A \mathbb{E}[S_3|\mathcal{F}_2](\omega) d\mathbb{P}(\omega) = \int_A S_3(\omega) d\mathbb{P}(\omega), \quad A \in \mathcal{F}_2$$

- **Definition**: for $\sigma-$algebra $\mathcal{G} \subset \mathcal{F}$, the **conditional expectation** $\mathbb{E}[X|\mathcal{G}]$ is such that
    - $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}-$measurable
    - For any $A \in \mathcal{G}$,

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega)$$

- When $\mathcal{G} = \{\emptyset, \Omega\}$, $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$

# Conditional expectation properties

- **Linearity:** $\mathbb{E}[aX+bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}], \quad a,b\in\mathbb{R}$
- **Take out what is known:** if $X$ is $\mathcal{G}-$measurable, $\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$. Take $Y=1$

$$\mathbb{E}[X|\mathcal{G}] = X$$

- **Independence:** if $X$ is independent of $\mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$
- **Iterated conditioning:** if $\mathcal{G}\subset\mathcal{H}$,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}]$$

- Verify that $\mathbb{E}\left[\mathbb{E}[S_3|\mathcal{F}_2]|\mathcal{F}_1\right] = \mathbb{E}[S_3|\mathcal{F}_1]$
- Take $\mathcal{G}=\{\emptyset,\Omega\}$. It follows that for any $\sigma-$algebra $\mathcal{H}$

$$\mathbb{E}[\mathbb{E}[\mathbf{X}|\boldsymbol{\mathcal{H}}]] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}] \mathbf{=} \mathbb{E}[\mathbf{X}]$$

# Martingales

- Consider a stochastic process $\{X_t\}$ in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
- $X = \{X_t\}$ is **adapted** if $\forall t$, $X_t$ is $\mathcal{F}_t-$measurable
- $X = \{X_t\}$ is a **martingale** if $\forall s \leq t$, $\mathbb{E}[|X_t|] < \infty$, and

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s$$

- Properties of martingales
  - **Constant expectations**

$$\mathbb{E}[X_t] = \mathbb{E}[X_0], \quad \forall t > 0$$

  - Given an integrable r.v. $Y$, $X_t = \mathbb{E}[Y|\mathcal{F}_t]$ is a martingale

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s] = X_s, \quad \forall s \leq t$$

# Markov processes

- An adapted stochastic process $\{X_t\}$ is a **Markov process** if for any measurable function $f$ and $s \leq t$

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = g(X_s)$$

  for some measurable function $g$
- The future of the process only depends on the current value of the process, but not its past
- In **Example 3**, stock price process is Markov

$$\mathbb{E}[f(S_{n+1})|\mathcal{F}_n] = pf(2S_n) + (1-p)f(S_n/2), \quad n = 0, 1, 2$$

# Lévy processes

- An adapted stochastic process $X$ has **independent increments** if for any $0 \leq s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s$
- It has **stationary increments** if $X_t - X_s$ and $X_{t-s} - X_0$ have the same distribution
- An adapted process $X$ with $X_0 = 0$ is a **Lévy process** if it has stationary and independent increments and its sample paths are cadlag a.s.
- Two important Lévy processes: Poisson process, Brownian motion

# Poisson processes

- **Poisson process** $\{N_t, t \geq 0\}$: counts the number of events that have occurred up to time $t$
    - Let $\lambda$ be the arrival rate (or intensity)
    - Let $\{\tau_n, n \geq 1\}$ be i.i.d. with $\tau_n \sim \text{Exponential}(\lambda)$
    - Let $T_0 = 0$, $T_n = \sum_{i=1}^n \tau_i$. Define

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{T_n \leq t\}}$$

    - Sample paths of a Poisson process are cadlag
    - For any $t$, $N_t \sim$Poisson$(\lambda t)$: $\mathbb{E}[N_t] = \text{var}(N_t) = \lambda t$

$$\mathbb{P}(N_t = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, k = 0, 1, \cdots$$

- **Compound Poisson process**: let $\{N_t\}$ be a Poisson process with arrival rate $\lambda$, $\{X_n, n \geq 1\}$ a sequence of i.i.d. r.v.s.

$$Y_0 = 0, \quad Y_t = \sum_{n=1}^{N_t} X_n, \; t > 0$$

Then $\{Y_t, t \geq 0\}$ is a compound Poisson process

- **Example 5:** Claims to a car insurance company arrive according to a Poisson process with intensity $\lambda$. $X_n$ is the dollar amount payment of the $n$th claim (assume i.i.d.). Then the total amount paid by time $t$ is a compound Poisson process.

- **Finite market model**:
  - $N$ time periods: $0 = t_0, t_1, \cdots, t_N = T$
  - $d$ ($d = 2$ for illustration) risky assets in addition to risk free investment with rate $r$:

$$i\text{th asset price process:} \quad (S_0^i, S_1^i, \cdots, S_N^i)$$

    Risk free lending modeled as buying a risk free bond with initial price 1

$$\text{bond price process:} \quad (B_0, B_1, \cdots, B_N)$$

  - Denote $S_n = (B_n, S_n^1, S_n^2)$ the price vector at time $t_n$
  - Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\Omega$ contains finite possible outcomes (each with positive probability)

# Equivalent martingale measure

- Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ are **equivalent** if

$$\mathbb{P}(A) > 0 \Leftrightarrow \mathbb{Q}(A) > 0, \quad \forall A \in \mathcal{F}$$

  They agree on what occurs with positive probability, disagree on the magnitude of the probability

- $\mathbb{Q}$ is an **equivalent martingale measure (risk neutral measure)** in the finite market model if
  - $\mathbb{Q}$ and $\mathbb{P}$ are equivalent
  - Discounted asset price process $(S_n^{*i} = S_n^i / B_n)$, $n = 0, 1, \cdots, N$, is a martingale

$$\mathbb{E}^{\mathbb{Q}}[S_{n+1}^{*i} | \mathcal{F}_n] = S_n^{*i}, \quad n = 0, 1, \cdots, N-1$$

# Binomial model

- In the binomial model, with $0 < d < e^{r\delta} < u$, the risk neutral measure $\mathbb{Q}$ is defined via the risk neutral probability $p^* = (e^{r\delta} - d)/(u - d)$
- **Discounted asset price process** $\{e^{-rn\delta} S_n, 0 \leq n \leq N\}$ is a martingale under $\mathbb{Q}$ in the above binomial model

$$e^{-rn\delta} S_n = \mathbb{E}^{\mathbb{Q}}[e^{-r(n+1)\delta} S_{n+1} | \mathcal{F}_n]$$

**Proof**. $\mathbb{E}^{\mathbb{Q}}[S_{n+1}|\mathcal{F}_n] = p^* u S_n + (1 - p^*) d S_n = S_n e^{r\delta}$.

- **Self financing trading strategy**
  $\phi = (\phi_n, n = 0, 1, \cdots, N-1)$

  $$\phi_n = (\phi_n^0, \phi_n^1, \phi_n^2)$$

  $\phi_n^0$, $\phi_n^1$, $\phi_n^2$: number of bonds, asset 1, asset 2 held from $t_n$ to $t_{n+1}$

  $$\phi_n^0 B_{n+1} + \phi_n^1 S_{n+1}^1 + \phi_n^2 S_{n+1}^2 = \phi_{n+1}^0 B_{n+1} + \phi_{n+1}^1 S_{n+1}^1 + \phi_{n+1}^2 S_{n+1}^2$$

  denoted using dot product by

  $$\phi_n \cdot S_{n+1} = \phi_{n+1} \cdot S_{n+1}$$

# First fundamental theorem of asset pricing

- An **arbitrage opportunity** in the finite market model is a self financing trading strategy $\phi$ such that

$$V_0(\phi) = 0, \quad V_N(\phi) \geq 0, \quad \mathbb{P}(V_N(\phi) > 0) > 0$$

- **Theorem (first fundamental theorem of asset pricing)**: *The finite market model is arbitrage free if and only if there exists an equivalent martingale measure* $\mathbb{Q}$
**Proof.** $\Leftarrow$: *suppose* $\phi$ *is self financing,* $V_0(\phi) = 0$, $V_N(\phi) \geq 0$, *want to show that* $V_N(\phi) = 0$. *First, discounted value process* $V_n^*(\phi) = V_n(\phi)/B_n$ *is a martingale under measure* $\mathbb{Q}$

- *Proof continued:*

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[V^*_{n+1}(\phi)|\mathcal{F}_n] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\phi_{n+1}\cdot S_{n+1}}{B_{n+1}}|\mathcal{F}_n\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{\phi_{n}\cdot S_{n+1}}{B_{n+1}}|\mathcal{F}_n\right] \\
&= \phi_n \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}}{B_{n+1}}|\mathcal{F}_n\right] \\
&= \frac{\phi_n \cdot S_n}{B_n} = V^*_n(\phi)
\end{aligned}
$$

*So* $\mathbb{E}^{\mathbb{Q}}[V^*_N(\phi)] = V_0(\phi) = 0$ *and hence* $V_N(\phi) = 0$
$\Rightarrow$*: Williams, 2006, Introduction to the Mathematics of Finance*

- Suppose the finite market model is arbitrage free, then there exists a risk neutral measure $\mathbb{Q}$
- **Suppose self financing strategy $\phi$ replicates** the payoff of a European style derivative $f_N = V_N(\phi)$, then derivative value $f_n$ at time $t_n$ = the value of the replicating strategy at $t_n$: $V_n(\phi)$

$$\begin{aligned} f_n &= V_n(\phi) = B_n V_n^*(\phi) \\ &= B_n \mathbb{E}^{\mathbb{Q}}[V_N^*(\phi)|\mathcal{F}_n] = B_n \mathbb{E}^{\mathbb{Q}}[B_N^{-1} f_N|\mathcal{F}_n] \end{aligned}$$

In particular,

$$f_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT} f_N]$$

- The market is **complete** if all European style derivative payoffs can be replicated
- **Theorem (the second fundamental theorem of asset pricing):** *An arbitrage free finite market model is complete if and only if it admits a unique equivalent martingale measure*
  **Proof**. *Williams 2006.*
- With $0 < d < e^{r\delta} < u$, the binomial model
  – admits an equivalent martingale measure, no arbitrage
  – is complete, unique equivalent martingale measure
  $\Rightarrow$ unique price for any European style derivative (obtained through risk neutral pricing formula)