Lecture Note 3.1: No-Arbitrage Bounds on Options Prices

We now turn our attention to options markets.

Over the next weeks we will examine in detail the real-world details of pricing options by no-arbitrage. In doing so, we will review some familiar modelling approaches and examine their **robustness**. To start, we will consider the most fundamental properties of options prices that are extremely robust in that they do not depend on any stochastic modelling assumption.

We have seen that forwards and swaps can be replicated by static portfolios involving the underlying asset or assets, and risk-free borrowing and lending. The initial price of such a replicating portfolio then had to be the price of the derivative. Likewise. when a portfolio can be found that always paysoff *at least* as much as the derivative, its initial price is an upper no-arbitrage bound on the derivative's price. And, a portfolio always paying less than the derivative can't cost more, and so provides a lower bound.

This is the basis for the relationships we will examine today.

Outline:

I. Background, Notation, and Assumptions
II. Basic Arbitrage Relationships
III. Put-Call Parity
IV. Relationships with Dividends or Carry Costs
V. Summary

I. Background.

- About options markets.
  - ▶ I assume everybody is familiar with how options contracts work.
  - ▶ For today's purposes we will be thinking about an option on a traded underlying asset (for example a stock or a currency) that can be easily sold short.
  - ▶ There are very large and liquid markets, both exchange-traded and OTC, for such products in all the major economies.
    - ∗ In the U.S., exchange traded equity options are usually cross-listed, meaning that the same contract can be traded on multiple exchanges.
  - ▶ Sometimes the contracts specify physical settlement and sometime cash settlement.
    - ∗ As we have seen, the distinction doesn't matter as long as the cash-settlement determination process uses a spot price that corresponds to the true market price where long or short positions in the underlying can actually be liquidated.
- Assumptions:

(A) For this note and the next we are going to assume there are <u>no transactions costs</u>.

Note: For each of the conclusions we draw, you should convince yourself that you know how to modify them to incorporate the effect of, say, including different borrowing and lending rates.

(B) As in the earlier lecture notes, we need to assume that all carry costs are known in advance, where, besides interest rates, this includes payouts, borrowing fees, storage costs, and/or convenience yields. We start with the case where there are none of these, then add them back. But they still have to be predictable.

(C) We will also ignore counterparty risk. This is reasonable for U.S. exchange-traded products because they all use a very safe CCP (The Options Clearing Corporation, or OCC).

- Notation:

Current date	t
Maturity or expiration date	T
Price of the underlying asset	S_t
Current Price of a \$1 face-value bond that matures at T	$B_{t,T} = e^{-r_T(T-t)}$
Exercise (strike) price	K (or X)
Value of a European call	$c(S, K, B, t, T)$
Value of an American call	$C(S, K, B, t, T)$
Value of a European put	$p(S, K, B, t, T)$
Value of an American put	$P(S, K, B, t, T)$

II. Basic Arbitrage Relations:

- To start, consider an option on a share of stock. Assume there are no fees for borrowing it.

- All of the following relations were proved by Robert Merton in 1973.

- *Of these, (A)-(G) hold regardless of whether or not the stock pays dividends.*

(A) Options never have a negative value:

$$\begin{aligned} c(S, K, t, T) &\geq 0 \qquad & C(S, K, t, T) &\geq 0 \\ p(S, K, t, T) &\geq 0 \qquad & P(S, K, t, T) &\geq 0 \end{aligned}$$

(B) An American option is at least as valuable as the corresponding European one:

$$\begin{aligned} C(S, K, t, T) &\geq c(S, K, t, T) \\ P(S, K, t, T) &\geq p(S, K, t, T) \end{aligned}$$

(C) A call is never worth more than the stock and a put is never worth more than the exercise price

$$\begin{aligned} C(S, K, t, T) &\leq S_t \qquad & c(S, K, t, T) &\leq S_t \\ P(S, K, t, T) &\leq K \qquad & p(S, K, t, T) &\leq K \end{aligned}$$

Why? Take the bound for p. Suppose $p > K$. Sell the put, invest K, and still have cash left today. At T, the worst that can happen is the stock is worthless. You are obligated to pay K for something worth zero. But your investment is worth at least K. So you have left over cash at T too.

(D) European puts are never worth more than the present value of the exercise price:

$$p(S, K, t, T) \leq K \cdot B_{t,T}$$

(E) An American option is worth at least its exercised (or "intrinsic") value, i.e., how much you would make if you exercised today.

$$\begin{aligned} C(S, K, t, T) &\geq \max[0, S_t - K] \\ P(S, K, t, T) &\geq \max[0, K - S_t] \end{aligned}$$

(F) Calls with a lower strike are more valuable. Puts with a higher strike are more valuable; i.e., for $K_1 > K_2$,

$$\begin{aligned} C(S, K_2, T) > C(S, K_1, T), &\qquad c(S, K_2, T) > c(S, K_1, T) \\ P(S, K_2, T) < P(S, K_1, T), &\qquad p(S, K_2, T) < p(S, K_1, T) \end{aligned}$$

(G) American options are more valuable with more time to maturity i.e., for $T_2 > T_1$,

$$\begin{aligned} C(S, K, t, T_2) \geq &C(S, K, t, T_1) \\ P(S, K, t, T_2) \geq &P(S, K, t, T_1) \end{aligned}$$

Obvious? Then why doesn't it hold for <u>European</u> options as well? Or does it?

- Suppose you can take in cash today selling one-month options and buying two-month ones.
- In one month our short option is worth its intrinsic value.
- To TRULY have an arbitrage, we have to be CERTAIN we can sell our long option for at least its intrinsic value.
- What is our long position worth?

  Answer 1 Whatever the market wants to pay us for it.

  Answer 2 At least as much as its lower no-arbitrage bound!

- *Can European options (calls or puts) ever be worth less than their intrinsic value?*

...A (first) partial answer:

(H) For a stock that does *not pay dividends*:

$$c(S, K, t, T) \geq \max[0, S - K \cdot B_{t,T}]$$
$$C(S, K, t, T) \geq \max[0, S - K \cdot B_{t,T}]$$

Proof: To prove this, we only need to show (why?)

$$c(S, K, t, T) \geq S - K \cdot B_{t,T}.$$

We do the proof by contradiction. If $c < S - K \cdot B$, we have an arbitrage:

Transaction	Cash flow (at t)	Payoff (at T)
	$-c$ S_t $-K \cdot B$	$\max[0, S_T - K]$ $-S_T$ K
Combined	$S - K \cdot B - c$	$\max[0, S_T - K]$ $-[S_T - K]$

- **Example:** Assume EBAY stock is at $S_t = \$100$, and the price of an EBAY call with $K = \$90$ and 6 months to maturity is $\$11$. Assume that EBAY will not pay any dividend within the next 6 months and assume that the riskfree interest rate (c.c., annualized) is 10%. Is there an arbitrage?

- Since $c = \$11$ and $S - K \cdot B = 100 - 90e^{-.10 \times 0.5} = \$14.39$, we have an arbitrage.

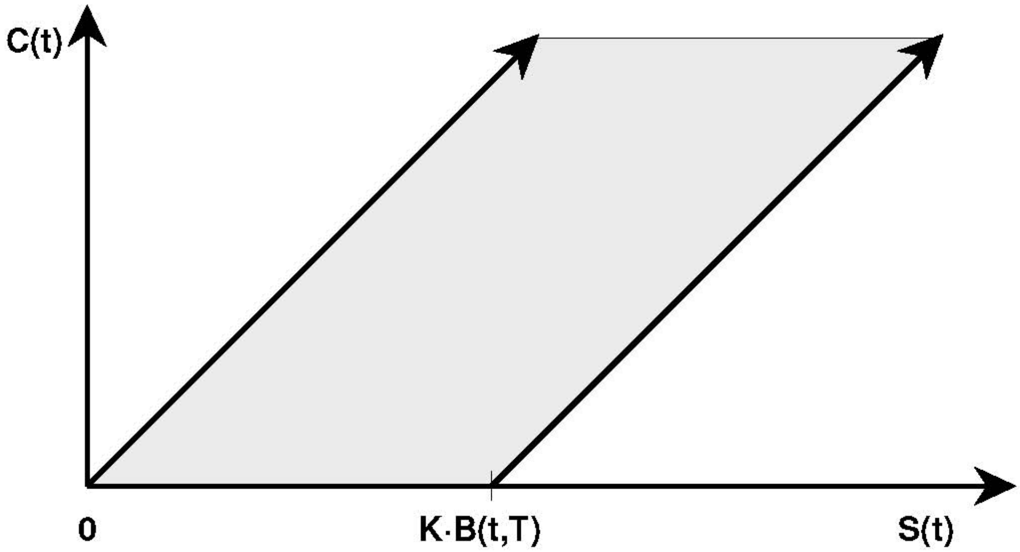
- Make sure you know what it is!

(I) For European puts on *non-dividend-paying* stocks, a similar arbitrage argument shows that:

$$p(S, K, t, T) \geq \max\left[0, K \cdot B - S\right],$$

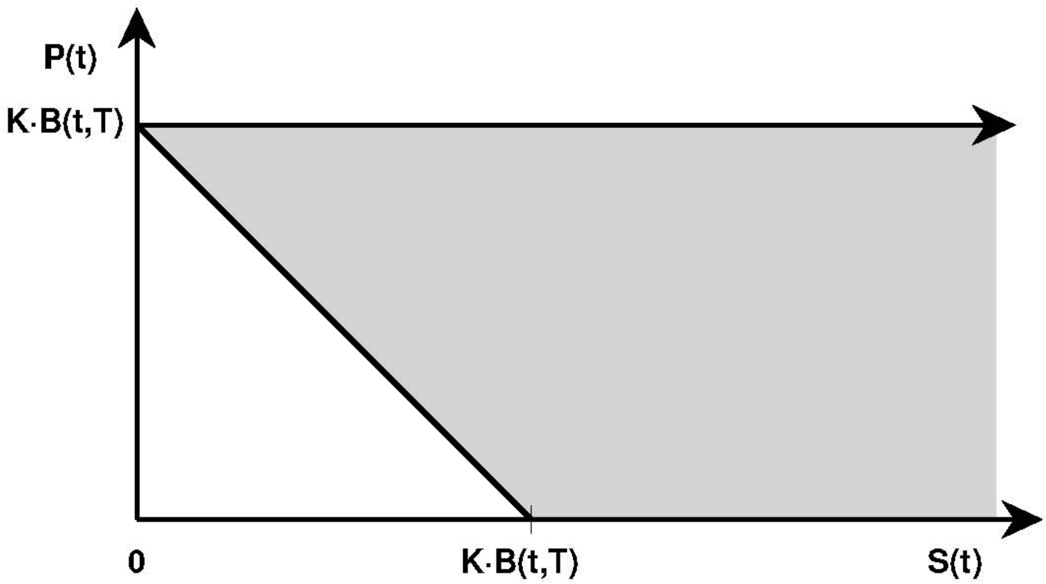
- Combining rules (C) and (H) implies that the value of a European call on a *non-dividend-paying* stock must lie in the shaded region:

$$\max\left[0, S_t - K \cdot B_{t,T}\right] \leq c(S, K, t, T) \leq S_t.$$



- Combining (D) and (I), we see that the value of a European put on a *non-dividend-paying* stock must lie within the shaded region:

$$\max\left[0, K \cdot B_{t,T} - S_t\right] \leq p(S, K, t, T) \leq K \cdot B_{t,T}.$$



(J) Note that, for European puts, the lower boundary is *worse* than the "intrinsic value", $max[K - S, 0]$. But we *can* now (sometimes) guarantee that European calls are worth more than their intrinsic value.

- So we can refine point **(G)** above:

  (G') For $T_2 > T_1$, $c(S, K, t, T_2) \geq c(S, K, t, T_1)$ <u>if</u> the underlying does not pay a dividend.

- Note that we <u>still</u> can't conclude this for European puts!

(K) For calls on a *non-dividend-paying* stock, we must have

$$C(S, K, t, T) = c(S, K, t, T).$$

- *<u>Proof.</u>* The options only differ by the right to exercise the American option before T. For exercise, we must have $C(t) < \max[0, S_t - K]$. But we just showed $C(t) \geq \max[0, S_t - B(t) \cdot K] > \max[0, S_t - K]$ **Thus, early exercise is never optimal for an American call on a non-dividend paying stock.** Hence its cash-flows are always equal to those of a European call. Hence they have the same value, by no arbitrage (again).

- We just learned one *optimal exercise policy* rule. To solve the general problem of when to optimally exercise American options on dividend paying stocks requires a model. However we can still deduce some contingent rules just by no-arbitrage.

(L) The only dates at which it might be optimal to exercise an American call on a dividend paying stock early, are the points immediately preceding an ex-dividend date.

- *Proof.* Suppose we are at t and t' is the last instant before the next ex-date. If early exercise were optimal at t, then it must be the case that $C(t) < S_t - K > 0$. If so, buy the call, short the stock, and put the net money $S_t - C(t) = K$ into the bank. Then exercise at t' and make $Ke^{r(t'-t)} - K > 0$.

(M) It is never optimal to exercise an American put on a dividend paying stock immediately preceding an ex-dividend date.

- *Proof.* Suppose we are at t', the last instant before the ex-date, and t'' is the ex-date, i.e. an instant after t'. If early exercise were optimal at t', then it must be the case that $P(t') = K - S(t') > 0$. If so, buy the put and the stock. Then collect the dividend and exercise at t''. You had to borrow $P(t') + S(t') = K$, but you repayed it an instant later, with no other cash-flows at t''. Hence the dividend represents your arbitrage profit.

- This rule is not too useful since it only rules out exercise at dates immediatetly preceding ex-dates. But we shouldn't feel too bad. The general problem of optimal early exercise of American puts still hasn't been solved analytically!

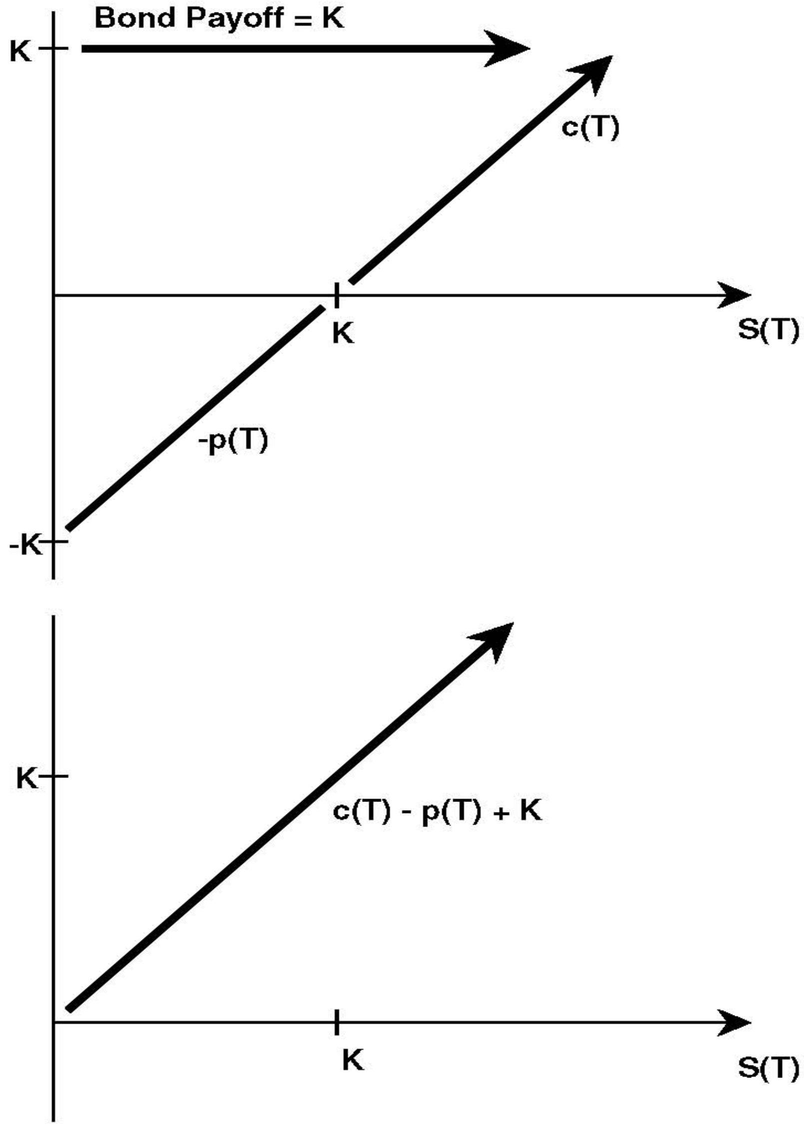
III. Put-Call Parity (no payouts)

(A) European case.

- For European options on non-dividend paying stocks:

$$S_t = c(S, K, t, T) - p(S, K, t, T) + K \cdot B_{t,T}.$$

- Intuition: a certain portfolio of bonds and options has the same payoff at maturity as a share of stock, so it must have the same initial price as a share of stock.



- If this relation does not hold there is an arbitrage opportunity.
- From the picture it is also clear that buying a call and selling a put at strike K is the same as going long a forward on S with forward price K.
  - ▶ So $(c - p) = B_{t,T}(F_{t,T} - K)$ which says the same thing,

- **Example:** What would you do if you saw these prices?

$$\begin{aligned} K &= 50, \qquad S = 50 \\ r &= 1.2\%, \qquad T - t = 1 \text{ month} \\ c &= 4.5, \quad p = 4.0 \end{aligned}$$

$$\Rightarrow \; S \neq c - p + K \cdot B$$

Transaction	Initial (t) Cashflow	Final (T) Cashflow	
		$S_T < 50$	$S_T > 50$
	$-\$50$	S_T	S_T
	$\$4.5$	0	$-[S_T - \$50]$
	$-\$4.0$	$\$50 - S_T$	0
	$\$49.95$	$-\$50$	$-\$50$
	$\$0.45$	0	0

- The Put-Call Parity relation tells us that we can make a synthetic stock, call, put, and bond for European options on a non-dividend-paying stock. It even tells us how to make it:

Synthethic stock:	$S = c - p + PV(K)$
Synthethic call :	$c = S + p - PV(K)$
Synthethic put:	$p = c - S + PV(K)$
Synthethic bond:	$PV(K) = S - c + p$

- **Question:** Which one has a higher price: an ATM European call or an ATM European put on the same non-dividend-paying stock?

- **Question:** Suppose you lived in a country with *usury laws*. Do you see how a bank could lend money to a company without ever charging interest?
  - ▶ This is an example of financial engineering that applies in Islamic countries.

(B) P/C/P – The American case (still no payouts to S).

- Although we can't use the same replication argument for American options, we can use our previous results to get a *pair of inequalities* that is pretty close to the *equality* above.

- Step 1: $P \geq p$ and $C = c$ and European put-call parity together give
$$P \geq C + B \cdot K - S.$$

- Step 2: By no-arbitrage, also have
$$c + K \geq P + S.$$

Proof. Suppose not. Sell the put and stock, buy the call, invest K in the bond and have cash left over. Now there are two possibilities:

1. The put we sold is not exercised early. So our payoff at T is:

Put plus share:	$-\max[K - S_T, 0] - S_T =$	$-\max[S_T, K]$
Call :	$\max[S_T - K, 0] =$	$\max[S_T, K] - K$
Bond:		K/B
Total:		$K/B - K > 0$

2. The put we sold is exercised early. So we buy back our stock for K from the put holder. But our bond is worth at least K. The put is gone, and we still have the call. So all together our position still has positive value.

So in neither case can you lose money. Hence you have an arbitrage.

- Step 3: Rewrite the inequalities from Steps 1 and 2, use $C = c$ again, and conclude

$$S - K \leq C - P \leq S - B \cdot K, \text{ or}$$

$$K \geq S - C + P \geq B \cdot K.$$

- Notice that this is almost as good as the European put-call parity when the options don't have long to maturity, or interest rates are low.

- **A caution:** The real assumption we are making about dividends and borrowing costs here is *not just* that they are zero, but also that we know they will stay zero between now and T.

  - When we come across apparent violations of p/c/p in real data, it is almost always because the markets know something about future dividend changes that we do not know.

Let's see how dividends affect things.

IV. Effects of Dividends and Carry Costs on Arbitrage Restrictions:

(A) Known dividends on a stock (discrete payouts).

- We can easily modify some of our conclusions today for the case of stocks whose dividends are known (with certainty) in advance.

  ▶ For any of the arbitrage strategies in the proofs that involved holding the stock until T, we just lower the cost by the amount we know we will take in from dividends.

  ▶ In other words, replace S_t everywhere with $S_t - PV(D)$. *As in Lecture 1, to avoid messy formulas involving the dividend dates, just write $PV(D)$ for the amount you would have to put in risk-free bonds today to replicate the payouts.*

- **Example** The lower bounds for options on dividend-paying stocks becomes:

$$C(K,T) \geq c(K,T) \geq \max[0, S_t - PV(D) - K \cdot B]$$

$$P(K,T) \geq p(K,T) \geq \max[0, K \cdot B - S_t + PV(D)]$$

  ▶ If either of these two rules is violated, one constructs an arbitrage by buying the option and shorting the right-hand side of the inequality.

  ▶ "Shorting" $PV(D)$ just means borrowing that amount today and repaying the Ds as they occur.

- The above laws allow us to immediately deduce a little bit more about the optimal early exercise problem for dividend paying stocks. In particular, we can say the following:

  1. If, at t we have $PV(D) < K - K \cdot B_{t,T}$, then it cannot be optimal to exercise an American call with strike K at that time.

  2. If, at t we have $PV(D) > K - K \cdot B_{t,T}$, then it cannot be optimal to exercise an American put with strike K at that time.

- Why?

  ▶ Just use the new lower bounds. Take the put case.

$$\begin{aligned} \text{If} \quad PV(D) &> K - K \cdot B \qquad \text{then} \\ K \cdot B - S + PV(D) &> -S + K \qquad \text{which implies} \\ P &> -S + K \end{aligned}$$

  which means you will be throwing away money if you exercise!

  ▶ The proof for the calls works similarly.

- **Question:** what does rule 2 tell us if interest rates are close to zero?

- Put-Call Parity For European optionss on stocks with known dividends becomes

$$c(S, K, t, T) - p(S, K, t, T) = S_t - PV(D) - K \cdot B_{t,T}$$

via the same argument of replacing S by $S - PV(D)$. (We are still assuming the stock can be freely shorted.)

- Put-Call Parity for *American* options on dividend paying stocks becomes a little trickier:

$$S - PV(D) - K \leq C - P \leq S - K \cdot B.$$

  ▶ The same substiution trick works on the left-hand inequality (the lower bound on $C - P$), but not on the upper bound.

  ▶ If the lower bound is violated, our arbitrage involves selling the stock and selling the put.

    ∗ The worst case outcome is that the put is not exercised until expiration and so we do have to hold the short position and pay all the dividends.

  ▶ If the upper bound is violated, then our enforcement leads us to buy the stock and sell the call.

    ∗ Then the worst case is the call is exercised immediately and we never get any of the dividends.

  ▶ Make sure you can formalize this argument yourself!

(B) Known percentage yield (continuous flow) to underlying.

- This case encompasses options on currencies, commodities, and any assets with borrowing fees. It is also not a bad approximation for baskets of stocks.

- Let's examine the case of currency options, where the foreign risk-free rate r_f could also be a dividend yield etc.

  ▶ Call the domestic rate r_d. So $B = e^{-r_d(T-t)}$.

  ▶ Now the trick is the following. For each inequality that we want to prove, re-do the same arguments but instead of buying (or selling) *one* unit of S, do the trade for $e^{-r_f(T-t)}$ units.

  ▶ Imagine we do this every $\Delta t = 1$ day, adjusting our position until T. Two things happen.

  1. The size of our position changes by

  $$e^{-r_f(T-t_1)} - e^{-r_f(T-t_0)} = e^{-r_f(T-t_0)}\left[e^{r_f\Delta t} - 1\right] \approx r_f\ \Delta t\ e^{-r_f(T-t_0)}.$$

  This says the percent change in our position size is $r_f \Delta t$.

  2. But we also get to receive the interest (or pay if we're short) equal to

  $$S_t e^{-r_f(T-t_0)}\left[e^{r_f\Delta t} - 1\right] \approx r_f\ \Delta t\ S_t\ e^{-r_f(T-t_0)}.$$

  The second part provides exactly the amount of money needed to do the first part!

▶ In other words, each period we will re-invest the flow received in time interval Δt back into our holdings of the underlying.

∗ The net cash-flow is obviously zero each period.

∗ The neat thing is, we maintain the same exponential holding quantity. In particular, our position will *equal exactly one unit at time t*.

- I will leave it as an exercise for you to fill in the details of the individual proofs.

▶ Intuitively, where before we replaced S by $S - PV(D)$, now we will try to replace S_t with $e^{-r_f(T-t)} \cdot S_t$ everywhere.

- The option lower bounds become

$$C(S,K,t,T) \geq c(S,K,t,T) \geq \max[0, S_t \cdot e^{-r_f(T-t)} - K \cdot e^{-r_d(T-t)}]$$
$$P(S,K,t,T) \geq p(S,K,t,T) \geq \max[0, K \cdot e^{-r_d(T-t)} - S_t e^{-r_f(T-t)}]$$

- European Put-Call Parity becomes

$$c - p = S_t \cdot e^{-r_f(T-t)} - K \cdot e^{-r_d(T-t)}$$

Or for a stock with (continuous) dividend yield d and borrowing rate y

$$c - p = S_t \cdot e^{-(d+y)(T-t)} - K \cdot e^{-r(T-t)}$$

- You should try to derive the American put-call parity *inequalities* yourself.

V. Summary

- Based on static dominating portfolios, we can prove a number of rules about option prices **without making any assumptions about the behavior of the underlying security over time.**

- To prove these rules, we show that if they fail, arbitrage opportunities will be present.
  - ▶ If we observe failures in real life, it may be because some subtle assumption we made is violated.

- Most important: Put-Call Parity.
  - ▶ Tells us how to synthesize either call, put, stock or bond using the other three.
  - ▶ Tells us we won't need separate theories to price (European) puts and calls.

- Methodological point: whenever analysing a new derivative, always start by deriving the "common sense" bounds that static domination requires.
  - ▶ Sometimes that's all you need anyway.
  - ▶ Can guide your intuition in deriving better results based on dynamic replication.
  - ▶ Provides sanity check for more "sophisticated" models.