Black-Scholes-Merton Model

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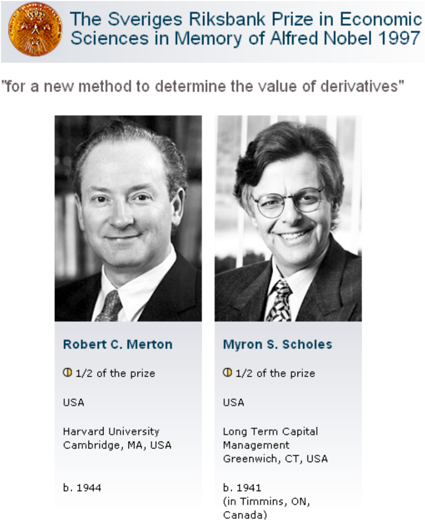
Readings: Shreve Section 4.5, Chapter 5

The Black-Scholes-Merton model

- A derivative pricing model based on geometric Brownian motion
- Originated from two papers: Black-Scholes (1973) and Merton (1973)
- Scholes and Merton won Nobel prize in Economics in 1997



Geometric Brownian motion

- Asset price process follows a GBM

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Instantaneous mean rate of return $\mu \in \mathbb{R}$, volatility $\sigma > 0$

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$

- $X_t = \ln(S_t/S_0)$ (the **log return process**),

$$X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$$

is a BM with drift. In differential form,

$$dX_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t$$

Option pricing in the BSM model

- Interested in pricing (European style) options in the BSM model
- First approach: finding a replicating portfolio so that option price = cost of replication (leads to a **partial differential equation**)
- Martingale approach: finding a risk neutral measure so that option price = discounted **risk neutral expectation** of option payoff (assuming the option payoff is replicable)

Black-Scholes-Merton equation

- Consider a **European style** derivative with payoff $V(S_T)$
- Binomial model: trading the underlying asset and investing at the risk free interest rate, one replicates the derivative payoff
- Value of the replicating portfolio X_t at time $t =$ derivative value at t
- Suppose at time t, Δ_t shares are held, the remaining $X_t - \Delta_t S_t$ is invested at the risk free rate r
- Find Δ_t so that the portfolio replicates the derivative payoff: comparing the dynamics of the replicating portfolio and the value process of the derivative

Dynamics of the replicating portfolio

- The dynamics of the replicating portfolio:
  - the value change of the stocks: $\Delta_t dS_t$
  - the value change of the risk free investment: $r(X_t - \Delta_t S_t)dt$

$$
\begin{aligned}
dX_t &= \Delta_t dS_t + r(X_t - \Delta_t S_t)dt \\
&= \Delta_t(\mu S_t dt + \sigma S_t dB_t) + r(X_t - \Delta_t S_t)dt \\
&= rX_t dt + \Delta_t S_t(\mu - r)dt + \sigma \Delta_t S_t dB_t
\end{aligned}
$$

The term $rX_t dt$ is the part of X_t that is earning risk free interest rate r. Can be removed by considering the discounted process $e^{-rt}X_t$

- By Itô product rule,

$$
\begin{aligned}
d(e^{-rt}X_t) &= -re^{-rt}X_t dt + e^{-rt}dX_t \\
&= -re^{-rt}X_t dt + e^{-rt}\left[rX_t dt + \Delta_t S_t(\mu - r)dt + \sigma \Delta_t S_t dB_t\right] \\
&= e^{-rt}\left[\Delta_t S_t(\mu - r)dt + \sigma \Delta_t S_t dB_t\right]
\end{aligned}
$$

Dynamics of the derivative price

- Denote the value of the derivative at time t by $V(t, S_t)$
- Dynamics of the derivative value process

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS_t \cdot dS_t \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 dt \\
&= (\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2)dt + \frac{\partial V}{\partial S}\sigma S_t dB_t
\end{aligned}
$$

- Want to compare $d(e^{-rt}X_t)$ and $d(e^{-rt}V)$

- By Itô product rule,

$$
\begin{aligned}
d(e^{-rt}V) &= -re^{-rt}Vdt + e^{-rt}dV \\
&= -re^{-rt}Vdt \\
&\quad + e^{-rt}\left[\left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S_t \frac{\partial V}{\partial S} dB_t \right] \\
&= e^{-rt}\left[\left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV \right) dt \right. \\
&\quad \left. + \sigma S_t \frac{\partial V}{\partial S} dB_t \right]
\end{aligned}
$$

Finding the replicating strategy

$$d(e^{-rt}X_t) = e^{-rt}\left[\Delta_t S_t(\mu - r)dt \qquad\qquad + \sigma S_t \Delta_t dB_t\right]$$

$$d(e^{-rt}V) = e^{-rt}\left[\left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 - rV\right)dt + \sigma S_t \frac{\partial V}{\partial S} dB_t\right]$$

- The trading strategy replicates the derivative value process if and only if

$$X_0 = V_0$$

$$\Delta_t = \frac{\partial V}{\partial S}(t, S_t)$$

$$\Delta_t S_t(\mu - r) = \frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 - rV$$

Black-Scholes-Merton equation

- European derivative value with payoff $V(S_T)$ solves

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad t \in [0, T)$$

- **Terminal condition**

$$V(T,S) = (S-K)^+ \text{ for calls}, \quad V(T,S) = (K-S)^+ \text{ for puts}$$

- Solve the Black-Scholes-Merton equation, one gets the Black-Scholes formula (transform the equation into a **heat equation** of the form $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ whose analytical solution is known (Wilmott, Dewynne and Howison 1994))

Black-Scholes formula (zero yield)

- European call and put option prices

$$c(t,S) = SN(d_+) - Ke^{-r(T-t)}N(d_-)$$

$$p(t,S) = -SN(-d_+) + Ke^{-r(T-t)}N(-d_-)$$

where

$$d_\pm = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

- The put-call parity holds

$$c + Ke^{-r(T-t)} = p + S$$

Hedging

- The replicating strategy requires $\Delta_t = \frac{\partial V}{\partial S}(t, S)$ stocks
- **Delta hedging**: one should hold Δ_t shares to hedge a short position
- For calls,
$$\Delta_t = \frac{\partial c}{\partial S}(t, S) = N(d_+) > 0$$
for puts,
$$\Delta_t = \frac{\partial p}{\partial S}(t, S) = N(d_+) - 1 < 0$$

Martingale method overview

- **Martingale method** in binomial model for **arbitrary** European derivatives (including path dependent)
  1. Under the risk neutral measure $\mathbb{Q}$, **discounted asset price process** is a martingale
  2. **Discounted value process of the replicating portfolio** for the European derivative is also a martingale under measure $\mathbb{Q}$
  3. **Risk neutral pricing formula** in the binomial model

$$f_n = \mathbb{E}^{\mathbb{Q}}[e^{-r(N-n)\delta} f_N | \mathcal{F}_n]$$

The value of the European derivative with payoff f_N is the risk neutral expectation of the discounted payoff

- **Martingale method** in the Black-Scholes-Merton framework
  1. Find the risk neutral measure (**equivalent martingale measure**) $\mathbb{Q}$ under which the **discounted asset price process** is a martingale
  2. Show that the **discounted value process of the replicating portfolio** for the European derivative is a martingale under $\mathbb{Q}$
  3. Derive the risk neutral pricing formula using the martingale property
  4. Establish the existence of a replicating portfolio
- Applicable to path dependent European derivatives

Change of measure

- Find the equivalent martingale measure $\mathbb{Q}$ in the BSM model
- Suppose $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures on a finite sample space Ω (so that $\mathbb{P}(\omega), \mathbb{Q}(\omega) > 0, \forall \omega \in \Omega$)
- **Radon-Nikodým derivative** of measure $\mathbb{Q}$ w.r.t. measure $\mathbb{P}$

$$Z(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}$$

For any event A,

$$\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A] = \mathbb{Q}(A) = \sum_{\omega \in A} \mathbb{Q}(\omega) = \sum_{\omega \in A} Z(\omega)\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_A Z]$$

- Basic properties of Radon-Nikodým derivative
  - Positive: $Z > 0$
  - Unit expectation:

$$\mathbb{E}^{\mathbb{P}}[Z] = \sum_{\omega} Z(\omega)\mathbb{P}(\omega) = \sum_{\omega} \mathbb{Q}(\omega) = 1$$

  - For any r.v. Y:

$$\mathbb{E}^{\mathbb{Q}}[Y] = \sum_{\omega \in \Omega} Y(\omega)\mathbb{Q}(\omega) = \sum_{\omega \in \Omega} Z(\omega)Y(\omega)\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}[ZY]$$

Radon-Nikodým theorem

- (**Radon-Nikodým**) *Let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be equivalent probability measures on* $(\Omega, \mathcal{F})$*. There exists an a.s. positive r.v.* Z *such that* $\mathbb{E}^{\mathbb{P}}[Z] = 1$*,*

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_A Z], \quad \forall A \in \mathcal{F}$$

- The Radon-Nikodým derivative Z is written as

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

- For any r.v. Y,

$$\mathbb{E}^{\mathbb{Q}}[Y] = \mathbb{E}^{\mathbb{P}}[ZY], \quad \mathbb{E}^{\mathbb{P}}[Y] = \mathbb{E}^{\mathbb{Q}}[Y/Z]$$

- Conversely, if Z is an a.s. positive r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}^{\mathbb{P}}[Z] = 1$, then

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_A Z], \quad \forall A \in \mathcal{F}$$

  defines an equivalent probability measure on $(\Omega, \mathcal{F})$
- Suppose $X \sim N(0,1)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and $Y = \mu + \sigma X$ (under measure $\mathbb{P}$, $Y \sim N(\mu, \sigma^2)$). Define a new measure $\mathbb{Q}$ via

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\theta X - \frac{1}{2}\theta^2}, \quad \text{where } \theta = \frac{\mu - \nu}{\sigma}$$

- $Y \sim N(\nu, \sigma^2)$ under the new measure $\mathbb{Q}$

$$\begin{aligned}
\mathbb{Q}(Y \le y) &= \mathbb{Q}\left(X \le \frac{y-\mu}{\sigma}\right) = \mathbb{E}^{\mathbb{P}}[1_{\{X \le \frac{y-\mu}{\sigma}\}} e^{-\theta X - \frac{1}{2}\theta^2}] \\
&= \int_{-\infty}^{\frac{y-\mu}{\sigma}} e^{-\theta x - \frac{1}{2}\theta^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
&= \int_{-\infty}^{\frac{y-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+\theta)^2} dx \\
&= \int_{-\infty}^{\frac{y-\mu}{\sigma}+\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \\
&= \int_{-\infty}^{\frac{y-\nu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = N\left(\frac{y-\nu}{\sigma}\right)
\end{aligned}$$

Girsanov theorem

- Girsanov theorem changes the drift of a Brownian motion
- **(Girsanov theorem)** *Given* $T > 0$, *let* $\{B_t, 0 \leq t \leq T\}$ *be a BM on a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *and* θ_t *be an adapted process, then*

$$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \theta_t^2 dt - \int_0^T \theta_t dB_t\right)$$

defines an equivalent measure $\mathbb{Q}$ *so that the process*

$$B_t^* = \int_0^t \theta_s ds + B_t \quad (\textit{in differential form } dB_t^* = \theta_t dt + dB_t)$$

is a standard BM under the measure $\mathbb{Q}$.

- When $\theta_t = \theta$ is a constant,

$$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\theta^2 T - \theta B_T\right).$$

  Then $B_t^* = \theta t + B_t$ is a standard BM under measure $\mathbb{Q}$.

- Recall that Z_t defined below is a martingale

$$Z_t = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s\right)$$

  satisfying $dZ_t = -\theta_t Z_t dB_t$. Therefore $\mathbb{E}^{\mathbb{P}}[Z_T] = Z_0 = 1$, $Z_T > 0$. So Z_T defines an equivalent measure $\mathbb{Q}$.

- $\{Z_t, 0 \leq t \leq T\}$ is known as the **Radon-Nikodým derivative process**

Proof of Girsanov theorem

- **Theorem (Lévy)**: *A continuous martingale M_t starting from zero is a BM if the quadratic variation $[M, M](t) = t$* (Shreve p.168)
- **Property I**: *suppose Y_t is $\mathcal{F}_t-$measurable,*

$$\mathbb{E}^{\mathbb{Q}}[Y_t] = \mathbb{E}^{\mathbb{P}}[Y_t Z_t]$$

In fact,

$$\mathbb{E}^{\mathbb{P}}[Y_t Z_t] = \mathbb{E}^{\mathbb{P}}[Y_t \mathbb{E}^{\mathbb{P}}[Z_T|\mathcal{F}_t]] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[Y_t Z_T|\mathcal{F}_t]]$$

$$= \mathbb{E}^{\mathbb{P}}[Y_t Z_T] = \mathbb{E}^{\mathbb{Q}}[Y_t]$$

- **Property II**: *suppose $s \leq t$ and Y_t is $\mathcal{F}_t-$measurable*

$$\mathbb{E}^{\mathbb{Q}}[Y_t|\mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}^{\mathbb{P}}[Y_t Z_t|\mathcal{F}_s]$$

By the definition of conditional expectation, for any $A \in \mathcal{F}_s$

$$\begin{aligned}
\int_A \frac{1}{Z_s}\mathbb{E}^{\mathbb{P}}[Y_t Z_t|\mathcal{F}_s] d\mathbb{Q} &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_A \frac{1}{Z_s}\mathbb{E}^{\mathbb{P}}[Y_t Z_t|\mathcal{F}_s]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_A \mathbb{E}^{\mathbb{P}}[Y_t Z_t|\mathcal{F}_s]\right] \quad \text{by property I} \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_A Y_t Z_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A Y_t] \quad \text{by property I} \\
&= \int_A Y_t d\mathbb{Q}
\end{aligned}$$

- The quadratic variation of B_t^* is t since

$$dB_t^* = \theta_t dt + dB_t, \quad (dB_t^*)^2 = dt$$

It suffices to show that B_t^* is a martingale under measure $\mathbb{Q}$. We first show that $B_t^* Z_t$ is a martingale under measure $\mathbb{P}$

$$\begin{aligned} d(B_t^* Z_t) &= B_t^* dZ_t + Z_t dB_t^* + dB_t^* \cdot dZ_t \\ &= B_t^* (-\theta_t Z_t dB_t) + Z_t(\theta_t dt + dB_t) \\ &\quad +(\theta_t dt + dB_t)(-\theta_t Z_t dB_t) \\ &= (1 - \theta_t B_t^*) Z_t \; dB_t \end{aligned}$$

Under measure $\mathbb{Q}$, $s \leq t$

$$\mathbb{E}^{\mathbb{Q}}[B_t^* | \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}[B_t^* Z_t | \mathcal{F}_s] = \frac{1}{Z_s} B_s^* Z_s = B_s^*$$

Equivalent martingale measure

- Asset price follows a (generalized) GBM over period $[0, T]$

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t,$$

μ_t and $\sigma_t > 0$ are adapted w.r.t. natural filtration $\mathbb{F} = \{\mathcal{F}_t\}$ of the BM

- Risk free interest rate is r_t, adapted w.r.t. $\mathbb{F}$
- *What is the EMM $\mathbb{Q}$ under which discounted asset price $S_t^* = D_t S_t$ is a martingale?* The **discount factor** is

$$D_t = e^{-\int_0^t r_s ds} \Leftrightarrow dD_t = -r_t D_t dt$$

- By the Itô product rule,

$$
\begin{aligned}
dS_t^* &= D_t dS_t + S_t dD_t \\
&= D_t(\mu_t S_t dt + \sigma_t S_t dB_t) - r_t D_t S_t dt \\
&= (\mu_t - r_t) D_t S_t dt + D_t \sigma_t S_t dB_t \\
&= \sigma_t S_t^* \left(\frac{\mu_t - r_t}{\sigma_t} dt + dB_t \right) \\
&= \sigma_t S_t^* \left(\theta_t dt + dB_t \right) \quad \text{where } \theta_t = \frac{\mu_t - r_t}{\sigma_t}
\end{aligned}
$$

Let $\quad B_t^* = \int_0^t \theta_s ds + B_t \quad \Leftrightarrow \quad dB_t^* = \theta_t dt + dB_t$

- By Girsanov theorem, B_t^* is a standard BM under measure $\mathbb{Q}$ defined by Radon-Nikodým derivative

$$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \theta_t^2 dt - \int_0^T \theta_t dB_t\right)$$

- Discounted asset price process $S_t^* = D_t S_t$ is a martingale under measure $\mathbb{Q}$

$$dS_t^* = \sigma_t S_t^* dB_t^* \Leftrightarrow S_t^* = S_0 + \int_0^t S_u^* dB_u^*$$

- $\mathbb{Q}$ is the desired risk neutral measure:
  - Discounted asset price process is martingale
  - By Ito product rule,

$$
\begin{aligned}
dS_t &= \mu_t S_t dt + \sigma_t S_t dB_t \\
&= \mu_t S_t dt + \sigma_t S_t (dB_t^* - \theta_t dt) \\
&= (\mu_t - \sigma_t \theta_t) S_t dt + \sigma_t S_t dB_t^* \\
&= r_t S_t dt + \sigma_t S_t dB_t^* \quad \text{since } \theta_t = \frac{\mu_t - r_t}{\sigma_t}
\end{aligned}
$$

- θ_t is known as the **market price of risk**: it is the excess rate of return per unit of volatility

Pricing European style derivatives

- Consider a European style derivative with payoff V_T ($\mathcal{F}_T-$measurable)
  - V may depend $S_u, u \leq T$ (e.g., lookback, Asian options)
- Consider a replicating strategy
  - with value X_t at time t
  - holding Δ_t shares
  - investing $X_t - \Delta_t S_t$ at the risk free interest rate
  - replicates the payoff of the derivative: $X_T = V_T$
  - *will show that such a strategy exists*
- First show that **discounted value process of the replicating portfolio** is a martingale under measure $\mathbb{Q}$

- Dynamics of X_t:

$$
\begin{aligned}
dX_t &= \Delta_t dS_t + r_t(X_t - \Delta_t S_t)dt \\
&= \Delta_t(\mu_t S_t dt + \sigma_t S_t dB_t) + r_t(X_t - \Delta_t S_t)dt \\
&= r_t X_t dt + \Delta_t S_t(\mu_t - r_t)dt + \sigma_t \Delta_t S_t dB_t
\end{aligned}
$$

- The $r_t X_t dt$ term will disappear when discounted value process is considered

- Dynamics of $X_t^* = D_t X_t$:

$$\begin{aligned} d(X_t^*) &= D_t dX_t + X_t dD_t \\ &= D_t(r_t X_t dt + \Delta_t S_t(\mu_t - r_t)dt + \sigma_t \Delta_t S_t dB_t) - r_t D_t X_t dt \\ &= D_t(\Delta_t S_t(\mu_t - r_t)dt + \sigma_t \Delta_t S_t dB_t) \\ &= \Delta_t S_t^* \sigma_t(\theta_t dt + dB_t) = \Delta_t S_t^* \sigma_t dB_t^* \end{aligned}$$

so $X_t^* = D_t X_t$ is a martingale under measure $\mathbb{Q}$

Risk neutral pricing

- **Risk neutral pricing** of European style (including path dependent) derivatives

$$D_t X_t = \mathbb{E}^{\mathbb{Q}}[D_T X_T | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t]$$

hence the value of the derivative at time t is given by

$$V_t = X_t = \frac{1}{D_t}\mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) V_T | \mathcal{F}_t\right]$$

Martingale representation theorem

- Existence of the replicating portfolio is guaranteed by the *martingale representation theorem*
- (**Martingale representation theorem**) *Let B_t be a BM in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and M_t a martingale w.r.t. $\mathcal{F}_t = \sigma(B_s, s \leq t)$. Then there exists an adapted process Δ_t such that*

$$dM_t = \Delta_t dB_t, \quad i.e., \quad M_t = M_0 + \int_0^t \Delta_s dB_s$$

- Define $\bar{V}_t$ by

$$\bar{V}_t = \frac{1}{D_t} E^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t], \quad 0 \leq t \leq T$$

then $\{D_t \bar{V}_t, 0 \leq t \leq T\}$ is a martingale under measure $\mathbb{Q}$: for $s \leq t$,

$$\mathbb{E}^{\mathbb{Q}}[D_t \bar{V}_t | \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t] | \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_s] = D_s \bar{V}_s$$

- By martingale representation theorem, there exists an adapted process Δ_t^*

$$d(D_t \bar{V}_t) = \Delta_t^* dB_t^*$$

- A trading strategy with initial value $X_0 = \bar{V}_0$, value X_t at t, holding Δ_t shares and invest $X_t - \Delta_t S_t$ at risk free rate will replicate the option payoff V_T

$$\Delta_t = \frac{\Delta_t^*}{S_t^* \sigma_t}$$

$$d(D_t X_t) = \Delta_t \sigma_t S_t^* dB_t^* = \Delta_t^* dB_t^* = d(D_t \bar{V}_t)$$

So $X_t = \bar{V}_t$ for any $t \in [0, T]$. In particular,

$$X_T = \bar{V}_T = V_T$$

Risk neutral pricing formula

- Derivative with payoff V_T: **risk neutral pricing formula**

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) V_T \middle| \mathcal{F}_t\right]$$

where r_t is the risk free interest rate. In particular,

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right) V_T\right]$$

- Under measure $\mathbb{Q}$, underlying asset price follows (μ_t not used)

$$dS_t = r_t S_t dt + \sigma_t S_t dB_t^*$$

- Black-Scholes formula follows with constants $r_t \equiv r, \sigma_t \equiv \sigma$

Derivatives on assets with continuous yield

- **Currency options**: a European call on GBP with strike K

$$\text{option payoff at maturity } T : (S_T - K)^+$$

  S_T is the exchange rate at T. A currency is an asset paying *continuous yield* (e.g., the GBP risk free interest rate)

- **Stock index options**: a European put on *S&P 500* with strike K

$$\text{option payoff at maturity } T : (K - S_T)^+$$

  S_T is *S&P 500* index value at T. Stock indices are considered as assets paying continuous dividend

- **Futures options**: a European call on copper futures with strike price $K = 3.15$ (per pound). Payoff at maturity T:

  $$(F_T - K)^+ \quad \textbf{plus} \quad \text{a long position in the futures contract}$$

  F_T is the futures price at time T of the futures contract (with maturity $T^* > T$)
- Why trade options on copper futures instead of copper itself?
  - Futures contracts are more liquid
  - Exercising of the futures option does not usually lead to delivery of the underlying asset
- *A futures option can be priced as an option on an asset paying continuous yield at the risk free interest rate*

Pricing derivatives on assets with continuous yield

- Suppose the underlying asset pays continuous yield at rate q_t ($\mathcal{F}_t$ adapted)

$$dS_t = (\mu_t - q_t)S_t dt + \sigma_t S_t dB_t$$

- Martingale method for the pricing of a European derivative with payoff V_T
    - Find the equivalent measure under which the discounted value of the replicating portfolio is a martingale
    - Obtain risk neutral pricing formula
- Suppose the risk free interest rate is r_t ($\mathcal{F}_t$ adapted)

- Consider the value X_t of a replicating portfolio with Δ_t shares, $X_t - \Delta_t S_t$ invested at the risk free rate
  - the value change of the underlying assets: $\Delta_t dS_t$
  - the value change of the risk free investment: $r_t(X_t - \Delta_t S_t)dt$
  - dividend: $q_t \Delta_t S_t dt$

$$\begin{aligned} dX_t &= \Delta_t dS_t + r_t(X_t - \Delta_t S_t)dt + q_t \Delta_t S_t dt \\ &= r_t X_t dt + (\mu_t - r_t)\Delta_t S_t dt + \Delta_t \sigma_t S_t dB_t \end{aligned}$$

- Discount factor

$$D_t = \exp\left(-\int_0^t r_s ds\right) \quad \text{(i.e., } dD_t = -rD_t dt\text{)}$$

- Discounted value process $X_t^* = D_t X_t$

$$\begin{aligned} d(X_t^*) &= D_t dX_t + X_t dD_t \\ &= r_t X_t^* dt + (\mu_t - r_t)\Delta_t S_t^* dt + \Delta_t \sigma_t S_t^* dB_t - r_t X_t^* dt \\ &= (\mu_t - r_t)\Delta_t S_t^* dt + \Delta_t \sigma_t S_t^* dB_t \\ &= \Delta_t S_t^* \sigma_t (\theta_t dt + dB_t) \quad \text{where } \theta_t = \frac{\mu_t - r_t}{\sigma_t} \\ &= \Delta_t S_t^* \sigma_t dB_t^* \quad \text{where } dB_t^* = \theta_t dt + dB_t \end{aligned}$$

- Find $\mathbb{Q}$ under which X_t^* is a martingale. It suffices to make B_t^* a standard BM under measure $\mathbb{Q}$

- Apply the Girsanov theorem with

$$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \theta_t^2 dt - \int_0^T \theta_t dB_t\right)$$

  Then B_t^* is a standard BM under measure $\mathbb{Q}$
- Under measure $\mathbb{Q}$, $X_t^* = D_t X_t$ is a martingale
- Dynamics of the asset price process under measure $\mathbb{Q}$

$$\begin{aligned} dS_t &= (\mu_t - q_t)S_t dt + \sigma_t S_t dB_t \quad (\text{note that } dB_t^* = \theta_t dt + dB_t) \\ &= (\mu_t - q_t)S_t dt + \sigma_t S_t(dB_t^* - \theta_t dt) \quad (\theta_t = \frac{\mu_t - r_t}{\sigma_t}) \\ &= (r_t - q_t)S_t dt + \sigma_t S_t dB_t^* \end{aligned}$$

- Since $D_t X_t$ is a martingale

$$D_t X_t = \mathbb{E}^{\mathbb{Q}}[D_T X_T | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[D_T V_T | \mathcal{F}_t]$$

- **Risk neutral pricing formula**

$$V_t = X_t = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_s ds \right) V_T \middle| \mathcal{F}_t \right]$$

- Existence of the replicating portfolio guaranteed by the martingale representation theorem

Black-Scholes formula (assets with continuous yield)

- Suppose $r_t \equiv r$, $q_t \equiv q$, $\sigma_t \equiv \sigma$ are constants

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t^*$$

For a call option with $V_T = (S_T - K)^+$, compute

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t\right]$$

where

$$S_T = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T + \sigma B_T^*\right)$$

has a log normal distribution under measure $\mathbb{Q}$

- **Black-Scholes formula** for European vanilla call and put options on assets with continuous yield

$$\begin{aligned} c(t, S_t) &= S_t e^{-q(T-t)} N(d_+) - K e^{-r(T-t)} N(d_-) \\ p(t, S_t) &= -S_t e^{-q(T-t)} N(-d_+) + K e^{-r(T-t)} N(-d_-) \end{aligned}$$

where

$$d_\pm = \frac{\ln(S_t/K) + (r - q \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

- Intuitively: replace S_t by $S_t e^{-q(T-t)}$ in the Black-Scholes formula on assets with no yield

- Alternatively, one obtains the following PDE for derivatives with payoff $V_T(S_T)$

$$\frac{\partial V}{\partial t} + (r - q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Numbers of shares needed for replicating/hedging

$$\Delta_t = \frac{\partial V}{\partial S} = \begin{cases} e^{-q(T-t)}N(d_+) & \text{for a vanilla European call} \\ e^{-q(T-t)}(N(d_+) - 1) & \text{for a vanilla European put} \end{cases}$$

- Index/currency options
    - Index options: q is the continuous dividend yield
    - Currency options: $q = r_f$ is the foreign risk free interest

Pricing futures options

- Consider an asset with price process (under risk neutral measure)

$$dS_t = rS_t dt + \sigma S_t dB_t^*$$

- Let F_t be the futures price of the asset with maturity $T^* \geq T$ (T is the option maturity)

$$F_t = e^{r(T^*-t)} S_t = e^{rT^*} S_t^*, \quad S_t^* = e^{-rt} S_t$$

- Dynamics of F_t

$$dF_t = e^{rT^*} dS_t^* = e^{rT^*} \sigma S_t^* dB_t^* = (r - r) F_t dt + \sigma F_t dB_t^*$$

Futures price evolves like an asset paying continuous yield at rate $q = r$

- Suppose option maturity is T, futures price at time t is F_t. Then European call/put prices are given by

$$c(t, F_t) = e^{-r(T-t)}[F_t N(d_+) - KN(d_-)]$$

$$p(t, F_t) = e^{-r(T-t)}[-F_t N(-d_+) + KN(-d_-)]$$

where

$$d_\pm = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

- **Black's model for pricing futures options** (Black 1976)

Greek Letters

- **Hedging** is important for derivative traders. E.g., writing a naked call option is very risky
- **Greeks letters** measure **sensitivities** of derivative price to asset price, volatility, etc.
- Consider a European derivative with value process $\{V_t, t \geq 0\}$ in the Black-Scholes-Merton model

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t^*$$

Delta

- **Delta** measures rate of change in derivative value as asset price changes

$$\Delta_t = \frac{\partial V}{\partial S}$$

- **Delta hedging** for a short position in a simple derivative (payoff V_T only depends on S_T)
  - Start with V_0 received from the short position
  - Long Δ_t shares at time t, invest $V_t - \Delta_t S_t$ at the risk free interest rate
  - The short position is completely hedged (the hedged position is **delta neutral**)

- Delta of a long forward contract
  - Value of a long forward contract with delivery price K (asset with no yield ($q = 0$))

$$V_t = e^{-q(T-t)} S_t - e^{-r(T-t)} K$$

  - Delta: $\Delta_t = e^{-q(T-t)} = 1$, a **static hedging**
- **Dynamic hedging**
  - Δ_t changes over time
  - Need to adjust number of shares held over time (e.g., daily)
  - How frequent one rebalances?
- Γ measures rate of change in delta as asset price changes

$$\Gamma = \frac{\partial^2 V}{\partial S^2}$$

Gamma

- Small gamma: delta is less sensitive to asset price changes; less frequent refinancing to keep a delta neutral position
- Gamma of the underlying asset is zero:

$$\frac{\partial^2 S}{\partial S^2} = 0$$

- Need an additional derivative (with non zero gamma) to create a **gamma neutral** position,
- A *delta- and gamma-neutral hedge* reduces the effect that transaction costs are not zero

- **Vega** measures the sensitivity of the derivative value to volatility

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$

Need two additional derivatives to hedge a certain derivative by constructing a *delta-, gamma-, and vega-neutral* portfolio

- **Risk management**: risk manager sets maximum allowed risk exposures (Δ, Γ, vega, etc.) for each trading desk