Brownian Motion

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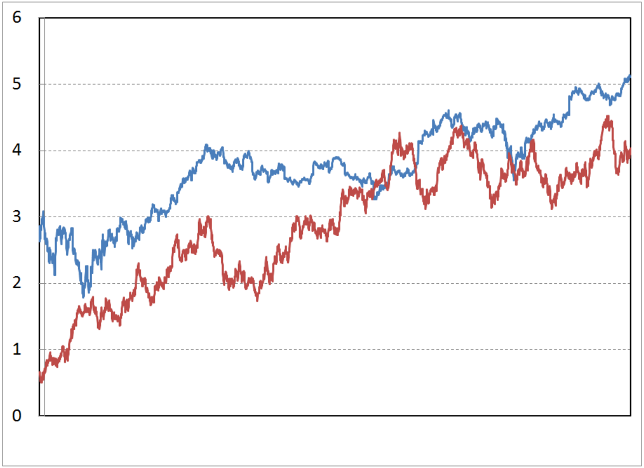
Readings: Shreve Chapter 3

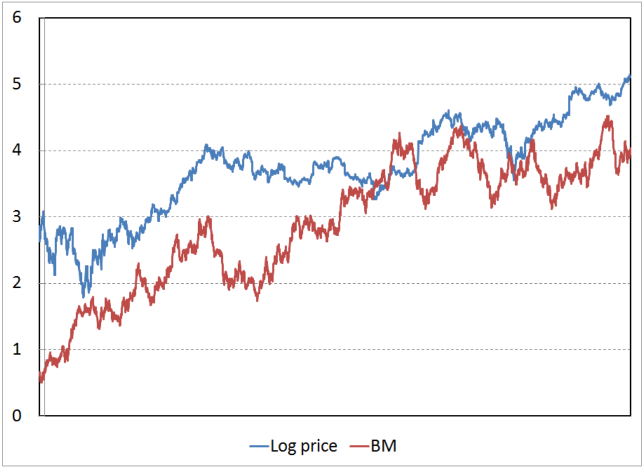
A brief history

- To price derivative securities, need to model dynamics of the underlying asset price process
- Black-Scholes-Merton option pricing model: Black and Scholes (1973), Merton (1973); Nobel prize in 1997; **Geometric Brownian motion**
- Brownian motion: **Robert Brown** observed very irregular movement of pollen particles in water (1827)
- Financial variables exhibit similar irregular paths

Amazon's stock



Amazon's stock



A brief history

- **Louis Bachelier:** the first who studied BM in his PhD thesis "*The Theory of Speculation*" (1900, supervised by Henri Poincaré)
- **Albert Einstein:** "*On the Motion – Required by the Molecular Kinetic Theory of Heat – of Small Particles Suspended in Stationary Liquid*" (one of 4 Annus Mirabilis papers of Einstein published in 1905)
- **Norbert Wiener**: mathematical construction of Brownian motion (1923)
- **Paul Samuelson**: rediscovered Bachelier's work in 1950s, modified and proposed geometric Brownian motion

Symmetric random walk

- Keep flipping a fair coin, let X_k be

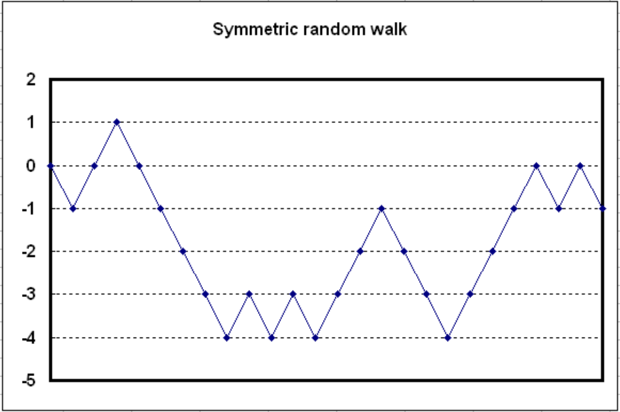
$$X_k = \begin{cases} 1, & \text{if H} \\ -1, & \text{if T} \end{cases}$$

- X_k's are i.i.d. The following defines a **symmetric random walk**

$$M_0 = 0, \quad M_n = X_1 + \cdots + X_n$$

$$M_{n+1} = \begin{cases} M_n + 1, & \text{with probability } 1/2 \\ M_n - 1, & \text{with probability } 1/2 \end{cases}$$

	A	B	C
1	n	Xk	Mn
2	0		0
3	1	-1	-1
4	2	1	0
5	3	1	1
6	4	-1	0
7	5	-1	-1
8	6	-1	-2
9	7	-1	-3
10	8	-1	-4
11	9	1	-3
12	10	-1	-4
13	11	1	-3
14	12	-1	-4
15	13	1	-3
16	14	1	-2
17	15	1	-1
18	16	-1	-2
19	17	-1	-3
20	18	-1	-4
21	19	1	-3
22	20	1	-2
23	21	1	-1
24	22	1	0
25	23	-1	-1
26	24	1	0
27	25	-1	-1



Scaled symmetric random walk

- Speed up the process, scale down step size, keep the same mean/variance

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

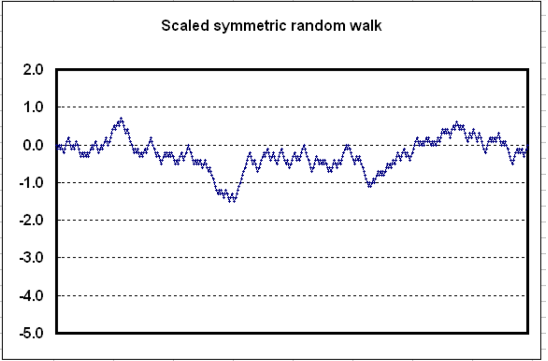
  For non-integer nt, use linear interpolation

- Consider $n = 100$ and t up to 4

$$W^{(n)}(0.01) = \frac{M_1}{\sqrt{n}}, \quad W^{(n)}(0.02) = \frac{M_2}{\sqrt{n}}, \quad \cdots, W^{(n)}(4) = \frac{M_{400}}{\sqrt{n}}$$

- n up and downs in 1 unit of time, step size is $1/\sqrt{n}$

	A	B	C	D	E
1	n	Xk	Mn	t	W
2	0		0	0.00	0.0
3	1	-1	-1	0.01	-0.1
4	2	1	0	0.02	0.0
5	3	-1	-1	0.03	-0.1
6	4	1	0	0.04	0.0
7	5	-1	-1	0.05	-0.1
8	6	-1	-2	0.06	-0.2
9	7	1	-1	0.07	-0.1
10	8	1	0	0.08	0.0
11	9	1	1	0.09	0.1
12	10	1	2	0.10	0.2
13	11	-1	1	0.11	0.1
14	12	-1	0	0.12	0.0
15	13	-1	-1	0.13	-0.1
16	14	1	0	0.14	0.0
17	15	-1	-1	0.15	-0.1
18	16	1	0	0.16	0.0
19	17	1	1	0.17	0.1
20	18	-1	0	0.18	0.0
21	19	-1	-1	0.19	-0.1
22	20	-1	-2	0.20	-0.2
23	21	-1	-3	0.21	-0.3
24	22	1	-2	0.22	-0.2
25	23	-1	-3	0.23	-0.3
26	24	1	-2	0.24	-0.2
27	25	-1	-3	0.25	-0.3
28	26	1	-2	0.26	-0.2
29	27	-1	-3	0.27	-0.3



“Independent increments”

- For any $0 \le s < t \le u < v$ such that ns, nt, nu, nv are integers, $W^{(n)}(v) - W^{(n)}(u)$ and $W^{(n)}(t) - W^{(n)}(s)$ are independent:

$$W^{(n)}(v) - W^{(n)}(u) = \frac{1}{\sqrt{n}}(M_{nv} - M_{nu}) = \frac{1}{\sqrt{n}}(X_{nu+1} + \cdots X_{nv})$$

only depends on $nu + 1$'th to nv'th coin flipping

$$W^{(n)}(t) - W^{(n)}(s) = \frac{1}{\sqrt{n}}(X_{ns+1} + \cdots X_{nt})$$

only depends on $ns + 1$'th to nt'th coin flipping

“Stationary increments”

- If in addition, $v - u = t - s$,

$$W^{(n)}(v) - W^{(n)}(u) = \frac{1}{\sqrt{n}}(X_{nu+1} + \cdots X_{nv})$$

and

$$W^{(n)}(t) - W^{(n)}(s) = \frac{1}{\sqrt{n}}(X_{ns+1} + \cdots X_{nt})$$

have the same distribution since X_k's are i.i.d.

Normality

- For fixed t, distribution of $W^{(n)}(t)$ converges to $N(0,t)$ as $n \to +\infty$.
- Central limit theorem: for i.i.d. $\{X_1, X_2, \cdots\}$ with mean μ and standard deviation σ,

$$\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

- In our case, $\mu = 0$, $\sigma = 1$; for t such that nt is an integer,

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} = \frac{1}{\sqrt{n}} \sum_{k=1}^{nt} X_k \sim N(0,t)$$

Standard Brownian motion

- Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A continuous time stochastic process $\{B_t : t \geq 0\}$, starting at $B_0 = 0$, is a **standard Brownian motion** if
  - (**Independent increments**) for any $0 \leq t_1 < t_2 \leq t_3 < t_4$, $B_{t_2} - B_{t_1}$ and $B_{t_4} - B_{t_3}$ are independent
  - (**Stationary increments**) for any $0 \leq s < t$, $B_t - B_s \sim N(0, t-s)$. In particular, $B_t \sim N(0, t)$ for any $t > 0$
- BM is also called a **Wiener processes**. Also denoted by $B(t), W(t), W_t$
- BM has continuous sample paths: $t \mapsto B_t$ is continuous

Filtration

- **Filtration** $\{\mathcal{F}_t, t \geq 0\}$: increasing time indexed sigma algebras, $\mathcal{F}_t$ contains information learned by observing the system up to time t
- Filtration for a Brownian motion
  - **Adaptivity**: B_t is $\mathcal{F}_t$ measurable
  - For $s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$
- **Natural filtration**: $\mathcal{F}_t = \sigma(B_s, s \leq t)$, the filtration generated by observing the Brownian motion itself

Finite dimensional distribution

- For any $s, t \geq 0$, $\text{cov}(B_t, B_s) = \min(s, t)$: suppose $s \leq t$,

$$\begin{aligned}
\text{cov}(B_s, B_t) &= \mathbb{E}[B_s B_t] - \mathbb{E}[B_s]E[B_t] \\
&= \mathbb{E}[B_s(B_t - B_s + B_s)] \\
&= \mathbb{E}[B_s^2] = \text{var}(B_s) = s = \min(s, t)
\end{aligned}$$

- **Finite dimensional distribution**: for $t_1 < \cdots < t_n$, $(B_{t_1}, \cdots, B_{t_n})$ is multivariate normal $N(0, \Sigma)$ with covariance matrix

$$\begin{pmatrix}
t_1 & t_1 & \cdots & t_1 \\
t_1 & t_2 & \cdots & t_2 \\
\vdots & \vdots & \ddots & \vdots \\
t_1 & t_2 & \cdots & t_n
\end{pmatrix}$$

First order variation of a C^1 function

- $f(t)$ differentiable on $[0, t]$; partition $\Pi = \{t_0 = 0,\ t_1, \cdots,$ $t_n = t\}$ with maximum step size $||\Pi|| = \max_{0 \leq j \leq n-1}(t_{j+1} - t_j)$
- **First order variation**

$$
\begin{aligned}
FV_t(f) &= \lim_{||\Pi|| \to 0} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \\
&= \lim_{||\Pi|| \to 0} \sum_{i=1}^{n} |f'(t_i^*)|(t_i - t_{i-1}) \\
&= \int_0^t |f'(u)| du < \infty
\end{aligned}
$$

Quadratic variation of a C^1 function

- **Quadratic variation**

$$
\begin{aligned}
[f,f](t) &= \lim_{||\Pi||\to 0} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2 \\
&= \lim_{||\Pi||\to 0} \sum_{i=1}^{n} |f'(t_i^*)|^2 (t_i - t_{i-1})^2 \\
&\leq \lim_{||\Pi||\to 0} ||\Pi|| \sum_{i=1}^{n} |f'(t_i^*)|^2 (t_i - t_{i-1}) \\
&= \lim_{||\Pi||\to 0} ||\Pi|| \int_0^t |f'(u)|^2 du < \infty \\
&= 0
\end{aligned}
$$

Quadratic variation of BM

- Sample paths of BM are nowhere differentiable
  – positive quadratic variation
  – unbounded first order variation
- BM has **positive quadratic variation**

$$[B,B](t) = \lim_{||\Pi||\to 0} \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 = t, \quad a.s.$$

(1). Mean of the summation is t

$$\mathbb{E}\left[\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2\right]=\sum_{i=1}^{n}\mathbb{E}\left[(B_{t_i}-B_{t_{i-1}})^2\right]=\sum_{i=1}^{n}(t_i-t_{i-1})=t$$

(2). We show

$$\operatorname{var}\left(\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2\right)\to 0$$

Note that for a normal r.v. $X\sim N(\mu,\sigma^2)$, $\mathbb{E}[(X-\mu)^4]=3\sigma^4$.

$$
\begin{aligned}
\operatorname{var}\left(\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2\right) &= \sum_{i=1}^{n}\operatorname{var}\left((B_{t_i}-B_{t_{i-1}})^2\right) \\
&= \sum_{i=1}^{n}\left(\mathbb{E}\left[(B_{t_i}-B_{t_{i-1}})^4\right]-(t_i-t_{i-1})^2\right) \\
&= \sum_{i=1}^{n}(3(t_i-t_{i-1})^2-(t_i-t_{i-1})^2) \\
&= 2\sum_{i=1}^{n}(t_i-t_{i-1})^2 \\
&\leq 2||\Pi||\sum_{i=1}^{n}(t_i-t_{i-1}) \to 0
\end{aligned}
$$

- BM has **unbounded first order variation**:

$$\lim_{||\Pi||\to 0} \sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}| = \infty, \quad a.s.$$

by noticing that

$$\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|^2 \leq \max_i |B_{t_i} - B_{t_{i-1}}| \cdot \sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|$$

Short hand notation

- For any $0 \le s < t$, $s = t_0 < \cdots < t_n = t$, QV over $[s, t]$

$$\lim_{||\Pi|| \to 0} \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 = t - s$$

denote this fact by

$$dB_t \cdot dB_t = dt \text{ or } (dB_t)^2 = dt$$

BM accumulates QV at rate 1 per unit time

- Does't mean that $(B_{t_i} - B_{t_{i-1}})^2 \approx t_i - t_{i-1}$ or $\frac{(B_{t_i} - B_{t_{i-1}})^2}{t_i - t_{i-1}} \approx 1$

$$\frac{(B_{t_i} - B_{t_{i-1}})^2}{t_i - t_{i-1}} \sim N(0,1)^2$$

- **Cross variation** of B_t and t

$$\lim_{||\Pi|| \to 0} \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})(t_i - t_{i-1}) = 0$$

denote this by

$$dB_t \cdot dt = 0$$

- Quadratic variation of $f(t) = t$ is 0, denote this by $dt \cdot dt = 0$ or $(dt)^2 = 0$

Martingales associated with BM

- **Natural filtration** for a Brownian motion $\mathcal{F}_t = \sigma(B_s, s \leq t)$
- $B = \{B_t\}$ is adapted w.r.t. natural filtration $\mathbb{F} = \{\mathcal{F}_t\}$. For $s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$
- $B = \{B_t\}$ **is a martingale**: for any $s \leq t$,

$$\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s|\mathcal{F}_s] + \mathbb{E}[B_s|\mathcal{F}_s] = B_s$$

- Martingales associated with BM
  - $B_t^2 - t$ is a martingale
  - $\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$ is a martingale – **exponential martingale**

- Let $Z_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$. For $0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}[\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)|\mathcal{F}_s] \\
&= \mathbb{E}[\exp(\sigma(B_t - B_s)) \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)|\mathcal{F}_s] \\
&= \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)\ \mathbb{E}[\exp(\sigma(B_t - B_s))] \\
&= \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)\ \phi(-i\sigma) \\
&= \exp(\sigma B_s - \frac{1}{2}\sigma^2 t) \exp(\frac{1}{2}\sigma^2 (t-s)) = Z_s
\end{aligned}$$

where $\phi(\xi) = \mathbb{E}[e^{i\xi(B_t - B_s)}] = \exp(-\frac{1}{2}\xi^2(t-s))$ is the characteristic function of $B_t - B_s \sim N(0, t-s)$

Markov property

- BM is a **Markov process**: for any Borel-measurable function f and any $0 \leq s < t$,

$$\mathbb{E}[f(B_t)|\mathcal{F}_s] = g(B_s)$$

  for some Borel-measurable function g.

- **Lemma**: *given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *and a* $\sigma-$*algebra* $\mathcal{G} \subset \mathcal{F}$. *If* X *is* $\mathcal{G}-$*measurable and* Y *is independent of* $\mathcal{G}$, *then*

$$\mathbb{E}[f(X, Y)|\mathcal{G}] = g(X)$$

  where $g(x) = \mathbb{E}[f(x, Y)]$

- Markov property then follows

$$\begin{aligned}\mathbb{E}[f(B_t)|\mathcal{F}_s] &= \mathbb{E}[f(B_s + B_t - B_s)|\mathcal{F}_s] \\ &= g(B_s)\end{aligned}$$

- g has the following form

$$\begin{aligned}g(x) &= \mathbb{E}[f(x + B_t - B_s)] \\ &= \int_{\mathbb{R}} f(x+u)\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{u^2}{2(t-s)}\right)du \\ &= \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(y-x)^2}{2(t-s)}\right)dy \\ &= \int_{\mathbb{R}} f(y)p(s,x;t,y)dy\end{aligned}$$

Transition probability density

- **Transition probability density** (TPD) of BM

  $$p(s, x; t, y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right)$$

  describes the probability of the BM going from x to y from time s to time $t \geq s$
- **Space homogeneous** since it only depends on $y - x$
- **Time homogeneous** since it only depends on the length of the time interval $t - s$

First passage times of BM

- For any $a \in \mathbb{R}$, define

$$\tau_a = \inf\{t \geq 0 : B_t = a\}$$

- We show that τ_a is almost surely finite: $\mathbb{P}(\tau_a < \infty) = 1$; but $\mathbb{E}[\tau_a] = \infty$
- The first passage time is a **stopping time**: τ is a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for any t
- **Optional stopping theorem**: *a martingale stopped at a stopping time is a martingale: $\{M_t\}$ is a martingale and τ is a stopping time. Denote* $\min(t, \tau) = t \wedge \tau$. *Then $\{M_{t\wedge\tau}\}$ is a martingale.*

τ_a is almost surely finite

- Assume $a > 0$: by symmetry, τ_a and τ_{-a} have the same distribution
- Recall the exponential martingale $Z_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$. By the optional stopping theorem, $\{Z_{t\wedge\tau_a}\}$ is a martingale ($Z_0 = 1$)

$$\mathbb{E}\left[\exp\left(\sigma B_{t\wedge\tau_a} - \frac{1}{2}(t\wedge\tau_a)\sigma^2\right)\right] = 1$$

- First we note that:

$$\lim_{t\to\infty} \exp\left(\sigma B_{t\wedge\tau_a} - \frac{1}{2}(t\wedge\tau_a)\sigma^2\right) = \mathbf{1}_{\{\tau_a<\infty\}} \exp\left(\sigma a - \frac{1}{2}\tau_a\sigma^2\right)$$

- If $\tau_a < \infty$, $\mathbf{1}_{\{\tau_a<\infty\}} = 1$

$$\lim_{t\to\infty} \exp\left(\sigma B_{t\wedge\tau_a} - \frac{1}{2}(t\wedge\tau_a)\sigma^2\right) = \exp\left(\sigma a - \frac{1}{2}\tau_a\sigma^2\right)$$

- If $\tau_a = \infty$, $\mathbf{1}_{\{\tau_a<\infty\}} = 0$. Note that $B_t < a$ for any $t \geq 0$,

$$\lim_{t\to\infty} \exp\left(\sigma B_{t\wedge\tau_a} - \frac{1}{2}(t\wedge\tau_a)\sigma^2\right) = 0$$

- Change the order of expectation and limit: note that

$$\exp\left(\sigma B_{t\wedge\tau_a} - \frac{1}{2}(t\wedge\tau_a)\sigma^2\right) < \exp(\sigma a)$$

By the dominated convergence theorem

$$\begin{aligned} 1 &= \lim_{t\to\infty}\mathbb{E}\left[\exp\left(\sigma B_{t\wedge\tau_a} - \frac{1}{2}(t\wedge\tau_a)\sigma^2\right)\right] \\ &= \mathbb{E}[\mathbf{1}_{\{\tau_a<\infty\}}\exp\left(\sigma a - \frac{1}{2}\tau_a\sigma^2\right)] \end{aligned}$$

Let $\sigma\to 0$, by the monotone convergence theorem,

$$\mathbb{P}(\tau_a<\infty) = \lim_{\sigma\to 0}\mathbb{E}\left[\mathbf{1}_{\{\tau_a<\infty\}}\exp\left(-\frac{1}{2}\tau_a\sigma^2\right)\right] = \lim_{\sigma\to 0} e^{-\sigma a} = 1$$

First passage time distribution

- **Reflection principle**: Let τ be a stopping time.

$$\hat{B}_t = \begin{cases} B_t & t \leq \tau \\ 2B_\tau - B_t & t > \tau \end{cases}$$

Then $\hat{B}_t$ is also Brownian motion.

- Let $\tau = \tau_a$, $a > 0$, which is also the first passage time for $\hat{B}_t$

$$\begin{aligned} \mathbb{P}(\tau_a \leq t, B_t \leq b) &= \mathbb{P}(\tau_a \leq t, \hat{B}_t \geq 2a - b) \\ &= \mathbb{P}(\hat{B}_t \geq 2a - b) \\ &= \mathbb{P}(B_t \geq 2a - b), \quad a > 0, b \leq a \end{aligned}$$

With $b = a$: $\mathbb{P}(\tau_a \leq t, B_t \leq a) = \mathbb{P}(B_t \geq a), a > 0$

- Distribution of first passage time:

$$\begin{aligned}
\mathbb{P}(\tau_a \le t) &= \mathbb{P}(\tau_a \le t, B_t \le a) + \mathbb{P}(\tau_a \le t, B_t > a) \\
&= \mathbb{P}(B_t \ge a) + \mathbb{P}(B_t > a) \;=\; 2\mathbb{P}(B_t \ge a) \\
&= 2\int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \;=\; \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-\frac{y^2}{2}} dy
\end{aligned}$$

- For arbitrary $a \neq 0$, the cdf, pdf, and expectation of τ_a:

$$\mathbb{P}(\tau_a \le t) = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^\infty e^{-\frac{y^2}{2}} dy = 2(1 - N(|a|/\sqrt{t}))$$

$$p(t) = \frac{|a|}{t\sqrt{2\pi t}} e^{-\frac{a^2}{2t}}, \quad t \ge 0, \quad \mathbb{E}[\tau_a] = +\infty$$

Joint distribution of BM and its maximum

- Denote $M_t = \max\{B_s, 0 \le s \le t\}$
- For any $x > 0$ and $y \le x$, note that $M_t \ge x \Leftrightarrow \tau_x \le t$

$$\mathbb{P}(M_t \ge x, B_t \le y) = \mathbb{P}(\tau_x \le t, B_t \le y) = \mathbb{P}(B_t \ge 2x - y)$$

- Denote the joint pdf of (M_t, B_t) by $p(x, y)$

$$\int_x^\infty \int_{-\infty}^y p(u, v) dv du = \frac{1}{\sqrt{2\pi t}} \int_{2x-y}^\infty \exp(-u^2/2t) du$$

- Take derivative w.r.t. x and y on both sides

$$p(x, y) = \frac{2(2x - y)}{t\sqrt{2\pi t}} \exp\left(-\frac{(2x - y)^2}{2t}\right), \quad x > 0, y \le x$$