FIN516: Term-structure Models

Lecture 12: LIBOR market model - Essentials of calibration

Dr. Martin Widdicks

UIUC

Fall, 2015

Introduction

- Now we have developed the LIBOR market model, we need to fill in some of the gaps.
- in particular, we will need to select our instantaneous volatility function and instantaneous correlation function to match some market prices.
- It will be straightforward to match the prices of caps (caplets) and also possible to match the prices of swaptions also, depending upon the choice of functions.
- Once we have calibrated out model, we can go ahead and implement it in Lecture 13.

LMM and Black

- Recall that from the Black formula, the price of a cap is given by:

$$\sum_{k=1}^{n} \tau_k P(0, T_k) E_0^k[\max(F_k(T_{k-1}) - K, 0)]$$

where E^k denotes an expectation under the forward adjusted probability measure.

- Note that the joint dynamics of the forward rates are not involved in this payoff, as there are no expectations involving more than one rate, and so correlation doesn't matter.
- Also recall from the toolkit that caps are said to be at the money when $K = S_{T_0, T_n}(0)$
- If we were to try to value a cap using the LMM then we would need to evolve

$$dF_k(t) = \sigma_k(t) F_k(t) dW_k(t)$$

under the appropriate forward measure.

LMM and Black

- Also recall that the value of a caplet according to the Black model is given by:

$$Caplet^B(0, T_{k-1}, T_k, K, \nu_k) = \tau_i P(0, T_k)\mathrm{Black}(K, F_k(0), \nu_k)$$

where

$$\begin{aligned} \mathrm{Black}(K, F, \nu) &= FN(d_1) - KN(d_2) \\ d_1 &= \frac{\ln(F/K) + \nu^2/2}{\nu} \\ d_2 &= \frac{\ln(F/K) - \nu^2/2}{\nu} \end{aligned}$$

- Here

$$\nu_k^2 = T_{k-1}\sigma^2_{Black,k}$$

where

$$\sigma^2_{Black,k} = \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k(t)^2 dt$$

LMM and Black

- Thus, we will need our LMM volatility structures to ensure that the caplet volatilities are matched for all caplet volatilities.
- We will go through our possible descriptions to see how successful they are.

Quiz

How many parameters would we need to match M caplet prices

- Four
- Less than M
- M.
- More than M.

Quiz

How many parameters would we need to match M caplet prices

- Four
- Less than M
- M. CORRECT
- More than M. CORRECT

Function 2: Piecewise constant - $T_k - t$ dependent

- Here we can write down the Black volatilities as

$$\sigma^2_{Black,k} = \frac{1}{T_{k-1}} \sum_{j=1}^{k} \tau_{j-1} \eta^2_{k-j+1}.$$

In this case we can choose the parameters η to exactly fit the time scales caplet volatilities. This, however, leaves no volatility parameters to calibrate to swaptions prices.

- For example,

$$\begin{aligned}
\sigma^2_{Black,1} &= \eta_1^2 \\
\sigma^2_{Black,2} &= \frac{1}{T_1}\left(\eta_2^2\tau_0 + \eta_1^2\tau_1\right) \\
\sigma^2_{Black,3} &= \frac{1}{T_2}\left(\eta_3^2\tau_0 + \eta_2^2\tau_1 + \eta_1^2\tau_2\right)
\end{aligned}$$

Function 6: Parametric

- Now we have the standard hump shaped parametric form for volatility, and so to match the caplet volatilities we need to solve

$$\begin{aligned}\sigma^2_{Black,k} &= \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \left([b(T_{k-1} - t) + a]\, e^{-c(T_{k-1}-t)} + d \right)^2 dt \\ &= \frac{1}{T_{k-1}} I^2(T_{k-1}; a, b, c, d)\end{aligned}$$

  where I is the integral of the volatility function, which does have an analytic solution that can be calculated using matlab, mathematica, or by hand!

- Thus we have four parameters to match all of the caplet prices, so in general we will not be able to match them exactly, never mind matching the swaption prices as well.

Function 6: Parametric

- For example,

$$\sigma^2_{Black,1} = \frac{1}{T_0}\int_0^{T_0}\left(\left[b(T_0-t)+a\right]e^{-c(T_0-t)}+d\right)^2 dt$$
$$\sigma^2_{Black,2} = \frac{1}{T_1}\int_0^{T_1}\left(\left[b(T_1-t)+a\right]e^{-c(T_1-t)}+d\right)^2 dt$$
$$\sigma^2_{Black,3} = \frac{1}{T_2}\int_0^{T_2}\left(\left[b(T_2-t)+a\right]e^{-c(T_2-t)}+d\right)^2 dt$$

or in the alternative notation

$$\sigma^2_{Black,1} = \frac{1}{T_0}I^2(T_0; a,b,c,d)$$
$$\sigma^2_{Black,2} = \frac{1}{T_1}I^2(T_1; a,b,c,d)$$
$$\sigma^2_{Black,3} = \frac{1}{T_2}I^2(T_2; a,b,c,d)$$

Function 7: Parametric

- Now when we have the second parameterisation we have some extra degrees of freedom, now to match the caplet volatlities we use:

$$\begin{aligned}\sigma^2_{Black,k} &= \frac{\Phi_k^2}{T_{k-1}} \int_0^{T_{k-1}} \left([b(T_{k-1} - t) + a]\, e^{-c(T_{k-1}-t)} + d \right)^2 dt \\ &= \frac{\Phi_k^2}{T_{k-1}} I^2(T_{k-1}; a, b, c, d)\end{aligned}$$

- The advantage here is that we can use the Φ_k values to match the caplet volatilities precisely.
- Indeed, the time homogeneity of the forward rates can be checked by seeing the required values of Φ_k^2 given by

$$\Phi_k^2 = \frac{\sigma^2_{Black,k} T_{k-1}}{I^2(T_{k-1}; a, b, c, d)}$$

Some useful results: Function 7

- The critical valuation for all of the calibrations (and for our Monte Carlo method later) is the integration of the instantaneous volatility function.
- From the Peter Jackel handout we have that the indefinite integral is, when ρ_{ij} is constant across changing t:

$$
\begin{aligned}
\int \rho_{ij}\sigma_i(t)\sigma_j(t)dt \quad = \quad & \rho_{ij}\phi_i\phi_j\frac{1}{4c^3}\times\Big(4ac^2d\left[e^{c(t-T_{j-1})}+e^{c(t-T_{i-1})}\right] \\
& +4c^3d^2t-4bcde^{c(t-T_{i-1})}\left[c(t-T_{i-1})-1\right] \\
& -4bcde^{c(t-T_{j-1})}\left[c(t-T_{j-1})-1\right] \\
& +e^{c(2t-T_{i-1}-T_{j-1})}\left(2a^2c^2+2abc\left[1+c(T_{i-1}+T_{j-1}-2t)\right]\right. \\
& \left.+b^2\left[1+2c^2(t-T_{i-1})(t-T_{j-1})+c(T_{i-1}+T_{j-1}-2t)\right]\right)\Big)
\end{aligned}
$$

Some useful results: Function 2

- For function 2 this is a lot easier, where $T_k < T_{i-1}, T_{j-1}$:

$$\int_0^{T_k} \rho_{ij}\sigma_i(t)\sigma_j(t)dt = \rho_{ij}\sum_{l=1}^{k}\eta_{i-l+1}\eta_{j-l+1}\tau_{l-1}$$

when T_k is greater than one of the reset dates:

$$\int_0^{T_k} \rho_{ij}\sigma_i(t)\sigma_j(t)dt = \rho_{ij}\sum_{l=1}^{\min(k,i-1,j-1)}\eta_{i-l+1}\eta_{j-l+1}\tau_{l-1}$$

Which functions are the best?

- Functions 2 and 7 (as well as others in the Brigo and Mercurio book) allow for the exact calibration to the current market prices of caps.
- However, some formulations will be more useful than others. Below, we look at their ability to produce the appropriate term structure of volatility, but there are some other concerns.
- When it comes to calibrating to swaptions as well, function 7 will prove to be the most useful allowing for flexible fitting to data and volatility term structures that look acceptable.

Term structure of volatility

- Again, recall that we describe the term structure of volatility at time T_j is a graph of expiry times T_{k-1} against average volatilities of the forward rates $F_k(t)$ up until the expiry time, or for $t \in (T_j, T_{k-1})$. Or a graph of points $(T_{j+1}, V(T_j, T_{j+1}), \ldots$ where

$$V^2(T_j, T_{k-1}) = \frac{1}{T_{k-1} - T_j} \int_{T_j}^{T_{k-1}} \sigma_k^2(t)dt$$

  for $k > j + 1$.
- On the next page, is the typical term structure of volatility at time 0.
- Now one problem, is that regardless of the time at which the volatility term structure is observed, the shape of the term structure should be approximately the same - it is time homogeneous.
- Note that being able to observe the term structure of volatility directly using deterministic functions is a very attractive feature of the LMM, certainly in contrast to one-factor models (except, perhaps the BDT model, as discussed in lecture 6)

BM: Term structure of volatility

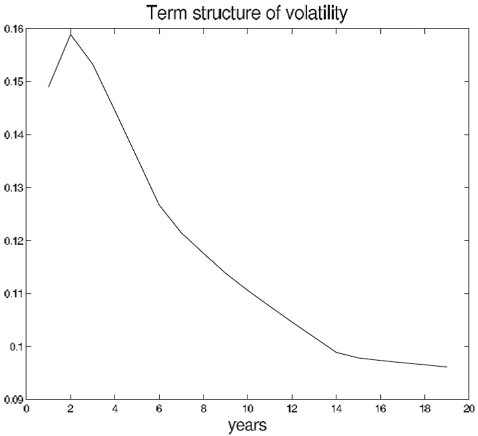

Fig. 6.1. Example of term structure of volatility, $T \mapsto v_{T-\text{caplet}}$, from the Euro market.

Term structures given by volatility choice

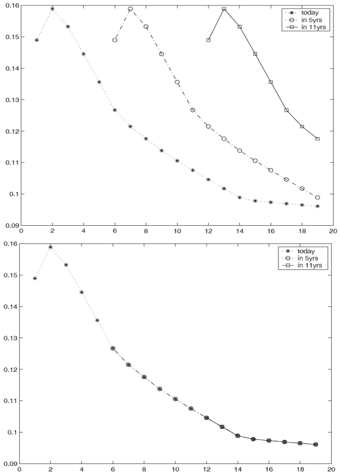

Fig. 6.2. Examples of evolution of the term structure of volatilities under Formulations 2 and 3.

Volatility structures given by functions 6 and 7

- Function 6 is the parametric analogue of function 2, and so it will behave similarly.
- Function 7 is the one most commonly used in practice and retains all of the features of function 5, and has a pleasant parametric form which is possible to integrate easily to generate the total volatility term required for Monte Carlo simulations.
- We will generally use this volatility structure, but you should be aware of the other possible choices.

Term structures given by Function 7

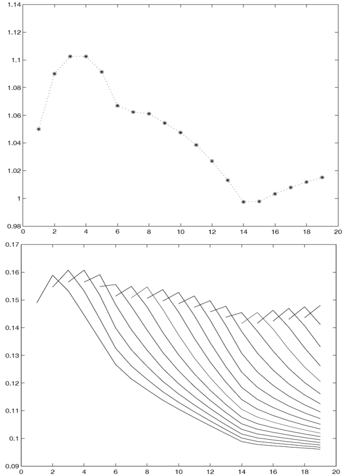

Fig. 6.3. An example of Φ (first graph) and the evolution of the term structure of volatilities (second graph) under Formulation 7.

Swaption matrix

A typical swaption matrix looks like this - and is easily available from Bloomberg, as we saw in our MIL session, where the rows are for each maturity of the swaption and the columns for the tenor of the underlying swap.

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	16.4	15.8	14.6	13.8	13.3	12.9	12.6	12. 3	12.0	11.7
2y	17.7	15.6	14.1	13.1	12.7	12.4	12.2	11.9	11.7	11.4
3y	17.6	15.5	...	...	...	...	...	...	...	...
4y	16.9	14.6	...	...	...	...	...	...	...	...
5y	15.8	13.9	...	...	...	...	...	...	...	...
7y	14.5	12.9	...	...	...	...	...	...	...	...
10y	13.5	11.5	...	...	...	...	...	...	...	...

Some initial thoughts

- The Black formula can easily value (European) swaptions and is the market standard. We will use swaption prices to provide an additional calibration of the LMM.
- Swaption prices are usually reported via their Black-implied volatilities and are usually reported in a matrix where each row indicates the maturity of the *swaption* T_0 and each column indicates the underlying swap length or the *tenor* of the swap, $T_n - T_0$.
- The table on the previous slide, shows a typical swaption matrix, produced from BM
- It may be possible that some of the implied volatilities in the swaption matrix may not have been updated as recently as others, this can cause troubles with calibration. Thus any calibrated prices that look remarkably different could be used to iron out these discrepancies.
- In general, be skeptical as to the quality of the quoted swaptions data.

The need for approximations

- The caplet fitting is nice and straightforward, however, we may want to be greedy and try to fit the data to European swaption prices as well.
- This will not be straightforward and there is a great deal of debate as to the best strategy.
- However, as part of the calibration we will need to compare our model swaption prices with market swaption prices. However, we should really use our market model and a full Monte Carlo to calculate the swaption prices.
- This would make calibration very time consuming and so we use analytic approximations to both the swaption price - well actually the implied volatility - of the swap rates in the LIBOR market model.

Rebonato's formula

- Recall the derivation for Black's formula for swaptions, where it is possible to write

$$dS_{T_0,T_n}(t) = \sigma_{0,n}(t) S_{T_0,T_n}(t) dW_t$$

as for the caplet formula and the LMM the crucial term is the Black swaption volatility

$$\sigma^2_{black}(T_0)T_0 = \int_0^{T_0} \sigma^2_{0,n}(t)dt = \int_0^{T_0} (d \ln S_{T_0,T_n}(t))(d \ln S_{T_0,T_n}(t))$$

- The aim of analytic approximations is to attempt to determine an approximation to $\sigma^2_{black}(T_0)$ when we are working in the LMM
- Of course, we know from Lecture 1 (and PS2) that

$$S_{T_0,T_n}(t) = \sum_{k=1}^{n} w_k(t) F_k(t)$$

Rebonato's formula

- Where,

$$S_{T_0,T_n}(t) = \sum_{k=1}^{n} w_k F_k(t) \tag{1}$$

where

$$w_k = \frac{\tau_k P(t, T_k)}{\sum_{j=1}^{n} \tau_j P(t, T_j)}.$$

and, so

$$w_k(t) = \frac{\tau_k \prod_{i=0}^{k} \frac{1}{1+\tau_i F_i(t)}}{\sum_{j=1}^{n} \tau_j \prod_{i=0}^{j} \frac{1}{1+\tau_i F_i(t)}}$$

Rebonato's formula

- This will make determining dS difficult, however, we can proceed by making the first of our heroic assumptions, by freezing the w values at time 0. (You can check how much the w values actually move around by running a Monte Carlo).
- So now,

$$S_{T_0,T_n}(t) \approx \sum_{k=1}^{n} w_k(0) F_k(t)$$

now we can consider dS (under any forward measure) as follows (recall PS2)

$$dS_{T_0,T_n}(t) \approx \sum_{k=1}^{n} w_k(0) dF_k(t) = (\dots)dt + \sum_{k=1}^{n} w_k(0)\sigma_k(t)F_k(t)dW_k(t)$$

Rebonato's formula

- Now, in our easy world, dropping the T_0 and T_n subscripts temporarily and assuming that the correlation is not time dependent,

$$\left(\frac{dS}{S}\right)\left(\frac{dS}{S}\right) \approx \frac{\sum_{i,j=1}^{n} w_i(0)w_j(0)F_i(t)F_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)}{S(t)^2}$$

now let's freeze all of the F values to give

$$(d\ln S(t))(d\ln S(t)) \approx \sum_{i,j=1}^{n} \frac{w_i(0)w_j(0)F_i(0)F_j(0)}{S(0)^2}\rho_{i,j}\sigma_i(t)\sigma_j(t)dt$$

and finally we get our approximation of the Black volatility,

$$\sigma_{black}^2(T_0)T_0 \approx \sum_{i,j=1}^{n} \frac{w_i(0)w_j(0)F_i(0)F_j(0)}{S_{T_0,T_n}(0)^2}\int_0^{T_0} \rho_{i,j}\sigma_i(t)\sigma_j(t)dt$$

- Then we can calibrate to the swaption volatility by choosing volatility and correlation functions that make this expression as close to the implied swaption volatility as possible.

Rebonato's formula

- For example, a one-year swaption on a one-year (semi-annual) swap has volatility, $\sigma^2_{black}(1)$, and two forward rates, $F_1(t) = F(t;1,1.5)$ and $F_2(t) = F(t;1.5,2)$, and so

$$\sigma^2_{black}(1) \approx \sum_{i,j=1}^{2} \frac{w_i(0)w_j(0)F_i(0)F_j(0)}{S_{T_0,T_n}(0)^2} \int_0^1 \rho_{i,j}\sigma_i(t)\sigma_j(t)dt$$

  where the parameters of $\sigma_1(t)$ and $\sigma_2(t)$ have been determined from the caplets and you can use this to determine the parameters in the correlation function. You will need to integrate the covariance function using the techniques described for caplets.

- Recall that

$$w_1(0) = \frac{\tau_1 P(0,1.5)}{\tau_1 P(0,1.5) + \tau_2 P(0,2)}$$

  and

$$w_2(0) = \frac{\tau_1 P(0,2)}{\tau_1 P(0,1.5) + \tau_2 P(0,2)}$$

How does it do? Can we do better?

- The two assumptions here - freezing the weights and freezing the forward rates are very extreme.
- As a result we would expect this approximation to be quite poor. On the contrary, it performs remarkably well (you could test this as part of your project as you will have a swaption valuation methodology!!).
- Of course, we can make this approximation a little better by explicitly determining the values of $\frac{\partial S_{T_0,T_n}(t)}{\partial F_k}$. In particular we find that (again dropping T_0, T_n)

$$
\begin{aligned}
dS &= \sum_{k=1}^{n}(w_k(t)dF_k(t) + F_k(t)dw_k(t)) + (\dots)dt \\
&= \sum_{k,h=1}^{n}\left(w_h(t)\delta_{\{h,k\}} + F_k(t)\frac{\partial w_k(t)}{\partial F_h}\right)dF_h(t) + (\dots)dt
\end{aligned}
$$

where δ is the Delta function where $\delta_{k,h} = 0$ unless $h = k$.

Hull and White's formula

- From the definition of w_k we can determine the partial derivatives as follows:

$$\frac{\partial w_k(t)}{\partial F_h} = \frac{w_k(t)\tau_h}{1+\tau_h F_h(t)} \left[\frac{\sum_{i=h}^{n} \tau_i \prod_{j=0}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=1}^{n} \tau_i \prod_{j=0}^{n} \frac{1}{1+\tau_j F_j(t)}} - 1_{k \geq h} \right]$$

- So if we set

$$\overline{w}_h(t) = w_h(t) + \sum_{k=1}^{n} F_k(t) \frac{\partial w_k(t)}{\partial F_h}$$

then we have that

$$dS = \sum_{h=1}^{n} \overline{w}_h(0) dF_h(t) + (\dots)dt$$

and now we can freeze the forward rates at time 0 to get a modified approximation of the Black swaption volatility.

Hull and White's formula

- The final formula is

$$\sigma^2_{black}(T_0)T_0 \approx \sum_{i,j=1}^{n} \frac{\overline{w}_i(0)\overline{w}_j(0)F_i(0)F_j(0)}{S_{T_0,T_n}(0)^2} \int_0^{T_0} \rho_{i,j}\sigma_i(t)\sigma_j(t)dt$$

this is often attributed to Hull and White, and can used to calibrate to swaption vols just like the Rebonato formula.

- We can simplify this equation a little bit and write the instantaneous volatility as

$$\sigma^{S_{0,n}}_{inst}(t) = \sum_{i,j=1}^{n} \zeta_i(t)\zeta_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)dt$$

where

$$\zeta_k(t) = \frac{\overline{w}_k(t)F_k(t)}{S_{T_0,T_n}(t)}$$

Calibrating volatility to swaption prices too

- Above we showed how we can exactly calibrate our volatility functions to caplet prices. Alternatively we could also calibrate them to swaption prices, or even to both.
- From above we can write the the calibration problem as follows,

$$[\sigma_{black}]^2 \, T_0 = \int_0^{T_0} \sigma_{inst}(t)^2 dt$$

  where the swap rate is S_{T_0,T_n}, T_0 is the maturity of the European swaption and naturally σ_{black} is the Black-implied volatility of the swap rate.
- Assuming that we know that instantaneous correlation and volatilities than we can write

$$\sigma_{inst}(t)^2 = \sum_{i,j} \zeta_i(t)\zeta_j(t)\rho_{ij}\sigma_i(t)\sigma_j(t)$$

  if we fix the weights to time 0 values then our approximation becomes

$$[\sigma_{black}]^2 \, T_0 = \sum_{i,j} \zeta_i(0)\zeta_j(0) \int_0^{T_0} \rho_{ij}\sigma_i(t)\sigma_j(t)dt$$

Calibrating volatility to swaption prices too

- Now, assuming a particular form for our correlation and instantaneous volatilities we can construct a calibration algorithm, we set up our model volatility as follows for each swaption in turn

$$
\begin{aligned}
\left[\sigma_{model}\right]^2 T_0 &= \int_0^{T_0} \sigma_{inst}^{S_{0,n}}(t)^2 dt \\
&= \sum_{i,j} \zeta_i(0)\zeta_j(0) \times \\
&\quad \int_0^{T_0} \rho_{ij}\phi_i\phi_j\psi(T_{i-1}-t;a,b,c,d)\psi(T_{j-1}-t;a,b,c,d)dt
\end{aligned}
$$

- Here, we are assuming that the instantaneous volatility has two parts: a time homogeneous function, $\psi(T-t;a,b,c,d)$ with parameters a, b, c, d, and a forward specific component ϕ_i.
- We can then choose parameters, either all at once, or one at a time to minimize the error between the Black volatility and our theoretical volatility.

How to calibrate

- We have to decide how many parameters we would like to use to calibrate the swaption prices. Typically we will use the Rebonato formula belwo to estimate the swaption volatilities and then vary our parameters accordingly to match with the markets volatilities. We could use our parametric function 7 to calibrate to caps and then use the correlation parameters to reproduce all of the swaption values.
- Alternatively we could use a, b, c and d and the correlation parameters values to get as close as possible to all of the swaption prices and then use the Φ values to ensure that the cap prices are also matched.
- Other calibrations use an exogenously determined correlation matrix (from historical data) and then use the volatility parameters to match as closely as possible to swaption prices.
- Some practitioners only calibrate to co-terminal swap prices, as we will see this makes sense for Bermudan swaption pricing, others do not calibrate to swaptions at all relying on the exogenous correlation matrix and the cap calibration

Fitting to Cap prices: Rebonato

- Rebonato has a detailed calibration algorithm, for the calibration to caps. This is based around the definition of the matrix S.
- To start with assume that we have defined a general volatility function of the form $\sigma_{inst}(t, T)$ which describes the volatility between times t and T, then

$$S(j,k) = \int_{T_{j-2}}^{T_{j-1}} \sigma_{inst}(u, T_{k-1})^2 du$$

where j denotes a price sensitive date and k the forward rate. This is similar to BM's volatility matrix

Inst. vol.	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$S(1,1)$	Dead	Dead	...	Dead
$F_2(t)$	$S(1,2)$	$S(2,2)$	Dead	...	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	Dead
$F_M(t)$	$S(1,M)$	$S(2,M)$	$S(3,M)$	...	$S(M,M)$

Fitting to Cap prices: Rebonato

- Now if we restrict the volatility function to be time homogeneous $\sigma_{inst}(t, T) = \sigma_{inst}(T - t)$, then this reduces to

Inst. vol.	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\dots$	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$S(1,1)$	Dead	Dead	$\dots$	Dead
$F_2(t)$	$S(1,2)$	$S(1,1)$	Dead	$\dots$	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	Dead
$F_M(t)$	$S(1,M)$	$S(1,M-1)$	$S(1,M-2)$	$\dots$	$S(1,1)$

- Now, let's write down a general volatility function

$$\sigma_{inst} = g(T)h(T - t)$$

and, still keeping things general, consider a general form to get the time homogeneous S matrix (S^{th}):

$$S^{th}(j,k) = \int_{T_{j-2}}^{T_{j-1}} h(T_{k-1} - u; a_1, \dots, a_m)^2 du$$

where the a values are parameters of the function, in function 6: $m = 4$ and $a_1 = a, \dots a_4 = d$.

Fitting to Cap prices: Rebonato

- Now let's perform a calibration on the time homogeneous part, let

$$\chi^2 = \sum_{k=1}^{M} e_k^2$$

and

$$e_k^2 = \left[\sigma_{Black,k}^2 T_{k-1} - \sum_{j=1}^{k} S^{th}(1, k-j+1) \right]^2$$

then we minimize the following

$$\min_{a_1,\ldots,a_4} \chi^2$$

Quiz

Will this match the caplet prices exactly?

- Yes
- No
- Perhaps

Quiz

Will this match the caplet prices exactly?

- Yes
- No CORRECT
- Perhaps

Fitting to Cap prices: Rebonato

- Now, this will not be a perfect match as we are only using 4 parameters to fit to all of the Black volatilities. Now we can add a time dependent parameter at each price sensitive event, this transforms our S matrix into

Inst. vol.	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$S_{th}(1,1)\epsilon_1^2$	Dead	Dead	...	Dead
$F_2(t)$	$S_{th}(1,2)\epsilon_1^2$	$S_{th}(1,1)\epsilon_2^2$	Dead	...	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	Dead
$F_M(t)$	$S_{th}(1,M)\epsilon_1^2$	$S_{th}(1,M-1)\epsilon_2^2$	$S_{th}(1,M-2)\epsilon_3^2$	...	$S_{th}(1,1)\epsilon_M^2$

and now we have a similar calibration algorithm

$$\chi^2 = \sum_{k=1}^{M} e_k^2$$

and

$$e_k^2 = \left[\sigma_{Black,k}^2 T_{k-1} - \sum_{j=1}^{k} S^{th}(1, k-j+1)\epsilon_j^2\right]^2$$

Fitting to Cap prices: Rebonato

- Then we minimize the following

$$\min_{\epsilon_1,\ldots,\epsilon_m} \chi^2$$

Aside: ϵ_m

- In the BM treatment we often did not have a time dependent parameter in our volatility function, but instead just used a forward rate specific parameter, ϕ. We will consider that shortly.
- Rebonato argues that it also makes sense to have a time specific parameter as well as forward rate specific as rates may become more or less volatile as time moves forward but the general time homogeneity stays.
- We need to be careful to have this parameter be relatively smooth, the proposed function is

$$\epsilon(t) = \left[\sum_{i=1}^{n} \epsilon_i \sin\left(\frac{t\pi i}{Mat} + \epsilon_{i+1}\right)\right] \exp(-\epsilon_7 t)$$

with $n = 2$ or 3 and Mat equal to the longest caplet maturity, ϵ_7 is another epsilon parameter.
- So our calibration need only be over the three or four parameters, rather than one for each time step in the general set-up.

Adding in the forward rate component

- The first stage ensured that we captured as much time homogeneity as possible. The second ensured that we first tried to explain discrepancies via a time dependent parameter. Finally we can ensure that the caplet prices match exactly.
- Me now assign a constant term k_i^2 for each caplet such that

$$\phi_k^2 = \frac{\sigma^2_{Black,k} T_{k-1}}{\sum_{j=1}^{k} S^{th}(1, k-j+1)\epsilon_j^2}$$

  and then we have our volatilities of the form

$$S(j,k) = S^{th}(1,k)\epsilon_j^2\phi_k^2$$

- If our volatility function was a good choice than ϕ_k values should all be close to 1.

Exogenous correlation matrix for Swaptions

- An alternative to using the correlation matrix to match swaption prices is to estimate the correlation matrix from historical data and then assume that the correlations will be similar going forward.
- If we then want to also match swaptions data and cap data then we will only have our volatility parameters to do so. There will also be other possible combinations of calibrations that we may wish to do.
- The data we would presumably use would be LIBOR curves, naturally observed under the real world measure. Each curve would give data about all of the current forward rates and so it would be possible to extract an approximation of the instantaneous correlations between the different rates.
- Note that we can do this because according to the Girsanov theorem, when changing measure the instantaneous correlations and volatilities are unchanged.

Estimating the exogenous correlation matrix

- Usually the market data that we have is of the form:

$$P(t, t+Z), P(t+1, t+1+Z), \ldots, P(t+n, t+n+Z)$$

where the time to maturity is fixed as we move forward in time. To estimate our correlations we need forward rate data of the form, which requires discount factors of the form

$$P(t, T), P(t+1, T), \ldots, P(t+n, T)$$

where the maturity is fixed as we move forward in time.

- So we will have to interpolate from the data to obtain the appropriate T fixed values.
- From this data we can obtain daily log-returns of the annualized forward rates then we assume a distribution for the returns on the forward rates where we have rates for times $T_1, \ldots, T_n$:

$$\left[\ln \frac{F_1(t+\Delta t)}{F_1(t)}, \ldots, \ln \frac{F_n(t+\Delta t)}{F_n(t)}\right] \sim N(\mu, V)$$

Estimating the exogenous correlation matrix

- Where Δt is the time period for observations (probably one day) and we estimate μ and V as follows:

$$\begin{aligned}\hat{\mu}_k &= \frac{1}{m}\sum_{i=1}^{m-1} \ln\left(\frac{F_k(t_{i+1})}{F_k(t_i)}\right)\\ \hat{V}_{j,k} &= \frac{1}{m}\sum_{i=0}^{m-1}\left[\left(\ln\left(\frac{F_j(t_{i+1})}{F_j(t_i)}\right) - \hat{\mu}_j\right)\left(\ln\left(\frac{F_k(t_{i+1})}{F_k(t_i)}\right) - \hat{\mu}_k\right)\right]\end{aligned}$$

  of observed days of the forward rate, and then we can estimate each of the entries in our correlation matrix as:

$$\hat{\rho}_{j,k} = \frac{\hat{V}_{j,k}}{\sqrt{\hat{V}_{j,j}}\sqrt{\hat{V}_{k,k}}}$$

Exogenous correlation matrix

- Sometimes the full correlation matrix may be too large, in that we will not be using as many stochastic factors as there are forward rates, in which case we will have to reduce the dimension of the instantaneous correlation matrix.
- If we did not have to, then if we have calibrated our volatility matrix to some market prices, caplets and swaptions perhaps then we have all that we need to perform our Monte Carlo simulations and to value the derivatives.
- If, however, we would like to reduce the factors then we will have to use the principal component analysis approach,

Co-terminal swaption calibration

- Of course, rather than use an exogenous matrix we can adjust our correlation matrix to try to match swaption data. Later we will look at a global calibration but there is a elegant way of calibrating our parameters to Co-terminal swaption prices.
- Co-terminal swaptions are swaptions where the underlying swaps will all matrure on the same date. So that we have the one-year swaption on a nine-year swap, the two year swaption on an eight year swap and so on.

Co-terminal swaption calibration

- Co-terminal swap rates S_i are forward swap rates $S_{T_i,T_n}(t)$ where T_n is fixed and $T_i \geq t$ is the maturity of the swaption.
- Let's assume that we choose function 7 for the volatility, and the simplest of the correlation structures:

$$\rho_{ij} = e^{-\beta|T_i - T_j|}$$

  or, alternatively, an exogenous correlation matrix.
- Now let's consider the elements of covariance matrix for swap rates, assuming that we know the covariance matrix for the forward rates with entries C^{FR}_{ij}. Then

$$\begin{aligned}
C^{SR}_{ij} &= \langle \frac{dS_i}{S_i} \cdot \frac{dS_j}{S_j} \rangle \\
&= \sum_{h=i+1}^{n} \sum_{l=j+1}^{n} \frac{\partial S_i/\partial F_h \cdot \partial S_j/\partial F_l}{S_i \cdot S_j} \cdot F_h F_l \cdot \langle \frac{dF_h}{F_h} \frac{dF_l}{F_l} \rangle \\
&= \sum_{h=i+1}^{n} \sum_{l=j+1}^{n} \frac{\partial S_i}{\partial F_h} \frac{F_h}{S_i} \cdot C^{FR}_{hl} \cdot \frac{F_l}{S_j} \frac{\partial S_j}{\partial F_l}
\end{aligned}$$

Co-terminal swaption calibration

- Now. let's define a new matrix called

$$Z_{ih}^{FR\to SR} = \frac{\partial S_i}{\partial F_h}\frac{F_h}{S_i}$$

then we can generate the covariance matrix for the co-terminal swap rates as

$$C^{SR} = Z^{FR\to SR}\cdot C^{FR}\cdot Z^{FR\to SR\prime}$$

- Now we can use our earlier results where we have determined that

$$S_i(t) = \sum_{h=i}^{n} w_h(t)F_h(t)$$

and

$$\frac{\partial w_i(t)}{\partial F_h} = \frac{w_i(t)\tau_h}{1+\tau_h F_h(t)}\left[\frac{\sum_{j=h}^{n}\tau_j\prod_{k=0}^{j}\frac{1}{1+\tau_k F_k(t)}}{\sum_{j=1}^{n}\tau_j\prod_{k=0}^{n}\frac{1}{1+\tau_k F_k(t)}} - 1_{i\geq h}\right]$$

Co-terminal swaption calibration

- And so, for $h \geq i + 1$

$$Z_{ih} = \left(w_i(t) + \sum_{i=1}^{n} \frac{\partial w_i(t)}{\partial F_h} F_i(t) \right) \frac{F_h}{S_i}$$

  and $Z_{ih} = 0$ for $h \leq i$, or use Rebonato's Formula:

$$Z_{ih} = w_i(0) \frac{F_h(0)}{S_i(0)}$$

  for $h \geq i + 1$ and zero otherwise.

- Notice how similar this is to the variance calculations above, but is a useful way of thinking about swaption calibration.

Co-terminal swaption calibration

- Now, the algorithm proceeds as follows: calibrate the volatility structure to cap prices and determine the terminal covariance of forward rates, C^{FR}, where

$$C^{FR}_{ij} = \frac{\int_t^T \sigma_i(s)\sigma_j(s)\rho_{ij}(s)ds}{T - t}$$

- Next, convert this into the swap rate covariance matrix

$$C^{SR} = Z \cdot C^{FR} Z'$$

- From here we use the Black implied volatilities of the swap rates given by the co-terminal swaption prices.
- Let the market volatilities on swaptions expiring at T_i be σ_i^{black} and then define the diagonal matrix, D by

$$D_{gi} = \frac{\hat{\sigma}_i^{black}}{\hat{\sigma}_i(0, T_i)}\delta_{gi}$$

Co-terminal swaption calibration

- Where $\delta_{gi} = 0$ unless $g = i$ and the $\hat{\sigma}$ from the calibrated volatility and correlation functions, and either the Rebonato or Hull and White approximations.
- Now we have
$$C^{SR}_{calib} = DZC^{FR}Z^T D$$
- Now let's define the calibration matrix M
$$M = Z^{-1}DZ$$
and so the forward rate covariance matrix is now
$$C^{FR}_{calib} = MC^{FR}M^T$$
- **The matrix M depends only on current yield curve and known volatility parameters but not on the size of time step and so is the same for all time steps!**

Co-terminal swaption calibration

- We can then use this to simulate forward rates (making sure to sclae by time appropriately) and it will match the co-terminal swaption prices and retain most of the original structure from the original caplet prices and the nice functional correlation form.
- In particular, we will need to take a square root to do the simulations, so
$$C^{FR}_{calib} = A^{FR}_{calib} A^{FR'}_{calib}$$
- Rebonato, Section 10.4 works through an example of this calibration.

Overview

- Now, we have seen the key methods for calibration.
- Calibrating to cap/caplet prices is easy. There are a variety of volatility functions that will calibrate the prices exactly. The most common choice is a parametric one.
- You can also choose parameters to calibrate to swaption volatilities also - this can be done with the aid of the Rebonato or Hull-White approximations.
- Typically, it is not advised to use cap and swaption calibrations together.
- We saw that we could calibrate the correlation matrix if we so wished and also how to write down an exogenous correlation matrix.
- Finally, we saw a neat way of calibrating to cap prices and also to co-terminal swaptions.