FIN516: Term-structure Models

Lecture 11: LIBOR market model - Monte-Carlo

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Introduction

- Here we will look at implementing the LMM.
- First we will discuss how to set up a Monte-Carlo method with state dependent drift, looking at a variety of predictor-corrector schemes.
- Then we will look at particular products and introduce three types of Monte-Carlo schemes: short jump, long jump and very long jump that can be used to value different types of interest rate derivatives.

Monte-Carlo simulation: set-up

- Now that we have calculated our drift terms, specified our instantaneous volatility function and our instantaneous volatility function we now need to carry out a Monte-Carlo simulation. In general, it will not be possible to use recombining trees because of the state dependency in the drift of the forward rates, so we are left with Monte Carlo simulations.
- First, let us consider the simulation of one forward rate where $P(t, T_M)$ is numeraire, and $T_M > T_k$:

$$dF_k = \sigma_k(t)F_k\left(-\sum_{j=k+1}^{M}\frac{F_j\tau_j\rho_{k,j}\sigma_j(t)}{1+F_j\tau_j}dt + dW_k\right)$$

or if we prefer

$$dF_k = \sigma_k(t)F_k\left(-\sum_{j=k+1}^{M}\frac{F_j\tau_j\rho_{k,j}\sigma_j(t)}{1+F_j\tau_j}dt + \sum_{i=1}^{m}b_{ki}(t)dW_i\right)$$

where now we have independent Brownian motions. m is equal to the number of forward rates, M or less.

Monte-Carlo simulation: set-up

- Taking logs removed the F_k term from the front:

$$d \ln F_k = -\sigma_k(t) \sum_{j=k+1}^{M} \frac{F_j \tau_j \rho_{k,j} \sigma_j(t)}{1 + F_j \tau_j} dt - \frac{1}{2} \sigma_k(t)^2 dt + \sigma_k(t) dW_k$$

or

$$d \ln F_k = -\sigma_k(t) \sum_{j=k+1}^{M} \frac{F_j \tau_j \rho_{k,j} \sigma_j(t)}{1 + F_j \tau_j} dt - \frac{1}{2} \sigma_k(t)^2 dt + \sigma_k(t) \sum_{i=1}^{m} b_{ki}(t) dW_i$$

- Now, let's consider a discrete period of time Δt

$$F_k(t+\Delta t) = F_k(t) \exp\left(-\sigma_k(t) \sum_{j=k+1}^{M} \frac{F_j \tau_j \rho_{k,j} \sigma_j(t)}{1 + F_j \tau_j} \Delta t - \frac{1}{2} \sigma_k(t)^2 \Delta t + \sigma_k(t) \sqrt{\Delta t} \phi_k \right)$$

Monte-Carlo simulation: set-up

- where $\phi_k \approx N(0,\rho)$ from the multivariate normal distribution or

$$F_k(t+\Delta t) = F_k(t)\exp\left(-\sigma_k(t)\sum_{j=k+1}^{M}\frac{F_j\tau_j\rho_{k,j}\sigma_j(t)}{1+F_j\tau_j}\Delta t - \sigma_k(t)^2\Delta t + \sigma_k(t)\sum_{i=1}^{m} b_{ki}(t)\sqrt{\Delta t}\phi_i\right)$$

  where $\phi_i \approx N(0,1)$ from a univariate normal distribution.

- Now for Black-Scholes we could take very long steps and simulate stock prices at time T with just one step. This was because the only information required for the path was known at time t. Our problem is that some of our parameters are time dependent.
- The volatility is time dependent but it is deterministic so this does not cause us any difficulty, as we can replace $\sigma_k^2(t)\Delta t$ with $\int_t^{t+\Delta t}\sigma_k^2(s)ds$, similarly with the variance term as $b_{ki}(t)$ is also deterministic once we have specified an instantaneous correlation function (as we will see).
- However, once we have considered the drift term , we will see that it is more natural to work with the covariance matrix.

Monte-Carlo simulation: Predictor corrector

- The problem come from the drift term,as the drift term is state dependent and thus will depend upon the path taken by all of the forward rates between t and $t + \Delta t$. Thus for our Monte Carlo technique to work we will need to use very short time steps.
- This can be solved by a **Predictor Corrector** method. In general, for a multidimensional process, $\mathbf{X}$

$$dX_k = X_k \mu_k(\mathbf{X}(t)) + X_k(t) \sum_{j=1}^{m} \sigma_{kj} dW_j$$

then we can rewrite the drift as

$$\hat{\mu}(\mathbf{X}(t)) = \frac{1}{2} \left(\mu_k(\mathbf{X}(t)) + \mu_k \left(\hat{\mathbf{X}}(t + \Delta t) \right) \right)$$

where

$$\hat{X}_k(t + \Delta t) = X_k \exp \left(\mu_k(\mathbf{X}(t)) \Delta t - \frac{1}{2} \Sigma_{kk} \Delta t + \sum_{j=1}^{\sigma} {}_{kj} \phi_j \sqrt{\Delta t} \right)$$

Monte-Carlo simulation: Predictor corrector

- Now, let's put this in terms of the LMM. The drift of an individual forward rate F_k depends upon the values of all other forward rates. The predictor-corrector technique says, let's freeze the drift at its time t value

$$\mu_k(\mathbf{F}, t) = -\sigma_k(t) \sum_{j=k+1}^{M} \frac{F_j(t)\tau_j \rho_{k,j}\sigma_j}{1 + F_j \tau_j}$$

and then estimate the value of $\hat{F}_k(t + \Delta t)$ for all $k = 1, \dots, M$ as

$$\hat{F}_k(t + \Delta t) = F_k(t) \exp\left(\mu_k(\mathbf{F}, t)\Delta t - \frac{1}{2}\sigma_k(t)^2 \Delta t + \sigma_k(t) \sum_{i=1}^{m} b_{ki}(t)\sqrt{\Delta t}\phi_i \right)$$

now this gives us an estimate for the drift at time $t + \Delta t$ which is

$$\hat{\mu}_k(\mathbf{F}, t) = -\sigma_k(t) \sum_{j=k+1}^{M} \frac{\hat{F}_j(t + \Delta t)\tau_j \rho_{k,j}\sigma_j(t)}{1 + \hat{F}_j(t + \Delta t)\tau_j}$$

Monte-Carlo simulation: Predictor corrector

- Then finally we have a new predictor-corrector drift

$$\mu_k^{new}(\mathbf{F}, t) = \frac{1}{2}\left(\mu_k(\mathbf{F}, t) + \hat{\mu}_k(\mathbf{F}, t)\right)$$

and then

$$F_k(t+\Delta t) = F_k(t)\exp\left(\mu_k^{new}(\mathbf{F}, t)\Delta t - \frac{1}{2}\sigma_k(t)^2\Delta t + \sigma_k(t)\sum_{i=1}^{m} b_{ki}(t)\sqrt{\Delta t}\phi_i\right)$$

- Note that we can use the same ϕ_i draws for both $\hat{F}(t+\Delta t)$ and then for the true simulation $F(t+\Delta t)$. This means that we have not needed to draw any more (normally distributed) random numbers.

Monte-Carlo simulation: Iterative Predictor corrector

- This form of the predictor corrector actually jumps all of the forward rates together to get the corrector drift. However, there is an iterative form that performs slightly better.
- Assuming that we choose the longest dates zero coupon bond $P(t, t_M)$ as the numeraire. This means that the forward rate $F_M(t)$ is driftless, and so its time $t + \Delta t$ value can be determined analytically.
- Then the next forward rate $F_{M-1}(t)$ has a drift that depends only upon $F_M(t)$ and so, its drift can be calculated using the known t and $t + \Delta t$ values of $F_M(t)$. As we move through our forward rates, we can calculate the predictor corrector drift iteratively:

$$\mu_k^{new}(\mathbf{F}, t) = -\frac{1}{2} \sum_{j=k+1}^{M} \left(\frac{F_j(t)\tau_j}{1 + F_j(t)\tau_j} + \frac{F_j(t+\Delta t)\tau_j}{1 + F_j(t+\Delta t)\tau_j} \right) \rho_{kj}\sigma_j(t)\sigma_k(t)$$

Monte-Carlo simulation: Iterative Predictor corrector

- Here we update the forward rates one at a time, and then use them to calculate the drift for the next, one-period shorter, forward rate.

Monte-Carlo simulation: How long should our jumps be?

- If there are early exercise or path dependent features, then we will need to evolve the forward rates to each possible exercise date ('long jumps'), usually these will be reset dates for the forwards.
- For European options, such as swaptions, there is nothing to stop us from taking one 'very long jump', from time 0 until the expiry of the option.
- In this case, we will have to be careful with the forward rates that are still live and ensure that we have estimated the volatility correctly over the lifetime just of the option.

Monte-Carlo simulation: How long should our jumps be?

- Now let's think about how we would evolve our forward rates over time steps larger than Δt.
- Assuming that we can use a predictor corrector scheme to deal with the stochastic F term then all of the other terms will depend upon integrating combinations of the volatilities of the forward rates and the correlation between them. We assume that these are deterministic functions, which are either smooth functions or piecewise constant.
- The drift will depend upon the evaluation of

$$\int_0^T \rho_{ij}\sigma_i(t)\sigma_j(t)dt$$

Monte-Carlo simulation: How long should our jumps be?

- This looks remarkably similar to the covariance between the forward rates at time T. So if we write our LMM model in terms of covariances we have

$$dF_k(t) = \mu_k(\mathbf{F}, t)F_k(t) + \sum_{i=1}^{m} \sigma_{ki}(t)F_k(t)dW_i(t), \quad t \leq T_{k-1}$$

where matrix σ is a square root of the covariance matrix, where the covariance matrix will be full of expressions of the form $\rho_{ij}\sigma_i(t)\sigma_j(t)$.

Monte-Carlo simulation: Long jumps

- To be precise, when we are considering long time steps in a Monte Carlo simulation, then our general Monte Carlo algorithm becomes:

$$F_k(T) = F_k(S) \exp\left(\int_S^T \mu_k(\mathbf{F}, t)dt - \frac{1}{2}\int_S^T \sigma_k^2(t)dt + \sum_{i=1}^{m} \int_S^T \sigma_{ki}(t)dW_i \right)$$

where now σ_{ij} is the square root of the *covariance* matrix but now of course, we use the predictor corrector for part of the drift to get

$$\mu_k(\mathbf{F}, t) = -\sum_{j=k+1}^{M} \frac{1}{2}\left[\frac{F_j(S)\tau_j}{1+F_j(S)\tau_j} + \frac{\hat{F}_j(T)\tau_j}{1+\hat{F}_j(T)\tau_j} \right] \rho_{kj}\sigma_j(t)\sigma_k(t)$$

or the iterative predictor corrector where $\hat{F}_j(T)$ is replaced by $F_j(T)$ and then over a long jump we have

$$\int_S^T \mu_k(\mathbf{F}, t)dt = -\sum_{j=k+1}^{M} \frac{1}{2}\left[\frac{F_j(S)\tau_j}{1+F_j(S)\tau_j} + \frac{F_j(T)\tau_j}{1+F_j(T)\tau_j} \right] \int_S^T \rho_{kj}\sigma_j(t)\sigma_k(t)dt$$

Monte-Carlo simulation: Long jumps

- Now create C - the covariance matrix between time S and T, where the instantaneous covariance matrix has $C_{ij} = \sigma_i(t)\sigma_j(t)\rho_{ij}$ then over a longer period of time, say from S to T the covariance is deterministic and given by

$$C_{ij} = \int_S^T \sigma_i(t)\sigma_j(t)\rho_{ij}(t)dt.$$

  Then we take the square root of the covariance matrix populated with C_{ij} to find σ_{ij}.

- Also, this matrix gives us

$$-\frac{1}{2}\int_S^T \sigma_k^2(t)dt = -\frac{1}{2}C_{kk}$$

  and the overall process becomes

$$F_k(T) = F_k(S)\exp\left(\int_S^T \mu_k(\mathbf{F},t)dt - \frac{1}{2}C_{kk} + \sum_{i=1}^{m}\sigma_{ki}\phi_i\right)$$

Monte-Carlo simulation: Long jumps

- Where, conveniently,

$$\int_S^T \mu_k(\mathbf{F}, t)dt = -\sum_{j=k+1}^{M} \frac{1}{2}\left[\frac{F_j(S)\tau_j}{1+F_j(S)\tau_j} + \frac{F_j(T)\tau_j}{1+F_j(T)\tau_j}\right] C_{jk}$$

Quiz

Can we calculate these integrals analytically?

- For certain choices of $\sigma_j(t)$ and ρ_{ij}
- Always
- Never

Quiz

Can we calculate these integrals analytically?

- For certain choices of $\sigma_j(t)$ and ρ_{ij} Correct
- Always
- Never

Note

- We have been a little unrigorous here.
- As are drifts are not time dependent then the true covariance between the forward rates cannot be calculated as simply as we have done here.
- However, by freezing the drifts or using a predictor-corrector scheme we are implicitly imposing time dependency on our forward rates and so we are able to use the simple formulas for the covariance as we can assume that the forward rates are approximately lognormal.
- If you are unhappy with this process, you could always use short steps in the Monte Carlo process and thus avoid having to use these approximate integrals.
- Rebonato gives some justification for using this approach in his book.

What about the dead forward rates

- Assume that we are simulating three forward rates, $F_1(t)$, $F_2(t)$, and $F_3(t)$ and that we are using $P(t, T_3)$ as the numeraire.
- Of course when discounting back to time 0 we will need to know the values of any payoffs at time T_3.
- Usefully, we will know all of the LIBOR rates from T_0 to T_3 and so if we have a payoff at T_1, say, we can move it forward to T_3 by multiplying by $(1 + \tau_2 F_2(T_1))(1 + \tau_3 F_3(T_2))$.
- So, essentially we can move all of the payoffs forward until time T_3 and then discount using $P(0, T_3)$.

What about the dead forward rates

- Note that this process is what we essentially do when we scale the payoffs by the value of the numeraire. For example if we have a payoff at T_1 then to discount this payoff to $t = 0$ then we need to calculate

$$P(0, T_3)\frac{Payoff(T_1)}{P(T_1, T_3)}$$

but this is equal to:

$$P(0, T_3)\frac{Payoff(T_1)}{P(T_1, T_3)} = P(0, T_3)Payoff(T_1)((1+\tau_2 F_2(T_1))(1+\tau_3 F_3(T_2))$$

Rebonato's TOTC matrix

- Rebonato uses a covariance matrix called TOTC or the total covariance matrix to simulate his paths. It is not really any different from the standard covariance matrix that we constructed above but makes the issue of dead forward rates more clear.
- Given forward M forward rates and price sensitive time events occurring at times k the we need to know the drift of all of these rates and their marginal covariances, given by

$$\begin{aligned} C(i,j,k+1) &= \int_{T_k}^{T_{k+1}} \sigma_i(u)\sigma_j(u)\rho_{i,j}du \\ & \quad \text{for } 0 \leq k \leq M-1 \text{ and } k \leq i,j \leq M \\ C(i,j,k) &= 0 \text{ for } i < k \text{ and/or } j < k \end{aligned}$$

- In particular, we know that

$$\sigma^2_{Black}(T_i)T_i = \int_0^{T_i} \sigma_i^2(u)du = \sum_{r=1}^{i} C(i,i,r)$$

Quiz

If we have M forward rates and k denotes the number of price sensitive events we have stepped through, how many non-zero entries are in the matrix at time T_k?

- M^2
- $M(M-1)$
- $(M-k)^2$
- $M^2 - k$

Quiz

If we have M forward rates and k denotes the number of price sensitive events we have stepped through, how many non-zero entries are in the matrix at time T_k?

- M^2
- $M(M-1)$
- $(M-k)^2$ Correct
- $M^2 - k$

Rebonato's TOTC matrix

- Then the matrix TOTC can be defined as

$$TOTC(i,j) = \sum_{k=1}^{\min(i,j)} C(i,j,k)$$

  this will be an $M \times M$ matrix that can be orthogonalized with either rank M or lower rank if you prefer.

- The next slide, shows the construction of a 3×3 TOTC matrix and the constituent parts.

A 3×3 TOTC matrix

$$\begin{pmatrix} C(1,1,1) & C(1,2,1) & C(1,3,1) \\ C(2,1,1) & C(2,2,1) & C(2,3,1) \\ C(3,1,1) & C(3,2,1) & C(3,3,1) \end{pmatrix}$$

$$\begin{pmatrix} C(2,2,2) & C(2,3,2) \\ C(3,2,2) & C(3,3,2) \end{pmatrix}$$

$$(C(3,3,3))$$

$$\begin{pmatrix} C(1,1,1) & C(1,2,1) & C(1,3,1) \\ C(2,1,1) & C(2,2,1)+C(2,2,2) & C(2,3,1)+C(2,3,2) \\ C(3,1,1) & C(3,2,1)+C(3,2,2) & C(3,3,1)+C(3,3,2)+C(3,3,3) \end{pmatrix}$$

The very long jump

- Let's take the square root, σ, of the the TOTC matrix, C, and then in principle we can evolve all of our M forward rates until the maturity of the final one using the following equation:

$$F_k(T_{M-1}) = F_k(0)\exp\left[\int_0^{T_{M-1}} \mu_k(\mathbf{F}, t)dt - \frac{1}{2}C_{kk} + \sum_{i=1}^{M}\sigma_{ik}\phi_i\right]$$

  (alternatively we could calculate the square root via other techniques)

- Of course, by T_{M-1} all of the forward rates will have reset, or alternatively that after they have reset their drifts and volatilities become zero and they remain at the same level.
- Of course, a variety of C values can give rise to the same TOTC matrix so some information is lost in using the TOTC matrix but if we are only interested in knowing the value of forward rates on their own reset date then there is no problem with this approach - called the very long jump process.

Recap: short jump

- Here the equation of motion is:

$$F_k(t+\Delta t) = F_k(t)\exp\left(\mu_k(\mathbf{F},t)\Delta t - \sigma_k(t)^2\Delta t + \sum_{i=1}^{m}\sigma_{ki}\phi_i\right)$$

  where $\mu_k(\mathbf{F},t)$ is determined by its formula and a simple Euler technique in this case.

- The σ_{ki} values come from the square root of a covariance matrix populated with terms:

$$C_{ij} = \rho_{ij}\sigma_i(t)\sigma_j(t)\Delta t$$

  where i and j count the number of 'alive'forward rates.

- To simulate N paths here you will need: Number of forward rates, M x Steps per reset date x number of reset dates x N draws of a random number.

Recap: short jump

Output from short jump will give us the following rates:

$t=0$	$t=dt$	$\dots$	$t=T_0$	$t=T_0+dt$	$\dots$	$T=T_1$	$T=T_1+dt$	$\dots$
$F_1(0)$	$F_1(dt)$	$\dots$	$F_1(T_0)$	-	-	-	-	-
$F_2(0)$	$F_2(dt)$	$\dots$	$F_2(T_0)$	$F_2(T_0+dt)$	$\dots$	$F_2(T_1)$	-	-
$F_3(0)$	$F_3(dt)$	$\dots$	$F_3(T_0)$	$F_3(T_0+dt)$	$\dots$	$F_3(T_1)$	$F_3(T_1+dt)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F_M(0)$	$F_M(dt)$	$\dots$	$F_M(T_0)$	$F_M(T_0+dt)$	$\dots$	$F_M(T_1)$	$F_M(T_1+dt)$	$\dots$

Recap: long jump

- Here the equation of motion is:

$$F_k(T_i) = F_k(T_{i-1}) \exp\left(\int_{T_{i-1}}^{T_i} \mu_k(\mathbf{F}, t)dt - \frac{1}{2}C_{kk} + \sum_{i=1}^{m} \sigma_{ki}\phi_i\right)$$

  where $\mu_k(\mathbf{F}, t)$ is determined by its formula and a predictor corrector technique in this case (see below)

- The C_{kk} and σ_{ki} values come from the coavariance matrix and its square root, respectively. The covariance matrix is populated with terms:

$$C_{ij} = \int_{T_{i-1}}^{T_i} \rho_{ij}\sigma_i(t)\sigma_j(t)\ dt$$

  where i and j count the number of 'alive'forward rates.

- To simulate N paths here you will need: Number of forward rates, M x reset dates, usually M x N draws of a random number.

Recap: long jump

- For the predictor corrector scheme, start with $k = M$ using $P(t, T_M)$ as numeraire and then the drift for $k = M - 1$ is given by:

$$\int_{T_{i-1}}^{T_i} \mu_{M-1}(\mathbf{F}, t)dt = -\frac{1}{2}\left(\frac{F_M(T_{i-1})\tau_M}{1 + F_M(T_{i-1})\tau_M} + \frac{F_M(T_i)\tau_M}{1 + F_M(T_i)\tau_M}\right) C_{M-1,M}$$

  where $F_M(T_i)$ is calculated easily as it is a martingale.

- And in general,

$$\int_{T_{i-1}}^{T_i} \mu_k(\mathbf{F}, t)dt = -\frac{1}{2}\sum_{j=k+1}^{M}\left(\frac{F_j(T_{i-1})\tau_j}{1 + F_j(T_{i-1})\tau_j} + \frac{F_j(T_i)\tau_j}{1 + F_j(T_i)\tau_j}\right) C_{jk}$$

- Note here that the predictor corrector techniques requires the evaluation of integrals of the form:

$$C_{jk} = \int_{T_{i-1}}^{T_i} \rho_{jk}\sigma_j(t)\sigma_k(t)\ dt$$

  and so you only have to evaluate these integrals once, for both the covariance matrix and also the predictor corrector technique.

Recap: long jump

- Also note that as you move forward in time the number of alive forward rates reduces by one each time, making the covariance matrix smaller. In the original covariance matrix (to time T_0) the following entry is (using formulation 7):

$$C_{12} = \phi_1 \phi_2 \rho_{12} \int_0^{T_0} \psi(T_0 - t)\psi(T_1 - t)dt$$

when we move to time T_0 this term becomes

$$C_{12} = \phi_2 \phi_3 \rho_{23} \int_{T_0}^{T_1} \psi(T_1 - t)\psi(T_2 - t)dt$$

so if the time between reset dates is the same, then the integral that you have to evaluate is always the same, and so you only have to do it once.

Recap: long jump

Output from long jump will give us the following rates:

$t=0$	$t=T_0$	$t=T_1$	$t=T_2$	$T=T_3$	$\dots$	$T=T_{M-1}$
$F_1(0)$	$F_1(T_0)$	-	-	-	-	-
$F_2(0)$	$F_2(T_0)$	$F_2(T_1)$	-	-	-	-
$F_3(0)$	$F_3(T_0)$	$F_3(T_1)$	$F_3(T_2)$	-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F_M(0)$	$F_M(T_0)$	$F_M(T_1)$	$F_M(T_2)$	$F_M(T_3)$	$\dots$	$F_M(T_{M-1})$

Recap: very long jump

- Here the equation of motion is:

$$F_k(T_{k-1}) = F_k(0)\exp\left(\int_0^{T_{k-1}} \mu_k(\mathbf{F},t)dt - \frac{1}{2}C_{kk} + \sum_{i=1}^{m}\sigma_{ki}\phi_i\right)$$

  where $\mu_k(\mathbf{F},t)$ is determined by its formula and a predictor corrector technique in this case.

- The C_{kk} and σ_{ki} values come from the coavariance matrix and its square root, respectively. The covariance matrix is populated with terms:

$$C_{ij} = \int_0^{T_{\min(i-1,j-1)}} \rho_{ij}\sigma_i(t)\sigma_j(t)\ dt$$

  where the integral is only calculated until the shortest lived forward rate has expired. This gives us the TOTC matrix from Rebonato.

- To simulate N paths here you will need: Number of forward rates, M x N draws of a random number.

Recap: very long jump

- There is one problem in using the iterative predictor-corrector here. For the values $j > k$ used in the calculation, they have longer reset dates and so will be evaluated at times longer than the T_{k-1} that we require.
- A simple adjustment is to use a truncated iterative predictor-corrector, where:

$$\int_0^{T_{k-1}} \mu_k(\mathbf{F}, t)dt = -\frac{1}{2}\sum_{j=k+1}^{M}\left(\frac{F_j(0)\tau_j}{1+F_j(0)\tau_j} + \frac{\hat{F}_j(T_{k-1})\tau_j}{1+\hat{F}_j(T_{k-1})\tau_j}\right)C_{jk}$$

  and now:

$$\ln \hat{F}_j(T) = \frac{T_{j-1} - T}{T_{j-1} - 0}\ln F_j(0) + \frac{T - 0}{T_{j-1} - 0}\ln F_j(T_{j-1})$$

  where T_{j-1} is the reset date for forward rate for $F_j(t)$.
- This is simple log-linear interpolation and can be improved (see Joshi paper on drift on COMPASS).

Recap: very long jump

Output from long jump will give us the following rates:

$t = 0$	$t = T_0$	$t = T_1$	$t = T_2$	$T = T_3$	$\dots$	$T = T_{M-1}$
$F_1(0)$	$F_1(T_0)$	-	-	-	-	-
$F_2(0)$	$-$	$F_2(T_1)$	-	-	-	-
$F_3(0)$	-	-	$F_3(T_2)$	-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F_M(0)$	-	-	-	-	$\dots$	$F_M(T_{M-1})$

Value of the numeraire asset

- In general our derivative value V_0 will be calculated by estimating the expectation:
$$V_0 = N_0 E_0 \left[\frac{V_T}{N_T} \right]$$
If we choose the longest numeraire $P(t, T_M)$ then we will need to know its value at all possible reset dates to calculate possible early payoffs.
- We will know $P(0, T_M)$ from initial data and so N_0 is straightforward. We also know that $P(T_M, T_M) = 1$. Then,
$$P(T_{M-1}, T_M) = \frac{1}{1 + F_M(T_{M-1})\tau_M}$$
which we will have in all of out Monte Carlo methods. In general, in the short or long jump:
$$P(T_i, T_M) = \frac{1}{\prod_{j=i+1}^{M}(1 + F_j(T_i)\tau_j)}$$

Value of the numeraire asset

- And in the very long jump:

$$P(T_i, T_M) = \frac{1}{\prod_{j=i+1}^{M}(1 + F_j(T_{j-1})\tau_j)}$$

- This is equivalent to moving all of the payoffs to the maturity of the numeraire, T_M where we know $P(T_M, T_M) = 1$, by using the forward rates. So if we know $Payoff_{T_i}$ then we can convert this to:

$$Payoff_{T_M} = Payoff_{T_i} \prod_{j=i+1}^{M} (1 + F_j(T_i)\tau_j)$$

in the case of the long jump method and

$$Payoff_{T_M} = Payoff_{T_i} \prod_{j=i+1}^{M} (1 + F(T_{j-1})\tau_j)$$

in the case of the very long jump method.

Conclusion

- We have looked at the LIBOR market model, this is defined by its drift, instantaneous volatility function and instantaneous correlation function.
- We have defined some possible forms for these functions and next we will determine how to calibrate these functions to market data.
- We also looked at how to implement a Monte Carlo technique, in particular how to determine the drift over large time steps and how to discretize the problem.
- Finally, we looked at types of products that we could value using the LMM, and some efficient ways of valuing them.