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Lecture Note 5.2: Extending The Formula

- In this note, we are going to solve some complicated derivatives problems in our continuous-time framework *without having to solve any partial differential equations.*

- The key step will be to derive a better version of the Black-Scholes formula: one which prices much more general options.This will allow us to greatly extend the range of situations in which we can apply the formula.

- We can also to some degree generalize the assumption of constant volatility using a very useful result found by Merton.

Outline:

I. Black-Scholes when the Underlying has Payouts

II. The Option to Exchange One Asset for Another

III. Applications:

   (A) Random Payouts

   (B) Black's Formula

IV. Deterministic Volatility

V. Summary

I. Black-Scholes with Payouts to S.

- We saw in the last note that it is easy to get the partial differential equation satisfied by a derivative when the underlying pays a fixed continuous yield.
  - ▶ But what we want to do now is get the *solution* for a European call, without having to solve that PDE.

- Suppose the underlying asset is a commodity with yield y per unit time. (We are at time t, and the option expires at T.)

- The **trick** is to realize that if we form a new asset which is a fund that starts off with $e^{-y(T-t)}$ units of the asset *and always reinvests the interest*, then it will have exactly 1 unit of the asset at T.
  - ▶ See Lecture Note 3.1 (p18).

- The value of the fund at t is $V_t = S_t \cdot e^{-y(T-t)}$ (i.e. price times quantity).

- Now consider three statements
  1. A European option on the fund has the same payoff as one on the asset;

  Why? The right to buy the fund for K at time T (not before) *is* the right to buy 1 unit of the asset then for K.

2. the fund has the same volatility as the asset;

Why? The volatility is the standard deviation of the percent changes in V_t. But

$$\frac{\Delta V_t}{V_t} = \frac{\Delta S_t}{S_t} + y\Delta t.$$

The volatility of the first term is σ. The volatility of the second is zero (it's not random).

3. The fund pays no dividends/has no yield.

Why? Because we said it reinvests all payouts.

- **Conclusion: Black-Scholes applies, using the original σ but with $\mathrm{V_t}$ replacing $\mathrm{S_t}$.**

- Price of call on S = price of call on V =

$$V\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) =$$

$$Se^{-y(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \qquad (1)$$

$$d_1 \equiv \frac{\ln\left(\frac{V}{Ke^{-r(T-t)}}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and}$$

$$d_2 \equiv \frac{\ln\left(\frac{V}{Ke^{-r(T-t)}}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)}.$$

- Note that the definition of d_1 and d_2 changed due to V too.

We can rewrite, for example,

$$d_1 \equiv \frac{\ln\left(\frac{S}{K}\right) + (r-y)(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and} \tag{2}$$

$$d_2 \equiv \frac{\ln\left(\frac{S}{K}\right) + (r-y)(T-t) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}. \tag{3}$$

- For currency options, just put $r = r_d$ and $y = r_f$.
- Keep in mind that *we did assume y was constant here,* as well as r.
- The same trick of defining a fund with no payouts lets us likewise deal with known, certain dividends.
  - ▶ The fund initially borrows $PV(D)$ which it repays as the dividends arrive. Hence it has no net payouts.
  - ▶ Adding the borrowing again doesn't change the volatility of the position.
  - ▶ Value of fund today: $V_t = S_t - PV(D)$.
  - ▶ Result: we can use Black-Scholes for European calls on dividend paying stocks with V replacing S everywhere.
- For both types of payouts, we can then use the relevant version of put-call parity to get European puts too.
  - ▶ Recall, e.g.

$$p = c(S, K, t, T) - S_t + K \cdot B(t, T) + PV(D).$$

II. Option to Exchange One Asset for Another

- Now consider the right to give up 1 share of Ford to get 1 share of GM. (This is what investors in Ford would have if GM made a stock-financed takeover bid.)
  - ▶ *Note: this is not the right to exchange either for the other.* It only goes one way.

- Suppose we are still in the Black-Scholes world, r is fixed, and each share satisfies the Black-Scholes type SDE:

$$\frac{dS^{GM}}{S^{GM}} = \mu^{GM}\, dt \;+\; \sigma^{GM}\, dW^{GM}$$

  (and likewise for Ford).

- Now we have two different random innovation series, dW^{GM} and dW^{F}, and they might be correlated.
  - ▶ Call their instantaneous correlation ρ.
  - ▶ That is, over time interval dt the expected product $dW^{GM} \cdot dW^{F}$ is $\rho\ dt$.

- Also, they can have (known, fixed) yields y^{GM} and y^{F}.

- We can write the payoff on the option as: $m(T, S_T^{GM}, S_T^{F}) = \max[S_T^{GM} - S_T^{F}, 0]$. **What is it worth?**

- This looks like it's well beyond the scope of what we've done so far because there are clearly two sources of randomness in the economy.

- Oddly enough, <u>it's not.</u>

- Suppose instead of calling the assets GM and Ford, we had called them Lira and Pesos respectively. Then, to a Dollar investor, it's price might not be obvious. But, to a Peso investor, the option is merely an ordinary call (with strike price 1), except quoted in a third currency.

- Mathematically, just re-write the payoff.

$$\max[S_T^{Lira} - S_T^{Peso}, 0] \textit{ (in dollars) } =$$

$$S_T^{Peso} \textit{ (in dollars/peso) } \cdot \max[\frac{S_T^{Lira}}{S_T^{Peso}} - 1, 0] \textit{ (in pesos)}$$

- In other words, the payoff is the Peso value of **a call with strike price=1** times the final Dollar-per-Peso exchange rate.
  - ▶ A dollar investor would get this payoff by buying Pesos today, buying FX calls denominated in Pesos, and holding this portfolio to expiration.

- So any time before T the portfolio's value must be the value of the call to a Mexican investor, times the spot exchange rate into Dollars.
  - ▶ And we have got a formula for <u>that</u> option.

- If E_t is the Peso/Lira rate, then the option, m, is worth

$$m(S_t^{Lira}, S_t^{Peso}, t, T) \text{ in Dollars} =$$

$$S_t^{Peso}\left(E_t \cdot e^{-r^{Lira}(T-t)}\mathcal{N}(d_1) - 1 \cdot e^{-r^{Peso}(T-t)}\mathcal{N}(d_2)\right)$$

$$= \quad S_t^{Lira} e^{-r^{Lira}(T-t)}\mathcal{N}(d_1) - 1 \cdot S_t^{Peso} e^{-r^{Peso}(T-t)}\mathcal{N}(d_2). \quad (4)$$

where now,

$$d_1 = \frac{\ln\left(\frac{S_t^{Lira}/S_t^{Peso}}{1}\right) + (r^{Peso} - r^{Lira})(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}.$$

- Notice two things about this pricing formula
  - ▶ If the original exchange ratio had been K-for-one instead of one-for-one, we just make the obvious adjustments.
  - ▶ The Dollar interest rate doesn't appear anywhere.
- This last fact is especially important. In fact, we didn't have to assume anything about the Dollar interest rate:

  We only assumed the Black-Scholes assumptions held for the Mexican (Peso) investor.
- The equation (4) for the right to exchange is known as Margrabe's formula.
- Why did we want it?
- Because we are now going to put back Ford and GM for Pesos and Lira, and we have the answer to the question we started with.

- When we do this, Margrabe's formula tells us that the option to exchange Ford shares for GM *still doesn't depend on interest rates.*
  - ▶ The only rates that appear are the yields (if any) on the assets involved.

- Notice an important point though: the σ in the formula is the volatility of the Peso/Lira exchange rate. Or, now, it's the volatility of the **ratio** of Ford's price to GM's.

- So we only assumed this exchange rate (or this ratio) behaves according to the type of model that Black-Scholes requires.
  - ▶ We have to assume that E has constant volatility.

- The individual prices in Dollars could've been more complicated. However, if we did want to model the components as lognormal, then the ratio automatically is.

- You can prove for yourself (using Itô's lemma) that if

$$\begin{aligned} dX/X &= \mu_X\,dt \;+\; \sigma_X\,dW_X \;\; \text{and} \\ dY/Y &= \mu_Y\,dt \;+\; \sigma_Y\,dW_Y \;\; \text{then for } Z \equiv X/Y, \\ \\ dZ/Z &= \mu_Z\,dt \;+\; \sigma_Z\,dW_Z \;\; \text{where} \\ \\ \sigma_Z^2 &\equiv \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y \;\; \text{and} \\ W_Z & \quad \text{is still a Brownian motion process.} \end{aligned} \tag{5}$$

- This tells us how to adjust the individual volatilities to get the ratio's volatility.

Example. Suppose Ford and GM shares each have 25% volatility but there returns have a correlation of 80%. What is the volatility of S_t^{GM}/S_t^{Ford}? What is the volatility of S_t^{Ford}/S_t^{GM}?

Answer: First note that the formula (5) above is symmetrical in X and Y. So the two ratios have the same variance. And

$$\sigma_{ratio} = \sqrt{\sigma_{Ford}^2 + \sigma_{GM}^2 - 2\,\rho\,\sigma_{Ford}\,\sigma_{GM}}$$

$$= \sqrt{2 \cdot (0.25)^2 - 2 \cdot (0.8)(0.25)(0.25)}$$
$$= 0.25\sqrt{2 - 1.6} = .158$$

- Positive correlation means the ratio volatility is <u>lower</u> than the simple average. Negative means higher.

- The important point to remember is that *the option to exchange A for B is just like an ordinary call when we are measuring prices in units of A.*
  - ▶ Hence the volatility we need is the volatility of B **in those units**.

III. Applications

Margrabe's formula has some very deep and practical applications, as I hope to convince you with a few examples.

(A) Random Rates

1. Black-Scholes again.

- Start by going back to the original Black-Scholes problem (one underlying asset, no dividends) and think of our plain European call in a new way.

- Isn't it just **the right to exchange $\mathbf{K}$ zero-coupon bonds maturing at $\mathbf{T}$ for 1 share**?
  (Hint: the answer is yes.)

- So what happens if we apply Margrabe's formula to it?

- Well, that means we need the Black-Scholes assumptions to apply in a world where things are priced in **units of the bond.** So we need that

a. the stock price divided by the bond price obeys a log-normal (constant volatility) law, and

b. the two securities have fixed payouts.

- Let's accept the first one. What about the second?

- Both securities do have fixed payouts: **zero.**

- So Margrabe says

$$c(S_t, B_t, t, T) = S_t \mathcal{N}(d_1) - B_t K \mathcal{N}(d_2). \tag{6}$$

where now,

$$
\begin{aligned}
d_1 &\equiv \frac{\ln\left(\frac{S_t}{B_t K}\right) + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and} \\
d_2 &= d_1 - \sigma\sqrt{(T-t)}.
\end{aligned}
$$

- So what? you say. This is just the Black-Scholes formula. But it isn't. **Why?**
- *We never had to assume the interest rate was constant.*
- Notice r doesn't appear anywhere. Instead we have B_t which was allowed to be a completely separate random process, not just $e^{-r(T-t)}$.
- This is a very significant result. It almost seems like magic. We don't need one of the main Black-Scholes assumptions after all!
  - ▶ How did we get away with this?
- We changed the volatility assumptions slightly. But our new assumption is really no more restrictive than our original one. That was not what was really important.
- What we really did was this: We allowed ourselves to trade in a new security whose value is proportional to S/B. The value of our hedge position in the new security exactly offset the cost of our option, so we had no instantaneous financing cost.

- **Example:** Value a 5-year call on Kyocera shares where $S = ¥9,300, K = ¥6,000, \sigma = 0.60$ and there are no dividends.

  - ▶ Trouble: The overnight Japanese interest rate is 0.25%, but the 5-year zero-coupon rate is 1.75%.

  - ▶ Not obvious which (if either) we should plug into (ordinary) Black-Scholes since the replication argument involved cash-flows at all times.

  - ▶ Now we know: use the 5-year rate (here 1.75%).

  - ▶ Model can now be consistent with **any** yield curve

  - ▶ As long as the ratio of stock-to-bond prices has constant volatility.

  - ▶ Assume 5-year bond price has volatility 12% and 10% correlation with stock. Then

    $$\sigma^2_{S/B} = (0.60)^2 + (0.12)^2 - 2 \cdot (0.10) \cdot (0.12) \cdot (0.60)$$

    So $\sigma_{S/B} = \sqrt{0.36} = 0.60$ is the adjusted volatility.

  - ▶ Conclusion: $B = 0.9162, d_1 = 1.0627, d_2 = -0.2789$, and $c = 5816$. (With the wrong rate you get about 200 yen less.)

2. Random payouts.

- The last formula we derived was kind of asymmetrical. We freed the payout to *cash* (i.e. the interest rate) from the restriction that it be constant. But what about the payout of the *underlying*?

- We know that, for currency options anyway, there's really no distinction. A call is just the right to exchange one thing for another.

- So what if we applied Margrabe's formula to the right to exchange K Peso discount bond for 1 Lira discount bonds?

  ▶ Then neither underlying asset has any payouts.

- Margrabe will give us the price...in Dollars.

- And if we multiply by the spot Pesos per Dollar we will have converted everything back to the Mexican investor's perspective and we get a new formula for currency options.

- Let's just be careful with our notation.

  ▶ The two assets we are applying the formula to are

$$
\begin{aligned}
S^{(1)} \text{ in Dollars} &= B_t^{Lira} \cdot S_t^{Lira} \text{ and} \\
S^{(2)} \text{ in Dollars} &= B_t^{Peso} \cdot S_t^{Peso}
\end{aligned}
$$

  ▶ It's the ratio of these that we need to have constant σ.

▶ The exchange rate that is the underlying for the call is

$$E_t \text{ in Pesos per Lira } = S_t^{Lira}/S_t^{Peso}.$$

- And the formula says

$$c(S_t^{Lira}, S_t^{Peso}, t, T) \text{ in Pesos} =$$

$$\frac{1}{S_t^{Peso}}\left(S^{(1)}\mathcal{N}(d_1) - KS^{(2)}\mathcal{N}(d_2)\right)$$

$$= E_t B_t^{Lira}\mathcal{N}(d_1) - KB_t^{Peso}\mathcal{N}(d_2).$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{E_t B_t^{Lira}}{KB_t^{Peso}}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and} \\ d_2 &= d_1 - \sigma\sqrt{(T-t)}. \end{aligned}$$

- And now we have freed both interest rates.

- This formula can be simplified by noticing that $E_t B_t^{Lira}/B_t^{Peso}$ is just the forward price for Lira, $F_{t,T}$.

▶ Also notice that this ratio is precisely the thing we assumed was lognormal: $S^{(1)}/S^{(2)}$.

▶ In other words, our assumption is now **that the FORWARD price has constant volatility.**

▶ And that is the σ we need to plug in to our formula.

- Let's dispense with these particular currencies and make the notation generic. We have deduced

$$c(E_t, B_t, t, T) = B_t \left[F_{t,T} \mathcal{N}(d_1) - K \mathcal{N}(d_2) \right]. \qquad (7)$$

where $B_t = B_{t,T}$ is the domestic zero-coupon risk-free bond maturing at T, and

$$d_1 = \frac{\ln\left(\frac{F_{t,T}}{K}\right) + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and}$$
$$d_2 = d_1 - \sigma\sqrt{(T-t)}.$$

- Now we've done more magic. **We made the foreign bond and the spot exchange rate dissapear entirely.**

- The first time we re-wrote Black-Scholes, freeing up the interest rate, we had to introduce another security: B. Likewise, now we freed the other payout by assuming that there is a third security, F, which we also assume we can trade.

  ▶ The forward price impounds all the information about payouts that is needed to price the options.

- The argument that brought us to this formula was specific to the FX market (because we applied Margrabe to the two countries' zero-coupon bonds).

- But looking at the formula, nothing specific to FX appears in it.

- Might we be able to use it for *any* underlying that has traded forwards?

- **Yes!**
- We just have to use a little bit of imagination in designing the right two assets.
- **So our formula applies to underlying securities with arbitrary, random payoffs** such as coupons, dividends, or convenience yields.
- What if it doesn't have traded forwards?
  - If the payouts are known (call them D) then we know we could just synthesize a forward, and, if it were quoted, its price would be

    $$F_{t,T} = (S - PV(D))/B_{t,T}.$$

    So we could just plug that in to (7).
  - If the payouts are unknown then we're out of luck. We need the forward.

- **Example:** Use Black-Scholes to value 6-month European calls on the CAC-40. $S = 6200, K = 6500, B = 0.975$
  - ▶ Trouble: 40 different dividends to keep track of. None of them certain!
  - ▶ Solution: Price the off the forwards: $F = 6455$.
  - ▶ Forward volatility = 19% (e.g. from data on forwards).
  - ▶ Result: $c = B_{t,T} \cdot [F_{t,T}\mathcal{N}(d_1) - K\mathcal{N}(d_2)] = 388.$

- As you can see, we have stretched the range of situations in which we can use the Black-Scholes formula beyond its original narrow base.
  We have used Margrabe to relax a major assumption.

- Next we show that we can apply the formula to *another whole class of derivatives.*

(B) Black's Formula

- Fischer Black realized that you could extend our arguments to price options on futures or forward contracts.

- **How they work:** A call on a time-T' forward with strike K gives the holder the right to *go long a forward* whose forward-price is K.

  ▶ The expiration can be any time $T \leq T'$.

  ▶ Holders will exercise at T if and only if K is below the then-prevailing forward price $F_{T,T'}$.

  ▶ If they do, they can immediately offset their long forward by selling forward, and lock-in

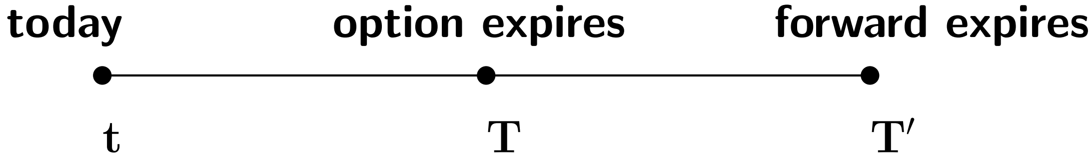
$$[S_{T'} - K] - [S_{T'} - F_{T,T'}]$$

  to be received at time T'.

  ▶ So the payoff function at T is

$$B_{T,T'} \cdot \max[F_{T,T'} - K, 0].$$

- The timing situation is this:



- Note some significant differences with the options we are used to.
  - ▶ The "underlying" is not a spot price.
  - ▶ And K is not an amount of money paid for anything upon exercise.
- The trick to pricing these is that we can still view their payoff as that from exchanging K zero-coupon bonds for the proceeds of a fund with zero yield.
  - ▶ Let's see how.
- The first step is to design the right synthetic fund. It has to be worth $B_{T,T'} \cdot F_{T,T'}$ for certain at date T.
- More tricks!
  - ▶ We can always make a fund worth $B_{T,T'} \cdot F_{T,T'}$ by buying one forward to date T' for $F_{t,T'}$ and also buying $F_{t,T'}$ zero-coupon bonds maturing at T'.
  - ▶ The value of this fund at any intermediate t' is:

    $$B_{t',T'} \cdot [F_{t',T'} - F_{t,T'}] + B_{t',T'} \cdot F_{t,T'}. = B_{t',T'} \cdot F_{t',T'}$$
  - ▶ So at $t' = T$ this gives us exactly what we wanted.
- Now we want to consider the right to exchange bonds that mature <u>at T'</u> for our fund.
  - ▶ Payoff: $\max[B_{T,T'} \, F_{T,T'} - B_{T,T'} \, K, 0]$
    $= B_{T,T'} \cdot \max[F_{T,T'} - K, 0]$.

- So we can immediately apply Margrabe again and get **Black's formula:**

$$c(F_{t,T'}, B_{t,T'}) = \\ B_{t,T'}\left[F_{t,T'}\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right]. \tag{8}$$

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{F_{t,T'}}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and} \\ d_2 &= d_1 - \sigma\sqrt{(T-t)}. \end{aligned}$$

- This looks almost the same, but there are now a couple of Ts to keep track of. The earlier formula is a special case of this one if we set $T = T'$. *A European option on spot expiring at T is the same as a European option expiring at T on a time-T forward.*

- Check the assumptions again:

  - ▶ interest rates and payouts **are** allowed to be completely random;
  - ▶ now σ is the <u>time-T' forward's volatility</u>;
  - ▶ only that forward price is assumed to be lognormal;

- Notice equation (8) prices European calls on forwards on any underlying.

- Options on **futures** work almost the same as those on forwards.
  - But now if you exercise a long call you get a future with futures price K.
  - This means that you will realize a profit $f_{T,T'} - K$ *immediately*, when the future is marked to market.
  - Payoff: $\max[f_{T,T'} - K, 0]$. The money is there today.

- Earlier in the course, we saw that $f = F$ *when interest rates are predictable.*
  - Under the same assumption, you can do a similar construction with funds.
  - The key is that you know in advance what the discount factor at T to time T' will be. In fact,

$$\frac{1}{B_{T,T'}} = \frac{B_{t,T}}{B_{t,T'}}$$

  - Then this is the number of forwards you hold in your fund.
  - I'll let you fill in the details.

- The end result is almost the same:

$$c(f_{t,T'}, B_{t,T}) = B_{t,T}\left[f_{t,T'}\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right]. \tag{9}$$

Only the bond price out front changes.

IV. Deterministic Volatility.

- One last very useful result that lets us get still more milage out of our closed-form solutions is a way to handle *time-varying volatility.*
  - ▶ We know the Black-Scholes PDE applies even if $\sigma = \sigma(S, t)$.
  - ▶ The neat thing is the Black-Scholes formula also applies when $\sigma = \sigma(t)$, in other words when the instantaneous (or spot) volatility changes predictably.
  - ▶ All we have to do is modify the vol we plug in!
- In his 1973 paper, Merton proved the following result.

  If the Black-Scholes assumptions hold, and c() is the Black-Scholes value of a European call, then, if σ is a function only of time, the value of a European call is

$$c(S, K, t, T, \bar{\sigma})$$

  where

$$\bar{\sigma}^2 = \frac{1}{T-t}\int_t^T \sigma^2(s)\,ds.$$

  i.e. just average the future values of the instantaneous variance.

- This result has a number of very useful applications.

(A) Real-World patterns.

- In some markets there *are* obvious and predictable temporal patterns to volatility.
  - ∗ There is less volatility over weekends and holidays.
  - ∗ There is more volatilities over periods of scheduled announcements, in particular, macroeconomic statistics and company earnings reports. Elections work similarly.
  - ∗ Bonds become predictably less volatile as they approach maturity. This is no mystery: their duration falls and their yield volatility doesn't change (much).
  - ∗ Some commodity prices have stong seasonal patterns in volatility. Electricity is about twice as volatile in the summer.
- Merton's adjustment rule allows us to use Black-Scholes in all these cases.

(B) Merton plus Margrabe.

▶ The decay pattern in bond volatility might have occurred to you earlier in the lecture when we wanted to assume that forward prices have constant volatility.

▶ You might not have liked the assumption that, for example, $S_t/B_{t,T}$ had constant vol, because of the denominator.

▶ But we can deal with that! We can compute the time averaged forward vol and plug that into the various versions of Margrabe that we derived.

▶ **Example:**

∗ Suppose we have a non-dividend-paying stock whose returns are uncorrelated with interest rates.

∗ Assume the interest rate to maturity, $r_{t,T}$ has constant volatility, σ_r.

∗ Then, from Itô,

$$\frac{dB_{t,T}}{B_{t,T}} = \text{ stuff } \; dt - (T-t)dr_{t,T}$$

∗ So the bond's volatility is just $(T-t)\sigma_r$.

∗ So, from our earlier formula, $F = S/B$ has squared volatility

$$\sigma_F^2 = \sigma_S^2 + (T-t)^2\sigma_r^2$$

* So Merton tells us to average this over the life of the option:

$$\bar{\sigma}_F^2 = \frac{1}{T-t}\int_t^T [\sigma_S^2 + (T-s)^2\sigma_r^2]\,ds.$$

$$= \sigma_S^2 + \frac{1}{(T-t)}\frac{(T-t)^3}{3}\sigma_r^2$$

or

$$\bar{\sigma}_F = \sqrt{\sigma_S^2 + \frac{(T-t)^2}{3}\sigma_r^2}.$$

* And we can plug this straight into the formula!

- For a stock with 30% vol and for an interest rate with 20% vol, and a one year option, this gives a corrected vol of

$$\sqrt{(0.3)^2 + \frac{(1)^2}{3}(0.2)^2} = 32.15\%$$

(C) Forward implied vols.

- As another application of Merton's rule, we can now give a consistent interpretation of the fact that, in real life, on any given day, Black-Scholes implied volatilities are not constant for all maturities of options.
  - ∗ There is always a *term-structure of implied volatilities.*
  - ∗ Recall, these BSIVs are computed by inverting the constant vol Black-Scholes formula.
- Now we can see that non-constant term structures could result from a particular deterministic pattern of the evolution of actual (spot) volatility.
  - ∗ That is, we should be able to find a projected path of σ that fits any given curve, once we use the Merton adjustment.
  - ∗ In fact we can! And this curve is called the term-structure of **forward implied volatilities.**
- Here's how it goes:
  - ∗ Given two dates, T_1, T_2, with $T_1 < T_2$, and given the BSIVs to those two dates, $\bar{\sigma}_1, \bar{\sigma}_2$, define the forward implied $\bar{\sigma}_{12}$ to be the level of volatility which, if it held between T_1 and T_2, with $\bar{\sigma}_1$ holding before T_1, would yield $\bar{\sigma}_2$ for the whole period to T_2.

* Using Merton's formula, it's easy to find this:

$$\bar{\sigma}_2^2 = \frac{1}{T_2 - t}\left[\int_t^{T_1} \sigma^2(s)\,ds + \int_{T_1}^{T_2} \sigma^2(s)\,ds\right]$$
$$= \frac{T_1 - t}{T_2 - t}\cdot \bar{\sigma}_1^2 + \frac{T_2 - T_1}{T_2 - t}\cdot \bar{\sigma}_{12}^2$$

  So, we can solve for $\bar{\sigma}_{12}$:

$$\bar{\sigma}_{12} = \sqrt{\frac{(T_2 - t)\bar{\sigma}_2^2 - (T_1 - t)\bar{\sigma}_1^2}{T_2 - T_1}}$$

* This is analogous to the way you find forward interest rates, given yields to any two dates.

* To build up the whole forward term structure, we would start with short-dated options (for example $T_1 = 1$ week, $T_2 = 2$ weeks), and then move successively farther out.

V. Summary

- We showed that we could use variants of the Black-Scholes formula in a number of situations in which the original assumptions don't apply.
  - ▶ We showed how, the assumption of known and fixed payouts and interest rates can be eliminated.

    Instead we allowed underlying assets with arbitrary, random payouts and arbitrary random term-structures.

  - ▶ We showed how to price options on forwards and futures.

    These seem quite different since no cash is paid on exercise and the underlying always has zero value .

  - ▶ We showed how to handle deterministic volatility.

    This allows us to model time- and level-dependent patterns.

- All the other Black-Scholes assumptions were maintained. In particular
  - ▶ There was still allowed to be only one source of randomness: the asset price ratio.
  - ▶ We had to be able to continuously trade in both S and B.
  - ▶ We still only dealt with European options.

Lecture Note 5.2: Summary of Notation

Symbol	Page	Meaning
y	p2	*continuous payout on asset whose price is* S
V_t	p2	*value at time* t *of fund holding* $e^{-y(T-t)}$ *units of asset which reinvests the payout*
$PV(D)$	p4	*present value at* t *of dividends to be paid before expiration*
$\mu^\bullet, \sigma^\bullet$	p5	*expected growth rate and volatility of asset whose price is* $S^\bullet$*, where on this page* $\bullet$ *is Ford, GM, Pesos or Lira*
dW^F, dW^{GM}	p5	*random normal innovations in prices of Ford and GM*
ρ	p5	*correlation between* dW^F, dW^{GM}
y^F, y^{GM}	p5	*continuous dividend yield payed by Ford and GM*
r^{Peso}, r^{Lira}	p6	*riskless rate for Pesos and Lira, assumed constant*
E_t	p6	*the peso-per-lira exchange rate* $= S^{Lira}/S^{Peso}$
$m(\ )$	p6	*value in dollars of a call on* E *with strike price 1*
σ	p6	*volatility of* E
σ_Z	p8	*volatility of the ratio* X/Y *when* X *and* Y *have constant volatilities* σ_X *and* σ_Y *and correlation* ρ *with each other*
B_t	p10	*value of a riskless 0-coupon bond maturing at the expiration of the call*
$S^{(1)}$	p13	*dollar value of lira denominated 0-coupon bond,* B^{Lira}
$S^{(2)}$	p13	*dollar value of peso denominated 0-coupon bond,* B^{Peso}
σ	p13	*volatility of* $S^{(1)}/S^{(2)}$
$F_{t,T}$	p14	*date-*T *forward peso-per-lira exchange rate*
T'	p18	*Forward contract settlement date of an option expiring at* T
$\bar{\sigma}^2$	p22	*average variance of returns until maturity*
$\bar{\sigma}_{1,2}$	p27	*forward implied vol between* $T1$ *and* $T2$