FIN516: Term-structure Models

Lecture 5: One-Factor models: Introduction

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Fall, 2015

Introduction

- The simplest term structure models are the 'one-factor'models.
- These specify a stochastic process for the instantaneous spot rate $r(t)$, and build the entire term structure from one stochastic factor - hence the one-factor model.
- Historically, these form the earliest term structure models, but the early models are not even able to calibrate the current term structure precisely. The later one-factor models that we will look at in detail can be calibrated to the current term structure as well as caplet prices, but it will not be possible to also match to swaption prices.
- Generally, derivatives are priced using tree methods and we will look at the implementation of these methods.

The Vasicek model

- The simplest one-factor model is the Vasicek model, developed in 1977. The most useful feature of the Vasicek model, is that it admits analytic solutions for Bond prices and for options on bonds. However, it has serious calibration problems.
- In this model, the spot rate follows:

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t)$$

where k is the rate at which the spot rate reverts back to the long-run mean rate θ. dW is a Brownian motion term and σ is the constant volatility term. This process is also called an **Ornstein-Uhlenbeck** or **O-U** process and exhibits mean reversion.

The Vasicek model: spot rates

- The SDE can be rewritten as:

$$d(e^{kt}r(t)) = e^{kt}dr(t) + ke^{kt}r(t)dt = \theta k e^{kt}dt + e^{kt}\sigma dW(t)$$

and so

$$e^{kt}r(t) = r(0) + \theta k \int_0^t e^{ks}ds + \sigma \int_0^t e^{ks}dW(s)$$

and so, in general

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)}dW(u)$$

- The spot rate (conditional on information at time s) is then normally distributed. The expected value of r_t at some future time t is:

$$E\left[r(t)|\mathcal{F}_s\right] = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

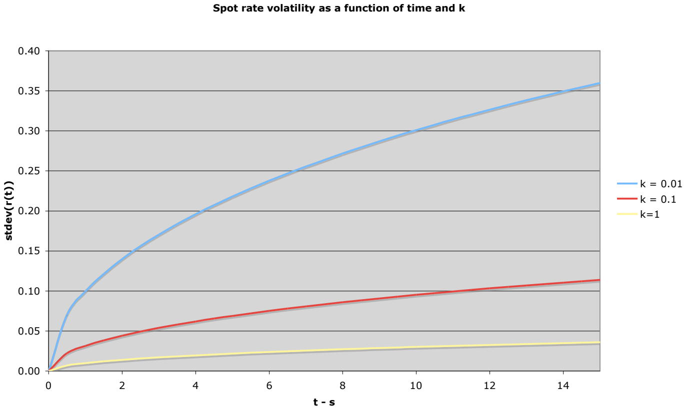
and the variance of the future spot rate is

$$\mathrm{Var}\left[r(t)|\mathcal{F}_s\right] = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right]$$

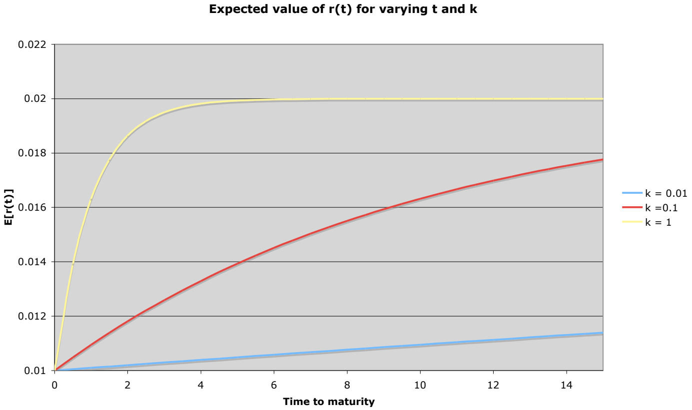
The Vasicek model: spot rates

- These results indicate that the short rate can take negative value with non-zero probability and illustrate the mean reversion, such that as $t \to \infty$ the expected value of $r(t)$ goes to θ.
- The volatility has a negative exponential form and the larger the mean reversion rate the smaller the spot rate volatility as the rates are drawn back to the long run average. The next page shows the shape of the short rate volatility for different mean reversion parameters as time increases. Also depicted is the expected value of the short rate for two different θ values.

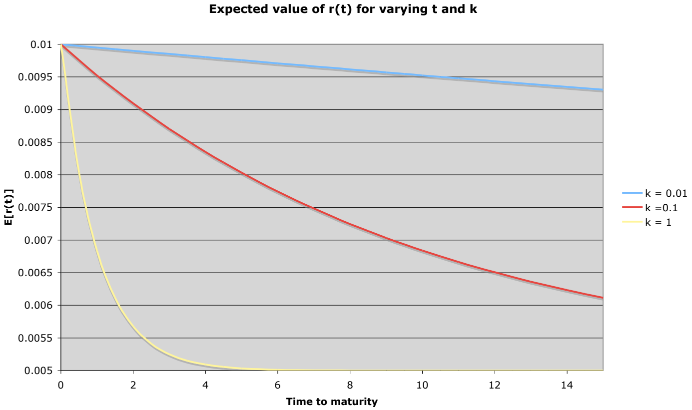
Spot rate volatility



Expected spot rate when $\theta = 0.02$, $r_0 = 0.01$



Expected spot rate when $\theta = 0.005$, $r_0 = 0.01$



The Vasicek formulas: bond

- Given the spot rate, recall that we can write the bond price as:

$$P(t, T) = E_t\left[e^{-\int_t^T r(s)ds}\right]$$

- There are three methods of deriving the bond price formula and details are in Mamon (2004) posted onto COMPASS. You have seen similar derivations in previous classes and so we just present the results here (see also PS4).
- The price of a zero coupon bond is given by

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

where

$$\begin{aligned} A(t, T) &= \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t, T) - T + t] - \frac{\sigma^2}{4k}B(t, T)^2\right\} \\ B(t, T) &= \frac{1}{k}\left[1 - e^{-k(T-t)}\right] \end{aligned}$$

The Vasicek formulas: continuously compounded rate

- From above, we know the zero coupon bond price:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

- By definition, the continuously compounded interest rate $R(t,T)$ is such that

$$P(t,T) = e^{-R(t,T)(T-t)}$$

  and so,

$$R(t,T) = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T)}{T-t}r(t)$$

- So the continuously compounded rate can be written as a function of deterministic functions and the spot rate at time t, we will return to this point.

Quiz

How can we adapt this to price coupon bonds?

- Not easily
- Derive a new formula using the same approach
- As a sum of zero-coupon bond formulas

Quiz

How can we adapt this to price coupon bonds?

- Not easily
- Derive a new formula using the same approach
- As a sum of zero-coupon bond formulas CORRECT

The Vasicek formulas: options

- Jamshidian (1989) (on COMPASS) derives the value of call and put options $ZBC(t,T,S)$ and $ZBP(t,T,S)$ at time t, expiring at time T with an exercise price Xon a zero coupon bond maturing at time S to be

$$\begin{aligned} ZBC(t,T,S,X) &= P(t,S)N(d_1) - XP(t,T)N(d_2) \\ ZBP(t,T,S,X) &= XP(t,T)N(-d_2) - P(t,S)N(-d_1) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{P(t,S)}{XP(t,T)}\right)}{\sigma_p} + \frac{\sigma_p}{2} \\ d_2 &= d_1 - \sigma_p \\ \sigma_p &= \sigma\sqrt{\frac{1-e^{-2k(T-t)}}{2k}}B(T,S) \end{aligned}$$

The Vasicek model: real world distribution

- The dynamics above and the pricing formulas are derived under the risk-neutral measure. It is interesting to see what the real world dynamics look like.
- To do this, we add a term to the drift so that

$$dr(t) = [k\theta - (k + \lambda\sigma)r(t)]dt + \sigma dW^0(t)$$

  this λ term will contribute to the market price of risk (as μ does in the model for stock returns).
- This choice of real world dynamics ensures that when changing the measure that the model is tractable both in the real world and risk-neutral world.
- When changing measure, $dW = dW^0 - \nu dt$ is still a Brownian motion, so

$$dr(t) = [k\theta - (k + \lambda\sigma)r(t) - \nu\sigma]dt + \sigma dW(t)$$

  and so choosing $\nu = \lambda r(t)$ reduces the process to that above - this is the market price of risk term.

The Vasicek model: real world distribution

- There is no reason why the market price of risk should necessarily be $\lambda r(t)$ the case but this formulation ensures that parameters in the model - noticeably the diffusion term does not change and so this can be estimated from real data and then used in the pricing of derivatives.

Quiz

What is the market price of risk in the model for stock returns:

- μ
- $\mu - r$
- $\frac{\mu - r}{\sigma}$

Quiz

What is the market price of risk in the model for stock returns:

- μ
- $\mu - r$
- $\frac{\mu-r}{\sigma}$ CORRECT

What's wrong with Vasicek?

- Vasicek allows for negative spot rates $r(t)$ which is the usual reason for discarding it as a viable model. However, we will see later that there are more sophisticated models that also allow for negative spot rates.
- Vasicek, has the attractive feature of analytic tractability which will make it relatively straightforward to value more advanced derivative products, but the limited number of parameters it is generally impossible to fit the Vasicek model to the initial term structure of rates. Essentially you will need to match all $P(t, T)$ values using only three parameters.
- This can be easily rectified by making some of the model parameters time varying, giving more degrees of freedom and allowing the model to fit the initial term structure. Hull and White (1990) choose to make θ, k and σ time varying allowing enough degrees of freedom to match the initial term structure.
- Brigo and Mercurio (2005) detail the more usual adaptation where only the long term average θ is time varying and are able to calibrate the model to the initial term structure.
- First, we will see a simple way to make rates non-negative.

Affine models in general

- The nice feature of Vasicek is that the continuously compounded rate can be written as a relatively simple function of the instantaneous spot rate. This is because the Vasicek model is part of a class of models called **Affine term structure models**.
- These are interest rate models where $R(t, T)$ is an affine function in the short rate:

$$R(t, T) = -\alpha(t, T) + \beta(t, T)r(t)$$

- In the case where the zero-coupon bond price is of the form

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

then these models are affine models.

Quiz

What are α and β when $P(t, T) = A(t, T)e^{-B(t,T)r(t)}$?

- ?

Quiz

What are α and β when $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$?

- ? $\alpha(t,T) = -\frac{\ln A(t,T)}{T-t}, \beta(t,T) = \frac{B(t,T)}{T-t}$

Forward rates in Affine models

- We have already seen that the forward rate $f(t, T)$ is related to the zero coupon bond price as follows:

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T}$$

- If $r(t)$ is described by an affine model then

$$f(t, T) = -\frac{\partial \ln A(t, T)}{\partial T} + \frac{\partial B(t, T)}{\partial T} r(t)$$

- And so the forward rate is described by the following process:

$$df(t, T) = (\dots)dt + \frac{\partial B(t, T)}{\partial T} \sigma dW(t)$$

where σ is the diffusion coefficient in the short rate dynamics. From this we can extract the volatility of the instantaneous forward rate for affine term structure models as:

$$\sigma_f(t, T) = \frac{\partial B(t, T)}{\partial T} \sigma$$

CIR model

- An easy fix to the problems of Vasicek is to ensure that as the short rate gets close to zero then the volatility reduces to zero. One way that this is achieved is to have a square root term in the diffusion term. This was developed by Cox, Ingersoll, Ross (1985) and is as follows:
$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$
where $2k\theta > \sigma^2$ to ensure positivity.
- This is also an affine model and so
$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$
where $A(t, T)$ and $B(t, T)$ are deterministic functions of time.
- This was once the preferred term structure model as the short rate did not go negative but still exhibited mean reversion and had analytic formulas for the bond and option on bond prices. However, it has the same difficulties in fitting market date.

Improving Vasicek

- The main practical problem with CIR and Vasicek is the inability to recreate the initial term structure (or match the prices of all the zero-coupon bonds in the market). This can be relatively easily overcome by making the parameters time varying - allowing for a different parameter for each of the zero-coupon bond prices.
- The Hull and White (1990) is an extension to Vasicek so that now:

$$dr = [\theta(t) - a(t)r(t)] + \sigma(t)dW(t)$$

- In some ways, this is overkill having three time varying parameters when only one is required and so it is often presented in a somewhat reduced form:

$$dr = [\theta(t) - ar(t)] + \sigma dW(t) \tag{1}$$

- Fitting this model to observed market prices is theoretically possible (we will see how in Lecture 6). If the market bond prices are $P^M(0,T)$ and the associated instantaneous forward rate at time 0 for maturity T is $f^M(0,T)(= f^M(0;T,T+dt))$, so

$$f^M(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T}$$

Quiz

How could we estimate $f^M(0, T)$

- It comes directly from LIBOR rates.
- Assume a functional form from zero coupon bond prices.
- Very tedious calibration exercise.

Quiz

How could we estimate $f^M(0, T)$

- It comes directly from LIBOR rates.
- Assume a functional form from zero coupon bond prices. CORRECT
- Very tedious calibration exercise.

Calibration and formulas

- As a result, it is possible to show that

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

- This may not appear useful at the moment as if requires the estimation of the derivatives of the instantaneous forward rates. However, in calibrating our data we often assume a functional form for these instantaneous rates and so we can estimate the derivatives from there.

- From equation 1 we can use this result to rewrite $r(t)$ as:

$$\begin{aligned} r(t) &= r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma \int_s^t e^{-a(t-u)}dW(u) \\ &= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW(u) \end{aligned}$$

where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

Calibration and formulas

- We can think of $\alpha(t)$ as an adjustment to the drift (or expected level) of the short rate to ensure that the zero coupon bond prices are matched at each maturity. Note that, consistent with this interpretation, $\alpha(t)$ depends upon the instantaneous forward rate at different maturities.
- The idea of $\alpha(t)$ is seen more clearly when we return to the Hull-White model when constructing trinomial trees in Lecture 7.

Calibration and formulas

- As with the standard Vasicek model, the Hull and White model leads to normally distributed short rates, now the mean and variance are given by

$$
\begin{aligned}
E[r(t)|\mathcal{F}_s] &= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} \\
\mathrm{Var}[r(t)|\mathcal{F}_s] &= \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right]
\end{aligned}
$$

- As the rates are normally distributed then of course this can lead to negative $r(t)$ values (see PS4) and as we know the mean and variance of $r(t)$ we can calculate this probability. It can be large when current rates are low (like they are today).
- As mentioned, we do not have to actually back out the $\theta(t)$ values to price options/bonds, we will instead end up with modified bond and option formulas that depend on the market prices of bonds.

Formulas

- We can then derive the formulas for the zero coupon bond,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where now

$$\begin{aligned} A(t,T) &= \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{ B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2 \right\} \\ B(t,T) &= \frac{1}{a}\left[1 - e^{a(T-t)}\right] \end{aligned}$$

- Of course, when $t = 0$ then we already know the value of $P(0,T)$ as it is the market price of the bond, so this formula is only of use calculating future $P(t,T)$ which we need for LIBOR rates.

Formulas

- However, the bond option pricing formulas are more useful:

$$
\begin{aligned}
ZBC(t,T,S,X) &= P(t,S)N(d_1) - XP(t,T)N(d_2) \\
ZBP(t,T,S,X) &= XP(t,T)N(-d_2) - P(t,S)N(-d_1)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{P(t,S)}{XP(t,T)}\right)}{\sigma_p} + \frac{\sigma_p}{2} \\
d_2 &= d_1 - \sigma_p \\
\sigma_p &= \sigma\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}B(T,S)
\end{aligned}
$$

Quiz

Do these results look familiar?

- They are the Black-Scholes option formulas
- They are the Vasicek option formulas
- They are the CIR option formulas

Quiz

Do these results look familiar?

- They are the Black-Scholes option formulas
- They are the Vasicek option formulas CORRECT
- They are the CIR option formulas

Comments

- So far the Hull and White model does not seem to be a radical improvement on the Vasicek model. When valuing European derivatives then the term structure fitting parameter is not very useful as it is only the volatility that is important. It will be far more important when considering path dependent or American style derivatives, where the path taken by $r(t)$ (and thus the drift) is more important.
- The volatility structures available are the same as for the Vasicek model, and are driven by the two constant parameters a and σ. It is possible to try to select these parameters to match either market prices or the full term structure of volatility, although this may be challenging with only two parameters. We will see an example next.
- The analytic formulas, although useful are mainly used to calibrate the model to market prices. As we will see below, caps and European swaptions can be priced using variations of the zero coupon bond formula.

Cap prices

- Recall that a cap is simply a series of caplets, which themselves are an adaptation of a zero-coupon bond option, with the strike price being a function of the cap rate.
- For a given cap rate X with payments made at $\mathcal{T} = \{T_1, \ldots, t_n\}$ on a nominal payment N, then the payoff of each caplet is given by:

$$\begin{aligned}
Cpl(T_i, t_{i-1}, t_i, N, X) &= N\tau_{i-1} \max(L(T_{i-1}, T_i) - X, 0) \\
Cpl(T_{i-1}, T_{i-1}, T_i, N, X) &= NP(T_{i-1}, T_i) \max\left(\frac{1}{P(T_{i-1}, T_i)} - (1 + X\tau_{i-1}), 0\right) \\
&= N \max(1 - (1 + X\tau_{i-1})P(T_{i-1}, T_i), 0) \\
&= N(1 + X\tau_{i-1}) \max\left(\frac{1}{1 + X\tau_{i-1}} - P(T_{i-1}, T_i), 0\right)
\end{aligned}$$

thus,

$$Cpl(0, T_{i-1}, T_i, N, X) = N(1 + X\tau_{i-1})ZBP\left(0, T_{i-1}, T_i, \frac{1}{1 + X\tau_{i-1}}\right)$$

Cap prices

- Combining together gives

$$
\begin{aligned}
Cap(0,\mathcal{T},N,X) &= N\sum_{i=1}^{n}\left[P(0,T_{i-1})N(-d_2)-(1+X\tau_{i-1})P(0,T_i)N(-d_1)\right] \\
&= N\sum_{i=1}^{n}(1+X\tau_{i-1})ZBP\left(0,T_{i-1},T_i,\frac{1}{1+X\tau_{i-1}}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{P(0,T_i)(1+X\tau)}{P(T,t_{i-1})}\right)}{\sigma_p}+\frac{\sigma_p^i}{2} \\
d_2 &= d_1-\sigma_p^i \\
\sigma_p^i &= \sigma\sqrt{\frac{1-e^{-2k(T_{i-1})}}{2k}}B(T_{i-1},T_i)
\end{aligned}
$$

Swaption prices

- A swaption can be expressed an option on a coupon bearing bond. First, let us explicitly write down the value of the coupon bond option.
- A bond paying n coupons $c = \{c_1, \ldots, c_n\}$ at times $\mathcal{T} = \{T_1, \ldots, T_n\}$ where $T_1 > T$ where T is the expiry of the option. Then
$$CB(T, \mathcal{T}, c) = \sum_{i=1}^{n} c_i P(T, T_i)$$
if we have a short rate zero coupon bond pricing formula where $P(T, T_i) = \pi(T, T_i, r(T)$ then
$$CB(T, \mathcal{T}, c) = \sum_{i=1}^{n} c_i \pi(T, T_i, r(T))$$
- Now, consider a coupon bon option, with payoff
$$CBO(T, T, \mathcal{T}, c, X) = \max(X - CB(T, \mathcal{T}, c), 0).$$

Swaption prices

- Let r^* be the value of the spot rate at time T for which the coupon bearing bond equals the strike price and so,

$$\begin{aligned} CBO(T,T,\mathcal{T},c,X) &= \max\left(\sum_{i=1}^{n} c_i\pi(T,T_i,r^*(T)) - \sum_{i=1}^{n} c_i\pi(T,T_i,r(T)),0\right) \\ &= \sum_{i=1}^{n} c_i \max(\pi(T,T_i,r^*(T)) - \pi(T,T_i,r(T)),0). \end{aligned}$$

- Now let X_i be the time T value of a pure discount bond maturing at T_i when the spot-rate is r^* thus,

$$CBO(T,T,\mathcal{T},c,X) = \sum_{i=1}^{n} c_i \max(X_i - \pi(T,T_i,r(T)),0)$$

and at earlier times,

$$CBO(t,T,\mathcal{T},c,X) = \sum_{i=1}^{n} c_i ZBO(t,T,T_i,X_i)$$

Quiz

What is the strike price when we write a swaption as an option on a coupon bond?

- c_i
- $\sum_{i=1}^{n} c_i P(0, T_i)$
- 1
- Swap rate, X

Quiz

What is the strike price when we write a swaption as an option on a coupon bond?

- c_i
- $\sum_{i=1}^{n} c_i P(0, T_i)$
- 1 CORRECT
- Swap rate, X

Swaption prices

- Now consider a payer swaption with strike rate X, maturity T and nominal value N which gives the holder the right to enter into a swap at $T_0 = T$ where she pays X and receives LIBOR set in arrears. $c_i = X\tau_{i-1}$ and $c_n = 1 + X\tau_{n-1}$.
- $r^*(T)$ is now the value of the spot rate at time T for which

$$\sum_{i=1}^{n} c_i A(T, T_i) e^{-B(T,T_i) r^*(T)} = 1$$

where $c_i = X\tau_i$ for $i < n$ and $c_n = 1 + X\tau_i$ and then

$$X_i = A(T, T_i) e^{-B(T,T_i) r^*(T)}$$

then

$$PS(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i ZBP(t, T, T_i, X_i)$$

Calibration example

- Assume that we have the yield curve (i.e $P(0, T)$ for all T), and market cap prices, typically at-the-money, we can now try to calibrate the Hull and White model to this data.
- The cap prices are quoted as Black implied volatilities and so the first thing that we need to do is to calculate the cash prices of the caps. To do this, recall that a cap is a sum of caplets, and the Black volatility is the volatility of the forward rate for each of the caplets that make up the cap. So from this data we can use our Black formula for the caps to determine the cash value of the caps.
- The data is reported from June, 2011 on a 7% cap, and there are cap volatilities for the next three years. The first cap matures in November, 2011 (based upon the rate set in August, 2011), and pays out three-month LIBOR and the resets are at quarterly intervals. The price of zero coupon bonds are also given, this enables us to calculate the forward rates at time 0.

Calibration example

- We should now have all of the parameters required to calculate the Black value of the caps. Note that you need to be careful, and for the second cap, you cannot simply calculate the new caplet with the new volatility and add it to the old one. Instead, you must calculate all caplet values using the market implied volatility, as it is flat or cap volatility.
- Next we need to calculate the Hull and White cap prices. Assuming initial values for a and σ, then we can value the first cap/caplet using the above formula. Then we can move on to the next caplet, and the value of the cap maturing on Feb 2012, will be the sum of the first two caplets valued using the Hull and White formula (there is no need to recalculate the first caplet now as it is entirely defined by a, σ and other fixed parameters).

Calibration example

- Once we have the Hull and White value for each of the caps we can calibrate, we want to minimize

$$sse = \sum_i \left(\frac{Cap_i^B - Cap_i^{HW}}{Cap_i^B} \right)$$

the sum of the squared percentage errors.

Black Cap prices

Date	*26-Jun-11*
Cap rate	*7%*
Principal	*1*

Date	Reset date (T)	Cap Black vol ($\sigma^2 T$)	P(0,T)	Spot r(0,T)	Forward F(0; T, T+tau)	Cap maturity (T+tau)	cap price	Face value (1+Xt)
28-Jun-11	0		1					
28-Aug-11	0.17	0.1525	0.9898	0.060505828	0.066336399	0.43	**0.000116**	1.0179
28-Nov-11	0.43	0.1725	0.9733	0.063677458	0.072896048	0.68	**0.001342**	1.0179
28-Feb-12	0.68	0.1725	0.9555	0.066887288	0.078975454	0.93	**0.003718**	1.0175
28-May-12	0.93	0.175	0.937	0.069928116	0.080118947	1.19	**0.006474**	1.0179
28-Aug-12	1.19	0.18	0.9182	0.071949454	0.080905523	1.44	**0.009511**	1.0179
28-Nov-12	1.44	0.18	0.8996	0.073390792	0.080347234	1.70	**0.012419**	1.0179
28-Feb-13	1.70	0.18	0.8815	0.074315712	0.083362279	1.94	**0.015776**	1.0173
28-May-13	1.94	0.18	0.8637	0.075358179	0.08092444	2.20	**0.01885**	1.0179
28-Aug-13	2.20	0.1775	0.8462	0.075908882	0.080223998	2.46	**0.021713**	1.0179
28-Nov-13	2.46	0.175	0.8292	0.076273533	0.07993665	2.71	**0.024512**	1.0179
28-Feb-14	2.71	0.175	0.8126	0.076542896	0.082798674	2.96	**0.027777**	1.0173
28-May-14	2.96	0.175	0.7963	0.076995813	0.080236698	3.21	**0.030769**	1.0179
28-Aug-14	3.21	0.1725	0.7803	0.077188984	0.079826701	3.47	**0.033499**	1.0179
28-Nov-14	3.47		0.7647					

HW Cap prices

Calibration

alpha	-0.28745
sigma	0.009782

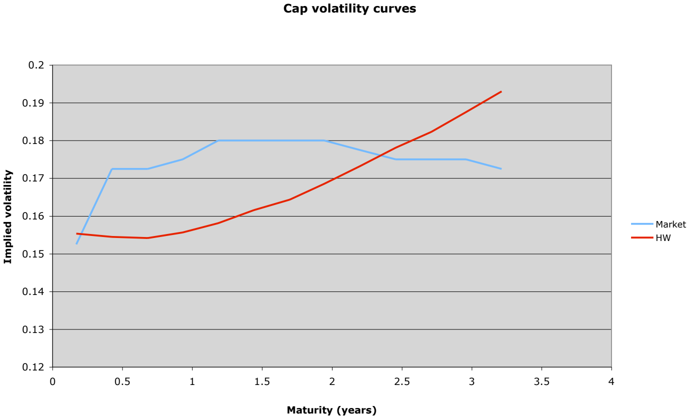
Hull White Caps / ***Implied vols from HW caps***

B(T,S)	sigma_p	d1	d2	HW	SE	*BLACK*	*IV*	*SE*
0.265176009	0.001094	0.841475	0.840381	0.000121	0.002242837	*0.00012112*	*0.15532*	*3.8751E-09*
0.265176009	0.0018	-0.40294	-0.40474	0.001229	0.007027578	*0.0012295*	*0.15445*	*4.6487E-09*
0.259201761	0.002314	-0.95082	-0.95314	0.003533	0.002457217	*0.00353345*	*0.15418*	*4.879E-09*
0.265176009	0.002877	-0.88037	-0.88325	0.006187	0.001959698	*0.00618745*	*0.15567*	*1.0965E-08*
0.265176009	0.003383	-0.80664	-0.81003	0.009062	0.002225059	*0.00906198*	*0.15811*	*7.3624E-09*
0.265176009	0.003886	-0.66562	-0.6695	0.01192	0.001612586	*0.01191981*	*0.16155*	*7.2573E-09*
0.256218212	0.00425	-0.7607	-0.76495	0.015253	0.001098765	*0.01525282*	*0.16433*	*7.9414E-09*
0.265176009	0.004908	-0.5556	-0.56051	0.018379	0.000622376	*0.01837925*	*0.16856*	*9.5744E-11*
0.265176009	0.005455	-0.46725	-0.4727	0.021501	9.4974E-05	*0.02150107*	*0.17321*	*2.3819E-08*
0.265176009	0.006026	-0.41045	-0.41648	0.024692	5.36008E-05	*0.02469159*	*0.17809*	*5.226E-10*
0.256218212	0.006403	-0.4818	-0.48821	0.028263	0.000306531	*0.02826295*	*0.1823*	*1.1217E-08*
0.265176009	0.007239	-0.35094	-0.35818	0.031723	0.000962651	*0.03172332*	*0.18753*	*3.3217E-09*
0.265176009	0.007909	-0.30759	-0.3155	0.035261	0.002767783	*0.03526094*	*0.19302*	*5.2511E-09*
				SUM	0.023431657		***SUM***	*9.1156E-08*

Calibration example

- Now we need to vary a and σ to find the best possible match between the prices. The values are $a = -0.2875$ and $\sigma = 0.009782$.
- This is interesting as, a is the speed of mean reversion which appears to be negative. That means that these cap prices require the spot rate to be mean fleeing rather than mean reverting to fit to the data. This is a surprisingly common problem with the Hull and White model as to fit the hump shaped volatility structure requires the rates to change very fast for short maturities.
- Another, final step can be to see what Black implied volatilities are obtained from the Hull and White calibration. To do this we need to find what volatilities would have to be entered into the Black formula to return the cap values from the Hull and White formula. This can be done by the use of solver, again by minimizing squared errors. This allows us to see the cap volatility curve created by the Hull and White model.
- We compare the market cap volatility curve with the one from the calibrated Hull and White model and note that it does not match very well from this set of data.

HW implied volatilities



HW comments

- Notice that with only one time varying constant, the cap prices are not matched precisely, as there are only two degrees of freedom to match the prices.
- The natural extension is to make a and σ both time varying also. This was proposed in the original paper. However, not all market quoted volatilities are useful as some may come from illiquid prices.
- Most importantly, the volatility structures implied by the HW model do not resemble those from the market and so over fitting would lead to an incorrect volatility structure. We will discuss this later in the course.
- To price derivatives where the path taken by $r(t)$ is important requires the use of numerical methods which we will study next class.

HW comments

- From our earlier example we saw the market caplet volatilities and the caplet volatilities implied by the Hull and White model.
- The market prices show the familiar hump shaped volatility that we have observed before.
- The model volatilities do not recreate the hump even though they give cap prices that match the market prices reasonably well.
- We will return to this idea later, to see which one-factor model can match the term structure of volatility the best.
- You will have a chance to practice a calibration on PS4.

Conclusion

- We have seen three standard one-factor models: Vasicek, CIR, Hull and White.
- Vasicek and CIR are not practically useful as they cannot be guaranteed to reproduce the current term structure of interest rates, so we need to at the very least introduce a time dependent parameter that will allow us to recreate the initial term structure.
- If we assume that short rates are normally distributed then we can write down analytic formulas for zero coupon bonds and also options on zero coupon bonds (and thus caps and European swaptions).
- It is then possible to try to calibrate our models to market prices. However, the limited numbers of parameters means that this is not always successful.