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The Term Structures of Oil Futures Prices

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EXECUTIVE SUMMARY

In recent years, there has been a massive development of derivative financial products in oil markets. The main interest came from large energy end-users who found in them a welcome opportunity to lock in fixed or maximum prices for their supplies over a period of time. Oil companies and oil traders were able to provide tailor-made swaps or options for the specific needs of the end-users. It all started with refined petroleum product swaps traded only for short maturities. Soon after, a new interest in long-term swaps and options on crude oil emerged. At the same time, several financial institutions launched oil-linked bonds and option warrants. Most of these long-term instruments were based on the price of the West Texas Intermediate (WTI) because of the liquidity of the futures and option contracts traded in the New York Mercantile Exchange (NYMEX).

Unlike swaps on refined petroleum products, long-term instruments on crude oil raise complicated problems of pricing and hedging. The pricing of these sophisticated instruments can always be reduced to the determination of the term structures of futures prices and volatilities. The NYMEX, with its WTI contracts, and the International Petroleum Exchange (IPE), with its Brent contracts provide satisfactory mechanisms for the discovery of short-term prices and volatilities. But the discovery of futures prices and volatilities for maturities exceeding exchange-traded maturities is far more problematic.

Market participants generally use basic extrapolations of the price and volatility curves. However, it would be preferable to construct a complete model of the term structures built on a few sensible hypotheses on oil markets. The resulting model could be used for longer maturities if validated by the movements of exchange-traded maturities.

The theory of storage provides an interesting approach as it introduces the important notion of convenience yield which accrues to the owner of a physical commodity but not to the owner of a contract for future delivery. This convenience yield on inventory can justify the seemingly abnormal situations of backwardation.

In this paper, we present a two-variable model of the term structures of futures prices and volatilities assuming that the spot and long-term prices of oil are stochastic, and are the main determinants of the convenience yield function. Although the resulting convenience yield is stochastic, the model admits an analytic formulation under some restrictions.

A simplified form of the model proved to be a rather good tool for the description of the futures prices and volatilities through time. With this simplified formulation, the term structures of futures prices and volatilities are entirely determined by a reduced set of state variables and parameters: the spot and long-term prices of oil, the volatilities and correlation of their stochastic processes and a single parameter β. This shows that the term structures of futures prices and volatilities can be reduced to a few parameters. It is then much easier to monitor these parameters in order to understand and follow in a satisfactory manner the movements of the term structures.

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1 INTRODUCTION

In recent years, there has been a massive development of derivative financial products in oil markets. The main interest came from large energy end-users who found in them a welcome opportunity to lock in fixed or maximum prices for their supplies over a period of time. Oil companies and oil traders were able to provide tailor-made swaps or options for the specific needs of the end-users. It all started with refined petroleum product swaps traded only for short maturities. Soon after, a new interest in long-term swaps and options on crude oil emerged. At the same time, several financial institutions launched oil-linked bonds and option warrants. Most of these long-term instruments were based on the price of the West Texas Intermediate (WTI) because of the liquidity of the futures and option contracts traded on the New York Mercantile Exchange (NYMEX).

Unlike swaps on refined petroleum products, long-term instruments on crude oil raise complicated problems of pricing and hedging. The pricing of these sophisticated instruments can always be reduced to the determination of the term structures of futures prices and volatilities. The NYMEX, with its WTI contracts, and the International Petroleum Exchange (IPE), with its Brent contracts, provide satisfactory mechanisms for the discovery of short-term prices and volatilities. But the discovery of futures prices and volatilities for maturities exceeding exchange-traded maturities is far more problematic.

Various simple methods can be advanced to achieve this goal. Market participants generally use basic extrapolations of the price and volatility curves. However, it would be preferable to construct a complete model of the term structures of prices and volatilities built on a few sensible hypotheses on oil markets. The resulting model could be used for longer maturities if validated by the price movements of exchange-traded maturities.

The theory of storage[1] provides an approach to the construction of such models as it explores the determinants of the term structure of the futures prices of a commodity in order to explain the seemingly abnormal situations of backwardation.[2] Indeed, the cost of carry of a commodity, including the cost of storage and the financial cost of holding this commodity, being always positive, one could expect the market to be in contango. The theory of storage introduces the important notion of convenience yield which accrues to the owner of the physical commodity but not to the owner of a contract for future delivery. This convenience yield on inventory can justify backwardation situations.

Brennan and Schwartz (1985) used this key notion in the valuation of commodity mines and Fama and French (1988) to understand the relation between business cycles and the behaviour of metal prices. More recently, Brennan (1989) proposed a one-variable oil futures pricing model using a convenience yield. Gibson and Schwartz (1990) went further with a two-variable model featuring stochastic spot price and convenience yield.

[1]For a complete discussion on price determinants, see Selected Writings of H. Working (1977).

[2]When the spot price of a commodity enjoys a premium over the futures prices, the market is said to be in backwardation. The opposite situation is called contango.

In this paper, we construct a two-variable model of the term structure of crude oil markets. We assume that the spot price of oil is the first state variable. The main contribution of this paper is to introduce a stochastic long-term price of oil as the second state variable. The convenience yield function is entirely determined by the spot price and the long-term price of oil and, therefore, is stochastic.

The rest of the paper is organized as follows. Section 2 describes some interesting features of price and volatility movements in the NYMEX WTI futures market. In particular, an useful analogy is drawn between the WTI futures market and the mechanics of a cantilever. In section 3, we turn to the literature on the term structure of interest rates. This literature is a valuable source of ideas and techniques on similar theoretical questions. Section 4 constitutes the main part of this paper as we progress gradually through the construction of the two-variable model of the term structure of prices and volatilities. In section 5, a simplified form of the model is tested on actual market data, and remarks are made on the dynamic properties of the model.

2 FEATURES OF PRICE FORMATION ON THE NYMEX WTI FUTURES MARKET

2.1 Analogy between the NYMEX WTI Futures Market and a Cantilever

The striking feature of movements in the price structure of NYMEX WTI futures contracts in recent years is the difference between the price behaviour of first nearby contracts and deferred contracts. The movements in the prices of the prompt contracts are large and erratic, while the prices of outer month contracts tend to remain relatively still. Any factor which affects the nearby contract price has an impact on the prices of subsequent contracts in the term structure that seems to decrease as maturity increases.

Figure 1 shows the responses of the different NYMEX WTI futures contracts to the changing situation in the physical market during July 1990 and the invasion of Kuwait by Iraq on 2 August, 1990. Between 20 June, 1990 and 10 October, 1990, the price of the first nearby contract moved from $ 15.30 to $ 38.69 per barrel and, in the same time, the price of the last futures contract in the term structure only moved from $ 19.78 to $ 24.99 per barrel.

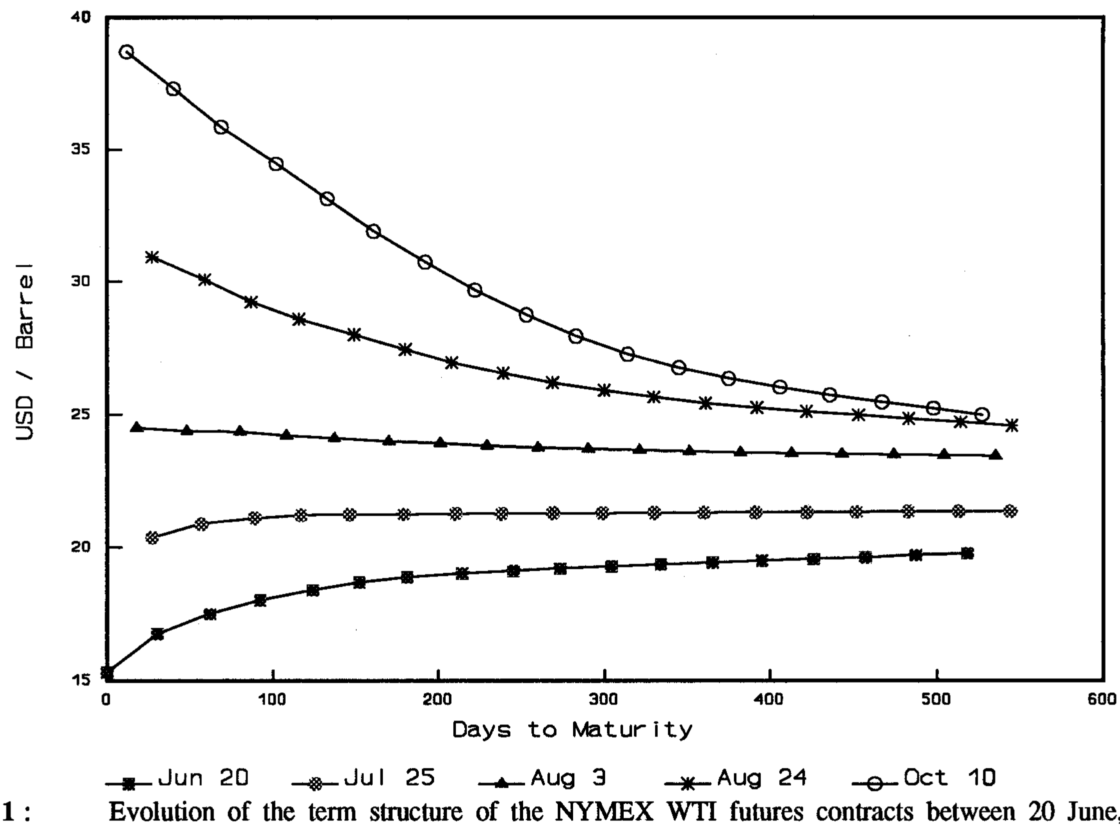

Figure 1 : Evolution of the term structure of the NYMEX WTI futures contracts between 20 June, 1990 and 10 October, 1990.

This phenomenon, to borrow an example familiar to engineers, can be likened to the effects of a force acting at the free end of a cantilever.[3] A concentrated force applied at the free end of the cantilever causes a deflection which is at its maximum at the free end and which declines to zero as we move towards the fixed end. The comparison of the WTI term structure with a cantilever[4] can be helpful in modelling the behaviour of the market.

To understand clearly the analogy, one has to view futures prices as single points of a continuous structure. Thus, on day one, the price of a futures contract that expires on day n is represented by the level of the beam at the distance n (measured in days) from the free extremity. On day two, the price is then represented by the level at the distance n-1 and so on. Hence the set of dots representing the term structure of futures prices moves every day along the beam. On the expiry date of the first nearby contract, the closing price of this contract corresponds to the deflected level of the cantilever at its free end, which in this analogy also represents the spot price for WTI since the futures price converges to the spot price at maturity. On the following trading day, the closest dot to the free end of the cantilever is around 30 days away. It is important to understand that the expiration of the first nearby contract does not induce any shift or discontinuity in the term structure, but simply causes the set of observation points to move forward. In this approach there is no discontinuity in the price series; a discontinuity only appears when the prices of a nearby contract are considered in isolation from the time-framework that leads the price movements from day one of the contract to its expiration.

The level at the free extremity of the cantilever corresponds to the spot physical price. The physical market is responsible for the price disturbance which affects the cantilever at its free end. In mechanics, the set of forces applied to a beam determines its shape in equilibrium. The WTI term structure of futures prices takes on a large variety of shapes, some of them displaying an inflexion point and a local minimum. Figure 2 illustrates different types of shapes observed on the market. It is difficult to provide a deterministic formulation for the set of forces which operate in the spot market. We shall, therefore, adopt a stochastic approach when studying the physical oil market and the movements of spot prices.

The analogy with a cantilever suggests that there exists a fixed oil price at the farthest end of the term structure. We consider that the level at which the cantilever is fixed corresponds to the price of a contract for delivery at the end of an infinite period of time; we call this price the long-term price of oil. As this type of contract is not traded, its price has to be inferred from the prices of traded futures contracts.

We do not propose to push the analogy any further as this may distort the analysis of the complex oil futures market. Nevertheless, the simple model of a cantilever subjected to a force that deflects its mean axis can be usefully kept in mind as it illustrates the idea of a long-term price of oil holding together a term structure subject to significant movements at the short end.

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[3]A cantilever is a beam fixed at one end to a column or a wall and hanging free at the other end.

[4]The term structure being in two dimensions only, time to maturity and price, the cantilever is supposed to remain in a vertical plan.

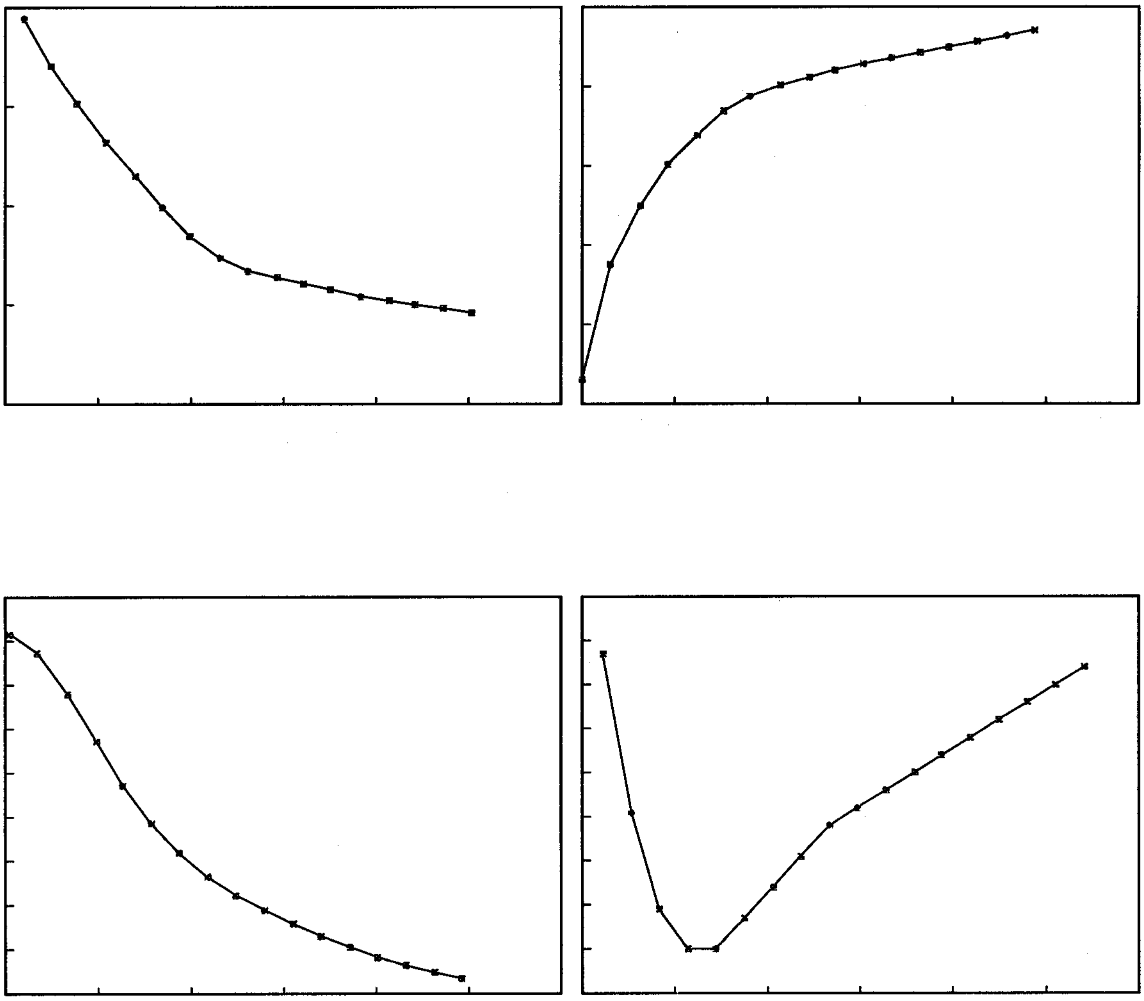

Figure 2 : Term Structures[5] of NYMEX WTI futures contracts on 2 January, 1990, 20 June, 1990, 14 November, 1990 and 26 February, 1991.

2.2 Other Properties of the NYMEX WTI Futures Market

The NYMEX WTI futures market has significantly expanded and developed since its creation in March 1983. Futures contracts are now traded for maturities up to 36 months forward compared to maturities of 6-9 months in earlier years. The traded volume has increased dramatically to a daily average of 80,000-100,000 lots.[6] The outer month contracts were not

[5]The horizontal axis corresponds to the time to maturity. The vertical axis corresponds to the price of the futures contracts. But more important is the overall shape of the price curves.

[6]One lot represents 1,000 barrels of crude oil. So the average volume is now 80,000,000-100,000,000 barrels a day.

actively traded in the early years of NYMEX, and consequently, their prices used to follow the price movements of the first month contracts. In recent years, outer month liquidity benefited from the development of over-the-counter (OTC) oil derivative markets. As liquidity and maturity increased, the prices of deferred contracts acquired more meaning and relevance. Market participants used these enhanced trading possibilities to assess and discern a long-term WTI price and engaged in attempts to stabilize it. As a result, the WTI futures market has become an important mechanism for oil price discovery not only in the short term but also in the long term.

This suggests that the analogy with the cantilever may hold only in the very short term. Since the prices of deferred futures contracts display some autonomy, the movements of the long-term oil price may also be stochastic. Indeed, if we enlarge the period of observation, we can observe price configurations with similar spot price and completely different term structure of futures prices. Figure 3 illustrates such situations observed over a longer period of time (between 16 June, 1989 and 25 July, 1990) than that shown in Figure 1.

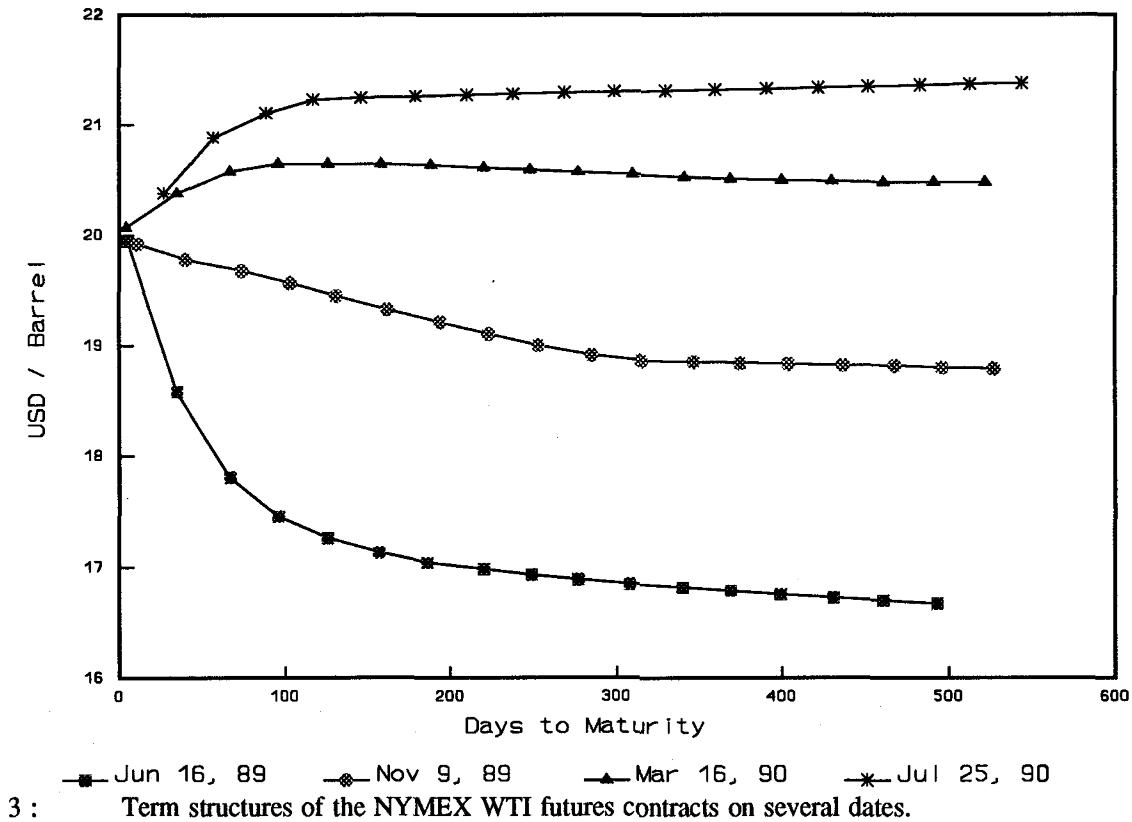

Figure 3 : Term structures of the NYMEX WTI futures contracts on several dates.

From now on, the long-term price of oil will be considered as stochastic. However, most of the time, the futures market features a decreasing pattern of volatilities along the term structure. Consequently, we can expect the volatility of the long-term price of oil to be much smaller than the spot price volatility.

Neither historical volatility nor implied volatility[7] are good measures of the instantaneous volatility in the term structure because both measures rely on the observation of futures contract prices over a period of time.[8] Since futures contracts are not continuously traded with respect to time to maturity, it is not possible to evaluate perfectly the instantaneous volatility at any point of the term structure. Figure 4 presents historical volatilities[9] of futures prices in a contango situation.

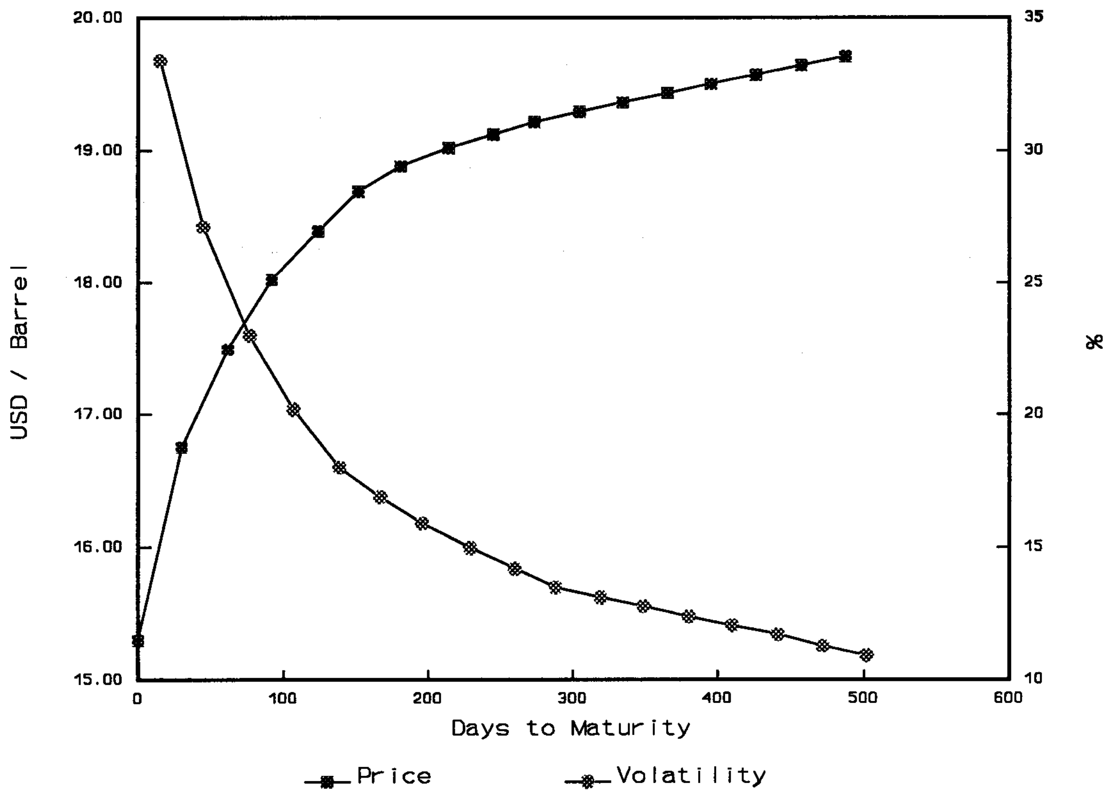

Figure 4 : Term structure of prices and volatilities of the NYMEX WTI futures contracts on 20 June, 1990.

[7]By implied volatility, we understand the volatility which can be inferred from the prices of NYMEX WTI option contracts and from a widely used option pricing model like the model of Black.

[8]Historical volatility measures the volatility of a determined futures contract price over a past period of time during which the time to maturity of this futures contract has been reduced by the length of this period. Implied volatility measures the expected volatility of a determined futures contract from the present time to the expiration of the option.

[9]Historical volatilities have been computed with the prices of the 20 trading days preceding 20 June, 1990. 20 trading days represent one month's time on average. On 20 June, 1990, we know the price and the historical volatility of all the futures contracts. Let us consider a futures contract expiring in n days on 20 June, 1990. The historical volatility gives a measure of its volatility over one month from the date when it was to expire in n+30 days. Therefore, in Figure 4 the historical volatility will arbitrarily represent the volatility of the term structure of futures prices at the point n+15.

The impact of information on price movements in oil markets partly depends on contents. The news that an oilfield will be closed for a three-month planned maintenance may not affect the term structure of futures prices in the same way as the unexpected discovery of a large oilfield.The former will certainly not affect the long-term equilibrium price of WTI; the latter may. It is thus important to treat the long-term equilibrium price of oil as a stochastic variable as this approach may capture the price effects of certain exogenous factors which have no perceptible influence on the short term physical market.

The analogy with a cantilever should not be taken too far. Although it may be tempting to compare the spreading of a deformation along a beam from its free towards its fixed extremity with the response of futures prices to a change in the spot price, this would be unwarranted as there is an important difference. The spreading of a deformation along the cantilever is not instantaneous, and, in any case, its velocity is lower than the velocity of light. This spreading resembles the transmission of a piece of news along a row of individuals. In the WTI futures market, assuming perfect economic efficiency, all futures contracts would react instantaneously to a change in the spot price. If the row of individuals mentioned above were taken to represent the market, we would say that in an economically efficient market they would all be receiving the piece of news at the same time. This means that we should not include waves in a dynamic of the term structure of futures oil prices. There are waves in the response to a sequential release of news if, for example, one part of the term structure begins to react to a second item at a time when another part is continuing to adjust to a first piece of news. This is ruled out by the assumption made here above.

This does not mean however that the interpretation of news and facts cannot change over time, or that markets react to information in an immediate and definitive way. Market participants may take some time in processing the implications of certain facts, and assessing their possible impact on prices. There are lags, rather than overlaps.

To exclude waves from the dynamic modelling of the term structure does not mean either that the settlement prices assessed by the NYMEX do not involve undulatory features. When spot price movements are erratic and large, market participants focus on the first futures contracts as they need to respond quickly to such changes. In such situations outer month contracts may not be traded at all in certain sessions and NYMEX will then report differentials based on trading of outer months on the previous day. These situations will be automatically corrected as soon as outer month contracts are traded again, but these delayed adjustments of deferred month contract prices resulting from the assessment system do cause undulations to appear.

2.3 Lessons from the NYMEX WTI Futures Markets

The analogy with a cantilever and the notion of a stochastic long-term price of oil throw light on some relevant features of the NYMEX WTI futures market and suggest some useful hypotheses for the modelling of a term structure.

The analogy suggests that an upward move of the spot price tends to increase the backwardation (or to reduce the contango as the case may be), and that a downward move increases the contango (or reduces the backwardation) because of the relative stillness of the long-term price of oil. This pattern is not systematic but frequently observed on the WTI futures market.[10] Going further along this line of reasoning, the apparent mean-reverting pattern of the term structure can be explained by this property of the long-term price combined with a cyclic motion of the spot price of oil. Indeed, historical series show successive periods of low and high prices. The long-term price of oil then provides a limit price for the change of state from backwardation to contango (or from contango to backwardation). The stochasticity of the long-term price allows the description of a much more complicated market than one reduced to a cantilever.

Although this long-term price is not the price of a traded asset, it could still be used as a state variable to capture exogenous actions on the term structure which do not affect the physical market. This variable, along with the spot price of oil could constitute a representative set of state variables to describe the motion of the whole term structure of futures prices and volatilities.

[10]Figure 1 shows that the movements of the spot price are likely to be responsible for the change from contango to backwardation.

3 THE TERM STRUCTURE OF INTEREST RATES

Though interest rates and oil markets are fundamentally different, the development of derivative securities on both raise the same theoretical questions. The same goal is pursued by researchers in both areas: the modelling of the term structure of prices and volatilities of the underlying products.

3.1 Review of the Literature

The literature on this subject has been abundant in the last fifteen years. Many avenues have been explored but two main approaches deserve attention.

A first approach, the traditional one, is to be found in the works of Vasicek (1977) and Cox, Ingersoll and Ross (1985), the references in this field. In the same vein, Brennan and Schwartz (1982) and Shaeffer and Schwartz (1984) designed two-variable models to describe the current yield curve of interest rates. Basically, these models assume a stochastic process followed by the short-term interest rate and derive a general formulation for the yield curve. The different works try to improve the assumptions so that the model can be fitted to market data as closely as possible.

These models provide a relatively simple and generally analytical formulation for the term structure of interest rates but not a general framework for the pricing of interest rate derivative securities. In particular, the volatilities of the different instruments may not be consistent.

A second approach, developed recently, constructs no-arbitrage models that fit perfectly the market data. Most of the works involve discrete-time one-factor models based on binomial or trinomial lattices. Ho and Lee (1986) first proposed such models providing an exact fit to the term structure of interest rates. Heath, Jarrow and Morton (1987) gave a general framework for this kind of model. Then Black, Derman and Toy (1990) and Hull and White (1990b) suggested one-factor models that fit not only the term structure of interest rates but also the structure of volatilities. Most recently, Hull and White (1990c) provided a general method to fit a model to market data including the current term structure of interest rates, the current volatilities of discount bond yields and the future volatility of short rate.

Basically, the method is to extract from market data the stochastic process of the short-term interest as implied in the term structure of prices and volatilities. Once the lattice has been constructed from market data, all kinds of options and interest rate derivative securities can be valued.

These models give a consistent pricing for different types of interest rate derivative securities at a given time but the estimation of their parameters and functions may display poor dynamic properties.

3.2 Comparison with Oil Markets

The literature on the term structure of interest rates provides several useful techniques to construct a term structure model for oil futures markets. The basic principles and methods used in the two approaches surveyed above remain valid for our purpose here.

Unfortunately, oil markets are running far behind interest rate markets in terms of sophistication and liquidity. The number of oil derivative securities remains relatively small and consequently, there is only a limited set of data available on these markets. NYMEX WTI futures contracts are traded up to 36 months, some contracts being hardly active and liquid, and represent the main source of data on the term structure of oil prices. NYMEX WTI option contracts are traded up to 9 months and consequently, the information set available on volatilities is even smaller than the one on prices.

Modelling the term structure of interest rate is a problem of interpolating data over a range of maturities. US Dollar bonds are traded up to 25 years. All kinds of prices and volatilities are accessible : spot prices, forward prices, spot volatilities, future spot volatilities, forward volatilities. The second approach briefly presented in Section 3.1 enables the construction of a lattice for a very long period of time and thus the pricing of all types of securities. Even if the construction of a lattice for oil markets is interesting, it does not provide the valuation of long-term oil-indexed derivative products because of the lack of data.

The problem is then to extract from the available information the determinants of longer-term prices and volatilities. However, the necessary information for this purpose may not be implied in the shorter-term markets and thus, one could doubt the relevance of trying to compute a price for a 7-year WTI swap based on eighteen price observations of futures contracts.

Yet, if we were able to construct a pricing function which fits as closely as possible to the market data and which is based on a reduced set of sensible hypotheses, we could then consider these hypotheses as relevant for the valuation of longer term instruments, provided that the behaviour of long-term prices and volatilities is consistent with our expectations.

In Section 4, we will construct different futures pricing models in order to describe as closely as possible the available market data.

4 MODELLING OF THE TERM STRUCTURE OF OIL FUTURES PRICES

4.1 The Role of the Convenience Yield

In this section, different futures pricing models will be examined in order to prove the need of the convenience yield.

4.1.1 A Futures Pricing Model with Cost of Carry

A first approach for deriving a very simple model for futures pricing is to assume that the futures price depends only on the spot price and the cost of carry of the physical oil. The physical oil is presumed to be a traded asset. Therefore, the expected growth rate of the spot price and the investor's attitude towards the spot price risk are irrelevant to the pricing of the futures contract. This important question, whether oil is or is not a traded security, will be discussed in Section 4.4.

We shall be using the following notations in this section:

S	Spot price
F	Futures price
t	Current time
T	Maturity time
τ	Time to maturity (τ=T-t)

We assume that the spot price of oil follows a diffusion process specified as:

$$dS = \mu(S)dt + \sigma(S)dz \tag{1}$$

where the functions μ(S) and σ(S) are the mean and standard deviation, respectively, of the instantaneous rate of growth of the spot price, and dz is a Wiener process.

In order to simplify this basic model, the functions μ and σ are considered independent of time. As the futures price F is a function of S and τ, we can use Itô's lemma to define its instantaneous price change as follows:

$$dF = F_S\, dS - F_\tau\, dt + \frac{1}{2} F_{SS}\, (dS)^2$$

Using expression (1) in the previous equation, we come to the following expression:

$$dF = [\mu\, F_S - F_\tau + \frac{1}{2}\sigma^2 F_{SS}]\, dt + \sigma\, F_S\, dz \tag{2}$$

We can simply rewrite this expression as follows:

$$dF = \alpha(S,\tau)\, dt + \gamma(S,\tau)\, dz \tag{3}$$

Then consider a portfolio consisting of one futures contract expiring in τ_1 and x futures contracts expiring in τ_2. We can choose x such that this portfolio is riskless. Thus its instantaneous return must equal zero (as no investment is required for futures contract). The no-risk and no-return conditions imply that the following system is verified for all couples (τ_1,τ_2):

$$\forall(\tau_1,\tau_2)$$

$$\gamma(S,\tau_1) + x\ \gamma(S,\tau_2) = 0$$
$$\alpha(S,\tau_1) + x\ \alpha(S,\tau_2) = 0$$

Hence there must exist a linear relationship between the drift and the variance of the futures contract:

$$\exists\lambda\ /\ \alpha(S,\tau) = \lambda(S)\ \gamma(S,\tau) \tag{4}$$

This relation expresses the fact that λ ,which is the market price per unit of spot price risk, is at most a function of S and is independent of τ. The value of λ is common to all futures contracts and all securities depending upon the spot price of oil S.

Now consider a portfolio consisting of one unit of physical oil and x futures contracts expiring in τ. Again, we can choose x such that this portfolio is riskless. But the holder of this portfolio has to carry the physical oil at a marginal cost of carry C_C. This cost of carry includes at least a cost of storage of the oil and is supposed to be constant and strictly positive. Therefore the instantaneous return of such a portfolio must be equal to the riskless interest rate r. Thus we have the following system:

$$\sigma + x\ \gamma = 0$$

$$\frac{\mu - C_C\ S + x\ \alpha}{S} = r$$

This system provides an expression for λ:

$$\lambda = \frac{\mu - (r+C_C)\ S}{\sigma} \tag{5}$$

Using the expression of λ and the values of α and γ from the identification of expressions (2) and (3) in the linear relationship (4), we can obtain the following partial differential equation governing the futures price:

$$(r+C_C)\ S\ F_S - F_\tau + \frac{1}{2}\ F_{SS}\ \sigma^2 = 0$$

subject to the initial condition:

$$F(S,0) = S$$

Assuming that the spot price of oil has a lognormal-stationary distribution, we can specify the form of the function σ[11]:

$$\sigma(S) = \sigma S$$

where σ is constant.

The partial differential equation becomes:

$$(r+C_C)\, S\, F_S - F_\tau + \frac{1}{2} F_{SS}\, \sigma^2 S^2 = 0 \tag{6}$$

subject to the same initial condition. Thus the futures price is given by:

$$F(S,\tau) = S\, e^{(r+C_C)\tau} \tag{7}$$

and is independent of the stochastic process of the spot price.

This basic model is built on the two assumptions that the futures price depends only upon the spot price of oil and that the structure of the futures prices reflects only the cost of carry of the physical oil. Even with some important restrictions like the constancy of the marginal cost of carry, we can draw some interesting and useful conclusions from this first model. Since the riskless interest rate r and the marginal cost of carry C_C are always positive, it is obvious from expression (7) that the price for delivery at infinite time is infinite and that only a term structure in contango can be described while backwardation is precluded. Unfortunately the actual market is more complicated and changing than that and for instance, NYMEX WTI futures contracts are in backwardation most of the time. So, we must improve this very simple pricing model in order to capture the full variety of observed WTI term structures.

In Section 4.1.2, we point out that the cost of carry is not sufficient for an extensive description of the term structure of futures prices and introduce the notion of convenience yield.

4.1.2 Introducing the Convenience Yield

When a futures market is in backwardation, one may wonder why some people hold physical stocks of oil and support a positive cost of carry since they could buy oil for a future maturity at a discount to the spot price. The reply lies in the fact that in some circumstances, there is

[11]There is no need to specify the function μ. Since the market price of spot price risk is known, the expected rate of growth of the spot price is unconnected to the pricing of a futures contract.

a convenience in holding physical oil instead of holding contracts for future oil, which results in an additional yield for the owner. This yield is called convenience yield.[12]

For instance, when the general level of stocks is low, people carrying physical oil may profit from an unexpected surge in demand, or from a sudden disruption of supply. The convenience of holding stocks when a local shortage occurs can be huge for an oil-dependent industry. This kind of situation may be more common for petroleum products which follow clear cycles of successive high-demand and low-demand periods. The low elasticity of not very substitutable products such as gasoil and gasoline led to the steep backwardation observed during the winter 1990-1991 for European gasoil and during the summer 1989 for American gasoline.

The theory of storage postulates an inverse relationship between the level of inventories and the net convenience yield. But one has to be careful when assessing the level of inventories. Different kinds of inventories can have disparate effects on the term structure of oil prices depending on the identity of the holder and the degree of availability. In February 1991, aggregate world crude oil stocks were at high levels not seen since 1981. Still the first futures contracts of both the NYMEX WTI futures market and the IPE crude oil futures markets were in steep backwardation. In fact, commercial stocks were very low compared to standard levels but producer states were holding large floating stocks close to the consumer centres. As a result of this situation, supply was tight and backwardation was present in spite of the large stocks.

The term structure reflects not only the current convenience yield but also its expected changes. If market participants expect a modification in the supply/demand pattern, backwardation can occur whatever the levels of inventories. The backwardation observed on both the NYMEX and the IPE markets at the beginning of 1991 was also explained by the fear of market participants that the Gulf war could cause important and long-lasting disruptions to crude oil supplies from the area.

Thus, a large convenience yield (i.e. backwardation) must simply be interpreted as the preference for the present time whatever the reasons are.

In Section 4.1.3, we use the notion of convenience yield in order to improve the properties of the simple model constructed in Section 4.1.1.

4.1.3 A Futures Pricing Model with Convenience Yield

The discussion in Section 4.1.2 thus allows us to introduce the notion of convenience yield in a futures pricing model. In the following, the convenience will be viewed as a continuous dividend yield accruing to the owner of physical oil but not to the owner of futures contracts.

[12] "The convenience yield of a commodity is defined as the flow of services which accrues to the owner of a physical inventory but not to the owner of a contract for future delivery.", Brennan (1989).

Let us note C_Y the marginal convenience yield (excluding the cost of storage and the financial cost) and suppose that it remains constant. With the notations of the previous section, the system expressing the return of the riskless portfolio changes to:

$$\sigma + x\,\gamma = 0$$

$$\frac{\mu + (C_Y - C_C)\,S + x\,\alpha}{S} = r$$

The same technique as before leads to the following formulation of the futures price:

$$F(S,\tau) = S\,e^{(r+C_C-C_Y)\tau} \tag{8}$$

This enhanced model is still simple and involves some very restrictive assumptions like the constancy of the convenience yield. Nevertheless, it can now describe backwardation states as well as contango states, depending on the sign of $r+C_C-C_Y$. The quantity $r+C_C-C_Y$ represents the difference between the cost of storage plus the financial cost of holding the physical oil and the convenience yield. Thus we find the intuitive result that if the convenience of holding physical oil is lower (greater) than the induced costs, the market must be in contango (backwardation).

The convenience yield must not be seen as a magic feature which comes from nowhere. It simply results from the identity between the term structure observed on the actual market and the one expected from a cost of carry theory. But it constitutes a crucial and useful notion for a realistic modelling of the term structure of commodities prices.

Remembering the analogy between the NYMEX WTI futures market and a cantilever, our simple model presents some inadequate properties.

First, if the market is in contango (backwardation), the limit value of the futures price for an infinite maturity is infinite (zero). The behaviour of the model for infinite maturity is not satisfactory. Moreover, there is a discontinuity when changing from one state to the other. None of those properties are consistent with the existence of a fixed long-term price of oil.

$$\textit{Contango} : r+C_C-C_Y > 0 \Rightarrow \lim_{\tau \to +\infty} F(S,\tau) = +\infty$$

$$r+C_C-C_Y = 0 \Rightarrow \lim_{\tau \to +\infty} F(S,\tau) = S$$

$$\textit{Backwardation} : r+C_C-C_Y < 0 \Rightarrow \lim_{\tau \to +\infty} F(S,\tau) = 0$$

Secondly, in contango, the first derivative of the futures price with respect to the spot price is always greater than 1. This contradicts the attenuation pattern of the deformations along the beam. Indeed, simple differentiation of expression (8) with respect to S gives :

$$\frac{\partial F}{\partial S}(S,\tau) = e^{(r+C_C-C_Y)\tau}$$

$$Contango : r+C_C-C_Y > 0 \Rightarrow \frac{\partial F}{\partial S}(S,\tau) > 1$$

Thirdly, the expression of the futures prices provides the volatility of the futures price. It is simple to prove from expression (8) that this volatility is equal to the volatility of the spot price. Thus, this feature of the model is inconsistent with the pattern of decreasing volatilities.

$$\gamma(S,\tau) = \sigma\ S\ F_S$$
$$= \sigma\ F$$

In Section 4.2, we examine a two-variable futures pricing model featuring a stochastic convenience yield proposed by Gibson and Schwartz.

4.2 A Futures Pricing Model with a Stochastic Convenience Yield

Gibson and Schwartz (1990) developed a two-factor model for pricing financial and real assets contingent on the price of oil.

The two state variables are the spot price of oil S and the instantaneous convenience yield δ. They assume that the spot price of oil has a lognormal-stationary distribution and that the convenience yield follows a mean reverting process. This joint diffusion process is specified as follows:

$$\begin{aligned} dS/S &= \mu\ dt + \sigma_1\ dz_1 \\ d\delta &= k(\alpha-\delta)\ dt + \sigma_2\ dz_2 \end{aligned} \tag{9}$$

dz_1 and dz_2 are two correlated Wiener processes such that:

$$dz_1\ dz_2 = \rho\ dt$$

where ρ denotes the correlation coefficient between the two Wiener processes.

Assuming that the futures price of oil depends only upon the spot price S, the convenience yield δ and the time to maturity τ, in the absence of arbitrage, the futures price must satisfy the following partial differential equation:

$$(r-\delta)SF_S + (k(\alpha-\delta)-\lambda\sigma_2)F_\delta - F_\tau + \frac{1}{2}\sigma_1^2S^2F_{SS} + \frac{1}{2}\sigma_2^2F_{\delta\delta} + \rho\sigma_1\sigma_2SF_{S\delta} = 0 \tag{10}$$

subject to the initial condition:

$$F(S,\delta,0) = S$$

λ is defined as the market price per unit of convenience yield risk and is at most a function of S, δ and t but not of τ. This parameter measures the excess return required by investors for bearing the risk of convenience yield. It appears because the convenience yield is not the price of a traded asset.

The partial differential equation (10) has no analytical solution. Therefore, the authors use a numerical technique to compute the futures price of oil. They estimate the parameters of the joint diffusion processes σ_1, k, α, σ_2 and ρ, and then the market price of convenience yield risk λ.

The underlying assumptions of this model and its features require several comments. First, the mean reverting pattern of the convenience yield is specified independently of the spot price of oil. This means that the spot price being stable, the convenience yield has a strong tendency to come back to its long-term mean. Although it is difficult to observe a period of stability for the spot price of oil, the parallelism with the cantilever hints at a different behaviour of the convenience yield. Indeed, the observation of the NYMEX WTI futures market suggests that the mean reverting pattern of the convenience yield should rather be explained by the successive periods of rises and falls of the spot price of oil, the convenience yield rising and falling with the spot price.

Secondly, the choice of proxies for the spot price and the convenience yield can be discussed. The spot price is taken as that of the first nearby futures contract. The time before delivery of the first nearby futures contract changes over time. It can be as low as a few days but also as long as a month. This method leads to large discontinuities in the observation of the spot price of oil. From one week to the other (the authors use weekly data), the change of the variable is explained not only by the change in the true spot price but also by the change in the time before delivery. This impact can be especially important in a situation of strong contango or backwardation, and also when an expiry occurs during the week.

The instantaneous convenience yield has been computed from a formulation of the futures price derived under no convenience yield uncertainty, namely:

$$F(S,\tau) = Se^{(r-\delta)\tau}$$

The authors do not give any information about the sensitivity of their results to that procedure. It seems rather odd to input values computed from a non-stochastic convenience yield model into a model with stochastic convenience yield. Moreover, the use of nearby contracts to determine the convenience yield induces discontinuities in the observation of the convenience yield.

Thirdly, this model does not provide any closed-form solution. Thus, it is rather difficult to implement and use for the valuation and the hedging of oil derivative securities.

In Section 4.3, we present a two-variable futures pricing model introducing a different pair of state variables : the spot price of oil and the long-term price. The convenience yield is still included in the model but not as a state variable.

4.3 A Futures Pricing Model with a Long-Term Price of Oil

In order to improve the simple model constructed in Section 4.1, we introduce a second state variable. We make the assumption that there is a price of oil L for delivery at infinite time. We call L the long-term price of oil. This variable is not the price of a traded asset but this hypothesis is consistent with the model of the cantilever where L represents the elevation at the fixed extremity. Our purpose here is not to discuss the existence of such a long-term price but to prove its usefulness in modelling the term structure of oil prices. However, we shall see that the assumption that the price of oil for delivery at infinite time is finite involves strong implications about the behaviour of market participants towards the long-term price risk.

We have two state variables: S, the spot price of oil and L, the long-term price of oil. We make the assumption concerning their joint stochastic process that both S and L follow a diffusion process specified as:

$$\begin{aligned} dS &= \mu_S(S,t)\, dt + \sigma_S(S,t)\, dz_1 \\ dL &= \mu_L(L,t)\, dt + \sigma_L(L,t)\, dz_2 \end{aligned} \qquad (11)$$

dz_1 and dz_2 are two correlated Wiener processes such that:

$$dz_1\, dz_2 = \rho(t)\, dt$$

where ρ denotes the correlation coefficient between the two Wiener processes.

In order to allow time-dependent parameters, we make the assumption that the futures price depends upon S, L, t and T (and not only on τ). We have the two following limit conditions for the futures price:

$$F(S,L,T,T) = S$$

$$\lim_{T \to +\infty} F(S,L,t,T) = L$$

showing that the price of a futures contract with immediate delivery must be equal to the spot price and that L is the price of oil with delivery at infinite time.

By using Itô's lemma, we can derive the instantaneous change of the futures price as follows:

$$dF = F_S dS + \frac{1}{2}F_{SS}(dS)^2 + F_L dL + \frac{1}{2}F_{LL}(dL)^2 + F_{SL}dSdL + F_t dt$$

Replacing dS and dL by their expressions in (11), we can write the instantaneous change:

$$dF = \left[\mu_S F_S + \frac{1}{2}\sigma_S^2 F_{SS} + \mu_L F_L + \frac{1}{2}\sigma_L^2 F_{LL} + \rho\sigma_S\sigma_L F_{SL} + F_t\right]dt + \sigma_S F_S dz_1 + \sigma_L F_L dz_2 \qquad (12)$$

We can rewrite this expression in a more general form:

$$dF = \alpha(S,L,t,T)dt + \gamma_1(S,L,t,T)dz_1 + \gamma_2(S,L,t,T)dz_2 \qquad (13)$$

By constituting a portfolio with one futures contract expiring at T_1, x futures contracts expiring at T_2 and y futures contracts expiring at T_3, and choosing x and y such that this portfolio is riskless, we can prove that there does exist a linear relationship between the functions α, γ_1 and γ_2 which is independent of T:

$$\exists\ \lambda_S, \lambda_L\ /$$
$$\alpha(S,L,t,T) = \lambda_S(S,L,t)\gamma_1(S,L,t,T) + \lambda_L(S,L,t)\gamma_2(S,L,t,T) \qquad (14)$$

λ_S (λ_L) is the market price per unit of spot price (long-term price) risk.

From now on, to simplify the notations, we enlarge the notion of convenience yield so that it includes all yields and costs resulting from holding physical oil except for the financial costs. The marginal convenience yield is still viewed as a dividend accruing continuously, and is noted C_Y. The convenience yield C_Y is a function of S, L and t.

Because the spot price is the price of a traded asset and the long-term price is the limit price of traded assets when maturity goes to infinity, we can express λ_S and λ_L.

By constituting a portfolio with one unit of physical oil and two positions in futures contract such that the resulting portfolio is riskless, it is easy to show:

$$\lambda_S = \frac{\mu_S + (C_Y - r)S}{\sigma_S} \qquad (15)$$

Given the specifications of L, we have:

$$\lim_{T \to +\infty} \alpha(S,L,t,T) = \mu_L(L,t)$$

$$\lim_{T \to +\infty} \gamma_1(S,L,t,T) = 0$$

$$\lim_{T \to +\infty} \gamma_2(S,L,t,T) = \sigma_L(L,t)$$

By taking the limit of the linear relationship (14) when T goes to infinity, we obtain the following expression:

$$\lambda_L = \frac{\mu_L}{\sigma_L} \tag{16}$$

This shows that the limit condition on the futures prices when maturity goes to infinity is very restrictive.[13] It implies the relation (16) which expresses that the variable L is considered to be the price of a traded asset although this is clearly not the case. Thus, market participants are assumed to have the same behaviour towards the long-term price risk as if a contract for delivery at infinite time was traded. Another consequence of this hypothesis is that the expected rate of growth of L is unconnected to the pricing of futures contracts.

Using expressions (15) and (16) and the values α, γ_1 and γ_2 from the identification of expressions (12) and (13), we come to the following partial differential equation satisfied by the futures price:

$$(r-C_Y)SF_S + \frac{1}{2}\sigma_S^2 F_{SS} + \frac{1}{2}\sigma_L^2 F_{LL} + \rho\sigma_S\sigma_L F_{SL} + F_t = 0$$

subject to the following conditions:

---

[13]Under the same specifications of the stochastic processes and the form of the convenience yield function which will be adopted, a more general model with no limit condition as maturity time goes to infinity can be derived. It is then possible to prove that the convergence of the futures price for infinite maturity time is equivalent to the following inequality:

$$\mu_L - \lambda_L\sigma_L \leq 0$$

Let us note λ^* the value of λ for which there is equality in the above inequality. λ^* represents the market price of long-term price risk if this kind of futures contract was actually traded. Therefore, the long-term price of oil has a finite value if, and only if, the market price of long-term price risk is inferior or equal to λ^*.

$$F(S,L,T,T) = S$$

$$\lim_{T \to +\infty} F(S,L,t,T) = L$$

Assuming that both the spot price and the long-term price of oil have a lognormal-stationary distribution, we can specify the form of the variance of their stochastic processes[14]:

$$\sigma_S(S,t) \equiv \sigma_S(t)S$$

$$\sigma_L(L,t) \equiv \sigma_L(t)L$$

The partial differential equation then becomes:

$$(r-C_Y)SF_S + \frac{1}{2}\sigma_S{}^2S^2F_{SS} + \frac{1}{2}\sigma_L{}^2L^2F_{LL} + \rho\sigma_S\sigma_L SLF_{SL} + F_t = 0 \tag{17}$$

subject to the same limit conditions.

The futures price then depends on the precise form of the marginal convenience yield function C_Y. We consider the following function for C_Y:

$$C_Y(S,L,t) = \beta(t) \ln \frac{S}{L} + \delta(t) \tag{18}$$

This form of the marginal convenience yield is motivated by the flexibility it affords and also because, under this specification and with constant parameters, there exists a closed-form solution to the partial differential equation. The conditions associated with the partial differential equation (17) induce conditions on the functions ß and δ which will be detailed later. We assume that the function ß is uniformly positive: the higher the ratio S over L, the greater the marginal convenience yield. This ratio is supposed to be the main determinant of the convenience yield function and consequently, of the global shape of the term structure of prices and volatilities. The underlying idea is that the term structure depends largely on the relative level of the spot price.

The form of the convenience yield function being specified, the futures prices are then fully determined by the equation (17) and the associated conditions.

Now consider the futures price function:

$$F(S,L,t,T) = A(t,T)S^{B(t,T)}L^{1-B(t,T)} \tag{19}$$

[14]There is no need to specify the functions μ_S and μ_L. See note 1.

This function satisfies the partial differential equation (17) and the associated conditions when:

$$A_t + (r-\delta(t))AB + \frac{1}{2}\nu(t)AB(B-1) = 0$$
$$B_t - \beta(t)B = 0 \tag{20}$$

where:

$$\nu(t) = \sigma_S(t)^2 + \sigma_L(t)^2 - 2\rho(t)\sigma_S(t)\sigma_L(t)$$

and with:

$$A(T,T) = 1 \qquad \lim_{T\to+\infty} A(t,T) = 1$$

$$B(T,T) = 1 \qquad \lim_{T\to+\infty} B(t,T) = 0$$

We shall see later that the functions T $\to$ A(0,T) and T $\to$ B(0,T) are defined from the term structure of futures prices and volatilities.[15] Then, we can determine A(t,T), B(t,T), ß(t) and δ(t) in terms of A(0,T), B(0,T) and υ(t).

By differentiating the equations (20) with respect to T, we come up with the following solutions for A and B:

$$B(t,T) = \frac{B(0,T)}{B(0,t)}$$

$$\ln A(t,T) = \ln A(0,T) - B(t,T)\ln A(0,t) \tag{21}$$
$$- \frac{1}{2} B(t,T)\,[B(t,T)-1]\,B(0,t)^2 \int_0^t \frac{\nu(u)}{B(0,u)^2}du$$

Substituting solutions (21) into equations (20), we obtain the parameter functions of the marginal convenience yield:

[15]Hull and White (1990b) use a similar technique for the pricing of interest-rate derivative securities.

$$\beta(t) = -\frac{B_T(0,t)}{B(0,t)}$$

$$\delta(t) = r - \frac{A_T(0,t)}{A(0,t)} - \beta(t) \ln A(0,t) + \frac{1}{2} B(0,t)\, B_T(0,t) \int_0^t \frac{\nu(u)}{B(0,u)^2}\, du \tag{22}$$

If we denote by $\sigma_F(t,T)$ the volatility at time t of the futures contract expiring at T, the variance of the instantaneous change of the price of this futures contract is:

$$\begin{aligned} var\ (dF) &= \sigma_F^2 F^2 dt \\ &= \left[\sigma_S^2 B^2 + \sigma_L^2 (1-B)^2 + 2\rho\sigma_S\sigma_L B(1-B)\right] F^2 dt \end{aligned} \tag{23}$$

Equation (23) enables us to compute B(0,T) for all T from the current term structure of futures price volatilities and from the parameters of the joint stochastic process of S and L: σ_S, σ_L and ρ. Equation (19) enables us to determine A(0,T) from B(0,T), the spot and long-term prices and the current structure of futures prices. Assuming that the function ν is constant through time, it is possible to compute the term structure of the marginal convenience yield implied in the market data from (22).

This approach is similar in its techniques and purposes to the second approach in interest rate markets presented in Section 3.1. Instead of extracting the implied stochastic process of the spot price of oil (drift and volatility), we derive the implied time-dependent convenience yield function. Then, all kinds of securities with maturity for which the convenience yield function is known can be valued in a consistent framework of prices and volatilities.

The central question pointed out by Hull and White (1990b) is whether the functions A and B estimated by the above techniques remain unchanged over time. In other words, is it possible to rely on the functions constructed at a certain time to describe the future term structures of prices and volatilities? Hull and White note that developing a model which describes the motion of the term structures of futures prices and volatilities with satisfactory accuracy is a different goal than developing a model that adequately values most of the derivative securities.

Therefore, if valuations of a large panel of oil derivative securities are to be achieved, a pricing model based on the general form of the functions ß and δ is required. This model will provide price and volatility consistency through the panel of securities. However, it will principally remain a static description of the term structures with poor dynamic properties. Indeed, the functions A and B (or ß and δ) appear to lack stability over time. If the changes in these functions are generally small from one day to the next, the functions A and B show significant modifications after a few days.

In order to obtain a model with adequate dynamic behaviour, we need to reduce the number of degrees of freedom of the parameters. In practice, the functions ß and δ will take constant or very simple forms. The model will then lose its abilities to fit exactly the market prices and volatilities in exchange for enhanced dynamic properties.

If the functions ß and δ are known, we can determine the functions A and B directly from equations (20) and their associated conditions:

$$B(t,T) = \exp\left[-\int_t^T \beta(u)du\right]$$
$$\ln A(t,T) = \int_t^T \left[[r-\delta(u)]B(u,T) + \frac{1}{2}\nu(u)B(u,T)[B(u,T)-1]\right]du \tag{24}$$

Then, the following conditions on the functions ß and δ arise:

$\forall t$

$$\lim_{T\to+\infty} \int_t^T \beta(u)du = +\infty$$
$$\lim_{T\to+\infty} \int_t^T \left[[r-\delta(u)]B(u,T) + \frac{1}{2}\nu(u)B(u,T)[B(u,T)-1]\right]du = 0 \tag{25}$$

If all parameters are supposed constant, expressions (24) give the formulation of the futures price:

$$F(S,L,\tau) = A(\tau)\, S^{B(\tau)}\, L^{1-B(\tau)}$$
$$A(\tau) = \exp\left[\frac{\nu}{4\beta}\left(e^{-\beta\tau}-e^{-2\beta\tau}\right)\right] \tag{26}$$
$$B(\tau) = e^{-\beta\tau}$$

The first condition in (25) is realized since ß is constant and positive. The second condition results in the following relation:

$$\delta = r - \frac{\nu}{4}$$

In that case, the term structures of futures prices and volatilities are entirely defined by the set of variables and parameters: S, L, σ_S, σ_L, ρ and ß. The limit condition is very strong since it reduces the set of dependent parameters and fixes the value of the constant component of the marginal convenience yield. It is worth noting that the value of δ removes the interest rate r in the expression of the futures price.

Although this model can describe several kinds of term structures of futures prices, it can be improved substantially by allowing the constant component of the convenience yield to be time dependent. The form of this term is not left completely opened as in the general formulation (18), but specified as follows:

$$C_Y(S,L,t) = \beta \ln \frac{S}{L} + \delta_\infty + \theta e^{-\eta t} \tag{27}$$

where ß, θ and η are constant and:

$$\delta_\infty = r - \frac{\nu}{4}$$

This form of the convenience yield function introduces more flexibility and enhances the properties of the model. The new term can be considered as the result of a shock due to the physical market which happened at time t=0. The parameters θ and η can be viewed as the amplitude and the inverse of a characteristic time of this shock. Indeed, market participants can still have a belief in the long-term price of oil, but the market for prompt deliveries can be affected by a short-term and non-lasting situation.

Under these specifications, conditions (25) are both realized and expressions (24) lead to the following formulation:

$$\begin{aligned}
F(S,L,t,T) &= A(t,T)\, S^{B(t,T)}\, L^{1-B(t,T)} \\
A(t,T) &= \exp\left[\frac{\nu}{4\beta}\left(e^{-\beta(T-t)} - e^{-2\beta(T-t)}\right) - \frac{\theta}{\beta-\eta} e^{-\eta T}\left(1 - e^{-(\beta-\eta)(T-t)}\right)\right] \qquad (28) \\
B(t,T) &= e^{-\beta(T-t)}
\end{aligned}$$

Without being as complex as the general model with time-dependent parameters, this model can generate several interesting curves for the term structures of prices and volatilities. It offers enough flexibility to accurately describe disparate situations. Figures 5 and 6 show normal backwardation and contango situations described by the model. Figures 7 and 8 show the impact of a short-term effect on backwardation and contango situations. Figure 9 presents several term structures of futures price volatilities which can be obtained depending on the correlation coefficient between the spot price and the long-term price. The range of volatility curves produced by models (26) and (28) could be significantly extended by allowing the function ß to have a similar form to the function δ specified in (27). This would enable short-term effects on the term structure of futures price volatilities. The instantaneous futures price volatility is only determined by the value at time t of the parameter functions σ_S, σ_L and ρ but not by their future evolutions.

The models (26) and (28) correspond to the first approach in interest rate markets described in Section 3.1. These models cannot fit perfectly the observed structures of prices and volatilities but may be used in a satisfactory way to extrapolate futures prices and volatilities beyond the exchange-traded maturities. The underlying hypothesis, namely the form of the convenience yield function, can be verified closely enough on short-term data to hold for the valuation of long-term instruments.

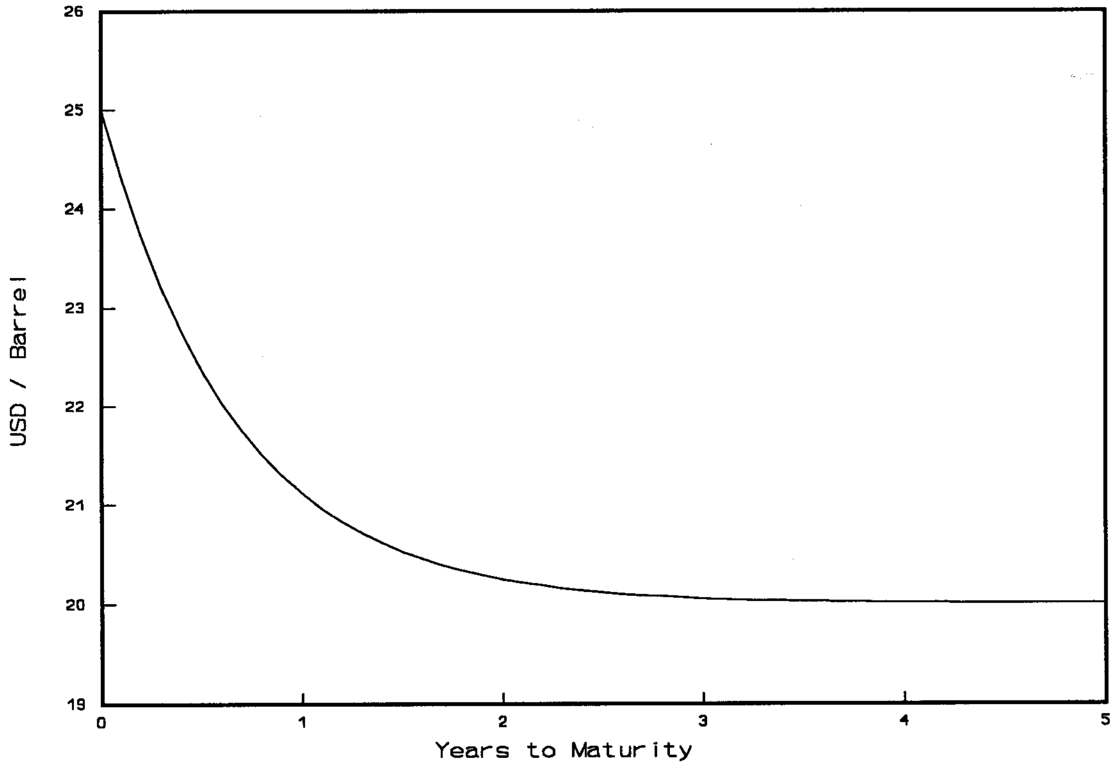

Figure 5 : Backwardation without any short-term effect. S=25, L=20, ν=0.158 (σ_S=40%, σ_L=10%, ρ=0.15), ß=1.5, θ=0.

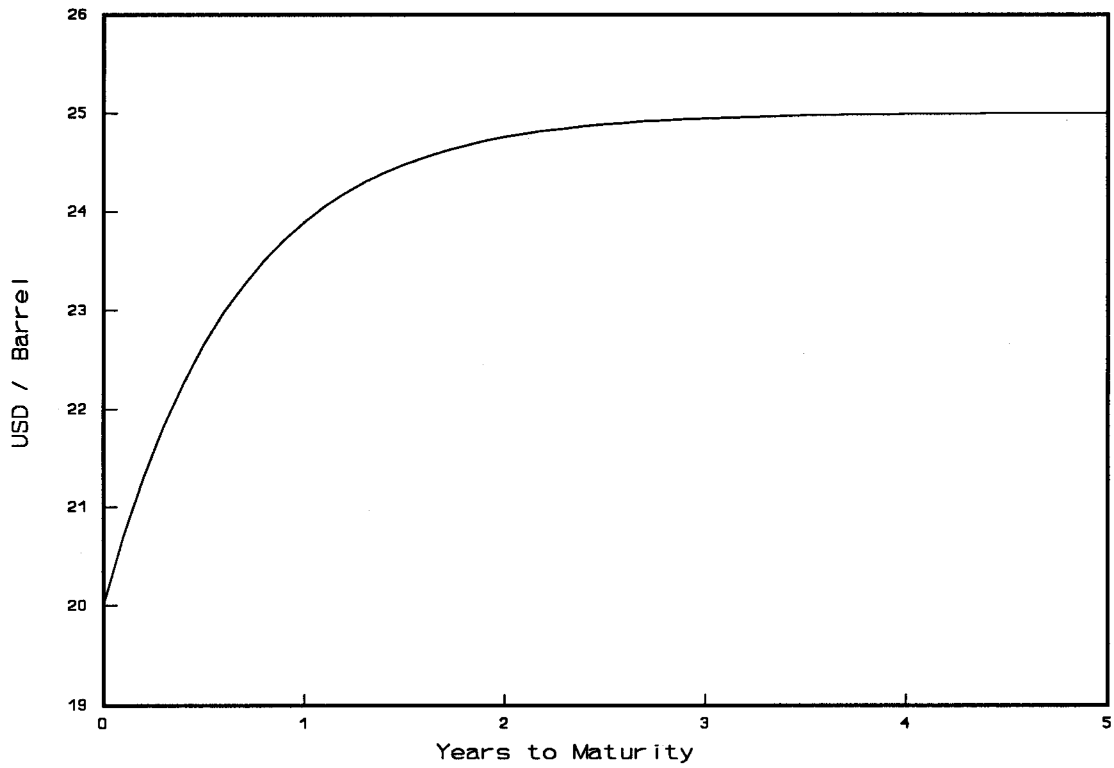

Figure 6 : Contango without any short-term effect. S=20, L=25, ν=0.158, β=1.5, θ=0.

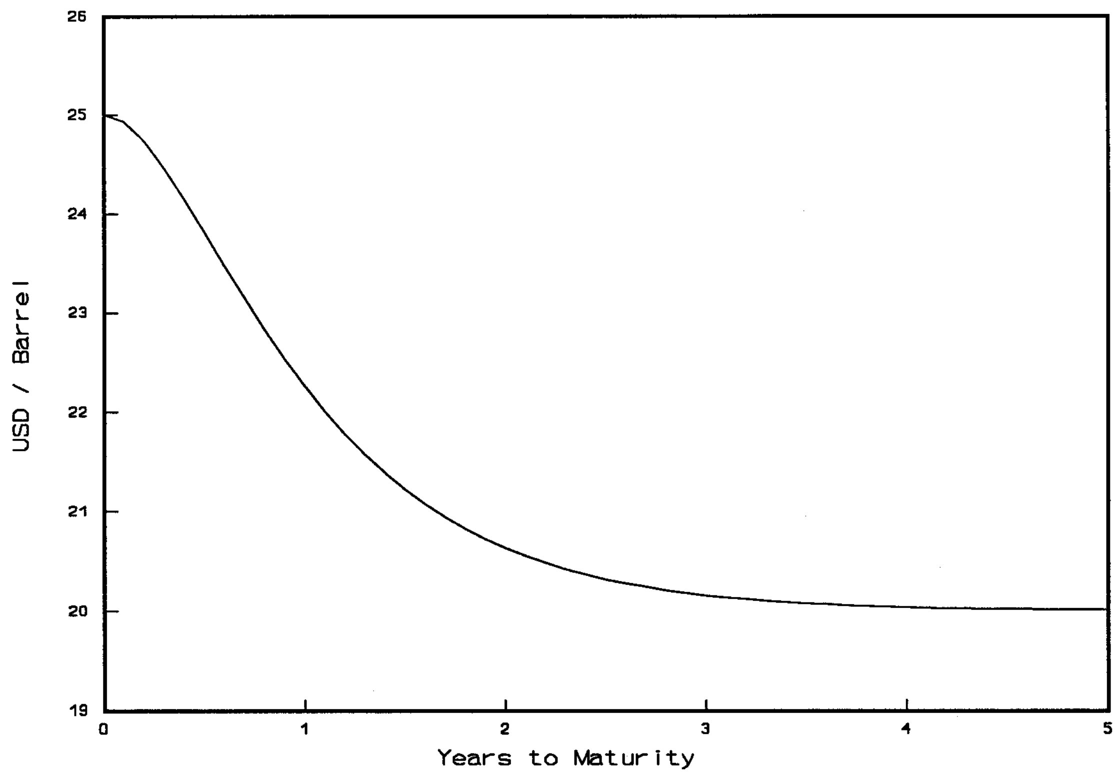

Figure 7 : Backwardation with a short-term effect. S=25, L=20, ν=0.158, β=1.5, θ=-0.3, η=2.

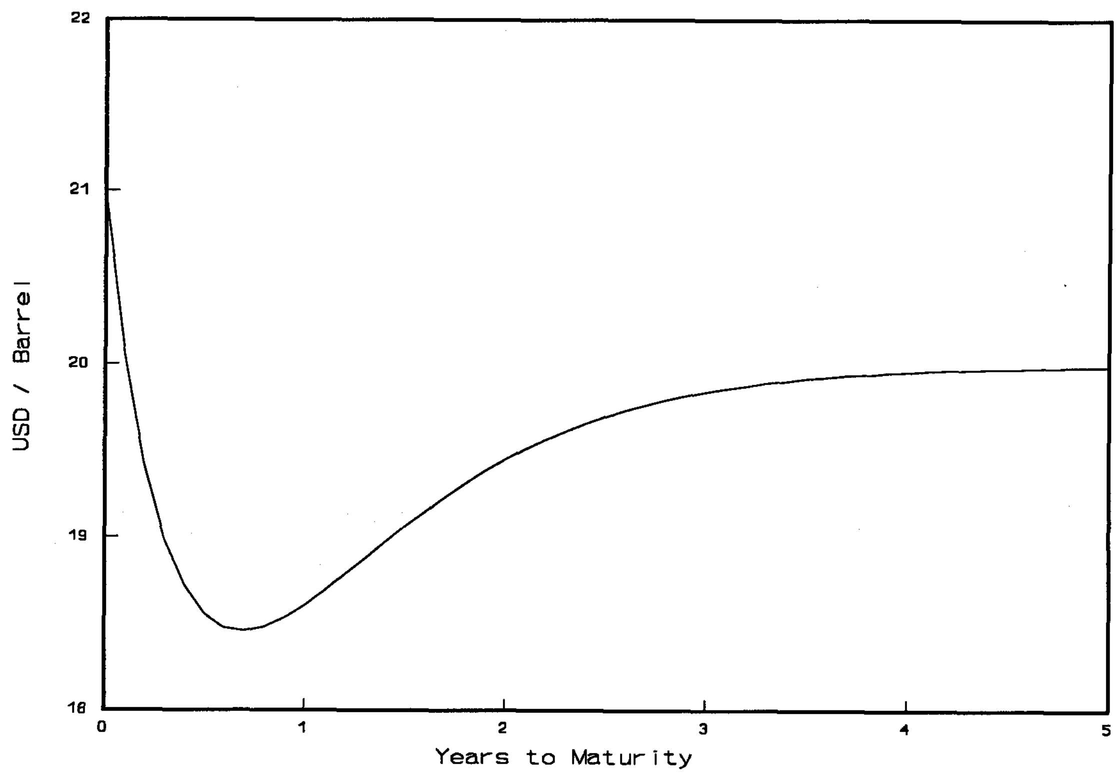

Figure 8 : Contango with a short-term effect. S=21, L=20, ν=0.158, β=1.5, θ=0.5, η=2.

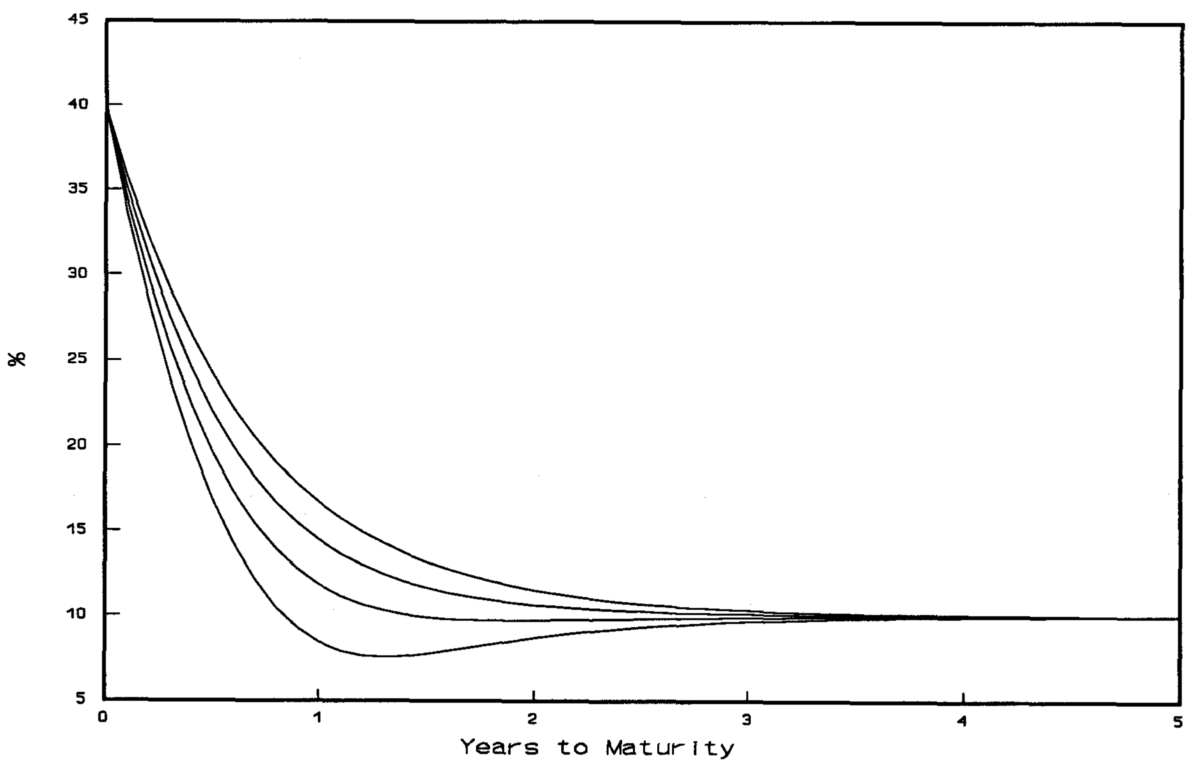

Figure 9 : Term structures of futures price volatilities. σ_S=40%, σ_L=10%, ρ=1 (upper curve), 0.5, 0 and -0.5 (lower curve), β=1.5.

In Section 4.4, we show how the model (26) can be constructed with different underlying hypotheses and without any reference to the notion of convenience yield of physical oil.

4.4 The Market Price of Risk vs the Convenience Yield

The models developed in Section 4.3 rest on the assumption that physical oil is a traded asset and that consequently the market price of the spot price risk is known. But some authors have pointed out that commodities are not traded assets and that their market price of risk is connected to the pricing of derivative securities.[16] However, the purpose of this section is not to decide whether physical oil is or is not a traded asset, but to examine the formulation which results from considering the market price of spot price risk.

Under the assumption that physical oil is not a traded asset, the expected rate of growth of its price has to enter into the pricing of a futures contract. In this section, we shall make the assumption that the spot price follows a mean-reverting diffusion process in which L represents the long-run mean value of the spot price. L also follows a diffusion process and their joint stochastic process is specified as:

$$\begin{aligned} \frac{dS}{S} &= k \ln \frac{L}{S}\, dt + \sigma_S\, dz_1 \\ \frac{dL}{L} &= \mu_L\, dt + \sigma_L\, dz_2 \end{aligned} \tag{29}$$

dz_1 and dz_2 are two correlated Wiener processes such that:

$$dz_1\, dz_2 = \rho\, dt$$

It is not claimed that the process specified above represents the best description of the spot price behaviour. In the absence of empirical results on the form of the spot price stochastic process, this specification serves only as an example.

For the purpose of simplification, all parameters will be considered constant. k is assumed to be strictly positive in order to ensure an actual mean reverting pattern. The formulation of the stochastic process followed by the spot price means that if S is greater (lower) than L, its expected growth of rate will be negative (positive) and S will tend to go back to L.

The futures price depends upon S, L and τ. By using expression (29) and the same technique as before, we come to the following partial differential equation:

[16]"Most commodities are held primarily for consumption and cannot be considered as traded securities.", Hull (1989), p.167.

$$\left(k \ln \frac{L}{S} - \lambda_S \sigma_S\right) S F_S + \frac{1}{2}\sigma_S^2 S^2 F_{SS} + \left(\mu_L - \lambda_L \sigma_L\right) L F_L + \frac{1}{2}\sigma_L^2 L^2 F_{LL} + \rho \sigma_S \sigma_L S L F_{SL} - F_\tau = 0 \tag{30}$$

subject to the initial condition:

$$F(S,L,0) = S$$

λ_S (λ_L) is the market price per unit of spot price (long-run mean value) risk. Assuming that the futures price has a finite limit value as maturity goes to infinity, the market price of risk of L can be calculated as if L was the price of a traded asset. Thus, we have:

$$\lambda_L = \frac{\mu_L}{\sigma_L}$$

Then the partial differential equation (30) is simplified and has the following closed-form solution:

$$\begin{aligned} F(S,L,\tau) &= A(\tau)\, S^{B(\tau)}\, L^{1-B(\tau)} \\ A(\tau) &= \exp\left[\frac{\lambda_S \sigma_S}{k}\left(e^{-k\tau} - 1\right) + \frac{\nu}{4k}\left(2e^{-k\tau} - e^{-2k\tau} - 1\right)\right] \\ B(\tau) &= e^{-k\tau} \end{aligned} \tag{31}$$

The model (31) is very close to the model (26) of Section 4.3. The parameter k plays a symmetric role to β. The limit value of the futures price when maturity goes to infinity is:

$$\lim_{\tau \to +\infty} F(S,L,\tau) = \exp\left[-\frac{\lambda_S \sigma_S + \frac{\nu}{4}}{k}\right] L$$

It is interesting to note that the limit value of the futures prices is not equal to L. The market price of spot price risk and the stochasticity of the spot price S and the long-term mean value L induce a skew in the price of oil for delivery at infinite time.

There is no reference to the convenience yield of physical oil in this section but we can still derive a model comparable to the models of Section 4.3. This shows that more important than the notion of convenience yield is the role of the expected rate of growth of the spot price. Indeed, the convenience yield used in the previous section operates in the same way as the expected rate of growth of S.

5 NUMERICAL APPLICATIONS

5.1 Estimation of the Model with a Non-Stochastic Long-Term Price of Oil

The main problem when estimating the models of Section 4.3 is that the variable L is not observable on the market. Its value cannot be easily deduced from the market data every single day because the futures price given by model (26) or (28) is also determined by the parameters of the stochastic process followed by the long-term price of oil.

In a first approach, we divided the historical data into calendar month periods and then considered the long-term price of oil as being non-stochastic in these periods. We used model (26) described in Section 4.3 with constant parameters and the non-stochasticity of L.[17] The market data are the daily settlement prices of the NYMEX WTI futures contracts from 1 June, 1989 to 28 February, 1991. All the futures contracts for which a settlement price is published by the NYMEX[18] have been used, regardless of their liquidity or representativeness. Any day where one or more futures contracts were limit-up or limit-down at the close of the NYMEX have been removed from the sample. All futures prices have been associated with their exact time to maturity in order to describe a continuous structure. The proxy for the spot price has been computed from the price of the futures contracts as follows:

$$S = F_1 - \tau_1 \frac{\partial F}{\partial \tau}(S,L,\tau_1)$$

where the partial derivative of F with respect to τ has been calculated by the following approximation:

$$\frac{\partial F}{\partial \tau}(S,L,\tau_1) = \frac{F_2 - F_1}{\tau_2 - \tau_1}$$

where F_n denotes the price of the n^{th} nearby futures contract maturing in τ_n years.

The annualized volatility of S has been computed on a daily basis for each period. The actual number of trading days has been used in order to neutralize the effect of the large number of limit-up or limit-down situations during the months of August, September, November and October 1990.[19]

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[17]When L is non-stochastic, $\sigma_L = 0$ and $\rho = 0$. Thus, the model (26) with constant parameters can be used with the small change that $\nu = \sigma_S^2$.

[18]As soon as a futures contract is traded, the NYMEX publishes a settlement price for this contract. However, the NYMEX does not publish any settlement price for a futures contract opened for trading which has not actually been traded.

[19]The number of trading days used for the estimation ranges between 16 and 23 depending on the circumstances of the relevant months.

The estimated values of both β and L are significant for every period. However, the width of the confidence intervals and the root mean square error increase substantially during the periods of sharp changes on the market. The uncertainty attached to the estimates of L is large when the market changes from contango to backwardation, or the other way round. The behaviour of the long-term price estimates is rather satisfactory and consistent with the model of the cantilever, and is fairly stable through time. The annualized volatility of the estimated values calculated on a monthly basis is 17.64 per cent over the whole period. Considering the average of the spot prices in each calendar month, the volatility of this average calculated on a monthly basis is 46.39 per cent over the whole period. The spot price and the long-term price are positively correlated; the value of the correlation coefficient ρ is 0.2054. Figure 10 shows the evolution of the monthly spot price average and the long-term price of oil. The crossing of the two curves indicates the changes of state: backwardation to contango or contango to backwardation. The long-term price being rather stable, the changes of state are mainly explained by the fall and the rise of the spot price.

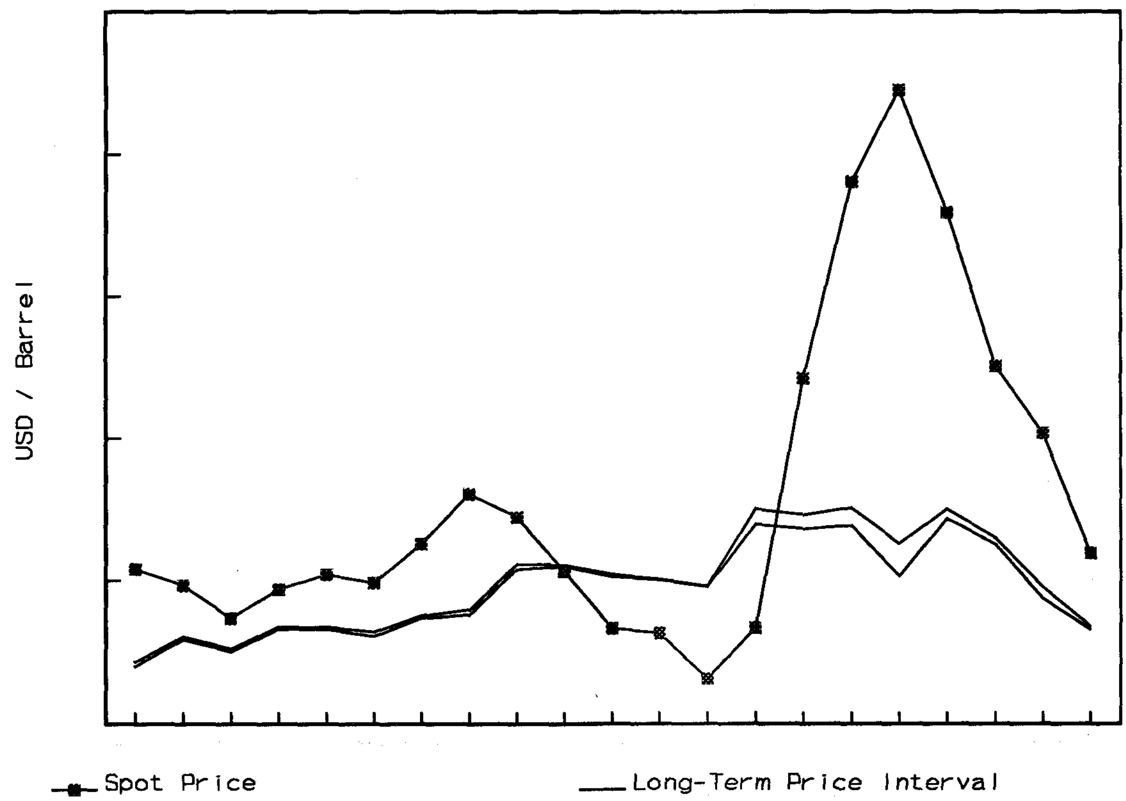

Figure 10 : Monthly averaged spot price and long-term price confidence interval from June 89 to February 91.

The estimates of β remain in the range [0.911,6.930]. The assumption that β is an inter-temporal constant is hardly justified. However, the evolution of β through time is rather smooth and does not present very large jumps. β remains surprisingly stable, its confidence interval being narrow between July and December 90, the most troubled period of the sample. But this period also presents large root mean square errors, and the model therefore may be less representative. January 1991 shows the highest root mean square error because the estimation suffers a lot from the non-stochasticity of L. The term structures of futures prices that emerged in January and February 1991 are poorly described by model (26). Model (28) would be more appropriate for these periods. We shall discuss this problem later in the next section.

In Section 5.2, we shall reintroduce the stochasticity of L in order to improve the results of the model.

5.2 Estimation of the Model with a Stochastic Long-Term Price of Oil

The results of Section 5.1 show that although L is stable relatively to the spot price, it still displays fairly large fluctuations over time. In order to reintroduce the stochasticity of L, we need to construct a daily series of values for L. The only way to do so is to extrapolate L from the daily term structure of prices of the traded futures contracts. However, the technique chosen for the extrapolation highly influences the quality of the constructed series.

We use the following form of function for the extrapolation:

$$f(\tau) = \exp\left(a_1 e^{-a_2 \tau} + a_3\right)$$

This function can be viewed as a simplified form of model (26). Moreover, fixing τ, the values $f(\tau)$, $f'(\tau)$ and $f''(\tau)$[20] will explicitly determine the parameters a_1, a_2 and a_3.

Then, on every single day of the period considered, we have calculated the values of a_1,a_2 and a_3, so that the function f has the same value and the same first and second derivatives as the observed structure of futures prices at the furthest traded maturity. Then L is the limit of the function f when τ goes to infinity. When a_2 is positive, we have:

$$a_2 > 0 \;\Rightarrow\; L = \lim_{\tau \to +\infty} f(\tau) = \exp(a_3)$$

If p denotes the rank of the last traded contract, we computed the parameters such that:

[20] $f'(\tau)$ and $f''(\tau)$ denote, respectively, the values at τ of the first and second derivatives of f with respect to τ.

$$\tau = \tau_p$$

$$f(\tau) = F_p$$

$$f'(\tau) = \frac{f(\tau) - F_{p-1}}{\tau - \tau_{p-1}}$$

$$f''(\tau) = \frac{f'(\tau) - \dfrac{F_{10} - F_8}{\tau_{10} - \tau_8}}{\tau - \tau_9}$$

Unlike the proxy for the first derivative as stated above, the proxy chosen for the second derivative of the observed term structure of futures prices measures the change in the first derivative in between the 9[th] nearby contract and the last traded contract. This choice is arbitrary and has been made in order to embrace a larger section of the term structure of futures prices. Even with such a choice for the second derivative, some delicate problems arise. On some dates of the sample, the second derivative appears to be non-negative, the first derivative being non zero. Consequently, the resulting function f has an infinite limit at infinite time which is not an acceptable feature. Therefore, on any such dates, we used the most recent second derivative with a negative value. This method used to avoid obtaining an infinite value of L may yield peculiar values when the second derivative is held constant while the first derivative displays a large change. All considered, this method remains the most sensible for our purposes.

The series of long-term prices resulting at this stage from the extrapolation has some unsatisfactory features as regards volatility. The annualized long-term price volatility calculated on a daily basis is 58.10 per cent over the whole period, while the spot price volatility is 62.61 per cent. This long-term price volatility is relatively high and not very consistent with our expectations and the results of Section 5.1 We can isolate two main causes responsible for this phenomenon, and this will provide us with a method for filtering the series of long-term prices.

First, the NYMEX sometimes assesses the prices of the outer futures contracts using a constant monthly increment. It is then difficult to decide whether this monthly increment should remain constant after the end of the exchange-traded maturities, or whether it should decrease to zero as maturity increases to infinity, as featured in our model. Our purpose here, which is to determine a finite value of L, requires the monthly increment to decrease to zero and consequently, as explained above, the non-negative second derivatives have been replaced by more adequate values.

Secondly, the extrapolation is very sensitive to the values of the first and second derivatives. On some dates, the market participants or the NYMEX may not have adjusted quickly enough in terms of derivatives, and consequently, the extrapolated value of L will show a large jump. These situations are normally corrected in the following days and therefore, those isolated values have been ignored.

In order to locate these atypical situations, we monitored the series of the logarithm of the daily returns of the long-term price, namely $\ln(L_t/L_{t-1})$. We then isolated 8 days (out of 423 days in the sample) where the computed value of L induces very large values of the logarithm of the return which cannot be explained by noticeable changes in the term structure of futures prices and which are completely reversed in the following days. In these cases, the mispriced values of L have been replaced by the linear interpolations between the previous and the next values. The volatility of the filtered series is then reduced to 33.65 per cent. However, this lower value of the long-term price volatility induces a higher value of the correlation coefficient. The value of ρ over the whole period is 0.4217 and therefore shows an important positive correlation between the spot price and the long-term price of oil.

The characteristics of the long-term price series are not completely satisfactory in terms of volatility and correlation with the spot price, the values of both parameters being too high. The more basic method used in Section 5.1 provides only monthly figures but presents superior qualities, which is to say lower volatility of the long-term price relatively to the spot price and lower correlation coefficient.[21] However, it seems difficult to justify a stronger filtering technique in order to improve substantially the qualities of the long-term price series.

Figure 11 compares the movements of the spot price and the long-term price of oil. The overall aspect of the curves is similar to the curves of Figure 10. By looking closely at the long-term price curve, it appears, as explained above, that some narrow peaks due to the slow adjustment of derivatives remain. However, the long-term price of oil has an overall satisfactory behaviour, especially during the period August 1990-January 1991 when it remained fairly stable in spite of erratic movements of the spot price. It is worth noting that the extension by the NYMEX at the end of November 1990 of the traded maturities[22] did not induce any jump in the extrapolation of the long-term price.

Figure 12 presents the spot price and long-term price volatilities calculated on a daily basis for each month of the sample. There are three periods of time where the long-term price volatility was greater than the spot price volatility, namely August and September 1989, February 1990 and July 1990. During the first of these periods, the term structure of futures prices was fairly flat. The second period, February 1990, saw the change from the backwardation of the beginning of that year to a contango. At last, in July 1990, the previous state reversed and the term structure of prices went back to backwardation.

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[21]In Section 5.1, the spot price volatility was 46.39 per cent and the long-term price volatility 17.64 per cent on the same basis. The correlation coefficient was 0.2054.

[22]At the end of November 1990, the NYMEX opened for trading maturities up to 36 months forward. WTI futures contracts are now listed with 18 consecutive forward months plus 4 additional contracts that are initially 21, 24, 30 and 36 months forward. When one month passes, the first nearby contract will expire and the remaining contracts will roll forward one month.

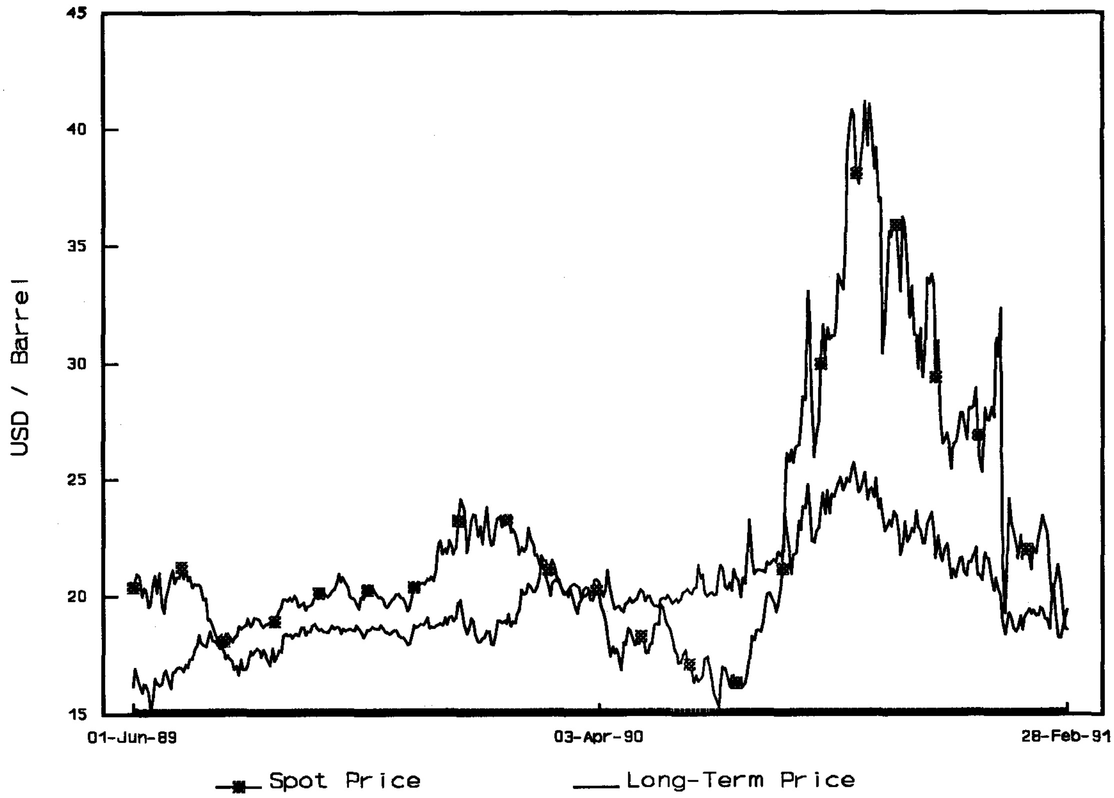

Figure 11 : Evolution of the spot and long-term prices of oil between 1 June 1989 and 28 February, 1991.

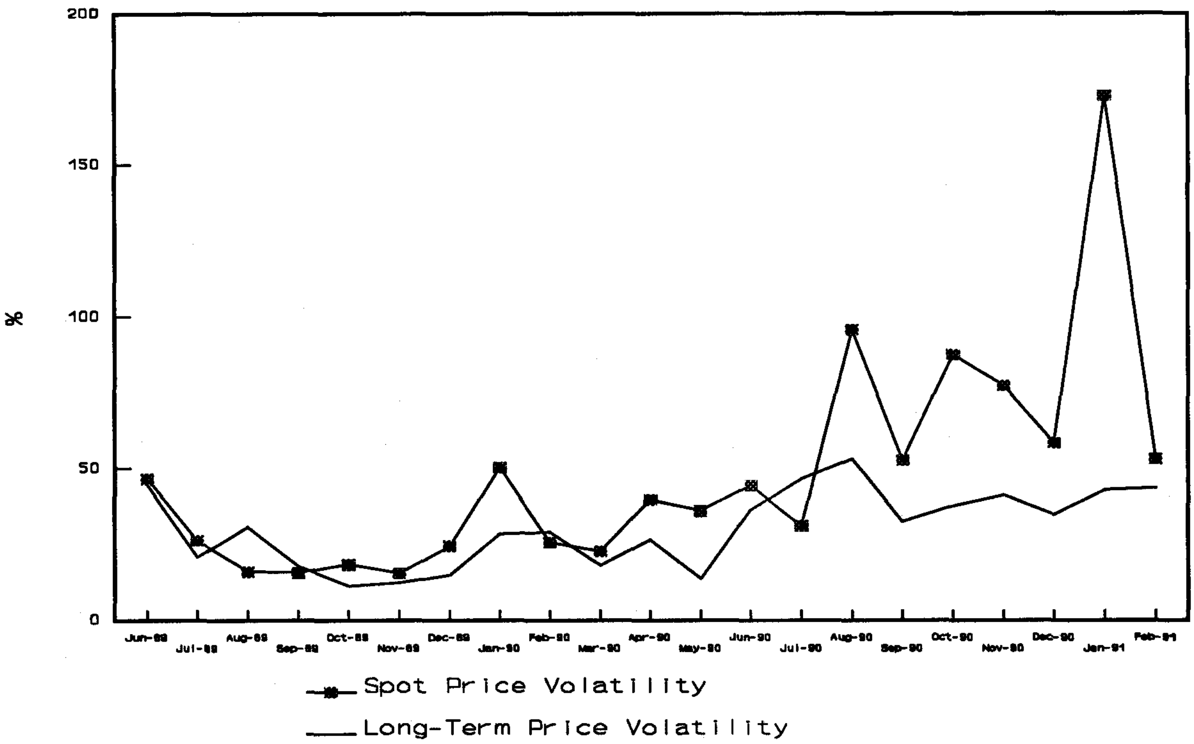

Figure 12 : Spot price and long-term price volatilities from June 1989 to February 1991.

Figure 13 compares the monthly values of L estimated in Section 5.1 with the daily series based on extrapolation. The underlying methods to compute those values of L differ widely: while the first method tends to minimize the sum of squares of the difference between the theoretical prices and the observed prices of the whole term structure, the second focuses only on the first and second derivatives at the end of traded term structure. Nevertheless, the resulting values are rather close most of the time. Significant differences appear in September and October 1990 when the slope of the term structure of futures prices was at its maximum. The daily series show an important spike in February 1991 due to the peculiar pattern of the first and second derivatives along the term structure of prices.

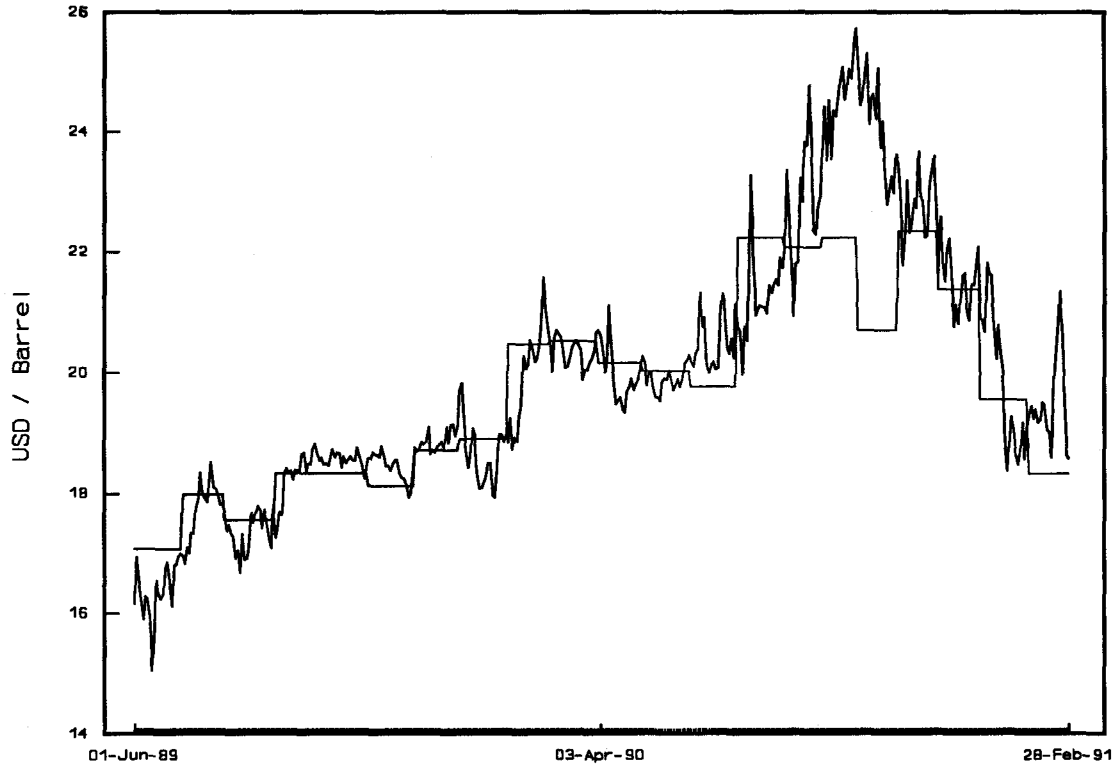

Figure 13 : Monthly estimates and daily values of the long-term price of oil between 1 June, 1989 and 28 February, 1991.

The annualized volatility of L and the correlation coefficient between S and L have been computed on a daily basis for each period. If we except April and May 1990 and January and February 1991, the estimates of β are bounded in the interval [1.152,2.637]. The interval for the same periods, L being considered non-stochastic, was [0.911,4.400]. Logically, the stochasticity of L absorbs some variability of the parameter β. Figure 14 compares the different values of β in two cases: a stochastic and a non-stochastic L.

In April and May 1990, we witnessed a period of steep contango on the first part (first 6 months on average) of the term structure of futures prices. The high values of β in these periods are due to inabilities of model (26) to describe adequately situations where a strong short-term effect operates. Should we look closer at the estimation, we would see that the first part of the term structure of prices is correctly approximated while the end is very poorly estimated.

In January and February 1991, the circumstances were different but the same causes continue to explain validly the high values of the parameter β. After 17 January, 1991, these months were characterized by backwardation in the short-term structure while the long-term structure was in strong contango. Again, only model (28) featuring a short-term effect can describe such situations. In January and February 1991, as in April and May 1990, the short-term effect was far more important than the long-term component of the convenience yield due to the relative positions of S and L. As a result of this peculiar state, the estimation error of model (26) was very large and the confidence interval of β particularly wide in February 1991.

Nevertheless, the utilization of a daily series for L enhances the representativeness of model (26) and proves that the long-term price should be construed as a stochastic variable. Figure 15 compares the root mean square errors obtained when L is considered as stochastic and as non-stochastic. The results of the estimation improve particularly during periods of change from backwardation to contango, or contango to backwardation as the case may be; also during the periods where the spot price of oil endures large and erratic movements.

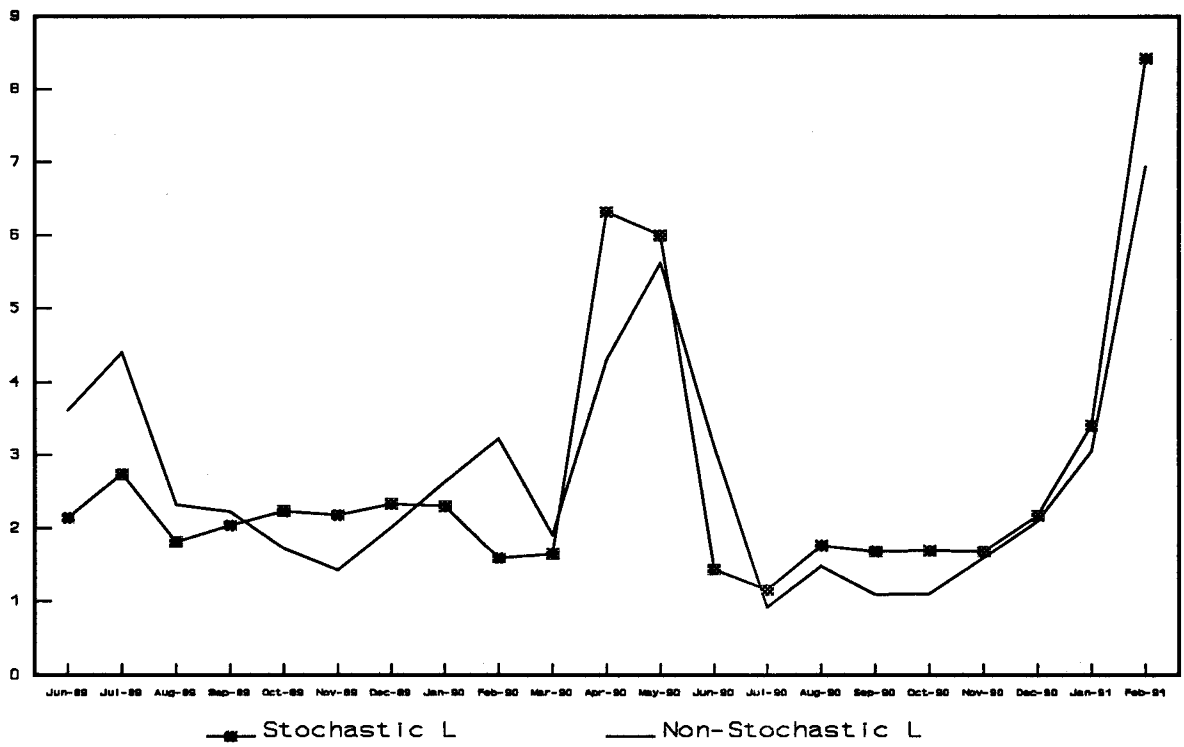

Figure 14 : Estimates of the parameter β of the model (26) when L is stochastic or not between June 1989 and February 1991.

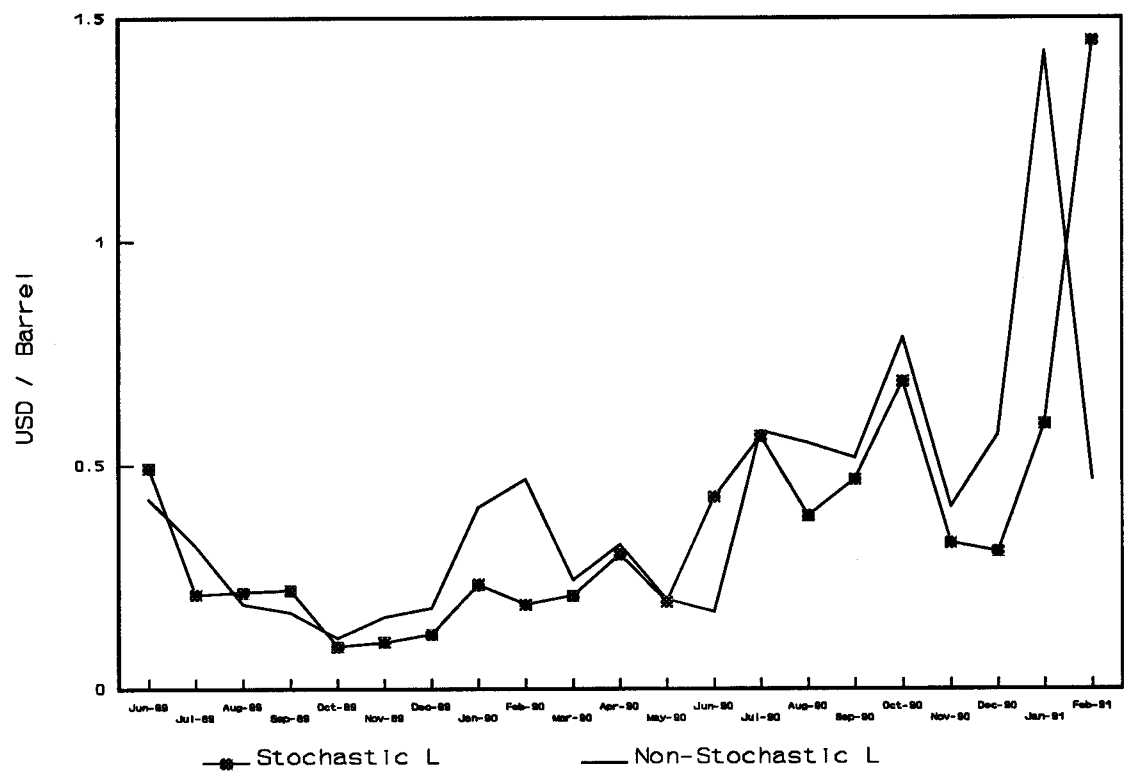

Figure 15 : Root mean square errors of the estimation of the model (26) when L is stochastic or not between June 1989 and February 1991.

6 CONCLUSION

In this paper, we have presented a two-variable model of the term structure of futures prices assuming that the spot and long-term prices of oil are stochastic and are the main determinants of the convenience yield function. Although the resulting convenience yield is stochastic, the model admits an analytic formulation under some restrictions.

A simplified form of the model proved to be a rather good tool for the description of the futures prices through time. With this simplified formulation, the futures price is entirely determined by a reduced set of state variables and parameters: the spot and long-term prices of oil, the volatilities and correlation of their stochastic processes and a single parameter β. This shows that the term structure of futures prices can be reduced to a few parameters. It is then much easier to monitor these parameters in order to understand in a satisfactory manner the movements of the term structure.

In order to describe as closely as possible the observed term structures of prices and volatilities, it may be necessary to increase the number of parameters, for instance, by allowing a shock on the convenience yield. A perfect fit to market data would even require totally time dependent parameters. Increasing the number of stochasticity sources (through the state variables) would also provide enhanced description properties of such a model. However, this would lead to higher complexity, and it is very unlikely that analytic formulation could be developed.

The term structure of prices on the NYMEX WTI futures market suggests the existence of a finite price of oil for delivery at infinite time. As we pointed out, this constitutes a very strong hypothesis. Indeed, it requires that the market participants attach the same price to the long-term price risk as if this long-term price was the price of a traded asset. The validity of such an assumption remains difficult to evaluate since risk aversion of market participants is involved. Moreover, petroleum products with strong seasonal patterns cannot be efficiently described by this couple of state variables. In that case, the seasonal effect of the convenience yield function overpasses the long-term price influence.

Further developments should enclose a more complex form of the convenience yield function, together with spot and long-term prices, in order to describe all kinds of products.

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