# Volatility Smiles and Alternative Models

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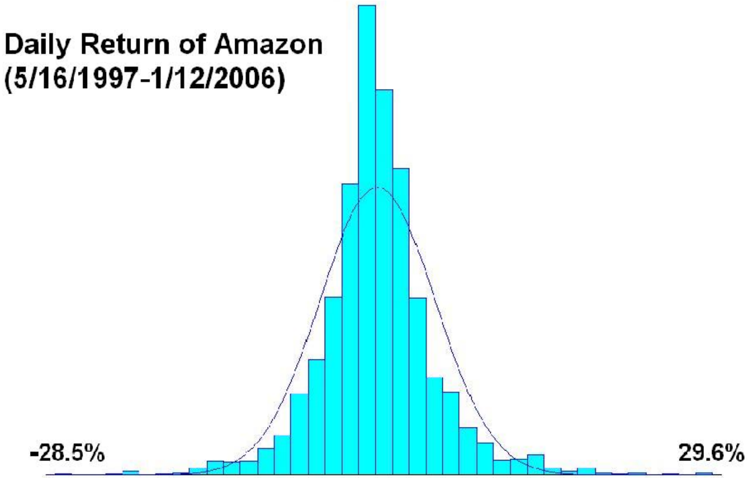
# Inadequacy of the BSM Model

- Very successful since its was published in 1973
- The BSM model is **unable to capture extreme price movements**
- Oct 19, 1987, DJIA dropped by more than 20%
- It assumes normal log-returns (with very thin tails)

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

$$\ln(S_t/S_0) \sim N((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

- Empirical financial data exhibit much fatter tails


Daily Return of Amazon
(5/16/1997-1/12/2006)
-28.5%
29.6%

# Implied volatility

- In the Black-Scholes-Merton model, European call option price is an **increasing** function of $\sigma$

$$c(\sigma)$$

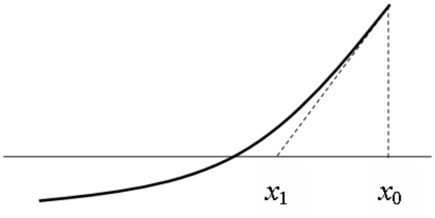
- 1-1 correspondence b/w option price and volatility
  - Given $\sigma$, can compute option price
  - Observe market option price, can compute $\sigma$ (**implied volatility**)

$$c_M = c(\sigma)$$

- The above can be solved using the **Newton-Raphson** method

# Newton-Raphson method for solving $f(x) = 0$

- Geometric interpolation




- The tangent line: $y = f'(x_0)(x - x_0) + f(x_0)$. $x_1$ solves

$$f'(x_0)(x_1 - x_0) + f(x_0) = 0 \quad \Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

# Volatility smiles

- Implied volatilities for options on an asset with different strikes
- If options were priced using the BSM model,
  - Options on the same underlying with different strikes should have the same implied volatility
  - Plot implied volatility vs strike: expect **a flat straight line**
- In practice, implied volatility exhibits **smile/skew** curves

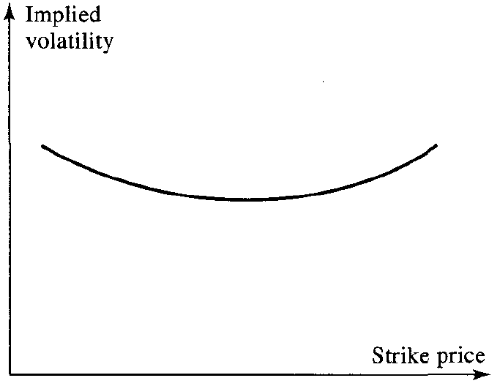
- Volatility skew (puts on Amazon's stock)

- Amazon's example: the BSM model underprices **out-of-the-money puts**
  - Current stock price around 44
  - Put with strike 35 is deep out-of-the-money
  - Much higher volatility needed to match the market price and BS price

# Crash-o-phobia

- Investors are concerned about potential market crashes (Crashophobia)
- More significant after 1987 market crash
- Traders now understand that normal distribution for the asset return understates probability of extreme movements

- Smiles in foreign currency markets

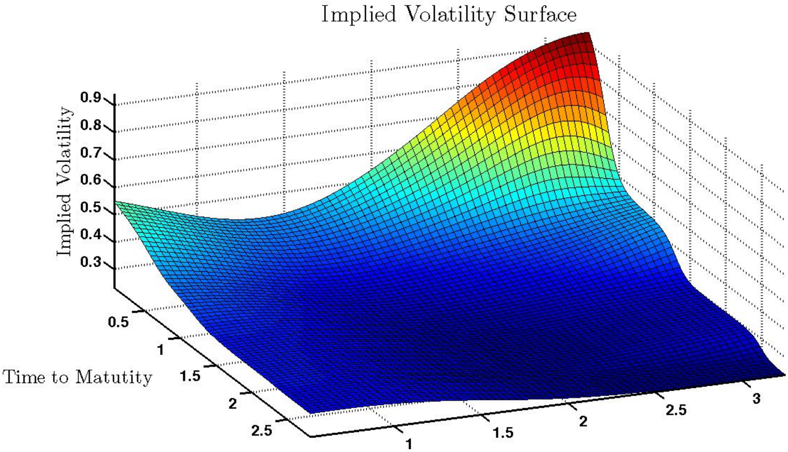


# Implied volatility term structure

- Implied volatilities for options on an asset with different maturities
- Implied volatilities are different for different maturities (**volatility term structure**)
- Implied volatility is function of maturity and strike: **implied volatility surface**

$$\sigma^{IV} = \sigma^{IV}(strike, maturity)$$

- Not flat as predicted by the BSM model



Implied Volatility Surface
Implied Volatility
0.9
0.8
0.7
0.6
0.5
0.4
0.3
Time to Matutity
0.5
1
1.5
2
2.5
1
1.5
2
2.5
3

# Alternative models

- Restrictive assumptions of the Black-Scholes-Merton model:
  - Asset price process is continuous
  - Volatility is constant
- Alternative models
  - Allow stochastic volatility: **stochastic volatility models**
  - Allow jumps in the asset price process: **jump models**
  - Combination: **stochastic volatility jump models**
  - These make extreme events more possible
  - More difficult to handle

# CEV model

- In CEV (constant elasticity of variance, Cox (1975)), asset price follows
$$\frac{dS_t}{S_t} = (r - q)dt + \sigma S_t^{\beta} dB_t^*$$
- When $\beta = 0$: BSM; When $\beta < 0$: volatility $\sigma(S_t) = \sigma S_t^{\beta}$ increases as asset price decreases. This makes even smaller asset price more likely and produces a distribution with **fatter left tail**

- Elasticity: percentage change in one variable due to percentage change in another variable

$$\frac{d\sigma(S)}{\sigma(S)} \Big/ \frac{dS}{S} = \beta$$

- European vanilla option pricing formulas are available analytically in terms of **non-central $\chi^2$ cdf**

# Local volatility model

- Local volatility model of Dupire (1994)
- Asset price process follows

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t)dB_t^*$$

  where $\sigma(S, t)$ is a deterministic function of $t$ and $S$
- Given European vanilla option prices for all maturities and strikes, can determine $\sigma$ so that model price and market price match perfectly

- **Dupire equation**: call price $c(K,T)$ in the local volatility model solves

$$c_T(K,T)+(r-q)Kc_K(K,T)-\frac{1}{2}\sigma^2(K,T)K^2c_{KK}(K,T)+qC(K,T)=0$$

$$\sigma^2(K,T)=\frac{2(c_T(K,T)+(r-q)Kc_K(K,T)+qC(K,T))}{K^2c_{KK}(K,T)}$$

- Observing market price $c(K,T)$, one obtains the local volatility $\sigma(\cdot,\cdot)$

# Heston's stochastic volatility model

- Asset price follows (here $B_{1t}$ and $B_{2t}$ are correlated)

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-q)dt + \sqrt{V_t}dB_{1t} \\ dV_t &= \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dB_{2t} \end{aligned}$$

- Variance process $V_t$ follows a **mean reverting** CIR process: $\theta-$ long run variance level, $\xi-$ volatility of volatility, $\kappa-$ mean reverting factor

- c.f. known analytically; **Fourier transform method** can be used for pricing European options
- PDE for path independent European option price: use 2-d Feynman-Kac
- Works well for long term contracts

# Jump diffusion models

- Merton (1976) normal jump diffusion model
- BM superimposed with jumps: $X_t = \ln S_t$ follows

$$X_t = \ln S_0 + \mu t + \sigma B_t^* + \sum_{n=1}^{N_t} Z_n$$

$N_t-$ PP$(\lambda)$; $\{Z_n\}-$ i.i.d. $N(m, s)$; $\mu$ determined so that discounted gain process is a martingale

$$\mu = r - q - \frac{1}{2}\sigma^2 + \lambda\left(1 - e^{m+s^2/2}\right)$$

- pdf of $X_t$ available in terms of infinite series; c.f. of $X_t$

$$\phi_t(\xi) = \exp\left( i\xi(\mu t + \ln S_0) - \frac{1}{2}\sigma^2 t \xi^2 - \lambda t(1 - e^{im\xi - s^2\xi^2/2}) \right)$$

- Partial **integro**-differential equations for European style contracts
- Kou (2002) double exponential jump diffusion model: $Z_n$ are double exponentially distributed

# Lévy process models

- **Lévy process models**
  - A Lévy process $X_t$ has **independent stationary increments**
  - Lévy process models for the underlying asset price

$$S_t = S_0 e^{X_t}$$

  - Special cases: BSM (no jumps); Merton's and Kou's jump diffusion models (finite jumps in any finite time interval)
  - Allows infinitely many jumps in any finite time interval; without a BM term: **pure jump model**
  - c.f. typically available, **Fourier transform method** can be used
- Work well for short term contracts

# Stochastic volatility jump diffusion model

- Adding jumps to the asset price process in Heston's model
- **Duffie-Pan-Singleton** (2000): adding jumps to both the asset price and variance processes
- c.f. available; Fourier transform method can be used
- Or solve 2-d partial integro-differential equations numerically
- Challenging numerically