FIN516: Term Structure Models

Lecture 3: Introduction to binomial/trinomial pricing

Dr. Martin Widdicks

UIUC

Fall, 2015

Overview

- It is possible with many of the term structure models to analytically value bonds and other vanilla products such as caps and European swaptions.
- However, in ALL models we consider it will not be possible to value interest rate derivatives with early exercise features with an analytic approach. As a result numerical methods will have to employed at some point.
- Within our set of possible short rate models, all will most naturally be represented by a binomial or trinomial tree for the short rate. Later in the course we will see how to calibrate these models so that the binomial tree matches the term structure of both LIBOR and, potentially, its volatility.
- We will briefly discuss how to construct Monte Carlo models to value interest rate derivatives

The basics

- As you have seen with equity option valuation it is possible to discretize the process taken by a stochastic variable - here this will be the short rate.
- Here we will assume that the process followed by the short rate in the risk-neutral world is known and then show how it is possible to value interest rate derivatives from the binomial tree.
- Unlike with equity options, the largest challenge when using the binomial tree is the construction of the tree. In this lecture we assume that this has already been achieved but we shall be returning to this in detail when we consider the short rate models next.

The tree

- Assume that we need to know the term structure of interest rates for a total of T years. For interest rate derivatives, T is not necessarily the maturity of the derivative but could be the maturity of the underlying asset in the derivative contract (the swap in a swaption for example)
- If we construct a binomial tree with N steps then the length of each time period is denoted by $\Delta t = T/N$.
- So we will be modeling the short rate from $i\Delta t$ until $(i+1)\Delta t$, denoted by $r(i, i+1)$ where $0 \leq i \leq N$.
- Naturally we do not know the future short rate at any given time $i\Delta t$, $i > 0$ and so we need to build in some uncertainty. This will be the volatility of the short rate. Of course we will have the prices of the zero coupon bonds (or equivalent) and the prices of caps and floors which we will use to calibrate our binomial tree.

Quiz

Which products will be the most useful for calibrating the volatility of the short rate?

- Swaps
- Caps
- FRAs

Quiz

Which products will be the most useful for calibrating the volatility of the short rate?

- Swaps Possibly
- Caps CORRECT
- FRAs

The tree

- This uncertainty is created by letting the short rate to one of two possible states of the world, the "up" state and the "down" state. Of course this can be trivially adapted to a trinomial tree where there are three states.
- To simplify the notation, let j denote the number of up jumps that the short rate has made through time and so we can denote the short rate as $r_{i,j}$ to be the rate from time i to $i+1$ after j jumps where $0 \leq j \leq i$.
- Let the value of the derivative be $V_{i,j}$ and as we are in the risk-neutral world we have the standard formula for valuing options

$$V_{i,j} = \frac{1}{1 + r_{i,j}\Delta t}\left[\pi V_{i+1,j+1} + (1-\pi)V_{i+1,j}\right]$$

  where π is the risk-neutral probability (all of our models will be in the risk-neutral world) of an up movement.
- The main difference between this equation and the standard binomial pricing equation is that the $r_{i,j}$ value is changing from time period to time period.

Adding more steps in the tree

- In the derivations above (and below) we can consider time steps representing reset dates for the underlying interest rate derivatives (i.e. 3 months, 6 months etc).
- However, to make the dsitribution of future rates more realistic we may wish to have more steps in the tree to more closely recreate the possible evolution of the short rate.
- When using very small steps in the tree, then it is reasonable to discount at a continuous rate is used as it is the instantaneous short rate than is being modeled and so the general option pricing formula is:

$$V_{i,j} = e^{-r_{i,j}\Delta t}\left[\pi V_{i+1,j+1} + (1-\pi)V_{i+1,j}\right]$$

Quiz

As $\Delta t \to 0$ what is $1 + r_{i,j}\Delta t$ the limit of

- Simple interest rate
- Continuously compounded interest rate
- Annualized interest rate
- All of the above

Quiz

As $\Delta t \to 0$ what is $1 + r_{i,j}\Delta t$ the limit of

- Simple interest rate
- Continuously compounded interest rate
- Annualized interest rate
- All of the above CORRECT

Zero coupon bonds

- The simplest product is the zero coupon bond. The notional principal (assumed to be \$1) on the bond is received at time T and this value is then discounted back through the tree at the appropriate rates until time 0.
- This is of course the discrete time equivalent of

$$P(0,T) = E_0\left[e^{-\int_0^T r(s)ds}\right]$$

that we will use for continuous models.

- In the future when we consider calibration, the choices of $r_{i,j}$ and π will allow us to match the market price for zero coupon bonds with the binomial tree price.

Example: tree

Assume* we have derived the following tree for the short rate, where $\pi = 0.5$

j	$t = 0$	1	2	3	4
4	$r_{i,j}$				25.15%
3				18.4%	17.90%
2			12.84%	12.84%	12.74%
1		8.41%	8.78%	8.96%	9.07%
0	5.00%	5.64%	6.01%	6.25%	6.46%

Where the simple interest rates are for the current period, which we assume here to be one year, so $\Delta t = 1$ which simplifies the calculations slightly.

* We will learn how to do this for particular short rate models in later classes.

Example: zero coupon bond pricing

From this we can value the zero coupon bond, expiring in any year from 1 to 5. Below is the valuation of the five-year bond. $P(0,5) = 0.6499$.

j	$t = 0$	1	2	3	4	5
5						1
4	$V_{i,j}$ $r_{i,j}$				0.7990 25.15%	1
3				0.6956 18.4%	0.8482 17.90%	1
2			0.6489 12.84%	0.7689 12.84%	0.8870 12.74%	1
1		0.6378 8.41%	0.7339 8.78%	0.8278 8.96%	0.9168 9.07%	1
0	0.6499 5.00%	0.7271 5.64%	0.8024 6.01%	0.8735 6.25%	0.9393 6.46%	1

Coupon bonds

- The valuation of coupon bonds is just as simple, now there is a final cash flow of $N + c$ and the coupon c will be paid at particular points in time.
- One way to value such a contract is to separate the coupon bond into a series of zero coupon bonds, so that if coupons are payable at time $T_1, T_2, T_3, \ldots, T_n = T$ then the price of the coupon bond is

$$CB(0, T_1, \ldots, T, N, c) = \sum_{i=1}^{n} cP(0, T_i) + NP(0, T)$$

- Recall how similar the expression for a coupon bond is to the fixed payoffs from a swap. We will use this result later to value swaps in a short rate model.

Example: coupon bond pricing

Consider a five-year bond paying an annual 10% coupon (c) on a principal (N) of \$100. Here we could calculate each of the zero five bonds separately as above and combine the five prices together but this would require valuing five separate zero coupon bonds. We have two solutions - the first is to simply adjust our pricing formula for the bond, below:

$$V_{i,j} = \frac{1}{1 + r_{i,j}\Delta t}\left[\pi V_{i+1,j+1} + (1-\pi)V_{i+1,j}\right] + cN$$

Example: coupon bond pricing

j	$t = 0$	1	2	3	4	5
5						110
4	$V_{i,j}$ $r_{i,j}$				97.89 25.15%	110
3				94.96 18.4%	103.30 17.90%	110
2			97.91 12.84%	103.43 12.84%	107.57 12.74%	110
1		105.07 8.41%	108.21 8.78%	110.23 8.96%	110.85 9.07%	110
0	105.43 5.00%	116.34 5.64%	116.46 6.01%	115.50 6.25%	113.33 6.46%	110

State or Arrow-Debreu prices

- The second solution is one that we will use extensively when calibrating trees later. This involves using Arrow-Debreu or **state prices**. The state price, which we define as $\hat{\pi}_{i,j}$ is the $t = 0$ price of an asset that a future time $i\Delta t$ pays out \$1 in state j.
- So the payoff of $\hat{\pi}_{i,j}$ is

$$\text{Payoff}(\hat{\pi}_{i,j}) = \begin{cases} 1 & \text{if node}(i,j) \text{ is reached} \\ 0 & \text{otherwise} \end{cases}$$

- By definition $\hat{\pi}_{0,0} = 1$, all other prices $\hat{\pi}_{i,j}$ must be determined by no arbitrage arguments as in the fundamental theorem of finance.
- The useful result is that once we have calculated the $\hat{\pi}_{i,j}$ values then they can be easily used to value all types of different interest rate products, including derivatives (and coupon bonds!)
- Via no arbitrage arguments a zero coupon bond paying out \$1 at time $i\Delta t$ will have current value

$$P(0, i\Delta t) = \sum_j \hat{\pi}_{i,j}$$

Quiz

- Under what 2 conditions is it possible to write the price of an asset as $P = D\pi$ where D is the payoff matrix in various states and π is the vector of Arrow-Debreu prices in those states of the world?

Quiz

- Under what 2 conditions is it possible to write the price of an asset as $P = D\pi$ where D is the payoff matrix in various states and π is the vector of Arrow-Debreu prices in those states of the world?
  Complete markets, no arbitrage

Building a state price tree

- If we know the state prices for all states and at all future dates then it will be simple to value zero-coupon bonds at varying maturities, which will make our coupon bond calculations far simpler.
- However, determining the state prices, seems just as time consuming as determining all of the zero-coupon bond prices. There is a trick to solving them all in one forward sweep through the tree.
- First, note that it is possible to write bonds maturing at time $(i+1)\Delta t$ in terms of the $\hat{\pi}_{i,j}$ and $r_{i,j}$ values as follows:

$$P(0,(i+1)\Delta t) = \sum_j \frac{\hat{\pi}_{i,j}}{1 + r_{i,j}\Delta t}$$

- We can then use this result to determine $\hat{\pi}_{i,j}$ from $\hat{\pi}_{i-1,j}$ as follows. First note that by definition $\hat{\pi}_{0,0} = 1$. Then in the next states

$$\hat{\pi}_{1,1} = \pi \frac{\hat{\pi}_{0,0}}{1 + r_{0,0}\Delta t} \quad \hat{\pi}_{1,0} = (1-\pi)\frac{\hat{\pi}_{0,0}}{1 + r_{0,0}\Delta t}$$

Quiz

What equation is being solved here?

- Black-Scholes
- Feynman-Kac
- Forward Kolmogorov
- Backward Kolmogorov

Quiz

What equation is being solved here?

- Black-Scholes
- Feynman-Kac
- Forward Kolmogorov CORRECT
- Backward Kolmogorov

Building a state price tree

- This can then be extended to future time steps by the following algorithm:

$$\hat{\pi}_{i,0} = (1-\pi)\frac{\hat{\pi}_{i-1,0}}{1+r_{i-1,0}\Delta t} \quad (1)$$

$$\hat{\pi}_{i,j} = (1-\pi)\frac{\hat{\pi}_{i-1,j}}{1+r_{i-1,j}\Delta t} + \pi\frac{\hat{\pi}_{i-1,j-1}}{1+r_{i-1,j-1}\Delta t} \text{ for } 0 < j < i \quad (2)$$

$$\hat{\pi}_{i,i} = \pi\frac{\hat{\pi}_{i-1,i-1}}{1+r_{i-1,i-1}\Delta t} \quad (3)$$

- Thus it is possible to step through the tree forward to build the $\hat{\pi}_{i,j}$ tree. On the next slide there is the state price tree for the short rate process seen earlier. It also lists the zero-coupon bond prices for each maturity.

The state price tree

j	$t=0$	1	2	3	4	5
5						0.0164
4	$\hat{\pi}_{i,j}$ $r_{i,j}$				0.0411 25.15%	0.0906
3				0.0973 18.4%	0.1749 17.90%	0.1967
2			0.2196 12.84%	0.3019 12.84%	0.2764 12.74%	0.2109
1		0.4762 8.41%	0.4450 8.78%	0.3108 8.96%	0.1927 9.07%	0.1118
0	1.0000 5.00%	0.4762 5.64%	0.2254 6.01%	0.1063 6.25%	0.0500 6.46%	0.0235
$P(0, i\Delta t)$	1.0000	0.9524	0.8900	0.8163	0.7350	0.6499

Coupon bond revisited

- Now we can value the coupon bond as follows:

$$\begin{aligned} CB(0,1,\ldots,5,100,0.1) &= 10P(0,1)+10P(0,2)+\cdots+110P(0,5) \\ &= 105.43 \end{aligned}$$

- Naturally, the coupon bond valuation was straightforward without this approach but the state price tree will be very useful when valuing bond options or swaptions, where the payoff is known at some future time.
- It will be most useful in calibration as it means that we do not have to revalue each zero coupon bond separately when we change parameters in the tree. Once a parameter changes then the state price tree will change and automatically value all zero coupon bonds. We will see this in great detail when we consider the Black-Derman-Toy model later in the course.

Bond options

- We can use our state price tree to value options on bonds. Consider a European option expiring at time T on a bond (potentially coupon bearing) that matures at T_B where $T_B > T$. Choose an appropriate Δt such that there are nodes at T and T_B, say i_T and i_B.
- To value this option we first use the short rate tree to value the bond as all possible states at T, this will provide is with the payoff from the option:

$$V_{i_T,j} = \max(CB_{i_T,j} - K, 0)$$

  for a call option.
- Then we can directly use the state price tree to value the option at time 0

$$V(0,0) = \sum_j \hat{\pi}_{i_T,j} V_{i_T,j}$$

- The next two slides show the valuation of a four-year European call option ($T = 4$) on a five-year 10% coupon bearing bond on a principal of \$100 and an exercise price of \$110

Bond option: payoff

j	$t = 0$	1	2	3	4	5
5						110
4	$CB_{i,j}$ $r_{i,j}$				97.89 25.15%	110
3				18.4%	103.30 17.90%	110
2			12.84%	12.84%	107.57 12.74%	110
1		8.41%	8.78%	8.96%	110.85 9.07%	110
0	5.00%	5.64%	6.01%	6.25%	113.33 6.46%	110

Bond option: Option valuation

- From using our state price tree from earlier we can value the bond option as follows

$$
\begin{aligned}
V(0,0) &= \sum_j \hat{\pi}_{4,j} \max(CB_{4,j} - K, 0) \\
&= 0 + 0 + 0 + 0.1927 \times 0.8527 + 0.0500 \times 3.3252 \\
&= 0.3306
\end{aligned}
$$

- We can also consider American bond options. Here we naturally have to check for early exercise at each point in time. This cannot be achieved using the state price tree, so we have to use the *brute force* approach.

$$
V_{i,j} = \max\left(CB_{i,j} - K, \frac{\pi V_{i+1,j+1} + (1-\pi) V_{i+1,j}}{1 + r_{i,j}\Delta t} \right)
$$

- The next slide shows the valuation of the American call option on the same bond. The value is 3.059

American bond option: valuation

j	$t=0$	1	2	3	4	5
5						110
4	$CB_{i,j}$ $V_{i,j}$ $r_{i,j}$				97.89 0 25.15%	110 -
3				94.96 0 18.4%	103.30 0 17.90%	110 -
2			97.91 0 12.84%	103.43 0 12.84%	107.57 0 12.74%	110 -
1		105.07 0.0829 8.41%	108.21 0.1798 8.78%	110.23 0.3913 8.96%	110.85 0.8527 9.07%	110 -
0	105.43 3.0586 5.00%	116.34 6.3401 5.64%	116.46 6.4645 6.01%	115.50 5.4955 6.25%	113.33 3.3252 6.46%	110 -

Swaptions

- We can now also value swaptions using our binomial model. Although we already have analytic formulas using Black's model, the binomial tree can support short rate models rather tahn the forward rate model used for the Black formula.
- The swaption is an option to enter into a swap of length $T_n - T$ that starts at the option maturity T.
- To value a swaption we use the fact that the swap can be decomposed into the fixed rate leg and the floating rate leg. The fixed rate leg at T can be valued as:

$$FixedLeg_T(T; T, \ldots, T_n) = N \sum_{i=0}^{n-1} K\tau_i P(T, T_{i+1}) + NP(T, T_n)$$

where the notional principal is also exchanged at the maturity of the swap

- This is clearly the same as a coupon bearing bond that we have already considered.

Quiz

How can we include the notional principal and still value the swaption correctly?

- It has value 0.
- It is actually the notional principal from the floating leg
- We simply subtract it from the final price
- It is usually included in the fixed payments of the swap

Quiz

How can we include the notional principal and still value the swaption correctly?

- It has value 0.
- It is actually the notional principal from the floating leg CORRECT
- We simply subtract it from the final price
- It is usually included in the fixed payments of the swap

Swaptions as coupon bond options

- The floating leg is

$$\begin{aligned} Float_T &= N\left(P(T,T) - P(T,T_n)\right) + NP(T,T_n) \\ &= N \end{aligned}$$

  where now the principal is also paid (to cancel out with the one from the coupon bond). This is then a floating rate note that *trades at par*.

- So a swaption can be modeled as an option on a coupon bond that comes into existence at the option maturity with an exercise price of N.
- The receiver swaption will be a call option on the coupon bond with a strike price equal to the notional principal, the payer swaption will be a put option.
- Thus, as for the options on coupon bonds, once the payoff at time T has been calculated then we can use the state price tree to immediately determine the value of the swaption. The only slight difference is that the coupon bond is issued at time T not time 0.

Quiz

At time 0 do we know the value of the coupon bond at T with certainty?

- Yes
- No
- In certain situations

Quiz

At time 0 do we know the value of the coupon bond at T with certainty?

- Yes
- No CORRECT
- In certain situations

Swaptions: example

- Consider a two-year European swaption which, if the holder wished, exercises into a three year payer annual swap with a fixed rate of 10% on a principal of \$100.
- The first stage is to value a three year bond paying an annual coupon of 10% on a principal of \$100 that comes into existence at $t = 2$. This is shown on the next slide.
- Once we have the value of the bond, $CB_{2,j}$, then we can calculate the payoff of the swaption as:

$$V_{2,j} = \max(100 - CB_{2,j}, 0)$$

and then

$$\begin{aligned} V_{0,j} &= \sum_j \hat{\pi}_{2,j} V_{2,j} \\ &= 12.09 \times 0.2196 + 1.79 \times 0.4450 + 0 \\ &= 2.6547 + 0.7962 \\ &= 3.4509 \end{aligned}$$

Swaption: valuation

j	$t = 0$	1	2	3	4	5
5						110
4	$CB_{i,j}$ $\hat{\pi}_{i,j} V_{i,j}$ $r_{i,j}$				97.89 0 25.15%	110 -
3				94.96 - 18.4%	103.30 - 17.90%	110 -
2			87.91 2.6547 12.84%	103.43 - 12.84%	107.57 - 12.74%	110 -
1		8.41%	98.21 0.7962 8.78%	110.23 - 8.96%	110.85 - 9.07%	110 -
0	5.00%	5.64%	106.46 0 6.01%	115.50 - 6.25%	113.33 - 6.46%	110 -

American/Bermudan swaptions

- There are different types of American swaptions: one where there is the option to enter into a new swap of fixed tenor at any point in time, and the other where there is the option to enter into an existing swap with fixed maturity (and thus reducing tenor)
- The second is the most common and what we will generally term an American swaption. Recall, this is the built in swaption in a mortgage.
- More precisely they are actually Bermudan swaptions as they can only be exercised at pre-specified points in time, when the swap rates reset, rather than at *any* time.
- Given a binomial or trinomial tree for the short rate it is straightforward to value both types of Bermudan swaptions.

Bermudan swaption into new (fixed tenor) swap

- Consider a Bermudan option, exercisable at times $T_0 = 0, T_1, \ldots, T_n = T$ with expiry T, where $T_i = i\Delta t$. At a given exercise time T_E the holder of the option can choose to enter into a swap making payments at times $T_E + i\Delta t$ until the end of the swap $T_E + T_S$.
- By the same method as above we can value the payer swap at each node as $S_{i,j} = N - CB_{i,j}$ (the receiver swap is $S_{i,j} = CB_{i,j} - N$) where CB is **now** the value of a coupon bond making coupon payments of K on a notional principal N at dates $T_{i+1}, \ldots, T_{i+1} + T_S$.
- First, we need the payoff of the swaption at time T given by $V_{n,j} = \max(S_{n,j}, 0)$ then at each point in time we compare the continuation value, CV:

$$CV_{i,j} = \frac{\pi V_{i+1,j+1} + (1-\pi)V_{i+1,j}}{1 + r_{i,j}}$$

Bermudan swaption into new swap: Example

- with the early exercise value, EE

$$EE_{i,j} = S_{i,j}$$

to give the option value as

$$V_{i,j} = \max(CV_{i,j}, EE_{i,j})$$

- Consider a two-year Bermudan option on a three-year receiver swap making fixed payments of 10% on a principal of \$100. This option can be exercised now, after one year, or at expiry.
- At expiry, we have already calculated the value of the option above in the European case. At t = 1, we can determine the value of the swap $S_{1,j}$ and compare it with the value of continuing.
- The calculations are on the next page, with early exercise of the swaption shown in red. The value of the swaption is \$8.22.

Swaption: valuation

j	$t = 0$	1	2	3	4	5
5						
4	$CB_{i,j}$ $V_{i,j}$ $r_{i,j}$				25.15%	-
3				- 18.4%	- 17.90%	-
2			87.91 $\max(0, -12.09) = 0$ 12.84%	- 12.84%	- 12.74%	-
1		98.53 $\max(0, -1.47) = 0$ 8.41%	98.21 $\max(0, -1.78) = 0$ 8.78%	- 8.96%	- 9.07%	-
0	108.22 $\max(3.38, 8.22) = 8.22$ 5.00%	107.10 $\max(3.06, 7.10) = 7.10$ 5.64%	106.46 $\max(0, 6.46) = 6.46$ 6.01%	- 6.25%	- 6.46%	-

Quiz

How do we calculate the following?

- $CB_{0,0}$
- $CB_{1,j}$
- $CB_{2,j}$

Quiz

How do we calculate the following?

- $CB_{0,0}$ Bond maturing at T = 3, paying coupons of \$0.10 at years 1, 2, and 3.
- $CB_{1,j}$ Bond maturing at T = 4, paying coupons of \$0.10 at years 2, 3, and 4.
- $CB_{2,j}$ Bond maturing at T = 5, paying coupons of \$0.10 at years 3, 4, and 5.

Quiz

What property of these short rate dynamics make it optimal to early exercise at $t = 0$ and $t = 1$?

- Rates are more volatile as time increases.
- Rates are generally higher as time increases.
- There are more possible rates as time increases.

Quiz

What property of these short rate dynamics make it optimal to early exercise at $t = 0$ and $t = 1$?

- Rates are more volatile as time increases.
- Rates are generally higher as time increases. CORRECT
- There are more possible rates as time increases.

Quiz

What do you imagine would be true of the corresponding payer swaption?

- You exercise early but only in the up states.
- It is never optimal to exercise early.
- The exercise strategy is the same as for the receiver swap.

Quiz

What do you imagine would be true of the corresponding payer swaption?

- You exercise early but only in the up states.
- It is never optimal to exercise early. CORRECT
- The exercise strategy is the same as for the receiver swap.

Bermudan swaption into existing (fixed maturity) swap

- The most common type of Bermudan swaption gives the holder the right to receive the remaining cashflows of a swap that has a fixed maturity date $T_E + T_S$, thus as T_E increases the tenor of the swap (T_S) decreases. This is again Bermudan as the holder is typically only allowed to exercise on a reset date.
- The valuation process is very similar to above, now the early exercise value is determined by the time $i\Delta t$ value of the underlying swap, rather than the value of a new swap. The maturity of the swap will be at least as large as the maturity of the option.
- To illustrate how this works, consider a T-year Bermudan swaption on $T+1$ year swap, where it is possible to exercise the option at any year up until year T. We divide the lifetime of the option into n steps, $T_0, T_1, \ldots, T_n = T$.
- At maturity, then the value of the swaption is $V_{n,j} = \max(N - CB_{n,j}, 0)$ (or $\max(CB_{n,j} - N, 0)$) where $CB_{i,j}$ is now the value of a coupon bond paying annual coupons and expiring at $T+1$.

Bermudan swaption into existing swap: Example

- Then the option can be valued by comparing the early exercise value with the continuation value as before where:

$$CV_{i,j} = \frac{\pi V_{i+1,j+1} + (1-\pi) V_{i+1,j}}{1 + r_{i,j}}$$

$$EE_{i,j} = N - CB_{i,j}$$

$$V_{i,j} = \max(CV_{i,j}, EE_{i,j})$$

- Using the same short rate tree, consider a 4-year Bermudan swaption on a five year annual swap an a principal of $100 with a fixed rate of 10%.
- The calculations are shown on the next page, and the value of the Bermudan swaption is $4.43 and there is optimal early exercise at some states at $T = 2, 3$, indicated in red in the tree.

Swaption: valuation

j	$t=0$	1	2	3	4	5
5						110
4	$CB_{i,j}$ $V_{i,j}$ $r_{i,j}$				87.89 12.10 25.15%	110 -
3				84.96 15.03 18.4%	93.30 6.70 17.90%	110 -
2			87.91 12.09 12.84%	93.44 6.56 12.84%	97.57 2.43 12.74%	110 -
1		95.07 7.20 8.41%	98.21 3.53 8.78%	100.23 1.11 8.96%	100.85 0 9.07%	110 -
0	105.43 4.34 5.00%	106.34 1.92 5.64%	106.46 0.53 6.01%	105.50 0 6.25%	103.32 0 6.46%	110 -

Quiz

How do we calculate the following?

- $CB_{0,0}$
- $CB_{1,j}$
- $CB_{2,j}$

Quiz

How do we calculate the following?

- $CB_{0,0}$ Bond maturing at T = 5, paying coupons of \$0.10 at years 1, 2, 3, 4, and 5.
- $CB_{1,j}$ Bond maturing at T = 5, paying coupons of \$0.10 at years 2, 3, 4, and 5.
- $CB_{2,j}$ Bond maturing at T = 5, paying coupons of \$0.10 at years 3, 4, and 5.

Types of features

- The backward induction method employed in Binomial and trinomial trees is perfect for valuing derivatives with American features, as we have seen above. As is well known, we will have more difficulties when using Monte Carlo methods.
- Path dependency is a less natural extension for binomial trees but in the one-dimensional short rate tree it is possible to price derivatives with path dependent features, such as barriers, lookback and Asian options.
- These are best illustrated by considering adaptations to the cap contract, rather than swaps.
- Recall that a cap on a notional principal N provides a payoff of

$$N\tau \max(L(t, t+\tau) - K, 0)$$

at $t + \tau$Where Δt is now the reset frequency of the cap.

Quiz

If we choose time steps smaller than τ how will we estimate $L(t, t+\tau)$?

- Calculate the bond prices $P(t, t+\tau)$ and back it out.
- Multiplying the current $r_{i,j}$ by $\tau/\delta T$.
- Adding up the individual $r_{i,j}$.

Quiz

If we choose time steps smaller than τ how will we estimate $L(t, t+\tau)$?

- Calculate the bond prices $P(t, t+\tau)$ and back it out. CORRECT
- Multiplying the current $r_{i,j}$ by $\tau/\delta T$.
- Adding up the individual $r_{i,j}$.

LIBOR rate problem

- As before, time to maturity T is divided into n steps of length Δt, giving times and the short rate $r_{i,j}$ denotes the rate from time $i\Delta t$ to $(i+1)\Delta t$. If Δt coincides with a reset date τ then it is natural to model $L(t, t+\Delta t)$ as $r_{i,j}$. Ensuring, of course, that the process for $r_{i,j}$ matches the prices of the appropriate vanilla instruments.
- If the two periods do not match then the appropriate rate $L(t, t+\tau)$ then $L(t, \tau)$ can be backed out from zero coupon bond prices, so

$$L(i, j, i+\tau) = \frac{1}{\tau}\left[\frac{1}{P(i, j, i+\tau)} - 1\right]$$

where $L(i, j, i+\tau)$ and $P(i, j, i+\tau)$ are the LIBOR rate and bond prices at time i in state j over a period of τ.

Valuing a barrier caplet

- Down-and-out barrier caps are caps where the caps expire worthless if the reference rate falls below a certain level, H, known as the barrier level.
- To value a caplet with a down-and-out barrier is simple. The payoff of the caplet at time T_n is given by

$$V_{n,j} = \begin{cases} P(n,j,n+\tau)N\tau\max(L(n,j,n+\tau)-K,0) & \text{if } L(n,j,n+\tau) > H \\ 0 & \text{if } L(n,j,n+\tau) < H \end{cases}$$

- Working back through the tree, the value of the caplet is

$$V_{i,j} = \begin{cases} \frac{\pi V_{i+1,j+1}+(1-\pi)V_{i+1,j}}{1+r_{i,j}} & \text{if } L(i,j,i+\tau) > H \\ 0 & \text{if } L(i,j,i+\tau) < H \end{cases}$$

- This values an individual caplet, the cap will be the sum of the individual caplets.

Quiz

- How is this complicated by $\Delta t < \tau$?
- How do we fix this?

Quiz

- How is this complicated by $\Delta t < \tau$? we no longer observe $L(t, t+\tau)$
- How do we fix this? Use an A-D tree, or standard tree to value $P(t, t+\tau)$ and rearrange to find $L(t, t+\tau)$

Lookbacks and Asians

- Derivatives with other types of path dependent features can be valued using adapted binomial trees.
- Options with lookback features where, for example, the cap rate at time t is determined by the **maximum** or **minimum** level the LIBOR rate has achieved, can be valued by including an extra dimension in the binomial tree that monitors the maximum level reached.
- Options with Asian features where, for example, the cap rate at time t is determined by the average level of LIBOR over a particular period, again can be valued by including an extra dimension for the average level.
- These methodologies are very similar to those used for valuing path dependent options on equity options and so techniques used for valuing then for equity options can be applied to interest rate derivatives too.

Monte Carlo for short rates: Payoffs

- We will be using the Monte Carlo method extensively when we consider the LIBOR market model , but it is also possible to use the Monte Carlo method when using short rate models. Typically, as is usual for Monte Carlo it is best used when there are multiple stochastic factors or when there are many path dependent features.
- Assume that we need the current $t = 0$ price of a path dependent derivative with European exercise features. This payoff can be a function of the short rates $r(t_i)$ at specific, pre-determined times $0 < t_1 < t_2, \cdots < t_m = T$ where T is the final maturity of the option.
- If we can have payoffs H at different times then the current value of the derivative is

$$V_0 = \sum_{j=1}^{m} \exp\left[-\int_0^{t_j} r(s)ds\right] H(r(t_i), \ldots, r(t_j))$$

Monte Carlo for short rates: Getting simple rates from short rates

- Usually the nature of the payoff depends on simple spot rates such as $L(t_i, t_i + \tau; r(t_i))$ where $\tau = 0.5$ that depend upon the current short rate $r(t_i)$. *If* we are using a short rate model that has an analytic zero coupon bond price formula then we can use this to back out $L(t_i, t_i + \tau)$.
- If we have a short rate formula without an analytic bond price then we will need to use trees to value it for each $r(t_i)$ at each possible point in time. This is computationally intensive, but less so than running a series of mini Monte Carlo simulations to the desired accuracy.
- As usual we then sample p paths for the short rate, at $q + 1$ sampling times $0 = s_0 < s_1 < s_2 < \cdots < s_q = T$ which can simply be the important times t_j or more frequent than those. Then $\Delta s_i = s_{i+1} - s_i$.

Monte Carlo for short rates: Discounting

- Within a Monte Carlo framework it is usual to approximate the discounting term
$$\exp\left[-\int_{s_k}^{s_j} r(s)ds\right]$$
by
$$exp\left(-\sum_{i=k}^{j-1} r(s_i)\Delta s_i\right)$$
but this is only accurate when the Δs_i values are small - again adding to the computational burden.
- A more sensible approach is to use $P(t, T)$ as the numeraire asset (we will learn how this affects the drift later):
$$E\left[\exp\left(-\int_0^{t_j} r(s)ds\right) H(r(t_i), \ldots, r(t_j))\right] = P(0, T)E^T\left[\frac{H(r(t_i), \ldots, r(t_j))}{P(t_j, T)}\right]$$

Monte Carlo for short rates: modeling the short rate

- This again only works well if the Bond price formula is analytic for the choice of short rate model, otherwise this modified formula is still computationally intensive.
- Now all that remains is to simulate the short rate. If it is of a nice form where we know the transition density function then we can take sample from the exact distribution of $r(s_{i+1})$ given $r(s_i)$.
- If we do not know the transition density function then we will need to approximate the change in the Brownian motion term which is simple as it is Normally distributed with mean 0 and variance Δs_i and independent of all other drawn variables. Thus at the start of the scheme we draw $p \times q$ Normally distributed random numbers.
- Then we can approximate the SDE for the short rate by an Euler discretization.

Monte Carlo for short rates: Valuation

- Finally we can value the derivative by determining the adjusted payoff on each path $1 \leq i \leq p$

$$h_i = \sum_{j=1}^{m} \frac{H(r(t_1), \ldots, r(t_j))}{P(t_j, T)}$$

  and then averaging and discounting them back to today

$$V_0 = P(0, T) \frac{\sum_{i=1}^{p} h_i}{p}$$

- In general, short rate models only have one or two stochastic factors and so most derivatives can be valued by using trees (or finite difference methods). There is rarely any advantage to using Monte Carlo methods unless there are a series of path dependent factors. However, Monte Carlo techniques are generally easy to implement and can serve as a useful benchmark.
- Finally, we will learn how to implement Monte Carlo methods in general for market models.

Conclusion

- We have seen that if it is possible to build a tree of the short rate then it will be possible all bonds, swaps and other vanilla products using the binomial tree. Correspondingly, given market bond prices it may also be possible to ensure that the tree reproduces these prices.
- Once we have the short rate tree then it is possible to value a wide array of derivatives, particularly those with early exercise, and also those with path dependent features. We need to be careful how we carry out the discounting when our time steps are shorter than the reference rates.
- Finally, we saw that it is also possible to use Monte Carlo methods for pricing interest rate derivatives, these will become more important as we move to models with multiple stochastic factors.