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# Lecture Note 9.2: CDOs as Derivatives

## Introduction:

Building on the analysis from last time, today we will look in more detail at modelling CDOs. Our motivation for doing so is to begin to understand how securitization works – or does not work.

Securitization is, and will continue to be, a huge part of finance. However its success hinges on the ability to accurately model its risks. The 2007-2009 financial crisis which involved great losses to many holders of CDOs, also claimed as casualties the models which people thought were telling them how to value and hedge these products. Until confidence in the risk modelling is restored, the market will not fully recover.

Our examination today will focus on those CDOs that are the easiest to model in a no-arbitrage framework.

## Outline:

**I.** Idealized CDOs.

**II.** High-Dimensional Models.

**III.** Hedging and Arbitrage.

**IV.** Conclusions.

**I.** Idealized CDOs.

- At its simplest, a CDO is just a mechanical rule for sharing cash-flows from a pool of risky assets between several tiers (tranches) of claimants of differing seniority.

- In an idealized case, the underlying assets are all bonds or loans maturing at the termination date of the CDO. The CDO is then totally passive.

- Even more idealized, the underlying bonds are all <u>tradeable</u> assets that can be used to hedge.

- Many CDOs *aren't like this.*
  - ▶ The assets are not separately traded (or are very illiquid).
  - ▶ The assets may have differing lives.
  - ▶ The CDO manager may have the right to change the assets by trading in the market.
  - ▶ Some of the tranches may be callable.
  - ▶ There may be additional guarantees of repayment by third parties.

  *CDOs like this are not derivatives.* We should not expect no-arbitrage models to apply to them.

- But the idealized CDOs do exist and also do constitute a large segment of the market. So we will first study them.

- Actually we can distinguish between two types of such securities.

  **Cash.** These are what we just described. They hold tradeable corporate bonds maturing at $T$. The tranches are defined by their upper and lower attachment points (and their coupon interest rates). When a bond defaults, the principal amount of the lowest tranche is reduced by the amount of the loss, $(1-R)$. The amount $R$ is paid to the highest tranche (and its remaining principal is lowered by that amount.)

  **Synthetic.** These own riskless bonds maturing at $T$ and *short positions* in CDSs (to date $T$) on a collection of risky names. The fees they receive plus the riskless interest makes up the interest payments to the tranches.

- In our idealized case, the two structures are economically identical. The cashflows are a function only of the history of default losses.

- A further sub-case of a synthetic CDO is **<u>unfunded</u>**.
  - ▶ Here there are no riskless bonds at all, so the only cash flows to tranche holders come from the short CDS positions.
  - ▶ When an investor goes "long" a tranche, he pays nothing. He receives his share of the CDS fees, and nothing at maturity.
  - ▶ When a default occurs, the lowest tranche must <u>pay</u> $(1-R)$ to settle the CDS – until its notional principal is exhausted. The top tranche's principal is reduced by $R$, but it receives no cash.

- We could go further and construct a purely theoretical unfunded synthetic CDO today by selecting a set of names and *imagining* that we sell protection on them and then tracking their collective cashflows until $T$.

- In fact, the market does this. There are several standard indexes of CDS portfolios.
  - For example, the **CDX, NA, IG** index tracks the performance of a portfolio of 125 equally-weighted North-American, investment-grade, cash-settled, 5-year CDSs. A new version is created every six months.
  - We can then define a hypothetical CDO tranche by specifying upper and lower attachment points. We can then track the hypothetical cash-flows of any such tranche over time.
  - Also, people can then trade these purely theoretical index tranches by doing what amount to **return swaps**: one side paying a fee (as a percentage of remaining principal), and the other paying the default losses for which the tranche of the portfolio is responsible.
  - We then have market-determined prices (fees) for each tranche over time.

  Notice that no one actually ever has to create this CDO as a legal entity and there can still be trade in the synthetic tranches. And there is no constraint on their size.

- We are going to model the CDO tranches of securities like this as a function of (a) the prices of the underlying bonds, or equivalently, (b) their CDS rates, or (c) the prices of zero-coupon, zero-recovery bonds of the same entities.
  - If the upper and lower attachment points for a particular tranche are $U$ and $L$, and there are $N$ names, we want to find something like

$$P^{UL}(\widehat{B}_t^{(1)}, \widehat{B}_t^{(2)}, \ldots \widehat{B}_t^{(N)}, )$$

    or

$$P^{UL}(\varphi_t^{(1)}, \varphi_t^{(2)}, \ldots \varphi_t^{(N)}, )$$

  - Actually we have to start by recognizing a more fundamental question: *when can we do this?*
    - Under what conditions are the CDO's risks spanned by the available assets?

  - Today, I will assume the riskless rate is fixed.
  - I will also assume the recovery rates, $R^{(i)}$ are known in advance.

**II.** More About High Dimensional Models.

- Last time we considered models in which default correlation can be induced between two entities by means of models of Poisson jumps to default.
  - ▶ Two entities can have common jump components.
  - ▶ And/or they can have correlated stochastic jump intensities.
- However there is a practical issue with this approach: with $N > 2$ entities it can involve a lot of computer power.
  - ▶ Most CDOs have more than 100 names in their asset portfolio. So, with stochastic intensities, you'd have to simulate, e.g., 100 correlated $\lambda$s for perhaps 1000 time steps (e.g. 200/year times 5 years).
  - ▶ Then, to get accurate values for the low-risk tranches (which don't experience default often), we might need 100000 paths.
  - ▶ This would be very slow, especially if we wanted to try lots of different input values to see how the answer changed.
- In fact, all the work generates information we don't really need, namely, the entire path of each firm's experience. *All we really need to know are the <u>default dates</u> of each name.*
  - ▶ Let me re-emphasize this point. If we are trying to compute

$$\mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[\int_{t}^{T} B_{t,\tau}\Gamma^{UL}(u)du]$$

where $\Gamma^{UL}$ are the cash flows to a given tranche, then the amount of those cash-flows is solely determined by which of the underlying reference entities are alive at each date.

- Thus, conditional on the *realized* default times of all of our entities (call them $\tau^{(1)}, \tau^{(2)}, \ldots \tau^{(N)}$), our valuation is just a deterministic exercise in mapping the cash-flows to the tranches.

- An efficient model, then, would be one that told us directly the distribution of the default dates.
  - Then we could just draw one random number for each entity from that distribution for each simulated outcome.

- We have seen that we can infer the risk-neutral default distribution from market prices of single-name credit protection (assuming known recovery rate).

- If $H(s)$ is the implied probability that $\tau \leq s$, then we can draw random numbers from this distribution by the so-called *inverse cdf* method.

  **(i)** Draw $u$ from a uniform random number generator.

  **(ii)** Compute $H^{-1}(u)$, i.e. find the value, $t^*$ on the horizontal axis such that $H(t^*) = u$.

  **(iii)** Repeat.

  By construction $t^*$ has the right distribution:

  $$\mathrm{Prob}(t^* \leq s) = \mathrm{Prob}(H^{-1}(u) \leq s) = \mathrm{Prob}(u \leq H(s)) = H(s)$$

  since $u \sim U([0,1])$.

- Often, the distributions extracted from CDS prices can be fit pretty well by a simple exponential function:
$$H(s,t) \doteq 1 - e^{-\hat{\lambda}^Q (s-t)}.$$
  - ▶ If that holds, then $H^{-1}(u) = -\log(1-u)/\hat{\lambda}^Q$.
  - ▶ Or, since $1-u$ also $\sim U([0,1])$, we can just put $t^* = -\log(u)/\hat{\lambda}^Q$.

  (Warning: we saw last time that a model with constant jump intensity $\lambda$ would in fact lead to exactly this type of $H$ cdf, BUT, if you want to model the intensity as stochastic, the cdf won't be exponential. And it will change over time.)

- Returning to our multivariate world, we really want is to simulate default times from an $N$-dimensional process which has specified *marginal* default intensities.
  - ▶ For this we need some specification for how the marginals are composed into the joint distribution.
- It is crucial to realize that the CDS market tells us nothing about this.
- If we just apply our inverse cdf method to each of the marginals independently, then that corresponds to the very strong assumption that the default events are independent – which we don't believe.
- It is not obvious, even for the case of (constant intensity) exponential random variables, how one should even define correlated defaults.

- One way of constructing correlated exponential outcomes is to do the following.

  **(i)** Draw one normally distributed vector $Z \sim \mathsf{N}(0, \mathcal{R})$ using some correlation matrix $\mathcal{R}$ .

  **(ii)** Turn these variables into uniform variates (on $[0, 1]$) by applying the one-dimensional normal CDF: $\nu^{(n)} = \mathcal{N}(Z^{(n)})$.

  **(iii)** Turn these into exponentially distributed draws via $\tau^{(n)} = -\log(\nu^{(n)})/\hat{\lambda}^{Q,(n)}$

  This is called the "normal Gaussian copula" method (NGC).

- If the normal draws for two entities are highly correlated, then the default times (scaled by respective intensities) will be highly correlated.

- Notice the last step is just the inverse cdf method. The second step might be called the inverse-inverse cdf method for generating uniform numbers.

- The first step is the one we really have to pay attention to.

- There is no particular reason to use the multivariate normal distribution in the first step. It is just a distribution for which correlated variates are easy to simulate. But so is the Student-$t$ distribution, for example.

  ▶ In other words, the use of normal copula is not a consequence of any model about the real world, or a required condition for no-arbitrage pricing. It is purely for convenience.

- It's not true, for example, that a default component model like the one last week (where every entity is exposed to a common systematic default event) will yield a joint distribution that corresponds to a Gaussian copula.
  - Moreover, if the $\lambda^{Q,(i)}$ are changing, then the default times won't be exactly exponential.

- It is not at all obvious that any reasonable model model of risk factors and their market prices of risk can be combined to yield this type of risk-neutral distribution.

- To believe in a given specification for a "pricing measure," it must be the case that you can identify the parameters required for the risk-adjustment via the returns of traded assets.

- Financial engineers who decided they were free to specify an arbitrary pricing process to suit their computational needs seem to have lost touch with the underlying foundations of the theory.

- But....

- Notice how easy the NGC approach makes life!

- ▶ One single draw from an $N$-dimensional normal gives us all the information we need for an entire default history of all our $N$ names.
- ▶ So evaluating expected outcomes by averaging over $M = 100000$ histories can be done in a few microseconds.

- There is an even further simplification one can do that still preserves the marginal distributions.
  - ▶ Assume that the normal variate $Z$ is the sum of two normal terms, one systematic and one idiosyncratic :
$$Z^{(n)} = \sqrt{\rho}\; Z^{syst} + \sqrt{1-\rho}\; Z^{idio,(n)},$$
    where the idiosyncratic components are independent across names.
  - ▶ This means every two firms have the same pair-wise correlation $\rho$.

- This version of the NGC used to be called the "market standard model".

- Notice that, like the Black-Scholes model, the market standard model now only depends on one unobservable parameter, $\rho$.
  - ▶ And, like the Black-Scholes formula, one can invert the model to find an *implied correlation*.
    - ∗ One tries different $\rho$ until the implied model price matches the observed market price of a given CDO tranche.

- It is remarkable that the simulation step is fast enough to permit a such a search computation.

- But note that this parameter has a murky meaning, even under the model.

  - It is <u>not</u>, for example, equal to the correlation in firms' firm values or bond prices.

  - It is not even the correlation between their intensity processes.

  So one must be extremely careful in interpreting statements about it.

- Like implied vol, however, one can always transform market prices into these units and use them to talk to other traders – even if no one accepts or believes the model.

- <u>If</u> the model were right, and you had the "true" $\rho$, then one could arbitrage mispriced tranches by delta-hedging using the exposures to each of the underlying bond prices (hedge ratios) implied by the model.

- For many years the "market standard model" was used by quants to price CDOs.

- It cannot be over-emphasized, however, that there is no economic underpinning at all to this model. It is merely a conjecture about what the risk neutral joint distribution might look like – given the marginals.

- While people are now more sensitive to the dangers and inconsistencies of this model, it can still be useful, at least as a benchmark, for simulating outcomes.

- If I have an $M$-vector $lambdaQ$ of marginal default probabilities with copula correlation $rho$, and I want to simulate a number $pools$ of pool histories, it's three lines of MATLAB code:
  - ▶ Note: everything to the right of a % sign is a comment.

```
%-----------------------------------------------------------------
M=125;  pools=100000;

systematic    = repmat( randn(1,pools), M,1);
ncop          = stdn_cdf(  sqrt(1-rho)*randn(M,pools) + sqrt(rho)*systematic );
default_times =-log( ncop )./lambdaQ;

% That's it! Now you have one matrix of 125 X 100k default times.
%-----------------------------------------------------------------
```

- Now if we want to price a tranche, all we have to do is compute its discounted payouts for each of these outcomes, and average across the pool simulations.

- For an unfunded synthetic CDO, we just compute the PV of the tranche's losses.

- If we want to express the price as a fee, then – just like for a CDS – we have to value the fee-side as the discounted value of the coupon rate times the surviving principal amount (call it the coupon multiplier) at each date.

- Then the fair fee would be the ratio of the (PV) expected tranche losses to the (PV) of the cumulative expected coupon multiplier.

- For example, if we wanted to compute these quantities for a tranche with $L = 3$ and $U = 7$, then we might do something like this in MATLAB:
  - Note: $.*$ means element-by-element matrix multiplication.

```
%-----------------------------------------------------------------
% Define CDO tranche

r=.02;  % riskfree rate
T = 5;  % maturity


lower_pt  =  3 ;      % attachment points for a given tranche
upper_pt  =  7 ;
tr_width  = upper_pt-lower_pt;




recovery = .60;
loss_per_def = (100/M)*(1-recovery);  % express these as
rec_per_def  = (100/M)*recovery;      % percentage of pool notional value


%--------------------------------------------------------------------
% got the default time matrix from copula step above


def_times = sort(default_times);            % sort() operates columnwise

% Not all defaults happen before the CDO matures. Set others to zero.

def_times       = min(def_times, T);
before_T        =  (def_times<T);        % boolean operator

pool_loss       = loss_per_def.*before_T;
```

```
CPL = cumsum( pool_loss );              % cumulative sum operator

pool_rec          = rec_per_def.*before_T;
CPR = cumsum( pool_rec );


% Build matrix of tranche experiences at each default time

tr_losses =  max(0,  min(CPL,upper_pt) - max(CPL-pool_loss,lower_pt) );

tr_loss_pct = tr_losses./tr_width;

pv_tranche_pct_loss_pct  = exp(-r*def_times) .*  tr_loss_pct;


% Now have to tabulate principal outstanding at each date to compute PV of coupon

% Tranche may experience lowering of principal from notional recovery

tr_recovery  = max(0, min(CPR, 100-lower_pt) - max(CPR-pool_rec, 100-upper_pt) );

tr_recovery_pct   = tr_recovery./tr_width;


% Outstanding notional is face value minus losses minus recoveries

tr_principal = 1.00 - cumsum(tr_loss_pct) - cumsum(tr_recovery_pct);



% Build present value factors for fees paid between each default

delta_t =def_times - [zeros(1,pools) ; def_times(1:end-1,:)];

pv_coupon_multiplier   = ((exp(-r*(def_times-delta_t))-exp(-r*def_times))/r) ...
                         .* tranche_principal.*before_T;



% done!

fee =  mean(sum(pv_tranche_pct_loss_pct))/mean(sum(pv_coupon_multiplier));

%----------------------------------------------------------------------
```

**III.** Hedging, Arbitrage, and Risk Control.

- We can view CDOs as pure derivatives because their cash-flows are replicable by trading in underlying bonds or CDSs – under some assumptions.
  - ▶ The big assumption is that we have to believe we have the right risk-neutral dependency structure.

- Let's think a little more about the role of the models in doing CDO arbitrage.

- I have pointed out that the normal Gaussian copula approach and the market standard model were not derived from an underlying model of actual default intensities and risk premia, and therefore do not seem particularly believable.
  - ▶ Indeed, they seem to require us to believe in constant exponential default intensities, which is counterfactual.
  - ▶ Anytime a model is wrong it will give us wrong hedge ratios.

- We might fix the asumption of exponential intensities by using the correct market implied $H$ densities in the third step.
  - ▶ We still have to believe in the constant copula correlation $\mathcal{R}$.

- Let us put the convenience of the NGC approach aside for a minute and go back to thinking about the default intensity type models that we studied last time.

- Assume that we have generated a joint distribution of defaults whose foundations we are comfortable with.

- How do we hedge?

- Here there is a crucial observation stemming from the jump-to-default models:
  - ▶ *For each underlying name, there are <u>two</u> hedge ratios.*

- If we thought default intensities were constant, then we could write our tranche value as:

$$P^{UL}(\widehat{B}_t^{(1)}, \widehat{B}_t^{(2)}, \dots \widehat{B}_t^{(N)}, )$$

  and its hedge ratio with respect to the first bond would be the change upon default:

$$\frac{P^{UL}(R, \dots \widehat{B}_t^{(N)}) - P^{UL}(\widehat{B}_t^{(1)}, \dots \widehat{B}_t^{(N)})}{R - \widehat{B}_t^{(1)}}.$$

  - ▶ In fact, under constant intensities, nothing besides a default can change the CDO value.

- Or, suppose we forget about jumps and go back to diffusion models of the bonds, then we'd compute:

$$\partial P^{UL}(\widehat{B}_t^{(1)}, \dots \widehat{B}_t^{(N)}) / \partial \widehat{B}_t^{(1)}.$$

- But with the jump-to-default model AND stochastic intensities we have to consider <u>both</u> default risk and diffusive risk of credit deterioration prior to default.

- This is the same situation as in option valuation with stochastic volatility.
  - ▶ Even though the tranche's <u>cash-flows</u> only depend on the names' survival status, their prices also depend on another parameter.

- To put it another way, we should write our tranche value as:

$$P^{UL}(\hat{\lambda}^{Q,(1)}_{t,T}, \ldots \hat{\lambda}^{Q,(N)}_{t,T};\ 1_{\{\tau^{(1)} \leq t\}}, \ldots, 1_{\{\tau^{(N)} \leq t\}}).$$

- Once we recognize this dual dependence, it follows immediately that we need more than one hedging instrument for each name in the pool.

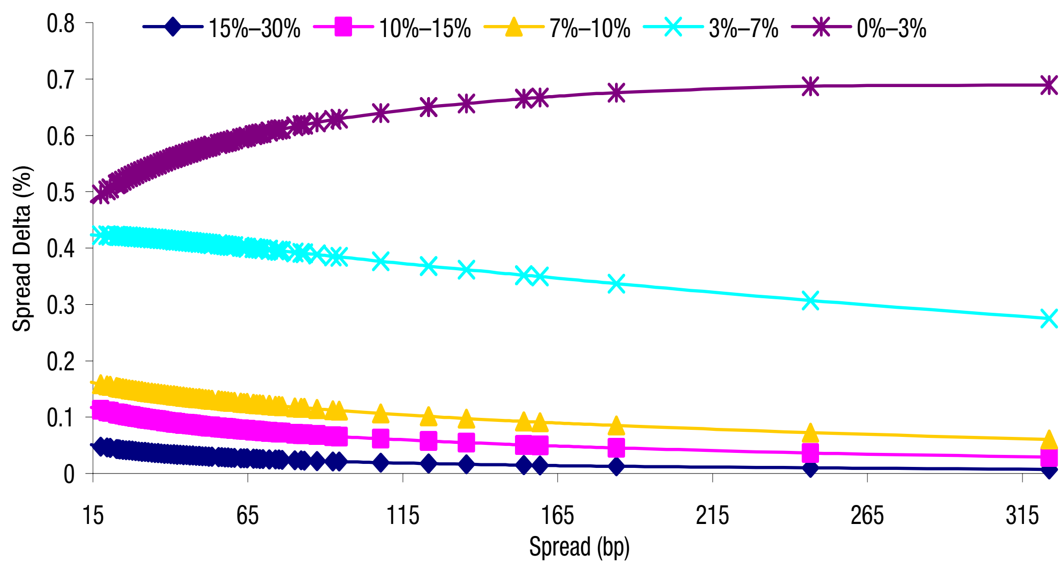
- So complete risk control means that we might not be able to perfectly hedge with just the underlying CDSs alone.
  - ▶ If we use them to eliminate the jump risk of a default event, then we may need something else – perhaps stock or options – to hedge the risk of credit deterioration *prior to* default.
  - ▶ Another analogy is to view this as like having to do delta and gamma hedging of a complex derivative: a single option generally won't let you hedge both.

- In practice, market participants *do* compute just a single tranche $\delta$s with respect to individual credit spreads.
  - This will work as long as the underlying bonds can only deteriorate diffusively.
- Here the simplicity of the NGC-model can lead people to overlook potentially dangerous misspecification.
- That said, one can use it to illustrate some important intuitions.
- Tranche loss deltas with respect to individual names are a function of the level of the risk of that entity.
  - For equity tranches, for low initial values of default probability, hedge ratios initally rise as each name deteriorates, creating effectively negative gamma. You have to sell more protection in down markets.
  - Intuitively, as a name deteriorates, the amount by which it will affect your losses converges to a fixed number.

- I have been neglecting a small point so far in hedging mechanics: the fee side of a CDO tranche is also exposed to the default risk.
  - ▶ Synthetic CDO tranches receive the fees from the underlying CDSs, and those fees stop when the individual names die.
  - ▶ That was why my Monte Carlo code above had to adjust the present value of the fee stream at the default times.
- The total value of a tranche position (the "mark-to-market" or MtM) is then the net difference between the value of the fee leg and the value of the protection leg.
- And it is the change in that value that we want a hedge ratio to quantify.
  - ▶ Moreover we can express this hedge in a variety of equivalent ways, including
    - ∗ Change in an existing CDO's value (MtM) per change in an individual CDS fee in basket.
    - ∗ Change in an existing CDO's value (MtM) per change in an individual CDS MtM value.

    The latter is the same as the number of CDS to sell per dollar exposure of the pool to that name.
    - ∗ That's what this graph shows.

**Figure 3. Individual Credit Spread Deltas as a Function of Individual Credit Spread Levels**




Source: Citigroup.

- The vertical axis is a fraction (not a percentage). The horizontal axis is the current spread level of the name being hedged.

- Notice that the higher tranches have positive convexity in that they become less exposed to an individual name as it deteriorates, because if it defaults, it's somebody else's problem.
  - ▶ Of course, this calculation was done for a *new* CDO in which none of the tranches have yet suffered a loss of principal.

- Besides the deltas, it might have occurred to you that we could hedge other inputs to the model that we are uncertain about, for example, $\rho$.

- It is certainly true that correlation has a huge effect on tranche valuation. But here we again run into the consistency issue.

▶ The model does not think we face any instantaneous risk from changing correlation.

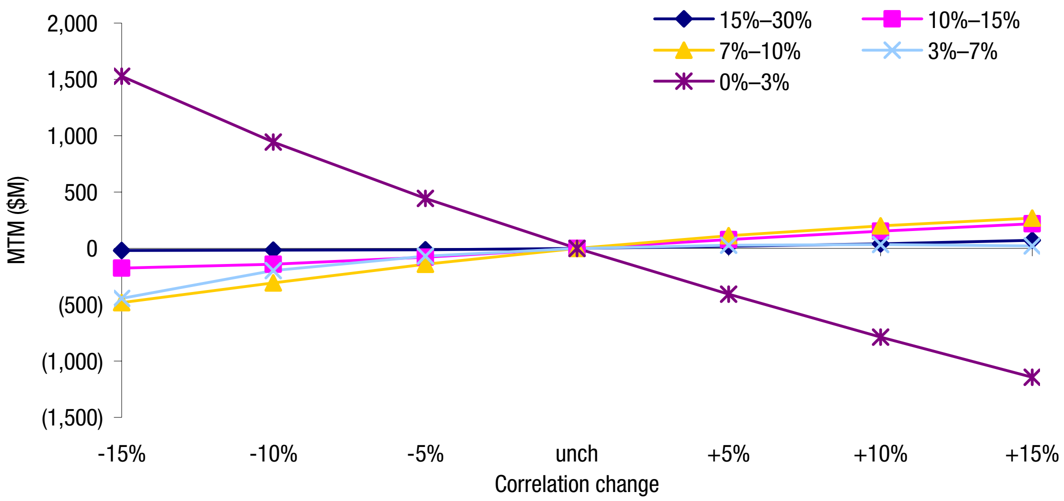
▶ Recall the correlations come about through the joint evolution of the $d\lambda^Q$.

- While the model does not think we need to hedge changes in $\rho$ that does not mean we cannot do it.
  ▶ In fact, a lot of traders do try to maintain correlation neutral positions.

  ▶ Others use CDOs precisely in order to bet on correlation.

- In either case, quantifying sensitivity to $\rho$ is something that is very easy to do by just re-running the Monte Carlo.
  ▶ The picture below shows tranche mark-to-market <u>losses</u> as $\rho$ changes.

**Figure 6. Mark-to-Market Value Change Versus the Correlation Change**




Source: Citigroup.

**IV.** Summary.

- CDOs are now widely regarded as *inherently dangerous* and threatening to the financial system.

- However, I do not think this view applies to the pure derivative CDOS we have studied today.
  - ▶ Although they were a huge part of the market, the synthetic ones are in net zero supply. So there were as many losers as winners for every position.

  - ▶ It is not the case that CDOs comprised of ordinary corporate bonds contributed significantly to the bank losses that were behind the credit crisis.

- But this is not to say that they are safe or easy to hedge.

- We have seen that they can be analyzed in a no-arbitrage framework.

- However, the models required to do so involve a lot of parameters and a lot of assumptions, so there is a very serious risk of model misspecification – particularly with regard to the dependency structure.

- While the computational simplicity of valuation via the direct simulation of correlated default times is appealing, this approach obscures the connection to the underlying hedging instruments – and may lead to naive and misleading hedge ratios.

## Lecture Note 9.2 Summary of Notation

| SYMBOL | PAGE | MEANING |
|---|---|---|
| $R$ | p3 | recovery value of a bond for CDS settlement |
| $P^{UL}$ | p5 | price of CDO tranche with |
| | | upper and lower attachment points $U$ and $L$ |
| $N$ | p5 | number of bonds or credits in CDO pool |
| $\widehat{B}^{(i)}_{t,T}$ | p5 | zero-coupon bond of entity $i$ |
| $\varphi^{(i)}_{t,T}$ | p5 | CDS fee of entity $i$ |
| $\Gamma^{UL}(u)$ | p6 | CDO tranche cash-flow at time $u$ |
| $\tau^{(i)}$ | p7 | default time of entity $i$ |
| $H(s)$ | p7 | RN probability of default before time $s$ |
| $U$ | p7 | uniform distribution |
| $\mathsf{N}(0,\mathcal{R})$ | p9 | normal with mean zero and correlation matrix $\mathcal{R}$ |
| $Z$ | p9 | randomly-drawn multivariate normal variate |
| $\nu$ | p9 | transformed uniform variate for copula |
| $\tau^{(n)}$ | p9 | simulated default date from random draw $Z^{(n)}$ |
| $Z^{syst}, Z^{idio,i}$ | p11 | systematic and idiosyncratic normal variates |
| $\rho$ | p11 | correlation between any two normal draws in copula |
| $1_{\{A\}}$ | p17 | indicator function $=1$ if A happens, 0 otherwise |