FIN516: Term-structure Models

Lecture 13: LIBOR market model - how to price derivatives

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Introduction

- Now we will go through the types of products that you have to value.
- We will start with caplets and caps even when we have the wrong numeraire.
- We will then look at products such as swaptions that can be priced using a single jump and also the very long jump technique.
- Then we will look at more complex products that require the use of the long jump technique.
- Finally we will look at products that require consideration of early exercise.

Types of products LMM can value

- Rebonato classes the products that can be handled into four categories:
  1. Path-dependent derivatives.
  2. Single look options
  3. Multi-look or Path-dependent compound products

1. Path dependent derivatives

- A path dependent derivative is one where the payoff at a time T_M depends upon the realization of a series of forward rates on reset dates $T_{k-1}, 1 \leq k \leq M$.
- Path dependent derivatives can also be split into two further categories: payoffs that depend on the joint realization of a series of forward rates, which by the time of the price sensitive event have all come to their own reset. These types of products can be valued using the very long jump procedure.
- Or, alternatively, payoffs where some might not have reset, these products must be valued using a long jump procedure.
- In general, these products must have payoffs that can be determined by market observable quantities rather than the discounted future expectations of values, and that they only depend upon forward rates at a finite set of observation points.

2. Single-look simple or compound derivatives

- Here there is only one price sensitive event that can be determined given the knowledge of a set of spanning forward rates at a single future time.
- Note that in this type of derivative at most one forward rate will have come to its own reset by the maturity of the derivative.
- These derivatives will be subject to the choice of the instantaneous volatility function as most of the forward rates will have not reset and so we cannot simply use the Black volatility for the volatility of these rates.
- We will see two examples below: the swaption and the caption.

3. Multi-look/path-dependent compound derivatives

- The most important derivative in the multi-look category is the Bermudan swaption. In this case we have a series of dates when price sensitive events occur.
- Now, however, the exercise strategy depends upon realizations of future expectations of all of the forward rates.
- Ideally we would like to use a tree for these types of products but such a tree would be non-recombining (or 'bushy') and this can be very time consuming for large numbers of stochastic factors and exercise dates.
- We will discuss more below.
- The path dependent options are easily dealt with in a Monte Carlo setting but problems occur when there is compounding as this requires knowledge of an early exercise boundary, causing difficulties.

Which length of jump?

- I have broken the products down into three basic types:
  1. **Simple products.** These can be values using either the very long jump or the long jump, but in either case only one time step is required for valuation.
  2. **Long jump products.** These are the most challenging to value as they require multiple steps in a Monte Carlo model. Here we need to know many forward rates at different points in time, this will include products with early exercise or call features.
  3. **Very long jump products.** These are relatively straightforward to value once you have set up your covariance matrix correctly. These are generally path dependent products but whose payoffs depend only on forward rates on reset dates.

Caplets and caps

- To value a caplet using an adaptation of Black's model, with a numeraire asset $P(t, T_k)$ and a forward rate $F_k(t)$ you just simulate one forward rate with one jump until its reset date T_{k-1}.
- You also need to know $P(0, T_k)$ to complete the valuation:

$$Caplet_0 = P(0, T_k) E_0 \left[N \tau_k \max(F_k(T_{k-1}) - K, 0) \right]$$

  where the expectation is calculated by averaging across all of the paths.
- If you wish to value a caplet using a longer (or shorter) numeraire then you will need to simulate more forward rates.
- Let's choose a caplet maturing in 10 years and 3 months and the numeraire is $P(t, 11.25)$.

Caplets and caps

- Choose $T_0 = 10, T_1 = 10.25, \ldots, T_4 = 11.25$.
- We will need to simulate the following forward rates: $F_1(t), F_2(t), \ldots F_5(t)$, We can do this in one long jump until $t = T_0 = 10$.
- At this point we can write the value of the caplet on the path p as

$$
\begin{aligned}
V_p &= P(0, 11.25) E_0 \left[\frac{Payoff}{1 + F_1(10)\tau_1} \times \frac{1}{P(10, 11.25)} \right] \\
&= P(0, 11.25) E_0 \left[N\tau_1 \max(F_1(10) - K, 0) \prod_{i=2}^{5} (1 + F_i(10))\tau_i \right]
\end{aligned}
$$

Caplets and caps

- where
$$P(0, 11.25) = \frac{1}{\prod_{i=0}{}^{5}(1 + F_i(0)\tau_i)}$$
and then, the caplet value , V, is
$$V = \frac{1}{N_p}\sum_{p=1}^{N_p} V_p$$
where N_p is the number of paths.

Quiz

What about a cap maturing at 11.25 with quarterly payments?

- One long jump until 11.
- Long jumps to 0.25, 0.5, . . . , 11.
- Short jumps until $\Delta t, 2\Delta t$,etc.

Quiz

What about a cap maturing at 11.25 with quarterly payments?

- One long jump until 11. CORRECT
- Long jumps to 0.25, 0.5, . . . , 11.
- Short jumps until $\Delta t, 2\Delta t$,etc.

Forward start swap

- A forward start swap is a swap that does not begin until some future time $T_0 > 0$.
- There are two ways in which we can think of a forward start swap. First, we can evolve the forward rates up to the start of the swap, a long jump, and then value the swap using the standard result. Secondly, we can actually evolve the forward rates all the way until the end of the swap, generating all of the cash flows using a very long jump.
- Let's consider the second method. The important dates for the swap are at $T_0 < T_1 < T_2 < \cdots < T_M$. Assume that the swap rate pays a fixed rate of K.
- We wish to determine the expectation of the swap value to the numeraire. Choosing, $P(t, T_M)$ as the numeraire then we can write its value at a general time T_j (using the very long jump) as

$$P(T_j, T_M) = \prod_{k=j+1}^{M} \frac{1}{1 + F_k(T_{k-1})\tau_k}$$

Forward start swap: Very Long Jump

- Thus, as we receive $N(F_j(T_{j-1}) - K)\tau_j$ at time T_j then the ratio of the of the payment to numeraire is

$$Q_j = \frac{N(F_j(T_{j-1}) - K)\tau_j}{\prod_{k=j+1}^{M} \frac{1}{1+F_k(T_{k-1})\tau_k}}$$

- Therefore for one particular path, p, of Monte Carlo the value of the discounted payoff is $\sum_{j=1}^{M} Q_j$ and the value of the swap is:

$$V_p = P(0, T_M) \sum_{j=1}^{M} Q_j$$

  and then, the swap value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

  where N_p is the number of paths.

- The numeraire here is the *natural* choice as it is the shortest maturity bond that does not mature before the swap does.

Forward start swap: Very Long Jump

- How about if we choose a short dated bond, such as $P(t, T_1)$. This appears to be an impossible choice as it expires before the swap. Well let's assume that when it expires we buy a new one-year zero coupon bond all the way up until T_M.
- Now at T_1 the value of our bond is 1 and so the first ratio or payoff to numeraire value is

$$Q_1 = N\tau_1(F_1(T_0) - K)$$

  the numeraire is reinvested in a money market account and is worth $(1 + F_2(T_1)\tau_2)$ at T_2.
- At T_2 the ratio is now

$$Q_2 = \frac{N\tau_2(F_2(T_1) - K)}{1 + F_2(T_1)\tau_2}$$

  and in general

$$Q_j = \frac{N\tau_j(F_j(T_{j-1}) - K)}{\prod_{i=2}^{j}(1 + F_j(T_{j-1})\tau_j)}$$

Forward start swap: Very Long Jump

- So, for each path, p, the discounted value of the swap is given by

$$V_p = P(0, T_1) \sum_{j=1}^{n} Q_j$$

and then, the swap value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Forward start swap: Long jump

- Taking the long jump method, where we only need to evolve the forward rates up until T_0. First let's choose a numeraire asset, say $P(t, T_M)$.
- Then at T_0 we can value the swap as we would usually, as we now know the discount rates for all of the swap payments. Mathematically, for each swap payment date $j = 1, \ldots M$

$$P(T_0, T_j) = \frac{1}{\prod_{k=1}^{j}(1 + F_k(T_0))\tau_k}.$$

- This formula then gives us the value of the swap and also value of the numeraire asset $P(T_0, T_M)$ and so the ratio of the two can be easily calculated for a given path, p,

$$V_p = P(0, T_M)\frac{N(1 - P(T_0, T_M) - \sum_{j=1}^{M} K\tau_j P(T_0, T_j))}{P(T_0, T_M)}$$

European Swaption: Long jump

- Let's choose $P(t, T_M)$ to be the numeraire where the maturity of the underlying swap is at T_M and the swaption matures at T_0.
- We need to simulate $F_1(t), \dots F_M(t)$ in one long jump until T_0. Given these forward rates we can determine the value of the swap at T_0 as above, and thus the payoff of the swaption $swaption(T_0) = \max(swap(T_0), 0)$.
- Then, for a given path, p

$$V_p = P(0, T_M) \frac{\max(N(1 - P(T_0, T_M) - \sum_{j=1}^{M} K\tau_k P(T_0, T_k)), 0)}{P(T_n, T_M)}$$

where

$$P(T_0, T_j) = \frac{1}{\prod_{k=1}^{j}(1 + F_k(T_0)\tau_k)}.$$

European Swaption: Long jump

- and then, the swaption value, V, is given by

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

European Swaption: Very Long jump

- In this case, we observe forward rates, F_k on their reset dates, T_{k-1} and so,

$$Q_j = \frac{N(F_j(T_{j-1}) - K)\tau_j}{\prod_{k=j+1}^{M} \frac{1}{1+F_k(T_{k-1})\tau_k}}$$

and for each path, p, the discounted value of the swaption is given by

$$V_p = P(0, T_M) \max\left(\sum_{j=1}^{M} Q_j, 0\right)$$

and then, the swaption value , V, is

$$V = \frac{1}{N_p}\sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Constant maturity swap (CMS)

- This is a hybrid between an FRA and a swap. It is a series of payments, each one based upon a fixed length of swap rate rather than the prevailing LIBOR rate in an ordinary swap.
- Thus at each reset time, T_j, we compute,

$$N(S_{T_j, T_j + \text{CMS length}}(T_j) - K)\tau_{j+1}$$

  which we receive at T_{j+1}. The swap rate is a fixed rate length, CMS length, which is observable on that day (e.g. the ten-year rate).
- There may also be payments based upon the difference between constant maturity swap rates - see later.
- Let's consider a CMS with quarterly payments based upon the 10 year swap rate. Payments start at $T_0 = 0.5$ and continue until $T_n = 2$ based on semi-annual payments

Constant maturity swap (CMS)

- We need to be able to calculate the ten year swap rate at all dates, T_j, and so we will need to simulate $F_k(t)$ for $k = j+1, \ldots, M$ where $T_M = 12$.
- At each reset date we need to know the values of $F_k(T_j)$ for $k = j+1, \ldots j+20$ so that we can calculate $S_{T_j, T_j+10}(T_j)$. The payments at T_{j+1} are simply

$$N(S_{T_j, T_j+10}(T_j) - K)\tau_{j+1}$$

- If we take $P(t, T_M)$ as the numeraire asset then the value of $P(T_j, T_M)$ can be calculated from the known forward rates at T_j, so

$$P(T_j, T_M) = \frac{1}{\prod_{k=j+1}^{M} 1 + F_k(T_j)\tau_k}$$

Constant maturity swap (CMS)

- From here the valuation is straightforward. We can record the ratios Q_j at each swap payment date,

$$Q_j = \frac{N(S_{T_{j-1},T_{j-1}+10}(T_{j-1}) - K)\tau_j}{P(T_j, T_M)}$$

or perhaps, more simply as

$$Q_j = N(S_{T_{j-1},T_{j-1}+10}(T_{j-1}) - K)\tau_j \prod_{i=j+1}^{M} (1 + F_i(T_{j-1})\tau_i)$$

- Then for each path, p, the discounted value of the swap is given by

$$V_p = P(0, T_M) \sum_{j=1}^{M} Q_j$$

Constant maturity swap (CMS)

- The swap value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Bermudan swaption (known exercise strategy)

- Assume that we have a Bermudan swap into an existing swap, thus on any of the swaps reset dates you have the option to enter into the swap.
- The value of the swaption will be maximum attainable by any exercise strategy. In fact our trigger swap will be one such exercise strategy, where we exercise as soon as the reset rate is above B.
- Of course, the optimal exercise strategy will also want to have a trigger that varies over time, so that it is higher at the start of the option and lower at the end.
- Assume that we have a known exercise strategy, then all the information is as a function of the forward rates to form a set of rules to exercise or not exercise at each point in time.
- Then we can value the Bermudan swaption by checking for exercise at each point in time and then moving forward if it has not been exercised.
- Of course, choosing the exercise strategy is the hard/fun part.

Bermudan swaption

- Let's imagine that we have a Bermudan swaption with two years to maturity on an existing 5-year swap making quarterly payments with a fixed rate K ($T_M = 7$). You can enter into this swap at $T_0 = 0.25, T_1 = 0.5, \ldots T_n = 2$.
- To value this we need to use the long jump technique.
- First simulate across *ALL* paths to obtain $F_k(T_i)$ at times $i = 1, \ldots, n$, for $k = i+1, \ldots, M$. This is the long jump technique.
- At maturity, T_n record the value of the swaption as

$$V^p(T_n) = \sum_{j=n+1}^{M} N\tau_j(F_j^p(T_n) - K)P(T_n, T_j)$$

- For each path, p, now move back in time and at each time T_i record the early exercise value, $EE^p(T_i)$ the value of the swap from the known forward rates at that time

$$EE^p(T_i) = \sum_{j=i+1}^{M} N\tau_j(F_j^p(T_i) - K)P(T_i, T_j)$$

Bermudan swaption

- These values require knowing $P(T_i, T_j)$, which is

$$P(T_i, T_j) = \frac{1}{\prod_{k=i+1}^{j} 1 + F_k^p(T_i)\tau_k}$$

- Now we need to compare the early exercise value with a continuation value. Starting at $t = T_{n-1}$, The continuation value is:

$$CV^p(T_{n-1}) = E_{T_{n-1}}\left[P(T_{n-1}, T_n)V^p(T_n)\right]$$

where $V^p(T_n)$ is the option value at maturity on this path and

$$P(T_{n-1}, T_n) = \frac{1}{1 + F_n^p(T_{n-1})\tau_n}$$

- Note that strictly speaking the discounting is

$$\begin{aligned} V^p(T_{n-1}) &= P(T_{n-1}, T_M)\frac{V^p(T_n)}{P(T_n, T_M)} \\ &= P(T_{n-1}, T_n)V^p(T_n) \end{aligned}$$

Bermudan swaption

- To prevent problems of perfect foresight we usually estimate $\hat{CV}^p(T_{n-1})$ using a regression equation.
- The inputs into the regression are up to you, if will be of the form:

$$Y = a + b_1 f_1(X_{n-1}) + b_2 f_2(X_{n-1}) + \ldots b_n f_n(X_{n-1}) + \epsilon$$

  where Y is $P(T_{n-1}, T_n)V(T_n)$ and the X_{n-1} values can be the swap rate, or the remaining forward rates or other parameters described in Lecture 9.
- Once the regression has been performed then,

$$V^p(T_{n-1}) = \begin{cases} EE^p(T_{n-1}) \text{ if } & \hat{CV}^p(T_{n-1}) < EE^p(T_{n-1}) \\ P(T_{n-1}, T_n)V^p(T_n) \text{ if } & \hat{CV}^p(T_{n-1}) > EE^p(T_{n-1}) \end{cases}$$

- We then repeat the algorithm at all T_i back to $t = 0$.

Bermudan swaption

- And then, the swaption value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V^p(0)$$

where N_p is the number of paths.

Real world products: CMS coupon payments

- The real world products that you can choose to value are bonds where size of the coupon payments depend upon functions of CMS rates.
- This does not create too much of a theoretical challenge as we have seen a simple CMS product above. Annoyingly the calculation of CMS rates requires us to use the long jump technique as we will need to know all the forward rates required to calculate the swap rates on every possible coupon date.
- Once we have all of the coupons and the principal then it is straightforward to value the bond, as we can scale each of the coupons by the numeraire at that point, sum them up and that gives us the expected value of the bond.

Real world products: callable

- Annoyingly, most of the real world products are also callable. This creates problems for a forward looking method like Monte-Carlo.
- As with the Bermudan swaption we will have to develop a rule for the paths in which the issue will call the bond back.
- For each path, at maturity (T_M, say) the value of the bond is known as the principal plus the final coupon.
- We will also know the call schedule on all of the possible call dates, $Call(T_i)$.
- At the coupon date before maturity the continuation value is $E_{T_{M-1}}[P(T_{M-1}, T_M)V(T_M)]$ where,

$$P(T_{M-1}, T_M) = \frac{1}{1 + F_M(T_{M-1})\tau_M}$$

- As with a Bermudan swaption we will determine this expectation as a function of information at time T_{M-1} using the bond values across all of the paths.

Real world products: callable

- Then, once we can estimate the continuation value, $CV(T_{M-1})$ then the value of the bond is simply

$$V(T_{M-1}) = \begin{cases} Call(T_{M-1}) \text{ if} & CV(T_{M-1}) > Call(T_{M-1}) \\ P(T_{M-1}, T_M)V(T_M) \text{ if} & CV(T_{M-1}) < Call(T_{M-1}) \end{cases}$$

- Once the bond has been called all future payments are zero.
- Along this path also store the value of any coupons at dates T_i, scaled by the value of the numeraire $P(T_i, T_M)$, the sum of these plus $V(0)P(0, T_M)$ is the value of the bond at time zero on this path, usually denoted V_p.
- And then, the bond value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Trigger Swap

- A trigger swap is a swap that doesn't take effect until a reference rate, usually LIBOR, hits a trigger level on one of a number of dates.
- We will consider an up-and-in swap based on a five year swap which starts in two years with quarterly payments. Let the swap payment dates be $T_j = 2 + j/4$, i.e $T_0 = 2$, $T_1 = 2.25, \ldots, T_M = T_{20} = 7$.
- Let's assume that the trigger rate is B and the reference rate is three-month LIBOR and it can be observed on the setting dates of the swap, T_j.
- We could now perform the long jump taking steps to each of the reset dates. So first we go to T_0 and check the value of $F_1(T_0)$, if it is above B then we value the swap in the usual way and take the ratio with the numeraire as we did above.
- If the rate is below B the we progress to T_1, repeating until the barrier is reached or until T_{20}.

Trigger Swap

- Actually, we could do this even more quickly. As we only need to know the values of the forward rates on their reset dates then we can take one *very long jump* until $T_{19} = 6.75$ or the reset date of the last forward.
- Ok, so we will need to simulate $F_1(t), \dots F_{20}(t)$ with a very long jump until each of the reset dates. This will give us $F_1(T_0), F_2(T_1), \dots, F_{20}(T_{19})$. Assume that the numeraire is $P(t, T_{20})$
- This gives is the swap payments at each date, $T_j, j = 1, \dots 20$:

$$swap_j = N\tau_j(F_j(T_{j-1}) - K)$$

and the value of the numeraire

$$P(T_j, T_{20}) = \frac{1}{\prod_{k=j+1}^{20} 1 + F_k(T_{k-1})\tau_k}$$

Trigger Swap

- This means that we can calculate the 'discounted 'value of the payoffs at each point as

$$Q_j = \frac{swap_j}{P(T_j, T_{20})}.$$

- Recall that to receive these payment the trigger has to be hit. So, move forward in time, if $F_1(T_0) < B$ then $Q_1 = 0$ and so on until $F_k(T_{k-1}) > B$ at which point $Q_j = \frac{swap_j}{P(T_j, T_{20})}$ for $j > k$.
- On each path, p,

$$V_p = P(0, T_{20}) \sum_{j=1}^{20} Q_j$$

and then, the Trigger swap value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Ratchet cap

- A ratchet cap is where the strike is updated at each caplet date, in particular the strike price of the ith caplet is given by $K_i = L(T_{i-2}, T_{i-1}) + X$ $(= F_{i-1}(T_{i-2}) + X)$ where X is a margin that can be either positive or negative.
- The ratchet cap can become sticky by changing the strike to be

$$K_i = \max(L(T_{i-2}, T_{i-1}) + X, K_{i-1} + X)$$

- The valuation of the ratchet cap requires knowledge of the previous forward rate at its reset data and so can be easily valued using Monte-Carlo methods.
- Let's consider a ratchet cap with caplets maturing at $T_1, \ldots T_M$ with $X = 0$. We will use the very long jump technique with $P(t, T_M)$ as numeraire.

Ratchet cap

- The very long jump technique gives us $F_k(T_{k-1})$ for all $k = 1, \ldots M$. We can also determine $P(T_k, T_M)$ for all k values using the simulated forward rates.
- Now the payoff at the maturity of each caplet is:

$$N\tau_k \max(F_k(T_{k-1}) - F_{k-1}(T_{k-2}), 0)$$

but these values are both known and we can determine the payoff, and also

$$P(T_k, T_M) = \frac{1}{\prod_{j=k+1}^{M} 1 + F_j(T_{j-1})\tau_j}$$

so,

$$Q_k = \frac{N\tau_k \max(F_k(T_{k-1}) - F_{k-1}(T_{k-2}), 0)}{P(T_k, T_M)}$$

Ratchet cap

- On each path, p,

$$V_p = P(0, T_M) \sum_{k=1}^{M} Q_k$$

and then, the Trigger swap value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Captions and floortions

- At some point in time (usually the expiry of the first caplet) we have the option to purchase the cap (with an exercise price or rate, K) for a pre-agreed price X.
- As the option is on all of the remaining caplets at one, we cannot decompose it into mini options on caplets (nor would this be interesting).
- Ok, let's say that the maturity of the caption is T_0 and the underlying cap has payoffs at $T_1, \dots, T_M$.
- So, we will need to simulate forward rates $F_k(t)$ for $k = 1, \dots, M$ from time 0 until T_0.

Captions and floortions

- At time T_0 then we can value each of the caplets using an adaptation of the Black-Formula.
- The 'current 'forward rate is $F_k(T_0)$, the time to maturity is $T_k - T_0$, the exercise price is K and the Black volatility σ_{black} is given by:

$$\sigma_{black} = \sqrt{\frac{1}{T_{k-1} - T_0} \int_{T_0}^{T_{k-1}} \sigma_k^2(t)dt}$$

  where $\sigma_k(t)$ is the instantaneous volatility function related to that forward rate.
- Now we can calculate the value of the cap. Then the payoff of the caption is $\max(cap - X, 0)$. If we choose $P(t, T_M)$ as numeraire then,

$$P(T_0, T_M) = \frac{1}{\prod_{k=1}^{M} 1 + F_k(T_{k-1})\tau_k}$$

Captions and floortions

- So on each path

$$V = P(0, T_M)\frac{\max(cap - X, 0)}{P(T_0, T_M)}$$

and then, the caption value , V, is

$$V = \frac{1}{N_p}\sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

LIBOR in-arrears caplet

- As we have seen before the payoff from a LIBOR in-arrears FRA or caplet occurs when the forward rate is reset T_{k-1} rather than the end of its reference period T_k.
- To value this, we simply take $P(t, T_k)$ as the numeraire, and the evolve the forward rate $F_{k-1}(t)$ to then (it now has non-zero drift), when we can easily compute the payoff and the ratio of payoff to numeraire value.

$$Q = \frac{Payoff(T_{k-1})}{P(T_{k-1}, T_k)} = N\tau_k \max(F_k(T_{k-1}) - K, 0)(1 + F_k(T_{k-1})\tau_k)$$

- On each path, p,

$$V_p = P(0, T_k)Q$$

LIBOR in-arrears caplet

- and then, the in-arrears caplet value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Cash settled swaptions

- In the Euro market true swaptions are less traded than the cash settled swaption.
- This is because on the main reasons to enter into a swaption is for hedging purposes, to potentially try to mitigate swap rate volatility risk.
- When the trade is made like this then the holder does not actually want to enter into the swap but just receive the cash for however much it is worth.
- However, the value of the swap depends upon the discount factors of the swap payments which are not directly observable in the market.
- The cash-settled swaption was designed to overcome this. When the annuity (or the fixed leg) is calculated the swap rate itself is used as the interest rate across future periods. Thus if there are M swap payments then the payoff from the cash-settled swaption at maturity, T_0, is:

Cash settled swaptions

- $$Payoff(T_0) = \max(S_{T_0,T_M}(T_0) - K, 0)\sum_{j=1}^{M} \frac{\tau_j}{(1 + S_{T_0,T_M}(T_0))^{T_j - T_0}}$$
- So, to price this we evolve the relevant forward rates up until T_0 using a long jump, or the expiry of the swap, T_m, using a very long jump, and then use them to compute the swap rate.
- What numeraire should we use? We would like to use $\sum_{j=1}^{N} \frac{\tau_j}{(1+S_{T_0,T_n}(T_0))^{T_j-T_0}}$ but this is not tradable and so instead we use the zero coupon bond that matures at the expiry date of the swap, $P(t, T_0)$, and then on a given path, p

$$V_p = P(0, T_0)Payoff(T_0)$$

Cash settled swaptions

- and then, the cash settled swaption value , V, is

$$V = \frac{1}{N_p} \sum_{p=1}^{N_p} V_p$$

where N_p is the number of paths.

Conclusion

- We looked at types of products that we could value using the LMM, and some efficient ways of valuing them.
- The simplest products (like swaptions) can be valued using just one long or very long jump.
- Options with early exercise or call features must be valued using a long jump technique, using a regression approach to estimate early exercise.
- Then, finally, we have some path dependent options (like ratchet caps) that can be valued using the very long jump technique.