FIN516: Term-structure Models

Lecture 7: Numerical methods for one-factor models: HW/BK

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Introduction

- We have seen that the BDT model is well designed for calibration to market data and is relatively easy to construct. As we will see from a future analysis of the models, its convenient structure is also its main problem, as the rate of mean reversion is stuck dependent upon $\sigma(t)$.
- Here we will revisit the other two standard one-factor models, Hull and White and Black and Karasinski. These models both display mean reversion where the mean reverting parameter can be selected.
- To cope with the mean reversion parameter, a trinomial tree is typically used where the tree can be adapted to ensure that the mean reversion can occur.

Hull and White model

- The full Hull and White model is as follows:

$$dr = [\theta(t) - a(t)r(t)] + \sigma(t)dW(t)$$

but is more usually presented with only $\theta(t)$ being time varying. We will begin with the most reduced form:

$$dr = [\theta(t) - ar(t)] + \sigma dW(t).$$

It is possible to allow a to also time vary. Notice that it is possible to derive analytic formulas for many derivatives using this format.

Hull and White: Trinomial tree

- There are two possible ways to build a trinomial tree for the Hull and White model, we will focus on that presented by the authors themselves in their 1994 article in the Journal of Derivatives.
- First assume that the rate, R, over a non-instantaneous period of time Δt follows the same stochastic process as the instantaneous rate, that is:
$$dR = [\theta(t) - aR]\, dt + \sigma dW$$
- Next, consider a slightly different process where the $\theta(t)$ term is zero (we will add this back later),
$$dR^* = -aR^* dt + \sigma dW$$
this is a symmetric process and the initial value of R^* is set to be zero. R^* is simple to understand and $E[R^*(t+\Delta t) - R^*(t)] = e^{-aR^*(t)\Delta t} - 1$ or $-aR^*(t)\Delta t$ for very small Δt.

Quiz

What is $\mathrm{Var}[R^*(t+\Delta t) - R^*(t)]$ for small Δt

- $\sigma \Delta t$
- $\sigma^2 \Delta t$
- $\frac{\sigma^2}{2a}$
- $\frac{\sigma^2}{2a}\left(1 - e^{-2a\Delta t}\right)$

Quiz

What is $\mathrm{Var}[R^*(t+\Delta t) - R^*(t)]$ for small Δt

- $\sigma \Delta t$
- $\sigma^2 \Delta t$ Correct for very small Δt
- $\frac{\sigma^2}{2a}$
- $\frac{\sigma^2}{2a}\left(1 - e^{-2a\Delta t}\right)$ Correct for small Δt

Building the tree from time zero

- We will construct a trinomial tree in a slightly different way from the binomial tree that we used for the BDT model.
- Here instead of fixing the probability and then adjusting the size of the up and down movements, we will fix the size of the movements and then adjust the probabilities.
- As the trinomial tree is essentially an explicit finite difference method we will fix the changes in the spot rate to the size of the time step to ensure the stability of the method.
- As before, let's divide the time to maturity T into N time steps, so that the time step size $\Delta t = N/T$ and a given time step is $t = i\Delta t$. The node at a given time step n, is indexed by j.
- Now the rate R^* is defined by $i\Delta R^*$ where ΔR^* is given by

$$\Delta R^* = \sigma\sqrt{3\Delta t}$$

Branching conventions

- As we have set the size of the spacings of R^* then the probabilities will be calculated when calibrating the tree to the yield curve data.
- It transpires that if we use the natural trinomial branchings where R^* can increase by ΔR^*, decrease by ΔR^* or remain at the same level then the probabilities can become negative for sufficiently large or small i values.
- For large i, an alternative branching is used where R^* stays the same, goes down by ΔR^* or goes down by $2\Delta R^*$, with an analogous extension for low i values.
- Hull and White suggest that for any j values greater than $j_{\max} = 0.184/(a\Delta t)$ that the different branching be used as well as for any $j < -j_{\max}$. This only works for positive values of a.

Quiz

Does calibration always produce positive values of a?

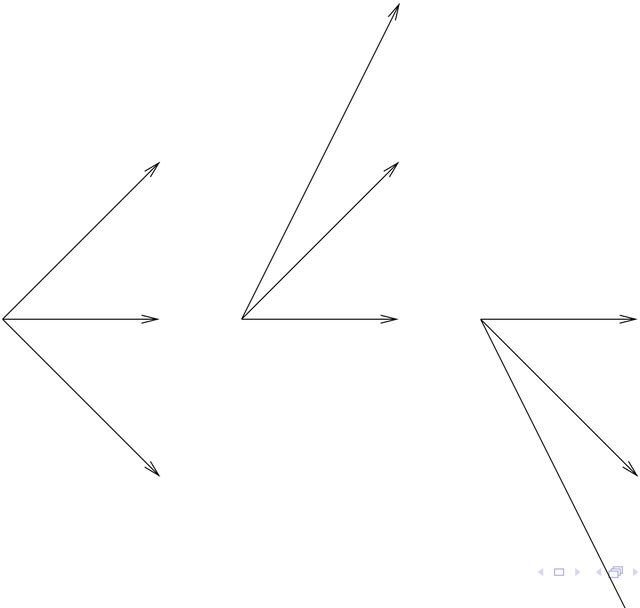
- Yes
- No
- Yes, it it's done correctly.

Quiz

Does calibration always produce positive values of a?

- Yes
- No Correct
- Yes, it it's done correctly.

Branching conventions



Probabilities

- For each of the three different scenarios we will have slightly different probabilities.
- The probabilities are chosen so that the mean and the variance of the rates in the trinomial tree matches that for the stochastic process and that they sum to 1.
- Hull and White calculate the following probabilities:

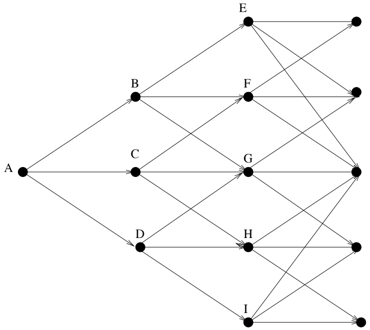
$$
\begin{array}{ll}
-j_{\max} < j < j_{\max} & p_u = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2} \\
 & p_m = \frac{2}{3} - a^2 j^2 \Delta t^2 \\
 & p_d = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2} \\
j < -j_{\max} & p_u = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2} \\
 & p_m = -\frac{1}{3} - a^2 j^2 \Delta t^2 - 2aj\Delta t \\
 & p_d = \frac{7}{6} + \frac{a^2 j^2 \Delta t^2 + 3aj\Delta t}{2} \\
j > j_{\max} & p_u = \frac{7}{6} + \frac{a^2 j^2 \Delta t^2 - 3aj\Delta t}{2} \\
 & p_m = -\frac{1}{3} - a^2 j^2 \Delta t^2 + 2aj\Delta t \\
 & p_d = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2}
\end{array}
$$

Simple example

- Let's suppose that $\sigma = 0.01$, $a = 0.1$, and $\Delta t = 1$. Here $\Delta R^* = 0.0173$.
- The smallest integer greater than $0.184/0.1$ is $j_{\max} = 2$. The tree and table below show the probability of each movement of R^*. Note carefully that the probabilities depend only upon the state j and so will be the same once the top node ($j_{\max}$) has been reached.
- In particular, the probabilities at node B are the same as for node F. Also note that the tree is symmetrical and so the probabilities at node D and the same as at node B.

Simple example

Node	A	B	C	D	E	F	G	H	I
R^*	0.000%	1.732%	0.000%	-1.732%	3.464%	1.732%	0.000%	-1.732%	-3.464%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867



Adding back the drift

- If we now define $\alpha(t)$ to be the difference between the real spot rate and R^* then we get

$$\alpha(t) = R(t) - R^*(t)$$

and we also know that

$$dR = [\theta(t) - aR]dt + \sigma dW$$

and

$$dR^* = -aR^* dt + \sigma dW$$

and so

$$d\alpha = [\theta(t) - a\alpha(t)]dt.$$

- $\alpha(t)$ can be written as

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

Adding back the drift

- It is then relatively simple to create the tree for R by simply adding $\alpha(t)$ onto the values of R^*, as all of the probabilities remain the same as this is a drift adjustment.
- However, this is an analytic formula for R based upon the continuous time distribution, and we have a discrete trinomial distribution and so it will not precisely value the zero-coupon bonds.
- To mechanically calculate $\alpha(t)$ we have to do so numerically using Arrow-Debreu prices as we did in the BDT trees.
- Define $\alpha(i)$as the value of R at time $i\Delta t$. Then, as before $\hat{\pi}(i,j)$ is the present value of a security that pays off 1 if node (i,j) is reached and zero otherwise.
- First, set $\alpha(0)$ to be the Δt period rate of interest, so that,

$$\alpha(0) = \frac{-ln(P(\Delta t))}{\Delta t}$$

Adding back the drift

- As we know $P(\Delta t)$ then we can also calculate the $\hat{\pi}$ values when $i = 1$ by using and adaptation to the formula we had before

$$\begin{aligned}\hat{\pi}_{(1,-1)} &= \hat{\pi}_{(0,0)} p(0,-1) e^{-(\alpha(0)\Delta t)} \\ \hat{\pi}_{(1,0)} &= \hat{\pi}_{(0,0)} p(0,0) e^{-(\alpha(0)\Delta t)} \\ \hat{\pi}_{(1,1)} &= \hat{\pi}_{(0,0)} p(0,1) e^{-(\alpha(0)\Delta t)}\end{aligned}$$

  where now $p(k,j)$ is the probability of moving from state k at $t = 0$ to state j at $t = \Delta t$.

- Now the price of $P(\Delta t, 2\Delta t)$ at node (1,1) is $e^{-(\alpha(1)+R^*(1,1))}$ and similarly for the other nodes at time 1 then we can write the value of $P(2\Delta t)$ as

$$P(2\Delta t) = \hat{\pi}_{(1,1)} e^{-(\alpha(1)+R^*(1,1))\Delta t} + \hat{\pi}_{1,0} e^{-(\alpha(1)+R^*(1,0))\Delta t} + \hat{\pi}_{1,-1} e^{-(\alpha(1)+R^*(1,-1))\Delta t}$$

  where the only unknown is $\alpha(1)$ and so we can solve for $\alpha(1)$

$$\alpha(1)\Delta t = \ln\left(\frac{\hat{\pi}_{(1,1)} e^{-R^*(1,1)\Delta t} + \hat{\pi}_{1,0} e^{-R^*(1,0)\Delta t} + \hat{\pi}_{1,-1} e^{-R^*(1,-1)\Delta t}}{P(2\Delta t)}\right)$$

Quiz

Why was $p(k,j)$ not in the formulas for the BDT model?

- It was always $\frac{1}{2}$. Correct
- It was not necessary with two states.
- It was not necessary without a drift adjustment.

Moving through the tree

- Now we move forward to $i = 2$ where we will need to find $\alpha(2)$ that ensures that our model correctly values $P(3\Delta t)$.
- As before we can write this as the sum of the Arrow-Debreu securities at $i = 2$, in general we will have

$$P((i+1)\Delta t) = \sum_{j=-m_i}^{m_i} \hat{\pi}_{i,j} e^{-(\alpha(i)+i\Delta R^*)\Delta t}$$

where m_i is the number of up (and down) steps at time i which we can then solve to find $\alpha(i)$ as follows:

$$\alpha(i) = \frac{\ln \sum_{j=-m_j}^{m_j} \hat{\pi}_{i,j} e^{i\Delta R^* \Delta t} - \ln P(i+1)}{\Delta t}.$$

- Once we have $\alpha(i)$then we can calculate the A-D security prices at the next time step:

$$\hat{\pi}_{i+1,j} = \sum_{k=j-2}^{j+2} \hat{\pi}_{i,k} p(k,j) \exp\left[-(\alpha(i) + k\Delta R^*)\Delta t\right]$$

Example: Adding back the drift

- Let's assume that we have the following continuously compounded zero rates:

Maturity	**Rate (%)**
0.5	3.430
1.0	3.824
1.5	4.183
2.0	4.512
2.5	4.812
3.0	5.086

- The value of $\hat{\pi}_{0,0}$ is 1 and then the value of $\alpha(0)$ is chosen to give the right price for a zero-coupon bond maturing at Δt

Quiz

What is $\alpha(0)$ for the above tree?

- 0.03430
- 0.03824
- 0.05086
- 3.430

Quiz

What is $\alpha(0)$ for the above tree?

- 0.03430
- 0.03824 Correct
- 0.05086
- 3.430

Example: Adding back the drift

- Thus, $\alpha(0) = 0.03824$. From the $\alpha(0)$ value we can calculate the $\hat{\pi}_{1,i}$ values.
- There is a probability of 0.1667 that node $(1,1)$ is reached, and the discount rate for this step is $R(0,0) = 0.03824$ and so,

$$\begin{aligned}
\hat{\pi}_{1,1} &= 0.1667e^{-0.03824} = 0.1604 \\
\hat{\pi}_{1,0} &= 0.6666e^{-0.03824} = 0.6417 \\
\hat{\pi}_{1,-1} &= 0.1667e^{-0.03824} = 0.1604
\end{aligned}$$

- Now we need to calculate $\alpha(1)$ using the formula

$$\begin{aligned}
\alpha(1) &= \frac{\ln\left(0.1604e^{-0.01732} + 0.6417e^{0} + 0.1604e^{0.01732}\right) - \ln 0.9137}{1} \\
&= 0.05205
\end{aligned}$$

Example: Adding back the drift

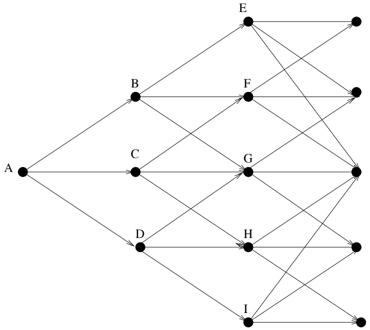
- Now we need to calculate the $\hat{\pi}_{2,i}$ values. Consider $\hat{\pi}_{2,1}$ as an example, the only nodes that can get to $(2,1)$ are $(1,1)$ and $(1,0)$ and so

$$\begin{aligned}\hat{\pi}_{2,1} &= 0.1604 \times 0.6566e^{-(0.05205+0.1732)} + 0.6417 \times 0.1667e^{-(0.05205+0)} \\ &= 0.1998\end{aligned}$$

- The other $\hat{\pi}_{2,i}$ values can be calculated in a similar way, then the $\alpha(2)$ values and so on.
- The next slide shows the tree and the values of $R(i,j)$ after the calibration exercise.

Simple example: with drift

Node	A	B	C	D	E	F	G	H	I
R^*	3.824%	6.937%	5.205%	3.473%	9.716%	7.984%	6.252%	4.520%	2.788%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867



Set-up

- The Hull and White spot rate process has the obvious problem (especially with low interest rates) that interest rates can become negative. This can be avoided by modeling the log of the spot rate:

$$d \ln r = [\theta(t) - a \ln r] \, dt + \sigma dW$$

- We treat a and σ as constants initially but they can be considered as time dependent also.
- Here we start by modeling the process for $\ln r$ and then eventually transform back to r. If we let $x = \ln r$ then the process becomes

$$dx = [\theta(t) - ax] \, dt + \sigma dW$$

- As before, we then remove the drift ($\theta(t) = 0$) and model the process followed by x^*, and then we add back the drift

Algorithm

- $\hat{\pi}_{0,0} = 1$ and $\alpha(0) = \ln R(0)$ and the price of the zero coupon bond maturing at time $(i+1)$ is given by

$$P(i+1) = \sum_{j=-m_i}^{m_i} \hat{\pi}_{i,j} e^{-\exp(\alpha(i)+j\Delta x)\Delta t}$$

- Unlike in the Hull and White model, this cannot be solved analytically and must be solved using an iterative procedure such as Newton's method.
- Once we have solved to find $\alpha(i)$ then we can determine the A-D security prices at $i+1$ now using

$$\hat{\pi}_{i+1,j} = \sum_k \hat{\pi}_{i,j} p(k,j) e^{-\exp(\alpha(i)+k\Delta x)\Delta t}$$

BK example

- Let's consider a tree spanning four years with $\Delta t = 1$, so there are three time steps. Assume that $a = 0.15$ and $\sigma = 0.1$. We will first construct the tree for $x = \ln r$
- $\Delta t = 1, \Delta x = \sigma\sqrt{3\Delta t} = 0.173$. With regular branching $p_u = 0.167, p_m = 0.667, p_d = 0.167$, with two down branches: $p_u = 0.762, p_m = 0.177, p_d = 0.062$, with two up branches: $p_u = 0.062, p_m = 0.177, p_d = 0.762$. The point at which the branching changes is at $j = 2$ and $j = -2$ as $2 > (0.184/0.15) > 1$
- The data given is that the continuously compounded annual zero rates are $R(0,1) = 0.05, R(0,2) = 0.0575, R(0,3) = 0.0625, R(0,4) = 0.0675$. We have that $x(0,0) = 0, R(0,0) = 0.05, \hat{\pi}_{(0,0)} = 1$

BK example

- Trivially, $\alpha(0) = \ln(0.05) = -2.9957$.
- Then we can calculate $\hat{\pi}_{1,i}$ as follows:

$$
\begin{aligned}
\hat{\pi}_{1,-1} &= \hat{\pi}_{0,0} p(0,-1) \exp[-\exp(\alpha(0) + x(0,0))\Delta t] \\
&= 0.167 \exp(-0.05) \\
&= 0.1585 \\
\hat{\pi}_{1,0} &= 0.6342 \\
\hat{\pi}_{1,1} &= 0.1585
\end{aligned}
$$

- From this we can write down the formula for $P(2)$

$$
\begin{aligned}
P(2) &= \sum_{j=-1}^{1} \hat{\pi}_{1,j} \exp[-\exp(\alpha(1) + j\Delta x)\Delta t] \\
&= 0.1585 \exp[-\exp(\alpha(1) - 0.173) \times 1] + 0.6343 \exp[-\exp(\alpha(1) + \\
&\quad +0.1585 \exp[-\exp(\alpha(1) + 0.173) \times 1]
\end{aligned}
$$

and then use a numerical procedure to find that $\alpha(1) = -2.7380$

BK example

- We can the update the values of x or $\ln r$ to get

$$\begin{aligned}
\ln R(1,1) &= 0.1732 - 2.738 = -2.5648 \\
\ln R(1,0) &= 0.0000 - 2.738 = -2.7380 \\
\ln R(1,-1) &= -0.1732 - 2.738 = -2.9112
\end{aligned}$$

  which can be converted to
  $R(1,1) = e^{-2.5648} = 0.0769, R(1,0) = 0.0647, R(1,-1) = 0.0544.$

- Next we calculate the new $\hat{\pi}$ values, followed by the formula for $P(3)$ which will give us the value of $\alpha(2)$ and then $R(2,-2), R(2,-1), R(2,0), R(2,1), R(2,2)$.

Matching volatility structures

- In this set up, there are only two parameters to try to match the term structure of volatility. Typically these would be chosen to most closely match a set or market prices - either caps or swaptions depending upon what instrument was going to be priced.
- It is possible to make $a(t)$ and $\sigma(t)$ time dependent in either the HW or BK set-up, this can match the current volatility term structure (as for the BDT tree) but is typically not advised.
- One option is to use a piecewise constant function for $\sigma(t)$ as we did for the BDT model. Sadly, the nice feature of the BDT model (that we could simply use the cap volatilities) does not hold for either the HW or BK models, but we can still use the same approach.
- A nice calibration method to use is the Levenberg-Marquardt procedure which ensures that even the piecewise constant volatility function is well behaved. If $U(n)$ is the current value of the time n calibrating instrument (say a cap) and $V(n)$ is the model value.

Matching volatility structures

- Now assume that σ_i is the volatility between i and $i+1$ then the choice of parameters $a, \sigma_0, \sigma_1, \ldots$ to minimize the following objective function:

$$\sum_{i=1}^{N}(U(i)-V(i))^2+\sum_{i=1}^{N} w_{1,i}(\sigma_i-\sigma_{i-1})^2+\sum_{i=1}^{N} w_{2,i}(\sigma_{i-1}+\sigma_{i+1}-2\sigma_i)^2$$

- The idea here is that the weights w_1 and w_2 are chosen to penalize volatility functions that lead to large changes in slope or curvature from time period to time period.

Matching with analytic values

- The Hull and White model also has analytic formulas for the zero coupon bonds, which can be used as part of the calibration and valuation procedure (as we will see, this is a very useful feature for Bermudan swaptions).
- First note that in the discrete, trinomial tree setting the interest rate is not actually the instantaneous short rate r but rather the Δt rate R and so the zero coupon bond price formula is

$$P(t, T) = \hat{A}(t, T)e^{-\hat{B}(t,T)R(t)}$$

where

Matching with analytic values

- $$\begin{aligned} B(t,T) &= \frac{1-e^{a(T-t)}}{a} \\ \hat{B}(t,T) &= \frac{B(t,T)}{B(t,t+\Delta t)}\Delta t \\ \ln \hat{A}(t,T) &= \ln\frac{P(0,T)}{P(0,t)} - \frac{B(t,T)}{B(t,t+\Delta t)}\ln\frac{P(0,t+\Delta t)}{P(0,t)} \\ &\quad -\frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)[B(t,T)-B(t,t+\Delta t)] \end{aligned}$$

Variable time steps

- In practice we are more interested in small time steps close to time 0 where the spot rate is typically changing fastest and less interested as t increases.
- There is nothing preventing us from using unequal time steps in our binomial or trinomial tree construction. Now however, Δt depends upon the current time, so we have $\Delta t_i = t_{i+1} - ti$ and so will the steps in R or x so that now $dx_i = 3\sqrt{\Delta t_i}$ which in turn will make the probabilities p_u, p_m, p_d time dependent as they will now depend upon Δt_i.
- The branching conventions will also depend upon Δt_i and this will be checked at each step through the tree.
- Typically, a short Δt (e.g. $1/365$) is used for the first year (although we need to ensure that the times correspond to reset dates) and then a longer Δt (e.g. $1/50$) is used in later years.

Conclusion

- We have seen how to build trees for more realistic short rate models that exhibit mean reversion.
- The trick is to use a trinomial tree and to allow for different possible branchings as the spot rate gets too high or too low.
- We have seen that it is straightforward to construct a tree that matches the initial term structure of zero rates, and how to do this for the Hull and White model and the Black and Karasinski model.