FIN516: Term-structure Models

Lecture 8: Volatility modeling and limitations of Short rate models

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Introduction

- We have seen that it is straightforward to construct one-factor models that can recreate the current term structure of zero rates.
- However, for these models to be useful in valuing option like products (caps and swaptions for example) then they will also need to match the term structure of volatility.
- Here we will see what we mean by the term structure of volatility, and see how well one-factor models can match it.
- We will see the limited ways in which short rate models can model volatility and discuss general problems with short rate models.

What do we mean by volatility?

- Volatility is a general term that can be used to cover a wide range of things. When talking about the term structure of volatility we need to be very careful to define what volatility we are referring to.
- Typically by the term structure of volatility we refer to implied volatilities of caplets, calculated by Black's formula.
- However, there are many other options: the value of the σ parameter in the HK/BDT/BK models. The standard deviation of the short rate, $s.d(r(T))$, the integral of $\sigma(t)$ over a certain period of time.
- If we assume that it is Black implied caplets that we are interested in calibrating to, is it not straightforward to simply allow $\sigma(t)$ to vary and thus match the term structure of volatility precisely?

Market (Black) caplet volatilities

- As in lecture 2 the **market cap volatility** is the value of σ that is inputted into the Black cap formula to obtain the market value of the cap.
- The **market caplet volatility** is the value of σ that is inputted into the Black caplet formula to obtain the market value of the caplet.
- To be more precise, consider the caplet resetting at T and paying out the LIBOR rate $L(t, T+\tau) = F(T; T, T+\tau)$ at $T+\tau$, then the T-expiry caplet volatility is

$$\begin{aligned} \nu^2_{T-caplet} &= \frac{1}{T}\int_0^T (d\ln F(t, T, T+\tau))(d\ln F(t; T, T+\tau)) \\ &= \frac{1}{T}\int_0^T \sigma(t; T, T+\tau)^2 dt. \end{aligned}$$

- $\sigma(t; T, T+\tau)$ is the instantaneous volatility of the forward rate $F(t; T, T+\tau)$, this is deterministic in the Black model and so $\nu^2_{T-caplet}$ is also deterministic and equal to σ^2.

Quiz

Why might the cap and caplet volatilities be different for contracts with the same maturity?

- Market imperfections.
- The cap volatility assumes all caplets have the same value, so is an average.
- The caplet is a riskier product.

Quiz

Why might the cap and caplet volatilities be different for contracts with the same maturity?

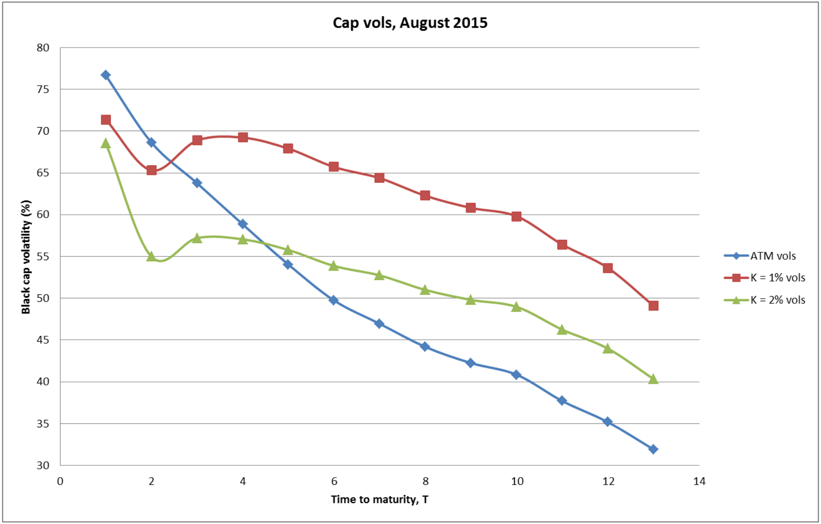
- Market imperfections.
- The cap volatility assumes all caplets have the same value, so is an average. Correct
- The caplet is a riskier product.

Term structure of caplet volatilities

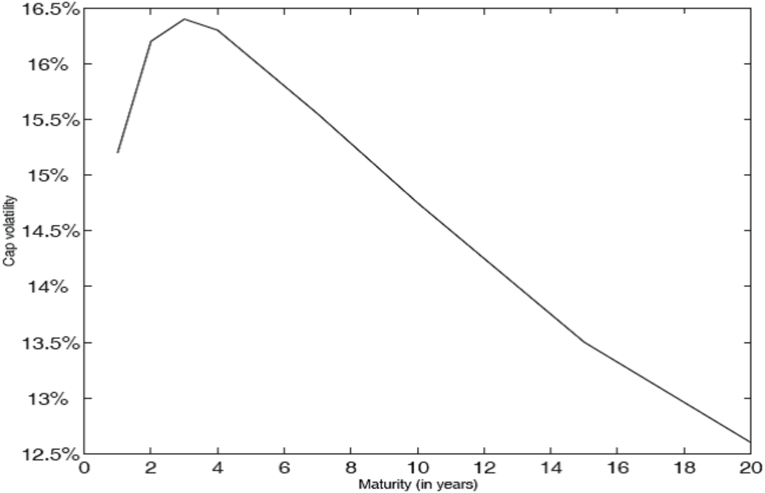
- The term structure of caplet volatilities is then the mapping $T \mapsto \nu_{T-caplet}$.
- It is this graph that typically exhibits the hump like shape, or decreasing trend.
- When we are now considering the term structure of volatility for one factor models, do we simply calculate the stochastic process for dF and then extract the integral of this function?
- We can consider a one-factor model to see how we could potentially do this. Consider an Affine model (e.g. Vasicek or CIR), then $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$ the forward rate is defined as:

$$F(t;T,T+\tau) = \frac{1}{\tau}\left[\frac{P(t,T)}{P(t,T+\tau)} - 1\right]$$

Current cap vols



Traditional cap vols



Term structure of caplet volatilities

- By Îto's lemma, under the forward measure, the driftless dynamics of $F(t; T, T+\tau)$ are:

$$\begin{aligned} dF(t;T,S) &= \sigma(t,r(t))\frac{A(t,T)}{A(t,T+\tau)}(B(t,S)-B(t,T)) \\ &\quad \times \exp\left\{-(B(t,T)-B(t,T+\tau))r(t)\right\} dW(t) \end{aligned}$$

where $\sigma(t, r(t))$ is the diffusion term of $r(t)$.

- This can be rearranged so that the dependence is on $F(t; T, T+\tau)$ rather than $r(t)$ by rearranging the formula for the forward rate.
- Even so the diffusion term will typically be dependent upon $F(t; T, T+\tau)$ which is itself stochastic.

Model implied caplet volatility

- This typically leads to random caplet volatilities but the market caplet volatilities are clearly not random. How do we overcome this?
- Now compute the value of the caplet using the desired model, $Cpl(0, T, T+\tau, F(0; T, T+\tau))$ and then invert the Black caplet formula to find the implied volatility that gives the *model* price.
- Thus define the **model implied T-caplet volatility**, $\nu_{T-caplet}^{MODEL}$ as the solution to the following formula:

$$Cpl(0, T, T+\tau, F(0; T, T+\tau)) = P(0, T+\tau)\tau F(0; T, T+\tau)\left(2N\left(\frac{\nu_{T-caplet}^{MODEL}\sqrt{T}}{2}\right) - 1\right)$$

for an at-the-money caplet $(K = F(0; T, T+\tau))$

- We can also define the model implied cap volatility in an analogous way, where ν_{T-cap}^{MODEL} is the volatility inputted into the Black cap formula that recreates the cap value obtained from the model.

Term structure of volatilities

- From now on the term structure of caplet (cap) volatilities implied by a one-factor model is the graph of of the model-implied T-caplet (cap) volatility against T.
- In practice, these term structures will exhibit humped (or downward sloping) shapes if the volatilities of instantaneous forward rates $\sigma_f(t, T)$ also exhibit humped (or downward sloping) shapes, although small humps in the caplet curve are possible even with monotonically decreasing instantaneous-forward volatilities.
- So, we need to be careful which values are exhibiting hump (or downward sloping) shapes: the cap volatilities, the caplet volatilities or $\sigma_f(t, T)$. Some models allow for humps in all of these structures, whereas others only allow for humps in the caplets or caps.
- Typically, only the caplet (or cap) volatilities are observed and so the calibration will typically be to match these as closely as possible.

Extension to Hull and White

- Now that we are focussed on recreating a hump shaped (or downward sloping) cap/caplet volatility curve we can try to adapt our one-factor models to do so.
- First we can specify a form for the instantaneous forward rate volatility:
$$\sigma_f(t, T) = \sigma[1 + \gamma(T - t)]e^{-\lambda(T-t)}$$
which we will see later in the LMM and creates the required humped shape.
- We can get close to this by the following choice of one-factor model:

$$\begin{aligned} dr(t) &= [\theta(t) - \beta(t)r(t)]dt + \sigma dW(t) \\ \beta(t) &= \lambda - \frac{\gamma}{1 + \gamma t} \end{aligned}$$

Extension to Hull and White

- This leads to
$$\sigma_f(t, T) = \sigma \frac{1 + \gamma T}{1 + \gamma t} e^{-\lambda(T-t)}$$
which is close to the desired functional form and leads to analytic formulas for zero coupon bonds and options on bonds.
- Note that the $\beta(t)$ value is deterministic and so it is possible to build a trinomial tree as it is for the Hull and White model, only the a value (now called $\beta(t)$) will be changing at each step - although this made lead to branching problems.

Extension to BK

- We can also create an analogous model to the BK model, by considering the log of the rates. This does not lead to analytic formulas for bond and option prices but does lead to a lognormal distribution rather than a normal distribution.
- Here the dynamics are described by the following

$$
\begin{aligned}
dx(t) &= [\theta(t) - \beta(t)x(t)]dt + \sigma dW(t) \\
\beta(t) &= \lambda - \frac{\gamma}{1+\gamma t} \\
r(t) &= e^{x(t)}
\end{aligned}
$$

- This model typically fits the cap-volatility curve better than the BK model, and can be implemented with a trinomial tree, again with a time varying a term (now called $\beta(t)$).

Extensions to CIR

- As we extended Vasicek to the Hull and White model, so that it could recreate the current term structure then we can do the same for the CIR model, and retain its analytic tractability, now the CIR model becomes

$$\begin{aligned} dx(t) &= k(\theta - x(t))dt + \sigma\sqrt{x(t)}dW(t), \quad x(0) = x_0 \\ r(t) &= x(t) + \psi(t) \end{aligned}$$

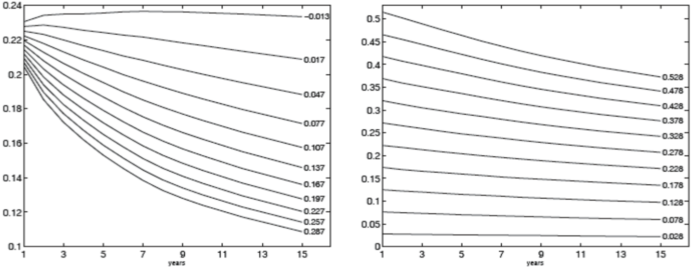
  where $\psi(t)$ is chosen to match the current term structure - details of how to do this are in Brigo and Mercurio (2001).
- This is called the CIR++ model and is a rival for BK as it does not allow negative rates but it does have analytic formulas for bond and option prices, and can be fitted to the term structure

Implied volatility curves

- Brigo and Mercurio compare the BK model with the CIR++ model by varying parameters and observing the implied cap volatility curves that results. The initial parameters are

BK	CIR++
$a = 0.077$	$x_0 = 0.0059$
$\sigma = 0.2286$	$k = 0.3945$
	$\theta = 0.2714$
	$\sigma = 0.0545$

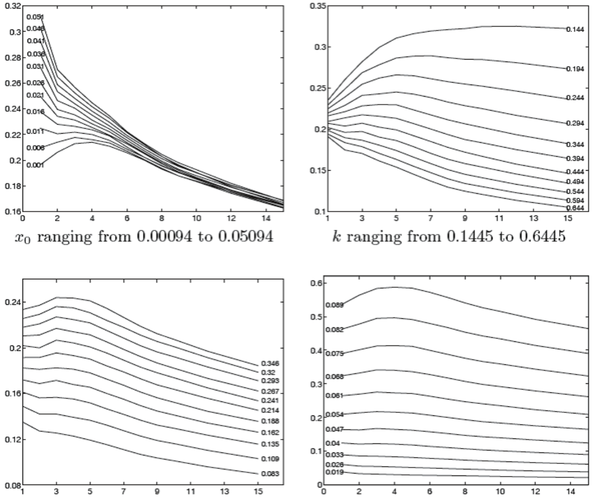
Implied volatility curves: BK



a ranging from -0.01289 to 0.28711 σ ranging from 0.0286 to 0.5286

Fig. 3.8. Cap volatility curves implied by the Black and Karasinski (1991) model.

Implied volatility curves: CIR++



x_0 ranging from 0.00094 to 0.05094

k ranging from 0.1445 to 0.6445

$\bar{\theta}$ ranging from 0.0831 to 0.3467

σ ranging from 0.01951 to 0.08951

Fig. 3.10. Cap volatility curves implied by the CIR++ model.

Quiz

Which model best creates the hump shaped volatility curve

- BK
- CIR++
- Both
- Neither

Quiz

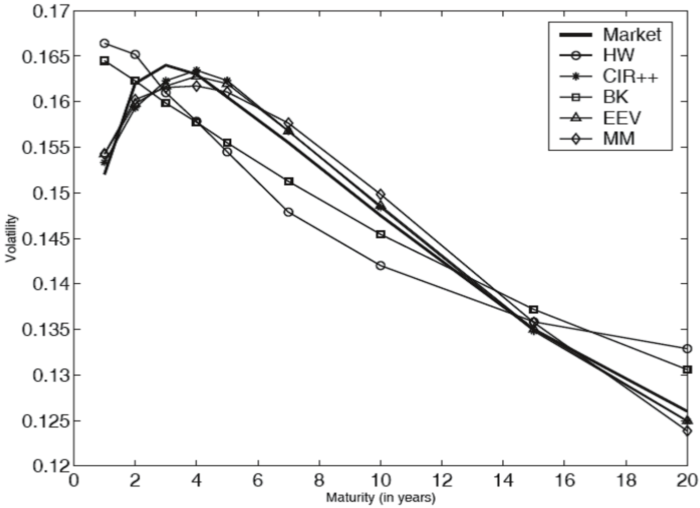
Which model best creates the hump shaped volatility curve

- BK
- CIR++ Correct
- Both
- Neither

Calibration to real data from Brigo Mercurio

- Finally, we can try to match some of our models to real data - you will have a chance to do this as part of your latest problem set.
- The calibration is to market cap prices, and is carried out for the HW, BK, CIR++, the extended HW above (MM), and an extended Vasicek analogous to CIR++ (EEV).
- The results for HW and BK are disappointing as the hump shape becomes a simple decreasing structure, and as in our simplified earlier example the mean-reversion parameter a is negative in the HW model, suggesting mean fleeing. This is not always the case, but HW has difficulty with calibration.
- Adding extra parameters as in MM, EEV and CIR++ seems to bring an improvement as now we have three or four parameters that can be used to calibrate to caps,

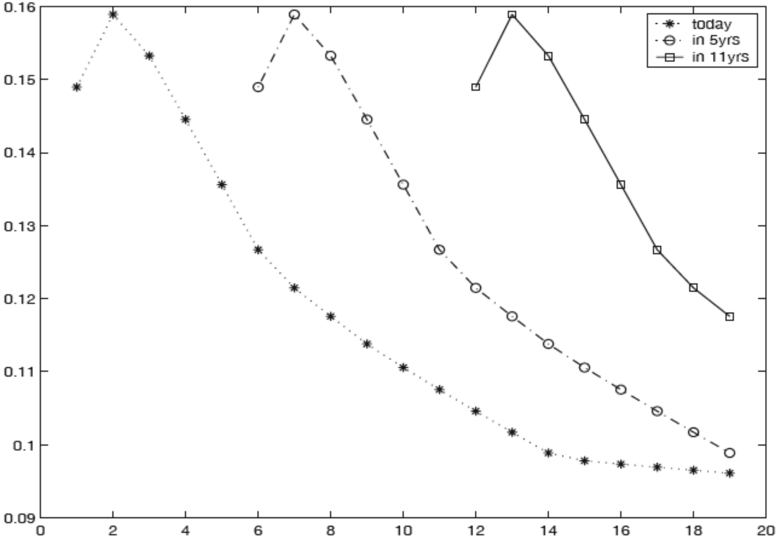
Calibration



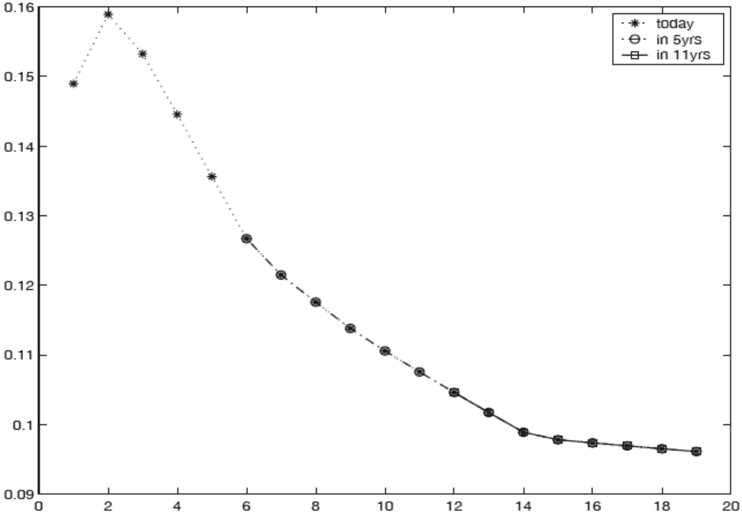
Time homogeneity

- We have seen that it is possible to generate volatility structures that closely match the volatility structures of current market caps.
- However, the volatility curves should be time homogeneous in that regardless of when we start from the volatility curves should have the same humped (or downward sloping) shape.
- To do this we simply move forward one year, and then calculate the new caplet volatilities from our model. They should again be humped shaped (or downward sloping) from a similar initial level, as the volatilities of the forward rates starting from some future time should have the same volatility structure.
- However, if we calibrate our model to the existing cap/caplet volatilities then typically we see time heterogeneity, the hump shape typically appears for only the current cap prices or maybe even those starting in one year from now, but they very rarely capture the full time homogeneity - we will need multifactor models for this.

Time homogeneous vols (what we observe)



Time heterogeneous vols (what our short rate models gives us)



Volatility structure

- If we have interest rate products that only depend upon the volatility from now until reset or maturity then this is not a problem. For example, we can price caplets and caps perfectly accurately.
- However, anything that has an early exercise or call feature, or any kind of path dependency then the incorrect modeling of the volatility structure will potentially be a more serious problem.
- So, short rate models fail to recreate the true volatility structures. How about changing yield curves?

Correlation

-

- Let us return to an idea from PS4, and consider an affine one factor model such as the Vasicek model, where the bond price is given by

$$P(0,T) = A(0,T)\exp(-B(0,T)r(0))$$

and, the continuously compounded rate is

$$R(0,T) = -\frac{\ln A(0,T)}{T} + \frac{B(0,T)}{T}r(0) = a(0,T) + b(0,T)r(0)$$

Now let's consider two continuously compounded rates at $T_1 = 1$ and $T_{10} = 10$ years.

$$\begin{aligned} Corr(R(t,T_1), R(t,T_2)) &= Corr(a(t,T_1) + b(t,T_1)r(t), a(t,T_2) + b(t,T_2)r(t)) \\ &= 1 \end{aligned}$$

- This should be the same result that you found on PS4.

Correlation

- Clearly both the one- and ten-year yields are dependent upon the same stochastic factor $r(t)$ and so when $r(t)$ changes they all change by the same amount.
- This is clearly unrealistic, as this says that the thirty year continuously compounded rate is perfectly correlated with the three month continuously compounded rate, so any changes to any rate, will automatically effect all of the other rates in the same direction.
- Note, that this is not a problem if we are not interested in products that do not explicitly depend upon modeling the correlation between different rates, such as a caplet.
- However, products whose values do not depend on correlations are those where the payoffs are only dependent upon one rate - such as swaptions, especially Bermudan ones. As soon as the payoff depends upon more than one rate then correlations between the rates matter and short rate models may well give very inaccurate valuations.

Quiz

Which of these products depends upon correlation between rates

- FRA
- Cap
- Swaption

Quiz

Which of these products depends upon correlation between rates

- FRA No
- Cap No
- Swaption Yes

Conclusion

- We have analyzed what we mean by the term structure of volatilities and we have tested our one factor models to see how well they reproduce the term structure of volatilities.
- Some one-factor models do well and it is generally possible to create something close to the humped volatility shape using more advanced volatility models.
- However, they cannot recreate time homogeneous volatility structures where the forward rate volatilities always exhibit humped shapes, regardless of when they start from and most damningly of all, the correlations between rate are always 1 and so if we wish to consider rates moving in different directions or by different factors then we are unable to do this with one-factor models.