# FIN516: Term-structure Models

## Lecture 9b): Toolkit III

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# Overview

- We now look at one extensions to Monte Carlo techniques that will be useful for Market Models: pricing American options using the Longstaff and Schwartz (2001) Monte Carlo method.
- Later we will need one more adaptation to Monte Carlo methods - the predictor/corrector method. This makes more sense when we know more about the LMM¿

# Overview on Monte Carlo methods

- One of the key unanswered questions in finance is how to value options with early exercise features by using a Monte Carlo method.
- Recall that the American option value $V_0$ is

$$V_0 = \max_{\tau} E_0^Q[e^{-r(\tau)} \max(S_\tau - K, 0)]$$

- The problem comes from the fact that Monte Carlo is a forward looking method. To use the sample paths we would have to test early exercise at each point in time on each sample path in order to determine what the optimal exercise strategy would be.
- This is incredibly time consuming and not practical, and simplistic approaches, such as the perfect foresight method where you simply choose the highest early exercise value during the lifetime of the option do not give acceptable approximations to the option value.

# Graphical explanation of perfect foresight

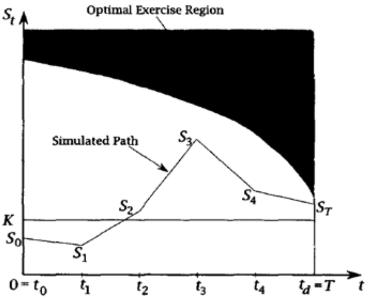


Fig. 1. Sample simulated path.

If you follow the optimal strategy (shaded) then you should only exercise at expiry, but you can get a better payoff from exercising at $t_3$. This is perfect foresight and does not value the option correctly

# Quiz

Why is it important to be able to value options with early exercise features with Monte Carlo?

# Recap on problem

- The key calculations when valuing an American option are determining the early exercise value and the continuation value.
- The option value at any point in time will be the larger of these two values.
- The early exercise value is simple to calculate at any point in time. The continuation value is typically determined by a backward iteration approach, where the value now is the discounted expected option value at the next instance in time.
- The key information we need to know is given the share price at time $t$, what is the expected option value at time $t + \Delta t$.
- This is simple to determine in binomial models but less so in Monte Carlo approaches.

# Longstaff and Schwartz (2001)

- The Longstaff and Schwartz method, essentially estimates the conditional expected option value at the next time step by simulating lots of paths and then carrying out a regression of the future realized option value as a function of the current value of the underlying asset.
- This gives an approximation for the continuation value that can then be compared to the early exercise value and then we know the option value at each point in time on each path.
- In terms of Monte Carlo pricing, all we actually need to know is the rule for early exercising, so we know when we receive the cash flows and the value of the option is the average of the discounted payoffs for each path.
- We will explain the method via an example and then describe the general method.

# Example

- We will attempt to value a Bermudan put option where exercise is possible now and at three future dates. $S_0 = 1, K = 1.1, r = 0.06$.
- The first step is to simulate some paths, the table below denotes the results:

Stock price paths

| Path | $t=0$ | $t=1$ | $t=2$ | $t=3$ |
|---|---|---|---|---|
| 1 | 1.00 | 1.09 | 1.08 | 1.34 |
| 2 | 1.00 | 1.16 | 1.26 | 1.54 |
| 3 | 1.00 | 1.22 | 1.07 | 1.03 |
| 4 | 1.00 | .93 | .97 | .92 |
| 5 | 1.00 | 1.11 | 1.56 | 1.52 |
| 6 | 1.00 | .76 | .77 | .90 |
| 7 | 1.00 | .92 | .84 | 1.01 |
| 8 | 1.00 | .88 | 1.22 | 1.34 |

- We need to use this information to determine the continuation value at each point in time for each path. To do this we will construct a Cash Flow Matrix at each point in time.

# Continuation value at t=2

- The table below denotes the cash flows at $t = 3$ assuming that we held the option that far:

Cash-flow matrix at time 3

| Path | $t=1$ | $t=2$ | $t=3$ |
|---|---|---|---|
| 1 | — | — | .00 |
| 2 | — | — | .00 |
| 3 | — | — | .07 |
| 4 | — | — | .18 |
| 5 | — | — | .00 |
| 6 | — | — | .20 |
| 7 | — | — | .09 |
| 8 | — | — | .00 |

- The next step is to attempt to find a function that describes the continuation value at time 2 as a function of the value of $S$ at time 2.
- To do this we use a regression technique, that takes the values at time 2 as the "$X$" values and the discounted payoff at time 3 as the "$Y$" values.

# Continuation value at t=2

Regression at time 2

| Path | $Y$ | $X$ |
|---|---|---|
| 1 | $.00 \times .94176$ | 1.08 |
| 2 | — | — |
| 3 | $.07 \times .94176$ | 1.07 |
| 4 | $.18 \times .94176$ | .97 |
| 5 | — | — |
| 6 | $.20 \times .94176$ | .77 |
| 7 | $.09 \times .94176$ | .84 |
| 8 | — | — |

- Note that the regression is only carried out on paths that are in the money at time 2.
- The regression here is simple where $Y$ is regressed upon $X$ and $X^2$ (the actual scheme is slightly more sophisticated). In this particular example: $Y = -1.070 + 2.938X - 1.813X^2$, and so we can use this to estimate the continuation value for each of the current share prices ($X$ in the regression).
- For example for path 1, where $S = 1.08$, the regression formula gives $Y$ (the continuation value) to be 0.0369.

# Quiz

Why dont we regress with out-of-the money paths?

- It makes the regression wrong
- The only interesting paths are the in the money ones so this saves time.
- Some other explanation

# Quiz

Why dont we regress with out-of-the money paths?

- It makes the regression wrong
- The only interesting paths are the in the money ones so this saves time. CORRECT
- Some other explanation

# Option value at t=2

Optimal early exercise decision at time 2

| Path | Exercise | Continuation |
|---|---|---|
| 1 | .02 | .0369 |
| 2 | — | — |
| 3 | .03 | .0461 |
| 4 | .13 | .1176 |
| 5 | — | — |
| 6 | .33 | .1520 |
| 7 | .26 | .1565 |
| 8 | — | — |

- This then allows you to decide at which points in time you would exercise and thus determine the cash flows at $t = 2$ (below). Notice that for each path, if you exercise at $t = 2$ then you do not also exercise at $t = 3$.

Cash-flow matrix at time 2

| Path | $t = 1$ | $t = 2$ | $t = 3$ |
|---|---|---|---|
| 1 | — | .00 | .00 |
| 2 | — | .00 | .00 |
| 3 | — | .00 | .07 |
| 4 | — | .13 | .00 |
| 5 | — | .00 | .00 |
| 6 | — | .33 | .00 |
| 7 | — | .26 | .00 |
| 8 | — | .00 | .00 |

# Quiz

Why are cash flows at subsequent times set to zero if you exercise at $t = 2$?

# Continuation value at t=1

- We can apply the same process to $t = 1$, for each of the paths that are in the money we regress the discounted future cash flows ($Y$) on the current value of the underlying asset ($X$), where $X$ and $Y$ are as given below:

Regression at time 1

| Path | $Y$ | $X$ |
|---|---|---|
| 1 | $.00 \times .94176$ | 1.09 |
| 2 | — | — |
| 3 | — | — |
| 4 | $.13 \times .94176$ | .93 |
| 5 | — | — |
| 6 | $.33 \times .94176$ | .76 |
| 7 | $.26 \times .94176$ | .92 |
| 8 | $.00 \times .94176$ | .88 |

- The regression equation here is $Y = 2.038 - 3.335X + 1.356X^2$ and again we use this to estimate the continuation value and decide on an early exercise strategy.
- The next table compares the two values and the final table denotes the early exercise or stopping rule.

# Stopping rule

Optimal early exercise decision at time 1

| Path | Exercise | Continuation |
|---|---|---|
| 1 | .01 | .0139 |
| 2 | — | — |
| 3 | — | — |
| 4 | .17 | .1092 |
| 5 | — | — |
| 6 | .34 | .2866 |
| 7 | .18 | .1175 |
| 8 | .22 | .1533 |

- The early exercise strategy is as follows:

Stopping rule

| Path | $t = 1$ | $t = 2$ | $t = 3$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1 |
| 4 | 1 | 0 | 0 |
| 5 | 0 | 0 | 0 |
| 6 | 1 | 0 | 0 |
| 7 | 1 | 0 | 0 |
| 8 | 1 | 0 | 0 |

- From this we can then value the option, by forming the final cash flow matrix from this rule.

# Option value

Option cash flow matrix

| Path | $t = 1$ | $t = 2$ | $t = 3$ |
|---|---|---|---|
| 1 | .00 | .00 | .00 |
| 2 | .00 | .00 | .00 |
| 3 | .00 | .00 | .07 |
| 4 | .17 | .00 | .00 |
| 5 | .00 | .00 | .00 |
| 6 | .34 | .00 | .00 |
| 7 | .18 | .00 | .00 |
| 8 | .22 | .00 | .00 |

- So the option value is the average of the discounted cash flows, so in this case:

$$\begin{aligned} V_0 &= \frac{1}{8}\big(0 + 0 + 0.07e^{-3r} + 0.17e^{-r} + 0 + 0.34e^{-r} \\ &\quad +0.18e^{-r} + 0.22e^{-r}\big) \\ &= 0.1144 \end{aligned}$$

# More sophisticated regression

- In general the regression here $Y = a_1 + a_2 X + a_3 X^2$ is not going to be satisfactory, especially as we will have far more than 8 paths when attempting to find the functional form of the continuation values.
- In fact the general form of $Y$ is:

$$Y = \sum_{j=0}^{M} a_j F_j(X)$$

- The user decides upon $M$ the number and the functional form $F_j(X)$ (in our example $F_j(X) = X^j$) where $F_j(X)$ is called a basis function.
- The suggestion of Longstaff and Schwartz is to choose basis functions described by Laguerre polynomials to provide the best fit. However, you are free to choice whichever basis functions you want (e.g. Polynomial as in the example, Chebyshev, Hermite etc.)

# More sophisticated regression

- The Laguerre polynomials are given by:

$$
\begin{aligned}
F_0(X) &= \exp(-X/2) \\
F_1(X) &= \exp(-X/2)(1-X) \\
F_n(X) &= \exp(-X/2)\frac{e^X}{n}\frac{d^n}{dX^n}(X^n e^{-X})
\end{aligned}
$$

- Then the least squares approach approximates the constants $a_j$, and when we have these values we can use them to predict the continuation value for each value of $S$ at every point in time.

# General procedure

1. Decide on the number of sample paths $N$, the number of basis functions for the regression, $M$, and the type of basis functions $F_j(X)$ and the number of observation dates $d$.
2. Draw $Nd$ Normally distributed random numbers and simulate the sample paths for the underlying asset at each point in time $S^n_{t_i}, \qquad 1 \leq i \leq d, \qquad 1 \leq n \leq N$
3. At expiry $t = t_d$, record the cash flow values $CF^n(t_d)$ which for a put are $\max(K - S^n_{t_d}, 0)$.
4. Move back to $t = t_{d-1}$ for each path where $S^n_{t_{d-1}} < K$ calculate the continuation value as $CV^n(t_{d-1}) = e^{-r(t_d - t_{d-1})} CF^n(t_d)$. From these values perform the regression to determine the functional form of the continuation value, $Y(S)$ where $S$ is the value of the underlying asset
5. Recalculate the continuation value as $CV^n(t_{d-1}) = Y(S^n_{t_{d-1}})$

# General procedure

6. For every path calculate the cash flow value where if the continuation value $CV^n(t_{d-1}) < K - S^n_{t_{d-1}}$ then $CF^n(t_{d-1}) = K - S^n_{t_{d-1}}$ and $CF^n(t_i) = 0$ for $i > d-1$ otherwise $CF^n(t_{d-1}) = 0$.
7. Repeat this process for the previous time step until you have $CF^n(t_i)$ for all $i$ and $n$. Note that in general to calculate $CV^n(t_i)$ before the regression

$$CV^n(t_i) = \sum_{k=i+1}^{d} e^{-r(t_k - t_i)} CF^n(t_k)$$

8. The option value $V_0$ is then

$$V_0 = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{M} e^{-rt_j} CF^i(t_j)$$

## Quiz

Why do steps 7 and 8 correctly describe the option value?

# More sources of uncertainty

- Naturally in our market models we will have a more complex set up then an American call option on a single underlying asset. However, the procedure is exactly the same.
- The only difficulties arise from specifying appropriate basis functions and with discounting under the appropriate numeraire asset.
- With a single source of uncertainty we have a wide range of choices of interpolating polynomials, but this is not the case when we have more sources of uncertainty. In the set up above the numeraire asset is the money market account with a constant interest rate $r$. We will need to adapt this to deal with any possible numeraire asset.

# Choice of basis functions

- In Longstaff and Schwartz's (2001) original paper they consider valuing an interest rate derivative using a string theory model. They use this to value a ten year bermudan swaption to enter into the swap that expires ten years after the option is issued (an option to enter into existing swap). Their choice of basis functions is: a constant, the zero coupon bond prices for the twenty six-month periods, and the current swap rate of the underlying swap, the square of this rate, the cube of this rate.
- At the first reset date this gives 22 basis functions and this will decrease by one at every reset date.
- In Brigo and Mercurio, they suggest using all of the underlying forward swap rates, their squares and a constant term. These basis functions will also decrease as time moves closer to the option maturity as the number of forward swaps available decreases.

# Choice of basis functions

- In Gatarek et al., they suggest using polynomials in the Brownian motion terms (they only consider a single factor. In general, the quality of the LSM model is closely linked to the selection of basis functions, it is preferable to use swap rates and functions of swap rates and as many as possible.

# Adjusting for different numeraire assets

- In the above explanation when using the input s for the continuation value we had the expression

$$CV^n(t_i) = \sum_{k=i+1}^{d} e^{-r(t_k - t_i)} CF^n(t_k)$$

the problem when we have a different numeraire is to discount properly across these times periods. Let $U^n(t_i)$ denote the value of the numeraire asset at time $t_i$ on path $n$, and so we change the above expression to give

$$CV^n(t_i) = \sum_{k=i+1}^{d} \frac{U^n(t_i)}{U^n(t_k)} CF^n(t_k)$$

- In general, all discounting will be performed in this way within the Monte Carlo methods where the numeraire asset is $U$

# How well does it perform?

- Longstaff and Schwartz provide proofs that as $M \to \infty$ and $N \to \infty$ the option value obtained from their scheme converges to the theoretical value.
- This isn't much use for practical considerations as you will be limited by how many basis functions you can calculate and how many simulations you can perform.
- There have been a few investigations into the method and the view of the methods performance are mixed: Broadie and Detemple (2004) say regression methods 'often incur unknown approximation errors and are limited by a lack of error bounds'.

# How well does it perform?

- A detailed appraisal of this technique by Moreno and Navas (2003) investigates the use of various polynomial fits and numbers of basis functions.
- It is not clear that increasing the number of basis functions actually increases the accuracy of the method and there is no real difference from using different types of basis functions (e.g. Chebyshev rather than Laguerre)
- For more complicated derivative pricing problems the trend is even less clear, sometimes errors can increase as you add more basis functions (too many essentially fits the stochastic values of $S$ exactly), see Brigo and Mercurio (p. 586) and Gatarek et al. (p. 252) for some discussion on basis functions in the LMM setting.
- In general, the method will provide good estimates but will be difficult to assess exactly how accurate it is.

# Conclusion

- We have looked at predictor-corrector techniques for Monte Carlo.
- We have seen the problems of valuing American options using Monte Carlo methods and various solutions.
- The most useful technique is the Longstaff and Schwartz (2001) technique.