FIN516: Term-structure Models

Lecture 10: LIBOR market model - set-up

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Introduction

- Recall from our work with the Black model (Lecture 2) that under the appropriate numeraire the forward rate and the swap rate could be lognormal with zero drift. So, after assuming a function of the volatility of the forward (or swap rate) then we can easily price caplets and swaptions.
- There are a few problems with this, notably that if forward rates are lognormal then swap rates cannot be lognormal and if we set the numeraire so that one forward rate is driftless then the other forward rates will not necessarily be driftless.
- Here we will start to address these problems by determining the drift of forward rates under different measures and also specifying some possible volatility functions.

Recall: Caplet pricing

- We have

$$F(t; T, T+\tau) = \frac{1}{\tau}\left(\frac{(P(t,T)}{P(t,T+\tau)} - 1\right) \quad (1)$$

$$= \frac{P(t,T) - P(t,T+\tau))/\tau}{P(t,T+\tau)} \quad (2)$$

and this is a portfolio of two zero coupon bonds (the *asset*) scaled by the zero coupon bond $P(t, T+\tau)$, thus by the fundamental theorem above the forward rate is a martingale under the $P(t, T+\tau)$ measure.

- If the forward rate $F(t; T+T+\tau)$ is lognormal with time-dependent volatility then

$$dF_t = \mu(F(t), t)F(t)dt + \sigma(t)F(t)dW(t)$$

(time notation supressed) but under the appropriate, $P(t, T+\tau)$ measure then $F(t)$ is a martingale and so

$$dF_t = \sigma(t)F(t)dW(t)$$

Quiz

Why is the discounted caplet price also a martingale under the $P(t, T + \tau)$ measure?

- Because the forward rate is a martingale.
- All traded assets are martingales with respect to the numeraire asset.
- Because its price can also be expressed as a fraction with $P(t, T + \tau)$ as the denominator.

Quiz

Why is the discounted caplet price also a martingale under the $P(t, T + \tau)$ measure?

- Because the forward rate is a martingale.
- All traded assets are martingales with respect to the numeraire asset. CORRECT
- Because its price can also be expressed as a fraction with $P(t, T + \tau)$ as the denominator.

Recall: Seven steps to the LMM

1. Write the derivative's payoff in terms of the behavior of forward rates (Lecture 12)
2. Observe the current forward rates and their market implied volatility (MIL session)
3. Choose a numeraire that makes the rates as driftless as possible (Lecture 12)
4. Compute the drifts of the forward rates in the appropriate measure (Today)
5. Choose or infer the instantaneous volatility curves for the forward rates (Today)
6. Choose or infer instantaneous correlations between different forward rates (Today)
7. Run a Monte Carlo simulation (Lecture 11)

LMM: formal set-up

- Now define, for brevity, $F_k(t) = F(t; T_{k-1}, T_k)$ and so $L(T_{k-1}, T_k) = F(T_{k-1}; T_{k-1}, T_k) = F_k(T_{k-1})$.
- Let $t = 0$ be the current time. Consider a set $\{T_0, \dots, T_M\}$ from which expiry-maturity pairs $\{T_{i-1}, T_i\}$ are taken.
- We will also denote $\{\tau_0, \dots, \tau_M\}$ as the corresponding year fractions so that τ_i is the year fraction associated with the expiry-maturity pair (T_{i-1}, T_i).
- τ_0 is the time from settlement until T_0 (i.e $T_{-1} = 0$). T_i are expressed in years from the current time.
- The forward rate $F_k(t)$, $k = 1, \dots, M$ is "alive" up to T_{k-1} when it becomes the simple compounded LIBOR rate $L(T_{k-1}, T_k)$.

LMM: More than one Brownian motion

- Consider the probability measure $\mathbb{Q}^k$ associated with the numeraire asset $P(\cdot, T_k)$ or the bond whose maturity coincides with the maturity of the forward rate.
- $\mathbb{Q}^k$ is usually termed the forward measure for the maturity T_k, Using simple compounding we know that

$$F_k(t)P(t, T_k) = \frac{P(t, T_{k-1}) - P(t, T_k)}{\tau_k}$$

  and so as $F_k(t)P(t, T_k)$ is the price of a tradable asset then it is a martingale under $\mathbb{Q}^k$ associated with the numeraire asset. But as $F_k(t)$ is the price of our trade asset scaled by the numeraire asset then $F_k(t)$ is a martingale.
- We assume the following driftless dynamics for F_k under $\mathbb{Q}^k$

$$dF_k(t) = \underline{\sigma}_k(t)F_k(t)dW(t), \quad t \leq T_{k-1}$$

LMM: More than one Brownian motion

Where $W(t)$ is an M-dimensional column vector Brownian motion (under $\mathbb{Q}^k$)

$$dW(t) = \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ \vdots \\ dW_M(t) \end{pmatrix}$$

and matrix $\rho = (\rho_{i,j})_{i,j=1,\dots,M}$

$$\rho = \begin{pmatrix} 1 & \rho_{1,2} & \dots & \rho_{1,M} \\ \rho_{2,1} & 1 & \dots & \rho_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{M,1} & \rho_{M,2} & \dots & 1 \end{pmatrix}$$

is defined so that

LMM: More than one Brownian motion

$$
\begin{aligned}
dW(t)dW(t)^T &= \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ \vdots \\ dW_M(t) \end{pmatrix} \begin{pmatrix} dW_1(t) & dW_2(t) & \dots & dW_M(t) \end{pmatrix} \\
&= \begin{pmatrix} dW_1(t)dW_1(t) & dW_1(t)dW_2(t) & \dots & dW_1(t)dW_M(t) \\ dW_2(t)dW_1(t) & dW_2(t)dW_2(t) & \dots & dW_2(t)dW_M(t) \\ \vdots & \vdots & \ddots & \vdots \\ dW_M(t)dW_1(t) & dW_M(t)dW_2(t) & \dots & dW_M(t)dW_M(t) \end{pmatrix} \\
&= \begin{pmatrix} dt & \rho_{1,2}dt & \dots & \rho_{1,M}dt \\ \rho_{2,1}dt & dt & \dots & \rho_{2,M}dt \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{M,1}dt & \rho_{M,2}dt & \dots & dt \end{pmatrix} \\
&= \rho dt
\end{aligned}
$$

LMM: Back to Black

- $\underline{\sigma}_k(t)$ is a horizontal M-vector volatility coefficient for the forward rate $F_k(t)$, however, as a default position we will assume that

$$\underline{\sigma}_k(t)] = [0 \quad 0 \quad \ldots \quad \sigma_k(t) \quad \ldots 0]$$

  or the only non-zero volatility entry is at position k.

- Notice that this choice for $\underline{\sigma}_k(t)$means that we can rewrite our SDE for F_k more simply as

$$dF_k(t) = \sigma_k(t) F_k(t) dW_k(t), \quad t \leq T_{k-1}$$

  as a one dimensional process - *but* where dW_k is correlated with all of the other dW_i values.

- Note that our matrix notation seems unnecessarily burdensome, however, in some scenarios it will become very useful. In particular $dW_i(t)dW_j(t) = \rho_{i,j}dt$

- Also, recall that this world, we can explicitly solve for $F_k(T)$ as

$$\ln F_k(T) = \ln F_k(0) - \int_0^T \frac{\sigma_k(t)^2}{2} dt + \int_0^T \sigma_k(t) dW_k(t)$$

LMM: Alternative descriptions: 1

- In full, if we have $1 \leq k \leq M$ forward rates, $F_k (= F_k(t))$, driven by M volatilities $\sigma_k (= \sigma_k(t))$ and M Brownian motions $dW_k (= dW_k(t))$ we have:

$$\begin{pmatrix} \frac{dF_1}{F_1} \\ \frac{dF_2}{F_2} \\ \cdots \\ \frac{dF_k}{F_k} \\ \cdots \\ \frac{dF_M}{F_M} \end{pmatrix} = \begin{pmatrix} \mu_1(F_1, \ldots, F_M, t) \\ \mu_2(F_1, \ldots, F_M, t) \\ \cdots \\ \mu_k(F_1, \ldots, F_M, t) \\ \cdots \\ \mu_M(F_1, \ldots, F_M, t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 & \ldots & 0 \\ & & \ddots & & \ddots & \\ \ldots & \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & \sigma_k & \ldots & 0 \\ & & \ddots & & \ddots & \\ \ldots & \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & \sigma_M \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \\ \vdots \\ dW_k \\ \vdots \\ dW_M \end{pmatrix}$$

  where the matrix ρ is described above and explains the relationship between the dW_k values.

- Here each forward rate is determined by only one (correlated) Brownian motion.

Quiz

What does the covariance matrix, $\Sigma_{i.j}$, look like?

- ρ_{ij}
- $\sigma_i \sigma_j \rho_j$
- $\sigma_i \sigma_j$

Quiz

What does the covariance matrix, $\Sigma_{i.j}$, look like?

- ρ_{ij}
- $\sigma_i \sigma_j \rho_j$ CORRECT
- $\sigma_i \sigma_j$

LMM: Alternative descriptions: 2

- Above we have written the process followed by F_k as
$$dF_k(t) = \sigma_k(t)F_k(t)dW_k(t), \quad t \leq T_{k-1}$$
which we could write as
$$dF_k(t) = \sum_{i=1}^{m} \sigma_{ki}(t)F_k(t)dW_i(t), \quad t \leq T_{k-1}$$
where now dW_i are **independent** Brownian motions and $m \leq M$, so we can also reduce the number of Brownian motions.
- This seems to have made our life more difficult as we now have the forward rate possibly depending upon $m \leq M$ Brownian motions. But recall, however, that it already did via the correlation between the Brownian motions.
- How, then should we interpret the $\sigma_{ki}(t)$ values? If we write the covariance matrix between forward rates as Σ which is an $M \times M$ matrix then we can interpret σ (a $M \times m$ matrix) as the square root of $\Sigma = \sigma\sigma^T$.
- Now we have an expression for each forward rate in terms of independent Brownian motions but the two expressions are equivalent in that $\sigma_k^2(t) = \sum_{i=1}^{m} \sigma_{ki}^2(t)$.

Quiz

How do we then find the matrix σ?

- .

Quiz

How do we then find the matrix σ?

- . Cholesky factorization or PCA

LMM: Alternative descriptions: 3

- Alternatively, we can simplify the $\sigma_{ki}(t)$ values some more.

$$
\begin{aligned}
dF_k(t) &= \sum_{i=1}^{m} \sigma_{ki}(t) F_k(t) dW_i(t), \quad t \leq T_{k-1} \\
&= \sigma_k(t) \sum_{i=1}^{m} \frac{\sigma_{ki}(t)}{\sigma_k(t)} F_k(t) dW_i(t) \\
&= \sigma_k(t) \sum_{i=1}^{m} \frac{\sigma_{ki}(t)}{\sqrt{\sum_{i=1}^{m} \sigma_{ki}^2(t)}} F_k(t) dW_i(t) \\
&= \sigma_k(t) \sum_{i=1}^{m} b_{ki}(t) F_k(t) dW_i(t)
\end{aligned}
$$

where $b_{ki}(t) = \frac{\sigma_{ki}}{\sqrt{\sum_{i=1}^{m} \sigma_{ki}^2(t)}}$ or b is the square root of the *correlation* matrix, $bb^T = \rho$ where b is an $M \times m$ matrix, of in the case where $m = M$ an $M \times M$ matrix.

LMM: Alternative descriptions: 4

- Instead of using a general square root for the covariance matrix, we could alternatively use principal components analysis, where we can write the equation for dF as:

$$dF_k = \sum_{i=1}^{m} \sqrt{\lambda_i} a_{ki}(t) F_k(t) dW_i(t)$$

- In general, I will prefer using the description with independent Brownian motions and b_{ki} or with σ_{ki} being the square root of the covariance matrix (this is good for Monte Carlo) but it will be possible to use any of the four alternative descriptions above.
- **For computational efficiency the covariance matrix representations will be superior as the integrals that we solve to fill in the covariance matrix will also be used in determining the drift.**

LMM: Correlation

- Recall that in one-factor models, all forward rates are instantaneously perfectly correlated.
- This level of instantaneous correlation is too high or *too correlated*.
- We can manage correlation by both defining the ρ matrix to have lower values, or by choosing appropriate volatility functions - recalling that overall (or terminal) correlation depends upon both the instantaneous correlation as well as how volatility changes over time.

Two forward rates (recap)

- Let us return to the LIBOR in-arrears case that cause us problems back in Lecture 2. Here assume the numeraire bond would be the bond maturing at time T_{k-1}, $P(0, T_{k-1})$ but we are interested in the forward rate $F_k(t)$.
- Now, $F_k(t)P(t, T_k)$ is tradable and so $F_k(t)\frac{P(t,T_k}{P(t,T_{k-1})}$ is a martingale in the forward measure, and so

$$F_k(t)\frac{P(t, T_k)}{P(t, T_{k-1})} = \frac{F_k(t)}{1 + F_k(t)\tau_k}$$

is a martingale. We also know that

$$\frac{P(t, T_k)}{P(t, T_{k-1})} = \frac{1}{1 + F_k(t)\tau_k}$$

is a martingale too.

Two forward rates (recap)

- By the product rule we have

$$d_t(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

and so if we choose, $X_t = F_k(t)$ and $Y_t = \frac{1}{1+F_k(t)\tau_k}$ then we of course know that the drift of Y_t and $X_t Y_t$ must be zero.

- Let's drop the time arguments and so

$$dF_k = \mu_k F_k dt + \sigma_k F_k dW_k$$

(where $F_k(t) = F_k$) and then

$$d\left(\frac{1}{1+F_k\tau_k}\right) = (\dots)dt - \frac{\tau_k \sigma_k F_k}{(1+F_k\tau_k)^2} dW_k$$

As it is a martingale then the drift term is zero.

Two forward rates (recap)

- Then by the product rule

$$
\begin{aligned}
d\left(\frac{F_k}{1+F_k\tau_k}\right) &= F\left(\frac{-\tau_k\sigma_k F_k}{(1+F_k\tau)^2}\right)dW_k + \frac{1}{1+F_k\tau_k}\left(\mu_k F_k dt + \sigma_k F_k dW\right) \\
&\quad + \left(\frac{-\tau_k\sigma_k F_k}{(1+F_k\tau_k)^2}\right)dW_k\left(\mu_k F_k dt + \sigma_k F_k dW_k\right) \\
&= \left(\frac{\mu_k F_k}{1+F_k\tau_k} - \frac{\sigma_k^2 F_k^2\tau_k}{(1+F_k\tau_k)^2}\right)dt + (\dots)dW_k
\end{aligned}
$$

- and so as this is also a martingale then the drift must be zero and so,

$$
dF_k = \frac{\tau_k F_k}{1+\tau_k F_k}\sigma_k^2 F_k dt + \sigma_k F_k dW_k
$$

is the required result.

- Note that this is not a very pleasant expression for the drift (and it certainly is worse than a zero drift) but it is a tractable expression, at least.

Quiz

What difficulty does the drift term create for a Monte-Carlo method?

- The process is now state dependent.
- There is now a volatility term in the drift.
- There is a fraction in the drift term.

Quiz

What difficulty does the drift term create for a Monte-Carlo method?

- The process is now state dependent. Correct
- There is now a volatility term in the drift.
- There is a fraction in the drift term.

General drift term for F_k, for $k > i$

- Now consider general forward rates $F_k = F_k(t)$ and let $P(t, T_k)$ be the zero coupon bond expiring at T_k. If we let $P(t, T_i)$ be the numeraire then we have that F_i is driftless. The above technique gave us the drift of F_{i+1}, how would we determine the drift of a general rate F_k, $k > i$?
- Now, in general we know that $F_k P(t, T_k)$ is tradable and that, naturally $P(t, T_k)$ is also tradable. So, as a result $\frac{F_k P(t,T_k)}{P(t,T_i)}$ and $\frac{P(t,T_k)}{P(t,T_i)}$ are martingales under the $P(t, T_i)$ measure.
- We can write the value of these zero coupon bonds in terms of forward rates with different expiry, in particular,

$$
\begin{aligned}
Z_k &= \frac{F_k}{\prod_{j=i+1}^{k}(1 + F_j \tau_j)} \\
Y_k &= \frac{1}{\prod_{j=i+1}^{k}(1 + F_j \tau_j)}
\end{aligned}
$$

General drift term for F_k, for $k > i$

- Now, assume a general form for the forward rate (dropping time arguments):

$$dF_k = \mu_k F_k dt + \sigma_k F_k dW_k$$

and let's determine the process followed by Y_k which by the product rule is of the form

$$dY_k = (\dots)dt - Y_k \sum_{j=i+1}^{k} \frac{\tau_j}{1 + F_j \tau_j} \sigma_j F_j dW_j$$

- As Y_r must be a martingale then the drift is equal to zero. Now we also have that

$$dZ_k = F_k dY_k + Y_k dF_k + dF_k dY_k$$

and recall that we write $dW^j dW^k = \rho_{j,k} dt$ where $\rho_{j,k}$ is the correlation between F_j and F_k.

General drift term for F_k, for $k > i$

- An implementation of Îto's lemma shows that

$$dZ_k = \left(Y_k \mu_k F_k - Y_k \sum_{j=i+1}^{k} \frac{\tau_j}{1 + F_j \tau_j} \rho_{j,k} F_k F_j \sigma_k \sigma_j \right) dt + \dots$$

thus, as Z_k is a martingale, we must have that

$$\mu_k = \sum_{j=i+1}^{k} \frac{\tau_j}{1 + F_j \tau_j} F_j \rho_{j,k} \sigma_k \sigma_j$$

- Finally we can write down the expression for the process followed by a general forward rate in full as:

$$dF_k = \sigma_k F_k \left(\sum_{j=i+1}^{k} \frac{F_j \tau_j \rho_{k,j} \sigma_j}{1 + F_j \tau_j} dt + dW_k \right)$$

where the forward rate has maturity k and the numeraire asset $(P(t, T_i))$ has maturity i when $k > i$.

General drift term for F_k, for $k < i$

- Of course, it is possible that $k < i$, indeed you saw this on your midterm and we will use this scenario the most often, in which case the process is similar and

$$dF_k = \sigma_k F_k \left(- \sum_{j=k+1}^{i} \frac{F_j \tau_j \rho_{k,j} \sigma_j}{1 + F_j \tau_j} dt + dW_k \right)$$

- Note that these results can be obtained by the use of the change of numeraire toolkit (for details see pp. 213-216 in BM)

Volatility functions

- To match theoretical caplet prices with market caplet prices, we are only interested in the *total volatility*, $\int_t^T \sigma^2(s)ds$, from t to the reset date of the underlying forward rate T. How the volatility changes in that period is of no interest for cap valuation.
- However, other products, notably swaptions, will depend upon knowing the volatility of the forward rate at some time before T and so we will need to specify how the volatility changes over time.
- Note that we also know that from caplet prices that volatilities seem to be largest when there are approximately two years remaining on the forward rate and lower for shorter and longer expiry dates - this is known as the hump shaped volatility function.
- Brigo and Mercurio propose 7 possible choices for volatility function. These choices encompass both of the most popular choices - which are functions 2, 6 and 7 which are reproduced below
- We will detail the three possible functions below.

Function 2: Piecewise constant - $T_k - t$ dependent

- The problem with the above specification is that there are potentially a large number of parameters to fit to market data. Instead we can conjecture that volatility is time homogeneous, in that the volatility depends only upon the time to expiry of the appropriate forward rate.
- This means that the volatility no longer depends on both t and T_k but simply on the time to maturity $T_k - T_{\beta(t)-1}$ where $\beta(t)$ is the index of the first forward rate that has not expired by time t - $F_{\beta(t)}$.
- So our table reduces by setting $\sigma_k(t) = \sigma_{k,\beta(t)} = \eta_{k-(\beta(t)-1)}$, see below:

Inst. vol.	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	η_1	Dead	Dead	...	Dead
$F_2(t)$	η_2	η_1	Dead	...	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	Dead
$F_M(t)$	η_M	η_{M-1}	η_{M-2}	...	η_1

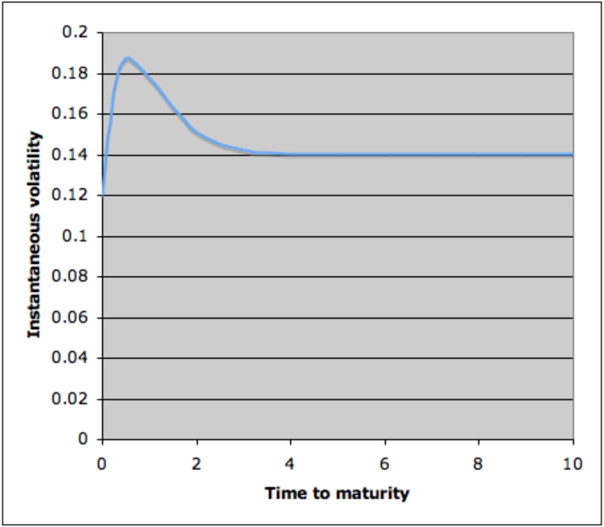
Function 6: Parametric

- This is the standard choice of parametric volatility. Here the volatilities of each forward rates are time homogeneous and just depend upon time remaining until expiry. Then a function is chosen to generate the typical hump shape.
- The standard choice is:

$$\sigma_i(t) = \psi(T_{i-1} - t; a, b, c, d) = [a(T_{i-1} - t) + d]\, e^{-b(T_{i-1}-t)} + c$$

Inst. vol.	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$\psi(1;\dots)$	Dead	Dead	...	Dead
$F_2(t)$	$\psi(2;\dots)$	$\psi(1;\dots)$	Dead	...	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	Dead
$F_M(t)$	$\psi(M;\dots)$	$\psi(M-1;\dots)$	$\psi(M-2;\dots)$	...	$\psi(1;\dots)$

Function 6: Diagram



Function 7: Parametric

- The problem with this is that we only have four parameters to match (at least) all of the caplet prices, and four parameters is not going to be enough to do that.
- There is an easy extension as we did in Function 4 above, so that we have an additional parameter for each forward rate so that (at least) all of the caplet prices can be exactly matched.

$$\sigma_i(t) = \Phi_i \psi(T_{i-1} - t; a, b, c, d) = \Phi_i \left([a(T_{i-1} - t) + d] \, e^{-b(T_{i-1} - t)} + c \right)$$

Inst. vol.	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$\Phi_1 \psi(1)$	Dead	Dead	...	Dead
$F_2(t)$	$\Phi_2 \psi(2)$	$\Phi_2 \psi(1)$	Dead	...	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	Dead
$F_M(t)$	$\Phi_M \psi(M)$	$\Phi_M \psi(M-1)$	$\Phi_M \psi(M-2)$	...	$\Phi_M \psi(1)$

Quiz

What values of Φ_i would lead to perfect time homogeneity?

- $\Phi_i = 1$ for all i
- $\Phi_i = 0$ for all i
- $\Phi_i < 1$ for all i
- $\Phi_i > 1$ for all i

Quiz

What values of Φ_i would lead to perfect time homogeneity?

- $\Phi_i = 1$ for all i Correct
- $\Phi_i = 0$ for all i
- $\Phi_i < 1$ for all i
- $\Phi_i > 1$ for all i

Volatility structures from data

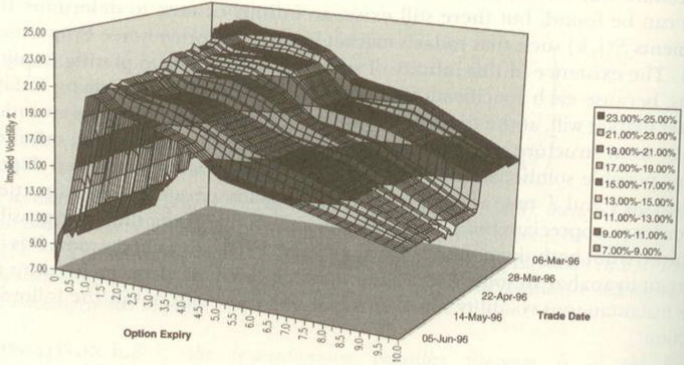


Figure 6.2 *Historical implied volatilities for different maturities collected over a period of approximately three months in 1996 for GBP. Notice the similarity over calendar time of the structural features of the surface, down to rather fine details such as the existence of a 'knuckle' around the five-year-maturity point. The implied volatilities have been shown only for a relatively short period (three months) for the sake of clarity, but these features persist over much longer periods. Occasionally, exceptional events (Russia crisis, Y2K year turn, etc.) disrupt this shape (see Figure 6.3), normally for short periods of time.*

Volatility structures from data

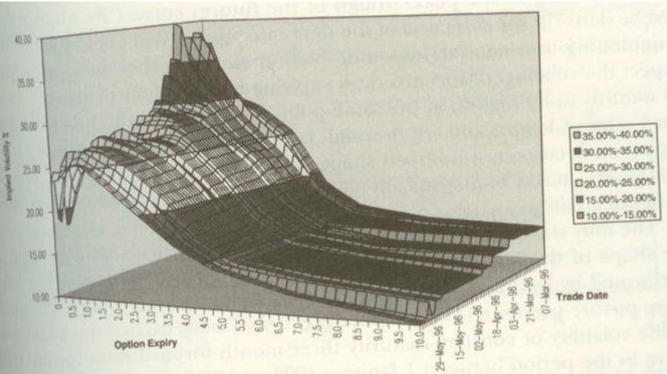


Figure 6.3 *Historical implied volatilities for different maturities collected over a period of approximately three months in 1996 for FRF, showing that the time-homogeneity constraint must not be accepted blindly. Note carefully that the figure shows the implied, rather than instantaneous, volatilities. At the short end, however, it is plausible to assume that the shape for the two functions should be closely related.*

Why are volatilities humped? (Rebonato)

- Let's think of the time to maturity of the forward rate as split into three parts. The first corresponds to the maturity period from spot until the maturity of the first one or two futures contracts. The second runs from the first until around 12-18 months from spot and the third is for longer maturities.
- In the short term then rates are in one or two modes. In normal conditions, the actions of the monetary authorities are not anticipated by the market and so the volatility is low here.
- In high uncertainty periods, the continuous analysis carried out by the market as to what will happen next can make a lot of changes at the short end. So we can be in either an initial high volatility or low volatility state.
- At the long end then most of the current news will not have and effect on the yield curve at say 20 years out, and so this volatility is also relatively low.

Why are volatilities humped? (Rebonato)

- In the middle (6 - 18 months) then this is the period when financial information has its most pronounced effect. For example, how long will QE last for? are more rate cuts/rises likely? So, in a normal current period it is perhaps likely that this period will be more volatile than the short period.
- Thus the hump only applies to the normal period and there is just a decreasing function in more tumultuous periods or when interest rates are very low.

Terminal correlation vs. Instantaneous correlation

- Consider payoffs that depend upon more than one rate - the most obvious being swaptions. Thus the price of a swaption will depend upon the correlation between the forward rates at the maturity of the option, T, these correlations are called *terminal correlations*.
- Terminal correlations do not just depend upon the instantaneous correlation functions (that we will see shortly) but also the choice of instantaneous volatilities. So when calibrating to swaption prices we will need to consider both of these functions.
- The *instantaneous* correlation is a quantity describing interrelations in changes of the forward rates, so that

$$\rho_{2,3} = \frac{dF_2(t)dF_3(t)}{Std(dF_2(t))Std(dF_3(t))}$$

where $Std(\cdot)$ is the standard deviation conditional on information at time t.

What determines terminal correlation

- In contrast the *terminal* correlation instead measures the degree of dependence between two different forward rates at a given terminal time instant. For example, the correlation between $F_2(T_1)$ and $F_3(T_1)$.
- In general, terminal correlation depends upon the instantaneous correlations *and*, more importantly, the time structure of the instantaneous volatilities. Note, so far that cap prices depend only upon total volatility it is for swaptions when the term structure of volatility really matters.
- Indeed the terminal correlation may also be dependent upon the choice of numeraire. Of course, instantaneous volatility and correlation are unchanged under measure changes but this will not necessarily be true of terminal correlation that can depend upon the drift rate (and thus the choice of measure).
- Brigo and Mercurio present an example as follows:

Terminal correlation: example

- Consider a payoff that depends upon two forward rates, $F_2(t)$ and $F_3(t)$ for $t \leq T_1$. Take $T_0 > 0$, $T_1 = 2T_0$, $\beta(t)$ measures time and can take values 1 (if $t < T_1$) and 2 (if $t > T_1$). The volatility is piecewise constant with one level for $\beta = 1$ and another from $\beta = 2$. Let's choose the forward measure so that F_2 is the martingale:

$$
\begin{aligned}
dF_2(t) &= \sigma_{2,\beta(t)} F_2(t) dZ_2(t) \\
dF_3(t) &= \frac{\tau_3 \sigma^2_{3,\beta(t)} F_3^2(t)}{1 + \tau_3 F_3(t)} dt + \sigma_{3,\beta(t)} F_3(t) dZ_3(t), \quad t \leq T_1
\end{aligned}
$$

- Now assume scenario when the payoff depends jointly on $\ln F_3(T_1) - \ln F_3(0)$ and on $F_2(T_1) - F_2(0)$ and we can show that

$$
\begin{aligned}
E^{\mathbb{Q}^2}\left[F_2(T_1) - F_2(0))(\ln F_3(T_1) - \ln F_3(0))\right] &= F_2(0)\rho_{2,3} \int_0^{T_1} \sigma_{2,\beta(t)} \sigma_{3,\beta(t)} dt \\
&= F_2(0)\rho_{2,3}(\sigma_{2,1}\sigma_{3,1} T_0 + \sigma_{2,2}\sigma_{3,2}\tau_1) \qquad (3)
\end{aligned}
$$

Terminal correlation: example

- First note, that any caplet price involving these forward rates would simply depend upon the average volatilities:

$$v_2^2 = T_0\sigma_{2,1}^2 + \tau_1\sigma_{2,2}^2$$

  and

$$v_3^2 = T_0\sigma_{3,1}^2 + \tau_1\sigma_{3,2}^2 + \tau_2\sigma_{3,3}^2$$

  no matter what values are taken by the individual σ values as long as the total variance is preserved.

- Let's put some numbers in. $\tau = 0.5$ in all cases, $F_2(0) = F_3(0) = 0.05$, $\rho_{2,3} = 0.75, \sigma_{3,3} = 0.1$ and there are two different cases:
  1. $\sigma_{2,1} = 0.5, \sigma_{3,1} = 0.1, \sigma_{2,2} = 0.1, \sigma_{3,2} = 0.5$
  2. $\sigma_{2,1} = 0.1, \sigma_{3,1} = 0.1, \sigma_{2,2} = 0.5, \sigma_{3,2} = 0.5$

- In both cases v_2 and v_3 from above have the same value and so the caplets will be priced correctly, whereas the correlation from equation 3 is 1.875×10^{-3} in case 1 and 2.34375×10^{-4} in case 2.

Some choices for instantaneous correlations

- As for instantaneous volatilities we must also come up with some possible choices for the structure of instantaneous correlations.
- In general, we would like our instantaneous correlations to exhibit three features:
  1. Positive, $\rho_{i,j} \geq 0$ for all i, j
  2. When moving away from a "1" diagonal entry along either a column or a row, we should have a monotonically decreasing pattern. This is because joint movements of far away rates are less correlated than those of close rates.
  3. When we move along the zero coupon curve, the larger the tenor the more correlated changes in the adjacent ranges become, so sub-diagonals should be increasing or the map $i \rightarrow \rho_{i+p,i}$ will be increasing for a fixed p
- W will need to generate an $M \times M$ full rank correlation matrix where M is the number of forward rates required. This correlation matrix is defined by $M(M-1)/2$ entries.

Full rank instantaneous correlations

- To recreate this, we have two choices. First, we can choose a number of parameters smaller than $M(M-1)/2$ to recreate the full rank matrix.
- The other problem we will have to consider is starting from a full rank matrix, decrease the rank to make the simulation more manageable.
- We start with looking at **four** possible full rank parameterizations.
- The first is a one parameter model where the correlations $\rho_{i,j}$ are given by

$$\rho_{i,j} = \rho(i-j) = \exp(-\beta|T_i - T_j|), \quad \beta \geq 0$$

  typically the β value is around 0.1.
- This can easily be extended to a two parameter model where

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp(-\beta|T_i - T_j|), \quad \beta \geq 0$$

  where ρ_∞ represents the lowest possible correlation in the matrix.

Full rank instantaneous correlations

- Finally, we can extend this to a three parameter model,

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp(-|T_i - T_j|(\beta - \alpha(\max(T_i, T_j) - 1))), \quad \beta \geq 0$$

- Now, this specification is not guaranteed to keep the resulting matrix positive semidefinite, but for practical purposes usually does, and is very straightforward to use.
- An alternative approach is guaranteed to be positive semidefinite and is only based on two parameters

$$\begin{aligned}\rho_{i,j} &= \exp\left[-\frac{|T_i - T_j|}{M-1}\left(-\ln \rho_\infty + \right.\right.\\ &\quad \left.\left. + \ \eta \frac{i^2 + j^2 + ij - 3Mi - 3Mj + 3i + 3j + 2M^2 - M - 4}{(M-2)(M-3)}\right)\right]\end{aligned}$$

where ρ_∞ is the first, and η is the second parameter; M is the size of the matrix.

Reduced rank instantaneous correlations: Example

- As the correlation matrix ρ is positive semidefinite then it can be written as $\rho = PHP'$ via the usual principal components analysis.
- This can lead to having to simulate fewer Brownian motion terms if required.
- To motivate this, consider a simple example from Rebonato, who first develop this factor reduction method.
- Imagine trying to simulate three forward rates, but with zero drift, typically we could write this out fully as

$$\frac{dF_t^i}{F_t^i} = \sigma_i(t)\left[b_{i1}dW_1(t) + b_{i2}dW_2(t) + b_{i3}dW_3(t)\right]$$

where dW_i are uncorrelated Brownian motions and the array given by b_{ik} is the 'square root' of the correlation matrix ρ, such that $bb^T = \rho$.

Reduced rank instantaneous correlations; example

- If we wanted to, we could alternatively write this in terms of *two* factors:

$$\frac{dF_t^i}{F_t^i} = \sigma_i(t)\,[b_{i1}dW_1(t) + b_{i2}dW_2(t)]$$

- Let's try to determine b_{ik} from a given correlation matrix. Assume that the instantaneous volatility function is piecewise constant over each relevant time interval (as are the correlations). This ensures that integrals like:

$$\int_{T_0}^{T_1} \sigma_i(u)\sigma_j(u)\rho_{ij}(u)du = \sigma_i(T_1)\sigma_j(T_1)\rho_{ij}(T_1)\Delta t$$

- Assume that are payoff depends upon evolving our forward rates to T_1 and we know the instantaneous volatilities and correlation between the forward rates as follows:

Reduced rank instantaneous correlations; example

- $$\begin{aligned} \sigma_1(T_0) &= 20.50\% \\ \sigma_2(T_0) &= 19.50\% \\ \sigma_3(T_0) &= 18.50\% \end{aligned}$$

  and
  $$\rho(T_0 \to T_1) = \begin{pmatrix} 1 & 0.904 & 0.818 \\ 0.904 & 1 & 0.904 \\ 0.818 & 0.904 & 1 \end{pmatrix}$$
- These correlation matrices are said to be the true correlation matrices, we will attempt to find simpler, lower rank versions.
- First let's create the *covariance* matrix
  $$\Sigma = \begin{pmatrix} 0.042 & 0.036 & 0.031 \\ 0.036 & 0.038 & 0.032 \\ 0.031 & 0.032 & 0.034 \end{pmatrix}$$

Reduced rank instantaneous correlations; exogenous ρ

- Now assume that we actually have the full correlation matrix ρ and we wish to use a reduced form for the purpose of simulation (so we don't need to generate as many correlated paths in our Monte Carlo techniques).
- We will now consider an approach based on principal components analysis.
- We know that it is possible to write $\rho = PHP^T$ where $PP^T = I$ and H is a diagonal matrix containing the eigenvalues of ρ. Now consider Λ which is the diagonal matrix whose entries are the square root of H, now letting $A = P\Lambda$ we have that $AA^T = \rho$ and $A^T A = H$.
- Now let's assume that we want to reduce the rank of ρ, define $\overline{\Lambda}^{(n)}$ to be the matrix $\overline{\Lambda}$ with the $M - n$ *smallest* diagonal terms set to zero.

Reduced rank instantaneous correlations; exogenous ρ

- Now define $\overline{B}^{(n)} = P^{(n)}\overline{\Lambda}^{(n)}$ and the new correlation matrix $\overline{\rho}^{(n)} = \overline{B}^{(n)}(\overline{B}^{(n)})^T$. The problem is, is that this modified correlation matrix does not have ones on the diagonal but we can achieve this by scaling to get

$$\rho_{i,j}^{(n)} = \frac{\overline{\rho}_{i,j}^{(n)}}{\sqrt{\overline{\rho}_{i,i}^{(n)}\overline{\rho}_{j,j}^{(n)}}}$$

which is an n-rank approximation of the true matrix.

Conclusion

- We have looked at the LIBOR market model, this is defined by its drift, instantaneous volatility function and instantaneous correlation function.
- We have defined some possible forms for these functions and eventually we will determine how to calibrate these functions to market data.
- Next we will look at how to implement the LMM.