FIN516: Term-structure Models

Lecture 4: Toolkit I

Dr. Martin Widdicks

UIUC

Fall, 2015

Overview

- Here we will cover some of the key tools and techniques that we will need to model the term structure and to value interest rate derivatives.
- The emphasis will be to develop practical methods rather than spending a lot of time on the theory.
- In this class we look at some practical tools required for future projects: constructing yield curves, *stripping* caplet volatilities from caps.
- We will revisit the toolkit later in the class to look at Cholesky factorization and principal components analysis, how to change measure, with a look at Girsanov's theorem and the very useful *Change of Numeraire Toolkit* developed by Brigo and Mercurio.
- Finally, we will revisit Monte Carlo methods, with particular emphasis on stepping algorithms and early exercise (also in Toolkit II)

Bond prices and forward rates data

- All of our models, whether short rate models or market models will at some point be calibrated to market data. Typically we have been assuming that we know one or all of the following pieces of information: zero coupon bond prices $P(0, T)$ at all maturities, $F(0; T, T+\tau)$ at all maturities for all tenors τ.
- Unfortunately we do not always have this information but have to extract it, or find it by interpolation from market data. The market data that we have available is
  - Short term deposit rates: overnight, one day, one week, up to 12 months.
  - Eurodollar contracts.
  - Forward rate agreements.
  - Spot starting swaps with maturities $2, 3, 4, 5, \ldots, 25$, and 30 years.
- From this we would like to build either the forward curve $F(0; T, T+\tau)$ for a fixed τ but varying T, the yield curve $P(0, T)$ for varying T, $SR_{0,T}(0)$ for all tenors T, the par swap curve.

Bond prices and forward rates data

- LIBOR is only quoted up to 12-months and there are no long dated LIBOR bonds from which we can obtain $P(0, T)$ for $T > 1$.
- So, we need to use swap and futures data to get these longer dated zero coupon bond prices.
- Futures data or FRA data can be used to extract rates of medium length. Typically the Eurodollar futures are used but these require a convexity adjustment. For details of how to do this you are referred to the useful paper by Ron (2000), available on COMPASS.

Zero bond prices from futures and swap rates

- We have already seen a brief introduction in PS 1 as to how to use swaps. We start from the equation

$$P(0,T_n) = \frac{1 - SR_{0,T_n}(0)\sum_{i=0}^{n-2}\tau_i P(0,T_{i+1})}{1 + \tau_{n-1}SR_{0,T}(0)} \tag{1}$$

- This is fine in theory, but usually we are given swap rates at most every year and we will need to know the semi-annual (or even quarterly) zero coupon bond prices to bootstrap to the next market data point.
- How these gaps are filled in varies from algorithm to algorithm. The simplest method is to simply linearly interpolate the par swap rates. This is a 'rough and ready ' approach and may lead to substantial errors.
- Alternatively, you could linearly interpolate the zero bond prices (or the log of zero bond prices or assume a function for the forward rate) and then back out the values of all of the zero coupon bond prices until the next maturity.

Example

- Let's imagine that we have annual zero coupon bond prices up to 15 years and the next known swap rate is at 20 years based on semi-annual payments, so we need to estimate the zero coupon bonds at 16, 16.5, 17, 17.5 ... 19.5 years and then extract the 20 year zero coupon bond price.
- We could assume that they are linear in that $SR_{0,T}(0) = \frac{20-T}{20-15} SR_{0,15}(0) + \frac{T-15}{20-15} SR_{0,20}(0)$ and then put these into equation 1 for the bond price.
- Alternatively, we could interpolate the zero coupon bond prices (or the log of zero coupon bond prices), where

$$\ln P(0,T) = \frac{20-T}{20-15} \ln P(0,15) + \frac{T-15}{20-15} \ln P(0,20)$$

for $T = 15.5, 16, \ldots .19.5$ but of course, here $P(0,20)$ is itself unknown and so must be chosen to match with the market swap rate.

$$SR_{0,20}(0) = \frac{1 - P(0,20)}{\sum_{i=0}^{39} \tau_i P(0,T_i)}$$

Quiz

Is the linear interpolation possible with more than one unknown zero coupon bond price?

- Yes, but you will have to solve a system of linear equations.
- Yes, all zero coupon bond prices will be functions of $P(0, T)$ and the previous known value $P(0, t)$, $t < T$.
- No, there are more unknowns than equations and so you will have to use a numerical algorithm to find the best fit.

Quiz

Is the linear interpolation possible with more than one unknown zero coupon bond price?

- Yes, but you will have to solve a system of linear equations.
- Yes, all zero coupon bond prices will be functions of $P(0, T)$ and the previous known value $P(0, t)$, $t < T$. CORRECT
- No, there are more unknowns than equations and so you will have to use a numerical algorithm to find the best fit.

The bootstrap, for the forward curve

- The standard way of extracting data is by using the bootstrap. From above we know the following bond prices

$$P(0, T_i)$$

for dates T_i, where they will include the coupon dates of the benchmark swaps.

- Then we can define,

$$P(T_{i-1}, T_i) = \frac{P(0, T_i)}{P(0, T_{i-1})}$$

which can then give us the forward rates using the standard expression

$$F(0; T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right)$$

- This however, only gives us forward rates between standard dates (e.g $F(0; 0.25, 0.5)$), it does not give use the 6-months forward rate after 1-month for example. To find this then we have to use interpolation.

Interpolation

- The most basic approach is to use linear interpolation of the zero coupon bond prices to fill in any possible gaps. For example, we may well have $P(0, 0.25)$, $P(0, 0.5)$ etc. but no intermediate points.
- So, for a time $T_{i-1} < T < T_i$ then

$$P(0, T) = \frac{T_i - T}{T_i - T_{i-1}} P(0, T_{i-1}) + \frac{T - T_{i-1}}{T_i - T_{i-1}} P(0, T_i)$$

- this can be easily adapted to linear interpolation of log bond prices

$$\ln P(0, T) = \frac{T_i - T}{T_i - T_{i-1}} \ln P(0, T_{i-1}) + \frac{T - T_{i-1}}{T_i - T_{i-1}} \ln P(0, T_i)$$

Interpolation

- Alternatively we can assume a functional form for the instantaneous forward rate prevailing at time t, $f(0, t) = F(0; t, t + dt)$. From Lecture 1, we know that

$$P(0, T) = P(0, T_{i-1}) \exp\left(-\int_{T_{i-1}}^{T} f(0, u) du\right)$$

  and so if we know the functional form of $f(0, t)$ then we can determine the zero coupon bond prices.

- In particular if we have n zero coupon bond prices we could fit an nth order polynomial to the data to interpolate.

- For example, we could assume that it is a constant $f(0, t) = c$ so that, from before

$$P(0, T_i) = P(0, T_{i-1}) exp(-c(T_i - T_{i-1}))$$

  or use this to determine c,

$$f(0, T_{i-1}) = c = \frac{1}{T_i - T_{i-1}} \ln P(T_{i-1}, T_i)$$

Interpolation

- where
$$P(T_{i-1}, T_i) = P(0, T_i)/P(0, T_{i-1})$$
- If we had two bond prices, we can assume that $f(0, t)$ is linear between the known bond prices and continuous throughout matching it to the known bond prices at each end of the period.
- We could even consider that the instantaneous rate is quadratic function of time and then use a bond prices to back out the parameters of the function by matching the instantaneous rate and its derivative.

Getting the $P(0, T_i)$ values

- For some this may also be too simplified a procedure, especially when there are large gaps in the available market data.
- Naturally, any type of linear interpolation, even of instantaneous forward rates will be simple to adapt to fill in the market gaps.
- It is possible to use more sophisticated interpolation procedures, even for obtaining the standard maturity rates. These can be left as unknowns and then one of the iterative procedures below will fit the curve as well as possible to the market swap rates and other key market data information.
- Once all of the standard rates are known we need to fit a functional form to the data to both piece the data together and to be able to work out rates and intermediate times.

Getting the $P(0, T_i)$ values

- Often the bond prices and, thus, forward rates are determined at the appropriate points and then a smoothing curve is placed to go as close as possible to all of the market data, to ensure that the zero curve or the forward curve is differentiable.
- There are many different methods for constructing the yield curve: Cubic splines, B-splines, forward monotone convex splines, Nelson and Siegel's (1987) formula, Svensson's formula (1994) etc. Some practical papers have been placed onto COMPASS if you are interested.
- Most financial software such as Bloomberg will have its own software for constructing the yield curve from spot rates, futures prices and swap rates. Typically, you just choose which products you wish to use for each maturity. We will have a chance to use this software later in the semester.

Caplet volatilities

- In order to calibrate our market models we will need to have the (Black implied) caplet volatilities for all the forward rates that we wish to model.
- The market data typically only lists the cap prices or more usually the (Black implied) cap volatility for the at-the-money cap. We wish to develop methods to extract the individual caplet volatilities from the cap prices.
- The market convention is to write the value of the cap as

$$Cap(0, T_0, \dots, T_n, K) = \sum_{i=0}^{n-1} \tau_i P(0, T_{i+1}) \mathrm{Black}(K, F(0; T_i, T_{i+1}), \sqrt{T_i} \sigma_{T_n}^{cap})$$

where the $\sigma_{T_n}^{cap}$ is the volatility across the whole period from 0 to T_{n-1} and, as before,

Stripping caplet volatilities from caps

$$\begin{aligned}\mathrm{Black}(K, F, \nu) &= FN(d_1) - KN(d_2)\\ d_1 &= \frac{\ln(F/K) + \nu^2/2}{\nu}\\ d_2 &= \frac{\ln(F/K) - \nu^2/2}{\nu}\end{aligned}$$

where $\nu = \sigma_{T_n}^{cap}\sqrt{T_i}$

- Of course, we usually consider a cap as the sum of individual caplets, each with their own volatility $\sigma_{T_i}^{caplet}$ for the period T_i, thus

$$Cap(0, T_0, \ldots, T_n, K) = \sum_{i=0}^{n-1} \tau_i P(0, T_{i+1}) \mathrm{Black}(K, F(0; T_i, T_{i+1}), \sqrt{T_i}\sigma_{T_{i+1}}^{caplet})$$

- To obtain $\sigma_{T_{i+1}}^{caplet}$ we have to use a so-called *stripping* algorithm. The market data typically consists of a list of cap volatilities $\sigma_{T_n}^{cap}$ for at-the-money caps with expiration after $1, 2, 3, \ldots, 10, 12, 15, 20$ years.

Stripping algorithm: what is at-the-money?

- These caps will consist of caplets reset every three months (with first payment after six months) for the entire duration of the cap. Alternatively, the caplets are reset every three months for the first year and then every six months in subsequent years. As with the yield curve bootstrap we will have the initial problem of missing data in our calibration in both scenarios.
- Before extracting the volatilities we need to determine what makes a cap at-the-money. Caps are said to be at-the-money if the exercise price K is equal to the *forward* swap rate, $S_{0.25,T_n}(0)$.
- It is the forward swap rate because a cap starting at $t = 0$ has a first reset date after three months and first payment after six-months. Thus from the swap formula it is possible to work out the strike price K for each cap.
- Note that from the yield curve we are able to obtain forward rates for each period as well as all of the zero coupon bond prices required to determine the at-the-money strike price.

The algorithm

- Let's start with the shortest dated $1 - year$ cap. We have quarterly reset dates and so the effective start is $t = 0.25$. The first volatility estimate we have is σ_1^{cap} and the strike price, K_1, say. There are two additional caps, with maturities $T = 0.5$ and $T = 0.75$ for which we have no volatility estimate but for which we know the at-the-money strike price.
- Initially, assume that the volatility is constant over the first year:

$$\sigma_{0.1}^{cap} = \sigma_{0.75}^{cap} = \sigma_{0.5}^{cap}$$

- So the first caplet volatility is then straightforward:

$$\sigma_{0.5}^{cap} = \sigma_{0.5}^{caplet}$$

- Moving to the next cap:

$$\begin{aligned} Cap(0, 0.25, 0.5, 0.75, K) \quad = \quad & 0.25P(0, 0.5)\mathrm{Black}(SR_{0.25,0.75}(0), F(0; 0.25, 0.5, \sqrt{0.25}\sigma_{0.75}^{cap}) \\ & +0.25P(0, 0.75)\mathrm{Black}(SR_{0.25,0.75}(0), F(0; 0.5, 0.75, \sqrt{0.5}\sigma_{0.75}^{cap}) \end{aligned}$$

The algorithm

- which allows us to work out the value of the unknown caplet from $t = 0.5$ to $t = 0.75$ as:

$$
\begin{aligned}
& 0.25P(0, 0.75)\mathrm{Black}(SR_{0.25,0.75}(0), F(0; 0.5, 0.75), \sqrt{0.5}\sigma_{0.75}^{caplet}) \\
= \quad & Cap(0, 0.25, 0.5, 0.75, K) \\
& -0.25P(0, 0.5)\mathrm{Black}(SR_{0.25,0.75}(0), F(0; 0.25, 0.5), \sqrt{0.25}\sigma_{0.5}^{caplet})
\end{aligned}
$$

  where we need to solve to find $\sigma_{0.75}^{caplet}$ as all other values are known.

- For later periods where we only have the cap volatilities each year then simply interpolate across volatility (either linearly or log-linearly) to obtain market implied volatilities for the intermediate stages.

- For example, we know the cap vol for $T_n = 4$ and $T_n = 5$ but for $T_n = 4.5$ there is no volatility, but we linearly interpolate between the volatilities for $T_n = 4$ and $T_n = 5$.

How do we model caplet volatility

- From Lecture 2 material we know that the estimate of the annual volatility $\sigma_{T_i}^{caplet}$ is equivalent to the root mean square volatility between time 0 and the start of the reset date T_{i-1},

$$\sigma_{T_i}^{caplet\,2} T_{i-1} = \int_0^{T_{i-1}} \sigma^2(t)dt$$

  where $\sigma^2(t)$ is the potentially time varying instantaneous volatility function.

- As part of our development of market models we will need to specify a type of function for this volatility. The most usual choice is to use a humped volatility function of the form:

$$\sigma(t) = (\alpha + \beta t)e^{-\lambda t} + \mu$$

  and variations on this, about which we shall discuss in great detail in the future.

Extra considerations for caplet volatility

- These caplet volatilities (or the volatilities of the appropriate forward rate) are only reported for the at-the-money caps. There are naturally more caps on the market with different cap levels (or exercise prices).
- One verification of the Black model (or the assumption of lognormal forward rates) is to see if caps with different exercise prices all return the same implied volatilities, as we also do with the Black-Scholes model.

Quiz

Would you expect the implied volatility to be the same for all caplets regardless of the exercise price, K?

- Yes.
- No.
- It depends.

Why?

Quiz

Would you expect the implied volatility to be the same for all caplets regardless of the exercise price, K?

- Yes.
- No.
- It depends. CORRECT

Why?

Extra considerations for caplet volatility

- As with the Black-Scholes model, we find a *volatility smile* effect for the implied volatilities of caplets. Usually this manifests as large volatilities for low strikes (K) with the volatilities generally decreasing as the strike price (K) increases.
- There is an intuitive explanation for this, as there is an absolute component to the movement of interest rates which does not exist in the equity markets setting. When interest rates are low they do move by smaller amounts but not proportionally. So, when interest rates are low, as they are now, they move around a lot more than would be predicted by a lognormal model.
- We will consider possible for models for this as extensions to the market model. Popular models include CEV models, SABR, and stochastic volatility models.

Conclusion

- We have looked at some technical details as to how to build a yield curve and also how to strip caplet volatilities from knowing the cap volatilities.