Stochastic Calculus

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Readings: Shreve Chapter 4

Motivation

- Want to evaluate integrals of the form

$$\int_0^T \Delta_t dS_t$$

  S_t - stock price process, Δ_t - a trading strategy. How do we compute such an integral when S_t is driven by a BM and is nowhere differentiable w.r.t. t?

- Construct the integral

$$\int_0^T X_t dB_t$$

  B_t is a standard BM, X_t is a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$ associated with B_t

- When $X_t \equiv 1$, we define

$$\int_0^T X_t dB_t = \int_0^T dB_t = B_T - B_0$$

- For indicator functions of the form ($A = (t_1, t_2)$ or $[t_1, t_2)$ or $(t_1, t_2]$ or $[t_1, t_2]$, where $0 \le t_1 < t_2 \le T$)

$$X_t = I_A(t) = \begin{cases} 1 & t \in A \\ 0 & \text{otherwise} \end{cases},$$

we define

$$\int_0^T X_t dB_t = \int_0^T I_A(t) dB_t = B_{t_2} - B_{t_1}$$

- When X_t is a **simple process** (linear combination of indicator functions), e.g.,

$$X_t = x_0 I_{[t_0,t_1)}(t) + x_1 I_{[t_1,t_2)}(t) + \cdots + x_{n-1} I_{[t_{n-1},t_n]}(t)$$

for $0 = t_0 < \cdots < t_n = T$ and r.v.'s $x_0, \cdots, x_{n-1}$ (x_i is $\mathcal{F}_{t_i}-$measurable), we define

$$\int_0^T X_t dB_t = \sum_{i=0}^{n-1} x_i (B_{t_{i+1}} - B_{t_i})$$

- Such integrals are called **Itô integrals**

Martingale property

- For any $0 \leq t \leq T$, suppose $t \in [t_k, t_{k+1}]$, then define

$$I_t = \int_0^t X_u dB_u = \sum_{i=0}^{k-1} x_i (B_{t_{i+1}} - B_{t_i}) + x_k (B_t - B_{t_k})$$

- I_t is an adapted stochastic process (Itô integral process)
- **I_t is a martingale**: for $0 \leq s < t \leq T$,

$$\mathbb{E}[I_t | \mathcal{F}_s] = I_s, \quad \text{or equivalently,} \quad \mathbb{E}[I_t - I_s | \mathcal{F}_s] = 0$$

- If $t_k \le s < t \le t_{k+1}$, $I_t - I_s = x_k(B_t - B_s)$

$$\mathbb{E}[x_k(B_t - B_s)|\mathcal{F}_s] = x_k\mathbb{E}[B_t - B_s] = 0$$

- If $s \in [t_j, t_{j+1})$, $t \in [t_k, t_{k+1}]$, $j < k$.

$$I_t - I_s = x_j(B_{t_{j+1}} - B_s) + x_{j+1}(B_{t_{j+2}} - B_{t_{j+1}}) + \cdots + x_k(B_t - B_{t_k})$$

For the first term,

$$\mathbb{E}[x_j(B_{t_{j+1}} - B_s)|\mathcal{F}_s] = x_j\mathbb{E}[B_{t_{j+1}} - B_s] = 0$$

- For the second term, since $s < t_{j+1}$ and hence $\mathcal{F}_s \subset \mathcal{F}_{t_{j+1}}$

$$
\begin{aligned}
\mathbb{E}[x_{j+1}(B_{t_{j+2}} - B_{t_{j+1}})|\mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[x_{j+1}(B_{t_{j+2}} - B_{t_{j+1}})|\mathcal{F}_{t_{j+1}}]|\mathcal{F}_s] \\
&= \mathbb{E}[x_{j+1}\mathbb{E}[B_{t_{j+2}} - B_{t_{j+1}}]|\mathcal{F}_s] \\
&= 0
\end{aligned}
$$

- For the last term, since $s < t_k$ and hence $\mathcal{F}_s \subset \mathcal{F}_{t_k}$

$$
\begin{aligned}
\mathbb{E}[x_k(B_t - B_{t_k})|\mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[x_k(B_t - B_{t_k})|\mathcal{F}_{t_k}]|\mathcal{F}_s] \\
&= \mathbb{E}\left[x_k\mathbb{E}[B_t - B_{t_k}]|\mathcal{F}_s\right] \\
&= 0
\end{aligned}
$$

Itô isometry

- What is the expectation of I_t?
- Variance of I_t

$$\mathrm{var}(I_t) = \mathbb{E}[I_t^2]$$

- I_t satisfies the following **Itô isometry**

$$\mathbb{E}[I_t^2] = \mathbb{E}\left[\left(\int_0^t X_u dB_u\right)^2\right] = \mathbb{E}\left[\int_0^t X_u^2 du\right]$$

- **Proof.** Assume $t \in [t_k, t_{k+1}]$.

$$I_t = \sum_{i=0}^{k-1} x_i (B_{t_{i+1}} - B_{t_i}) + x_k (B_t - B_{t_k})$$

Denote $D_i = B_{t_{i+1}} - B_{t_i}$, $D_k = B_t - B_{t_k} \Rightarrow I_t = \sum_{i=0}^{k} x_i D_i$

$$I_t^2 = \sum_{i=0}^{k} x_i^2 D_i^2 + 2 \sum_{0 \le i < j \le k} x_i x_j D_i D_j$$

- For $i < j$, $x_i x_j D_i$ is $\mathcal{F}_{t_j}-$measurable, D_j is independent of $\mathcal{F}_{t_j}$

$$\mathbb{E}[x_i x_j D_i D_j] = \mathbb{E}[x_i x_j D_i] \mathbb{E}[D_j] = 0$$

- x_i is $\mathcal{F}_{t_i}-$ measurable, D_i is independent of $\mathcal{F}_{t_i}$

$$
\begin{aligned}
\mathbb{E}[I_t^2] &= \sum_{i=0}^{k} \mathbb{E}[x_i^2 D_i^2] = \sum_{i=0}^{k} \mathbb{E}[x_i^2]\mathbb{E}[D_i^2] \\
&= \sum_{i=0}^{k-1} \mathbb{E}[x_i^2]\mathbb{E}[(B_{t_{i+1}} - B_{t_i})^2] + \mathbb{E}[x_k^2]\mathbb{E}[(B_t - B_{t_k})^2] \\
&= \sum_{i=0}^{k-1} \mathbb{E}[x_i^2](t_{i+1} - t_i) + \mathbb{E}[x_k^2](t - t_k) \\
&= \mathbb{E}\left[\sum_{i=0}^{k-1} x_i^2(t_{i+1} - t_i) + x_k^2(t - t_k)\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} X_u^2 du + \int_{t_k}^{t} X_u^2 du\right] = \mathbb{E}\left[\int_0^t X_u^2 du\right]
\end{aligned}
$$

Quadratic variation

- The **quadratic variation** of the process I_t is

$$[I, I](t) = \int_0^t X_u^2 du$$

Proof. it suffices to show that the quadratic variation of I_t on $[t_i, t_{i+1}]$ is

$$\int_{t_i}^{t_{i+1}} X_u^2 du$$

Consider a partition $t_i = s_0 < \cdots < s_n = t_{i+1}$

$$\sum_{j=0}^{n-1} |I_{s_{j+1}} - I_{s_j}|^2 = x_i^2 \sum_{j=0}^{n-1} |B_{s_{j+1}} - B_{s_j}|^2 \to x_i^2(t_{i+1} - t_i) = \int_{t_i}^{t_{i+1}} X_u^2 du$$

Differential form

- Itô integral process in **differential form**

$$I_t = \int_0^t X_u dB_u \quad \text{is written as} \quad dI_t = X_t dB_t$$

The differential form of the quadratic variation

$$[I, I](t) = \int_0^t X_u^2 du \quad \text{is written as} \quad dI_t \cdot dI_t = X_t^2 dt$$

Itô integral process accumulates quadratic variation at rate X_t^2

- Recall that the quadratic variation of BM

$$[B, B](t) = t \quad \text{or in differential form} \quad dB_t \cdot dB_t = dt$$

BM accumulates quadratic variation at rate 1

General case

- Itô integrals for general integrand X_t which is adapted to $\{\mathcal{F}_t\}$

$$I_t = \int_0^t X_u dB_u$$

  Approximate X_t by simple processes of the form

$$X_{t_0} I_{[t_0,t_1)}(t) + X_{t_1} I_{[t_1,t_2)}(t) + \cdots + X_{t_{n-1}} I_{[t_{n-1},t_n]}(t)$$

  I_t is defined as follows, where $||\Pi|| = \max\{t_{i+1} - t_i\}$

$$I_t = \int_0^t X_u dB_u = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} X_{t_i}(B_{t_{i+1}} - B_{t_i})$$

- When $\mathbb{E}[\int_0^t X_u^2 du] < \infty$, the Itô integral is well defined

- Properties of Itô integrals
  - I_t is continuous in t
  - I_t is adapted w.r.t. $\{\mathcal{F}_t\}$
  - I_t is a martingale
  - I_t satisfies the Itô isometry
  - Quadratic variation: $dI_t \cdot dI_t = X_t^2 dt$
  - Linearity: for $a, b \in \mathbb{R}$

$$\int_0^t (aX_u + bY_u)dB_u = a\int_0^t X_u dB_u + b\int_0^t Y_u dB_u$$

An example

- Compute the Itô integral

$$\int_0^T B_t dB_t$$

Wrong solution:

$$\int_0^T B_t dB_t = \frac{1}{2}\int_0^T dB_t^2 = \frac{1}{2}B_T^2$$

Compute by definition: divide $[0, T]$ into n equal intervals, with $\Delta = T/n$, we show that

$$\int_0^T B_t dB_t = \lim_{n\to\infty}\sum_{i=0}^{n-1} B_{i\Delta}(B_{(i+1)\Delta} - B_{i\Delta}) = \frac{1}{2}B_T^2 - \frac{1}{2}T$$

$$
\begin{aligned}
\sum &= -\sum_{i=0}^{n-1}(B_{i\Delta}^2 - B_{(i+1)\Delta}B_{i\Delta}) \\
&= -\frac{1}{2}\sum_{i=0}^{n-1}(2B_{i\Delta}^2 - 2B_{(i+1)\Delta}B_{i\Delta}) \\
&= -\frac{1}{2}\sum_{i=0}^{n-1}(2B_{i\Delta}^2 - 2B_{(i+1)\Delta}B_{i\Delta} + B_{(i+1)\Delta}^2 - B_{(i+1)\Delta}^2) \\
&= -\frac{1}{2}\sum_{i=0}^{n-1}((B_{(i+1)\Delta} - B_{i\Delta})^2 + B_{i\Delta}^2 - B_{(i+1)\Delta}^2) \\
&= -\frac{1}{2}\left(\sum_{i=0}^{n-1}(B_{(i+1)\Delta} - B_{i\Delta})^2 - B_{n\Delta}^2\right) \Rightarrow \frac{1}{2}B_T^2 - \frac{1}{2}T
\end{aligned}
$$

- **Chain rule in ordinary calculus**:

$$df(g(t)) = f'(g(t))dg(t)$$

- With $f(x) = \frac{1}{2}x^2$, $f'(x) = x$

$$\begin{aligned}\int_0^T g(t)dg(t) &= \int_0^T f'(g(t))dg(t) = \int_0^T df(g(t)) \\ &= f(g(T)) - f(g(0)) \\ &= \frac{1}{2}(g(T))^2 - \frac{1}{2}(g(0))^2\end{aligned}$$

- The chain rule from ordinary calculus doesn't apply when $g(t) = B_t$

$$\int_0^T B_t dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}B_0^2 - \frac{1}{2}T$$

- **Itô formula** is the chain rule in stochastic calculus
- **Itô formula case I.** *suppose $f(x)$ has continuous derivatives f', f''. Then*

$$f(B_T) - f(B_0) = \int_0^T f'(B_t)dB_t + \frac{1}{2}\int_0^T f''(B_t)dt$$

or in differential form

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

- Let $f(x) = \frac{1}{2}x^2$. $f'(x) = x$, $f''(x) = 1$, consider $f(B_t) = \frac{1}{2}B_t^2$.

$$df(B_t) = B_t dB_t + \frac{1}{2}dt, \quad \text{that is} \quad f(B_T) - f(B_0) = \int_0^T B_t dB_t + \frac{1}{2}T$$

Therefore,

$$\int_0^T B_t dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}T$$

- **Intuitively,** for any $t < s$, $\Delta t = s - t$, $\Delta B_t = B_s - B_t$,

$$f(B_s) - f(B_t) = f'(B_t)\Delta B_t + \frac{1}{2}f''(B_t^*)(\Delta B_t)^2$$

where B_t^* is between B_t and B_s. In differential form

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

Proof of Itô formula

- Consider a partition $0 = t_0 < \cdots < t_n = T$, denote $\Delta B_{t_i} = B_{t_{i+1}} - B_{t_i}$,

$$
\begin{aligned}
f(B_T) - f(B_0) &= \sum_{i=0}^{n-1} [f(B_{t_{i+1}}) - f(B_{t_i})] \\
&= \sum_{i=0}^{n-1} f'(B_{t_i})\Delta B_{t_i} + \frac{1}{2}\sum_{i=0}^{n-1} f''(B^*_{t_i})(\Delta B_{t_i})^2
\end{aligned}
$$

Note that $\displaystyle\sum_{i=0}^{n-1} f'(B_{t_i})\Delta B_{t_i} \to \int_0^T f'(B_t)dB_t$

It remains to show that

$$\sum_{i=0}^{n-1} f''(B^*_{t_i})(\Delta B_{t_i})^2 \Rightarrow \int_0^T f''(B_t)dt$$

- Note that

$$
\begin{aligned}
& \left| \sum_{i=0}^{n-1} f''(B^*_{t_i})(\Delta B_{t_i})^2 - \sum_{i=0}^{n-1} f''(B_{t_i})(\Delta B_{t_i})^2 \right| \\
\leq & \sum_{i=0}^{n-1} |f''(B^*_{t_i}) - f''(B_{t_i})|(\Delta B_{t_i})^2 \\
\leq & \max\{|f''(B^*_{t_i}) - f''(B_{t_i})|\} \sum_{i=0}^{n-1} (\Delta B_{t_i})^2 \Rightarrow 0
\end{aligned}
$$

Therefore, it suffices to show that

$$
\sum_{i=0}^{n-1} f''(B_{t_i})(\Delta B_{t_i})^2 \Rightarrow \int_0^T f''(B_t)dt \Leftarrow \sum_{i=0}^{n-1} f''(B_{t_i})\Delta t_i
$$

- It suffices to show that

$$\sum_{i=0}^{n-1} f''(B_{t_i})\left((\Delta B_{t_i})^2 - \Delta t_i\right) \Rightarrow 0$$

We show that the mean of the above is zero, the variance converges to 0

$$\mathbb{E}\left[\sum_{i=0}^{n-1} f''(B_{t_i})((\Delta B_{t_i})^2 - \Delta t_i)\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[f''(B_{t_i})((\Delta B_{t_i})^2 - \Delta t_i)\right]$$

$$= \sum_{i=0}^{n-1} \mathbb{E}[f''(B_{t_i})]\mathbb{E}[(\Delta B_{t_i})^2 - \Delta t_i]$$

$$
\begin{aligned}
\text{var} &= \mathbb{E}\left(\sum_{i=0}^{n-1} f''(B_{t_i})((\Delta B_{t_i})^2 - \Delta t_i)\right)^2 \\
&= \sum_{i=0}^{n-1} \mathbb{E}\left[(f''(B_{t_i}))^2((\Delta B_{t_i})^2 - \Delta t_i)^2\right] \\
&\quad +2 \sum_{0\le i<j\le n-1} \mathbb{E}\left[f''(B_{t_i})f''(B_{t_j})((\Delta B_{t_i})^2 - \Delta t_i)((\Delta B_{t_j})^2 - \Delta t_j)\right] \\
&= \sum_{i=0}^{n-1} \mathbb{E}\left[(f''(B_{t_i}))^2((\Delta B_{t_i})^2 - \Delta t_i)^2\right] \\
&= \sum_{i=0}^{n-1} \mathbb{E}[(f''(B_{t_i}))^2]\mathbb{E}[((\Delta B_{t_i})^2 - \Delta t_i)^2]
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
\text{var} &= \sum_{i=0}^{n-1} \mathbb{E}[(f''(B_{t_i}))^2]\text{var}((\Delta B_{t_i})^2 - \Delta t_i) \\
&= \sum_{i=0}^{n-1} \mathbb{E}\left[(f''(B_{t_i}))^2\right] 2(t_{i+1} - t_i)^2 \\
&\leq 2||\Pi||\mathbb{E}\left[\sum_{i=0}^{n-1}(f''(B_{t_i}))^2(t_{i+1} - t_i)\right] \\
&\Rightarrow 0 \cdot \mathbb{E}\left[\int_0^T (f''(B_t))^2 dt\right] = 0
\end{aligned}
$$

Itô formula For $f(t, x)$

- **Itô formula case II**: suppose $f(t, x)$ has continuous derivatives

$$\dot{f} = \frac{\partial f}{\partial t}, \quad f' = \frac{\partial f}{\partial x}, \quad f'' = \frac{\partial f^2}{\partial x^2}$$

Then

$$f(T, B_T) = f(0, 0) + \int_0^T \dot{f}(t, B_t)dt + \int_0^T f'(t, B_t)dB_t + \frac{1}{2}\int_0^T f''(t, B_t)dt$$

or in **differential form**

$$df(t, B_t) = \dot{f}(t, B_t)dt + f'(t, B_t)dB_t + \frac{1}{2}f''(t, B_t)dt$$

- **Intuitively**, for any $t < s$, $\Delta t = s - t$, $\Delta B_t = B_s - B_t$,

$$f(s, B_s) - f(t, B_t) = \dot{f}(t, B_t)\Delta t + f'(t, B_t)\Delta B_t + \frac{1}{2}f''(t, B_t)(\Delta B_t)^2$$

$$+\frac{\partial^2 f}{\partial t \partial x}(t, B_t)\Delta t \Delta B_t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t, B_t)\Delta t^2 + \cdots$$

or in differential form

$$\begin{aligned} df(t, B_t) &= \dot{f}(t, B_t)dt + f'(t, B_t)dB_t + \frac{1}{2}f''(t, B_t)(dB_t)^2 \\ &\quad + \frac{\partial^2 f}{\partial t \partial x}(t, B_t)dt \cdot dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t, B_t)(dt)^2 \end{aligned}$$

Recall that $(dB_t)^2 = dt$, $dt \cdot dB_t = (dt)^2 = 0$.

Itô processes

- Itô formula can be extended to more general processes: $\{X_t, t \geq 0\}$ is an **Itô process** if

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s$$

for some constant X_0 and adapted processes μ_t and σ_t. In differential form,

$$dX_t = \mu_t dt + \sigma_t dB_t$$

- **Example (Brownian motion with drift)** when $\mu, \sigma > 0$ are constants and $X_0 = x$,

$$X_t = x + \mu t + \sigma B_t \quad \Leftrightarrow \quad dX_t = \mu dt + \sigma dB_t$$

where μt is the **drift** term

- **Quadratic variation of an Itô process:** $dX_t = \mu_t dt + \sigma_t dB_t$

$$[X, X](t) = \int_0^t \sigma_s^2 ds, \quad \text{or in differential form,} \quad dX_t \cdot dX_t = \sigma_t^2 dt$$

Intuitively,

$$dX_t = \mu_t dt + \sigma_t dB_t$$

and hence

$$dX_t \cdot dX_t = \mu_t^2 (dt)^2 + 2\mu_t \sigma_t dB_t \cdot dt + \sigma_t^2 (dB_t)^2 = \sigma_t^2 dt$$

- **Integral w.r.t. Itô process**: $dX_t = \mu_t dt + \sigma_t dB_t$

$$\int_0^t Y_s dX_s = \int_0^t Y_s \mu_s dt + \int_0^t Y_s \sigma_s dB_s$$

Itô formula for Itô processes

- **Itô formula case III**: suppose $f(t,x)$ has continuous derivatives $\dot{f}(t,x)$, $f'(t,x)$, $f''(t,x)$ and X_t is an Itô process: $dX_t = \mu_t dt + \sigma_t dB_t$. Then

$$f(T,X_T) = f(0,X_0) + \int_0^T \dot{f}(t,X_t)dt + \int_0^T f'(t,X_t)dX_t$$

$$+\frac{1}{2}\int_0^T f''(t,X_t)\sigma_t^2 dt$$

or in **differential form**

$$\begin{aligned} df(t,X_t) &= \dot{f}(t,X_t)dt + f'(t,X_t)dX_t + \frac{1}{2}f''(t,X_t)dX_t \cdot dX_t \\ &= (\dot{f} + \mu f' + \frac{1}{2}\sigma^2 f'')dt + \sigma f' dB_t \end{aligned}$$

Exponential martingale example

- Recall that
$$Z_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$$
is a martingale with expectation $\mathbb{E}[Z_t] = Z_0 = 1$.
- More generally, for an adapted process θ_t,
$$Z_t = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s\right)$$
is a martingale with $\mathbb{E}[Z_t] = Z_0 = 1$
- Let $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, consider $f(X_t)$ where
$$dX_t = -\frac{1}{2}\theta_t^2 dt - \theta_t dB_t$$

- By Itô formula,

$$
\begin{aligned}
dZ_t &= df(X_t) \\
&= f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t \cdot dX_t \\
&= Z_t dX_t + \frac{1}{2}Z_t dX_t \cdot dX_t \\
&= Z_t(-\theta_t dB_t - \frac{1}{2}\theta_t^2 dt) + \frac{1}{2}Z_t\theta_t^2 dt \\
&= -\theta_t Z_t dB_t
\end{aligned}
$$

Therefore, Z_t is an Itô integral and hence a martingale

$$
Z_t = Z_0 - \int_0^t \theta_u Z_u dB_u = 1 - \int_0^t \theta_u Z_u dB_u
$$

Geometric Brownian motion

- Suppose the dynamics of a stock price is given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

for constants $\mu \in \mathbb{R}, \sigma \geq 0$.

- **When there is no stochastic term ($\sigma = 0$),**

$$dS_t = \mu S_t dt \quad \Leftrightarrow \quad S_t = S_0 e^{\mu t}$$

where μ the rate of return of the stock. Define the log return process $X_t = \ln(S_t/S_0)$

$$dX_t = \mu dt$$

- **If $\sigma > 0$,** consider $X_t = \ln(S_t/S_0) = f(S_t)$, $f(x) = \ln(x/S_0)$, $f' = 1/x$, $f'' = -1/x^2$,

$$\begin{aligned} dX_t &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)dS_t \cdot dS_t \\ &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t) - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t \end{aligned}$$

Therefore,

$$X_t = \ln(S_t/S_0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t,$$

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$

- σ is known as the **volatility** of the stock

$$\sigma^2 = \mathrm{var}(\ln(S_1/S_0))$$

Volatility = standard deviation of log return per unit time

- Note that

$$(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \sim N((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

Therefore, S_t has a **lognormal distribution**:

$$\mathbb{E}[S_t] = S_0 e^{\mu t}$$

μ is the mean rate of return

- S_t is said to follow a **geometric Brownian motion**

- More generally, let the dynamics of a stock price be given by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t$$

for adapted processes $\mu_t, \sigma_t > 0$.

- Similarly, $X_t = \ln(S_t/S_0)$ follows

$$dX_t = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB_t$$

$$X_t = \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s$$

S_t is given by the following

$$S_t = S_0 \exp\left(\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s\right)$$

Vasicek Interest Rate Model

- The dynamics of the interest rate in Vasicek's model is given by

$$dr_t = (a - br_t)dt + \sigma dB_t$$

for constants $a, b, \sigma > 0$. The solution for r_t is

$$r_t = \frac{a}{b} + e^{-bt}(r_0 - \frac{a}{b}) + \sigma e^{-bt} \int_0^t e^{bs} dB_s$$

- In fact, let $X_t = \int_0^t e^{bs} dB_s$ so that

$$dX_t = e^{bt} dB_t$$

and $f(t,x) = \frac{a}{b} + e^{-bt}(r_0 - \frac{a}{b}) + \sigma e^{-bt} x$, with

$$\dot{f}(t,x) = a - bf(t,x), \quad f'(t,x) = \sigma e^{-bt}, \quad f''(t,x) = 0$$

- By Itô formula,

$$\begin{aligned} df(t,X_t) &= \dot{f}(t,X_t)dt + f'(t,X_t)dX_t + \frac{1}{2}f''(t,X_t)dX_t \cdot dX_t \\ &= (a - bf(t,X_t))dt + \sigma e^{-bt} e^{bt} dB_t \\ &= (a - bf(t,X_t))dt + \sigma dB_t \end{aligned}$$

- r_t is **mean reverting**: $dr_t = b(a/b - r_t)dt + \sigma dB_t$
  - if $r_t > a/b$, the drift is negative
  - if $r_t < a/b$, the drift is positive
  - $\lim_{t\to\infty} \mathbb{E}[r_t] = a/b$
  - It can be shown that

$$X_t = \int_0^t e^{bs} dB_s$$

  is normally distributed with mean 0. Therefore,

$$\mathbb{E}[r_t] = \frac{a}{b} + e^{-bt}(r_0 - \frac{a}{b})$$

- Drawback: r_t is normally distributed and can be negative

Cox-Ingersoll-Ross (CIR) interest rate model

- The dynamics of the interest rate in the CIR model is given by

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dB_t$$

  for some $a, b, \sigma > 0$.
  - r_t is non-negative
  - Closed form solution not available
  - It is mean reverting

$$\mathbb{E}[r_t] = \frac{a}{b} + e^{-bt}(r_0 - \frac{a}{b}) \to \frac{a}{b}$$

Multi-dimensional Browinan motion

- d-dimensional Brownian motion: $B_t = (B_{1t}, \cdots, B_{dt})$
  - Each B_{it} is a standard Brownian motion
  - B_{it} and B_{jt} are independent for $i \neq j$
- Variations: $(dB_{it})^2 = dt$, $dB_{it} \cdot dt = 0$, $(dt)^2 = 0$. We show that $dB_{it} \cdot dB_{jt} = 0$ for $i \neq j$
- Consider a partition of $[0, T]$: $0 = t_0 < \cdots < t_n = T$,

$$\sum_{k=0}^{n-1} (B_{it_{k+1}} - B_{it_k})(B_{jt_{k+1}} - B_{jt_k}) = \sum_{k=0}^{n-1} \Delta B_{it_k} \Delta B_{jt_k}$$

has zero expectation because $\Delta B_{it_k} = B_{it_{k+1}} - B_{it_k}$ and $\Delta B_{jt_k} = B_{jt_{k+1}} - B_{jt_k}$ are independent.

- Its variance converges to 0 as $||\Pi|| \to 0$

$$
\begin{aligned}
\text{var} &= \sum_{k=0}^{n-1} \mathbb{E}\left[\Delta B_{it_k}^2 \Delta B_{jt_k}^2\right] + 2 \sum_{0 \le k < l \le n-1} \mathbb{E}\left[\Delta B_{it_k} \Delta B_{jt_k} \Delta B_{it_l} \Delta B_{jt_l}\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\left[\Delta B_{it_k}^2\right] \mathbb{E}\left[\Delta B_{jt_k}^2\right] \\
&= \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \le ||\Pi|| \sum_{k=0}^{n-1} (t_{k+1} - t_k) \Rightarrow 0
\end{aligned}
$$

Therefore, the cross variation of B_{it} and B_{jt} is zero:

$$
\text{cross variation} = \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} \Delta B_{it_k} \Delta B_{jt_k} = 0
$$

or in differential form, $dB_{it} \cdot dB_{jt} = 0$

Multi-dimensional Itô formula

- **Itô formula case IV**: suppose $f(t, x, y)$ has continuous derivatives $\dot{f}$, f_x, f_{xx}, f_y, f_{yy}, f_{xy}. X_t and Y_t are the following Itô processes:

$$dX_t = \mu_{1t} dt + \sigma_{11t} dB_{1t} + \sigma_{12t} dB_{2t}$$

$$dY_t = \mu_{2t} dt + \sigma_{21t} dB_{1t} + \sigma_{22t} dB_{2t}$$

Then

$$df(t, X_t, Y_t) = \dot{f} dt + f_x dX_t + f_y dY_t + \frac{1}{2} f_{xx} dX_t \cdot dX_t + f_{xy} dX_t \cdot dY_t + \frac{1}{2} f_{yy} dY_t \cdot dY_t$$

where

$$dX_t \cdot dX_t = (\sigma_{11t}^2 + \sigma_{12t}^2) dt$$

$$dY_t \cdot dY_t = (\sigma_{21t}^2 + \sigma_{22t}^2) dt$$

$$dX_t \cdot dY_t = (\sigma_{11t}\sigma_{21t} + \sigma_{12t}\sigma_{22t}) dt$$

- (**Itô product rule**): for two Itô processes X_t and Y_t,

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t$$

Follows from the Itô formula with $f(x, y) = xy$, $\dot{f} = 0$, $f_x = y$, $f_y = x$, $f_{xx} = f_{yy} = 0$, $f_{xy} = 1$

- **Multi-asset model:** the dynamics of two stocks are given by

$$\begin{aligned} dS_{1t} &= \mu_1 S_{1t} dt + \sigma_1 S_{1t} dB_{1t} \\ dS_{2t} &= \mu_2 S_{2t} dt + \sigma_2 S_{2t} (\rho dB_{1t} + \sqrt{1-\rho^2} dB_{2t}) \end{aligned}$$

where B_{1t} and B_{2t} are independent BM's, $\sigma_1 > 0, \sigma_2 > 0, \rho \in [-1, 1]$.

- $B_{3t} = \rho B_{1t} + \sqrt{1-\rho^2} B_{2t}$ is a BM, hence S_{1t} and S_{2t} are geometric BM's
- B_{1t} and B_{3t} are correlated with correlation coefficient ρ

$$\begin{aligned} \text{cov}(B_{1t}, B_{3t}) &= \mathbb{E}[B_{1t}B_{3t}] - \mathbb{E}[B_{1t}]\mathbb{E}[B_{3t}] \\ &= \mathbb{E}[B_{1t}(\rho B_{1t} + \sqrt{1-\rho^2} B_{2t})] \\ &= \rho\mathbb{E}[B_{1t}^2] = \rho t \end{aligned}$$

Therefore, the correlation coefficient is

$$\text{corr}(B_{1t}, B_{3t}) = \frac{\text{cov}(B_{1t}, B_{3t})}{\sqrt{t}\sqrt{t}} = \rho$$