FIN516: Term-structure Models

Lecture 1: Rates, Bonds, FRAs, Swaps and Other products

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Overview

- Interest rate derivatives are the most widely traded of all derivative contracts. We will briefly introduce models that will price these derivatives we will start by analyzing the types of products available.
- However, there is a second purpose. Most of the models we use are no-arbitrage (or relative) pricing models which rely on the **known** price of other assets (similar to underlying assets for equity derivatives) in order to value more complex products. Unlike for equity or foreign exchange derivatives, the 'underlying' asset is not so simple to define. We will generally rely on four fundamental assets (FRAs, Swaps, Caps and Swaptions) which we will introduce in detail here.
- There are certain basic definitions and notation which also need to be cleared up as well as a recap on some fixed income fundamentals (e.g short rates, bonds, discount rates, forward rates, LIBOR, etc.)

Types of rates

- **Treasury rates** are the interest rates for a country borrowing on its own currency. For example, US treasury rates are the rates at which the US government can borrow in US dollars. These rates are typically deduced by analyzing the prices of bonds issued by the government.
- We will typically not be considering Treasury rates as we will be considering decisions made by banks and corporations and not governments. The government rate is often viewed as risk-free rate, and it is typically not possible for Banks to borrow money at a rate as low as the government.
- Banks actively lend money to each other through the interbank market. These loans are typically 1-, 3-, or 6-months and the rates are known as the LIBOR (for the offer rates) or LIBID (for the bid rate). Typically we will consider this as one rate - **the LIBOR rate**.
- These rates are naturally higher than the treasury rates as banks are more likely to default on the loans than governments. Typically, we model LIBOR (or general interbank) rates.

Types of rates

- Unlike with treasury rates, for Interbank rates there are no underlying bonds but there are other products which will allow us to obtain information about the rates.
- There are also **repo** rates. Here the party wishing to borrow has some securities. They sell the securities and are willing to buy them back in the future at some slightly higher price. The difference in these prices is the repo rate. Often this is done for a very short period of time - such as the overnight repo.

Quiz

How are the LIBOR spot rates determined?

- From LIBOR zero coupon bonds
- From yields on bank bonds
- By a committee
- From prices of other traded derivatives

Quiz

How are the LIBOR spot rates determined?

- From LIBOR zero coupon bonds
- From yields on bank bonds
- By a committee CORRECT
- From prices of other traded derivatives CORRECT for long maturities

The short rate: r_t

- The most basic type of interest rate product is the bank or **money market account**. The value of the bank account at time t is denoted by $B(t)$.
- We assume that the money market account accrues interest at the risk-free rate r_t. Thus, the stochastic differential equation (SDE) for $B(t)$ is simple:

$$dB(t) = r_t B(t) dt, \quad B(0) = 1$$

where r_t is a positive function that may be dependent upon time. This SDE can be easily solved to find that

$$B(t) = \exp\left(\int_0^t r_s ds\right)$$

- In this formulation r_t is the *instantaneous* rate at which money is accrued. This rate is called either the instantaneous spot rate, or more commonly **the short rate**. There are many different short rate models. These are the basic term-structure models.

The short rate: r_t

- To verify this definition of the rate, expand B across a time period Δt

$$B(t + \Delta t) = B(t)(1 + r_t \Delta t)$$

or

$$r_t \Delta t = \frac{B(t + \Delta t) - B(t)}{B(t)}$$

or the money grows at a rate r_t at each point in time t.

- **In equity derivative pricing, this money market account appears in derivations of Black-Scholes, often with a constant rate r. Here it will play a central role as we are interested in products whose values explicitly depend upon r_t.**

Quiz

If r_t is a constant, r, then what is $B(t)$?

- $\ln r$
- e^{rt}
- $B(0)$
- $(1 + rt)$
- $(1 + r)^t$

Quiz

If r_t is a constant, r, then what is $B(t)$?

- $\ln r$
- e^{rt} CORRECT
- $B(0)$
- $(1 + rt)$
- $(1 + r)^t$

Discounting: $D(t, T)$

- From this definition of the short-rate we revisit discounting. Consider \$1 being available at time T. To determine the amount A that would have to be invested at time t we need to solve

$$1 = A \exp\left(\int_t^T r_s ds\right)$$

$$1 = A \frac{B(T)}{B(t)}$$

$$A = \frac{B(t)}{B(T)}$$

- This lead to the concept of a discount factor when interest rates are not deterministic, or a **stochastic discount factor**, $D(t, T)$

$$D(t, T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s ds\right)$$

- In interest rate derivatives, both discount factors and bank accounts, $B(t)$. will typically be modeled as stochastic processes.

Quiz

What is $D(0, t)$?

- $\frac{1}{B(t)}$
- $B(t)$
- 1
- e^{rt}

Quiz

What is $D(0, t)$?

- $\frac{1}{B(t)}$ CORRECT
- $B(t)$
- 1
- e^{rt}

Zero coupon bonds: $P(t, T)$

- A zero coupon (or discount) bond guarantees the holder of the bond the payment of \$1 at time T with no other payments being made.
- The value at time t of a zero coupon bond pays out \$1 at time T is denoted by $P(t, T)$ and $P(t, T) = E_t[D(t, T)]$
- When interest rates are deterministic then the value of the bond is simply $D(t, T)$ but when when $D(t, T)$ is stochastic then the two values do not necessarily match as $P(t, T)$, being the value of a contract, should have a deterministic value. In short rate models we will need to solve our SDEs in order to back out the value of the zero-coupon bond.
- Zero coupon bonds are the standard building blocks of interest rate modeling and we shall use them as such. However, note that in the LIBOR markets, zero-coupon bonds do not actually exist.

Compounding: Day counts

- So far we have simply used t and T without discussing what unit of time they measure. In interest rate products it is important to get the day count correct and there are a few different conventions. As with equity options we want t and T to be times measured in years.
- **Actual/365** convention. Here the year is 365 days long and the fraction of the year between T and t is the number of days divided by 365. The year fraction between January 5th 2014 and July 5th 2014 is $182/365 = 0.49863$
- **Actual/360** convention. Here the year is only 360 days long and so the fraction of the year between t and T is the number of days divided by 360. The year fraction between January 5th 2014 and July 5th 2014 is $182/360 = 0.50556$

Compounding: Day counts

- **30/360** convention. Here months are assumed to be 30 days long and the year is 360 days long. The fraction of time between t and T is now given by a messy formula

$$T - t = \frac{\max(30 - d_t, 0) + \min(d_T, 30) + 360(y_T - y_t) + 30(m_T - m_t - 1)}{360}$$

where d_t, m_t, y_t denote the day, month and year of each date. The year fraction between January 5th 2014 and July 5th 2014 is 0.5. Try it (or use the Excel command DAYS360(Date1, Date2)).

Compounding: Continuous compounding of spot rate

- Although zero-coupon bonds will be the building blocks, for LIBOR modeling there are quoted rates not bond prices. Thus we need to be able to move between rates and bonds. To do so, we have to consider the various ways of compounding.
- The standard approach in equity option pricing is to use **continuously compounded** rates. $R(t,T)$ denotes the continuously compounded rate quoted at time t until time T. This is the constant rate such that an investment of $P(t,T)$ at time t accrues interest continuously to be ultimately worth \$1 at time T. Note here that this rate is a **known constant rate**.
- Thus

$$\begin{aligned} P(t,T)\exp(R(t,T)(T-t)) &= 1 \\ R(t,T) &= -\frac{\ln(P(t,T))}{T-t} \end{aligned}$$

and

$$P(t,T) = \exp(-R(t,T)(T-t))$$

Compounding: Simple compounding of spot rate

- In our LIBOR world, we denote the simply compounded spot rate of interest from time t to time T by $L(t, T)$. This is a constant rate such that an initial investment of $P(t, T)$ accrues interest proportional to time over $T - t$. This again is a known constant rate
- Thus,

$$\begin{aligned} P(t,T)(1 + L(t,T)(T-t)) &= 1 \\ L(t,T) &= \frac{1 - P(t,T)}{P(t,T)(T-t)} \end{aligned}$$

and

$$P(t,T) = \frac{1}{1 + L(t,T)(T-t)}$$

- Typically LIBOR rates are converted to zero-coupon bond prices by using the actual/360 day count convention.

Quiz

What is the relationship between $R(0, T)$ and $L(0, T)$?

- $L(0, T) = \ln(R(0, T))$
- $L(0, T) = R(0, T)$
- $R(0, T) = \ln(1 + L(0, T))$
- $R(0, T) = \frac{1}{T} \ln(1 + L(0, T)T)$

Quiz

What is the relationship between $R(0,T)$ and $L(0,T)$?

- $L(0,T) = \ln(R(0,T))$
- $L(0,T) = R(0,T)$
- $R(0,T) = \ln(1 + L(0,T))$
- $R(0,T) = \frac{1}{T}\ln(1 + L(0,T)T)$ CORRECT

Compounding: Annual compounding of spot rate

- $Y(t, T)$ denotes the annually compounded interest rate quoted at time t until the future time T. It is the constant, known rate such that an investment of $P(t, T)$ accrues interest which is reinvested annually to be worth \$1 at time T.
- Thus,

$$
\begin{aligned}
P(t, T)(1 + Y(t, T))^{T-t} &= 1 \\
Y(t, T) &= \frac{1}{P(t, T)^{1/(T-t)}}
\end{aligned}
$$

and

$$P(t, T) = \frac{1}{(1 + Y(t, T))^{(T-t)}}$$

- Typically annual rates are converted to zero-coupon bond prices by using the actual/365 day count convention.
- It is also possible to compound the rate more than once per year, in the usual fashion.

Instantaneous rates

- Note that to be more precise we should use $\tau(t, T)$ to denote the time between t and T as this time depends on the precise day count convention.
- When the time period becomes short enough, each of these rates: $R(t, T), L(t, T), Y(t, T)$ converges to the short rate r_t, hence the short rate models do not explicitly specify which type of compounding will be used. As $t \Rightarrow T$ then

$$
\begin{aligned}
\exp(R(t,T)(T-t)) &\Rightarrow 1 + R(t,T)(T-t) \\
(1 + Y(t,T))^{T-t} &\Rightarrow 1 + Y(t,T)(T-t)
\end{aligned}
$$

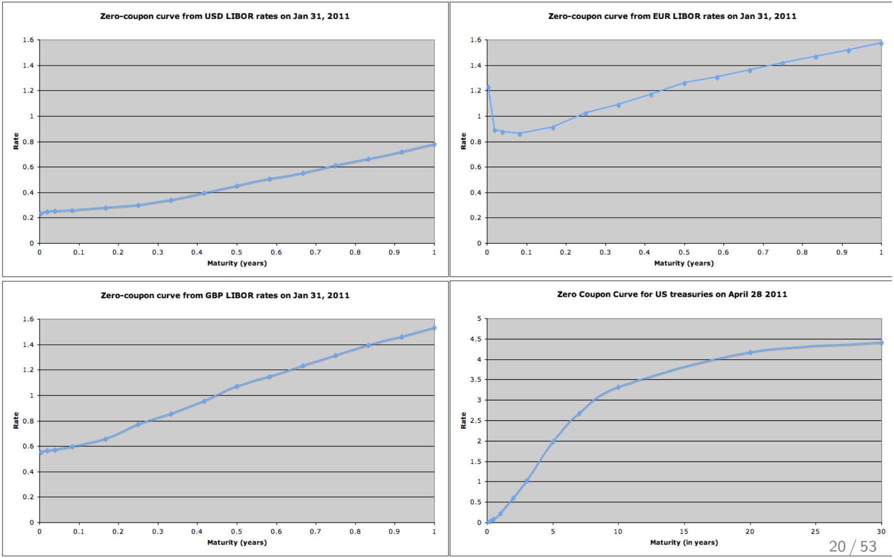
$$1 + L(t,T)(T-t) = 1 + L(t,T)(T-t)$$

- Typically, as we will be valuing LIBOR products we will generally consider only simple discounting. We will se this more clearly when considering Forward Rate Agreements and Swap contracts shortly.

Zero-coupon curve

- More commonly known as the *yield curve* the Zero-coupon curve at time t is the graph of either $L(t, T)$ or $Y(t, T)$ for longer than one-year. This also represents the **Term Structure of Interest Rates**. The Zero-coupon curve can have many different shapes, it can monotonically increase, decrease, be humped, flat or many other shapes.
- The next few slides show different curves. The $-LIBOR curve, the EUR-LIBOR, the GBP-LIBOR as well as the US treasury curve. As LIBOR rates only go to 12 months the zero-coupon LIBOR curves only extend for one year. However, from other interest rate derivative contracts it is possible to extend the curve out to longer maturities.
- Note that the Euro and Pound curves are non-monotonic, whereas the dollar curve is not. The fourth diagram of US treasury bond zero-coupon rates extends to 30 years and is generally monotonic.
- The aim of any model is to reproduce these zero-coupon curves as well as allowing for possible changes in the curves over time.

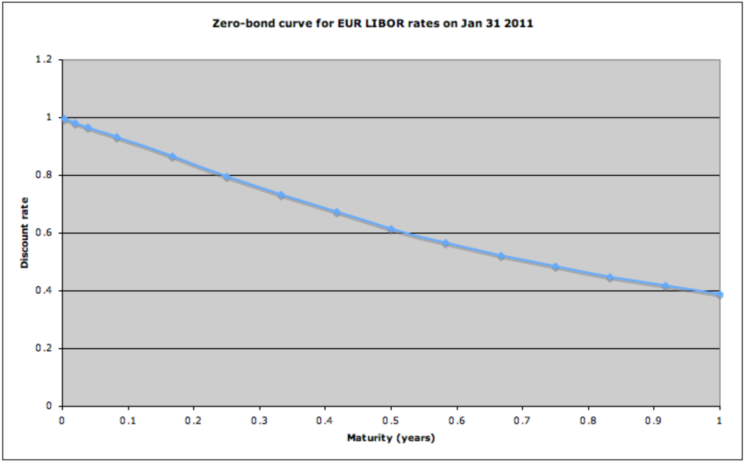
Figure 1.1


Zero-coupon curve from USD LIBOR rates on Jan 31, 2011
Zero-coupon curve from EUR LIBOR rates on Jan 31, 2011
Zero-coupon curve from GBP LIBOR rates on Jan 31, 2011
Zero Coupon Curve for US treasuries on April 28 2011
Rate
Maturity (years)
Maturity (in years)

Zero-bond curve

- The zero-bond curve is the graph of $P(t, T)$ across different maturities T. This curve can also be referred to as the term structure of discount factors.
- This is a less common curve than the zero-coupon curve and also less useful. The zero-coupon curve can have many different shapes whereas the zero-bond curve will be monotonically decreasing as long as the rates are positive.
- On the next page we have the term structure of discount factors for the Euro LIBOR - note the difference in the shape of the curves.

Figure 1.2


Zero-bond curve for EUR LIBOR rates on Jan 31 2011
Discount rate
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Maturity (years)

Quiz

Why is the zero-bond curve useful?

- It is easier to understand bond prices than rates
- It is more stable than the zero-coupon curve
- It is useful for discounting

Quiz

Why is the zero-bond curve useful?

- It is easier to understand bond prices than rates
- It is more stable than the zero-coupon curve
- It is useful for discounting CORRECT

Forward Rate Agreement (FRA)

- The size of the notional amounts on interest rate derivatives are enormous, but it is worth remembering that very rarely are the principals actually exchanged, more often it is the stream of payments. The most simple type of product in this category is the **Forward Rate Agreement (FRA)**.
- The forward rate agreement involves three distinct time instants. The current time, t, the expiry (or **reset** time $T > t$ and the maturity of the contract $T+\tau$. The basic idea is that the holder of the FRA receives an interest payment for the period between T and $T+\tau$. At the maturity $T+\tau$ a fixed payment based on a fixed rate K is exchanged for a floating rate payment of $L(T, T+\tau)$ that resets at T and matures at $T+\tau$.
- Formally the payoff is:

$$\text{Payoff}(FRA)_{T+\tau} = KN\tau - L(T, T+\tau)N\tau$$

where N is the notional principal of the FRA.

The Forward Rate $F(t; T, T+\tau)$

- Thus the value of the FRA at time T can be easily calculated by using the known rate $L(T, T+\tau)$ thus the value of the FRA at time T is
$$\mathrm{PV}(FRA)_T = \frac{KN\tau - L(T, T+\tau)N\tau}{1 + L(T, T+\tau)\tau}$$
where PV denotes the value of the FRA at the reset time T. This is possible at T because at the this time the rate $L(T, T+\tau)$ is known as it has just been reset.
- The interesting question is to determine the current (time t) value of the FRA . To do this we first consider the definition of a forward rate. The simply compounded forward rate at time t for the expiry T and maturity $T+\tau$ is denoted by $F(t; T, T+\tau)$. Thus far we know that
$$F(T; T, T+\tau) = L(T, T+\tau)$$

FRA: No arbitrage argument

- Thus, we can rewrite of FRA equations in terms of forward rates:

$$\begin{aligned}\mathrm{Payoff}(FRA)_{T+\tau} &= N(K\tau - F(T;T,T+\tau)\tau) \\ \mathrm{PV}(FRA)_T &= N\frac{K\tau - F(T;T,T+\tau)\tau}{1+F(T;T,T+\tau)\tau}\end{aligned}$$

- The interesting thing now is that $F(t;T,T+\tau)$ is a stochastic variable and so the value of the FRA will depend upon the trader's view of the LIBOR rates between t and T.
- If there exist zero coupon bonds with maturities that match the expiry (reset) and maturity dates of the FRA then it is possible to construct a portfolio at time t, $\pi(t)$ as follows:

$$\pi(t) = N\left[P(t,T) - P(t,T+\tau)\right]$$

and thus at $T+\tau$

$$\pi(T+\tau) = N\left[(1+L(T,T+\tau)\tau) - 1\right] = NF(T;T,T+\tau)\tau$$

FRA: No arbitrage argument

- The portfolio is thus equal to the floating leg of the FRA. By the principal of no arbitrage the value of the portfolio and the floating leg must be indentical at time t. Therefore we have a unique, no arbitrage value for the forward rate:

$$
\begin{aligned}
P(t,T) - P(t,T+\tau) &= F(t;T,T+\tau)\tau P(t,T+\tau) && (1)\\
F(t;T,T+\tau) &= \frac{1}{\tau}\left(\frac{P(t,T)}{P(t,T+\tau)} - 1\right) && (2)
\end{aligned}
$$

- The final step is to show that the current value of the FRA is given by

$$
\begin{aligned}
FRA(t,T,T+\tau,\tau,N,K) &= N\tau\left[K - F(t;T,T+\tau)\right]P(t,T+\tau)\\
&= NKP(t,T+\tau)\tau - N\left[P(t,T) - P(t,T+\tau)\right]
\end{aligned}
$$

- The final step comes from discounting the (now guaranteed) payoff at time $T+\tau$ back to time t by multiplying by the known bond price $P(t,T+\tau)$.

Quiz

What is $E_t[F(T; T, T+\tau)$?

- $L(t, T)$
- $L(t, T+\tau) - L(t, T)$
- $F(t; T, T+\tau)$
- $F(t; T, T+\tau)(T-t)$

Quiz

What is $E_t[F(T; T, T+\tau)$?

- $L(t, T)$
- $L(t, T+\tau) - L(t, T)$
- $F(t; T, T+\tau)$ CORRECT
- $F(t; T, T+\tau)(T-t)$

Instantaneous forward rates

- Thus the no-arbitrage time-t expectation of $F(T; T, T+\tau)$ is $F(t; T, T+\tau)$. Note that as for forward prices in equities and foreign exchange this is not the real world expectation of the forward rate but a risk-neutral estimate. Typically as there is risk then a risk-premium will be applied to estimate the true expected rate.
- Equation 1 also allows us to consider the limit of the the forward rate as $\tau \to 0$. Here we see that

$$\begin{aligned}
\lim_{\tau\to 0} F(t; T, T+\tau) &= -\lim_{\tau\to 0} \frac{1}{P(t, T+\tau)} \frac{P(t, T+\tau) - P(t, T)}{\tau} \\
&= -\frac{1}{P(t, T+\tau)} \frac{\partial P(t, T)}{\partial T} \\
&= -\frac{\partial \ln P(t, T)}{\partial T}
\end{aligned}$$

Instantaneous forward rates

- This leads a to the definition of the **instantaneous forward rate**, prevailing at current time t for a maturity $T > t$, $f(t, T)$

$$f(t,T) = \lim_{\tau \to 0} F(t; T, T+\tau) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{3}$$

- From this definition we can rewrite the price of a zero coupon bond as

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

Instantaneous forward rates

- Although this cannot be arrived at from actual traded contracts the instantaneous forward rate will be a useful tool in some later derivations. In particular the Heath, Jarrow and Morton (1992) framework focuses uses the set of $f(t,T)$ values as its fundamental building block.
- Note the important difference between the instantaneous short rate, r_t and the instantaneous forward rate $f(t,T)$. The short rate represents the actual rate from t to $t+\delta t$ whereas $f(t,T)$ denotes an expectation (under some measure that we are yet to see) of the rate from $T>t$ to $T+\delta T$ as both $\delta t, \delta T \to 0$.
- Traditional term structure models specify a process for the instantaneous short rate, whereas the LIBOR Market Model will model the forward rates instead.

Traders view by Rebonato

- The above analysis assumes that the basic building blocks of determining forward rates (and the value of FRAs) are zero coupon bonds of varying maturities. However, the typical trader will work the other way around. They will impute the value of the zero-coupon bonds, $P(t, T)$ from the observable values of the FRAs and the current spot LIBOR rate.
- In practice this matters as the traders will not actually hedge the positions of the FRAs with zero coupon bonds. This is partly because the notional principles on their trades are so large that they would not want to take on this possible credit risk. Instead they deal only in net flows from transactions (e.g. the payoffs from FRAs).
- So although traders will calculate the P values, they will typically hedge by balancing out the fixed and floating transactions and then hedging the small amount that is left over.

Traders view by Rebonato

- Thus, the derivation above is somewhat misleading as traders will actually trade on their expectations of future realizations of the forward rate (based on supply and demand) and then use this to agree on a price for the zero-coupon bond. In this way the zero-coupon bond prices can be considered 'virtual prices '. Note, however, that future derivations are far more elegant if the zero-coupon bond is taken to be the fundamental asset.

Interest rate swaps

- A swap is an exchange of a floating rate for a fixed rate (or vice versa). The rates are paid on a notional principal but only the interest payments are exchanged.
- A swap is a multi-period arrangement, potentially lasting for several years and will involve payments at regular intervals, typically 3 or 6 months.
- There are two types of swaps: **Payer** and **Receiver** swaps, where the paying or receiving relates to the fixed rate side of the swap
- There are also **spot-start** and **forward-start** swaps, where the *spot-* or *forward-* denotes when the fixed rate payments are set. Spot-start means that they are set now whereas forward-start means that they will be set at some future date. The typical traded swap contract is a spot-start.
- Let's suppose that the dates when the payments are exchanged are given by $T_1 < T_2 < \dots T_n$ and $\tau_i = T_{i+1} - T_i$, these payments are based on rates set at $T_0 < T_1 < \dots T_{n-1}$. Let $\mathcal{T} = \{T_0, \dots, T_{n-1}\}$ Note that the time interval does not have to be the same. It is also possible to have fixed and floating rates be paid out at different times.

Fixed and floating leg valuations

- The fixed leg is the simplest. This requires making payments at times T_{i+1} of value $NK\tau_i$ where K is the fixed interest rate and N is the notional principal. Since K is known then the current value of the fixed leg payments is:

$$FixedLeg_t(T_0, \ldots, T_{n-1}) = N \sum_{i=0}^{n-1} K\tau_i P(t, T_{i+1}) \tag{4}$$

- The floating leg is made up of a collection of payments determined by the reset of a particular LIBOR rate on a set of pre-specified dates and paid when that particular LIBOR rate is paid. Thus it looks like a series of FRAs, and so the current value of the floating leg payments is:

$$FloatLeg_t = N \sum_{i=0}^{n-1} \tau_i F(t; T_i, T_{i+1}) P(t, T_{i+1}). \tag{5}$$

Fixed and floating leg valuations

- From our earlier no arbitrage result (Equation 1) reduces to

$$
\begin{aligned}
FloatLeg_t &= N\sum_{i=0}^{n-1}[P(t,T_i) - P(t,T_{i+1})] \qquad (6)\\
&= N\left(P(t,T_0) - P(t,T_n)\right) \qquad (7)
\end{aligned}
$$

- Note that when $T_0 = t$ (a spot-start swap) then $P(t, T_0) = 1$.
- Thus, a payer floating swap has current value

$$
PFS(t,\mathcal{T},N,K) = N\left(P(t,T_0) - P(t,T_n)\right) - N\sum_{i=0}^{n-1} K\tau_i P(t,T_{i+1})
$$

Quiz

What does the fixed leg look similar to?

- A zero coupon bond
- A coupon bond
- A coupon only bond (no principal)
- A floating rate bond
- An FRA

Quiz

What does the fixed leg look similar to?

- A zero coupon bond
- A coupon bond
- A coupon only bond (no principal) CORRECT
- A floating rate bond
- An FRA

Aside: FRAs and swaps

- It is possible to consider swaps as a sum of FRAs.
- Consider the discounted payoff of a receiver swap as

$$\sum_{i=0}^{n-1} D(t, T_{i+1}) N\tau_i (K - L(T_i, T_{i+1}))$$

and so this is simply the sum of FRAs as follows:

$$\begin{aligned} RFS(t, \mathcal{T}, N, K) &= \sum_{i=0}^{n-1} FRA(t, T_i, T_{i+1}, \tau_i, N, K) \\ &= N \sum_{i=0}^{n-1} \tau_i P(t, T_{i+1})(K - F(t, T_i, T_{i+1})) \\ &= -NP(t, T_0) + NP(t, T_n) + N \sum_{i=0}^{n-1} \tau_i K P(t, T_{i+1}) \end{aligned}$$

Aside: Coupon bond and Floating note

- The two legs could be considered as an exchange of a coupon bearing bond (fixed) and a floating rate note (floating)
- A **coupon bearing bond** is a contract that ensures future payments at times $\mathcal{T} = \{T_0, \ldots, T_n\}$ of a deterministic amount $c_0, \ldots, c_N$. If we define the cash flows, $c_i = N\tau_i K$ and $c_n = N\tau_{n-1}K + N$ then the current value of this bond $CB(t, \mathcal{T}, c)$ is

$$CB(t, \mathcal{T}, N, c) = \sum_{i=0}^{n-1} c_i P(t, T_{i+1})$$

- A **floating rate note** is a contract ensuring the payment at times $T_1, \ldots, T_n$ of the LIBOR rates that reset at the previous instants $T_0, \ldots, T_{n-1}$. The note also makes a final payment consisting of the notional value. This note can be valued, $FRN(t, \mathcal{T})$ by considering a receiver swap with no fixed leg, and its value is $FRN(t, \mathcal{T}) = NP(t, T_0)$.
- Thus if $t = T_0$ then the floating rate note is equal to the nominal value, or a floating rate note always trades at par.

Swap rate

- From the equations for the fixed and floating legs it is possible to determine the equilibrium swap rate at time t, $S_{T_0,T_n}(t)$, which is given by

$$S_{T_0,T_n}(t) = \frac{P(t,T_0) - P(t,T_n)}{\sum_{i=0}^{n-1} \tau_i P(t,T_{i+1})} = \frac{P(t,T_0) - P(t,T_n)}{Fixed_t(T_0,\dots,T_n)} \tag{8}$$

- Alternatively, this can be written in terms of forward rates, using the first line of equation 6 above, to give

$$S_{T_0,T_n}(t) = \sum_{i=0}^{n-1} \omega_i F(t;T_i,T_{i+1}) \tag{9}$$

  where

$$\omega_i = \frac{\tau_i P(t,T_{i+1})}{\sum_{j=0}^{n-1} \tau_j P(t,T_{j+1})}.$$

- Both of these equations are important - showing the link between swap rates and zero-coupon bond prices and forward rates respectively.

Swap rates and the LIBOR curve

- Earlier we saw that as LIBOR rates are only quoted up to 12 months, it is not possible to extend the LIBOR curve out to longer maturities.
- However, it is possible to extract implied LIBOR rates for dates longer than 12 months from swap rates. This is typically how the LIBOR curve is constructed. So in the LIBOR market, swaps (and FRAs) will be seen as fundamental assets whose prices will be used to construct the term structure which in turn will be used to price other interest rate derivatives.
- Given swap dates $0 = T_0 < T_1 < T_2 < \cdots < T_n$ and swap rates $S_{T_0,T_i}(t)$ running from T_0 to T_n then $P(0,0) = 1$, $P(0,T_1) = \frac{1}{1+S_{T_0,T_1}(0)(T_1 - T_0)}$ and we can use equation 8 to iteratively find the remaining $P(0,T_i)$ values. From the $P(0,T_i)$ values it is possible to extract the appropriate $L(0,T_i)$ values to form the LIBOR curve.

Caplets/Floorlets

- From forward rate agreements, it is natural to consider an option on a forward rate. Call options on the forward rate are called **caplets**, put options are called **floorlets**.
- Typically for a forward rate $F(t; T, T+\tau)$ the expiry of the caplet/floorlet is typically the start or reset date of the FRA, time T. Thus at $T+\tau$ we receive

  $$\mathrm{Payoff}(Caplet)_{T+\tau} = N(\max(L(T, T+\tau) - K, 0))\tau$$

  or

  $$\mathrm{Payoff}(Floorlet)_{T+\tau} = N(\max(K - L(T, T+\tau), 0))\tau$$

  where N is the notional principal and K is the exercise price of the caplet/floorlet
- The pricing of caplets and floorlets will be similar to the valuation of equity options, it will depend upon assumptions as to the distribution of $L(T, T+\tau)$, in particular its volatility. We will look at simple pricing of caplets and floorlets when we see Black's model next lecture.

Caps/Floors

- As suggested from the naming conventions, a **Cap** will be a collection of caplets. A cap can also be considered as an interest rate swap where the exchange of fixed and floating payments only occurs if it has positive value.
- The cap discounted payoff is given by

$$\sum_{i=0}^{n-1} N\tau_i \max(L(T_i, T_{i+1}) - K, 0)P(t, T_{i+1})$$

  and the floor discounted payoff is given by

$$\sum_{i=0}^{n-1} N\tau_i \max(K - L(T_i, T_{i+1}), 0)P(t, T_{i+1})$$

  noting that $L(T_i, T_{i+1})$ is not known at time t.
- Caps and floors are valued as a sum of caplets and floorlets respectively, and will thus rely upon distributional assumptions about forward rates.

Quiz

When does a cap have a positive payoff?

- When LIBOR is high
- When LIBOR is low
- It depends

Quiz

When does a cap have a positive payoff?

- When LIBOR is high CORRECT
- When LIBOR is low
- It depends

How does a cap work?

- Practically speaking a cap provides a maximum rate that the borrower will have to pay. A floor will provide a minimum rate for a receiver.
- To see how this works consider a company has debt that must be repaid at the LIBOR rate every 3-months. The company is concerned about LIBOR rates increasing in the future and wishes to cap repayments at a rate K.
- They purchase a cap and so at each payment date they make a payment of L and receive $\max(L-K,0)$ from the cap. So the overall payment is $L-\max(L-K,0)=\min(L,K)$ and so they will pay L up to the point where it reaches K after which they will pay K.
- The floor will ensure that someone who is receiving floating rate payments will always receive an amount K.
- Caps are said to be in the money if $S_{T_0,T_n}(t)>K$ with obvious extensions.

Swaption

- The final fundamental interest rate derivative is the **swaption**.
- As the name suggests a swaption is an option on a swap. The payer swap is an option giving the holder the right but not the obligation to enter into a payer swap at some future time. This future time is the option maturity. Typically the swaption is designed so that the maturity conicides with the first reset date of the underlying swap.
- The length of the underlying swap is known as the **tenor** of the swaption
- From equations 4 and 5 the value of the payer swap at the option maturity, T, is

$$
\begin{aligned}
& N \sum_{i=0}^{n-1} \tau_i F(T; T_i, T_{i+1}) P(T, T_{i+1}) - N \sum_{i=0}^{n-1} K \tau_i P(T, T_{i+1}) \\
= \quad & N \sum_{i=0}^{n-1} P(T, T_{i+1}) \tau_i (F(T; T_i, T_{i+1}) - K)
\end{aligned}
$$

Swaption

- The option will only be exercised if this value is positive and so the payoff of the payer swaption is

$$\mathrm{Payoff}(Swaption)_T = \max\left(N \sum_{i=0}^{n-1} P(T, T_{i+1})\tau_i (F(T; T_i, T_{i+1}) - K), 0 \right)$$

or in terms of swap rates,

$$\mathrm{Payoff}(Swaption)_T = N \max\left(S_{T,T_n}(T) - K, 0\right) Fixed_T(T, \ldots, T_n).$$

where,

$$Fixed_T(T, \ldots, T_n) = \sum_{i=0}^{n-1} \tau_i P(T, T_{i+1})$$

Swaption

- Payer Swaptions are said to be in-the-money if $S_{T,T_n}(T) > K$ with natural extensions.
- Unlike the cap, the swaption cannot be decomposed into single dated products as the summation is inside the max operator not outside.

Quiz

How would you determine the $t < T$ value of a swaption?

American/Bermudan swaptions

- The natural extension to the swaption above is to consider a swaption that can be exercised at any time or at any time that the LIBOR rate resets. This is called a **Bermudan swaption**.
- We need to be a little careful with how the Bermudan swaption actually works. The option gives you a right to enter into either a receiver or payer swap, but the maturity of the swap is set when the option is issued and does not increase. So if you purchase a 30-year receiver Bermudan option on a 30-year swap and choose to exercise the option after 5-years then upon exercise you will obtain a 25-year receiver swap.
- As we will see valuing Bermudan swaptions is challenging as it will be tough to both determine the optimal early exercise strategy using our model of choice - The LIBOR Market Model.

Triggers, Captions and Floortions

- A **trigger FRA** is an FRA that comes into existence if and only if the forward rate is above a level H at the start of the FRA.
- A **trigger swap** is a swap that does not take effect until some reference rate passes a trigger level one one or a number of pre-specified dates. These are similar to barrier options for equity options but the barrier can be triggered by different values than the underlying asset.
- As with barrier options, there are many types of trigger options: *up-and-out*; *down-and-out*; *up-and-in*; *down-and-in*, etc.
- A **caption** is an option on a cap. At some time, perhaps the expiry of the first caplet there is the option to purchase a cap for a pre-agreed price, K.,
- Note that this is an option on all of the remaining caplets at once and, as with the swaption, so cannot be decomposed into a series of caplets.
- Naturally a floortion is an option on a floor.

Constant maturity swap (CMS) and others

- A CMS is a hybrid between a swap and a FRA. Rather then receiving the prevailing LIBOR rate at a particular point in time a swap rate is received instead.
- For example, a CMS could make payments every six-months where the 'floating 'rate is the 30-year swap rate at each six months period. So at each time $t + \tau$ the receiver receives $N(S_{t,t+30}(t) - K)\tau$, where in our example $S_{t,t+30}(t)$ is the 30-year swap rate at time t.
- There are a large number of interest rate products, such as floaters, inverse floaters, ratchet floaters, callable floaters and more traditional ones not mentioned here: bond options, bond futures, Eurodollar futures, convertible bonds, callable bonds etc. We will encounter some of these during the course but the most important products are described above.

Conclusion

- We have introduced the key rates - the short rate, the forward rate and the swap rate. The term structure models we see will develop models for one of these rates. The first set of models will focus on the short rate, the later models (such as the LMM) will consider modeling the forward and swap rates.
- As we will be focussing on no arbitrage prices we also need to define our underlying asset. As the forward and short rates are not traded themselves we need simple products that will proxy as underlying assets. The simplest is the zero-coupon bond, but we also consider the Forward Rate Agreement.
- Other 'vanilla' products such as caps, floors, swaps and swaptions will allow us to test our term structure models as well us allowing us to calibrate our model (in a manner akin to estimating volatilities from option prices).
- Once we have calibrated our models we will attempt to model more advanced interest rate derivatives such as Bermudan swaptions.