Introduction

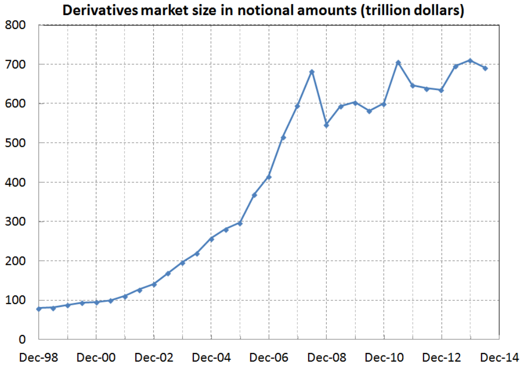
Liming Feng

Dept. of Industrial & Enterprise Systems Engineering
University of Illinois at Urbana-Champaign

Derivatives

- Pricing **derivative securities** (e.g., options)
- Derivative: a contract whose value depends on the values of other basic **underlying assets/variables**
- End of June 2014, global over-the-counter **derivatives market size** measured by notional amount outstanding was about $691.5 trillion (*Bank for International Settlements*)
- On Thursday 12/5/2013, a total of 3,702,255 contracts were traded on **Chicago Board Options Exchange**

Market size



- Accurate pricing and efficient risk management of derivatives important

Objectives of the course

- **Probabilistic modeling**: modeling dynamics of asset prices; deriving values of derivative securities
- **Numerical methods**: connection to numerical solution of PDEs, monte carlo simulation, Fourier transform (theoretical foundations for **IE 525**)
- **Computer implementation**: C/C++ implementation of the above methods

Basics of derivative securities

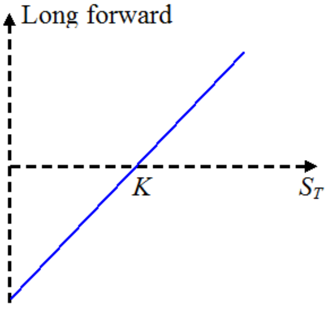
Comprehensive coverage in **Fin 513**

Forward contracts

- **Forward contract**: an agreement to buy or sell an asset at a future time T (maturity) for a certain price K (delivery price)
- **Long forward:** agree to buy, payoff at maturity $S_T - K$;
  short forward: agree to sell, payoff $K - S_T$, where S_T is the asset price at maturity
- **Forward price**: the delivery price that makes the forward contract zero cost initially (note the difference between forward price and the value of an existing forward contract)

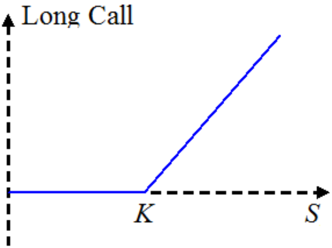
Forward payoff

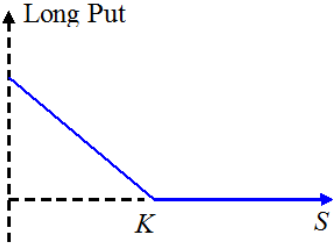
- Payoffs of long and short forward positions



Options

- **Call option**: gives the holder the right but not the obligation to buy an asset for price K (**strike price**) by a future time T
- **Put option**: gives the right to sell
- **Long call**: buy a call, payoff function $(S-K)^+ := \max(0, S-K)$; **long put**: buy a put, payoff $(K-S)^+$





Exercise styles

- **European:** can be exercised at maturity only; e.g., S&P 100 index option XEO on CBOE
- **American:** can be exercised at any time before maturity; e.g., most stock options, S&P 100 index option OEX on CBOE
  - American option = European option + right to exercise early
  - American option price $\geq$ European option price
- **Bermudan:** can be exercised at any time in a discrete set, e.g., at the end of each week for a weekly monitored Bermudan option

Intrinsic value

- **Intrinsic value** of an option at $t \leq T$

$$\text{call: } (S_t - K)^+, \quad \text{put: } (K - S_t)^+$$

- American option value $\geq$ its intrinsic value at any time $t \leq T$: **time value** $=$ option value $-$ intrinsic value
- At time $t \leq T$, an option is
  - **in the money** if intrinsic value > 0
  - **out of the money** if $S_t > K$ for a put (or $S_t < K$ for a call)
  - **at the money** if $S_t = K$

- Plain vanilla options vs **exotic options** with nonstandard payoffs
- Knock-out **barrier options**: the option is terminated whenever the underlying asset price crosses a lower or upper barrier
- **Lookback options**: payoff depends on the maximum or minimum asset price
- **Asian options**: payoff depends on average asset price

Types of derivative traders

- **Hedgers** use derivatives to reduce risk
- **Speculators** use derivatives to bet on the future direction of a market variable
- **Arbitrageurs** look for risk free profit
- **Market makers** maintain a two way market, execute trades for others and earn bid/ask spread

  Bid: price at which the market maker is willing to buy
  Ask: price at which the market maker is willing to sell

An example of speculation

Example (speculation using call options)

A trader believes that IBM stock price will increase in a month. Current IBM stock price is \$100. The investor decides to buy 1000 call options with one month maturity and strike price \$105. The call option price is \$1.5

	Initial investment \$1500	
	$S_T = 90$	$S_T = 110$
Buy 15 shares Return	$15(90-100) = -150$ -10%	$15(110-100) = 150$ 10%
Buy 1000 calls Return	-1500 -100%	$1000(110-105) - 1500 = 3500$ $3500/1500 = 233\%$

Leverage: use of options magnifies gain/loss

An example of arbitrage

Example (arbitrage)

The current price of a stock is \$1 per share. A call option to buy the stock in 1 year at \$0.8 per share is traded at \$2 per call. \$1 deposited for 1 year will earn \$0.05

- **Arbitrage:** a risk free trading strategy which requires no initial cost and yields non-negative payoff with probability one and strictly positive payoff with positive probability
- Arbitrageurs **buy low sell high**: sell call, buy stock, deposit 1
  - $S_T > 0.8$: sell stock to call option holder, gain \$1.05+0.8
  - $S_T \leq 0.8$: option not exercised, gain \$1.05+S_T
  - Market response: share price increases, call price decreases
  - Arbitrage opportunities are very **short-lived in liquid markets**
- In pricing derivatives, we assume **no arbitrage**

No arbitrage pricing

Assumptions

- No transaction costs
- No trading restrictions (such as short selling)
- No tax issues
- Market participants can borrow and lend at the same risk free interest rate
- **No arbitrage**

Determine forward price

- **Forward price**: THE delivery price K so that the value of the forward contract is 0
- Suppose the asset provides a known yield at rate q per year (continuous compounding): one unit of the asset at time 0 grows to e^{qT} units at T
- Suppose the risk free interest rate is r per year (continuous compounding)
- Forward price for the delivery of one unit of the asset at T:

$$F_0 = S_0 e^{(r-q)T}$$

- Suppose $F_0 > S_0 e^{(r-q)T}$, consider the following trading strategy

  At time 0

  - Buy e^{-qT} unit of the asset
  - Short forward (to sell one unit of the asset at T)
  - Borrow $e^{-qT} S_0$ at rate r

  At time T, e^{-qT} unit of the asset grows into one unit

  - Forward contract: sell the asset and receive F_0
  - Repay the loan: $e^{(r-q)T} S_0$

  A costless, riskless income $F_0 - e^{(r-q)T} S_0 > 0$, arbitrage!

- Suppose $F_0 < S_0 e^{(r-q)T}$, consider the following trading strategy

At time 0

  - Long forward (to buy one unit of the asset at T)
  - **Short sell e^{-qT} units** of the asset and receive $S_0 e^{-qT}$
  - Deposit $S_0 e^{-qT}$ at rate r

At time T

  - Deposit account: get $e^{(r-q)T} S_0$
  - Forward contract: buy the asset for F_0
  - Short selling account: **return one unit** of the asset

A costless, riskless income $e^{(r-q)T} S_0 - F_0 > 0$, arbitrage!

Valuing forward contracts

- Value of a forward contract with time to maturity T and delivery price K
- Decompose the (long) forward payoff
  - Payoff of the long forward contract at maturity

$$S_T - K = (S_T - F_0) + (F_0 - K)$$

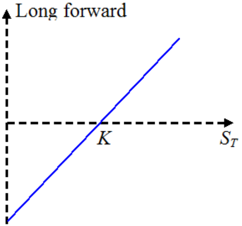
  - A forward contract with forward price F_0 as the delivery price
  - A contract with fixed payoff $F_0 - K$ at time T
- Value of the forward contract: **discount the payoff as if the asset price at maturity would be the forward price**

$$V_0 = e^{-rT}(F_0 - K) = S_0 e^{-qT} - K e^{-rT}$$

European put-call parity

- Long forward = long call + short put:

$$S_T - K = (S_T - K)^+ - (K - S_T)^+$$



Value of long forward contract $V_0 = c - p = S_0 e^{-qT} - Ke^{-rT}$

- Prices of a European put and a European call with the same maturity T and strike price K on an asset with continuous yield q satisfy the **European put-call parity**:

$$c + Ke^{-rT} = p + S_0 e^{-qT}$$

- Arbitrage opportunities exist if $c + Ke^{-rT} > p + S_0 e^{-qT}$: sell 1 call, buy 1 put, buy e^{-qT} unit of the underlying, borrow if necessary (reverse the strategy if $c + Ke^{-rT} < p + S_0 e^{-qT}$)

Bounds for call option price

- **European call** option price bounds (asset with continuous yield q)
$$(S_0 e^{-qT} - Ke^{-rT})^+ \leq c \leq S_0 e^{-qT}$$
- **Upper bound**: European call option payoff $(S_T - K)^+ \leq S_T$.
$$c \leq S_0 e^{-qT}$$
Arbitrage opportunities exist otherwise (if $c > S_0 e^{-qT}$, sell 1 call, buy e^{-qT} unit of the underlying, deposit)

- **Low bound**: from put-call parity

$$\begin{aligned} c &= p + S_0 e^{-qT} - K e^{-rT} \\ &\geq S_0 e^{-qT} - K e^{-rT} \end{aligned}$$

Call price is non-negative: $c \geq 0$. Therefore,

$$c \geq (S_0 e^{-qT} - K e^{-rT})^+$$

- American call on an asset with continuous yield q

$$(S_0 e^{-qt} - K e^{-rt})^+ \leq C \leq S_0, \quad \forall\ 0 \leq t \leq T$$

Bounds for put option price

- **European put** option price bounds

$$(Ke^{-rT} - S_0 e^{-qT})^+ \leq p \leq Ke^{-rT}$$

- **Upper bound**: European put option payoff $(K - S_T)^+ \leq K$; **lower bound**: $p = c + Ke^{-rT} - S_0 e^{-qT} \geq Ke^{-rT} - S_0 e^{-qT}$
- American put on an asset with continuous yield q

$$(Ke^{-rt} - S_0 e^{-qt})^+ \leq P \leq K, \quad \forall\, 0 \leq t \leq T$$

American calls on assets with no income

- **Early exercise never optimal:** suppose $S_0 > K$
- Exercise the call to buy the share and hold the share:

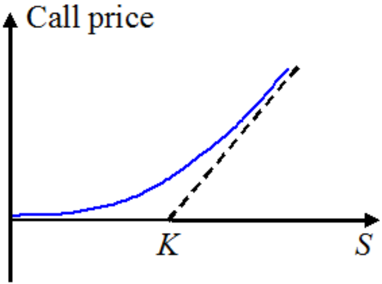
Better to exercise later to delay the cash payment K

- Exercise the call to buy the share, sell the share immediately, receive $S_0 - K$:

Better to sell the call

$$C \geq (S_0 - Ke^{-rT})^+ > S_0 - K$$

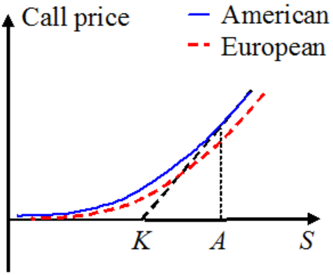
- Early exercise is not optimal: American call value = European call value
- Positive **time value** at any time before maturity



asset with no income

American calls on assets with income

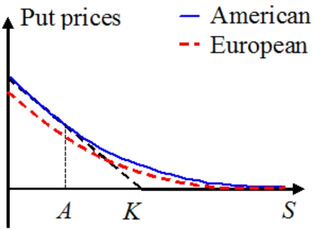
- May be optimal to early exercise an American call on an asset paying income



asset with income

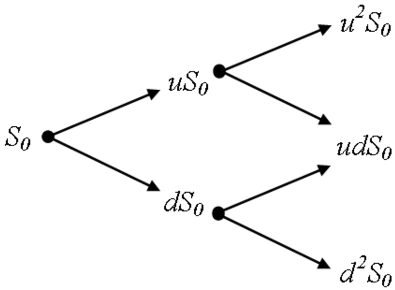
- Call should be exercised when $S \geq A$; **early exercise boundary**: collection of A's for varying option maturities

- **Early exercise may be optimal for American puts**
- When the underlying asset price is close to zero, early exercise is optimal
  - Exercise the put and receive K immediately
  - Exercise later to receive at most K
- American put should be exercised when $S < A$



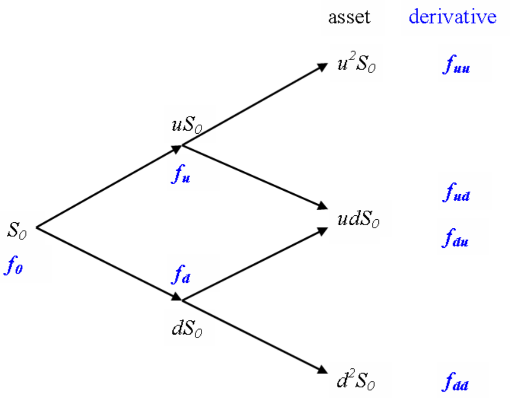
Two-step binomial model

- Two-step binomial model over time period $[0, T = 2\delta]$: asset price rises with probability p and drops with probability $1 - p$, $0 < p < 1$, $0 < d < u$



- Asset price process: $S = \{S_0, S_\delta, S_{2\delta}\}$, risk free investment: $1 \Rightarrow e^{r\delta} \Rightarrow e^{2r\delta}$

- Suppose the asset pays a continuous yield at rate q. The model is arbitrage free iff $d < e^{(r-q)\delta} < u$
- Price a European derivative with payoff f at maturity



No arbitrage pricing

- Construct a *replicating portfolio* by trading the underlying asset and risk free investment
- **Replicating portfolio**
  - Time 0: long Δ_0 units, borrow Ψ_0 (cost f_0)
  - Asset price increases at time δ: long Δ_u, borrow Ψ_u
  - Asset price decreases at time δ: long Δ_d, borrow Ψ_d
- **Select Δ's, Ψ's**

  $$\text{Time } 2\delta: \text{ value of the replicating portfolio } = \text{ derivative payoff}$$

- **Derivative price** $= f_0$ to avoid arbitrage

- *Asset price increases at time δ:*

$$\begin{aligned}
\text{derivative payoff} &= \text{portfolio payoff} \\
f_{uu} &= \Delta_u \cdot e^{q\delta} \cdot u^2 S_0 - \Psi_u e^{r\delta} \\
f_{ud} &= \Delta_u \cdot e^{q\delta} \cdot udS_0 - \Psi_u e^{r\delta}
\end{aligned}$$

$$\Delta_u = \frac{f_{uu} - f_{ud}}{uS_0 e^{q\delta}(u-d)}, \quad \Psi_u = \frac{df_{uu} - uf_{ud}}{e^{r\delta}(u-d)}$$

Portfolio value (amount needed to replicate derivative payoff)

$$f_u = \Delta_u \cdot uS_0 - \Psi_u = e^{-r\delta}\left[p^* f_{uu} + (1-p^*) f_{ud}\right]$$

where

$$p^* = \frac{e^{(r-q)\delta} - d}{u - d}$$

- *Asset price decreases at time δ:*

$$\begin{aligned} \text{derivative payoff} &= \text{portfolio payoff} \\ f_{du} &= \Delta_d \cdot e^{q\delta} \cdot udS_0 - \Psi_d e^{r\delta} \\ f_{dd} &= \Delta_d \cdot e^{q\delta} \cdot d^2 S_0 - \Psi_d e^{r\delta} \end{aligned}$$

$$\Delta_d = \frac{f_{du} - f_{dd}}{dS_0 e^{q\delta}(u-d)}, \quad \Psi_d = \frac{df_{du} - uf_{dd}}{e^{r\delta}(u-d)}$$

Portfolio value (amount needed to replicate derivative payoff)

$$f_d = \Delta_d \cdot dS_0 - \Psi_d = e^{-r\delta}\left[p^* f_{du} + (1-p^*) f_{dd}\right]$$

- *At time 0*:

$$\begin{aligned} \text{amount needed at } \delta &= \text{portfolio payoff} \\ f_u &= \Delta_0 \cdot e^{q\delta} \cdot uS_0 - \Psi_0 e^{r\delta} \\ f_d &= \Delta_0 \cdot e^{q\delta} \cdot dS_0 - \Psi_0 e^{r\delta} \end{aligned}$$

$$\Delta_0 = \frac{f_u - f_d}{S_0 e^{q\delta}(u-d)}, \quad \Psi_0 = \frac{df_u - uf_d}{e^{r\delta}(u-d)}$$

Portfolio value (amount needed to replicate derivative payoff)

$$f_0 = \Delta_0 S_0 - \Psi_0 = e^{-r\delta}\left[p^* f_u + (1-p^*) f_d\right]$$

Backward induction

- Starting from derivative payoff

$$f_{uu}, f_{ud}, f_{du}, f_{dd} \quad \Rightarrow \quad f_u, f_d \quad \Rightarrow \quad f_0$$

where

$$\begin{aligned} f_u &= e^{-r\delta}(p^* f_{uu} + (1-p^*)f_{ud}) \\ f_d &= e^{-r\delta}(p^* f_{du} + (1-p^*)f_{dd}) \\ f_0 &= e^{-r\delta}(p^* f_u + (1-p^*)f_d) \end{aligned}$$

where

$$p^* = \frac{e^{(r-q)\delta} - d}{u-d}$$

Risk neutral pricing

- Derivative price

$$f_0 = e^{-2r\delta}((p^*)^2 f_{uu} + p^*(1-p^*) f_{ud} + p^*(1-p^*) f_{du} + (1-p^*)^2 f_{dd})$$

- By no arbitrage condition: $0 < p^* < 1$, consider it as the probability of asset price going up in another world
- p^* is called **risk neutral probability**. In the **risk neutral world**, asset earns risk free interest rate

$$\mathbb{E}^*[e^{2q\delta} S_{2\delta}] = e^{2r\delta} S_0$$

- Derivative price = **risk neutral expected payoff discounted at the risk free rate**

- **Actual probability** p doesn't enter the pricing formula! It has been included in the asset price
- Summary

Physical world	Risk neutral world
Where we live	Where we price derivatives
Asset price goes up with prob p	Asset price goes up with prob p^*
Asset earns risk adjusted rate	Asset earns risk free rate

- Risk neutral pricing: (1). find risk neutral probability; (2). compute risk neutral expected payoff; (3). discount at the risk free rate

Delta hedging

- Hedge a short position in a derivative contract
- Sell a derivative, hold Δ_t shares, $t = 0, \delta$
  - At time 0,
$$\Delta_0 = \frac{f_u - f_d}{S_0 e^{q\delta}(u-d)}$$
  - Asset price goes up at time δ
$$\Delta_u = \frac{f_{uu} - f_{ud}}{uS_0 e^{q\delta}(u-d)}$$
  - Asset price goes down at time δ
$$\Delta_d = \frac{f_{ud} - f_{dd}}{dS_0 e^{q\delta}(u-d)}$$
- The hedged position is risk free

- Backward induction for an American put option: starting with payoff

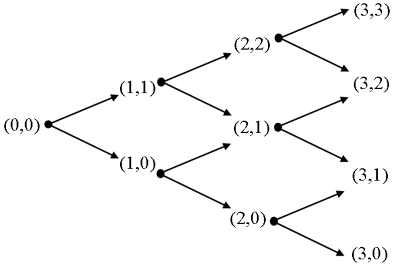
$$f_{uu}, f_{ud}, f_{du}, f_{dd}$$

Compute

$$\begin{aligned} f_u &= \max(K - uS_0, e^{-r\delta}(p^* f_{uu} + (1-p^*) f_{ud})) \\ f_d &= \max(K - dS_0, e^{-r\delta}(p^* f_{du} + (1-p^*) f_{dd})) \\ f_0 &= \max(K - S_0,\ \ e^{-r\delta}(p^* f_u + (1-p^*) f_d)) \end{aligned}$$

Multi-step binomial model

- Multi-step binomial model over time period $[0, T = N\delta]$



- Node (n, j): time $n\delta$, j is the number of up moves in the asset price, $0 \leq j \leq n$. Asset price at node (n, j): $S_{n,j} = u^j d^{n-j} S_0$

- Price a (path independent) European derivative with payoff $f(S_T)$ at maturity

$$f_{N,j} = f(S_{N,j}), \quad 0 \leq j \leq N$$

- **Risk neutral pricing formula**

$$f_0 = e^{-rT}\mathbb{E}^*[f(S_T)] = e^{-rT}\sum_{j=0}^{N}\left(\begin{array}{c} N \\ j \end{array}\right)(p^*)^j(1-p^*)^{N-j}f_{N,j}$$

where the number of paths leading to node (N, j) is

$$\left(\begin{array}{c} N \\ j \end{array}\right) = \frac{N!}{j!(N-j)!}$$

- **Backward induction**
  - Start with
$$f_{N,j}, \quad j = 0, 1, \cdots, N$$
  - For $n = N-1, N-2, \cdots, 0$
$$f_{n,j} = e^{-r\delta}(p^* f_{n+1,j+1} + (1-p^*) f_{n+1,j}), \quad j = 0, 1, \cdots, n$$
- American style derivatives:
$$f_{n,j} = \max\left(\text{intrinsic value}, e^{-r\delta}(p^* f_{n+1,j+1} + (1-p^*) f_{n+1,j})\right)$$

Path dependent derivatives

- Derivative payoff depends on the whole path of the asset price process (lookback, Asian options)

$$f(S_0, S_\delta, \cdots, S_{N\delta})$$

- Backward induction or risk neutral pricing: need to differentiate different paths

$$f_0 = \mathbb{E}^*[e^{-rT} f(S_0, \cdots, S_{n\delta})] \neq e^{-rT} \sum_{j=0}^{N} \binom{N}{j} (p^*)^j (1-p^*)^{N-j} f_{N,j}$$

CRR binomial model

- Given option maturity T. Divide $[0, T]$ into n equal intervals: $\delta = T/n$
- Select u and d in the binomial model as follows (**Cox-Ross-Rubinstein binomial model**)

$$u = e^{\sigma\sqrt{\delta}}, \quad d = e^{-\sigma\sqrt{\delta}}$$

  where σ is the volatility parameter in the Black-Scholes-Merton model
- The CRR model converges to the Black-Scholes-Merton model as n gets large

Continuous time models

- Binomial model is a discrete approximation to the Black-Scholes-Merton model
- **Limitations of binomial model**: slow; difficult for path dependent contracts; difficult to incorporate jumps and stochastic volatility
- **Continuous time models**: more general derivative securities; more realistic models
- **Stochastic calculus** provides tools for derivatives valuation in continuous time models; establishes the connection to PDE, monte carlo, transform approaches