# FIN516: Term-structure Models

## Lecture 9: Toolkit II

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# Overview

- Here we will cover some of the key tools and techniques that we will need to model the term structure and to value interest rate derivatives.
- The emphasis will be to develop practical methods rather than spending a lot of time on the theory.
- We will first take a careful look at how to change measure, with a look at Girsanov's theorem and the very useful *Change of Numeraire Toolkit* developed by Brigo and Mercurio. This is designed for dealing with term structure models where it is common to have to switch numeraires, when prices and rates can depend upon multiple Brownian motions.
- We will then look at Cholesky factorization and principal components analysis.

# Fundamental Theorem of Finance

- We start with a quick recap of the Fundamental Theorems of Finance.
- There are $K+1$ assets that are continuously traded from time 0 to $T$ and their prices are given by $S = \{S_t : 0 \leq t \leq T\}$ with components $S^0 = B, S^1, S^2, \dots, S^K$. These could be considered stock prices, or bond prices or both, each one will be driven by at least one Brownian motion.
- Where, in particular, $B_t$ is a bank account

$$dB_t = r_t B_t dt$$

  and the discount factor is $D(0,t) = 1/B_t$ - we use this a lot from now on.
- $B_t$ will be the numeraire in this set up.

# Trading strategies

- A **trading strategy** is a $(K+1$ dimensional$)$ process $\phi$ whose components are predictable and the **value process** associated with the trading strategy is

$$V_t(\phi) = \phi_t S_t = \phi_t^0 B_t + \sum_{k=1}^{K} \phi_t^k S_t^k, \qquad 0 \le t \le T$$

whilst the **gains process** associated with the trading strategy, $\phi$, is

$$G_t(\phi) = \int_0^t \phi_u dS_u = \int_0^t \phi_t^0 dB_t + \sum_{k=1}^{K} \int_0^t \phi_u^k dS_u^k, \qquad 0 \le t \le T$$

- The $\phi$ values are the holdings in each of the assets in the economy, where these values are known just before time $t$.
- The trading strategy is **self-financing** if $V(\phi) \ge 0$ and

$$V_t(\phi) = V_0(\phi) + G_t(\phi), \qquad 0 \le t < T$$

in that the value of the strategy only changes due to changes in the asset prices and there are no other cash flows.

# No arbitrage and equivalent measures

- An **equivalent martingale measure** $\mathbb{Q}$ is a new probability measure such that
  1. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, that is $Prob(A) = 0$ under $\mathbb{P}$ if and only if $Prob(A) = 0$ under $\mathbb{Q}$, for every event $A$.
  2. *Technical: The Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ is square integrable with respect to $\mathbb{P}$.*
  3. The **discounted asset price** process $D(0,\cdot)S$ is a martingale under $\mathbb{Q}$, i.e.

$$E_u^{\mathbb{Q}}\left(D(0,t)S_t^k\right) = D(0,u)S_u^k, \forall k = 0, 1, \dots, K \text{ and } 0 \leq u \leq t \leq T$$

- An **arbitrage strategy** is a self-financing strategy such that $V_0(\phi) = 0$ but $Prob(\{V_T(\phi) > 0\}) > 0$ under $\mathbb{P}$. It can be proved that the existence of an equivalent martingales measure implies the absence of arbitrage opportunities.

# Attainable payoffs and the fundamental theorems

- A contingent claim (like an option) is a positive random variable on. A contingent claim $H$ is **attainable** if there exists some self-financing trading strategy $\phi$ such that $V_T(\phi) = H$. Such a $\phi$ is said to generate $H$ and $\pi_t = V_t(\phi)$ is the price at time $t$ associated with $H$ (we often refer to $H$ as the payoff of the claim).
- The first fundamental theorem of finance follows. Assume that there exists an equivalent martingale measure $\mathbb{Q}$ and let $H$ be an attainable contingent claim. Then for each $0 \leq t \leq T$ there exists a *unique* price $\pi_t$ associated with $H$, i.e.,

$$\pi_t = E_t^{\mathbb{Q}}\left(D(t, T)H\right)$$

- Finally, to ensure that $H$ is attainable we state that a financial market is **complete** if and only if every contingent claim is attainable.
- The fundamental theorems of finance state that a financial market is arbitrage free and complete if and only if there exists a unique equivalent martingale measure, $\mathbb{Q}$.

## Quiz

If $r_t = r$ is a constant, what are acceptable simplifications for $\pi_t$

- $\pi_t = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds} H|\mathcal{F}_t\right)$
- $\pi_t = e^{-\int_t^T r_s ds} E^{\mathbb{Q}}\left(H|\mathcal{F}_t\right)$
- $\pi_t = e^{-r(T-t)} E^{\mathbb{Q}}\left(H|\mathcal{F}_t\right)$

Will this be true if $r_t$ is stochastic?

# Quiz

If $r_t = r$ is a constant, what are acceptable simplifications for $\pi_t$

- $\pi_t = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds} H | \mathcal{F}_t\right)$ CORRECT
- $\pi_t = e^{-\int_t^T r_s ds} E^{\mathbb{Q}}\left(H | \mathcal{F}_t\right)$ CORRECT
- $\pi_t = e^{-r(T-t)} E^{\mathbb{Q}}\left(H | \mathcal{F}_t\right)$ CORRECT

Will this be true if $r_t$ is stochastic?

# Example: Set-up

- Let's consider an option when there is one stock following GBM and the money market account. We need one more theorem to start...
- The **Martingale Representation Theorem** states that if $W_t$ is a Brownian motion and that $M_t$ is a continuous Martingale then there exists a predictable function $\phi$ such that

$$dM_t = \phi dW_t$$

- Predictable here means that its value for time $s$ is always known at time $t < s$, usually it is of a form $\phi(W_t, t)$ or $\phi(S_t, t)$. Crucially all the information that determines $M_t$ comes from $W_t$ .
- Let's think of our standard equity option, $C_t$. From above, using $B_t$ as a numeraire then then

$$M_t = \frac{C_t}{B_t}$$

is a martingale and so there exists a $\phi$ so that

$$d\left(\frac{C_t}{B_t}\right) = \phi dW_t$$

# Example - single stock

- If $S_t$ follows GBM then we can find $\phi$,

$$d\left(\frac{C_t}{B_t}\right) = \sigma \frac{S_t}{B_t} \frac{dC_t}{dS_t} dW_t$$

but we also know (EMM point 3) that

$$d\left(\frac{S_t}{B_t}\right) = \sigma \frac{S_t}{B_t} dW_t$$

as this can also be represented by a martingale and we can invoke the martingale representation theorem.

- Combining these two gives us

$$d\left(\frac{C_t}{B_t}\right) = \frac{\partial C_t}{\partial S_t} d\left(\frac{S_t}{B_t}\right)$$

or that we can hedge the random part of $C_t$ by holding $\frac{\partial C_t}{\partial S_t}$ of $dS$. This is, of course, delta hedging.

# Example - Self financing

- We start of with cash worth $C_0$ and we need to replicate $C_t$ at all times. We should hold $\frac{\partial C}{\partial S}(t, S_t)$ of stock and so we should be holding $C_t - \frac{\partial C}{\partial S}(t, S_t)$ of bonds.
- The self financing portfolio is then

$$\alpha_t B_t + \beta_t S_t$$

and

$$\begin{aligned} \alpha_t &= \frac{C_t}{B_t} - \frac{\partial C}{\partial S}(t, S_t)\frac{S_t}{B_t} \\ \beta_t &= \frac{\partial C}{\partial S}(t, S_t) \end{aligned}$$

- The current value is obviously $C_t$ and we need to verify that $dC_t = \alpha_t dB_t + \beta_t dS_t$ (or gains only come from changes in $B$ and $S$). However, we can write $dC_t$ as

$$dC_t = d\left(\frac{C_t}{B_t}\right)B_t + \left(\frac{C_t}{B_t}\right)dB_t + d\left(\frac{C_t}{B_t}\right)dB_t$$

# Example - Self financing

- $B_t$ is deterministic and so

$$dC_t = B_t \frac{\partial C_t}{\partial S} d\left(\frac{S_t}{B_t}\right) + \frac{C_t}{B_t} dB_t$$

and then again

$$d\left(\frac{S_t}{B_t}\right) = \frac{1}{B_t} dS_t - \frac{S_t}{B_t^2} dB_t$$

and so

$$dC_t = \frac{\partial C_t}{\partial S} dS_t + \left(\frac{C_t - S_t \frac{\partial C_t}{\partial S}}{B_t}\right) dB_t$$

and so the self-financing condition is satisfied. So in the simple equity option setting the option payoff is replicable (attainable) and so it can be priced using:

$$\pi_0 = E_0^{\mathbb{Q}}\left[\frac{C_T}{B_T}\right].$$

# Changing numeraire

- For equity option pricing the equivalent martingale measure $\mathbb{Q}$ is typically the most convenient measure with the bank account, $B$, as the numeraire, as payoffs can be easily discounted at the risk-free rate. There are times when this is not the most convenient choice.
- With interest rate derivatives, as $D(t, T)$ is modeled as stochastic the $\mathbb{Q}$ measure may not be the most natural choice. Can we derive the fundamental theorems for different measures?
- In lecture 2 we saw that this was possible when we derived Black's formula for caps and swaptions. Here we will be more rigorous and develop a general methodology for changing numeraire.
- A **Numeraire** is any, positive non-dividend-paying asset, in that it is identifiable with a self financing strategy.
- A numeraire is a reference asset that is chosen to normalize all other asset prices with respect to it, so if the numeraire is $N$ then the relative prices $B/N, S^1/N, \ldots S^K/N$ are considered.
- Geman, El Karoui, and Rochet (1995) showed that self financing strategies are still self financing with respect to any numeraire asset.

# Changing numeraire

- The self-financing condition (again) in differential form is

$$dV_t(\phi) = \sum_{k=1}^{K} \phi_t^k dS_t^k$$

which implies that

$$d\left(\frac{V_t(\phi)}{N_t}\right) = \sum_{k=1}^{K} \phi_t^k d\left(\frac{S_t^k}{N_t}\right)$$

- This means that an attainable contingent claim will be attainable under any numeraire. The following proposition generalizes the fundamental theorem of finance to any numeraire:
- There exists a numeraire $N$ and a probability measure $\mathbb{Q}^N$, equivalent to the initial $\mathbb{P}$ such that the price of any traded asset $X$ relative to $N$ is a martingales under $\mathbb{Q}^N$

$$\frac{X_t}{N_t} = E_t^{\mathbb{Q}^N}\left[\frac{X_T}{N_T}\right], \qquad 0 \leq t \leq T$$

# Changing numeraire: Radon-Nikodym

- If we consider a general numeraire asset $U$ then there also exists another probability measure $\mathbb{Q}^U$, equivalent to the initial $\mathbb{P}$ such that the price of any traded asset $X$ relative to $U$ is also a martingale under $\mathbb{Q}^U$

$$\frac{X_t}{U_t} = E_t^{\mathbb{Q}^U}\left[\frac{X_T}{U_T}\right], \qquad 0 \leq t \leq T$$

- Thus we know that, via rearranging

$$E_0^{\mathbb{Q}^N}\left[\frac{X_T}{N_T}\right] = E_0^{\mathbb{Q}^U}\left[\frac{X_T}{U_T}\frac{U_0}{N_0}\right]$$

as both sides are equal to $X_0/N_0$ but by definition

$$E_0^{\mathbb{Q}^N}\left[\frac{X_T}{N_T}\right] = E_0^{\mathbb{Q}^U}\left[\frac{X_T}{N_T}\frac{d\mathbb{Q}^N}{d\mathbb{Q}^U}\right]$$

and so

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}^U} = \frac{U_0 N_T}{U_T N_0} \tag{1}$$

# Numeraire toolkit: Fact 1

- **The price of any asset divided by a numeraire asset is a martingale (i.e. has no drift) under the probability measure associated with that numeraire.**
- As an example, consider the price any non-dividend paying asset $S$ scaled by the bank account $B$, and so

$$\frac{S_t}{B_t} = e^{-\int_0^t r_s ds} S_t$$

must be a martingale under the $\mathbb{Q}$ measure. Assume that $\mu(S_t, t)$ is the drift of $S$ under this measure then

$$\begin{aligned} d\left(e^{-\int_0^t r_s ds} S_t\right) &= S_t(-r_t e^{-\int_0^t r_s ds})dt + e^{-\int_0^t r_s ds} dS_t \\ &= -r_t S_t e^{-\int_0^t r_s ds} dt + e^{-\int_0^t r_s ds} \mu(S_t, t)dt + (\dots)dW_t \\ &= = e^{-\int_0^t r_s ds} \left(-r_t S_t + \mu(S_t, t)\right) dt + (\dots)dW_t. \end{aligned}$$

So, if this is to be a martingale then there must be no drift and $\mu(S_t, t) = r_t S_t$ which is the usual result when using the bank account as the numeraire asset.

# Quiz

When deriving Black's formula, the forward rate $F(t, T, T+\tau)$ can be decomposed into

$$F(t; T + T + \tau) = \frac{P(t, T) - P(t, T + \tau))/\tau}{P(t, T + \tau)}$$

What is the correct choice of numeraire to obtain Black's formula

- $P(t, T)$
- $P(t, T + \tau)$
- $P(t, T) - P(t, T + \tau)$

# Quiz

When deriving Black's formula, the forward rate $F(t, T, T+\tau)$ can be decomposed into

$$F(t; T + T + \tau) = \frac{P(t, T) - P(t, T + \tau))/\tau}{P(t, T + \tau)}$$

What is the correct choice of numeraire to obtain Black's formula

- $P(t, T)$
- $P(t, T + \tau)$ CORRECT
- $P(t, T) - P(t, T + \tau)$

# Numeraire toolkit: Fact 2

- **The time $t$ price, $\pi_t$ under the risk neutral measure $\mathbb{Q}$**

$$\pi_t = E_t^{\boxed{Q}}\left[\boxed{B_t}\frac{\text{Payoff}(T)}{\boxed{B_T}}\right]$$

**is invariant to a change of numeraire. So that for a numeraire $N$, we have**

$$\pi_t = E_t^{\boxed{Q^N}}\left[\boxed{N_t}\frac{\text{Payoff}(T)}{\boxed{N_T}}\right].$$

**If we substitute the symbols inside the boxes of the two equations then the price does not change.**

# Example of changing numeraire

- Let's look at a midterm question from last year to understand how changing the numeraire works. Consider forward rates,

$$\begin{aligned} F_1(t) &= F(t; 0.25, 0.5) \\ F_2(t) &= F(t; 0.5, 0.75) \\ F_3(t) &= F(t; 0.75, 1.0) \\ F_4(t) &= F(t; 1.0, 1.25) \\ &\vdots \end{aligned}$$

where:

$$dF_i = \mu_i(F_i, t) F_i dt + \sigma_i F_i dW_i$$

and where the Brownian motions can be correlated, $dW_i dW_j = \rho_{ij} dt$.

# Example of changing numeraire

i. What choice of numeraire means that the forward rate $F_3(t) = F(t; 0.75, 1.0)$ is a martingale?

- Solution: $P(t, 0.75)$
- Now consider **using** $P(t, 1.25)$ **as the numeraire**.

ii. Consider $A = F_3(t)P(t, 1)$ and $B = P(t, 1)$, what functions of $A$, $B$ and $P(t, 1.25)$ are martingales? What feature of $A$ and $B$ makes this the case?

- Solution:

$$\frac{A}{P(t, 1.25)}, \quad \frac{B}{P(t, 1.25)}$$

are martingales, as long as $A$ and $B$ are traded assets - as they are in this case.

# Example of changing numeraire

iii. By calculating the stochastic processes from part ii. (or otherwise) find $\mu_3(F_3, t)$ when $P(t, 1.25)$ is the numeraire. (Note: You can write your answer using $\sigma_i$ and $F_i$ values for $i \neq 3$)

- Solution:

$$
\begin{aligned}
F_3 \frac{P(t,1)}{P(t,1.25)} &= F_3(1 + F_4\tau) \\
\frac{P(t,1)}{P(t,1.25)} &= 1 + F_4\tau \\
dF_4 &= \sigma_4 F_4 dW_4 \\
dF_3 &= \mu_3(F_3, t)F_3 dt + \sigma_3 F_3 dW_3
\end{aligned}
$$

# Example of changing numeraire

- and so

$$d(1 + F_4\tau) = \tau\sigma_4 F_4 dW_4$$

as it is a martingale

$$d(F_3(1 + F_4\tau)) = (\mu_3 F_3(1 + F_4\tau) + \tau\sigma_4\sigma_3\rho_{34}F_4F_2)dt + (\dots)dW$$

by Ito, given $dW_4 dW_3 = \rho_{34}dt$. So, if this is to be a martingale, we require:

$$\mu_3 = \frac{-\tau\sigma_4\sigma_3\rho_{34}F_4}{1 + F_4\tau}$$

# Example of changing numeraire

iv. Write down the payoff of the caption at $T = 0.75$ in terms of $F_i(t), K, \sigma$ etc.

- Solution:

$$\begin{aligned} Caption_{0.75} &= 0.25N \max\left(P(0.75,1)\max(F_3(0.75)-K,0)\right. \\ &\quad \left.+P(0.75,1.25)\left(F_4(0.75)N(d_1)-KN(d_2)\right),0\right) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(F_4(0.75)/K)+\frac{1}{2}\sigma_4^2\times 0.25}{\sigma_4\sqrt{0.25}} \\ d_2 &= d_1-\sigma\sqrt{0.25} \end{aligned}$$

# Example of changing numeraire

v. Write down its current ($t = 0$) value value in terms of an expectation (you should choose an appropriate numeraire asset).

- Solution:

$$Caplet_0 = P(0, 1.25)E\left[\frac{Caplet_{0.75}}{P(0.75, 1.25)}\right]$$

keeping $P(t, 1.25)$ as the numeraire.

# Example of changing numeraire

vi. Briefly explain how you would value a caption. Be clear as to the stochastic processes followed by your underlying assets, any difficulties with valuation and what numerical scheme (if any) you would prefer to use.

- Solution: You will need to simulate two forward rates until $t = 0.75$ probably with Monte Carlo as one of the drifts is stochastic.
  The first problem is to determine the correlation between the rates, $\rho_{34}$.
  Next, only $F_4(t)$ will be martingale, $F_3$ will have a drift. If we choose $P(t, 1.25)$ as the numeraire, then we can use the drift for $F_3$ that we calculated above.
  Unfortunately, this drift is stochastic which will make it hard to calculate expectations, except via Monte Carlo methods.
  Finally, the forward rates are only evolving until 0.75 not the reset and so we may want to consider a non constant volatility as the caplets only give us volatility until the reset date.

# Girsanov's theorem: Part 1

1. **If $W$ is a Brownian motion under the probability measure $\mathbb{P}$, then there exists an equivalent measure $\mathbb{Q}$ such that $\tilde{W}_t = W_t + \nu t$ is a Brownian motion.**

- At this point in time we will not say how to convert between $\mathbb{P}$ and $\mathbb{Q}$, only that it is possible to do so.

# Example of part 1

- Suppose we have a process:

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where $W$ is a Brownian motion under $\mathbb{P}$ and we wished to change the measure so that the drift was $\gamma_t$instead.

- Under $\mathbb{Q}$ we have

$$dX_t = \mu_t dt - \nu\sigma_t dt + \sigma_t d\tilde{W}_t$$

and so if we choose:

$$\nu = \frac{\mu_t - \gamma_t}{\sigma_t}$$

then

$$dX_t = \gamma_t dt + \sigma d\tilde{W}_t$$

# Quiz

If we wished $dX_t$to have zero drift in the above example, what should $\nu$ equal?

- $\nu = 0$.
- $\nu = \mu_t$.
- $\nu = \mu_t / \sigma_t$
- $\nu = (\mu_t - r_t)/\sigma_t$

# Quiz

If we wished $dX_t$to have zero drift in the above example, what should $\nu$ equal?

- $\nu = 0$.
- $\nu = \mu_t$.
- $\nu = \mu_t / \sigma_t$Correct
- $\nu = (\mu_t - r_t)/\sigma_t$

# Quiz

When we apply Girsanov's theorem in the Black Scholes set-up what is the relationship between the real world Brownian motion, $dW$, and the risk-neutral Brownian motion, $d\tilde{W}$?

- $d\tilde{W} = dW$
- $d\tilde{W} = dW + \mu dt$
- $d\tilde{W} = dW + \frac{\mu}{\sigma} dt$
- $d\tilde{W} = dW + \frac{\mu - r}{\sigma} dt$

# Quiz

When we apply Girsanov's theorem in the Black Scholes set-up what is the relationship between the real world Brownian motion, $dW$, and the risk-neutral Brownian motion, $d\tilde{W}$?

- $d\tilde{W} = dW$
- $d\tilde{W} = dW + \mu dt$
- $d\tilde{W} = dW + \frac{\mu}{\sigma} dt$
- $d\tilde{W} = dW + \frac{\mu - r}{\sigma} dt$ CORRECT

# Real probs → R-N probs

- Consider a very simple two period random walk with, which does not necessarily recombine. To get from time 0 to time 2 there are four possible paths in the table below

| Path | Probabilities |
|---|---|
| $\{0, 1, 2\}$ | $p_1 p_2 = \pi_1$ |
| $\{0, 1, 0\}$ | $p_1(1 - p_2) = \pi_2$ |
| $\{0, -1, 0\}$ | $(1 - p_1)p_3 = \pi_3$ |
| $\{0, -1, -2\}$ | $(1 - p_1)(1 - p_3) = \pi_4$ |

- The $\pi$ values could denote the probability measure $\mathbb{P}$.
- We could also denote the same paths using a different probability measure $\mathbb{Q}$ with path probabilities $\pi_1', \pi_2', \pi_3', \pi_4'$

# Radon-Nikodym derivative

- Now form the ratios of the probabilities $\pi_i'/\pi_i$ which gives us an idea as to how to distort the $\mathbb{P}$ measure to get the $\mathbb{Q}$ measure.
- We write this mapping as $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and it is known as the **Radon-Nikodym derivative** of $\mathbb{Q}$ with respect to $\mathbb{P}$ up to time 2.
- Of course, knowing $\mathbb{P}$ (that is $\pi_i, \dots$) and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ enables us to be able to calculate $\mathbb{Q}$.
- The only problem is if any of the probabilities are 0 as it becomes impossible to use the Radon-Nikodym derivative to switch between probabilities.
- This gives us the standard definition: Two measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be **equivalent** if they agree on what is possible in the same sample space:

$$\mathbb{P}(A) > 0 \Leftrightarrow \mathbb{Q}(A) > 0$$

and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ only exists if the two measures are equivalent.

# Radon-Nikodym derivative

- The Radon-Nikodym derivative has some useful properties which are easiest to see in the discrete setting. Consider a claim (option) with payoffs $x_i$ on path $i$ then under $\mathbb{P}$ its expected value is:

$$E^{\mathbb{P}}[X] = \sum_i \pi_i x_i$$

  where $i$ ranges over the four paths. The expectation with respect to $\mathbb{Q}$ is

$$E^{\mathbb{Q}}[X] = \sum_i \pi_i' x_i = \sum_i \pi_i \left( \frac{\pi_i'}{\pi_i} x_i \right) = E^{\mathbb{P}} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} X \right]$$

- This simple result can be extended to give a useful general result:

$$E_0^{\mathbb{Q}}[X_T] = E_0^{\mathbb{P}} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} X_T \right]$$

- At intermediate times $0 < t < T$ we need to know more about the Radon-Nikodym derivative, which we can do by converting it into a stochastic process, by letting the time horizon vary.

# Radon-Nikodym derivative

- Let $\zeta_t$ be the Radon-Nikodym derivative taken up to the horizon $t$. Thus in our simple example $\zeta_1$ can take values $\pi_1'/\pi_1$ and $(1-\pi_1')/(1-\pi_1)$.
- In general we can write:

$$\zeta_t = E_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

- $\zeta_t$ is very useful as it represents the amount of change of measure so far up to time $t$. So if we wanted to know $E^{\mathbb{Q}}[X_t]$ it would be $E^{\mathbb{P}}[\zeta_t X_t]$ and, more importantly for us, we have:

$$E_s^{\mathbb{Q}}[X_t] = \frac{E_s^{\mathbb{P}}[\zeta_t X_t]}{\zeta_s}$$

# Changing measure in continuous time

- We extend our nice discrete results to continuous time.
- We need to think a little more deeply about probability measures where we have continuous times and states.
- To do this we need all of the marginal distributions at each time $t$ conditional on every history $\mathcal{F}_s$ for all times $s < t$ . We simplify it slightly by considering a large (but finite) set of times $0 = t_0, t_1, \ldots, t_n - 1, t_n = T$ and points $x_1, \ldots, x_n$.
- With one point , $x$and one time, $t_1$ we could use the normal density function:

$$f_1(x) = \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x^2}{2t_1}\right)$$

- Extending it to different points at different times we get

$$f_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \Delta t_i}} \exp\left(-\frac{\Delta x_i^2}{2\Delta t_i}\right)$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta t_i = t_i - t_{i-1}$.

# Changing measure in continuous time

- So in a non-elegant way we can define our Radon-Nikodym derivative as a limit:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \lim_{n\to\infty} \frac{f_n^{\mathbb{P}}(x_1, \ldots, x_n)}{f_n^{\mathbb{Q}}(x_1, \ldots, x_n)}$$

- This still satisfies our earlier conditions:

$$E_0^{\mathbb{Q}}[X_T] = E_0^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X_T\right]$$

$$E_s^{\mathbb{Q}}[X_t] = \frac{E_s^{\mathbb{P}}[\zeta_t X_t]}{\zeta_s}$$

# Changing measure in continuous time: In practice

- Consider a Brownian motion, $W_t$, under the probability measure $\mathbb{P}$. Now, as if by magic, define an equivalent probability measure, $\mathbb{Q}$ such that:
$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\nu W_T - \frac{1}{2}\nu^2 T)$$
- Before progressing, let's consider what a normally distributed random variable looks like. A random variable $X$ is normally distributed (with mean $\mu$ and variance $\sigma^2$, if and only if
$$E[\exp(\theta X)] = \exp(\theta\mu + \frac{1}{2}\theta^2\sigma^2)$$
- Given that, let's try to calculate the expectation of $\theta W_T$ under the measure $\mathbb{Q}$:
$$\begin{aligned} E^{\mathbb{Q}}[\exp(\theta W_T)] &= E^{\mathbb{P}}\left[\exp\left(-\nu W_T - \frac{1}{2}\nu^2 T + \theta W_T\right)\right] \\ &= \exp\left(-\frac{1}{2}\nu^2 T + \frac{1}{2}(\theta-\nu)^2 T\right) \end{aligned}$$

# Changing measure in continuous time: In practice

or,

$$E^{\mathbb{Q}}[\theta W_T] \quad = \quad \exp\left(-\theta\nu T - \frac{1}{2}\theta^2 T\right)$$

# Changing measure in continuous time: In practice

- This is the same as a normally distributed variable with mean $-\nu T$ and variance $T$. Thus the distribution of $W_T$ under $\mathbb{Q}$ is also normal, only now with mean $-\nu T$
- This also holds for $W_t$ which is Brownian motion with respect to $\mathbb{P}$ and a Brownian motion with constant drift $-\nu$ under $\mathbb{Q}$. So, $W_t$ is a Brownian motion (with the same volatility) under both measures!
- This seems remarkable, but it should not be surprising. The possible points on the path are the same (as the measures are equivalent) what the change of measure does is adjust the probabilities of a particular path being chosen. So, under $\mathbb{Q}$ the drift is negative and so paths that end up with negative values are more likely than they were under $\mathbb{P}$.
- These results are formalized in Girsanov's theorem, now the full version

# Girsanov's theorem

1. If $W$ is a Brownian motion under the probability measure $P$, then there exists an equivalent measure $\mathbb{Q}$ such that $\tilde{W}_t = W_t + \nu t$ is a Brownian motion.
2. Let $W$ be a Brownian motion under a probability measure $P$, then under an equivalent measure:

$$\mathbb{Q}(A) = E\left[1_A \frac{d\mathbb{Q}}{dP}\right]$$

$W$is a Brownian motion with drift $-\nu$. Or $\tilde{W} = W + \nu T$is a Brownian motion. (A is an arbitrary set of paths (possible values of $W_T$) and $1_A$ is an indicator which takes the value 1 only when path $A$ occurs.)

- In this case,

$$\frac{d\mathbb{Q}}{dP} = \exp\left(-\int_0^T \nu_t dW_t - \frac{1}{2}\int_0^T \nu_t^2 dt\right)$$

# Example of part 2

- Consider switching between measures:

$$P(W_T < x) = P(N(0,1) < x/\sqrt{T}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{T}} e^{-s^2/2} ds$$

transforming,

$$P(W_T < x) = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{x} e^{-s^2/2T} ds$$

and so,

$$\begin{aligned} \mathbb{Q}(W_T < x) = E[1_{W_T<x} M(T)] &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{x} e^{-\frac{s^2}{2T}} e^{-\frac{1}{2}\nu^2 T - \nu s} ds \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{x} e^{-\frac{(s+\nu T)^2}{2T}} ds \end{aligned}$$

- Simply changing $r = s + \nu T$ shows that this is the same as

$$P(W_T < x + \nu T).$$

# Example of part 2

- Thus the probability that $W_T < x$ in the new measure is equal to the probability that $W_T - \nu T$ is less than $x$ in the old measure. That is $W_T$ is a Brownian motion with drift $-\nu$ in the new measure, as required.

# Girsanov's theorem: Comprehension check!

- Recall that when,

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t \\
dB_t &= rB_t dt
\end{aligned}
$$

  then to make $S_t/B_t$ a martingale, we introduce a new measure $\mathbb{Q}$ under which $\tilde{W} = W_t - \nu t$ is a Brownian motion and $\nu = -\frac{\mu - r}{\sigma}$.

- More formally, there exists an exponential martingale $M_t$ where

$$dM = \nu M dW$$

  and

$$M_t = \exp\left(-\frac{1}{2}\nu^2 t + \nu W(t)\right) = \left(= \frac{d\mathbb{Q}}{d\mathbb{P}}\right)$$

  and if $W$ is a Brownian motion under $\mathbb{P}$ then under $\mathbb{Q}$ we have that under $Prob(A) = E\left[1_A M_t\right]$, under $\mathbb{Q}$. So, $W$ is Brownian motion with drift term $\nu$.

- This $M_t$ is also known as the Radon-Nikodym derivative (see above).

# Girsanov's theorem: Theorem

Consider a stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

under $\mathbb{P}$. Given a new drift $\mu^*(x)$ and assuming that $(\mu^*(x) - \mu(x))/\sigma(x)$ is bounded. Define the measure $\mathbb{Q}$ by

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} = \quad & \exp\left\{ -\frac{1}{2}\int_0^t \left( \frac{\mu^*(X_s) - \mu(X_s)}{\sigma(X_s)} \right)^2 ds \right. \\ & \left. + \int_0^t \frac{\mu^*(X_s) - \mu(X_s)}{\sigma(X_s)} dW_s \right\} \end{aligned}$$

then $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and, most importantly

$$dW_t^* = -\left[ \frac{\mu^*(X_t) - \mu(X_t)}{\sigma(X_t)} \right] dt + dW_t$$

is a Brownian motion under $\mathbb{Q}$ and

$$dX_t = \mu^*(X_t)dt + \sigma(X_t)dW_t^*$$

# Numeraire toolkit: Fact 3

- There are two important equations here

$$\mathrm{drift}_{\mathrm{asset}}^{\mathrm{Num\ 2}} = \mathrm{drift}_{\mathrm{asset}}^{\mathrm{Num\ 1}} - \mathrm{vol}_{\mathrm{asset}} \mathrm{Corr} \left( \frac{\mathrm{Vol}_{\mathrm{Num\ 1}}}{\mathrm{Num\ 1}} - \frac{\mathrm{Vol}_{\mathrm{Num\ 2}}}{\mathrm{Num\ 2}} \right)^T .$$

or

$$\mu_t^U(X_t) = \mu_t^N(X_t) - \sigma_t(X_t)\rho \left( \frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t} \right)^T$$

where there are $n$ Brownian motions driving the asset prices and

- $\mathrm{vol}_{\mathrm{asset}}$ ($\sigma_t(X_t)$) is the $1 \times n$ vector corresponding to that asset. Or an $n \times n$ matrix if you are considering all assets simultaneously.
- $\mathrm{Corr}$ ($\rho$) is the $n \times n$ correlation matrix explaining the movements between the $n$ Brownian motions.
- $\mathrm{Vol}_{\mathrm{Num\ i}}$ ($\sigma_t^U$ or $\sigma_t^N$) is the $1 \times n$ vector of volatilities for that numeraire asset.
- $\mathrm{Num\ i}$ ($U_t$ or $N_t$) is the current value of that numeraire asset.

# Numeraire toolkit: Fact 3

- So that we can calculate the drift of an asset when moving between numeraire 1 and numeraire 2. This will depend upon the asset volatility (unchanged under the change of numeraire) as well as the instantaneous correlations and the volatility and values of the two numeraire assets.
- This can be rewritten in terms of the random, Brownian motion, shocks. If each of the assets feature the same multi-dimensional Brownain motion shock under a given measure. Then as we change measure the random shock will no longer be driftless but will acquire a drift as follows:

$$\text{Brownian Shocks}^{\text{Num 2}}_{\text{Corr}} = \text{Brownian Shocks}^{\text{Num 1}}_{\text{Corr}} - \text{Corr}\left(\frac{\text{Vol}_{\text{Num 2}}}{\text{Num 2}} - \frac{\text{Vol}_{\text{Num 1}}}{\text{Num 1}}\right)^T.$$

or

$$\boxed{CdW^U_t = CdW^N_t - \rho\left(\frac{\sigma^U_t}{U_t} - \frac{\sigma^N_t}{N_t}\right)^T dt}$$

# Numeraire toolkit: Fact 3

- This fact requires a little more explanation than Facts 1 and 2.
- Consider a general numeraire $N$ and an $n$-vector process for assets $X$ under $\mathbb{Q}^N$

$$dX_t = \mu_t^N(X_t)dt + \sigma_t(X_t)CdW_t^N$$

  where here $\mu_t^N$ is an $n \times 1$ vector, $\sigma_t$ is a $n \times n$ diagonal matrix, $W^N$ is a $n$-dimensional standard Brownian motion $(n \times 1)$ and $C$ is a $n \times n$ matrix used to model the correlation between the Brownian motions, where $\rho = CC^T$.

- Now we wish to write the process followed by the assets $X$ under the new numeraire $U$

$$dX_t = \mu_t^U(X_t)dt + \sigma_t(X_t)CdW_t^U$$

- To do this, we will need to resort to Girsanov's theorem.

# Aside: proof of Fact 3

- Now extend Girsanov's theorem to multiple assets and Brownian motions

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}^U} = \exp\left\{-\frac{1}{2}\int_0^t \left|(\sigma_s(X_s)C)^{-1}[\mu_s^N(X_s) - \mu_s^U(X_s)]\right|^2 ds + \int_0^t \left\{(\sigma_s(X_s)C)^{-1}[\mu_s^N(X_s) - \mu_s^U(X_s)]\right\}^T dW_s^U\right\}$$

assuming that $C$ is an invertible or full-rank matrix.

- The process $M_t = \frac{d\mathbb{Q}^N}{d\mathbb{Q}^U}$ will be an exponential martingale,

$$dM_t = \alpha_t M_t dW_t^U \tag{2}$$

where

$$\alpha_t = [\mu_s^N(X_s) - \mu_s^U(X_s)]^T(\sigma_s(X_s)C)^{-1\,T}$$

- From earlier, equation 1, we also know that

$$M_T = \frac{U_0 N_T}{N_0 U_T}$$

# Aside: proof of Fact 3

- and so

$$M_t = E_t^{\mathbb{Q}^U}[M_T] = E_t^{\mathbb{Q}^U}\left[\frac{U_0 N_T}{N_0 U_T}\right] = \frac{U_0 N_t}{N_0 U_t} \tag{3}$$

- Now differentiating

$$dM_t = \frac{U_0}{N_0} d\left(\frac{N_t}{U_t}\right) = \frac{U_0}{N_0}\sigma_t^{N/U} C dW_t^U \tag{4}$$

where, as $N/U$ is a martingale, we have assumed the following dynamics:

$$d\left(\frac{N_t}{U_t}\right) = \sigma_t^{N/U} C dW_t^U$$

where $\sigma_t^{N/U}$ is a $1 \times n$ vector which we will determine later. By comparing our two definitions of $M_t$ we have

$$\alpha_t M_t = \frac{U_0}{N_0}\sigma_t^{N/U} C$$

# Aside: proof of Fact 3

- By substituting the value of $M_t$ from equation 3 then we have that

$$\alpha_t = \frac{U_t}{N_t}\sigma_t^{N/U} C$$

or, by some arduous rearranging

$$\boxed{\mu_t^U(X_t) = \mu_t^N(X_t) - \frac{U_t}{N_t}\sigma(X_t)\rho(\sigma_t^{N/U})^T} \tag{5}$$

where, by definition, $\rho = CC^T$ and this gives the change in the drift of a stochastic proces when changing from numeraire $U$ to $N$.

# Aside: proof of Fact 3

- Now let's specify the processes followed by $U$ and $N$ under $\mathbb{Q}^U$ which, so far have been general.

$$\begin{aligned} dN_t &= (\dots)dt + \sigma_t^N C dW_t^U \\ dU_t &= (\dots)dt + \sigma_t^U C dW_t^U \end{aligned}$$

where $\sigma_t^N$ and $\sigma_t^U$ are $1 \times n$ vectors, $W_u$ is an $n$-dimensional driftless standard Brownian motion (under $\mathbb{Q}^U$) and $CC^T = \rho$ (all $n \times n$). Then the drift of $X$ under $U$ is

$$\boxed{\mu_t^U(X_t) = \mu_t^N(X_t) - \sigma_t(X_t)\rho \left( \frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t} \right)^T}$$

- To show this we need to look at equation 5 and

$$d\frac{N_t}{U_t} = \frac{1}{U_t} dN_t + N_t d\frac{1}{U_t} + dN_t d\frac{1}{U_t}$$

where

$$d\frac{1}{U_t} = -\frac{1}{U_t^2} dU_t + \frac{1}{U_t^3} dU_t dU_t.$$

# Aside: proof of Fact 3

- By comparing coefficients of the $CdW_t^U$ terms we have

$$\sigma_t^{N/U} = \frac{\sigma_t^N}{U_t} - \frac{N_t}{U_t}\frac{\sigma_t^U}{U_t}$$

- Thus

$$\begin{aligned} \mu_t^U(X_t) &= \mu_t^N(X_t) - \frac{U_t}{N_t}\sigma(X_t)\rho(\sigma_t^{N/U})^T \\ &= \mu_t^N(X_t) - \frac{U_t}{N_t}\sigma(X_t)\rho\left(\frac{\sigma_t^N}{U_t} - \frac{N_t}{U_t}\frac{\sigma_t^U}{U_t}\right)^T \\ &= \mu_t^N(X_t) - \sigma_t(X_t)\rho\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)^T \end{aligned}$$

  as required.

- Also, we can recover the second result from Fact 3

$$\boxed{CdW_t^U = CdW_t^N - \rho\left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^N}{N_t}\right)^T dt}$$

# Quiz

In the Vasicek model, under the standard risk-neutral measure (with $B(t)$ as numeraire)

$$dr = k[\theta - r(t)]dt + \sigma dW^B.$$

The bond price is given by

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

What is the process followed by $dr$ with the bond as numeraire?

- $dr = k[\theta - r(t)]dt + \sigma dW^P$
- $dr = [k\theta + \sigma B(t, T) - kr(t)]dt + \sigma dW^P$
- $dr = [k\theta - B(t, T)\sigma^2 - kr(t)]dt + \sigma dW^P$

What is the relationship between $dW^P$ and $dW^B$?

# Quiz

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- $dr = [k\theta + \sigma B(t, T) - kr(t)]dt + \sigma dW^P$
- $dr = [k\theta - B(t, T)\sigma^2 - kr(t)]dt + \sigma dW^P$ CORRECT

What is the relationship between $dW^P$ and $dW^B$?

# How do we model caplet volatility

- From Lecture 2 material we know that the estimate of the annual volatility $\sigma_{T_i}^{caplet}$ is equivalent to the root mean square volatility between time 0 and the start of the reset date $T_{i-1}$,

$$\sigma_{T_i}^{caplet\,2} T_{i-1} = \int_0^{T_{i-1}} \sigma^2(t) dt$$

  where $\sigma^2(t)$ is the potentially time varying instantaneous volatility function.

- As part of our development of market models we will need to specify a type of function for this volatility. Some non-parametric choices are to choose a piecewise constant functions depending only on time to maturity, piecewise constant functions depending upon the maturity time of the forward rate, or more complex piecewise constant functions.

# Quiz

Would you expect the implied volatility to be the same for all caplets regardless of the exercise price, $K$?

- Yes.
- No.
- It depends.

Why?

# Quiz

Would you expect the implied volatility to be the same for all caplets regardless of the exercise price, $K$?

- Yes.
- No.
- It depends. CORRECT

Why?

# Extra considerations for caplet volatility

- There are also parametric choices, the most usual to use a humped volatility function of the form:

$$\sigma(t) = (\alpha + \beta t)e^{-\lambda t} + \mu$$

  and variations on this, about which we shall discuss in great detail in the future.

- These caplet volatilities (or the volatilities of the appropriate forward rate) are only reported for the at-the-money caps. There are naturally more caps on the market with different cap levels (or exercise prices).

- One verification of the Black model (or the assumption of lognormal forward rates) is to see if caps with different exercise prices all return the same implied volatilities, as we also do with the Black-Scholes model.

# Extra considerations for caplet volatility

- As with the Black-Scholes model, we find a *volatility smile* effect for the implied volatilities of caplets. Usually this manifests as large volatilities for low strikes ($K$) with the volatilities generally decreasing as the strike price ($K$) increases.
- There is an intuitive explanation for this, as there is an absolute component to the movement of interest rates which does not exist in the equity markets setting. When interest rates are low they do move by smaller amounts but not proportionally. So, when interest rates are low, as they are now, they move around a lot more than would be predicted by a lognormal model.
- We will consider possible for models for this as extensions to the market model. Popular models include CEV models, SABR, and stochastic volatility models.

# Finding square roots of the covariance matrix

- We will eventually be modeling many correlated forward rates and we will need techniques to simulate or construct these correlated paths.
- This will require an entire vector of correlated Brownian motions from a given covariance matrix. To construct such a vector we start with the covariance matrix between the forward rates. In general, this will be

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{N1} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_{NN} \end{pmatrix}$$

- For example if we have two correlated processes, this could be written as

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

# Finding square roots of the covariance matrix: Cholesky factorization

- We wish to write the process followed by the two assets in the following form

$$\begin{aligned} dX_1/X_1 &= \mu_1 dt + a_{11} dW_1 + a_{12} dW_2 \\ dX_2/X_2 &= \mu_2 dt + a_{21} dW_1 + a_{22} dW_2 \end{aligned}$$

  where $dW_1$ and $dW_2$ are two independent Brownian motions.

- To determine $a_{ij}$ we need to find a square root of the covariance matrix. As $\Sigma$ is symmetric and generally strictly positive definite then there will be a solution to

$$CC^T = \Sigma$$

  in fact, there will be many solutions.

# Finding square roots of the covariance matrix: Cholesky factorization

- One solution is to restrict $C$ to be a lower triangular matrix, where it can be found by Cholesky factorization. The matrix will look like

$$C = \begin{pmatrix} c_{11} & 0 & \dots & 0 \\ c_{12} & c_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{1N} & c_{2N} & \dots & c_{NN} \end{pmatrix}$$

- The algorithm is that, $c_{11} = \sqrt{\sigma_{11}}$ and then

$$\begin{aligned} c_{ii} &= \sqrt{\sigma_{ii} - \sum_{k=1}^{i-1} c_{ki}^2} \\ c_{ij} &= \frac{\sigma_{ij} - \sum_{k=1}^{i-1} c_{ki} c_{kj}}{c_{ii}} \end{aligned}$$

# Cholesky factorization: return to our example

- For our simple $2 \times 2$ matrix we obtain

$$
\begin{aligned}
c_{11} &= \sigma_1 \\
c_{12} &= \rho\sigma_2 \\
c_{22} &= \sqrt{1-\rho^2}\sigma_2
\end{aligned}
$$

- Thus

$$
\begin{aligned}
dX_1/X_1 &= \mu_1 dt + \sigma_1 dW_1 \\
dX_2/X_2 &= \mu_2 dt + \rho\sigma_2 dW_1 + \sqrt{1-\rho^2}\sigma_2 dW_2
\end{aligned}
$$

as you have (presumably) seen before.

# Finding square roots of the covariance matrix: Principal Component Analysis

- Assume that the covariance matrix is an $N \times N$ matrix. From spectral theory then there exist a basis of eigenvectors (column) $\mathbf{e}_1, \ldots, \mathbf{e}_n$ such that

$$\Sigma \mathbf{e}_j = \lambda_j \mathbf{e}_j$$

for some $\lambda_j \geq 0$ (called eigenvalues), and the vectors $\mathbf{e}_j$ are orthonormal. Let $P$ be the matrix with the $j$th column equal to $\mathbf{e}_j$, then as $\mathbf{e}_j$ are orthonormal we have

$$PP^T = P^T P = I$$

and so

$$\Sigma = P \hat{\Sigma} P^T$$

where $\hat{\Sigma}$ is a diagonal matrix with the eigenvalues $\lambda_j$ on the diagonal.

# Principal Component Analysis: additional remarks

- We can rewrite this in order so that $\lambda_j \geq \lambda_{j+1}$ then if we have a collection of independent Brownian motions $W = W_1, W_2, \ldots, W_N$ then the random $(N \times 1)$ vector

$$B = P\sqrt{\hat{\Sigma}}W$$

- Where $B$ is a multidimensional Gaussian random variable with covariance matrix $\Sigma$ as in the case of Cholesky factorisation.
- When $\lambda_1$ (or even $\lambda_1, \lambda_2$, and $\lambda_3$) is much larger than the rest of the eigenvalues, then we can reduce the number of required Brownian motion terms, in this case the random vector $B$ above can be accurately represented by the random vector

$$\tilde{B} = P\sqrt{\hat{\Sigma}}W$$

where now $W$ is an $3 \times 1$ vector, $\sqrt{\hat{\Sigma}}$ is a $3 \times 3$ matrix and $P$ is a $N \times 3$ matrix with columns corresponding to the the three eigenvectors, where $\tilde{\Sigma} = P\hat{\Sigma}P^T$ approximates the covariance matrix.

# Principal Component Analysis: additional remarks

- This technique can be useful in reducing the number of independent processes that need to be modeled in computationally intensive cases like Bermudan swaption pricing.

# Conclusion

- We have developed techniques to change the numeraire of asset processes. Most importantly we have formulas to calculate the drifts under different measures. This will be vital for the LMM and also for two factor models.
- Finally, we have looked at how to determine square roots of covariance matrices and saw that it may be possible to reduce the rank of the matrix by using principal components analysis.