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Lecture Note 1: Forwards and Futures

This lecture note examines the pricing and hedging of forwards and futures contracts. For some of them, no arbitrage pricing is easy. But not always.

I know you are already familiar with these instruments. Our goal is to make sure you know how the markets work in practice and see precisely how and where the assumptions of the theory may be violated.

Outline:

I. Definitions and Contractual Features

II. Pricing Forward Contracts
  (**A**) currencies
  (**B**) stock indexes
  (**C**) commodities
  (**D**) anything ?

III. An Example

IV. More on Shorting Stocks.

V. Futures vs. Forwards
  (**A**) pricing
  (**B**) hedging

VI. Summary

I. Definitions.

(A) Forward Contracts

- A forward contract is an agreement between two parties to exchange something at a prespecified price on a prespecified date.
    - ▶ The person who agrees to buy is said to be **long**, and the person who agrees to sell is said to be **short**.
- Contract specifications must include:
    1. Amount and quality of good to be delivered; mechanics of delivery (when/where).
    2. Price (F) and currency;
    3. Time of delivery (T) also called settlement or maturity date;
    4. Remedies in the event of default; collateralization (if any).
- Actually, forward transactions are not intrinsically different from "spot" trades. In fact, most "spot" trades of assets *do* specify a few days ahead for settlement.
- **Q:** What "engineering" problem do you think forward contracts were designed to solve?

- Forward contracts are almost all traded in over-the-counter (OTC) markets, which means that there is no centralized exchange.
  - Instead, participants interact through telephone, web, and other communications systems, on a bilateral basis.
  - Regulators want this to change to a system based on "swap execution facilities" (SEFs) which are meant to look like electronic exchanges. More on this later.

- Properties of Forward contracts:
  - Customizable
  - No reporting of trades (for now)
  - No public prices (for now)
  - May not be assignable
  - No realization of gains/losses prior to T
  - Credit risk

(B) Futures Contracts

- Futures contracts are like forward contracts in that they are an agreement between two parties to buy (sell) something at a prespecified date for a pre-agreed price.
  - ▶ However, whereas forward contracts are privately arranged between, say, two corporations or a corporation and a bank (i.e., between two professional players), futures contracts are:
    - ∗ Exchange traded and regulated.
    - ∗ Standardized;
    - ∗ Marked to market;
    - ∗ Cleared via a single counterparty.
  - ▶ The point of this distinct form is:

    1. To enable numerous unacquainted agents to trade the same contract;

    2. "Out-source" credit/legal risk to the clearing house.
- The exchange specifies a standard set of delivery dates, and other legal details for all contracts.
  - ▶ Then all trades are reported and the market is transparent.
  - ▶ Rather than bilateral negotiation, prices are just set by all the public orders.

- The "clearing house" is a highly-capitalized, safe firm that assumes the other side of every trade.
  - Suppose you and I decide to trade 10000 bushels of soybeans for July delivery at a price of $f_0 = \$6.65$ per bushel, and that you are selling and I am buying.
  - Then, technically, that one trade is split into two trades: the clearing house agrees to buy from you at that price and sell to me.
  - Neither of us has to worry about the other's creditworthiness before doing the trade.
  - And the two of us can forget about each other after the trade. If I later unwind my position by going short (with someone else) my two trades (long and short) **net out**.

Q: Aren't futures a more efficient product than forwards? Why do you think both still exist?

- What is marking-to-market?
  - ▶ Every day the exchange uses the closing price to value everybody's account AND *winners get paid (and losers pay) immediately on the same day*.
  - ▶ So, continuing the above example, if July soybeans close today at $f_1 = 6.92$, then you have to pay the clearing house 0.27 X 10000 = 2700.
    - ∗ The settlement price of your position is also re-set to 6.92.
    - ∗ And this revaluation will continue for every day that you have an open position.
    - ∗ So if tomorrow's close is f_2, then you will have cash-flow of $-(f_2 - 6.92) \times 10000$ then.
  - ▶ This requirement is the clearing house's way of making sure that it is never exposed to too much risk of its customers defaulting.
    - ∗ If a customer defaults and they close out his position, they can lose at most one-day's profit/loss.

- Notice that marking to market may be *very* inconvenient if you are a farmer with a short position hedging your crop who *will have the soybeans* at harvest time but doesn't necessarily have any more cash today if prices go up.

  * You will then want to have access to someone who will loan you money.

  * Your broker may loan you money but you will still have to have some collateral – called "margin" – and he will want more margin when you have to pay out money to the exchange to cover mark-to-market losses.

  * But don't confuse your broker's "margin call" with the exchange's mark-to-market.

    - Putting money into your margin account is not a cash out-flow; it's just moving your own money around.

    - But mark-to-market cash-flows are actual gains and losses.

- About cash settlement:
  - ▶ On the very last day of a contract's life, somebody who is long receives the actual good and pays the previous day's marking price. That's called physical settlement.
  - ▶ If the spot price is S_T, then the economic value of this last cash-flow is $S_T - f_{T-1}$, which would be realized by selling the physical good after it is delivered.
  - ▶ A very natural and easy modification would be to eliminate the need for the physical transaction and simply pay the long position $S_T - f_{T-1}$ in money. (And minus that to the short side, of course.) That's cash settlement.
  - ▶ This makes the last day just like every other day in the contract's life, with the spot price replacing the futures closing price for marking-to-market purposes.
  - ▶ Since most people trade these instruments for their economic exposure properties (and not to actually receive/deliver the underlying product) this saves lots of hassle.
  - ▶ Note that, for the cash-settlement to truly work (that is, to be *economically* the same as physical settlement), it has to be the case that the reference price that is used really IS the price at which someone could sell/buy a position in the physical good.

- One you understand the mechanism of cash-settlement, however, the door is open to an entirely new class of futures contracts: **non-physical underlyings**.
  - ▶ We can trade in a contract on any index or statistic!
  - ▶ Such a statistic might be:
    - ∗ Daily temperature at the Tokyo airport.
    - ∗ Fed Funds rate.
    - ∗ Number of seats won by a particular party in an election.
    - ∗ Next month's inflation rate.
    - ∗ The total payouts to all home-owner insurance policies in Florida.
  - ▶ If the statistic, X_T, can be determined in a reliable and fair way, then we will just agree that the final cash-flow will be $X_T - f_{T-1}$ on date T.
  - ▶ Notice that there is no way you could have physical settlement for any of the above examples. There is no "good" to be delivered.

- Examples of cash settlement:

  Eurodollar futures: The eurodollar contract is used to hedge movements in LIBOR, the 90-day interbank dollar rate in London. On the settlement day of the futures contract, the exchange takes the average rate offered by a sample of large banks at noon.

  ▶ This is likely to change in the near future!

  Stock index futures: Contracts on indexes like the Dow-Jones 30 or the S&P 500 are used to bet on the overall movement in the market, without having to trade the component stocks. On settlement date, the exchange computes the final price of each stock in the index (usually averaged over some period like the last hour), and then weights these according to the index weights.

  Freight forwards: Shipping companies can hedge fluctuations in the costs of operating "Supramax" cargo ships on particular routes. The Baltic Shipping Exchange publishes indexes every month of actual costs on 50 different routes. These are used to settle *forward freight agreements.*

  ▶ In December 2012 the People's Bank of China authorized yuan denominated freight contracts to be settled by the Shanghai Shipping Exchange.

II. Forward Prices

- By convention, forward contracts (and futures) are entered into without either side paying the other anything up-front. Our task now is to describe mathematically the forward price, $F(\cdot)$, at which both sides will agree the contract is worth zero today.

- In otherwords, if soybeans today cost \$6.21 a bushel, what is the correct price for June-delivery soybeans?

- Let's answer this by looking at foreign exchange forwards first.

(A) Pricing currency forwards.

- Example (perfect markets):

  ▶ Spot ¥/\$ exchange rate is 120.00.
  One year (annually compounded) ¥interest rate is 0.15%.
  One year (annually compounded) \$ interest rate is 0.30%.

  ▶ ¥120.00 today = ¥120.18 in 1 year.
  ¥120.00 today = \$ 1.00 today
  \$ 1.00 today = \$ 1.0030 in 1 year.

  ▶ Notice that – even without a forward contract – we can already enter into trades that *transform future-dollars to future yen* at a guaranteed rate.

  1. Borrow dollars;
  2. Sell them for yen;
  3. Invest the yen.

  ▶ Here's the picture:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{¥}120.00 \arrow[r, shift right, "0.15\%"'] \arrow[d, shift right, "120.00\ \text{¥}/\$"'] & \text{¥}120.18 \arrow[l, shift right] \arrow[d, shift right, "?"'] \\
\$\ 1.00 \arrow[u, shift right] \arrow[r, shift right] & \$\ 1.0030 \arrow[l, shift right, "0.3\%"'] \arrow[u, shift right]
\end{tikzcd}
$$

Now **1 Year from now**

- If there is a forward contract, there would then be another *completely equivalent* way of converting future-dollars into future-yen.

- So it better be the case that forward rate ensures :
  $ 1.00 in 1 year = 120.00 X 1.0015/1.003 = ¥119.82 in 1 year.

  * The two methods better give the same result or something is seriously wrong.

- In math,

$$F_{t,T}(S_t, R^{¥}, R^{\$}) = S_t \times \left[\frac{1 + R^{¥}}{1 + R^{\$}}\right].$$

* Using continuously compounded interest rates, this expresssion is

$$F_{t,T}(S_t, r^{¥}, r^{\$}) = S_t \times e^{(r^{¥}-r^{\$})(T-t)}$$

  where $(T-t) = 1$ in the example.

* This formula assumes both S and F are quoted in yen-per-dollar. If we want to view FX rates like prices of any other commodity, this convention means we are viewing the dollar as the "good" and using yen as our numeraire.

▶ Note that using continuously compounded rates does not mean we are assuming the interest is actually *paid* continuously.

* The r s are the rates that apply for borrowing/lending to time T with all cash flows at the end.

* Also Note that we do not have to assume a flat term structure here. Sometimes I will distinguish rates at different maturities with notation like $r_{t,T}$

▶ We can also write the formulas above in terms of the prices of riskless zero coupon bonds maturing at T:

$$F_{t,T}(S_t, B^{¥}_{t,T}, B^{\$}_{t,T}) = S_t \cdot B^{\$}_{t,T} / B^{¥}_{t,T}$$

since, in either currency, $B_{t,T} = e^{-r_{t,T}\ (T-t)}$.

▶ Now, what did we mean precisely when we said the formula "better hold....or something is wrong"?

- The no-arbitrage argument formalized.
  - ▶ Suppose the forward price were *anything other* than 119.82. Call it 119.00. As long as this case, we can get free money. How?
    - ∗ Buy the thing that is too cheap (forward dollars).
    - ∗ Sell the thing that's too rich (spot dollars).
    - ∗ Put the yen in the bank.
    - ∗ Borrow the dollars to deliver. (Use yen as collateral).
    - ∗ Then . . . do nothing!
  - ▶ Arbitrage Table:

Transaction	Cashflow (at t)	Cashflow (at T)
Buy 1.00 dollars forward	0	-119.00 yen + 1 dollar
Lend 119/1.0015 yen	-118.82 yen	+119.00 yen
Borrow 1/1.003 dollars	+.9970 dollars	-1 dollar
Sell .9970 dollars spot @ 120.00	+119.64 yen - 0.9970 dollars	0
Total	+0.82 yen	0 yen , 0 dollars

  - ▶ Do this times 1,000,000,000,000,000,000,000!
  - ▶ If instead of 119.00, the forward price were 121.00, we'd reverse these trades and again make free money.
  - ▶ We could now formally **prove** our pricing law by similarly showing that ANY number other than $S\ [\frac{1+R^{¥}}{1+R^{\$}}] = 119.82$ likewise implies arbitrage profits.

▶ The strategy implemented in our table above is called a "cash and carry" arbitrage trade.

∗ We replicate a forward position with a "cash" position, and then just "carry" it into the future.

▶ Notice that the replicating strategy for the forward:

involves a spot position and two bond positions. The government bonds (or riskless bank deposits) and the foreign currency are the underlying securities.

is perfect. The cashflows from the underlying assets *exactly* off-set those from the forward *regardless of where spot is at settlement.*

is available. Nothing stops us from doing it.

is static. No intermediate transactions required.

has known price today. We're taking spot exchange rate and both interest rates as given.

- Problems with the argument?

▶ Are there any reasons why we might ever **not** observe

$$\text{Forward} = \text{Spot} \times e^{(r_{t,T}^{domestic} - r_{t,T}^{foreign})(T-t)}$$

in real life?

1. Borrowing/lending problems?

- If term-loan markets don't exist, the right side of our formula won't be meaningful (e.g. for small countries).
- Notice that the borrowing we did was fully collateralized. We did not make unrealistic assumptions about our credit.

2. Transactions costs?

Q: How big are they? Whose matter?

Q: What do they do to the argument?

⋆ "The Paradigm" (from last lecture) amended:

> Something's worth more than the least it costs *someone* to replicate it (=the most it costs to hedge it), and less than the most it costs *anyone* to replicate it.

Q: What can we say about what the forward price should be inside the no-arbitrage band?

3. Credit risk?

- We calculated the forward price for two counterparties who *knew with certainty* that each would fulfill his obligation.

4. Note some other implicit assumptions:

- Free capital flows.
- Enforcibility of contracts!

(B) Forward Price of a Stock Index.

- How would we determine the forward price for a basket of stocks that make up an index?
  - ▶ Let's suppose the forward is a physical delivery contract, and all the stocks are freely tradeable and liquid.
  - ▶ Then the basic argument is the same as for currencies: "cash and carry".
- This time, let's consider the upper and lower arbitrage limits separately.
- First we'll we figure out how much it will cost to buy and hold the basket till delivery
  - ▶ The (future value) cost of doing that must be the upper bound on F. (Call it $\overline{F}$.)
    - ∗ If someone ever wanted to pay more than $\overline{F}$, we would agree to go short to them, then buy all the shares and hold them, and make riskless profit.
- As usual, we start out neglecting transactions costs.

- **Buy-and-hold cost:**
  - ▶ The cost of buying a unit of the index in the spot market is assumed known: S_t.
    - ∗ I'm using S to denote the sum of all the individual stock prices times their weight in the index.

- We borrow to pay for it at r using the shares as collateral – just as we did with yen.
  - $*$ Net cash-flow today: 0.
  - $*$ Amount owed at T: $S_t \cdot e^{r(T-t)}$.
  - $*$ As in the currency example, r is being expressed in continuous compounded form for mathematical convenience but we are assuming the interest is only due at T.
    - So r here means the time-T rate we can lock in today.
    - We are not assuming this rate is constant over time or the same for all T.
- We also get any dividends paid between now and T.
  - $*$ For now, **approximate** the stream of dividends we get from all the individual stocks as a continuous **percentage** flow at the rate d per unit time.
  - $*$ This is a bit crude. We will relax it later to think about discrete payouts.
  - $*$ For large baskets of stocks, like the S&P 500, the dividends really are fairly smoothly spread out over time.
  - $*$ With this approximation d is just like the foreign interest rate: the continuous yield on holding a unit of the underlying good.
  - $*$ Note that we are also assuming d is known today.

- This is a reasonable assumption for U.S. companies up to about a year ahead.

▶ As with the currency case, we can use the yield we will get from the "good" to reduce the *amount* of it that we need to hold today.

▶ Instead of buying 1.0 units of the index, buy $1/e^{d(T-t)}$ units and reinvest in shares as the dividends are received.

∗ This ensures that we will have exactly 1.0 unit at T.

- Why? Because, if N_t denotes our position size, the dividends per unit time are worth $d\ S_t\ N_t\ dt$. Reinvesting means that our position size grows by $dN_t = d\ N_t\ dt$. And the solution to that equation is $N_t = e^{-d(T-t)}$.

∗ And buying fewer shares today reduces the amount of money we need to borrow.

∗ So the net amount owed at T is $S_t \cdot e^{(r-d)(T-t)}$.

▶ **Conclude** F cannot be more than the cost of transporting shares into the future:

$$\overline{F_{t,T}} = S_t\ e^{(r-d)(T-t)}$$

• Now what about a lower bound?

▶ For that, we have to figure out how much it will cost to sell and *borrow* one share till delivery.

▶ That (in future dollars) must be the lower bound on F. (Call it $\underline{F}$.)

▶ Notice that we are now considering a position that involves **short selling** the stocks: selling something we do not already own.

▶ We assumed above that we could borrow money using shares as collateral. And borrowing shares is just the same transaction from the other side.

- **Sell-and-borrow cost:**

  ▶ Find someone who owns the index shares and wants to finance their long position at rate r.

  ▶ Borrow shares from them today; sell shares in the market; No net cash flow.

  * Use the money received, S_t, to collateralize the stock loan.
  * The lender will still pay us interest on it.
  * So the cash we have at T is $S_t e^{r(T-t)}$.

  ▶ However, we also owe the stock lender any dividends we receive on his shares.

  * As above, we can initally borrow fewer shares, $1/e^{d(T-t)}$, and then, borrow and sell more shares at rate d per unit

time to generate the cash that we owe the lender.

* This is just our reinvestment step above, but going in reverse.
* When time T arrives we will have borrowed exactly 1.0 units of the index.

- Again, selling fewer shares initially (and earning less interest on the money) reduces the amount we will have at T to $S_t\ e^{(r-d)(T-t)}$.
- **Conclude:** lower bound for forward rate must be:

$$\underline{F_{t,T}} = S_t\ e^{(r-d)(T-t)}$$

  * If anyone were willing to sell us the index forward below this price, we would go long, and take the short-spot position in the individual stocks above.
  * At date T, we take delivery of the basket of shares from our forward purchase and return it to the lender.

- Of course, since we neglected all transactions costs, upper and lower bound are the same number.

$$F_{t,T} = S_t e^{(r-d)(T-t)}$$

(**C)** Pricing Commodity Forwards.

- Now let's return to soybeans.
- Can we use our "cash and carry" argument once again?
- At this point we can guess what it would say:

$$F_{t,T} = S_t \; e^{\left\{\left(\text{yield to holding cash}\right) - \left(\text{yield to holding commodity}\right)\right\}(T-t)}$$

since this is what we derived for currencies and stock indexes.

  - ▶ The yield for holding cash is just r.
  - ▶ The yield for holding the underlying good was its payout rate (assuming it to be continuous, for simplicity) plus a possible lending fee.
  - ▶ The difference between them is the net cost of carry.

- And, in fact, this formula sometimes remains true, just with different names for the yield terms.
  - ▶ Dividend yield (or foreign rate) corresponds to (negative) **storage costs.**
    - ∗ These include cost of spoilage, warehousing, etc.
    - ∗ If we assume that these are (1) proportional to the value of S, and (2) is incurred continuously at rate u, and (3)

incurred by the owner (not the borrower), then these are just like an interest rate.

▶ Lending fee often called **convenience yield.**

* Also referred to as the lease or rental rate.
* Represents all the non-monetary benefits of having physical access to the good.
* Again express this cost as a proportional yield, y.

- Then the pricing formula says:

$$F_{t,T} = S_t \cdot e^{(r-y+u)(T-t)}.$$

Q: Why did I say this only holds "sometimes"?

Q: What has to be true about the commodity to enforce the arbitrage?

- There are two reasons why the transactions in the cash-and-carry replication may not be possible for a commodity.
- First, to enforce the lower no-arbitrage bound, an arbitrageur would need to be able to hold a short position in the commodity until delivery.
  - ▶ In practice it is impossible to borrow many commodities – at any price.
  - ▶ More subtle: units of the commodity need to be perfectly interchangeable.
- Second, to enforce the upper no-arbitrage bound, an arbitrageur would need to be able to hold a long position in the commodity until delivery.
  - ▶ Not all goods are storable!
    - ∗ Fresh orange juice.
    - ∗ Electricity.
    - ∗ other examples?

(D) Forwards on Anything?

- Our analysis of commodities highlights the limitations of the cash-and-carry argument.

  Arbitrage can enforce the forward price formula if and only if the underlying good can be transported in to the future at a cost that is known today.

- Do you think the argument works for:
  - Real estate?
  - Beaujoulais nouveau?
  - Bitcoins?

- Note that there are forward contracts for these.
  - See `https://www.icbit.se/`
  - Just because we cannot price something by no-arbitrage doesn't mean people can't trade it!

- When the no-arbitrage argument fails, the forwards are *no longer derivatives*. Instead they become primary securities that help complete the market.

III. An Important Example

- Here are some closing prices for Dow Jones index forwards from one day earlier this winter:

	Date		$(T_n - t)$	Closing	$R_{S/A}$%
t	Dec 03	2014	–	17896	–
T_1	Dec	'14	0.048	17886	0.01
T_2	Jan	'15	0.121	17878	0.02
T_3	Feb	'15	0.216	17839	0.03
T_4	Mar	'15	0.293	17817	0.05
T_5	Jun	'15	0.543	17738	0.08
T_6	Sep	'15	0.789	17681	0.10
T_7	Dec	'15	1.041	17602	0.13
T_8	Jun	'16	1.540	17490	0.30
T_9	Dec	'16	2.041	17429	0.49

- The "Closing" column shows the end-of-day prices of forwards to different settlement horizons, T_n. (The first row is the index itself, or "spot").

- The right column shows the risk-free interest rate (semi-annually compounded) to each date.

- What can we learn from these prices?
  - ▶ Do people expect the Dow to fall over the next two years?
  - ▶ Does that represent "the market's forecast"?

- Here's another table:

	Date		r_{cc}	$\frac{1}{T_n - t}\log(\frac{F}{S})$	diff
T_1	Dec	'14	0.0001	-0.0128	0.0129
T_2	Jan	'15	0.0002	-0.0083	0.0085
T_3	Feb	'15	0.0003	-0.0147	0.0150
T_4	Mar	'15	0.0005	-0.0151	0.0156
T_5	Jun	'15	0.0008	-0.0164	0.0172
T_6	Sep	'15	0.0010	-0.0153	0.0163
T_7	Dec	'15	0.0013	-0.0159	0.0172
T_8	Jun	'16	0.0030	-0.0149	0.0179
T_9	Dec	'16	0.0049	-0.0129	0.0178

- The right-hand column is the difference between the third and fourth columns.
- What do those numbers mean?
  - ▶ These are very highly forcastable because the stocks in the index are large companies with very stable payout policies.
- The key point to realize is that there are **3 term-structures** in this data: the futures; the interest rates; and the underlying yield (dividends).
  - ▶ *Using the law of no-arbitrage, any two of them must imply the third.*
  - ▶ So, given two, **the third contains no information.**
  - ▶ In particular, holding interest-rates and total carry costs fixed, futures prices tell us <u>nothing</u> about "expected" spot prices in the future.

IV. More on Shorting Stocks.

- Selling short stocks is a key step in many arbitrage strategies that we will encounter.
- So it it worthwhile to take closer look at how it actually works in real life.
- For Now, I want to call your attention to two features of the "stock loan" market.

1. Borrowing fees.

▶ Sometimes it is difficult (or even impossible) to find a stock holder willing to lend shares that you might want to short. In this case, you may be willing to pay an extra fee for the privilege.

  * Usually, that fee is expressed as a continuously compounded percentage of the value of the stock. Call it w.
  * This is then just like an extra dividend stream that the owner of the stock can generate by loaning the shares out.

▶ In practice, we often see very high w values – sometimes over 100%! – for stocks that a lot of people want to short, like "hot" IPOs in the tech sector.

  * Sometimes this accounts for people's apparent willingness to hold stocks that "everybody knows are overvalued"

- But the reverse can happen too. There may be shares that no one really wants to short or to hold as collateral. Then we can observe *negative* fees.

  $*$ In that case, w is more like a storage cost for a commodity.

- As far as forward prices go, the modification to the formula is straightforward:

$$F_{t,T} = S_t \cdot e^{(r-(d+w))(T-t)}.$$

- Of course, like our assumption about the dividend stream, there is an assumption here that you can lock in the rate w for the full period until T.

  $*$ The common practice in the markets, however, is that either borrower or lender can rengotiate w – or even cancel the transaction – at any time.

2. Discrete dividends.

- I noted above that when stocks pay dividends the borrower has to pay the dividend to the lender.

  $*$ The lender no longer gets it from the company because, once you sell the stock in the market, the company sees the new buyer as the legal owner of those shares.

- For most companies, dividends are paid 1, 2 or 4 times a year. In the U.S., quarterly dividends are the most common.

- For a single stock, then, we should not assume the payments are continuous. How does that affect the forward pricing formula?

- Let's forget about borrowing fees now, and continue to assume the amount and timing of the dividends between t and T is known.

- Consider the net buy-and-hold forward cost $\overline{F}$.

  * Without the dividend, our cost at T is $S_t \, e^{r(T-t)}$ because we have to borrow S_t dollars at t.

  * But if we know the dividends with certainty, we can sell short zero-coupon bonds to each dividend date and that will generate some extra cash today.

  * To be specific, suppose there are N dividends of amount D_n to be paid at dates u_n between now and T.

  * Let $PV(D)$ stand for $\Sigma_{n=1}^{N} \, B_{t,u_n} \, D_n$.

  * This is the amount we would get today from shorting the zero coupon bonds, i.e., borrowing against the future dividends.

  * Then we pay off these short positions with the dividends at each time u_n.

  * This amount reduces the cash we have to borrow to go long the share, and hence the total cost of carrying the position.

- Of course, since we are still ignoring transactions costs, the argument works in reverse for the sell-and-hold cost.
- Pricing result:

$$F_{t,T} = [S_t - PV(D)]\, e^{r(T-t)}$$

- The argument works just as well for other assets with lumpy payouts – like coupon bonds.
- I will leave it for you to think about the impact on the argument of *uncertainty* about the amount and timing of dividends.
  - Sometimes there are surprises in real life.

V. Futures vs. Forwards.

- We already discussed the contractual differences between a forward and a future. Do those differences have important consequences for pricing and hedging?

(A) Pricing.

- **Question:** If the spot price (of something) is S and the forward price is F, what would be the futures price f such that you are indifferent between going long or short a futures contract at f?

- An obvious guess is $F = f$ since then going long a forward and short a future involves no cash flow now and no position in the underlying at T.

- What about intermediate cash flows?

  ▶ If you are long a forward and short a future and spot goes up, you will make money at T but you will have to pay out money *today*.

  ▶ Does this timing difference "wash-out" (it could work for you or against you)?

  ▶ Or does it make the long side of the future systematically better than the short side (or vice versa)?

- **Answer:** Usually, the timing difference has no net value to either contract.
  - ▶ It is not difficult to show that, if interest rates and carry costs etc are predictable (or not too random) then we <u>must</u> have $f = F$ or there will be arbitrage.
    - $*$ Imagine that there are two days until T and we know both the (simple) one-day interest rate (call it $R_{T-2,T-1}$) AND the one day interest rate that will hold tomorrow (i.e. the forward rate $R_{T-2,T-1,T}$).
    - $*$ Suppose we have both a two-day forward and a two-day futures contract on a non-dividend paying stock, with $f_{T-2,T} > F_{T-2,T}$.
    - $*$ With a little imagination, you can construct an arbitrage! (And likewise if you assume $f_{T-2,T} < F_{T-2,T}$.)
    - $*$ Then, by backwards induction, you can extend the argument back to any earlier date to show that we must have $f_{T-k,T} = F_{T-k,T}$ for all k.
  - ▶ Furthermore, the conclusion will hold even with random carry costs *as long as the rates aren't correlated with the spot price.*
  - ▶ What would happen if they were?

- Take the case of (extreme) negative correlation.
  - $*$ Suppose every time the spot price went UP interest rates went DOWN.
  - $*$ Would you rather be long a forward or long a future?
    - There is still no reason to think that the mark-to-market cash-flows from a futures position will be more positive or negative.
    - BUT NOW the reinvestment of those cashflows DOES result in a systematic pattern.
    - Can you see which way it goes?
    - What's an example of an underlying asset that has this type of correlation?

- Even with this extreme correlation, the reinvestement effect will still usually be quite small. Hence:
- **Main Conclusion:** $f = F$ **holds unless**

  1. net carry costs from now till T are unpredictable, **and**

  2. the spot price S is highly correlated with those costs.
- Secondary conclusion:
  - $*$ $F > f$ if and only if $\rho(S_t, (r_t - d_t)) < 0$.

(B) Hedging.

1. The **hedge ratio** or "delta" of one security with respect to another is just the increase (or decrease) in its value when the other one changes in value by $1 – holding everything else fixed.

2. So if the value of a derivative is $V = V(X,t)$, where X is the value of the underlying, we write

$$\delta = \frac{\partial V}{\partial X} \approx \frac{\Delta V}{\Delta X}.$$

3. If you have a formula for V, you can compute this sensitivity just by perturbing the input (X) by a little and seeing how much V changes (or by using calculus).

4. Consider the delta of a forward contract with respect to its underlying (a stock, for instance) after it is created.

a. First, **beware** a potential source of confusion:

- When we refer to a "stock price", it means – of course – the value of one share of stock.
- But we have been using "forward price" to mean the <u>settlement</u> price of a *new* forward contract whose <u>value</u> – we have seen – is zero by definition.
- Now we're considering the value of an *old* forward contract, i.e. one whose *settlement* price was agreed upon some time in the past.

- So let that old "price" be K, and let the contract's *value* at time t be $V^K = V(S_t, K, r, \ldots)$.

- Obviously this doesn't have to be zero, if, for instance, the underlying has gone up a lot since the date at which the contract was originated.

b. To calculate the value of an old forward contract with forward price K, let's imagine you off-set a (long) position by going short another forward at the current price $F_{t,T}$.

- Then you have a riskless position which will pay you $(F - K)$ at T.

- Since the new forward doesn't cost anything, the value of the old forward must be the (present) value of this position:
$$V^K = (F - K) \cdot e^{-r(T-t)}.$$

c. Now plug in the expression for $F = S_t e^{+r(T-t)}$ that we derived, and get

$$V^K = S_t - K \cdot e^{-r(T-t)}.$$

(This is for zero dividends/convenience yield, etc. Make sure you can get the general expression yourself.)

d. Conclude:

$$\frac{\partial V^K}{\partial S_t} = 1.$$

5. Now let's calculate the delta of a futures contract similarly (with no payouts, as above).

   a. At inception, or at the last mark-to-market date, the value is zero, by definition. Call the future's "price" then f_0.

   b. At any time (until the next mark-to-market) after that, the contract could be offset in the market with a new contract at futures price $f_t = S_t e^{r(T-t)}$.

   c. The profit/loss would be realized at the end of the end of the day when both contracts are marked-to-market at some price f_1 to net:

   $$(S_t e^{r(T-t)} - f_1) - (f_0 - f_1) = S_t e^{r(T-t)} - f_0.$$

   d. Thus the time-t value of the original contract, V^f is $S_t e^{r(T-t)} - f_0$.

   e. Hence,

   $$\frac{\partial V^f}{\partial S_t} = e^{r(T-t)} > 1.$$

   (Changing S by a dollar changes $aS + b$ by a dollars.)

6. **Conclusion: futures and forwards have different hedge ratios**.

- In fact, for a non-dividend paying stock (i.e. no payouts), we have just shown that the forward delta is one-for-one and the futures delta is always greater than this.
- So, when a given spot position is hedged with futures, one would go short slightly FEWER futures contracts.
  - ▶ This is called "tailing the hedge".
  - ▶ Intuitively, the futures price moves more than spot because of the implicit leverage in a futures position.
- If the underlying has payouts (e.g. a currency or a commodity) we will still get different deltas for forwards and futures, but either one might be bigger, depending on the sign of the net carry.

VI. Summary

- Cash-and-Carry replication yields arbitrage pricing of forwards and futures, given the spot price, and the relevant payouts and/or interest rates.

- Main assumption: carry costs are known in advance, or term borrowing/lending of cash and the underlying to settlement date are possible.

- Transactions costs turn the no-arbitrage *price* into upper and lower no-arbitrage *bounds*.

- Futures eliminate assignability and credit-risk problems with forwards, but may impose inconvenient standardization and cash-flow requirements.

- With otherwise identical terms and predictable carry costs, forward and futures prices are equal.

- The value of a forward or future is zero at initiation. Thereafter the value depends on the difference between the *current* forward/futures price and the one fixed in the contract.

- The *delta* or hedge ratio of forwards and futures with respect to spot may differ from each other, and may not be equal to one.

- IF no-arbitrage pricing applies, then there is *no extra information* in forward/future prices about where spot will be in the future.

Lecture Note 1: Summary of Notation

Symbol	Page	Meaning
F	p3	*delivery price of a forward contract (or "forward price")*
T	p3	*delivery date (or maturity) of contract*
f_n	p5,6	*futures price on day* n
S_T	p8	*spot market price of underlying asset at* T
X_T	p9	*value of a (non-traded) index* T
$F_{t,T}$	p12	*forward price at time* t *for delivery time* T
R	p12	*simple annual risk-free rate*
r	p12	*continuously-compounded risk-free rate per annum*
$r_{t,T}$	p13	*continuously-compounded rate per annum*
		at time t *for a risk-free payment at* T
$B_{t,T}$	p13	*price at* t *of a risk-free zero-coupon*
		bond maturing at T
$\overline{F}, \underline{F}$	p17	*upper and lower no-arbitrage forward price bounds*
d	p18	*dividend yield on a stock index*
u	p22	*storage cost for physical commodity*
y	p23	*borrowing fee or "convenience yield"*
		for a physical commodity
$R_{S/A}$	p26	*annual riskless rate (semiannually compounded form)*
w	p28	*percentage fee paid to borrow stock*
D_n	p30	*lump-sum dividend to be paid at time* u_n
$PV(D)$	p30	*present value of all remaining dividends between* t *and* T

Lecture Notation, continued

Symbol	Page	Meaning
$R_{T-2,T-1,T}$	*p33*	*today's one-day forward, one-day interest rate*
$\rho(S,(r-d))$	*p34*	*correlation between spot price and carry costs*
$V(X)$	*p35*	*value of some asset determined by a parameter* X
δ	*p35*	*the incremental change in* V *per unit change in* X
V^K	*p36*	*the value of an existing forward contract with delivery price* K *at time* t
$\frac{\partial V^K}{\partial S}$	*p36*	*the "delta" of a forward contract*
$\frac{\partial V^f}{\partial S}$	*p37*	*the "delta" of an equivalent futures contract*