Mean-Variance Investing

This version: 08-10-2012

Abstract

Mean-variance investing is all about diversification. Diversification considers assets holistically and exploits the interaction of assets with each other, rather than viewing assets in isolation. Holding a diversified portfolio allows investors to increase expected returns while reducing risks. In practice, mean-variance portfolios that constrain the mean, volatility, and correlation inputs to reduce sampling error have performed much better than unconstrained portfolios. These special cases include equal-weighted, minimum variance, and risk parity portfolios.

1. Norway and Wal-Mart

On June 6, 2006, the Norwegian Ministry of Finance announced that the Norwegian sovereign wealth fund, bureaucratically labeled "The Norwegian Government Pension Fund – Global" (GPFG), had sold Wal-Mart Stores Inc. on the basis of "serious/systematic violations of human rights and labor rights."[1] As one of the largest funds in the world and a leader in ethical investing, GPFG's decision to exclude Wal-Mart was immediately noticed. Benson K. Whitney, the U.S. ambassador to Norway complained that the decision was arbitrarily based on unreliable research and unfairly singled out American companies. A spokesperson from Wal-Mart disputed Norway's decision and the company sent two senior executives to plead its case before the Ministry of Finance.

[1] This is based on "The Norwegian Government Pension Fund:The Divestiture of Wal-Mart Stores Inc.," Columbia CaseWorks, ID#080301, 2010. The quote is from Ministry of Finance press release No. 44 in 2006.

Norway is the world's third-largest oil exporter after Saudi Arabia and Russia. Norway first found oil in the North Sea in 1969 and quickly found that its large discovery distorted its economy. During the 1970s and 1980s, Norway experienced many symptoms of the *Dutch disease* (see Chapter XX), with growing oil revenues contributing to a less competitive and shrinking manufacturing sector. When oil prices slumped in the mid-1980s, over-reliance on oil revenue contributed to a period of slow economic growth. Sensibly, Norway decided to diversify.

Norway's "Government Petroleum Fund" was set up in 1990 to channel some of the oil revenue into a long-term savings mechanism. The fund served two purposes: (1) it diversified oil wealth into a broader portfolio of international securities, improving Norway's risk-return trade-off and (2) it inoculated Norway from the Dutch disease by quarantining wealth overseas and only gradually letting the oil money trickle into the economy. In January 2006, the fund was renamed "The Norwegian Government Pension – Global," although it had no explicit pension liabilities. The new title conveyed the fund's goal of managing its capital to meet long-term government obligations as well as benefit future generations.

At first the fund was invested only in government bonds. In 1998 the investment universe was enlarged to allow a 40% allocation to equities and subsequently raised to 60% in 2007. In 2010 the fund was permitted to invest up to 5% of assets in real estate, and GPFG bought its first properties in London and Paris in 2011. While the asset universe of GPFG had gradually broadened, since the fund's inception the "reluctant billionaires of Norway" have always sought to meaningfully invest their fortune in line with the country's social ethos.[2] GPFG practiced

[2] Mark Landler, Norway Backs Its Ethics with Its Cash, New York Times, May 4, 2007.

socially responsible investing (SRI). In 2005, government regulation was passed making this formal. The regulation stated that:[3]

- The financial wealth must be managed so as to generate a sound return in the long term, which is contingent on sustainable development in the economic, environmental and social sense.
- The fund should not make investments which constitute an unacceptable risk that the Fund may contribute to unethical acts or omissions, such as violations of human rights, gross corruption or severe environmental damages.

The Ministry of Finance appointed an independent Council on Ethics, which issued recommendations on whether an investment constituted a violation of GPFG's ethical guidelines. If there was unacceptable risk, the Council would recommend the exclusion of a company. The Council continuously monitored all companies in the fund's portfolio to uncover possible violations using publicly available information, media sources, national and international organizations, and independent experts.

In April 2005 the Council began examining alleged unethical activities by Wal-Mart. These included many reported violations of labor laws and human rights, including reports of child labor, serious violations of working hour regulations, paying wages below the legal minimum, hazardous working conditions, and unreasonable punishment. The Council found widespread gender discrimination. Wal-Mart stopped workers from forming unions. There were reports of children performing dangerous work and the use of illegal immigrant labor.

[3] Section 8 of the Government Pension Fund Regulation No. 123, December 2005.

In September 2005, the Council sent a letter to Wal-Mart asking the company to comment on the alleged human rights violations. Wal-Mart acknowledged the letter but did not respond.

From January until March 2006, the Ministry conducted its own assessment. The Ministry found that exercising the fund's ownership rights through an activist approach would not be effective in influencing Wal-Mart's business practices. Divestment decisions were always considered the last resort, but in Wal-Mart's case the Ministry decided it was appropriate.

When the Ministry announced it had sold all its holdings in Wal-Mart on June 6, 2006, it quoted the report from the Council of Ethics:

> What makes this case special is the sum total of ethical norm violations, both in the company's own business operations and in the supplier chain. It appears to be a systematic and planned practice on the part of the company to hover at, or cross, the bounds of what are accepted norms for the work environment. Many of the violations are serious, most appear to be systematic, and altogether they form a picture of a company whose overall activity displays a lack of willingness to countervail violations of norms in its business operations.[4]

Excluding companies is not without cost: by shrinking its universe, GPFG's investment opportunities were smaller, it lost diversification benefits, and lowered its best risk-return trade-off. As more companies were excluded, there were further losses in diversification benefits. In January 2010, GPFG excluded all tobacco companies. What did these exclusions do to GPFG's maximum attainable risk-return trade-off? What did it cost to be ethical?

[4] Press release, Ministry of Finance, June 6, 2006.

In this chapter I cover mean-variance investing. This is by far the most common way to choose optimal portfolios. The main takeaway is that diversified portfolios should be selected because investors can reduce risk and increase returns. The underlying concept of diversification can be implemented in different ways, and many of the approaches popular at the time of writing, like risk parity and minimum variance portfolios, are special cases of unconstrained mean-variance portfolios. An advantage of mean-variance investing is that it allows diversification benefits (and losses) to be measured in a simple way. We will later use mean-variance investing concepts to estimate how much Norway is losing in choosing to be socially responsible—in other words, to answer the question, how much does it cost Norway to divest Wal-Mart?

2. Mean-Variance Frontiers

Mean-variance frontiers depict the best set of portfolios that an investor can obtain (only considering means and volatilities, of course!). Let's start by considering a U.S. investor contemplating investing only in U.S. or Japanese equities.

2.1 U.S. and Japan

In the 1980s Japan was poised to take over the world. Figure 1 plots cumulated returns of U.S. and Japanese equities from January 1970 to December 2011 using MSCI data. (I also use this data for the other figures involving G5 countries in this chapter.) Japanese returns are plotted in the solid line and U.S. returns are shown in the dashed line. Japanese equities skyrocketed in the 1980s. Many books were written on Japan's stunning success, like Vogel's (1979) "Japan as Number One: Lessons for America." Flush with cash, Japanese companies went on foreign buying binges. Japanese businesses bought marquee foreign companies: Universal Studios and Columbia Records were sold to Matsushita Electric and Sony, respectively. The Japanese also bought foreign trophy real estate. The Mitsubishi Estate Company of Tokyo purchased

Rockefeller Center in 1989. In 1990 the famous Peeble Beach golf course was sold to a Japanese businessman, Minuro Isutani. Figure 1 shows the U.S. did well during the 1980s too, but not nearly as well as booming Japan.

[Figure 1 here]

Then everything crashed. Isutani had bought Peeble Beach a year after the Nikkei had hit its peak in 1989, and he was later investigated for money laundering by the FBI.[5] Figure 1 shows that since 1990 Japanese stocks have been flat. But while Japan was languishing, the U.S. boomed. Even so, Japanese cumulated returns were higher at December 2011 than U.S. cumulated returns. Since 2000, Figure 1 shows that the U.S. and Japan have a greater tendency to move together. They jointly slowed during the early 2000s, experienced bull markets during the mid-2000s, and then crashed during the financial crisis of 2007-2008. Over the whole sample, however, Japan has moved very differently from the U.S.

The average return and volatility for the U.S. in Figure 1 are 10.3% and 15.7%, respectively. The corresponding numbers for Japan are 11.1% and 21.7%, respectively. We plot these rewards (means) and risks (volatilities) in mean-standard deviation space in Figure 2. The U.S. is represented by the square and Japan is represented by the circle. The *x*-axis is in standard deviation units and the *y*-axis units are average returns.

[Figure 2 here]

The curve linking the U.S. and Japan in Figure 2 is the *mean-variance frontier*. Like the literature, I use the terms mean-variance frontier and mean-standard deviation frontier interchangeably as the two can be obtained simply by squaring, or taking a square root, of the *x*-

[5] A Japanese Laundry Worth $1 Billion? Businessweek, May 24, 1993.

axis depending on whether one uses volatility or variance units. The mean-variance frontier for the U.S. and Japan represents all combinations of the U.S. and Japan. Naturally, the red square representing the U.S. is a 100% U.S. portfolio and the circle representing Japan is a 100% Japanese portfolio. All the other positions on the mean-variance frontier represent different portfolios containing different amounts of the U.S. and Japan.

The mean-variance frontier is a *parabola*, or a bullet. The top half of the mean-variance frontier is *efficient*: an investor cannot obtain a higher reward, or expected return, for a given level of risk measured by volatility. Investors will choose portfolios on the top, efficient part of the frontier. The U.S. sits on the underbelly of the bullet. You can achieve a higher expected return for the same volatility by moving onto the top half of the frontier. The U.S. is an *inefficient* portfolio. No one should hold a 100% U.S. portfolio.

2.2 Diversification

The fact that Japanese equities have moved differently to the U.S., especially over the 1980s and 1990s causes the mean-variance frontier to bulge outwards to the left. The correlation between the U.S. and Japan is 35% over the sample. (The U.S.-Japan correlation post-2000 is 59%, still far below one.) Mean-variance frontiers are like the Happy (or Laughing) Buddha: the fatter the stomach or bullet, the more prosperous the investor becomes. Notice that the left-most point on the mean-variance frontier in Figure 2 has a lower volatility than either the U.S. or Japan. This portfolio, on the left-most tip of the bullet, is called the *minimum variance portfolio*.

Starting from a 100% U.S. portfolio (the square in Figure 2), an investor can improve her risk-return trade-off by including Japanese equities. This moves her position from the U.S. (the square) to Japan (the circle) and the investor moves upwards along the frontier in a clockwise direction. Portfolios to the right-hand side of the circle (the 100% Japan position) represent *levered* portfolios. Portfolios on the top half of the frontier past the circle are constructed by shorting the U.S., like a -30% position, and then investing the short proceeds in a levered Japanese position, which would be 130% in the U.S. in this case. All of the efficient portfolios lying on the top half of the frontier – those portfolios with the highest returns for a given level of risk – contain Japanese equities. The minimum variance portfolio also includes Japan.

The American investor can improve her risk-return trade-off by holding some of Japan because Japan provides *diversification benefits*. This is the fundamental concept in mean-variance investing, and it corresponds to the common adage “don’t put all your eggs in one basket.” The U.S. and Japan held together are better than the U.S. held alone. The advantages of diversification imply that we cannot consider assets in isolation; we need to think about how assets behave together. This is the most important takeaway of this chapter.

Diversified, efficient portfolios of the U.S. and Japan have higher returns and lower risk than the 100% U.S. position. Why? When the investor combines the U.S. and Japan, the portfolio reduces risk because when one asset does poorly, another asset can potentially do well. The risk of the U.S. alone is partly offset by holding some of Japan. This is similar to an insurance effect (except that the purchaser of insurance loses money, on average): When the U.S. does relatively poorly, like during the 1980s, Japan has a possibility of doing well. Some of the risk of the U.S. position is avoidable and can be offset by holding Japan as insurance.

What about the opposite? During the 1990s Japan was in the doldrums and the U.S. did well. The U.S. investor would have been better off holding only the U.S. Yes, he would – *ex post*. But forecasting is always hard. At the beginning of the 1990s, the investor would have been better off on an *ex-ante* basis by holding a portfolio of both the U.S. and Japan. What if the roles were reversed so that in the 1990s Japan did take over the world and the U.S. swapped places with Japan? Holding Japan in 1990 diversified away some of this ex-ante risk.

The formal theory behind diversification was developed by Harry Markowitz (1952), who was awarded the Nobel Prize in 1990. The revolutionary Capital Asset Pricing Model (CAPM) is laid on the capstone of mean-variance investing and we discuss that model in Chapter XX. The CAPM pushes the diversification concept further and derives that an asset's risk premium is related to the (lack of) diversification benefits of that asset. This turns out to be the asset's beta.

Mathematically, diversification benefits are measured by covariances or correlations. Denoting r_p as the portfolio return, the variance of the portfolio return is given by

$$\begin{aligned} \mathrm{var}(r_p) &= w_{US}^2\,\mathrm{var}(r_{US}) + w_{JP}^2\,\mathrm{var}(r_{JP}) + 2w_{US}w_{JP}\,\mathrm{cov}(r_{US}, r_{JP}) \\ &= w_{US}^2\,\mathrm{var}(r_{US}) + w_{JP}^2\,\mathrm{var}(r_{JP}) + 2w_{US}w_{JP}\rho_{US,JP}\sigma_{US}\sigma_{JP,} \end{aligned} \tag{0.1}$$

where r_{US} denotes U.S. returns, r_{JP} denotes Japanese returns, and w_{US} and w_{JP} are the portfolio weights held in the U.S. and Japan, respectively. The portfolio weights can be negative, but they must sum up to one, $w_{US} + w_{JP} = 1$, as the portfolio weights total 100%. (This last constraint is called an *admissibility condition* on the portfolio weights.) The covariance, $\mathrm{cov}(r_{US}, r_{JP})$, in the first line of equation (1.1) can be equivalently expressed as the product of correlation between the U.S. and Japan ($\rho_{US,JP}$), and the volatilities of the U.S. and Japan (σ_{US} and σ_{JP}, respectively), $\mathrm{cov}(r_{US}, r_{JP}) = \rho_{US,JP}\sigma_{US}\sigma_{JP}$.

Large diversification benefits correspond to *low* correlations. Mathematically, the low correlation in equation (1.1) reduces the portfolio variance. Economically, the low correlation means that Japan is more likely to pay off when the U.S. does poorly and the insurance value of Japan increases. This allows the investor to lower her overall portfolio risk. The more Japan does not look like the U.S., the greater the benefit of the U.S. investor holding Japan. Mean-variance investors love holding things that act differently to what they currently hold. The more dissimilar, or the lower the correlation, the better.[6]

Figure 3 plots the U.S. and Japan mean-variance frontier for different correlation values. The solid line is the frontier with the 35.4% correlation in data. The dashed line is drawn with 0% correlation and the dotted line drops the correlation to -50%. As the correlation decreases, the tip of the mean-variance frontier pushes to the left – the bullet becomes more pointed. The lower correlation allows the investor to lower risk as Japan provides even more diversification benefits.

[Figure 3 here]

2.3 G5 Mean-Variance Frontiers

In Figures 4 and 5 we add the UK, France, and Germany.

[Figure 4 here]

First consider Figure 4, which plots the mean-variance frontier for the G3: U.S., Japan, and the UK. The G3 frontier is shown in the solid line. For comparison, the old U.S.-Japan frontier is in the dashed line. Two things have happened moving from the G2 (U.S. and Japan) to the G3 (U.S., Japan, and the UK):

1. The frontier has expanded.

[6] In this simple setting, there is only one period. In reality, correlations move over time and correlations increase during bear markets. Ang and Bekaert (2002) show that there are still significant diversification benefits of international investments when correlations increase during bad times.

The Happy Buddha becomes much happier adding the UK. The pronounced outward shift of the wings of the mean-variance bullet means that an investor can obtain a much higher return for a given standard deviation. (There is also a leftward shift of the frontier, but this is imperceptible in the graph.) Starting at any point on the U.S.-Japan frontier, we can move upwards on an imaginary vertical line and obtain higher returns for the same level of risk. Adding the UK to our portfolio provides further diversification benefits because now there is an additional country which could have high returns when the U.S. is in a bear market while before there was only Japan. There is also a chance that both the old U.S. and Japan positions would do badly; adding the U.K. gives the portfolio a chance to offset some or all of those losses.

2. All individual assets lie inside the frontier.

   Individual assets are *dominated*: diversified portfolios on the frontier do better than assets held individually. Now all countries would never be held individually. Diversification removes asset-specific risk and reduces the overall risk of the portfolio.

In Figure 5, we add Germany and France. The G5 mean-variance frontier is in the solid line and it is the fattest: the U.S.-Japan and the U.S.-Japan-UK frontiers lie inside the G5 frontier. There is still a benefit in adding Germany and France from the U.S.-Japan-UK, but it is not as big a benefit as adding the UK starting from the U.S.-Japan. That is, although Happy Buddha continues getting happier by adding countries, the rate at which he becomes happier decreases. There are *decreasing marginal diversification benefits* as we add assets. As we continue adding assets beyond the G5, the frontier will continue to expand but the added diversification benefits become smaller.

[Figure 5 here]

Figures 4 and 5 show that if we add an asset, the mean-variance frontier gets fatter. Conversely, if we remove an asset, the mean-variance frontier shrinks. Removing assets just as Norway did by divesting Wal-Mart can only cause the maximum Sharpe ratio to (weakly) decrease. We will later compute how the mean-variance frontier shrinks as we remove assets.

The mathematical statement of the problem to Figures 4 and 5 is:

$$\min_{\{w_i\}} \operatorname{var}(r_p)$$
$$\text{subject to } E(r_p) = \mu^* \text{ and } \sum_i w_i = 1, \tag{0.2}$$

where the portfolio weight for asset *i* is w_i. We find the combination of portfolio weights, $\{w_i\}$, that minimizes the portfolio variance subject to two constraints. The first is that the expected return on the portfolio is equal to a target return, μ^*. The second is that the portfolio must be a valid portfolio, which is the admissibility condition we have seen earlier. Students of operations research will recognize equation (1.2) as a *quadratic programming problem*, and what makes mean-variance investing powerful (but alas, misused, see below) is that there are very fast algorithms for solving these types of problems.

Figure 6 shows how this works pictorially. Choosing a target return of $\mu^* = 10\%$, we find the portfolio with the lowest volatility (or variance). We plot this with an X. Then we change the target return to $\mu^* = 12\%$. Again we find the portfolio with the lowest volatility and plot this with another X. The mean-variance frontier is drawn by changing the target return, μ^*, and then linking all the Xs for each target return. Thus, the mean-variance frontier is a *locus* of points, where each point denotes the minimum variance achievable for each expected return.

[Figure 6 here]

2.4 Constrained Mean-Variance Frontiers

So far, we have constructed unconstrained mean-variance frontiers. Often investors face constraints on what types of portfolios they can hold. One constraint faced by many investors is the inability to short. When there is a *no short-sales constraint*, all the portfolio weights have to be positive ($|w_i| \geq 0$), and we can add this constraint to the optimization problem in equation (1.2).

Adding short-sale constraints changes the mean-variance frontier, sometimes dramatically. Figure 7 contrasts the mean-variance frontier where no shorting is permitted, in the solid line, with the unconstrained frontier, drawn in the dotted line. The constrained mean-variance frontier is much smaller than the unconstrained frontier and lies inside the unconstrained frontier. The constrained frontier is also not bullet-shaped. Constraints inhibit what an investor can do and an investor can only be made (weakly) worse off. If an investor is lucky, the best risk-return trade-off is unaffected by adding constraints. We see this in Figure 7 in the region where the constrained and the unconstrained mean-variance frontiers lie on top of each other. But generally constraints cause an investor to achieve a worse risk-return trade-off. Nevertheless, even with constraints, the concept of diversification holds: the investor can reduce risk by holding a portfolio of assets rather than a single asset.

[Figure 7 here]

2.5 The Risks of Not Diversifying

Many people hold a lot of stock in their employers. Poterba (2003) reports that for large defined-contribution pension plans, the share of assets in own-company stock is around 40%. This comes, usually but not always, from discounted purchases and is designed by companies to encourage employee loyalty. For individuals, such concentrated portfolios can be disastrous – just as the employees of Enron (bankrupt 2001), Lucent (which spiraled downwards after it was spun off from AT&T in 1996 and then was bought at a pittance by Alcatel in 2006), and Lehman Brothers (bankrupt 2008) found out. Enron employees had over 60% of their retirement assets in company stock when Enron failed.[7] According to the Employee Benefits Research Institute, among 401(k) plan participants who had equity exposure in 2009, 12% had company stock as their only equity investment. For equity market participants in their 60s, 17% had equity exposure only through company stock.[8]

While the cost of not diversifying becomes painfully clear when your employer goes bankrupt, mean-variance investing reveals that there is a loss even when your employer remains solvent. Individuals can generate a higher risk-return trade-off by moving to a diversified portfolio. Poterba (2003) computes the cost of not diversifying a retirement account relative to simply investing in the diversified S&P 500 portfolio. Assuming that half of an individual's assets are invested in company stock, the certainty equivalent cost (see Chapter XX) of this concentrated position is about 80% of the value of investing in the diversified S&P 500 portfolio. This is a substantial reduction in utility for investors. An individual should regularly cash out company stock, especially if the stock is rising faster than other assets in his portfolio. As own-company stock rises relative to other assets, it represents even greater concentrated risk for that investor.[9] (Furthermore, your human capital itself is concentrated with that employer. See also Chapter XX.)

[7] See Barber and Odean (2011).

[8] Figure 28 from VanDerhei, J., S. Holden, and L. Alonso, 2010, 401(k) Plan Asset Allocation, Account Balances and Loan Activity in 2009, Employee Benefit Research Institute Issue Brief No. 350.

[9] Individual investors, unfortunately, tend to do exactly the opposite – individuals hold larger amounts in employer stocks which have had the strongest return performance over the last 10 years, as Benartzi (2001) shows.

The wealthy often do not diversify enough. JP Morgan's 2004 white paper, "Improving the Odds: Improving the 15% Probability of Staying Wealthy," identified excessive concentration as the number one reason the very wealthy lose their fortunes. The 15% probability in the study's title comes from the fact that in first Forbes 400 list of the richest 400 people in America, fewer than 15% of the original 400 were on that list one generation later. While the Forbes 400 tracks the mega-rich, the wealthy below them are also likely to lose their wealth. Kennickell (2011) reports that of the American households in the wealthiest 1% in 2007, approximately one-third of these households fell out of the top 1% two years later.

Diversification helps preserve wealth. Entrepreneurs and those generating wealth from a single business often find diversification counterintuitive.[10] After all, wasn't it concentrated positions that generated the wealth in the first place? This is the business they know best, and their large investment in it may be illiquid and hard to diversify. But diversification removes company-specific risk that is outside the control of the manager. Over time, prime real estate ceases to be prime and once great companies fail because their products become obsolete. While some companies stumble due to regulatory risk, macro risk, technological change, and sovereign risk, other companies benefit. Diversification reduces these avoidable *idiosyncratic* risks. JP Morgan reports that of the 500 companies in the S&P 500 index in 1990, only half remained in the index in 2000. This is testament to the need to diversify, diversify, and diversify if wealth is to be preserved.

Institutional investors also fail to sufficiently diversify. Jarrell and Dorkey (1993) recount the decline of the University of Rochester's endowment. In 1971, Rochester's endowment was $580 million making it the fourth largest private university endowment in the U.S. In 1992, the

[10] See "Stay the Course? Portfolio Advice in the Face of Large Losses," Columbia CaseWorks, ID #110309, 2011.

endowment totaled $620 million ranking it 20th among private university endowments. What happened? From 1970-1992, the endowment earned only 7% compared to a typical 60%-40% equities-bonds portfolio return of 11%. Had Rochester simply invested in this benchmark portfolio, the endowment would have ranked 10th among private university endowments in 1992. By 2011, Rochester had dropped to 30th place.[11] A big reason for the underperformance of the Rochester endowment was the excessive concentration held in local companies, especially Eastman Kodak. Kodak filed for bankruptcy in February 2012.

Boston University is another example. Over the 1980s and 1990s, Boston University invested heavily in Seragen Inc., then a privately held local biotech company. According to Lerner, Schoar and Wang (2008), Boston University provided at least $107 million to Seragen between 1987 to 1997 – a fortune considering the school's endowment in 1987 was $142 million. Seragen successfully went public, but suffered setbacks. In 1997, the University's stake was worth only $4 million. Seragen was eventually bought by Ligand Pharmaceuticals Inc. in 1998 for a total of $30 million.

Norway's sovereign wealth fund, in contrast, was created precisely to reap the gains from diversification. Through GPFG, Norway swaps a highly concentrated asset – oil – into a diversified financial portfolio and thus improves its risk-return trade-off.

2.6 Is Diversification Really a Free Lunch?

Diversification has been called the only "free lunch" in finance and seems too good to be true. If you hold (optimized) diversified portfolios, you can attain better risk-return trade-offs than holding individual assets. Is it really a free lunch?

[11] Counting only private university endowments using NACUBO data at 2011 fiscal year end.

Yes, if you only care about portfolio means and variances.

Mean-variance investing, by definition, only considers means and variances. Portfolio variances are indeed reduced by holding diversified portfolios of imperfectly correlated assets (see equation (1.1)). In this context, there is a free lunch. But what if an investor cares about other things? In particular, what if the investor cares about downside risk and other higher moment measures of risk?

Variances always decrease when assets with non-perfect correlations are combined. This causes improvements in returns and reductions in risk in mean-variance space. But other measures of risk do not necessarily diminish when portfolios are formed. For example, a portfolio can be more negatively skewed, and thus have greater downside risk, than the downside risk of each individual asset.[12] Investors care about many more risk measures than simply variance, so this may matter.

Diversification is not necessarily a free lunch when other measures of risk are considered. Nevertheless, from the viewpoint of characterizing the tails of asset returns, standard deviation (or variance) is the most important measure. Furthermore, optimal asset weights for a general utility function can be considered to be mean-variance weights to a first approximation, but we may have to change an investor's risk aversion in the approximation (see Chapter XX).[13] While variance is the first-order risk effect, in some cases the deviations from the mean-variance approximation can be large. You still need to watch the downside.

[12] The technical jargon for this is that higher moment risk measures are not necessarily subadditive. See Artzner et al. (1999).
[13] Technically, you can Taylor expand a utility function so that the first term represents CRRA utility, which is approximately mean-variance utility.

In the opposite direction, diversification kills your chances of the big lottery payoff. If you are risk-seeking and want to bet on a stock having a lucky break – and hope you become a billionaire from investing everything in the next Microsoft or Google – diversification is not for you. Since diversification reduces your idiosyncratic risk, it also limits the extremely high payoffs that can occur from highly concentrated positions. The risk-averse investor likes this because it also limits the catastrophic losses that could result from failing to diversify. Just ask the employees of Enron and Lehman, and the faculty and students of the University of Rochester and Boston University.

The overall message from mean-variance investing is that diversification is good. It minimizes risks that are avoidable and idiosyncratic. It views assets holistically, emphasizing how they interact with each other. By diversifying, investors improve their Sharpe ratios and can hold strictly better portfolios – portfolios that have higher returns per unit of risk, or lower risk for a given target return – than assets held in isolation.

3. Mean-Variance Optimization

We've described the mean-variance frontier and know that the best investment opportunities lie along it. Which efficient portfolio on the mean-variance frontier should we pick?

That depends on each investor's risk aversion. We saw in the previous chapter that we can summarize mean-variance preferences by *indifference curves*. Maximizing mean-variance utility is equivalent to choosing the highest possible indifference curve. Indifference curves correspond to the individual maximizing mean-variance utility (see equation (XX) in Chapter XX restated here):

$$\max_{\{w_i\}} E(r_p) - \frac{\gamma}{2} \mathrm{var}(r_p), \tag{0.3}$$
subject to any constraints.

The coefficient of risk aversion, γ, is specific to each individual. The weights, $\{w_i\}$, correspond to the risky assets in the investor's universe. Investment in the risk-free asset, if it is available, constitutes the remaining investment (all the asset weights sum to one).

3.1 Without a Risk-Free Asset

Figure 8 shows the solution method for the case without a risk-free asset.[14] The left graph in the top row shows the indifference curves. As covered in Chapter XX, one particular indifference curve represents one level of utility. The investor has the same utility for all the portfolios on a given indifference curve. The investor moves to higher utility by moving to successively higher indifference curves. The right graph in the top row is the mean-variance frontier that we constructed from Section 2. The frontier is a property of the asset universe, while the indifference curves are functions of the risk-aversion of the investor.

[Figure 8 here]

We bring the indifference curves and the frontier together in the bottom row of Figure 8. We need to find the tangency point between the highest possible indifference curve and the mean-variance frontier. This is marked with the *X*. Indifference curves lying above this point represent higher utilities, but these are not attainable – we must lie on the frontier. Indifference curves lying below the *X* represent portfolios that are attainable as they intersect the frontier. We can, however, improve our utility by shifting to a higher indifference curve. The highest possible

[14] The mathematical formulation of this problem corresponds to equation (1.3) with the constraint that the weights in risky assets sum to one, or that the weight in the risk-free asset is equal to zero.

utility achievable is the tangency point X of the highest possible indifference curve and the frontier.

Let's go back to our G5 countries and take a mean-variance investor with a risk aversion coefficient of $\gamma = 3$. In Figure 9, I plot the constrained (no shorting) and unconstrained mean-variance frontiers constructed using U.S., Japanese, UK, German, and French equities. The indifference curve corresponding to the maximum achievable utility is drawn and is tangent to the frontiers at the asterisk. At this point, both the constrained and unconstrained frontiers overlap. The optimal portfolio at the tangency point is given by:

	US	JP	UK	GR	FR
Optimal Portfolio	0.45	0.24	0.16	0.11	0.04

This portfolio is heavily weighted towards the U.S. and Japan with weights of 45% and 24% respectively. Note that by construction, this portfolio consists only of risky assets, so the portfolio weights sum to one. (The weights in this example are also fairly close to the market capitalization weights of these countries.) With a risk-free rate of 1%, the Sharpe ratio corresponding to this optimal portfolio is 0.669.

[Figure 9 here]

3.2 With a Risk-Free Asset

The addition of a risk-free asset expands the investor's opportunities considerably. Since there is only one period, the risk-free asset has no variance. Think of T-bills as an example of a security with a risk-free return. (There is some small default risk in T-bills which you should ignore for now; I cover sovereign default risk in Chapter XX.)

When there is a risk-free asset, the investor proceeds in two steps

1. Find the best risky asset portfolio

   This is called the *minimum variance efficient* (MVE) portfolio, or tangency portfolio, and is the portfolio of risky assets which maximizes the Sharpe ratio.[15]

2. Mix the best risky asset portfolio with the risk-free asset

   This changes the efficient set from the frontier into a wider range of opportunities. The efficient set becomes a *capital allocation line* (CAL), as I explain below.

The procedure of first finding the best risky asset portfolio (the MVE) and then mixing it with the risk-free asset is called two-fund separation. It was originally developed by Tobin (1958), who won the Nobel Prize in 1981. Given the limited computing power at the time, it was huge breakthrough in optimal portfolio choice.

Let's first find the best risky asset portfolio, or MVE. Assume the risk-free rate is 1%. Figure 10 plots our now familiar mean-variance frontier for the G5 and marks the MVE with an asterisk. The dashed diagonal line which goes through the MVE is the capital allocation line. (We have seen the CAL in the previous chapter.) The CAL starts at the risk-free rate, which is 1% in Figure 10 and is tangent to the mean-variance frontier. The tangency point is the MVE. The CAL is obtained by taking all combinations of the MVE with the 1% yielding risk-free asset. The MVE itself corresponds to a 100% position in only G5 equities and the intersection point of the CAL on the *y*-axis at 1% corresponds to a 100% risk-free position.

[Figure 10 here]

[15] I have personally found this terminology confusing because "mean variance efficient portfolio" sounds similar to the "minimum variance portfolio." Unfortunately this terminology is engrained, and I will also use it here.

The slope of the CAL represents the portfolio's Sharpe ratio. Since the CAL is tangent at the MVE, it represents the maximum Sharpe ratio that can be obtained by the investor. A line that starts from the risk-free rate of 1% on the *y*-axis but with a larger angle, which tilts closer to the *y*-axis, cannot be implemented as it does not intersect the frontier. The frontier represents the set of best possible portfolios of G5 risky assets, and we must lie on the frontier. A line that starts from the risk-free rate of 1% on the *y*-axis but with a lesser angle than the CAL, which tilts closer to the *x*-axis, intersects the frontier. These are CALs that can be obtained in actual portfolios but do not represent the highest possible Sharpe ratio. The maximum Sharpe ratio is the tangency point, or MVE.

The MVE in Figure 10 has a Sharpe ratio of 0.671. It consists of:

	US	JP	UK	GR	FR
MVE Portfolio	0.53	0.24	0.12	0.10	0.02

All the portfolios that lie on the CAL have the same Sharpe ratio, except for the 100% risk-free position that corresponds to the risk-free rate of 1% on the *y*-axis.

Now that we've found the best risky MVE portfolio, the investor mixes the risk-free asset with the MVE portfolio. This takes us off the frontier. Finding the optimal combination of the MVE with the risk-free asset is equivalent to finding the point at which the highest possible indifference curve touches the CAL. The tangency point is the investor's optimal portfolio. In Figure 11, we graph the CAL and show the optimal holding in the triangle for an investor with a risk aversion of $\gamma = 3$. The indifference curve that is tangent to this point – which corresponds to the maximum utility for this investor – is also plotted.

[Figure 11 here]

In Figure 11, the tangency point of the highest indifference curve and the CAL lies to the right of the MVE. This means the investor shorts the risk-free asset, or borrows money at 1%, and has a levered position in the MVE. The optimal positions corresponding to the triangle, which is the tangency MVE point, are:

	US	JP	UK	GR	FR	Risk-Free
MVE Portfolio	0.80	0.37	0.18	0.15	0.03	-0.52

The proportions of the risky assets relative to each other in this optimal portfolio are the same as the weights of the MVE. That is, the 0.53 MVE weight of the U.S. is the same as $0.80/(0.80+0.37+0.18+0.15+0.03)$. The optimal position for the $\gamma=3$ investor has a Sharpe ratio of 0.671, which is the same as the CAL as it lies on the CAL.

How much has the investor gained in moving from our previous constrained setting in Section 3.1 (no risk-free asset available) to the example with the risk-free asset included? The certainty equivalent of the tangency position with the short position in the risk-free rate is 0.085. The corresponding certainty equivalent restricting the investor to only risky asset positions obtained earlier is 0.077. Letting the investor have access to the (short) risk-free asset position represents a significant risk-free utility increase of 80 basis points.

3.3 Non-Participation in the Stock Market

Mean-variance investing predicts that with just equities and a risk-free asset, all investors should invest in the stock market except those who are infinitely risk averse. In reality, only half of investors put money in the stock market.[16] This is the *non-participation puzzle*.

Table 12 reports equity market participation rates by households in the U.S. calculated by Li (2009) using data from the Panel Study of Income Dynamics (PSID), a household survey conducted by the University of Michigan, and the Survey of Consumer Finances (SCF), which is conducted by the Federal Reserve Board. (Obviously you can't be poor to have some savings, and researchers go further and exclude those with very meager savings.)

In Table 12, the PSID and SCF stock holdings track each other closely and have hovered around 30%. The SCF also counts stocks included in pension plans and IRAs and when these retirement assets are included, the proportion of households holding stocks increases to around 50%. There has been a general increase in stock market participation when retirement assets are included from around 30% in the 1980s to 50% in 2005. Yet about half of U.S. households do not hold any equities. This is not just a U.S. phenomenon; Laakso (2010) finds that stock market participation in Germany and France, for example is well below 50% for both countries including both direct investments and those made indirectly through mutual funds and investment accounts. Italy, Greece, and Spain have stock participation rates at approximately 10% or below.

[Table 12 here]

[16] The first paper in this literature is Blume, Crockett and Friend (1974). The non-participation puzzle rose to economists' attention with Mankiw and Zeldes (1991) as an explanation for the equity premium puzzle, which I discuss in Chapter XX.

Several explanations have been proposed for the high non-participation in stocks markets. Among these are:

1. Investors do not have mean-variance utility.

   We covered many more realistic utility functions in Chapter XX. Utility functions that can capture the greater risk aversion investors have to downside losses can dramatically lower optimal holdings of equities. Investors with disappointment utility, in particular, will optimally not participate in the stock market, as shown by Ang, Bekaert and Liu (2005).

2. Participation costs.

   These costs include both the transactions costs of actually purchasing equities, but more broadly they include costs of becoming financially educated and "psychic" costs of over-coming fears of equity investments. Consistent with a participation cost explanation, Table 12 shows that more people have invested in stocks as these have become easier to trade since the 1980s with the arrival of online trading and easier access to mutual funds. According to Vissing-Jorgensen (2002), a cost of just $50 in year 2000 prices explains why half of non-stockholders do not hold equities. On the other hand, Andersen and Nielsen (2010) conclude that participation costs cannot be an explanation. In their somewhat morbid paper they examine households inheriting stocks due to sudden deaths. These households pay no participation costs to enter stock markets. Most of these households simply sell the entire stock portfolio and move it into risk-free bonds.

3. Social Factors.

   Several social factors are highly correlated with holding equities. Whether you invest in equities depends on whether your neighbor invests in equities. Non-equity holders may also have less trust in markets than their peers. Investors' expectations of returns are highly dependent on whether they have been burned by previous forays into the stock market.[17] Provocatively, Grinblatt, Keloharju and Linnainmaa (2012) find a link between intelligence and investing in stock markets (the more intelligent the investor, the larger the amounts of stocks held).

Whatever the reason, my advice is don't be a non-participant. Invest in equity markets. You'll reap the equity risk premium too (see Chapter XX). But do so as part of a diversified portfolio.

4. Garbage In, Garbage Out

Mean-variance frontiers are highly sensitive to estimates of means, volatilities, and correlations. Very small changes to these inputs result in extremely different portfolios. These problems have caused mean-variance optimization to be widely derided. The lack of robustness of "optimized" mean-variance portfolios is certainly problematic, but it should not take away from the main message of mean-variance investing that diversified portfolios of assets are better than individual assets. How to find an optimal portfolio mean-variance portfolio, however, is an important question given these difficulties.

4.1 Sensitivity to Inputs

Figure 13 showcases this problem. It plots the original mean-variance frontier estimated from January 1970 to December 2011 in the solid line. The mean of U.S. equity returns in this sample is 10.3%. Suppose we change the mean to 13.0%. The 13.0% choice is well within two standard error bounds of the data estimate of the U.S. mean. The new mean-variance frontier is drawn in the dashed line. There is a large difference between the two.

---

[17] See Hong, Kubik and Stein (2004) for social interaction, Guiso, Sapienza and Zingales (2008) for trust, and Malmendier and Nagel (2011) for investors being affected by past stock market returns.

[Figure 13 here]

The mean-variance frontier portfolios corresponding to a target return of 12% in the original case (U.S. mean is 10.3%) and the new case (U.S. mean is 13.0%) are:

	US mean = 10.3%	US mean = 13.0%
US	-0.0946	0.4101
JP	0.2122	0.3941
UK	0.4768	0.0505
GR	0.1800	0.1956
FR	0.2257	-0.0502

Previously, we didn't have negative weights in the U.S. because we worked with the (constrained) optimal portfolio for a risk aversion of $\gamma = 3$. This corresponds to a target return of 11.0%. The portfolio on the frontier corresponding to a target return of 13% involves a short U.S. position of –9%. This small change in the target return and the resulting large change in the portfolio weights itself showcases the lack of robustness of mean-variance optimization.

Changing the U.S. mean to 13.0% has caused the U.S. position to change from -9% to 41%, the UK position to move from 48% to approximately 5%, and the French position to shrink from 23% to -5%. These are very large changes for a small change in the U.S. mean. No wonder Michaud (1989) calls mean-variance portfolios "error maximizing portfolios."

4.2 What to Do?

Change Utility

My first recommendation is not to use mean-variance utility. Investors are fearful of risk, they care about relative performance (like catching-up-with-the-Joneses or habit utility), and they dread losses much more than they cherish gains. Unfortunately, we have a dearth of commercial optimizers (none that I know about at the time of writing) which can spit out optimal portfolios for more realistic utility functions, but there are plenty of very fancy mean-variance optimizers. If you must insist on (or are forced to use) mean-variance utility…

Use Constraints

Jagannathan and Ma (2003) show that imposing constraints helps a lot. Indeed, raw mean-variance weights are so unstable that practitioners using mean-variance optimization always impose constraints. Constraints help because they bring back unconstrained portfolio weights to economically reasonable positions. Thus, they can be interpreted as a type of *robust estimator* which *shrinks* unconstrained weights back to reasonable values. We can do this more generally if we…

Use Robust Statistics

Investors can significantly improve estimates of inputs by using robust statistical estimators. One class of estimators is *Bayesian shrinkage* methods.[18] These estimators take care of outliers and extreme values that play havoc with traditional classical estimators. They shrink estimates back to a *prior*, or model, which is based on intuition or economics. For example, the raw mean estimated in a sample would not be used, but the raw mean would be adjusted to the mean implied by the CAPM (see Chapter XX), a multifactor model, or some value computed from fundamental analysis. Covariances can also be shrunk back to a prior where each stock in an industry, say, has the same volatility and correlation – which is reasonable if we view each stock in a given industry as similar to the others.[19]

[18] Introduced by James and Stein (1961).

[19] See Ledoit and Wolf (2003) and Wang (2005), among others. Strictly speaking, the mean-variance solution involves an inverse of a covariance matrix, so we should shrink the inverse covariance rather than the covariance.

No statistical method, however, can help you if your data are lousy.

Don't Just Use Historical Data

Investors must use past data to estimate inputs for optimization problems. But many investors simply take historical averages on short, rolling samples. This is the worst thing you can do.

In drawing all of the mean-variance frontiers for the G5, or various subsets of countries, I used historical data. I plead guilty. I did, however, use a fairly long sample, from January 1970 to December 2011. Nevertheless, even this 40-year sample is relatively short. You should view the figures in this chapter as what has transpired over the last 40 years, and not as pictures of what will happen in the future. As the investment companies like to say in a small voice, past performance is no guarantee of future returns. The inputs required for mean-variance investing – expected returns, volatilities, and correlations – are statements about what we think will happen in the future.

Using short data samples to produce estimates for mean-variance inputs is very dangerous. It leads to pro-cyclicality. When past returns have been high, current prices are high. But current prices are high because future returns may actually be low. While predictability in general is very weak, Chapter XX provides evidence that there is some. Thus, using a past data sample to estimate a mean produces a high estimate right when future returns are likely to be low. These problems are compounded when more recent data is weighted more heavily, which occurs in techniques like exponential smoothing.

An investor using a sample where returns are stable, like the mid-2000s right before the financial crisis, would produce volatility estimates that are low. But these times of low volatilities (and

---

This is done by Kourtis, Dotsis, and Markellos (2009). Tu and Zhou (2011) show shrinkage methods can be used to combine naïve and sophisticated diversification strategies in the presence of estimation risk.

high prices) are actually periods when risk is high. Sir Andrew Crockett of the Bank of England says (with my italics):[20]

> The received wisdom is that risk increases in the recessions and falls in booms. In contrast, it may be more helpful to think of risk as *increasing* during upswings, as financial imbalances build up, and *materializing* in recessions.

The low estimates of volatilities computed using short samples ending in 2007 totally missed the explosions in risks that materialized in the 2008-2009 financial crisis.

Use Economic Models

I believe that *asset allocation is fundamentally a valuation problem.* The main problem with using purely historical data, even with the profession's best econometric toolkit, is that it usually ignores economic value. Why would you buy more of something if it is expensive?

Valuation requires an *economic framework*. Economic models could also be combined with statistical techniques. This is the approach of Black and Litterman (1991), which is popular because it delivers estimates of expected returns that are "reasonable" in many situations. Black and Litterman start with the fact that we observe market capitalizations, or market weights. The market is a mean-variance portfolio implied by the CAPM equilibrium theory (see Chapter XX). Market weights, which reflect market prices, embody the market's expectations of future returns. Black and Litterman use a simple model – the CAPM – to reverse engineer the future expected returns (which are unobservable) from market capitalizations (which are observable). In addition, their method also allows investors to adjust these market-based weights to investors' own beliefs using a shrinkage estimator. I will use Black-Litterman in some examples below in Section 6.

[20] Marry the Micro- and Macro-Prudential Dimensions of Financial Stability, speech at the Eleventh International Conference of Banking Supervisors, 20-21 September 2000, http://www.bis.org/speeches/sp000921.htm

An alternative framework in estimating inputs is to work down to the underlying determinants of value. In Part II of this book, I will build a case for thinking about the underlying factors which drive the risk and returns of assets. Understanding how the factors influence returns, and finding which factor exposures are right for different investors in the long run, enables us to construct more robust portfolios.

The concept of *factor investing*, where we look through asset class labels to the underlying factor risks, is especially important in maximizing the benefits of diversification. Simply giving a group of investment vehicles a label, like “private equity” or “hedge funds,” does not make them asset classes. The naïve approach to mean-variance investing treats these as separate asset classes and places them straight into a mean-variance optimizer. Factor investing recognizes that private equity and hedge funds have many of the same factor risks as traditional asset classes. Diversification benefits can be overstated, as many investors discovered in 2008 when risky asset classes came crashing down together, if investors do not look at the underlying factor risks.

Keep It Simple (Stupid)

The simple things always work best. The main principle of mean-variance investing is to hold diversified portfolios. There are many simple diversified portfolios, and they tend to work much better than the optimized portfolios computed in the full glory of mean-variance quadratic programming in equations (1.2) and (1.3). Simple portfolios also provide strong benchmarks to measure the value-added of more complicated statistical and economic models.

The simplest strategy – an equally weighted portfolio – turns out to be one of the best performers, as we shall now see.

5. Special Mean-Variance Portfolios

In this section I run a horse race between several portfolio strategies, each of which is a special case of the full mean-variance strategy. Diversification is common to all the strategies, but they build a diversified portfolio in different ways. This leads to very different performance.

5.1 Horserace

I take four asset classes: U.S. government bonds (Barcap U.S. Treasury), U.S. corporate bonds (Barcap U.S. Credit), U.S. stocks (S&P 500), and international stocks (MSCI EAFE), and track performance of various portfolios from January 1978 to December 2011. The data are sampled monthly. The strategies implemented at time t are estimated using data over the past five years, $t-60$ to t. The first portfolios are formed at the end of January 1978 using data from January 1973 to January 1978. The portfolios are held for one month, and then new portfolios are formed at the end of the month. I use one-month T-bills as the risk-free rate. In constructing the portfolios, I restrict shorting down to -100% on each asset.

Using short, rolling samples opens me up to the criticisms of the previous section. I do this deliberately because it highlights some of the pitfalls of (fairly) unconstrained mean-variance approaches. Consequently, it allows us to understand why some special cases of mean-variance perform well and others badly.

I run a horserace between:

Mean-Variance Weights where the weights are chosen to maximize the Sharpe ratio.

Market Weights which are given by market capitalizations of each index

Diversity Weights which are (power) transformations of market weights recommended by Fernholz, Garvy and Hannon (1998).

Equal Weights, or the 1/*N* rule, which simply holds one-quarter in each asset class. Duchin and Levy (2009) call this strategy the "Talmudic rule" since the Babylonian Talmud recommended this strategy approximately 1,500 years ago: "A man should always place his money, one third in land, a third in merchandise, and keep a third in hand."

Risk Parity is the strategy du jour and chooses asset weights proportional to the inverse of variance [*Risk Parity (Variance)*] or to the inverse of volatility [*Risk Parity (Volatility)*]. The term "risk parity" was originally coined by Edward Qian in 2005.[21] It has shot to prominence in the practitioner community because of the huge success of Bridgewater Associates, a large hedge fund with a corporate culture that has been likened to a cult.[22] Bridgewater launched the first investment product based on risk parity called the "All Weather" fund in 1996. In 2011 the founder of Bridgewater, Ray Dalio, earned $3.9 billion, which was approximately the GDP of Swaziland that year.[23] (I cover hedge funds in Chapter XX). Bridgewater's success has inspired many copycats. The original implementations of risk parity were done on variances, but there are fans of weighting on volatilities.[24]

Minimum Variance is the portfolio on the left-most tip of the mean-variance frontier which we've seen before.

---

[21] Qian, E., 2005, Risk Parity Portfolios: Efficient Portfolios Through True Diversification, PanAgora.
[22] Kevin Roose, Pursuing Self-Interest in Harmony with the Laws of the Universe and Contributing to Evolution is Universally Rewarded, New York Magazine, Apr 10, 2011,
[23] Compensation number from Alpha Magazine's Top-Earning Hedge Fund Managers list in 2011. Swaziland's GDP in 2011 was USD 4.00 billion as reported by the World Bank.
[24] Versions of risk parity where the weights are inversely proportional to volatility are advocated by Martellini (2008) and Choueifaty and Coignard (2008).

Equal Risk Contributions form weights in each asset position such that they contribute equally to the total portfolio variance.[25]

Kelly (1956) **Rule** is a portfolio strategy that maximizes the expected log return. In the very long run, it will maximize wealth. (I explain more in Chapter XX.)

Proportional to Sharpe Ratio is a strategy that holds larger positions in assets that have larger realized Sharpe ratios over the last five years.

Over this sample a 100% investment in U.S. equities had a Sharpe ratio of 0.35. This will turn out to be dominated by all the diversified portfolios in the horserace, except for the most unconstrained mean-variance portfolio. This is consistent with the advice from the example in Section 2 where no-one should hold a 100% U.S. equity portfolio.

Table 14 reports the results of the horserace. Mean-variance weights perform horribly. The strategy produces a Sharpe ratio of just 0.07 and it is trounced by all the other strategies. Holding market weights does much better, with a Sharpe ratio of 0.41. This completely passive strategy outperforms the Equal Risk Contributions and the Proportional to Sharpe Ratio portfolios (with Sharpe ratios of 0.32 and 0.45, respectively). Diversity Weights tilt the portfolio towards the asset classes with smaller market caps, and this produces better results than market weights.

[Table 14 here]

The simple Equal Weight strategy does very well with a Sharpe ratio of 0.54. What a contrast with this strategy versus the complex mean-variance portfolio (with a Sharpe ratio of 0.07)! The Equal Weight strategy also outperforms the market portfolio (with a Sharpe ratio of 0.41). De

[25] See Qian (2006) and Maillard, Roncalli and Teiletche (2010).

Miguel, Garlappi and Uppal (2009) find that the simple 1/*N* rule outperforms a large number of other implementations of mean-variance portfolios, including portfolios constructed using robust Bayesian estimators, portfolio constraints, and optimal combinations of portfolios which I covered in Section 4.2. The 1/N portfolio also produces a higher Sharpe ratio than each individual asset class position. (U.S. bonds had the highest Sharpe ratio of 0.47 in the sample.)

Risk Parity does even better than 1/*N*. The outperformance, however, of the plain-vanilla Risk Parity (Variance) versus Equal Weights is small. Risk Parity (Variance) has a Sharpe ratio of 0.59 compared to the 0.54 Sharpe ratio for Equal Weights. Risk Parity based on volatility does even better and has the highest out-of-sample Sharpe ratio of all the strategies considered, at 0.65. When risk parity strategies are implemented on more asset classes (or factor strategies, see Chapter XX) in practice, historical Sharpe ratios for risk parity strategies have often exceeded one.

The outperformance of the Minimum Variance portfolio versus standard mean-variance weights and the market portfolio has been known for at least 20 years.[26] One reason that minimum variance portfolios outperform the market is that there is a tendency for low volatility assets to have higher returns than high volatility assets, which I cover in Chapter XX, and the minimum variance portfolio overweights low volatility stocks. The last two strategies in Table 14 are the Kelly Rule and the Proportional to Sharpe Ratio strategies. Both also outperform the Mean-Variance Weights and the market portfolio in terms of Sharpe ratios. You would have been better off, however, using the simple 1/*N* strategies in both cases.

[26] At least since Haugen and Baker (1991).

Figure 15 plots cumulated returns of the Market Weights, Equal Weights, Risk Parity (Variance), and Mean-Variance strategies. All of these returns are scaled to have the same volatility as the passive market weight strategy. The dominance of the Equal Weights and Risk Parity strategies are obvious. In addition, Figure 15 shows that the Risk Parity strategy has the smallest drawdown movements of the four strategies.

[Figure 15 here]

5.2 Why Does Unrestricted Mean-Variance Perform So Badly?

The optimal mean-variance portfolio is a complex function of estimated means, volatilities, and correlations of asset returns. There are many parameters to estimate. Optimized mean-variance portfolios can blow up when there are tiny errors in any of these inputs. In the horserace with four asset classes, there are just 14 parameters to estimate and even with such a low number mean-variance does badly. With 100 assets, there are 5,510 parameters to estimate.[27] For 5,000 stocks (approximately the number listed in U.S. markets) the number of parameters to estimate is over 12,000. The potential for errors is enormous.

Let's view what happens when we move from the optimal mean-variance strategy and turn off some of the inputs, so that we are relieved from estimating means, volatilities, correlations, or combinations of all three. Table 16 lists some special cases as the restrictions are imposed.

[Table 16 here]

The minimum variance portfolio is a special case of full mean-variance that does not estimate means and in fact assumes that the means are all equal. Risk parity is a special case of mean-variance that does not estimate means or correlations; it implicitly assumes that all assets have

[27] For N assets, you have N means and $N \times (N+1)/2$ elements in the covariance matrix.

the same mean and all assets are uncorrelated. The equally weighted portfolio has nothing to estimate. It is also a special case of mean-variance and assumes all assets are identical.

I have included market weights in Table 16. Like equal weights, there are no parameters to estimate using market weights. The important difference between equal weights and market weights is that the equal weighted portfolio is an *active strategy*. It requires trading every period to rebalance back to equal weights. In contrast, the market portfolio is *passive* and requires no trading. The action of rebalancing in equal weights imparts this strategy with a *rebalancing premium*. Rebalancing also turns out to be the foundation of an optimal long-run investing strategy. I cover these topics in the next chapter.

As you go down Table 16 from full-blown mean-variance to equal weights or the market portfolio, you estimate fewer parameters and thus there are fewer things that can go wrong with the mean-variance optimization. The extreme cases are the equal weight or market weight positions which require no analysis of data (except looking at market capitalizations in the case of the market portfolio).

Long-run means are very tricky to estimate. Sampling at weekly or daily frequencies does not allow you to more accurately estimate means – only extending the sample allows you to pin down the mean more precisely.[28] For any asset like the S&P 500, the only way to gauge the long-run return is to look at the index level at the beginning and at the end of the sample and divide it by time. It doesn't matter how it got to that final level; all that matters is the ending index value. Hence, we can only be more certain of the mean return if we lengthen time. This makes

[28] This is shown in a seminal paper by Merton (1980).

forecasting returns very difficult. (I cover this further in Chapter XX.) The minimum variance portfolio outperforms mean-variance because we remove all the errors associated with means.

Volatilities are much more predictable than means. High frequency sampling allows you to estimate variances more accurately even though it does nothing for improving estimates of means. Higher frequency data also allows you to produce better estimates of correlations. But correlations can switch signs while variances can only be positive. Thus, variances are easier to estimate than correlations. Poor estimates of correlations also have severe effects on optimized mean-variance portfolios; small changes in correlations can produce big swings in portfolio weights.[29] Risk parity turns off the noise associated with estimating correlations. (More advanced versions of risk parity do take into account some correlation estimates.) In the horserace, risk parity produced higher Sharpe ratios (0.59 and 0.65 using variances and volatilities, respectively) than the minimum variance portfolio (which had a Sharpe ratio of 0.52).

In summary, the special cases of mean-variance perform better than the full mean-variance procedure because fewer things can go wrong with estimates of the inputs.

5.3 Implications for Asset Owners

I went from the full mean-variance case to the various special cases in Table 16 by adding restrictions. To practice mean-variance investing, the investor should start at the bottom of Table 16 and begin from market weights. If you can't rebalance, hold the market. (The horserace results in Table 14 show that you will do pretty well, and much better than mean-variance.)

[29] Green and Hollifield (1992) provide bounds on the average correlation between asset returns that are required for portfolios to be well balanced.

If you can rebalance, move to the equal weight portfolio. You will do better than market weights in the long run. Equal weights may be hard to implement for very large investors because when trades are very large, investors move prices and incur substantial transactions costs. It turns out that any well-balanced, fixed-weight allocation works well. Jacobs, Muller and Weber (2010) analyze more than 5000 different portfolio construction methods and find that any simple fixed-weight allocation thrashes mean-variance portfolios.

Now if you can estimate variances or volatilities, you could think about risk parity. My horserace only estimated volatility by taking the realized volatility over a past rolling sample. Ideally we want estimates of future volatility. There are very good models for forecasting volatility based on GARCH or stochastic volatility models, which I cover in Chapter XX.

Suppose the hotshot econometrician you've just hired can also accurately estimate correlations as well as volatilities. Now you should consider relaxing the correlation restriction from risk parity. Finally, and the hardest of all, is the case if you can accurately forecast means. If and only if you can do this, should you consider doing (fairly unconstrained) mean-variance optimization.

Common to all these portfolio strategies is the fact that they are diversified. This is the message you should take from this chapter. Diversification works. Computing optimal portfolios using full mean-variance techniques is treacherous, but simple diversification strategies do very well.

Warning on Risk Parity

The second panel of Table 14 reports the average weights in each asset class from the different strategies. Risk parity did very well, especially the risk parity strategy implemented with volatilities, because it over-weighted bonds during the sample. Risk parity using variances held, on average, 51% in U.S. Treasuries and 36% in corporate bonds versus average market weights of 14% and 8%, respectively. There were even larger weights on bonds when risk parity is implemented weighting by volatilities rather than variances. Chapter XX will show that interest rates trended downwards from the early 1980s all the way to the 2011 and bonds performed magnificently over this period. This accounts for a large amount of the out-performance of risk parity over the sample.

Risk parity requires estimates of volatilities. Volatilities are statements of risk. Risk and prices, which embed future expected returns, are linked in equilibrium (see Chapter XX). Howard Marks (2011), a hedge fund manager, says:

> The value investor thinks of high risk and low prospective return as nothing but two sides of the same coin, both stemming primarily from high prices.

Risk parity overweights assets that have low volatilities. Past volatilities tend to be low precisely when today's prices are high. Past low volatilities and high current prices, therefore, coincide with elevated risk today and in the future.[30] At the time of writing, Treasury bonds have record low yields and so bond prices are very high. Risk-free U.S. Treasuries can be the riskiest investment simply because of high prices. And at a low enough price, risky equities can be the safest investments. Risk parity, poorly implemented, will be pro-cyclical because it ignores valuations, and its pro-cyclicality will manifest over decades because of the slow mean reversion of interest rates.

[30] A contrary opinion is Asness, Frazzini and Pedersen (2012). They argue that investors are averse to leverage and this causes safe assets to have higher risk-adjusted returns than riskier assets. Risk parity allows some investors to exploit this risk premium.

6. Norway and Wal-Mart Redux

Diversification involves holding many assets. In Section 2 we saw that when we started with the U.S. and then progressively added additional countries to get to the G5 (U.S., Japan, UK, Germany, France), there were tremendous benefits from adding assets. Conversely, if we go backwards and remove assets from the G5 we decrease the diversification benefits.

Norway is excluding Wal-Mart on the basis of alleged violations of human rights and other ethical considerations. Removing any asset makes an investor worse off, except in the case when the investor is not holding that asset in the first place. When we are forced to divest an asset, what is the reduction in diversification benefits?

6.1 Loss of Diversification Benefits

When I teach my case study on Norway and its disinvestment of Wal-Mart in my MBA class on asset management, I ask the students to compute the lost diversification benefits from throwing out Wal-Mart. We can do this using mean-variance investing concepts. I will not do the same exercise as I give my students, but I will go through an experiment that removes various sectors from a world portfolio. This is also relevant to Norway because as of January 2010, GPFG no longer holds any stocks in the tobacco sector. Other prominent funds like CalPERS and CalSTRS are also tobacco-free.

I take the FTSE All World portfolio as at the end of June 2012. This index has 39 sectors and 2871 stocks at this date. What happens if we the eliminate tobacco? Let's use mean-variance concepts to quantify the loss of diversification benefits. In this exercise, I compute variances and correlations using a Bayesian shrinkage estimator operating on CAPM betas (see Ledoit and Wolf (2003)) and estimate expected returns using a variant of Black-Litterman (1991). I set the risk-free rate to be 2%. I compute mean-variance frontiers constraining the sector weights to be positive.

We start with all the sectors. Then, I'll remove tobacco. Next I'll remove the aerospace and defense sector. Norway has selectively divested some companies in this sector because it automatically excludes all companies involved in the manufacture of nuclear weapons and cluster munitions.[31] A final exclusion I'll examine is banks. Sharia law prohibits the active use of derivatives and debt as profit-making activities. Thus, it is interesting to see what diversification costs are routinely incurred by some Sharia compliant funds.

As we move from the full universe to the restricted universe, we obtain the following minimum standard deviations and maximum Sharpe ratios:

	Minimum Volatility	Maximum Sharpe Ratio
All Sectors	0.1205	0.4853
No Tobacco	0.1210	0.4852
No Tobacco and Aerospace & Defense	0.1210	0.4852
No Tobacco, Aerospace & Defense, and Banks	0.1210	0.4843

The increase in the minimum volatility is tiny – from 12.05% to 12.10%. Similarly, the reduction in maximum Sharpe ratio is negligible, moving from 0.4853 for the full universe to 0.4852 when tobacco is removed, to 0.4843 when all three sectors are removed. Thus, the loss in diversification for removing one, or a few, sectors is extremely small. Figure 16 plots the (constrained) mean-variance frontiers for each set of sectors. They are indistinguishable on the

[31] As of June 2012 there were 19 defense manufacturers excluded. A full list of GPFG's current exclusions is at http://www.regjeringen.no/en/dep/fin/Selected-topics/the-government-pension-fund/responsible-investments/companies-excluded-from-the-investment-u.html?id=447122

graph. Norway is effectively losing nothing by selling Wal-Mart.[32] It is also effectively losing nothing by excluding tobacco.

[Figure 16 here]

The extremely small costs of divestment are not due to the portfolios having zero holdings in the sectors that are being removed. In the full universe, the portfolio with the maximum Sharpe ratio contains 1.53% tobacco, 1.19% aerospace and defense, and 9.52% banks. Even removing an approximately 10% bank holding position has negligible cost in terms of diversification losses.

Diversification losses are so small because extra diversification benefits going from 38 to 39 sectors, or even 36 to 39 sectors, are tiny (recall there are *decreasing marginal diversification benefits*). In Section 2 when we added Germany and France to the G3 (U.S., Japan, and UK), there was a much smaller shift in the frontier compared with moving from the U.S.-Japan to the G3 (see Figures 4 and 5). In our sector example the small marginal diversification benefits come about because there are few opportunities for that lost sector to pay off handsomely when the other 38 sectors tank.

6.2 Socially Responsible Investing

From the mean-variance investing point of view, SRI must always lose money because it reduces diversification benefits. Could it make money as an active management (alpha) strategy? Studies like Kempf and Osthoff (2007) find that stocks that rank highly on KLD measures have high returns. The KLD ratings are constructed by MSCI KLD and consider various social and environmental criteria. While Norway has thrown Wal-Mart out, Wal-Mart gets high KLD

[32] This does not include the actual transactions costs of divestment. My case “The Norwegian Government Pension Fund:The Divestiture of Wal-Mart Stores Inc.”, Columbia CaseWorks, ID#080301, 2010, also estimates these transactions costs.

ratings partly because it has taken many steps to reduce its carbon footprint. On the other hand, Geczy, Stambaugh and Levin (2004) find that SRI mutual funds underperform their peers by 30 basis points per month. Harrison Hong, a Princeton academic and one of the leading scholars on socially responsible investing, shows in Hong and Kacperczyk (2009) that "sin" stocks like tobacco, firearms manufacturers, and gambling have higher risk-adjusted returns than comparable stocks.[33]

In his magnum opus written in 1936, Keynes says, "There is no clear evidence that the investment policy which is socially advantageous coincides with that which is most profitable." My reading of the SRI literature is that Keynes's remarks are equally applicable today.

I believe there is some scope for SRI in active management. There are some characteristics of firms that predict returns. Some of these effects are so pervasive that they are factors, as I discuss in Chapter XX. Many of the firms that rank highly on SRI measures are likely to be more transparent, have good governance, senior managers who are less likely to steal, efficient inventory management, use few accounting gimmicks, and respond well to shareholder initiatives. These are all characteristics that we know are linked to firm performance. A simple example is limiting the rents firm managers can extract from shareholders allows shareholders to take home more. Gompers, Ishii and Metrick (2003) create a governance index to rank companies from "dictatorships" to "republics." Companies that have many provisions to entrench management, anti-takeover provisions, and limit proxy votes, for example, would be

[33] Hong, Kubik and Scheinkman (2012) argue against the hypothesis of "doing well by doing good" and argue exactly the opposite. They show that corporate social responsibility (CSR) is costly for firms and firms only do good when they are not financially constrained. In this sense, CSR is costly for firms.

defined as dictatorships. They find that republics – which are also likely to rank high on SRI criteria – have higher returns than dictatorships.[34]

If you are able to pick firms based on particular properties and characteristics, and these are related to SRI, then you might be able to outperform. This method of SRI does not throw out companies; it actively selects companies on SRI criteria but does not limit the manager's investment opportunities by excluding companies.[35] Like all active strategies, it is hard to beat factor-based strategies (see Chapter XX).

SRI also serves an important role when it reflects the preferences of an asset owner. In Norway's case, practicing SRI gives the sovereign wealth fund legitimacy in the eyes of its owners – the Norwegian people.[36] SRI is the asset owners' choice.

The main message from mean-variance investing is to hold a diversified portfolio, which Norway does. Diversification benefits are a free lunch according to mean-variance investing. Doing SRI by exclusions is costly because it shrinks diversification benefits. But starting from a well-diversified portfolio (and some of the best-performing diversified portfolios are the most simple, like equal weighted and market weighted portfolios), the loss from excluding a few stocks is tiny. The cost of being socially responsible for Norway is negligible.

At the time of writing in 2012, Wal-Mart was still on Norway's excluded list.

[34] There is debate about whether this effect has persisted after the original Gompers, Ishii and Metrick (2003) study and whether the effect is about risk or mispricing (see Chapter XX). Cremers and Ferrel (2012) find stocks with weak shareholder rights have negative excess returns over 1978 to 2007 while Bebchuck, Cohen and Wang (2011) argue the original Gompers, Ishii and Metrick results disappear over the 2000s.
[35] A more aggressive form of doing this is shareholder activism. Shareholder activism by hedge funds adds significant value (see Brav et al. (2008)) even though the evidence for mutual funds and pension funds adding value for shareholders is decidedly mixed (see Gillan and Starks (2007). Dimson, Karakas and Li (2012) find that corporate social responsibility activist engagements generate abnormal returns.
[36] See Ang (2012b).

Figure 1

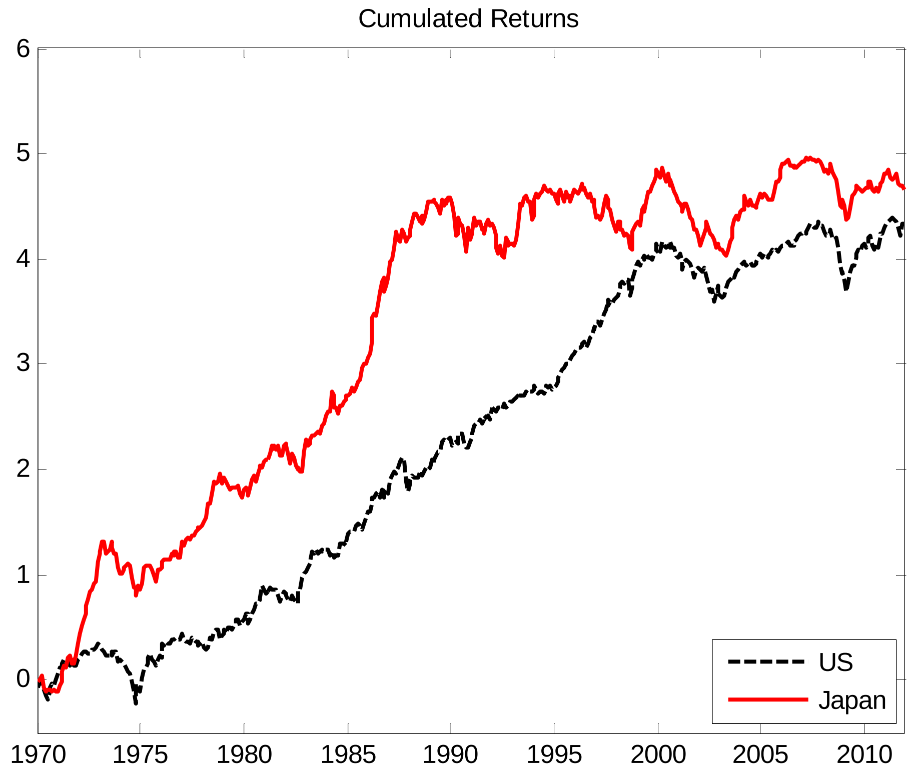

Cumulated Returns
6
5
4
3
2
1
0
1970
1975
1980
1985
1990
1995
2000
2005
2010
US
Japan

Figure 2

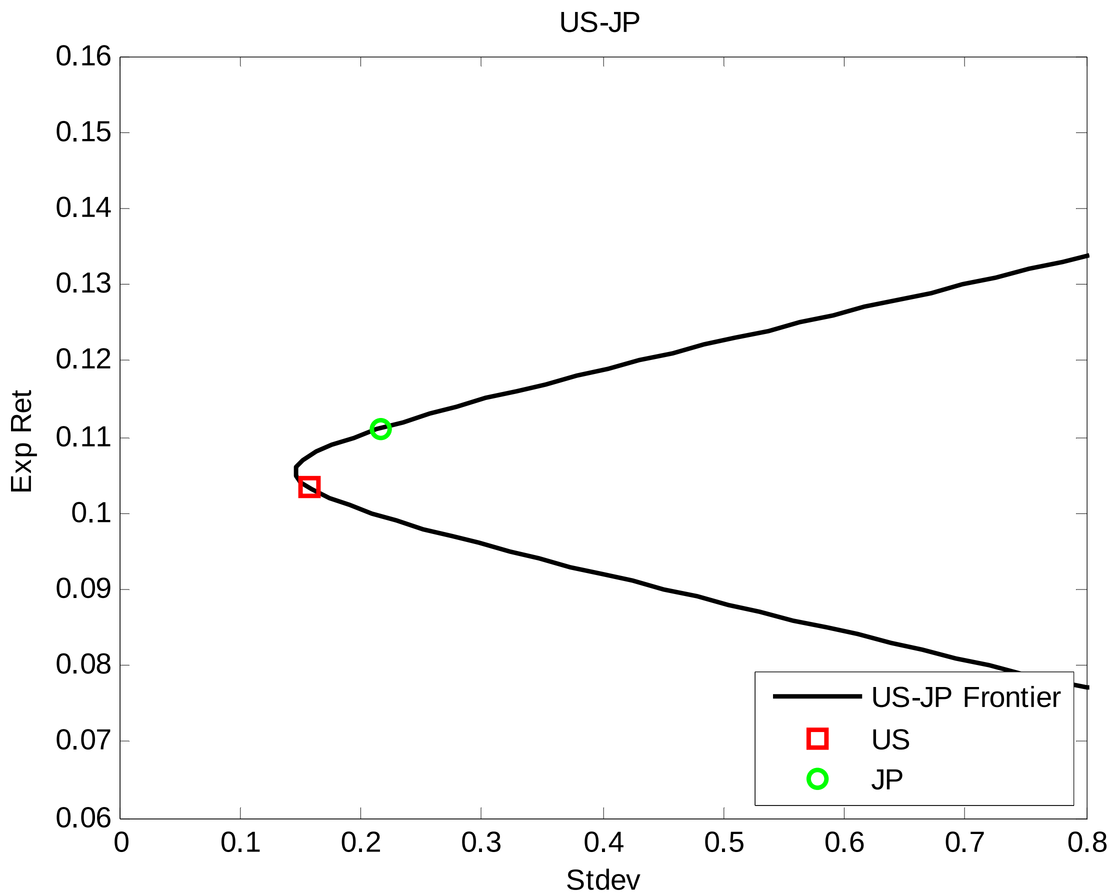

US-JP
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0.07
0.06
Exp Ret
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Stdev
US-JP Frontier
US
JP

Figure 3

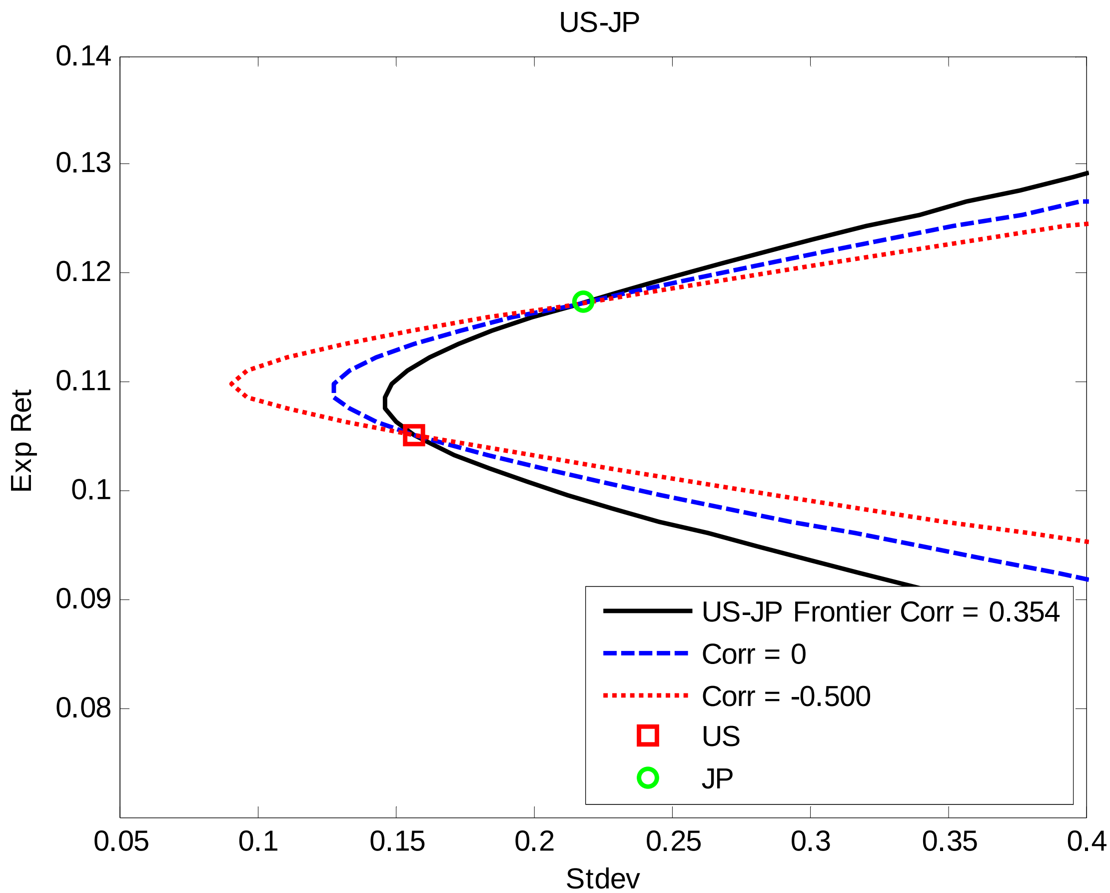

US-JP
0.14
0.13
0.12
0.11
0.1
0.09
0.08
Exp Ret
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Stdev
US-JP Frontier Corr = 0.354
Corr = 0
Corr = -0.500
US
JP

Figure 4

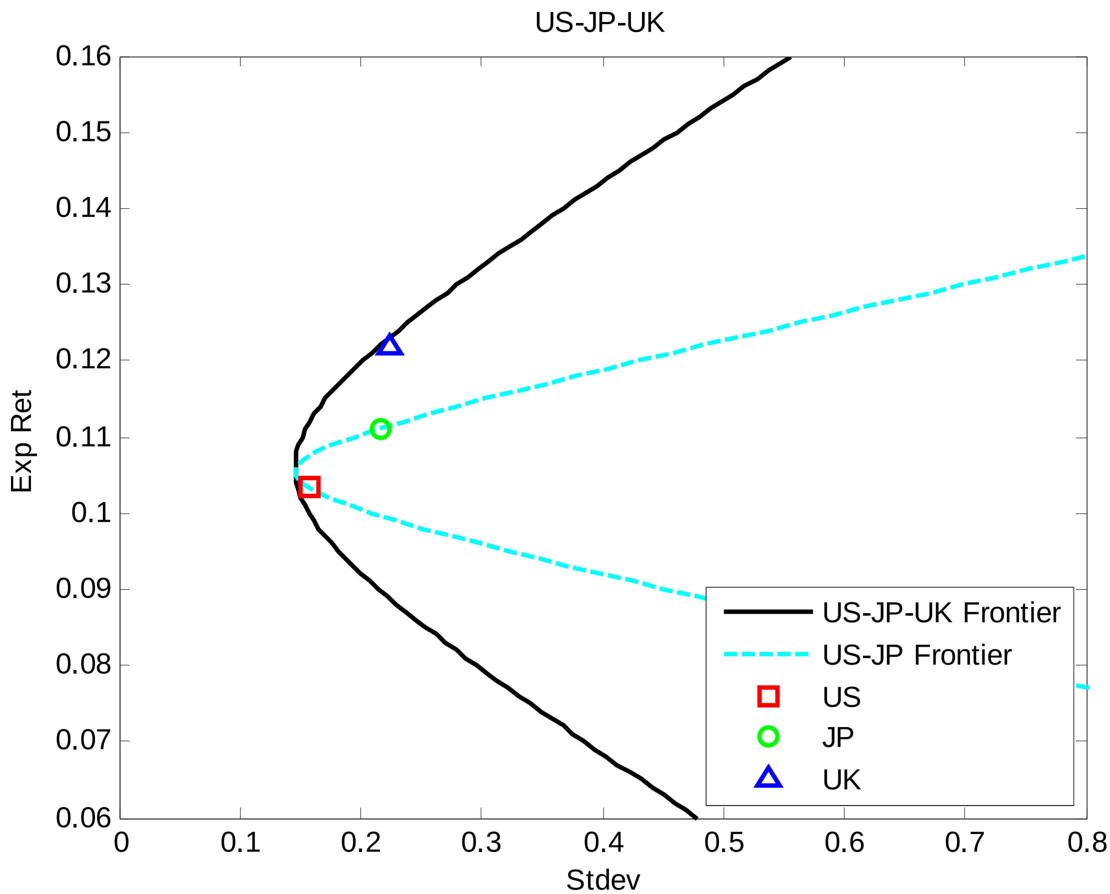

US-JP-UK
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0.07
0.06
Exp Ret
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Stdev
US-JP-UK Frontier
US-JP Frontier
US
JP
UK

Figure 5

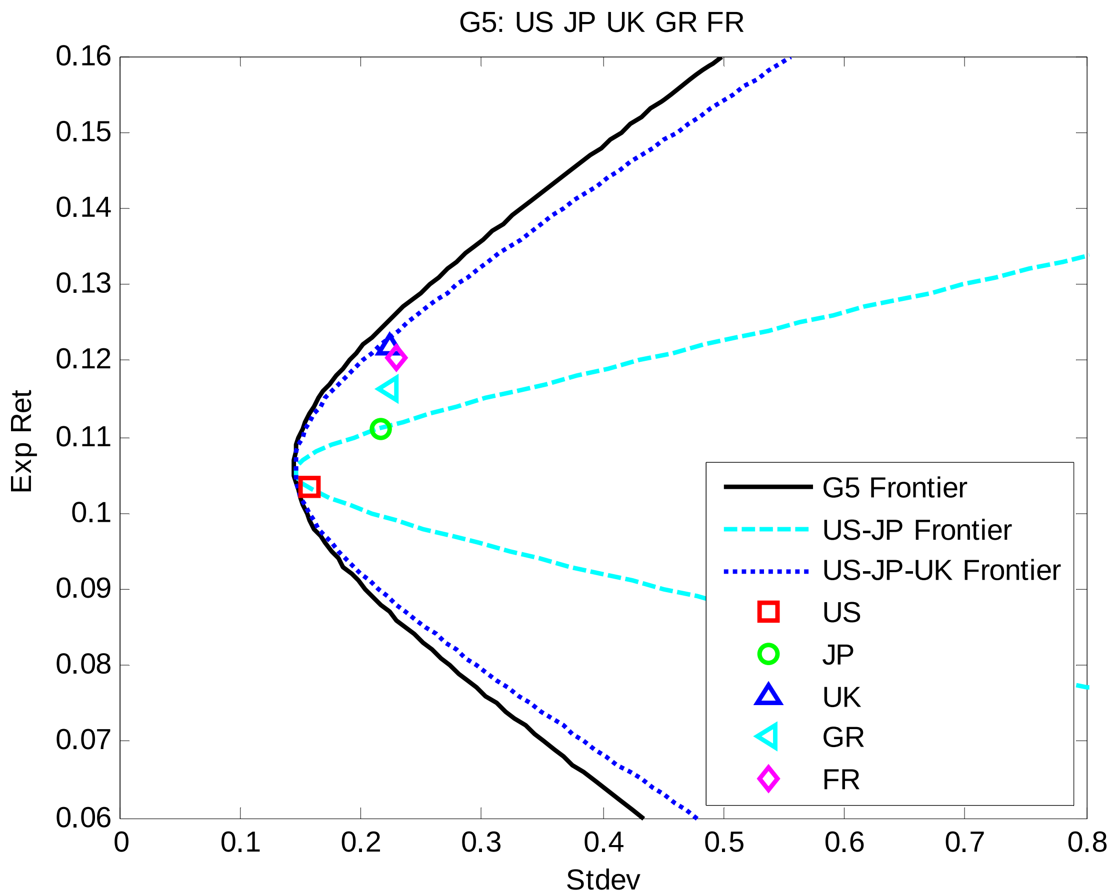

G5: US JP UK GR FR
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0.07
0.06
Exp Ret
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Stdev
G5 Frontier
US-JP Frontier
US-JP-UK Frontier
US
JP
UK
GR
FR

Figure 6

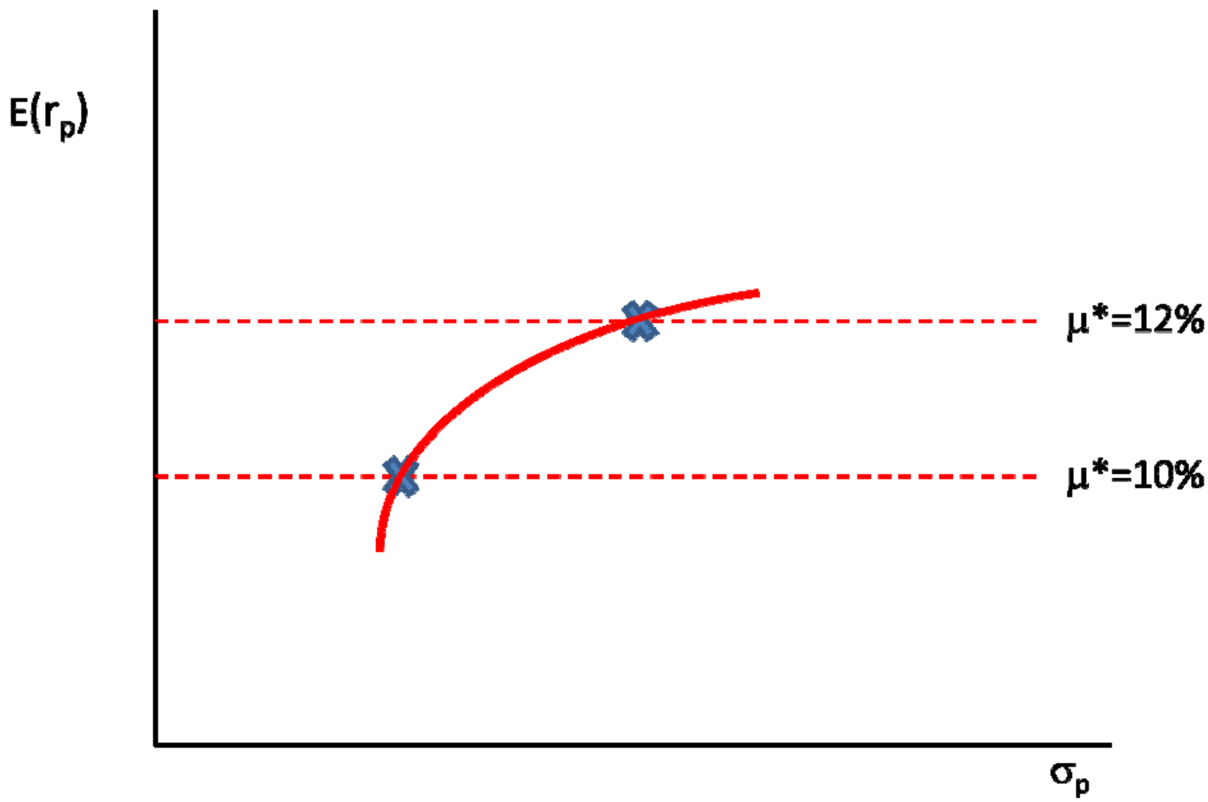


Figure 7

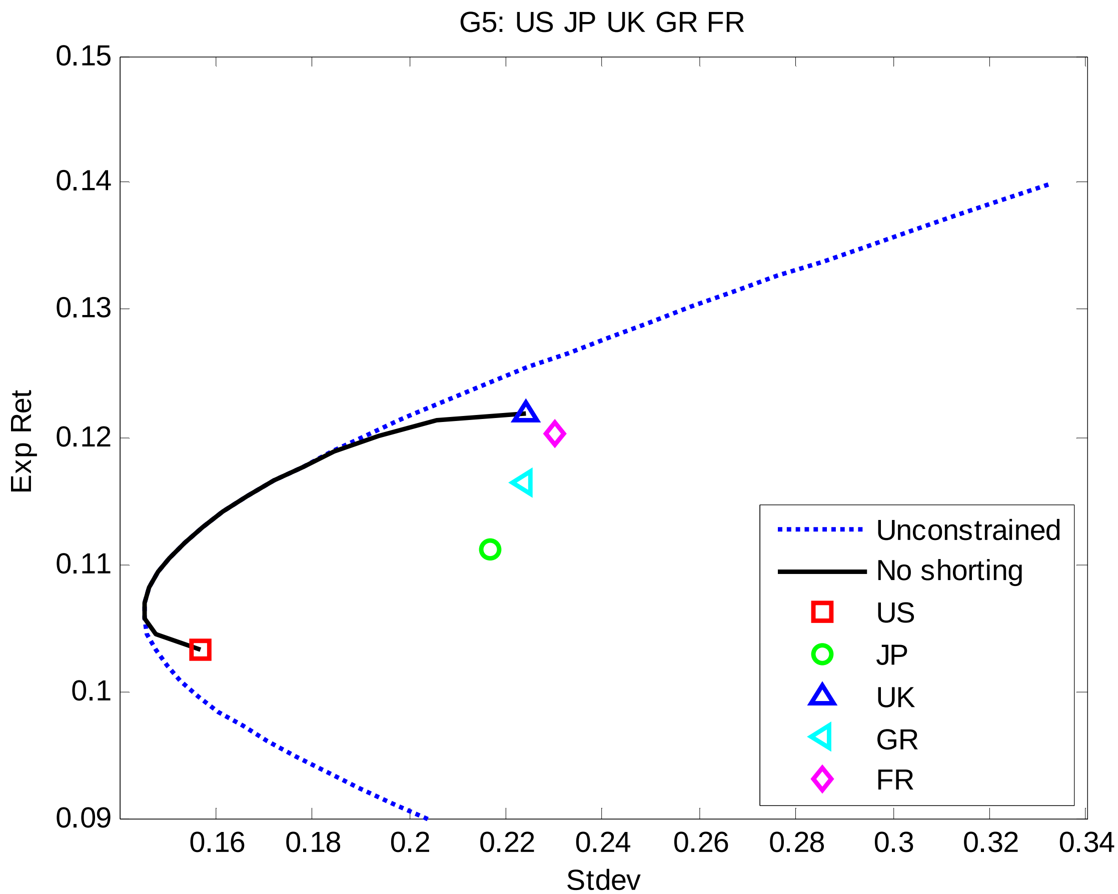

G5: US JP UK GR FR
Exp Ret
Stdev
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.16
0.18
0.2
0.22
0.24
0.26
0.28
0.3
0.32
0.34
Unconstrained
No shorting
US
JP
UK
GR
FR

Figure 8

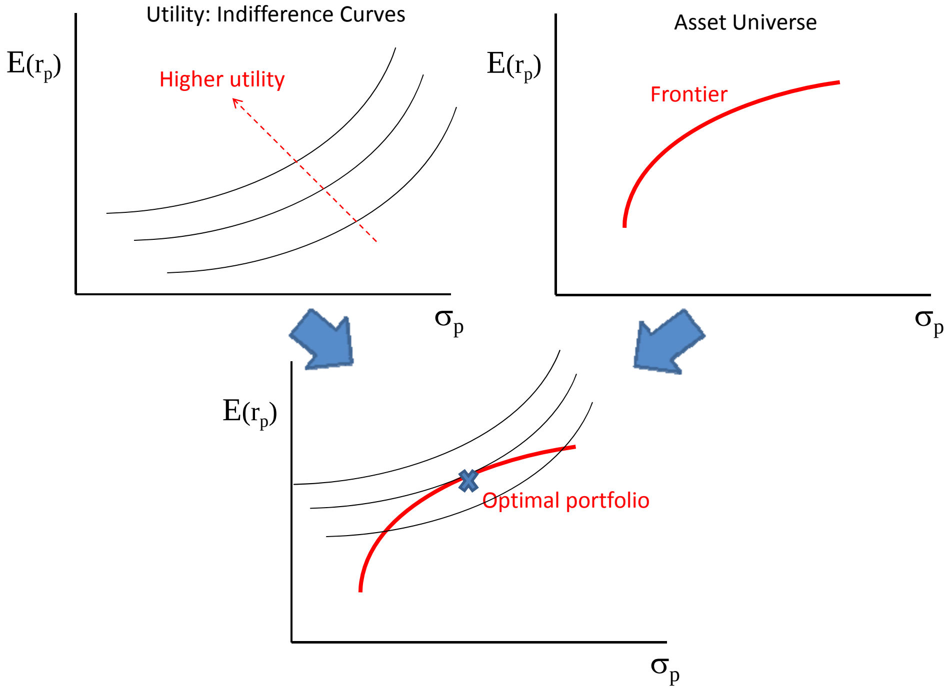

Utility: Indifference Curves
E(r_p)
Higher utility
σ_p
Asset Universe
E(r_p)
Frontier
σ_p
E(r_p)
Optimal portfolio
σ_p

Figure 9

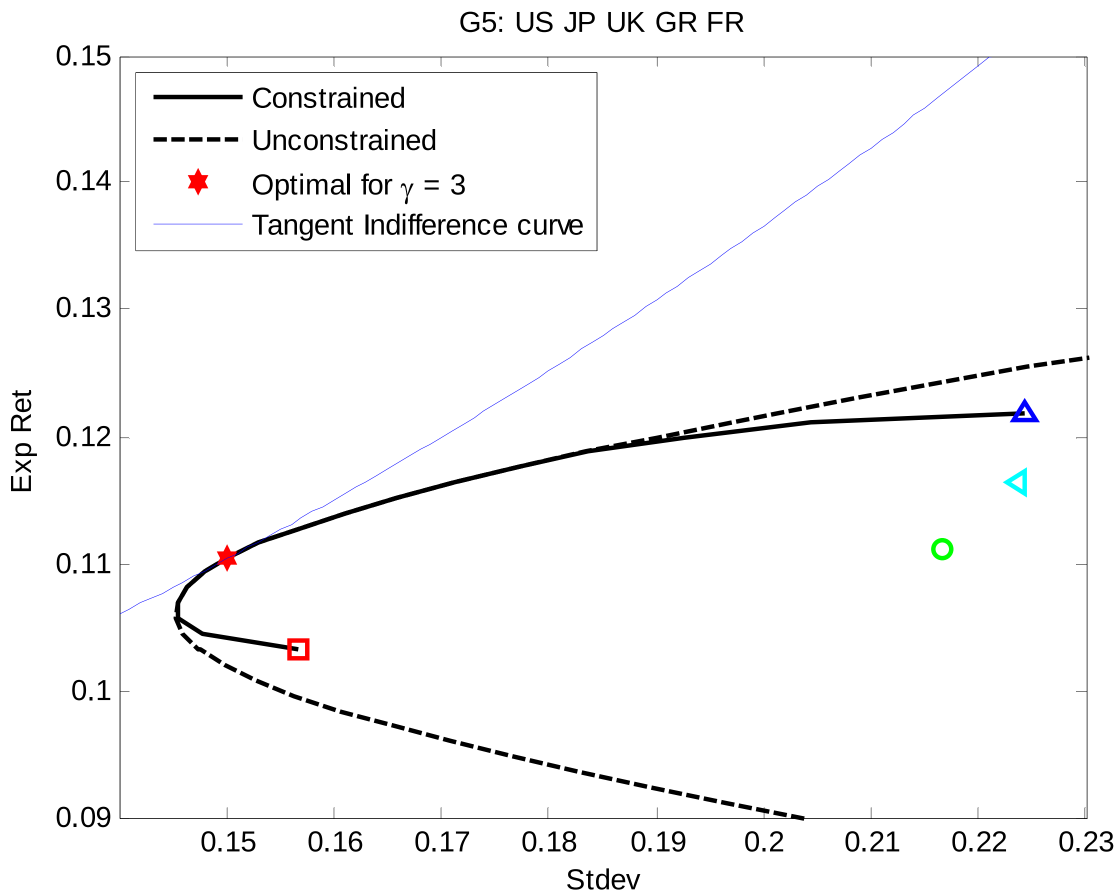

G5: US JP UK GR FR
Constrained
Unconstrained
Optimal for γ = 3
Tangent Indifference curve
Exp Ret
Stdev
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.15
0.16
0.17
0.18
0.19
0.2
0.21
0.22
0.23

Figure 10

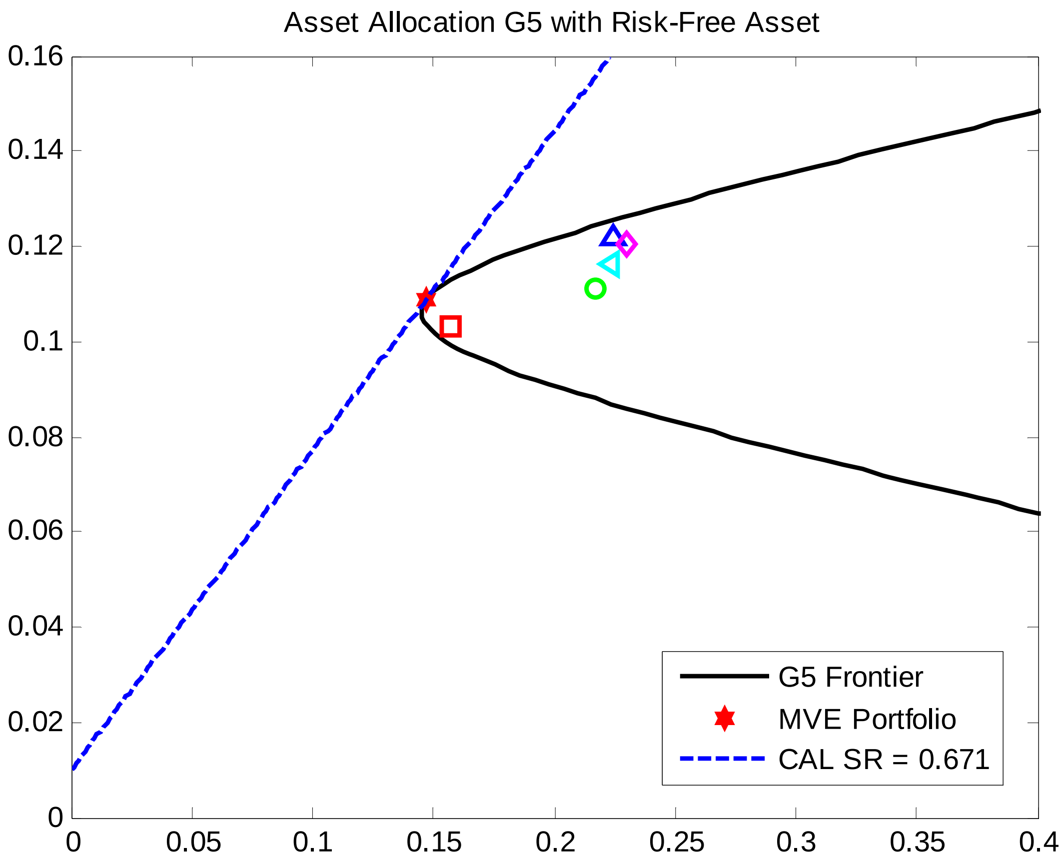

Asset Allocation G5 with Risk-Free Asset
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
G5 Frontier
MVE Portfolio
CAL SR = 0.671

Figure 11

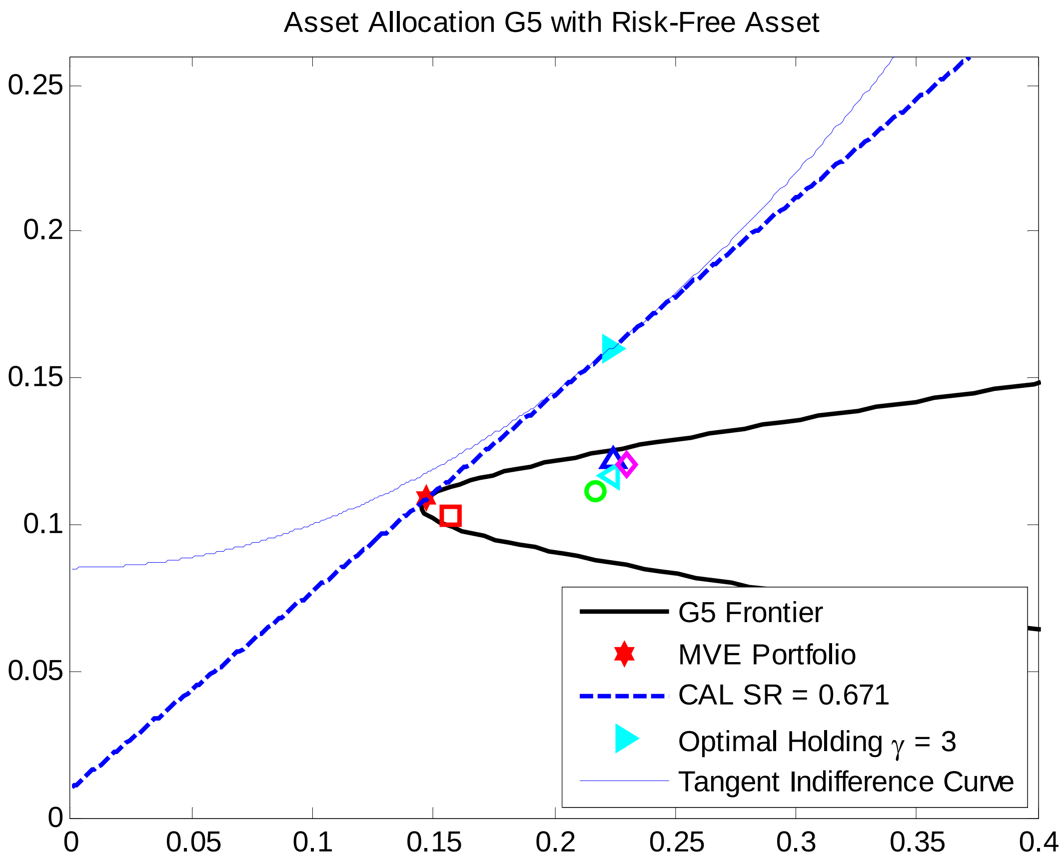

Asset Allocation G5 with Risk-Free Asset
0.25
0.2
0.15
0.1
0.05
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
G5 Frontier
MVE Portfolio
CAL SR = 0.671
Optimal Holding γ = 3
Tangent Indifference Curve

Table 12

Equity Market Participation Rates							
	1984	1989	1994	1999	2001	2003	2005
PSID	27%	31%	37%	28%	32%	29%	27%
SCF (excluding pensions and IRAs)	--	21%	23%	30%	32%	30%	29%
SCF (including pensions and IRAs)	--	32%	39%	50%	52%	51%	51%

Figure 13

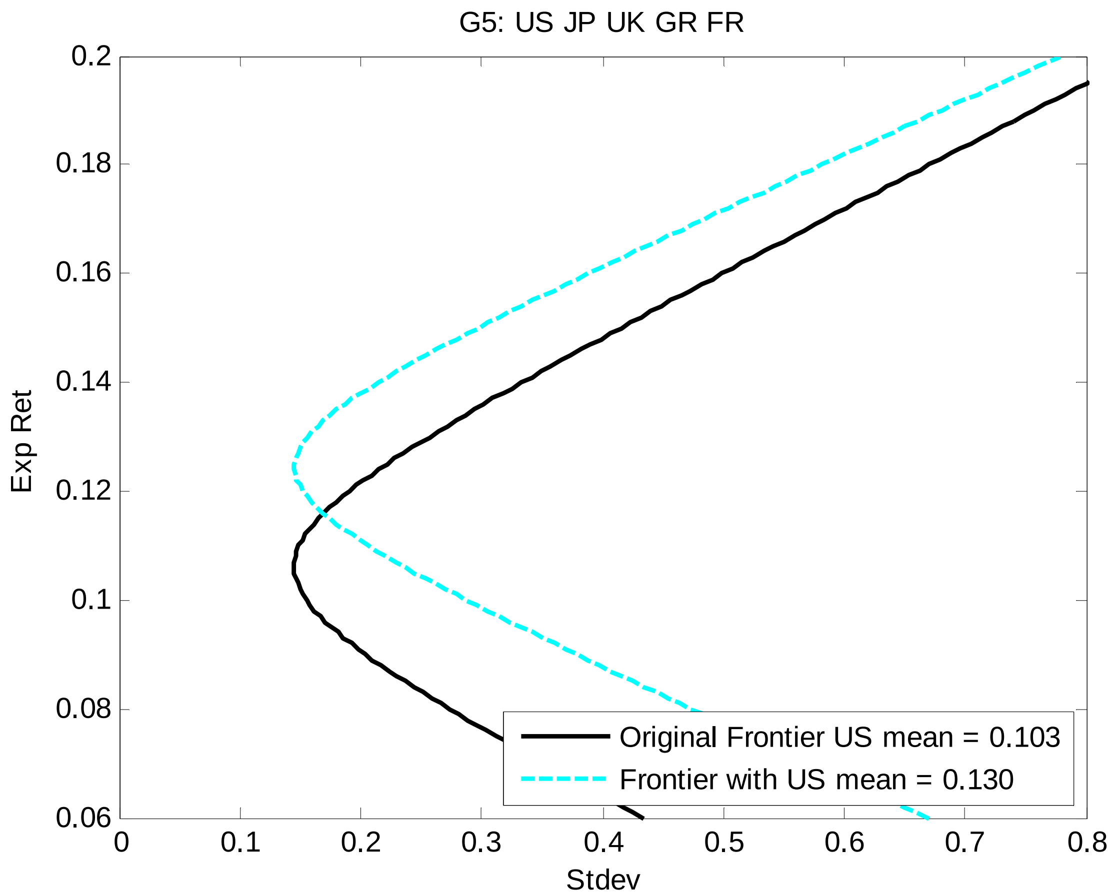

G5: US JP UK GR FR
Exp Ret
Stdev
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Original Frontier US mean = 0.103
Frontier with US mean = 0.130

Table 14

Portfolio Strategies Across US Government Bonds, US Corporate Bonds, US Stocks, International Stocks, 1978-2011

	Raw Return	Volatility	Sharpe Ratio	Comments
Mean-Variance Weights	6.06	11.59	0.07	Maximizes Sharpe ratio
Market Weights	10.25	12.08	0.41	
Diversity Weights	10.14	10.48	0.46	Uses a transformation of market weights
Equal Weights (1/4)	10.00	8.66	0.54	
Risk Parity (Variance)	8.76	5.86	0.59	Weights inversely proportional to variance
Risk Parity (Volatility)	9.39	6.27	0.65	Weights inversely proportional to volatility
Minimum Variance	7.96	5.12	0.52	
Equal Risk Contributions	7.68	7.45	0.32	Equal contribution to portfolio variance
Kelly Rule	7.97	4.98	0.54	Maximizes expected log return
Proportional to Sharpe Ratio	9.80	9.96	0.45	

Average Asset Weights

	US Govt Bonds	US Corp Bonds	US Stocks	International Stocks
Mean-Variance Weights	0.74	-0.05	0.06	0.25
Market Weights	0.14	0.08	0.41	0.37
Diversity Weights	0.19	0.15	0.33	0.32
Equal Weights (1/4)	0.25	0.25	0.25	0.25
Risk Parity (Variance)	0.51	0.36	0.07	0.06
Risk Parity (Volatility)	0.97	-0.30	0.17	0.16
Minimum Variance	1.41	-0.51	0.07	0.03
Equal Risk Contributions	0.50	0.42	0.25	-0.17
Kelly Rule	1.18	-0.29	0.07	0.04
Proportional to Sharpe Ratio	0.24	0.21	0.21	0.35

Figure 15

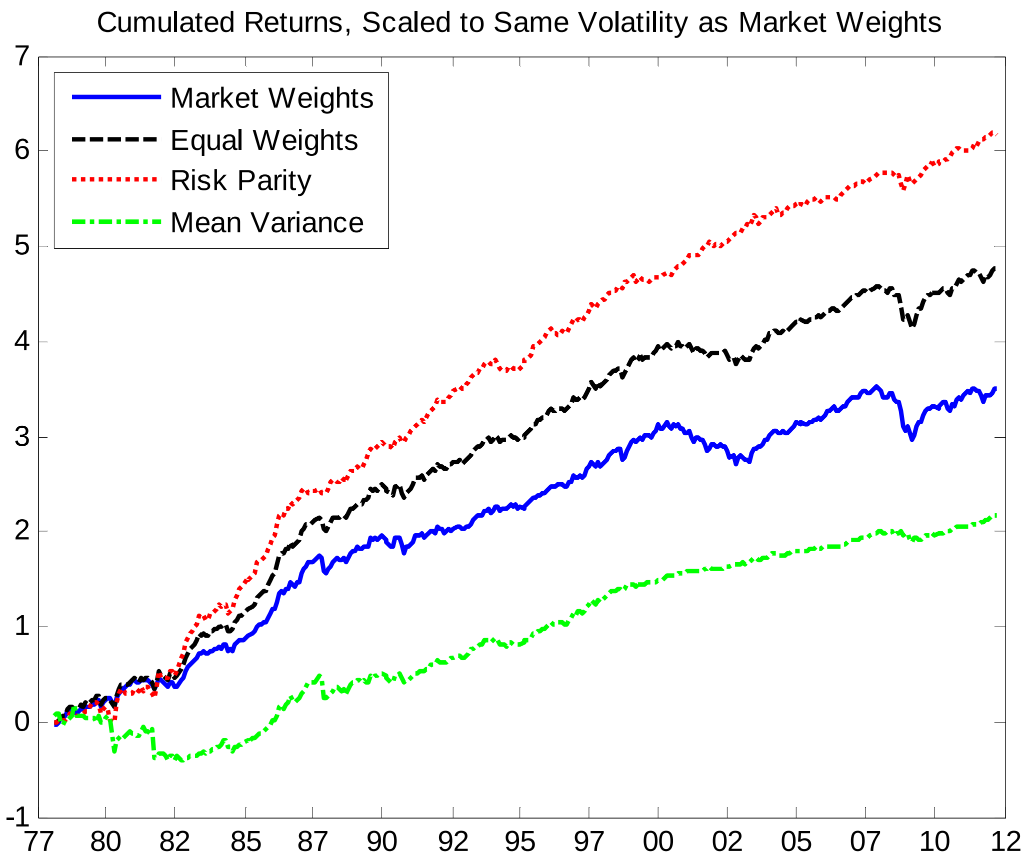
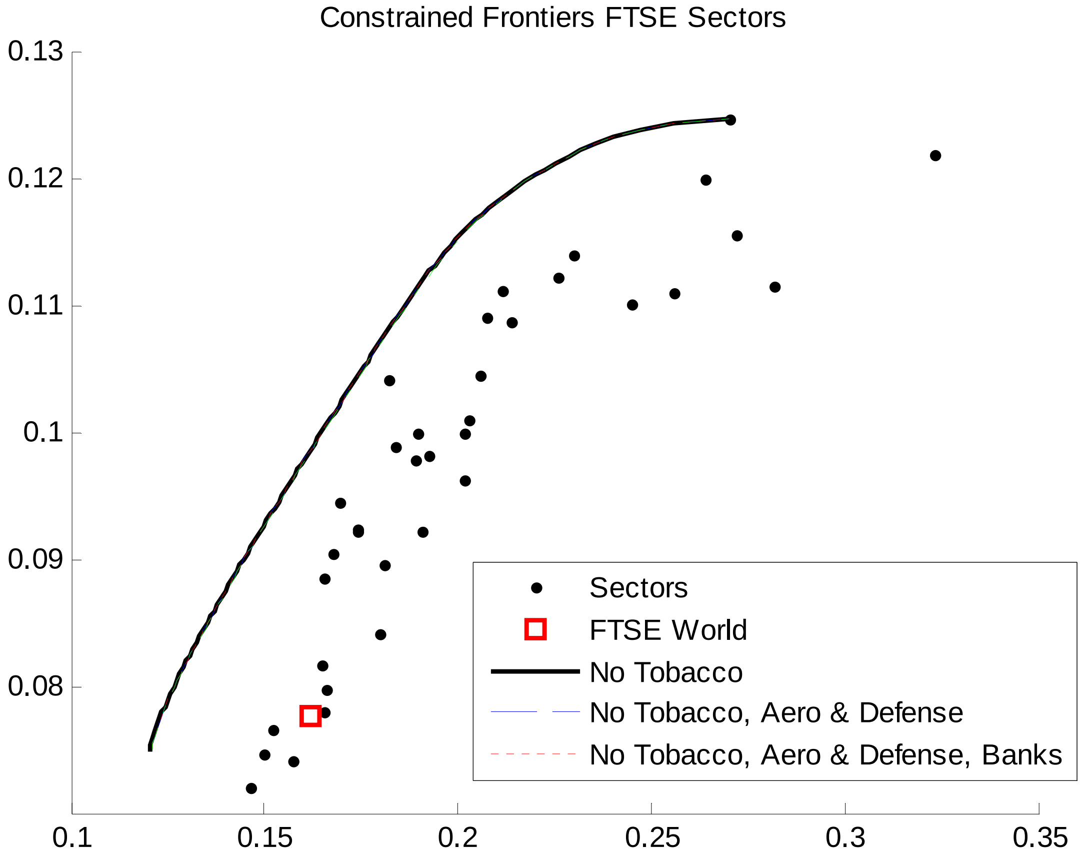

Cumulated Returns, Scaled to Same Volatility as Market Weights
Market Weights
Equal Weights
Risk Parity
Mean Variance
7
6
5
4
3
2
1
0
-1
77
80
82
85
87
90
92
95
97
00
02
05
07
10
12

Table 16

	Assumptions on Means	Assumptions on Volatilities	Assumptions on Correlations	Comments
Optimal Mean-Variance	Unconstrained	Unconstrained	Unconstrained	Most complex
Minimum Variance	Equal	Unconstrained	Unconstrained	No need to estimate means
Risk Parity	Equal	Unconstrained	Equal to zero	No need to estimate means or correlations
Equally Weighted (1/N Portfolio)	Equal	Equal	Equal	Most simple and active, Nothing to estimate
Market Weight	--	--	--	Observable and passive, Nothing to estimate

Figure 16


Constrained Frontiers FTSE Sectors
0.13
0.12
0.11
0.1
0.09
0.08
0.1
0.15
0.2
0.25
0.3
0.35
Sectors
FTSE World
No Tobacco
No Tobacco, Aero & Defense
No Tobacco, Aero & Defense, Banks