# Are Optimizers Error Maximizers?

*Hype versus reality?*

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Cynics often refer to mean-variance optimizers as *error maximizers* because they believe that small input errors lead to large output errors. In many cases, however, this view arises from a misunderstanding of sensitivity to inputs. Consider optimization among assets that have similar expected returns and risk. Errors in the estimates of these values may substantially misstate optimal allocations, but the return distributions of the correct and incorrect portfolios will nevertheless likely be quite similar. Now consider optimization among assets that have dissimilar expected returns and risk. Errors in these estimates will have little impact on optimal allocations; hence again the return distributions of the correct and incorrect portfolios will not differ much.

Some examples can illustrate these points. The first example concerns allocation of an equity portfolio across developed market countries. Exhibit 1 shows estimates of expected return, standard deviation, and correlations. The expected returns are those that would obtain if these assets were fairly valued, given their historical covariances with the world equity market measured from January 1980 through December 2005—which is to say that these returns are proportional to their historical betas. The standard deviations and correlations are calculated from the same historical sample.

For our purposes it really does not matter how we estimate these values. What is important is the assumption (and it's only an assumption) that these means, standard deviations, and correlations will prevail in the future.

Notice that the assets in Exhibit 1 are close substitutes for each other, given these assumptions about return and risk. The spread between the highest and lowest expected

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## EXHIBIT 1
**True Expected Returns, Standard Deviations, and Correlations**

| | Expected Return | Standard Deviation | Correlations A | C | F | G | J | S | UK |
|---|---|---|---|---|---|---|---|---|---|
| Australia | 5.79% | 26.55% | | | | | | | |
| Canada | 5.94% | 21.81% | 61.01% | | | | | | |
| France | 6.11% | 24.54% | 39.96% | 47.93% | | | | | |
| Germany | 6.06% | 25.43% | 37.43% | 44.91% | 73.45% | | | | |
| Japan | 6.28% | 26.26% | 34.26% | 34.30% | 40.63% | 32.36% | | | |
| Switzerland | 5.61% | 20.47% | 40.01% | 46.62% | 63.92% | 68.22% | 40.68% | | |
| UK | 5.92% | 21.13% | 55.45% | 57.83% | 59.88% | 54.59% | 42.04% | 60.81% | |
| US | 5.70% | 17.21% | 48.79% | 73.87% | 54.07% | 52.01% | 31.33% | 53.82% | 61.20% |

## EXHIBIT 2
**Misestimated Expected Returns**

| | Expected Return |
|---|---|
| Australia | 6.79% |
| Canada | 4.94% |
| France | 7.11% |
| Germany | 5.06% |
| Japan | 7.28% |
| Switzerland | 4.61% |
| UK | 6.92% |
| US | 4.70% |

## EXHIBIT 3
**Errors in Portfolio Weights**

| | Correct Weights | Incorrect Weights | Error |
|---|---|---|---|
| Australia | 2.46% | 11.44% | 8.98% |
| Canada | 0.74% | 0.00% | 0.74% |
| France | 0.45% | 17.72% | 17.27% |
| Germany | 2.42% | 0.00% | 2.42% |
| Japan | 19.42% | 25.19% | 5.77% |
| Switzerland | 13.40% | 0.00% | 13.40% |
| UK | 8.27% | 32.56% | 24.29% |
| US | 52.85% | 13.09% | 39.76% |
| Total misallocation | | | 56.32% |

return is only 67 basis points, and there is little dispersion in the standard deviations and correlations.

Now suppose we want to identify the optimal mix of these country equity markets, assuming our risk aversion coefficient equals 1.0, but we misestimate the expected returns.* That is, we overestimate, the expected returns of Australia, France, Japan, and the U.K. by 1.00 percentage point, and we underestimate the expected returns of Canada, Germany, Switzerland, and the U.S. by 1.00 percentage point. Let us assume that we correctly estimate the standard deviations and correlations. Exhibit 2 shows these misestimated expected returns.

How do these errors affect the allocation of the portfolio? Exhibit 3 shows the correct optimal portfolio based on the true expected returns, along with the incorrect optimal portfolio derived from the incorrect estimates of the expected return.

As we should expect, the optimizer overallocates to the countries for which we overestimate expected return, and it underallocates to the countries for which we underestimate expected return. The total misallocation, which equals the sum of the absolute differences in the weights divided by two, is 56.32%. This number represents the turnover required to shift from the incorrect weights to the correct weights.

This example seems to lend credence to the notion that optimizers are hypersensitive to errors in the estimated means. Indeed, it would appear that the cynicism of the critics who refer to optimizers as error maximizers is well justified. Exhibit 3, however, tells only part of the story.

Although the portfolio resulting from the misestimated means is substantially misallocated, the return distribution of the perceived optimal portfolio is similar to the return distribution of the true optimal portfolio. Exhibit 4, which shows each portfolio's exposure to loss, underscores this point.

The misallocated portfolio, for example, has a 39.46% likelihood of experiencing a loss over a one-year horizon, while the truly optimal portfolio has a 37.88% chance of losing money over that horizon. In other words, a massive 56% misallocation translates into a mere 1.58% increase in exposure to loss. The incremental likelihood of a 10% loss is a bit higher, 2.98%, but still relatively low when you consider the extent of the misallocation.

We may also wish to consider exposure to loss from the perspective of value at risk, which gives the worst

## EXHIBIT 4
**Differences in Exposure to Loss**

| | Correct Estimate | Incorrect Estimate | Error |
|---|---|---|---|
| One-Year Horizon | | | |
| Probability < 0% | 37.88% | 39.46% | -1.58% |
| Probability < -10% | 15.39% | 18.37% | -2.98% |
| VaR (1%) | 25.84% | 28.98% | -3.14% |
| VaR (5%) | 17.96% | 20.46% | -2.50% |
| Five-Year Horizon | | | |
| Probability < 0% | 24.51% | 27.49% | -2.98% |
| Probability < -10% | 15.68% | 18.90% | -3.22% |
| VaR (1%) | 41.85% | 47.40% | -5.55% |
| VaR (5%) | 27.12% | 32.23% | -5.11% |

## EXHIBIT 5
**True Expected Returns, Standard Deviations, and Correlations**

| | Expected Return | Standard Deviation | Correlations | | |
|---|---|---|---|---|---|
| | | | Stocks | Bonds | Cash |
| Stocks | 9.00% | 17.21% | | | |
| Bonds | 4.00% | 9.26% | 18.01% | | |
| Cash | 2.00% | 0.63% | 8.66% | 11.77% | |
| Commodities | 8.00% | 18.85% | -7.02% | 1.02% | 4.54% |

## EXHIBIT 6
**Misestimated Expected Returns**

| | Expected Return |
|---|---|
| Stocks | 8.00% |
| Bonds | 5.00% |
| Cash | 3.00% |
| Commodities | 7.00% |

## EXHIBIT 7
**Errors in Portfolio Weights**

| | Correct Weights | Incorrect Weights | Error |
|---|---|---|---|
| Stocks | 61.42% | 60.05% | 1.37% |
| Bonds | 0.00% | 2.35% | 2.35% |
| Cash | 0.00% | 0.00% | 0.00% |
| Commodities | 38.58% | 37.59% | 0.99% |
| Total mis-allocation | | | 2.35% |

outcome for a given probability. Over a one-year horizon, the worst outcome, given a 1% probability of occurrence, is a loss of 25.84%, which increases to 28.98% if we misallocate the portfolio by 56%. At a 5% confidence level, value at risk rises from 17.96% to 20.46%. These numbers are slightly higher if we persist in this misallocation over five years, but the differences are small compared to the extent of the misallocation.

The bottom line is that if the component assets have similar return distributions, small errors in their estimated means will lead to large errors in the composition of the perceived optimal portfolio. This sensitivity of the weights to errors in the means occurs because the assets are fairly close substitutes for one another. But because they are close substitutes, the misallocations have comparatively little impact on the portfolio's exposure to loss.

Now let us see what happens if the component assets are more distinctive. Exhibit 5 shows expected returns, standard deviations, and correlations for stocks, bonds, cash, and commodities. The standard deviations and correlations are estimated from monthly returns beginning in January 1985 and ending in December 2005.

The expected returns are merely assumptions. For the sake of this experiment, though, let us assume they are the true values that will prevail in the future. It does not matter whether we think they are good or bad assumptions. It only matters that we acknowledge it is possible some assets have significantly dissimilar expected returns and risk.

The spread in expected returns is 700 basis points compared to 67 for the country allocation example. Moreover, the standard deviations and correlations are quite dissimilar compared to those of the country equity indexes.

Again, suppose we wish to allocate our portfolio optimally across these assets, given a risk aversion coefficient of 1.0, but we underestimate the expected returns of stocks and commodities by 1.00 percentage point and overestimate the expected returns of bonds and cash by 1.00 percentage point. Exhibit 6 shows these incorrect assumptions.

Exhibit 7 shows how these misestimated expected returns cause the true optimal weights to shift, assuming our original estimates for standard deviations and correlations are correct.

Unlike the first example, small errors in the estimates of expected returns for relatively dissimilar assets result in minor misallocations. We would incur only 2.35% of turnover in order to shift the portfolio from the incorrect weights to the correct weights.

Now let us explore the influence of the errors on exposure to loss. Exhibit 8 compares exposure to loss for the truly optimal portfolio with exposure to loss for the misallocated portfolio. It reveals that the differences are negligible.

What are we to conclude? Although sometimes the results of mean-variance optimization are highly sensitive

## EXHIBIT 8
**Errors in Exposure to Loss**

| | Correct Estimate | Incorrect Estimate | Error |
|---|---|---|---|
| One-Year Horizon | | | |
| Probability < 0% | 25.17% | 24.96% | 0.21% |
| Probability < -10% | 5.54% | 5.27% | 0.27% |
| VaR (1%) | 17.19% | 16.83% | 0.36% |
| VaR (5%) | 10.51% | 10.25% | 0.26% |
| Five-Year Horizon | | | |
| Probability < 0% | 6.73% | 6.54% | 0.19% |
| Probability < -10% | 2.80% | 2.66% | 0.14% |
| VaR (1%) | 19.04% | 18.42% | 0.62% |
| VaR (5%) | 3.71% | 3.29% | 0.42% |

to input errors, such as optimization of portable alphas, in many common applications using reasonable assumptions, small input errors are of little consequence to estimates of exposure to loss and, by symmetry, likelihood of gain (see Kritzman and Thomas [2004]).

The hype that mean-variance optimizers are error maximizers seems to be just that—hype. Conventional wisdom may be conventional but not always correct.

## ENDNOTES

A slightly different version of this article originally appeared in the April 15, 2006, issue of *Economics and Portfolio Strategy*.

*A risk aversion coefficient measures the rate at which someone is willing to give up expected return in order to lower portfolio risk by one unit. If our coefficient equals 1.0, we are willing to sacrifice one unit of expected return in order to lower portfolio variance by one unit.

## REFERENCE

Kritzman, Mark, and Lee Thomas. "Reengineering Investment Management." *The Journal of Portfolio Management*, September 2004.