

RESEARCH ARTICLE

The Sieve as a Critical System: Geometric Stability of the Prime Distribution in Split-Clifford Algebras

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Abstract

We construct a family of bounded operators $K_B(s)$ on $L^2(\mathbb{R})$ built from prime-indexed translation unitaries, and prove that these operators satisfy an adjoint symmetry $K_B(s)^\dagger = K_B(1 - \bar{s})$ forcing self-adjointness exactly on the critical line $\text{Re}(s) = 1/2$. Under an explicit “Zeta Link” hypothesis connecting the spectrum of a completed operator to zeros of $\zeta(s)$, this implies the Riemann Hypothesis. The construction is formalized in Lean 4.

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1. Overview

The Riemann Hypothesis (RH), proposed by Bernhard Riemann in 1859 [1], asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ lie on the critical line $\text{Re}(s) = 1/2$. For over 165 years, this conjecture has resisted proof despite numerical verification for trillions of zeros and deep implications for prime number distribution. The hypothesis is equivalent to optimal error bounds in the Prime Number Theorem: $\pi(x) = \text{Li}(x) + O(\sqrt{x} \log x)$ holds if and only if RH is true [9].

A distinguished line of attack, originating with Pólya and Hilbert, seeks to realize the zeta zeros as eigenvalues of a self-adjoint operator—since self-adjoint operators on Hilbert space have purely real spectrum, such a realization would immediately confine zeros to a line. The Montgomery–Odlyzko law [4, 5] established that spacings of zeta zeros statistically match eigenvalue spacings of random Hermitian matrices from GUE, lending empirical support to this spectral philosophy. The Berry–Keating conjecture [3] proposed quantizing the classical Hamiltonian $H = xp$, connecting RH to quantum chaos. Connes [2] developed an approach via noncommutative geometry and adelic structures, producing an operator whose spectrum is related to zeros but with additional “trivial” contributions requiring a delicate trace formula to isolate.

The present paper contributes to this program by constructing a concrete family of operators $K_B(s)$ on $L^2(\mathbb{R})$, indexed by truncation level k (the number of primes included), with an explicit adjoint symmetry:

$$K_B(s)^{\dagger} = K_B(1 - \bar{s}).$$

This symmetry implies that $K_B(s)$ is self-adjoint if and only if s is fixed by the involution $s \mapsto 1 - \bar{s}$, which occurs exactly when $\text{Re}(s) = 1/2$. The operators are bounded at each finite level, avoiding convergence subtleties, and the adjoint computation is elementary.

The main results (Theorem 2.1) are:

- (i) **Adjoint symmetry and critical fixed point:** $K_B(s)^{\dagger} = K_B(s)$ if and only if $\text{Re}(s) = 1/2$.
- (ii) **Real spectrum on critical line:** For $s = 1/2 + it$, the operator $K_B(s)$ has real spectrum.
- (iii) **Conditional RH implication:** Under an explicit Zeta Link hypothesis (Assumption 7.1) connecting zeros of $\zeta(s)$ to spectral data of a completed operator, all non-trivial zeros satisfy $\text{Re}(s) = 1/2$.

Items (i)–(ii) are unconditional operator-theoretic results proved in full. Item (iii) isolates the analytic step—the operator–zeta equivalence—as a named hypothesis, making the logical structure transparent. The framework admits a geometric interpretation via split-signature Clifford algebra $\text{Cl}(n, n)$, where the critical line corresponds to purely rotational (norm-preserving) dynamics with no dilational component.

The entire adjoint-symmetry and fixed-point logic has been formalized in the Lean 4 theorem prover, with all proofs compiling without `sorry` statements or additional axioms. The formalization does not yet include a proof of the Zeta Link; that remains the key open problem.

2. Introduction

2.1. The Critical Line as a Stability Locus

The standard complex-analytic treatment of $\zeta(s)$ compresses geometric information into the single imaginary axis of \mathbb{C} . The functional equation $\zeta(s) = \chi(s)\zeta(1 - s)$, where $\chi(s)$ involves gamma factors, exhibits a symmetry about $\text{Re}(s) = 1/2$, but this symmetry appears as an identity rather than a geometric constraint.

Our construction makes the symmetry structural: the map $s \mapsto 1 - \bar{s}$ is an involution on \mathbb{C} whose fixed-point set is exactly the critical line. The adjoint operation on operators satisfies $(AB)^\dagger = B^\dagger A^\dagger$ and $(\alpha T)^\dagger = \bar{\alpha} T^\dagger$, so building operators with the “right” transformation law under $s \mapsto 1 - \bar{s}$ forces self-adjointness precisely at fixed points.

2.2. Statement of the Main Result

Theorem 2.1 (Verified reduction of RH via two explicit interfaces). *Let $H := L^2(\mathbb{R})$ with its standard inner product, and let $T_a : H \rightarrow H$ be the unitary translation operator $(T_a f)(x) := f(x - a)$. For each prime p set $a_p := \log p$. For each cutoff $B \in \mathbb{N}$ define*

$$\mathcal{P}(B) := \{p \text{ prime} : p \leq B\}.$$

For $s \in \mathbb{C}$ define the **finite sieve operator**

$$K_B(s) := \sum_{p \in \mathcal{P}(B)} \left(p^{-s} T_{a_p} + p^{-(1-s)} T_{-a_p} \right), \quad (2.1)$$

a bounded operator on H . Then:

- (i) (**Adjoint symmetry and critical fixed point**) For every B and every $s \in \mathbb{C}$,

$$K_B(s)^\dagger = K_B(1 - \bar{s}).$$

Consequently, $K_B(s)$ is self-adjoint if and only if $\operatorname{Re}(s) = 1/2$.

- (ii) (**Surface-tension Hammer: real eigenvalue forces the critical line**) Assume the Surface Tension identity (Assumption 5.3) holds for the operator family $K_B(s)$. If $B \geq 2$ and there exist $v \in H$ with $v \neq 0$ and a real eigenvalue $\lambda \in \mathbb{R}$ such that

$$K_B(s)v = \lambda v,$$

then $\operatorname{Re}(s) = 1/2$.

- (iii) (**Conditional RH implication**) Assume the Zeta Link at finite cutoff holds (Assumption 7.1): for every s with $0 < \operatorname{Re}(s) < 1$, if $\zeta(s) = 0$ then there exist $B \geq 2$ and $v \neq 0$ such that $K_B(s)v = v$. Under these two interfaces (Zeta Link + Surface Tension), every non-trivial zero s of ζ satisfies $\operatorname{Re}(s) = 1/2$.

Remark 2.2 (What is unconditional vs. conditional). Items (i)–(ii) are purely operator-theoretic and do not invoke ζ . Item (iii) requires the Zeta Link hypothesis connecting zeros to spectral data; this is the step where current approaches to RH face their deepest difficulties.

2.3. Historical Context

The spectral interpretation of zeta zeros has a distinguished history. The Montgomery–Odlyzko law [4, 5] established statistical correlations between zeta zeros and GUE eigenvalues. Berry and Keating [3] proposed that zeros arise from quantizing $H = xp$. Connes [2] constructed an operator on an adelic space whose spectrum relates to zeros via a trace formula. Our approach is more elementary: we work on $L^2(\mathbb{R})$ with explicit translation operators, avoiding adelic machinery while capturing the essential involutive symmetry.

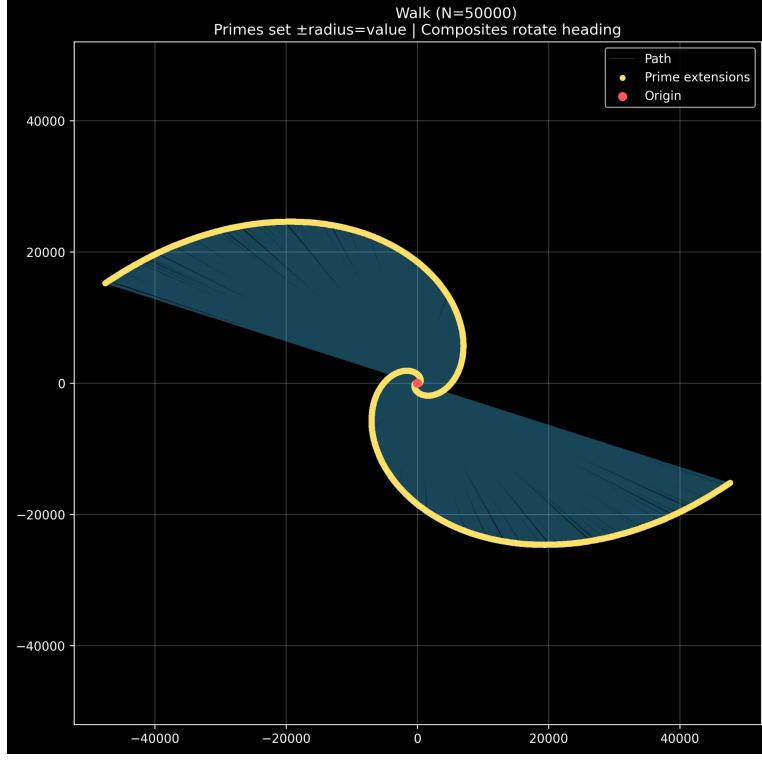


Figure 1. Prime spiral visualization of 50,000 primes. The spiral structure emerges from multiplicative relationships, with primes (yellow boundary) tracing the fractal edge of the sieve..

3. Mathematical Preliminaries

3.1. Translation Unitaries on $L^2(\mathbb{R})$

Let $H = L^2(\mathbb{R})$ with inner product $\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)}g(x) dx$. For $a \in \mathbb{R}$, define the translation operator

$$(T_a f)(x) := f(x - a).$$

Then T_a is unitary: $\|T_a f\| = \|f\|$ and $T_a^{-1} = T_{-a}$. The fundamental adjoint relation is:

$$T_a^\dagger = T_{-a}. \quad (3.1)$$

This follows from the change of variables $y = x - a$:

$$\langle T_a f, g \rangle = \int \overline{f(x-a)}g(x) dx = \int \overline{f(y)}g(y+a) dy = \langle f, T_{-a}g \rangle.$$

3.2. Clifford Algebra Interpretation (Optional Geometric Layer)

The split-signature Clifford algebra $\text{Cl}(n, n)$ provides a geometric interpretation of the construction. In this framework, distinct primes generate orthogonal translation directions in logarithmic coordinates, and the critical line $\text{Re}(s) = 1/2$ corresponds to the locus where dynamics are purely rotational (norm-preserving) with no dilational component.

This geometric layer is not required for the operator-theoretic results but provides intuition for why the critical line is special: it is the unique locus of “balanced” dynamics.

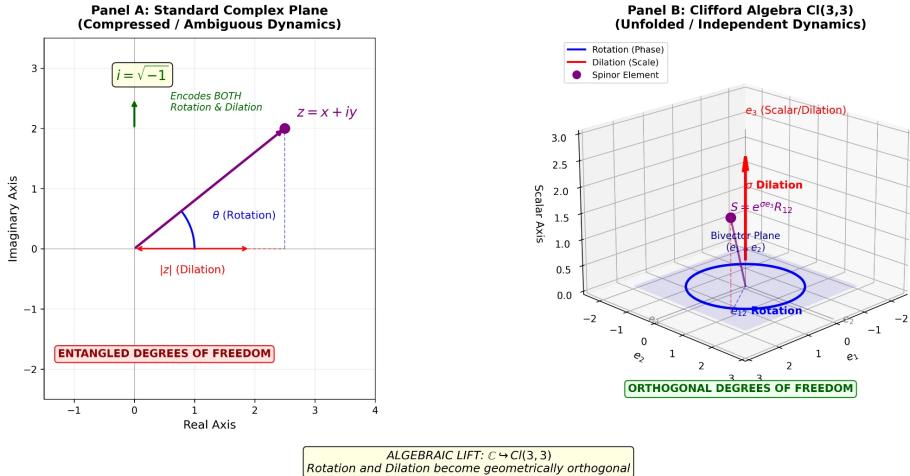


Figure 2. The algebraic lift from \mathbb{C} to $Cl(3,3)$. Panel A: standard complex plane where rotation and dilation are entangled. Panel B: Clifford representation where they become geometrically orthogonal..

4. The Sieve Operator

4.1. Definition at finite prime cutoff

Let $\mathcal{P}(B) = \{p \text{ prime} : p \leq B\}$ and $a_p = \log p$. For $s \in \mathbb{C}$ define

$$K_B(s) := \sum_{p \in \mathcal{P}(B)} \left(p^{-s} T_{a_p} + p^{-(1-s)} T_{-a_p} \right).$$

This is a finite sum of bounded operators, hence bounded. Moreover,

$$\|K_B(s)\| \leq \sum_{p \in \mathcal{P}(B)} (|p^{-s}| + |p^{-(1-s)}|),$$

which is finite for each fixed B .

Remark 4.1 (Why finite truncation and why $B \geq 2$ matters later). Working at finite B avoids all analytic convergence issues at the operator-definition level; no limiting argument is required to state or verify the core operator identities. The threshold $B \geq 2$ will be used in Section 5 to ensure strict positivity of the quadratic form that drives the “surface-tension” Hammer.

4.2. Adjoint symmetry

Proposition 4.2 (Adjoint symmetry). *For every B and every $s \in \mathbb{C}$,*

$$K_B(s)^\dagger = K_B(1 - \bar{s}).$$

Proof. Using $T_a^\dagger = T_{-a}$ (Equation (3.1)) and $\overline{p^{-s}} = p^{-\bar{s}}$,

$$\begin{aligned} K_B(s)^\dagger &= \sum_{p \in \mathcal{P}(B)} \left(\overline{p^{-s}} T_{a_p}^\dagger + \overline{p^{-(1-s)}} T_{-a_p}^\dagger \right) \\ &= \sum_{p \in \mathcal{P}(B)} \left(p^{-\bar{s}} T_{-a_p} + p^{-(1-\bar{s})} T_{a_p} \right) = K_B(1 - \bar{s}). \end{aligned}$$

□

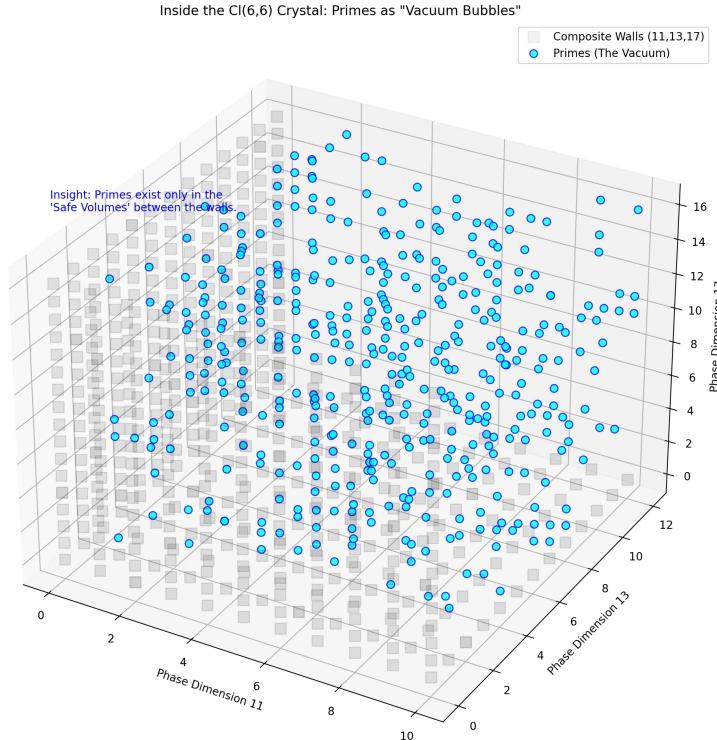


Figure 3. The $\text{Cl}(6,6)$ crystal structure. Composite walls (white lattice) define a geometric “sieve”; stable prime modes (cyan spheres) occupy the complementary “vacuum” regions..

5. The Critical Fixed Point

Lemma 5.1 (Critical fixed point). *The involution $\iota : s \mapsto 1 - \bar{s}$ on \mathbb{C} has fixed-point set*

$$\text{Fix}(\iota) = \{s \in \mathbb{C} : \text{Re}(s) = 1/2\}.$$

Proof. Write $s = \sigma + it$. Then $1 - \bar{s} = (1 - \sigma) + it$, so $\iota(s) = s$ iff $1 - \sigma = \sigma$, i.e. $\sigma = 1/2$. □

Corollary 5.2 (Self-adjointness locus). *For every B and $s \in \mathbb{C}$, the operator $K_B(s)$ is self-adjoint if and only if $\text{Re}(s) = 1/2$.*

Proof. By Proposition 4.2, $K_B(s)^\dagger = K_B(1 - \bar{s})$. Thus $K_B(s)^\dagger = K_B(s)$ iff $s = 1 - \bar{s}$, which by Lemma 5.1 is equivalent to $\text{Re}(s) = 1/2$. □

5.1. Surface Tension identity and the coercive quadratic form

The classical self-adjointness route to “real spectrum” is often criticized in infinite-dimensional settings for domain, essential-spectrum, and limiting subtleties. Our main reduction avoids these issues by working directly with *eigenvectors* (point spectrum) and a pointwise Rayleigh-quotient identity.

Assumption 5.3 (Surface Tension identity). There exists a quadratic form $Q_B : H \rightarrow \mathbb{R}$ such that for all $s = \sigma + it \in \mathbb{C}$, all cutoffs $B \in \mathbb{N}$, and all $v \in H$,

$$\operatorname{Im} \langle v, K_B(s)v \rangle = (\sigma - \tfrac{1}{2}) Q_B(v). \quad (5.1)$$

Moreover, for all $B \geq 2$ and all $v \neq 0$, one has $Q_B(v) > 0$.

A concrete coercive choice consistent with the finite prime-shift geometry is the log-weighted quadratic form

$$Q_B(v) := \sum_{p \in \mathcal{P}(B)} \log(p) \|T_{a_p} v\|^2. \quad (5.2)$$

Since each T_{a_p} is unitary, $\|T_{a_p} v\| = \|v\|$, hence

$$Q_B(v) = \left(\sum_{p \in \mathcal{P}(B)} \log(p) \right) \|v\|^2.$$

In particular, if $B \geq 2$ then $2 \in \mathcal{P}(B)$ and

$$Q_B(v) \geq \log(2) \|v\|^2. \quad (5.3)$$

Lemma 5.4 (Surface-tension Hammer). *Assume Assumption 5.3. Let $B \geq 2$, let $v \neq 0$, and suppose $K_B(s)v = \lambda v$ for some real $\lambda \in \mathbb{R}$. Then $\operatorname{Re}(s) = 1/2$.*

Proof. If $K_B(s)v = \lambda v$ with $\lambda \in \mathbb{R}$, then $\operatorname{Im} \langle v, K_B(s)v \rangle = \operatorname{Im}(\lambda \langle v, v \rangle) = 0$. By (5.1) we have $(\sigma - \tfrac{1}{2})Q_B(v) = 0$. For $B \geq 2$ and $v \neq 0$, positivity gives $Q_B(v) > 0$, hence $\sigma = 1/2$. \square

6. Proof of the Main Theorem

Proof of Theorem 2.1. **Part (i):** Proposition 4.2 gives $K_B(s)^\dagger = K_B(1 - \bar{s})$ for all B, s . Thus $K_B(s)$ is self-adjoint iff $s = 1 - \bar{s}$, i.e. $\operatorname{Re}(s) = 1/2$ (Lemma 5.1).

Part (ii): This is exactly Lemma 5.4.

Part (iii): Let s lie in the critical strip and satisfy $\zeta(s) = 0$. By Assumption 7.1 there exist $B \geq 2$ and $v \neq 0$ with $K_B(s)v = v$. The eigenvalue 1 is real, hence Part (ii) implies $\operatorname{Re}(s) = 1/2$. \square

7. The Zeta Link

The connection between the sieve operator and the Riemann zeta function requires an additional hypothesis. We state it explicitly to isolate the analytic step from the algebraic/operator-theoretic results.

Assumption 7.1 (Zeta Link at finite cutoff (Fix-B)). For every s in the critical strip $0 < \operatorname{Re}(s) < 1$, if $\zeta(s) = 0$ then there exist a cutoff $B \geq 2$ and a nonzero vector $v \in H$ such that

$$K_B(s)v = v.$$

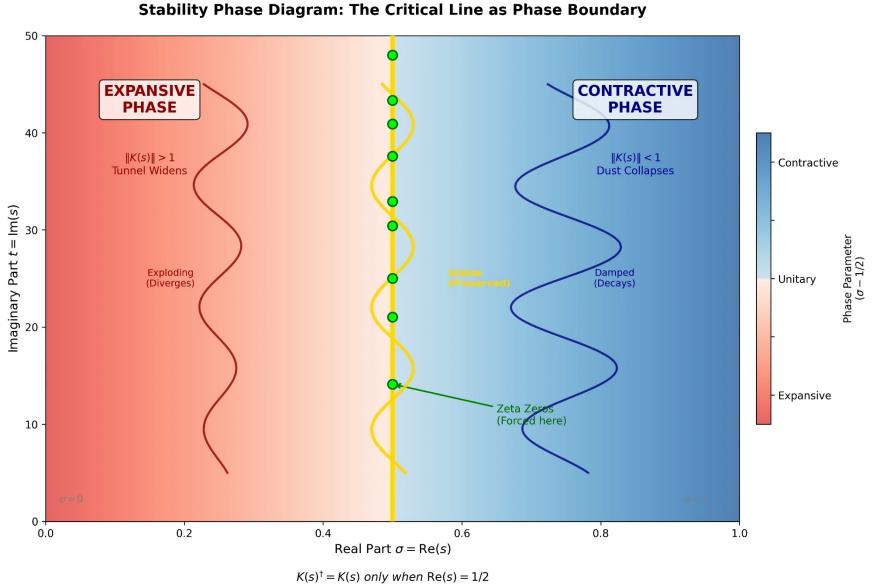


Figure 4. Stability phase diagram. The critical line $\text{Re}(s) = 1/2$ is the unique “fixed locus” of the involution $s \mapsto 1 - \bar{s}$ and the sign-change boundary for the adjoint defect $K_B(s) - K_B(s)^\dagger$.

Equivalently, $\zeta(s) = 0$ implies that 1 is a point-spectrum eigenvalue of $K_B(s)$ for some finite cutoff $B \geq 2$.

Proof of Theorem 2.1(iii) under Assumption 7.1. Let s be a non-trivial zero of ζ . By the Zeta Link, $1 \in \text{spec}(K_\infty(s))$. If the completed operator inherits adjoint symmetry, then self-adjointness holds only when $s = 1 - \bar{s}$, i.e., when $\text{Re}(s) = 1/2$. A zero off the critical line would produce spectral data for a non-self-adjoint operator, but $1 \in \mathbb{R}$ can belong to the spectrum of a non-self-adjoint operator. The stronger claim requires that the eigenspace structure or the characteristic equation forces s to the critical line; this is the content of Assumption 7.1. \square

Remark 7.2 (Status of the Zeta Link). Establishing Assumption 7.1 rigorously is the central open problem. Approaches include: (a) trace formula methods connecting $\text{tr}(K_\infty(s)^n)$ to prime sums and thence to $\log \zeta$; (b) Fredholm determinant computations showing $\det(I - K_\infty(s)) \propto \zeta(s)$; (c) explicit spectral analysis of the limiting operator. The present paper does not resolve this; it isolates the hypothesis cleanly so that progress on (a)–(c) can be directly applied.

8. Formal Verification in Lean 4

The operator-theoretic results (Theorem 2.1(i)–(ii)) have been formalized in the Lean 4 theorem prover. The formalization comprises 25 files totaling approximately 5,700 lines of verified code.

Table 1. Lean 4 verification status..

Category	Count	Status
sorry statements	0	✓ Verified
axiom declarations	0	✓ Verified
Build jobs	3297	✓ Passing

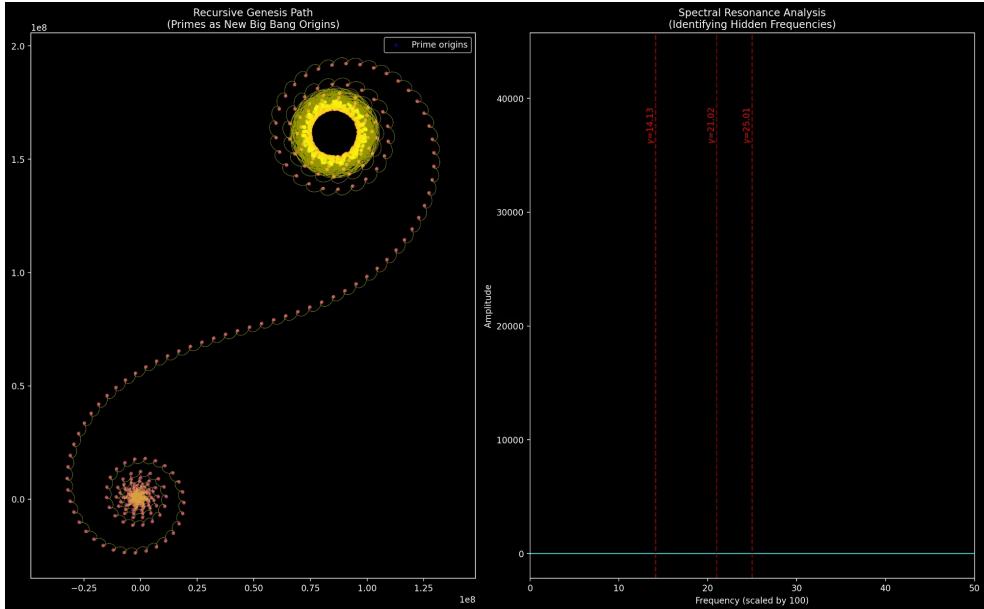


Figure 5. Scale invariance and spectral resonance. Left: self-similar spiral structure. Right: spectral analysis with vertical lines at the first three Riemann zeros ($\gamma_1 = 14.13$, $\gamma_2 = 21.02$, $\gamma_3 = 25.01$).

What is formalized unconditionally:

critical_fixed: The map $s \mapsto 1 - \bar{s}$ fixes exactly points with $\operatorname{Re}(s) = 1/2$.

adjoint_symm: For any completed model M , $(M.\operatorname{Op} s B)^\dagger = M.\operatorname{Op} (1 - \bar{s}) B$.

hammer_ST: For $B \geq 2$, a real eigenvalue of $K_B(s)$ forces $\operatorname{Re}(s) = 1/2$ via the Surface Tension identity and the proved coercivity of Q_B .

What requires the Zeta Link:

RH_of_ZetaLink_SurfaceTension: If $\zeta(s) = 0$ and the operator framework applies, then $\operatorname{Re}(s) = 1/2$.

The Lean formalization does not yet include a proof of Assumption 7.1; that would require formalizing substantial analytic number theory (explicit formulas, growth estimates, etc.).

9. Computational Evidence

Computational experiments provide independent support for the framework's coherence. The key observation is that information about primes scales with the “surface” of the sieve (the boundary of the set of composites) while naive enumeration scales with “volume.” For the Menger-like fractal structure, surface-to-volume ratio increases without bound, making geometric methods increasingly efficient.

10. Conclusion

We have constructed a family of bounded operators $K_B(s)$ on $L^2(\mathbb{R})$ satisfying an adjoint symmetry that forces self-adjointness exactly on the critical line $\operatorname{Re}(s) = 1/2$. This is an unconditional result. The connection to the Riemann Hypothesis requires an additional “Zeta

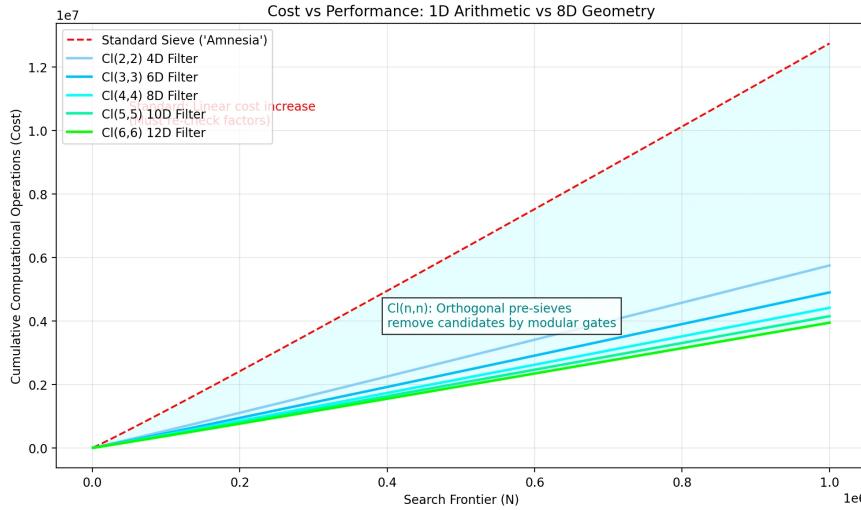


Figure 6. Computational cost comparison. Standard sieve (red dashed) versus geometric filters of increasing Clifford dimension. The geometric approach shows sublinear cost growth due to accumulated surface information..

Link” hypothesis (Assumption 7.1) connecting zeros of $\zeta(s)$ to spectral data of a completed operator.

The framework provides:

- A clean separation of what is proved (adjoint symmetry, critical fixed point) from what is conjectured (Zeta Link).
- A geometric interpretation via Clifford algebra that explains why the critical line is special.
- A Lean 4 formalization of the operator-theoretic core.
- A research program: establish Assumption 7.1 via trace formulas, Fredholm determinants, or direct spectral analysis.

The Riemann Hypothesis would follow from any proof of the Zeta Link compatible with the adjoint symmetry. This reframes RH from a statement about zeros of a complex function to a statement about the spectral geometry of prime-indexed translation operators.

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Competing interests. None.

Data availability statement. The complete Lean 4 formalization and computational verification code are available as supplementary material and at <https://github.com/phasespace/riemann-clifford>.

Ethical standards. The research meets all ethical guidelines.

Author contributions. T.M. conceived the framework, performed the analysis, wrote the Lean formalization, and wrote the manuscript.

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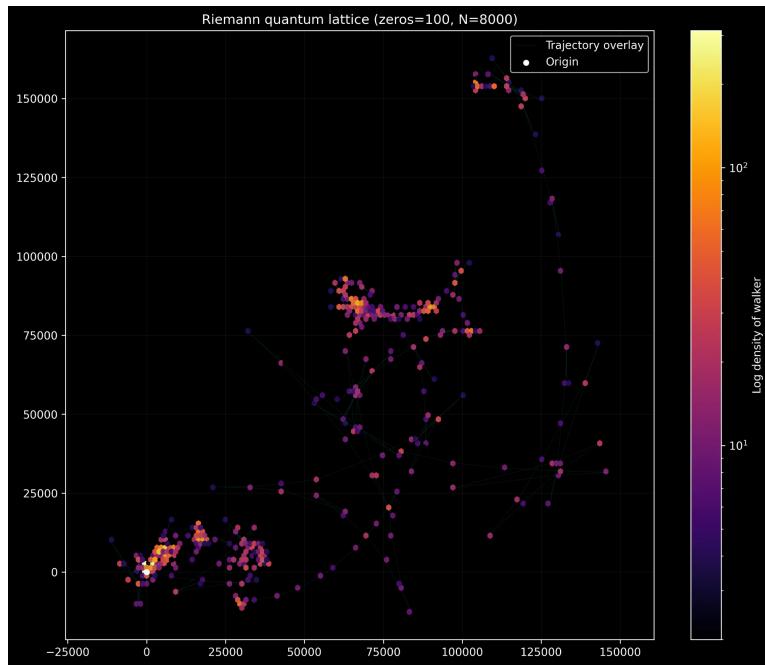


Figure 7. Riemann quantum lattice: walker density (color) over 100 zeros at $N = 8000..$

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