Algorithms for Wireless Communications II

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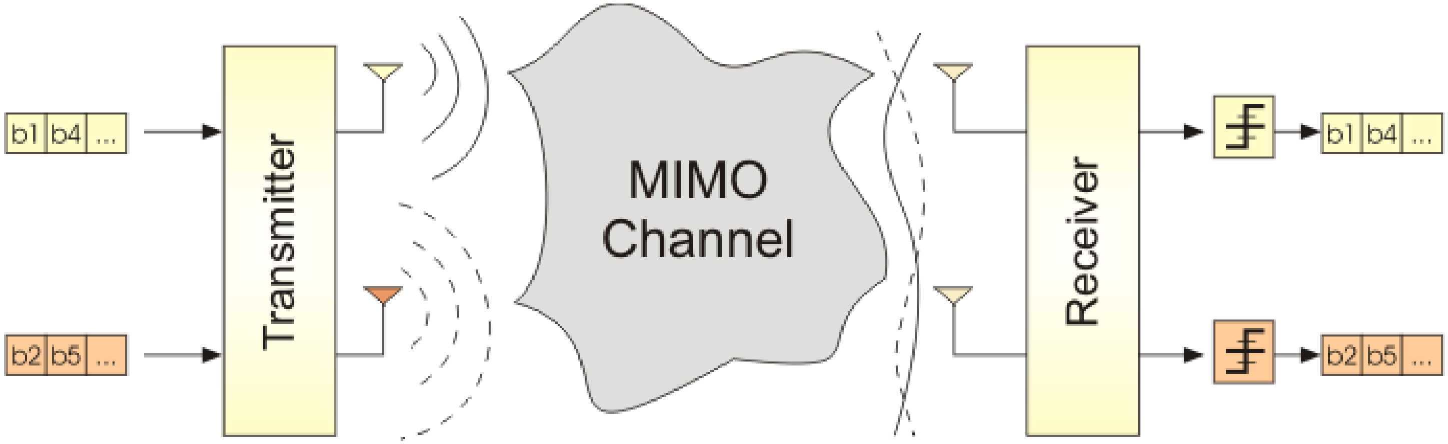
Basic MIMO Receivers

- System Model
- Maximum Likelihood Receiver
- Linear Receivers (MMSE & ZF)
- *Successive Cancellation Receivers*
- *Performance*

Reminder: MIMO Spatial Multiplexing



- Split the high rate data stream into N_T independent, lower rate streams
- Each stream is modulated independently using M-QAM constellations
- No transmit channel knowledge: each stream is transmitted from its own transmit antenna (at the same time in the same frequency band)

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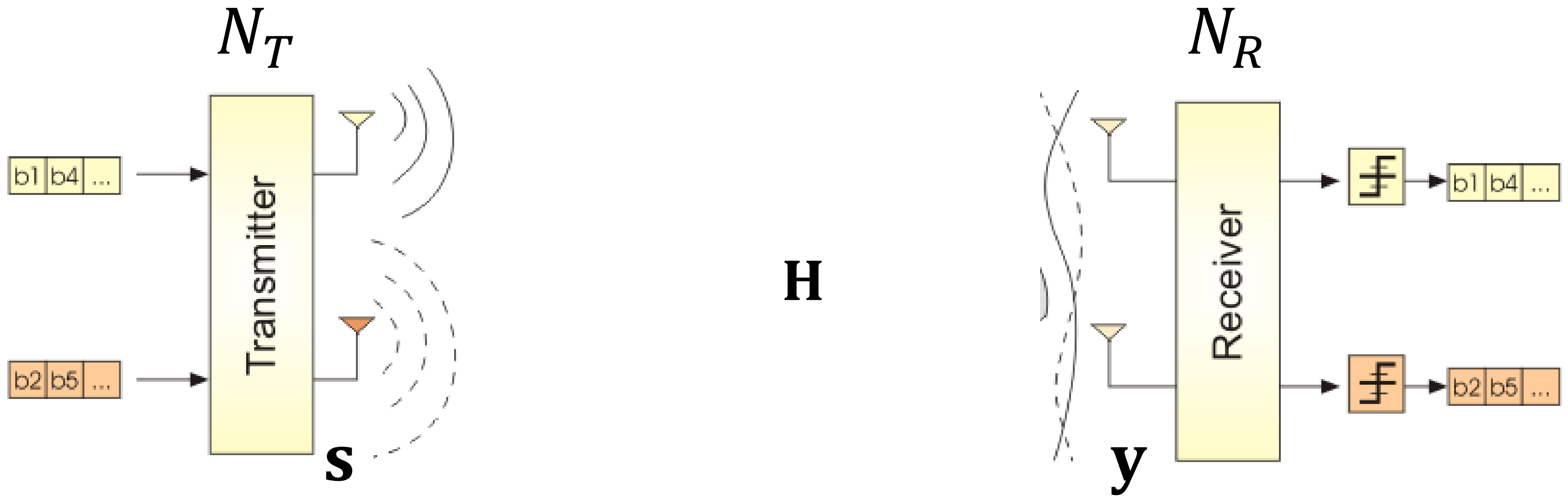
System Model for Spatial Multiplexing



$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}$$

- $\mathbf{s} = [s_1 \quad \cdots \quad s_{N_T}]^T$: Transmitted vector-symbol with $s_i \in \mathcal{O}$ ➔ $\mathbf{s} \in \mathcal{O}^{N_T}$
- $\mathcal{O}$: Set of constellation points
- $\mathbf{H} = [\mathbf{h}_1 \quad \cdots \quad \mathbf{h}_{N_T}]$: Channel matrix
- $\mathbf{y} = [y_1 \quad \cdots \quad y_{N_T}]^T$: Received vector
- $\mathbf{n} = [n_1 \quad \cdots \quad n_{N_T}]^T$: Gaussian noise with i.i.d. entries $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2)$

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Optimum MIMO Detection

Task of the MIMO detector: estimate the transmitted symbol vector $\mathbf{s}$ based on the received vector $\mathbf{y}$

- Assume we know the channel matrix $\mathbf{H}$

- Best possible estimate if obtained using Maximum Likelihood criterion

$$\hat{\boldsymbol{s}}_{\mathrm{ML}} = \arg\max_{\hat{\boldsymbol{s}} \in \mathcal{O}^{N_T}} Pr(\boldsymbol{y}|\boldsymbol{s} = \hat{\boldsymbol{s}})$$

- The noise is i.i.d. Gaussian, hence

$$Pr(\boldsymbol{y}|\boldsymbol{s} = \hat{\boldsymbol{s}}) \propto \boldsymbol{e}^{-\|\boldsymbol{y}-\boldsymbol{H}\hat{\boldsymbol{s}}\|^{2}}$$

  - Remember: taking the logarithm yields a sufficient statistics

- The ML detection rule for MIMO spatial multiplexing becomes

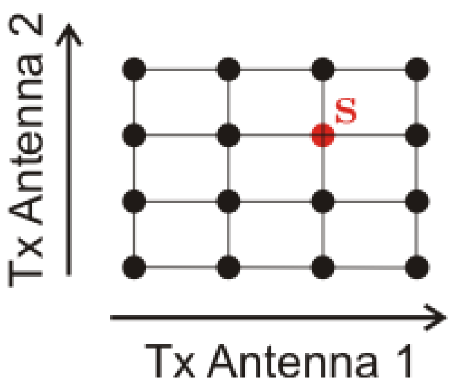
$$\hat{\boldsymbol{s}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{s}} \in \mathcal{O}^{N_T}} \|\boldsymbol{y} - \boldsymbol{H}\hat{\boldsymbol{s}}\|^{2}$$

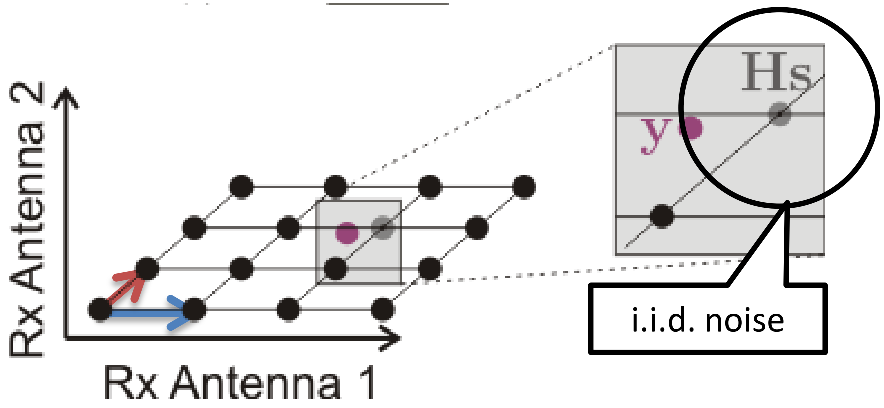
Graphical Interpretation

- Consider a system with $N_T = N_R = 2$ (2x2) with real-valued 4-PAM modulation on each antenna/stream

- Transmitter: streams lie along the orthogonal axis in the signal space
- Receiver: Signal components lie in the direction of the columns of $\mathbf{H}$
  ➔**Skewed constellation points**

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.3 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$





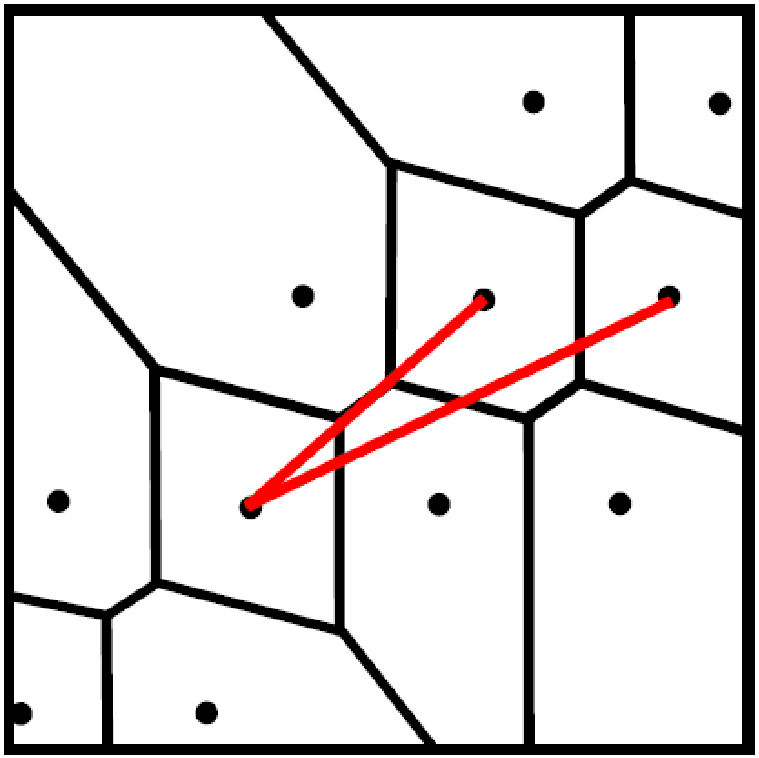
- The noise translates the received point $\mathbf{y}$ away from $\mathbf{Hs}$

ML: find the point among all $\mathbf{Hs}$ that lies closest to $\mathbf{y}$

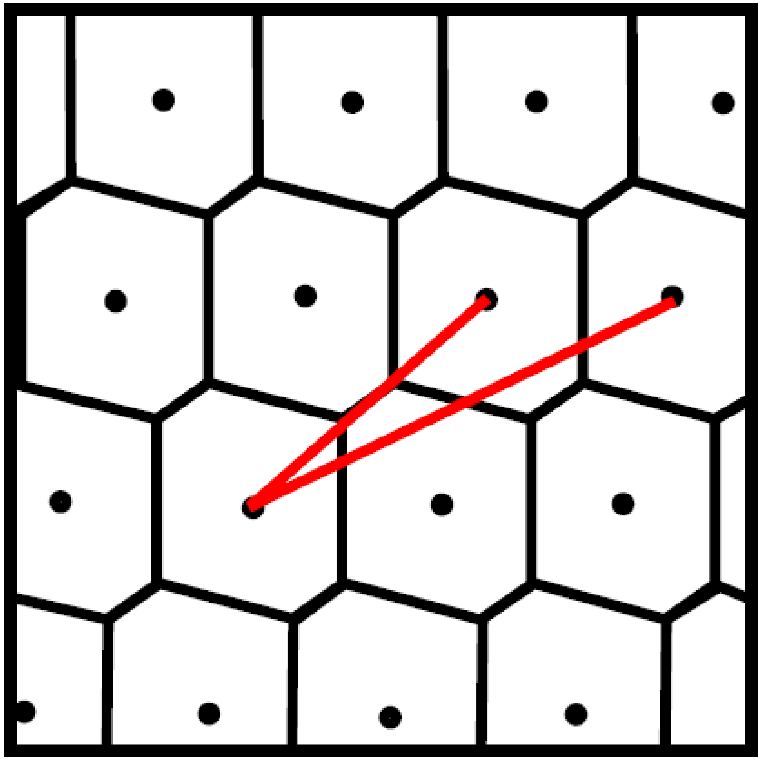
Voronoi Regions are Decision Regions of the ML Detector

Voronoi region: region around a reference constellation point in which all points lie closer to this point than to any other constellation point

Examples



Finite constellation alphabet



Infinite constellation alphabet

Complexity of ML Detection

- Straightforward approach to solving the ML detection problem: Evaluate $||\boldsymbol{y} - \boldsymbol{H}\hat{\boldsymbol{s}}||^2$ for all possible candidate symbols and find the minimum

- Number of possible candidates
  - For M-QAM: M possible candidates for each spatial stream
  - For N_T spatial streams: M^{N_T} candidates

- Assume $M = 2^q$: transmit q bits on each stream
- Number of symbols: 2^{qN_T}
- Spectral efficiency: qN_T bits per vector-symbol (bits/s/Hz)

Number of candidate symbols to check grows exponentially with the spectral efficiency

- Example: 4 streams, 64-QAM
  - Spectral efficiency is 24 bits/s/Hz
  - For each received symbol ML detection checks 16 million candidates

Linear MIMO Detection

- ML detection is complex since it considers all spatial streams jointly

A pragmatic solution

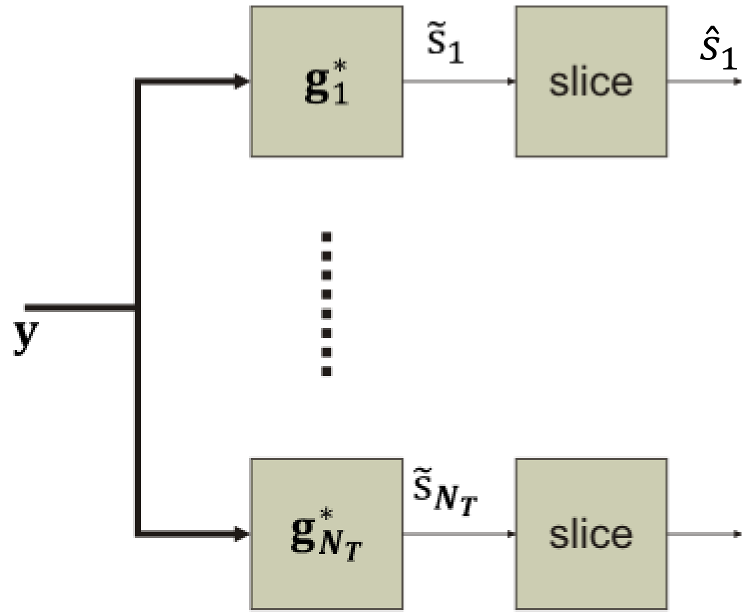
- Initially neglect the fact that constellation points are chosen from a limited set
- Use a linear estimator to recover the transmitted signal

$$\tilde{s}_k = \mathbf{g}_k^* \mathbf{y} = \mathbf{g}_k^* \mathbf{H}\mathbf{s} + \mathbf{g}_k^* \mathbf{n}$$

$$\tilde{\mathbf{s}} = \mathbf{G}\mathbf{y} = \mathbf{G}\mathbf{H}\mathbf{s} + \mathbf{G}\mathbf{n} \quad \text{with} \quad \mathbf{G}^{\mathrm{H}} = [\mathbf{g}_1 \quad \cdots \quad \mathbf{g}_{N_T}]$$

- Map each recovered stream to the nearest constellation point (slicing)

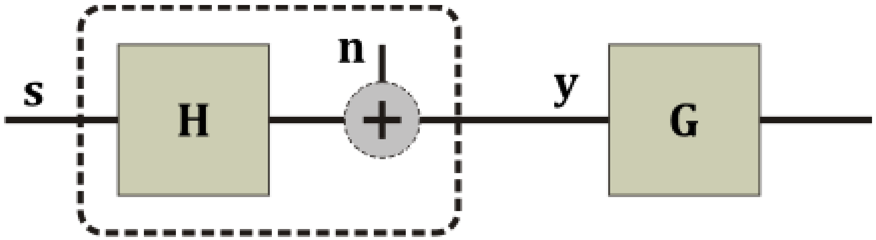
$$\hat{s}_k = Q(\tilde{s}_k)$$



How to choose G??

Linear Zero Forcing Detector

$$\tilde{\mathbf{s}} = \mathbf{G}\mathbf{y} = \mathbf{G}\mathbf{H}\mathbf{s} + \mathbf{G}\mathbf{n}$$



Choose the linear filter to undo the effect of the channel

$$\mathbf{G}\mathbf{H} = \mathbf{I} \quad \text{so that} \quad \mathbf{G}\mathbf{y} = \mathbf{s} + \mathbf{G}\mathbf{n}$$

- For $N_T = N_R$: $\mathbf{G} = \mathbf{H}^{-1}$
- For $N_T < N_R$: use the least-squares solution $\mathbf{G} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H = \mathbf{H}^\dagger$
  - Given by the pseudo inverse
  - For $N_T = N_R$, $\mathbf{H}^\dagger = \mathbf{H}^{-1}$
- For $N_T > N_R$: no solution exists (linear detection is not possible)

Performance of Zero Forcing with a Static Channel

- Consider $\mathbf{G} = \mathbf{H}^{\dagger}$

- The input of the slicers is given by
  - The transmitted symbol
  - The additive Gaussian noise multiplied with the zero-forcing equalizer

$$\boxed{\tilde{\mathbf{s}} = \mathbf{H}^{\dagger}\mathbf{y}} = \mathbf{s} + \underbrace{\mathbf{H}^{\dagger}\mathbf{n}}_{\tilde{\mathbf{n}}}$$

- The zero forcing equalizer completely eliminates interference between streams
- The noise is now correlated (colored)

$$\mathbb{E}\{\tilde{\mathbf{n}}\tilde{\mathbf{n}}^{H}\} = \sigma^{2}(\mathbf{H}^{H}\mathbf{H})^{-1}$$

- By decoding each stream independently, we ignore this noise correlation

Performance of Zero Forcing with a Static Channel

- Diagonal elements of the noise covariance matrix correspond to the noise variance on the individual streams
- Signal to noise ratio for each stream is given by

$$ZF - SNR_i = \frac{P_S/N_T}{\sigma^2[(\mathbf{H}^H\mathbf{H})^{-1}]_{ii}}$$

- The total signal to noise ratio is given by

$$ZF - SNR = \frac{P_S}{\sigma^2 \sum_{\boldsymbol{i=1}}^{N_T}[(\mathbf{H}^H\mathbf{H})^{-1}]_{ii}}$$

- With the SVD of $\mathbf{H} = \mathbf{U}\boldsymbol{\Lambda}\boldsymbol{V}^{\mathrm{H}}$ we find that $\sum_{\boldsymbol{i=1}}^{N_T}[(\mathbf{H}^H\mathbf{H})^{-1}]_{ii} = \sum_{\boldsymbol{i=1}}^{N_T}\frac{1}{\lambda_i^2}$
  - λ_i : Eigenvalues of $\mathbf{H}$

$$ZF - SNR = \frac{P_S}{\sigma^2 \sum_{\boldsymbol{i=1}}^{N_T}\frac{1}{\lambda_i^2}} < \frac{P_S\lambda_{N_T}^2}{\sigma^2}$$

- **If the channel has a small Eigenvalue the SNR degrades significantly**

Performance of Zero Forcing with a Static Channel

For comparison, we use a lower bound on the SNR without the zero forcing equalizer

- Transmitted signal has power P_S and $\lambda_{N_T}^2$ is the smallest Eigenvalue of the channel
- Total received signal power : $P_S \sum_{\boldsymbol{i=1}}^{N_T} \lambda_i^2 > P_S N_T \lambda_{N_T}^2$ (lower bound with smallest EV)
- Total noise power $\sigma^2 N_T$

$$SNR = \frac{P_S \sum_{\boldsymbol{i=1}}^{N_T} \lambda_i^2}{\sigma^2 N_T} > \frac{P_S \lambda_{N_T}^2}{\sigma^2}$$

Compare this to the SNR after zero forcing

$$ZF - SNR < \frac{P_S \lambda_{N_T}^2}{\sigma^2} < SNR$$

Noise enhancement: zero forcing degrades SNR

A Graphical Interpretation

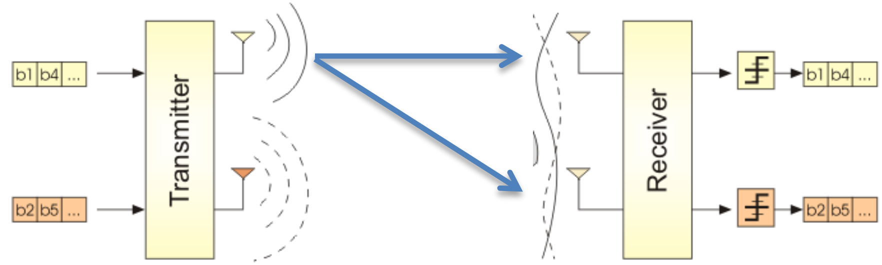
Assume we are interested in the 1st stream

$$\mathbf{y} = \mathbf{h}_1 s_1 + \sum_{i \neq k} \mathbf{h}_i s_i + \mathbf{n}$$

Desired stream

Interference (other streams)

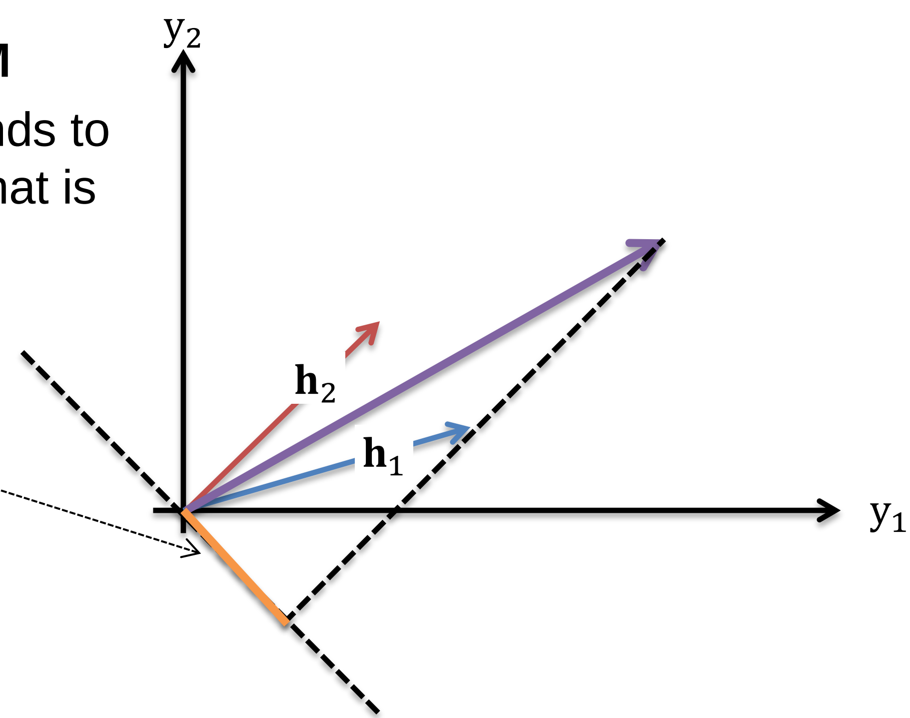
Noise



$$\mathbf{H} = [\mathbf{h}_1 \quad \cdots \quad \mathbf{h}_{N_T}]$$

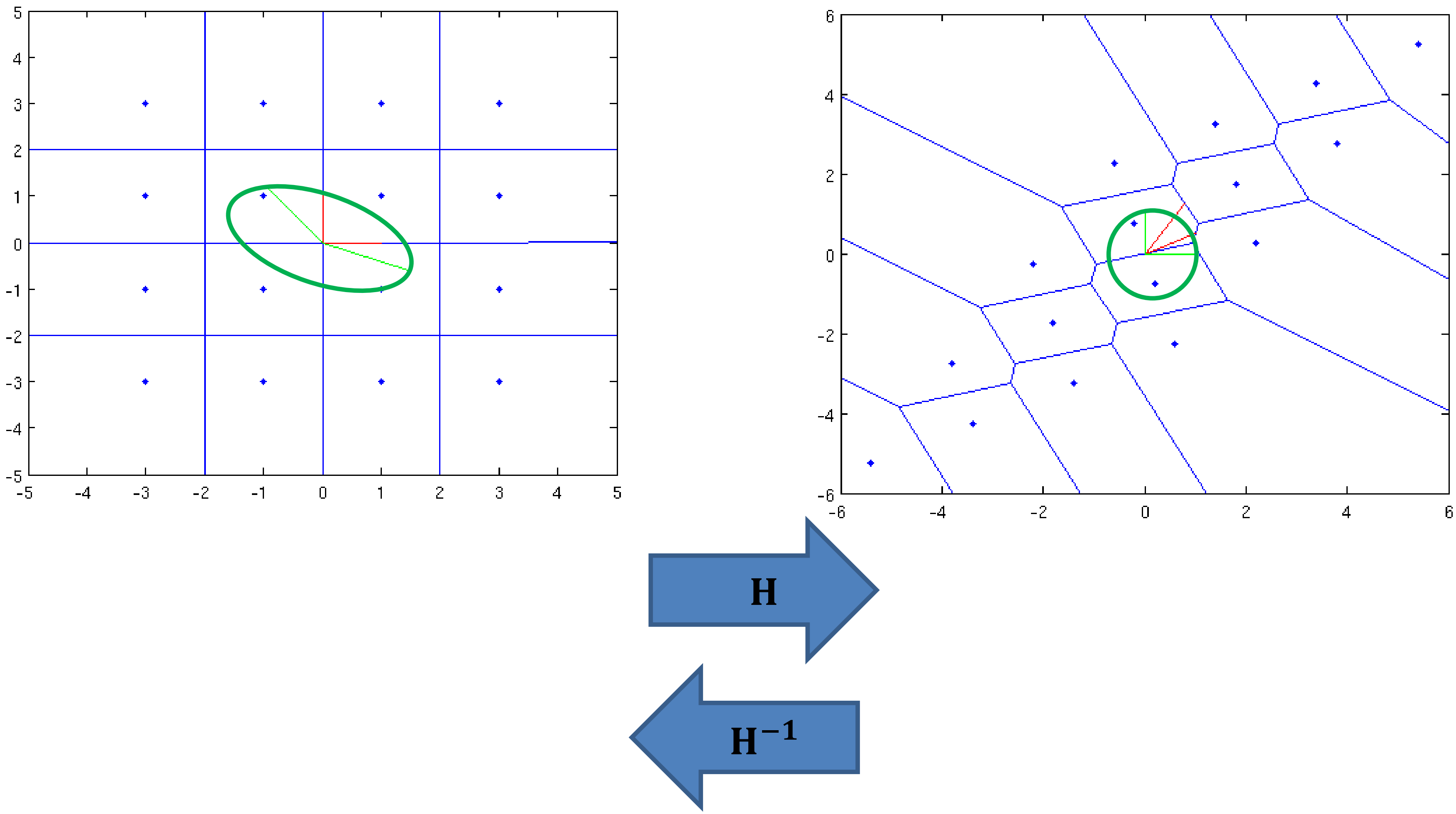
Consider again a 2x2 system with 4-PAM

- Nulling the interference of $\mathbf{h}_2 s_2$ corresponds to projecting the signal onto the subspace that is orthogonal to $\mathbf{h}_2$
- Only a small part of the energy of $\mathbf{h}_1 s_1$ lies in that subspace (unless $\mathbf{h}_1 \perp \mathbf{h}_2$)
- SNR is degraded since the noise power remain the same as along $\mathbf{h}_1$



Graphical Interpretation

Basis transformation and transformation of the noise



MMSE Receiver

- Zero interference comes at the cost of *noise enhancement*

Basic idea: balance the distortion due to noise and due to residual interference

- MMSE filter design criterion: determine the linear filter to minimize the mean squared error

$$\mathbf{W} = \arg\min_{\widetilde{\mathbf{W}}} \mathbb{E}\left\{\left\|\widetilde{\mathbf{W}}\mathbf{y} - \mathbf{s}\right\|^{\mathbf{2}}\right\}$$

- We obtain $\mathbf{W} = \boldsymbol{\Phi}_{\boldsymbol{ys}}\boldsymbol{\Phi}_{yy}^{-1}$ with $\boldsymbol{\Phi}_{\boldsymbol{ys}} = \frac{P_S}{N_T}\mathbf{H}^{\mathrm{H}}$ and $\boldsymbol{\Phi}_{\boldsymbol{yy}} = \left(\mathbf{H}\mathbf{H}^{\mathrm{H}} + \sigma^2\mathbf{I}\right)$

$$\mathbf{W} = \mathbf{H}^{\mathrm{H}}\left(\mathbf{H}\mathbf{H}^{\mathrm{H}} + N_T\frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T\frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}}\mathbf{H}^{\mathrm{H}}$$

A Note on Complexity

Since $N_R \geq N_T$ we know that $\mathbf{H}$ is either square of tall

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{bmatrix}$$

Option 1: $\mathbf{H}^{\mathrm{H}}\left(\mathbf{H}\mathbf{H}^{\mathrm{H}} + N_T \frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}}$ involves

- $\mathbf{Z} = \mathbf{H}\mathbf{H}^{\mathrm{H}} + N_T \frac{\sigma^2}{P_S}\mathbf{I}$: $N_R^2 N_T$ multiplications
- $inv(\mathbf{Z})$: Inversion of a $N_R \times N_R$ matrix (ill conditioned for $N_T \frac{\sigma^2}{P_S}$ small)
- $\mathbf{H}^{\mathrm{H}}\mathbf{Z}$: $N_T^2 N_R$ multiplications

Option 2: $\left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T \frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}}\mathbf{H}^{\mathrm{H}}$ involves

- $\mathbf{Z} = \mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T \frac{\sigma^2}{P_S}\mathbf{I}$: $N_T^2 N_R$ multiplications
- $inv(\mathbf{Z})$: Inversion of a $N_T \times N_T$ matrix
- $\mathbf{Z}\mathbf{H}^{\mathrm{H}}$: $N_T^2 N_R$ multiplications

Computationally less complex and numerically more stable for high SNR

Relationship Between Estimators

$$\mathbf{W} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T \frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}} \mathbf{H}^{\mathrm{H}}$$

For low-SNR ($\frac{\sigma^2}{P_S} \to \infty$)

- Additive noise is the dominant source of distortion

$$\mathbf{W} \to \left(N_T \frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}} \mathbf{H}^{\mathrm{H}}$$

- *MMSE converges to a matched filter*

For high-SNR ($\frac{\sigma^2}{P_S} \to 0$)

- Interference is the dominant source of distortion

$$\mathbf{W} \to \left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-\mathbf{1}} \mathbf{H}^{\mathrm{H}}$$

- *MMSE converges to the zero-forcing filter*

Unbiased MMSE Receiver

- Consider $\mathbf{W} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T \frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}} \mathbf{H}^{\mathrm{H}}$ and apply to the received vector

$$\tilde{\mathbf{s}} = \mathbf{W}\mathbf{y} = \underbrace{\mathbf{W}\mathbf{H}}_{\boldsymbol{R}}\,\mathbf{s} + \underbrace{\mathbf{W}\mathbf{n}}_{\tilde{\mathbf{n}}}$$

$$\tilde{s}_k = r_{kk} s_{\boldsymbol{k}} + \sum_{\boldsymbol{i} \neq \boldsymbol{k}} r_{ki} s_{\boldsymbol{i}} + \tilde{s}_k$$

- Note that
  $\mathbf{R} = \mathbf{W}\mathbf{H} \neq \mathbf{I}$ ➔ Residual interference (off-diagonal elements)
  $r_{kk} < 1$ ➔ Desired stream is attenuated (MMSE is a biased estimator)

- Biased MMSE estimate provides the lowest MSE, but not necessarily the best BER
  - Example: In the low SNR regime, σ^2 is large and $r_{kk} \ll 1$. Hence, constellations shrink
- Multiplying $\tilde{s}_k$ with a scalar does not change the SNR

Bias can be removed by slicing a scaled version of $\tilde{s}_k$: $Q(\tilde{s}_k / r_{kk})$

Summary of Linear Receivers

Three step process:

1. Compute a linear filter from the channel estimate $\mathbf{H}$

$$\mathbf{G} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-\mathbf{1}}\mathbf{H}^{\mathrm{H}} \qquad \textbf{(ZF)}$$

$$\mathbf{G} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T\frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}}\mathbf{H}^{\mathrm{H}} \qquad \textbf{(MMSE - Biased)}$$

2. Apply linear filter to the received vector $\mathbf{y}$

$$\tilde{\mathbf{s}} = \mathbf{W}\mathbf{y}$$

3. Quantize to the nearest constellation point

$$\hat{s}_k = Q(\tilde{\mathrm{s}}_k)$$

Linear ZF Detection with QR Decomposition

QR-decomposition of the channel matrix

$$\mathbf{H} = \mathbf{QR}$$

- $\mathbf{Q} = [\boldsymbol{q}_1 \quad \cdots \quad \boldsymbol{q}_{N_T}] \in \mathcal{C}^{N_R \times N_R}$: unitary matrix, i.e., $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}$
- $\mathbf{R} = \begin{bmatrix} r_{11} & \cdots & r_{1N_T} \\ \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & r_{N_T N_T} \end{bmatrix} \in \mathcal{C}^{N_T \times N_T}$: upper triangular
- MATLAB: `[Q,R]=qr(H)` yields the QR-decomposition of **H**

Some properties:

- The ith column of $\mathbf{H}$ is a linear combination of $[\boldsymbol{q}_1 \quad \cdots \quad \boldsymbol{q}_i]$

$$\mathbf{h}_i = \sum_{k=i}^{N_T} r_{ki}\boldsymbol{q}_i$$

- $\boldsymbol{q}_i$ contributes only to $[\mathbf{h}_i \quad \cdots \quad \mathbf{h}_{N_T}]$, but NOT to $[\mathbf{h}_1 \quad \cdots \quad \mathbf{h}_{i-1}]$ ➔ $\boldsymbol{q}_i \perp \mathbf{h}_j$, j<i

Linear ZF Detection with QR Decomposition

For example:

- $\boldsymbol{q}_{N_T}$ is orthogonal to $\mathbf{h}_1 \cdots \boldsymbol{h}_{N_T-1}$
- $\boldsymbol{q}_{N_T-1}$ is orthogonal to $\mathbf{h}_1 \cdots \boldsymbol{h}_{N_{T-2}}$
- and so on…

Basis transform: project the received vector onto the columns of $\mathbf{Q}$

$$\tilde{\mathbf{y}} = \mathbf{Q}^{\mathrm{H}}\mathbf{y} = \mathbf{Q}^{\mathrm{H}}\mathbf{Q}\mathbf{R}\boldsymbol{s} + \underbrace{\mathbf{Q}^{\mathrm{H}}\mathbf{n}}_{\tilde{\boldsymbol{n}}} = \mathbf{R}\boldsymbol{s} + \tilde{\boldsymbol{n}}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \vdots \\ \tilde{y}_{N_T} \end{bmatrix} = \begin{bmatrix} r_{11} & \cdots & \cdots & r_{1N_T} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{N_T N_T} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ \vdots \\ s_{N_T} \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \vdots \\ \vdots \\ \tilde{n}_{N_T} \end{bmatrix}$$

- $\tilde{y}_i$: projection of the received signal onto $\boldsymbol{q}_{N_T}$ ➔ only contributions from $s_i \cdots s_{N_T}$
- $\tilde{\boldsymbol{n}}$: i.i.d. Gaussian with same properties as $\mathbf{n}$ ($\mathbf{Q}^{\mathrm{H}}$ is unitary)

Linear ZF Detection with QR Decomposition

$$\begin{bmatrix} r_{11} & \cdots & \cdots & r_{1N_T} \\ 0 & \ddots & & \vdots \\ & & \ddots & \vdots \\ 0 & & 0 & r_{N_T N_T} \end{bmatrix} \begin{bmatrix} \hat{s}_1 \\ \vdots \\ \vdots \\ \hat{s}_{N_T} \end{bmatrix} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \vdots \\ \tilde{y}_{N_T} \end{bmatrix}$$

ZF detection: solve $\mathbf{R}\tilde{s} = \tilde{\mathbf{y}}$ with back substitution

- Start with the last stream and proceed backward

- For $i = N_T$ downto 1
  - Scaling : $\tilde{s}_i = \tilde{y}_i / r_{ii}$
  - Backsubstitution : $\tilde{\mathbf{y}} \leftarrow \tilde{\mathbf{y}} - \boldsymbol{r}_i \tilde{s}_i$
- End
- Quantize $\hat{s}_i = \mathrm{Q}(\tilde{s}_i)$

$$\tilde{\mathbf{s}} = \mathbf{R}^{-1}\tilde{\mathbf{y}}$$

MMSE Receiver Based on QR Decomposition

- We know that solving $\mathbf{R}\tilde{\boldsymbol{s}} = \mathbf{Q}^H\mathbf{y}$ in a LS sense with $\mathbf{H} = \mathbf{QR}$ through backsubstitution corresponds to the solution of $\tilde{\boldsymbol{s}} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{y}$

We are interested in $\tilde{\boldsymbol{s}} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T\frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}}\mathbf{H}^{\mathrm{H}}\boldsymbol{y}$

- Introduce the augmented channel matrix $\overline{\boldsymbol{H}} = \begin{bmatrix}\mathbf{H}\\ \sqrt{N_T\frac{\sigma^2}{P_S}}\mathbf{I}\end{bmatrix}$ and $\bar{\mathbf{y}} = \begin{bmatrix}\mathbf{y}\\ \mathbf{0}\end{bmatrix}$
- Note that $\mathbf{H}^{\mathrm{H}}\mathbf{H} + N_T\frac{\sigma^2}{P_S}\mathbf{I} = \overline{\boldsymbol{H}}^{\mathrm{H}}\overline{\boldsymbol{H}}$ and $\mathbf{H}^{\mathrm{H}}\boldsymbol{y} = \overline{\boldsymbol{H}}^{\mathrm{H}}\bar{\mathbf{y}}$
- We can therefore write $\tilde{\boldsymbol{s}} = \left(\overline{\boldsymbol{H}}^{\mathrm{H}}\overline{\boldsymbol{H}}\right)^{-\mathbf{1}}\overline{\boldsymbol{H}}^{\mathrm{H}}\bar{\mathbf{y}}$
- Introduce the QR decomposition of $\overline{\boldsymbol{H}} = \begin{bmatrix}\overline{\mathbf{Q}_1} & \overline{\mathbf{Q}_3}\\ \overline{\mathbf{Q}_2} & \overline{\mathbf{Q}_4}\end{bmatrix}\begin{bmatrix}\overline{\mathbf{R}}\\ \mathbf{0}\end{bmatrix}$ and solve

$$\begin{bmatrix}\overline{\mathbf{R}}\\ \mathbf{0}\end{bmatrix}\tilde{\boldsymbol{s}} = \begin{bmatrix}\overline{\mathbf{Q}_1}^{\mathrm{H}} & \overline{\mathbf{Q}_2}^{\mathrm{H}}\\ \overline{\mathbf{Q}_3}^{\mathrm{H}} & \overline{\mathbf{Q}_4}^{\mathrm{H}}\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \mathbf{0}\end{bmatrix} \overset{LS}{\rightarrow} \overline{\mathbf{R}}\tilde{\boldsymbol{s}} = \overline{\mathbf{Q}_1}^{\mathrm{H}}\mathbf{y}$$

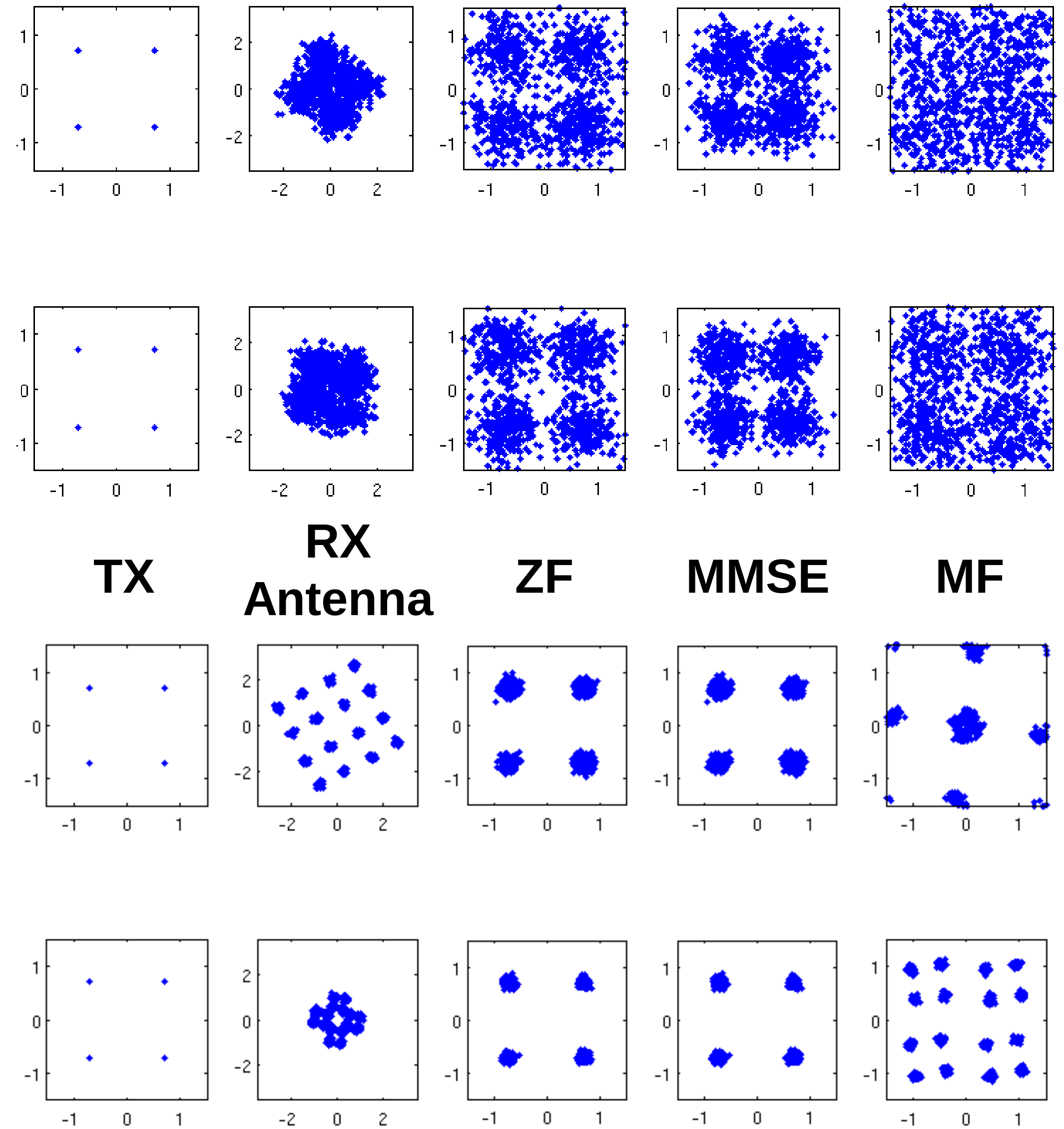
Illustrative Example

Low SNR: 6dB

- ZF: severe noise enhancement
- MMSE: good tradeoff between noise and interference, but signal is scaled (BIAS)
- MF: hard to distinguish between noise and interference

High SNR: 20dB

- ZF and MMSE perform almost equally well
- MF: Interference is clearly visible (no cancellation capability)



Successive Interference Cancellation (SIC)

Problem of linear detection:

- Detection of each stream requires nulling of the interference of $N_T - 1$ streams
- Since we always project into the null-space of $N_T - 1$ streams we loose all the energy in these dimensions (for all streams)

Interference cancellation

- Assume we already know $K - 1$ streams (signal components) s_k with $k = 1 \ldots K - 1$
- In this case, we can cancel the impact of s_k from the received vector $\mathbf{y}$

$$\mathbf{y}^{(K)} = \mathbf{y} - \sum_{k=1}^{K-1} \mathbf{h}_k s_k \qquad \boldsymbol{y} = \mathbf{Hs} + \mathbf{n} \rightarrow \mathbf{y}^{(K)} = \mathbf{H}^{(K)} \begin{bmatrix} s_K \\ \vdots \\ s_{N_T} \end{bmatrix} + \mathbf{n}$$

with $\mathbf{H}^{(K)} = [\mathbf{h}_K \quad \cdots \quad \mathbf{h}_{N_T}] \in \mathcal{C}^{N_R \times N_T - K + 1}$

After perfect interference cancellation of $K - 1$ streams we obtain a new system with $N_T - K + 1$ streams and N_R «receive antennas»

Successive Interference Cancellation (SIC)

Basic idea of SIC: perform the following two steps iteratively

- Linear detection on one stream only and assume the detected symbol is correct
- Perform interference cancellation with the detected symbol

SIC algorithm

Unit row vector

$$\boldsymbol{e}_1 = [1 \quad 0 \quad \cdots \quad 0]$$

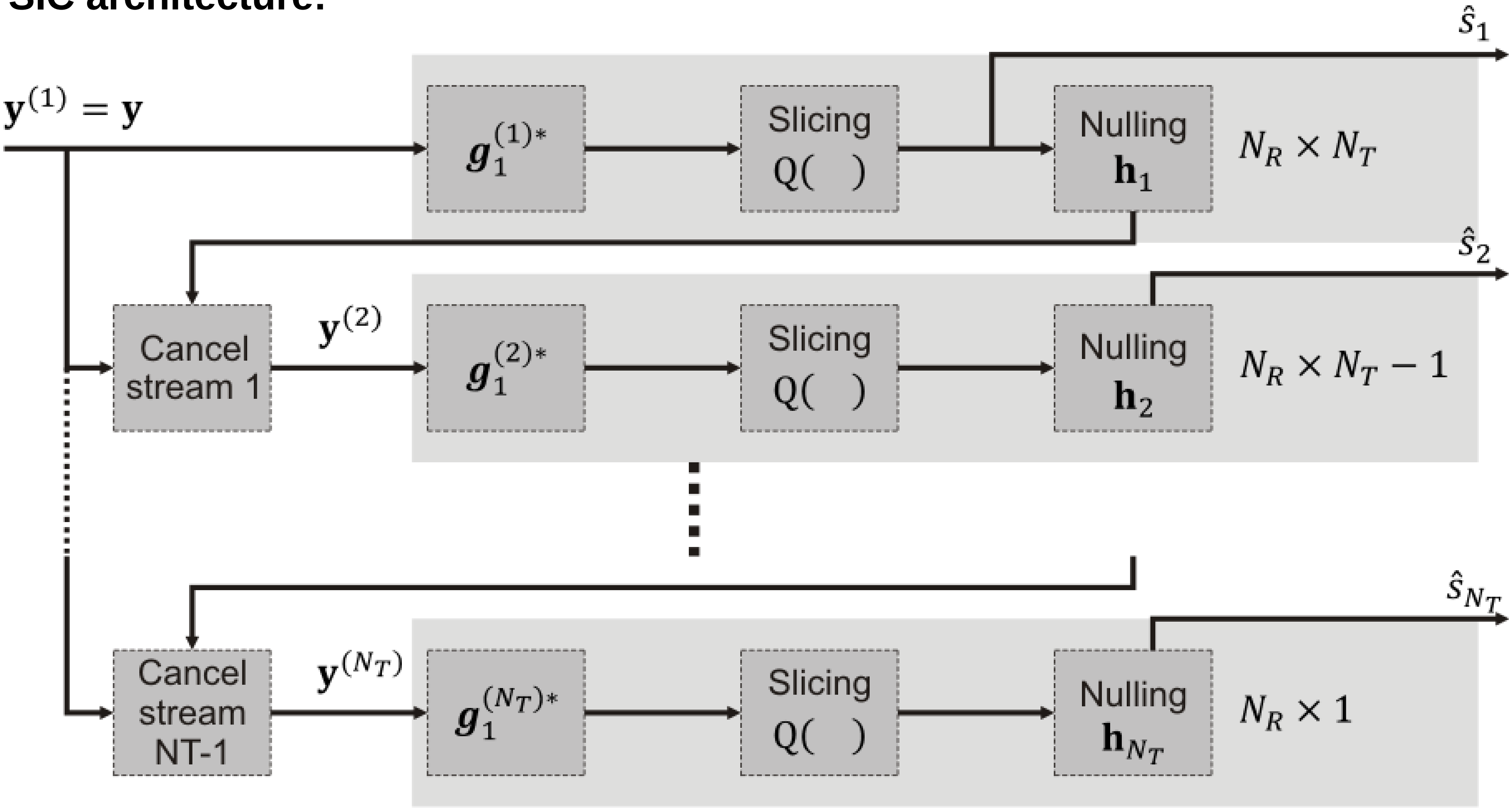
- Set $\mathbf{y}^{(1)} = \mathbf{y}$, $\mathbf{H}^{(1)} = \mathbf{H}$
- For $i = 1$ to N_T
  - Linear estimator for the first remaining stream: $\boldsymbol{g}_1^{(i)*} = \boldsymbol{e}_1 \mathbf{H}^{(i)\dagger}$ (first row of $\mathbf{H}^{(i)\dagger}$)
  - Symbol detection: $\hat{s}_i = \mathrm{Q}\left(\boldsymbol{g}_1^{(i)*} \mathbf{y}^{(i)}\right)$
  - ***Interference cancellation***: $\mathbf{y}^{(i+1)} = \mathbf{y}^{(i)} - \mathbf{h}_i \hat{s}_i$
  - Remove 1st column from $\mathbf{H}^{(i)} \to \mathbf{H}^{(i+1)}$ $\quad \mathbf{H}^{(i+1)} = \left[\mathbf{h}_2^{(i)} \quad \cdots \quad \mathbf{h}_{N_T}^{(i)}\right]$
- End

Note that

- $\mathbf{H}^{(i)} = [\mathbf{h}_i \quad \cdots \quad \mathbf{h}_{N_T}]$
- $\mathbf{y}^{(i)} = \mathbf{y} - \sum_{k=1}^{i-1} \mathbf{h}_k \hat{s}_k \approx \mathbf{y} - \sum_{k=1}^{i-1} \mathbf{h}_k s_k$

Successive Interference Cancellation (SIC)

SIC architecture:

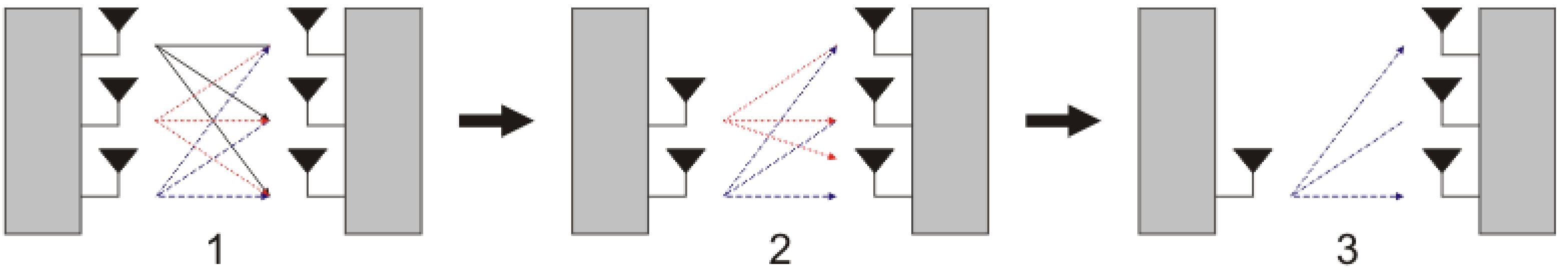


- Note: the number of interferers decreases with each stage

Performance of SIC

Observations:

- The numbers of interferers decreases with each stage, but the number of observations remains the same
- Intuitively, nulling becomes “easier” with each stream that we cancel
- Eventually, we arrive at a SIMO system with one stream and N_R observations



Unfortunately: performance gain is only marginal

- The error probability of the weakest stream determines the symbol-error probability
- Without “perfect” cancellation, error propagation limits the performance (damages/interfers with later detected streams)

Need to detect the most reliable streams first

Detection Order with SIC

- No particular reason forces us to perform detection and SIC in natural order
- In each stage of the SIC algorithm, we can choose which stream to detect
- In total there are $N_T!$ different orders for the SIC

- The detection order can be changed by
  - Permuting the columns of $\mathbf{H}$ and rearranging the order of the entries of the symbol-vector and accordingly

$$\mathbf{H} \leftarrow \mathbf{H}\boldsymbol{\Pi} \text{ and } \hat{\boldsymbol{s}} \leftarrow \boldsymbol{\Pi}^{\mathrm{H}}\hat{\boldsymbol{s}} \qquad \Rightarrow \qquad \mathbf{H}\boldsymbol{\Pi}\boldsymbol{\Pi}^{\mathrm{H}}\hat{\boldsymbol{s}} = \mathbf{H}\hat{\boldsymbol{s}}$$

  with the unitary permutation matrix $\boldsymbol{\Pi}$

  - Directly: detecting any stream and removing the corresponding column of $\mathbf{H}$

Ordered SIC

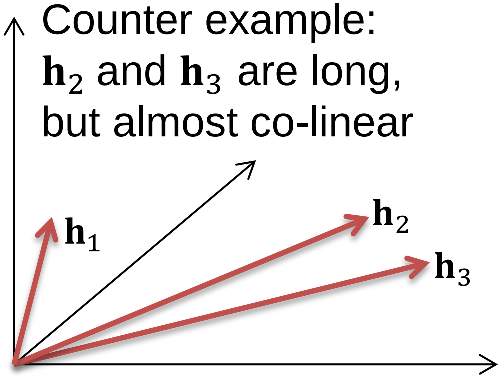
Objective: detect the more reliable streams as early as possible

Received Power based ordering

- Consider the contribution to the received power of each stream

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n} = \sum_{k=1}^{N_T} s_k\, \mathbf{h}_k + \mathbf{n}$$

Counter example:
$\mathbf{h}_2$ and $\mathbf{h}_3$ are long,
but almost co-linear



- Contribution of stream k to the received power $\|\mathbf{h}_k\|^2$

- **Column-norm ordering**: choose the detection order according to the norms of the columns of the channel $\|\mathbf{h}_k\|^2$

- **Unfortunately, the geometry of the channel is not accounted: If two or multiple streams are strong, but lie almost in the same subspace, separation is still difficult (noise enhancement)!**

V-BLAST Order

So, what is the right detection order?

- Ultimately, the quality of the decision depends on the SINR at the slicer

- Slicer input signal

$$\tilde{s}_k = \boldsymbol{g}_k^{(i)*}\mathbf{y}^{(i)} = s_k + \boldsymbol{g}_k^{(i)*}\mathbf{n}$$

$$\|\tilde{s}_k - s_k\|^2 = \left\|\boldsymbol{g}_k^{(i)*}\right\|^2 \|\mathbf{n}\|^2$$

- In each stage, the SINR is different for each stream

Easier to compute since $\boldsymbol{g}_k^{(i)*}$ is computed anyway, while $\left(\mathbf{H}^{(i)H}\mathbf{H}^{(i)}\right)^{-1}$ may not be explicitly available

$$SINR_k^{(i)} = \frac{P_S/N_T}{\sigma^2[(\mathbf{H}^{(i)H}\mathbf{H}^{(i)})^{-1}]_{kk}} \propto 1/\left\|\boldsymbol{g}_k^{(i)*}\right\|^2$$

- V-BLAST ordering: always select the stream with the best post-detection $SINR_k^{(i)}$, i.e., with the *smallest* equalizer gain $\left\|\boldsymbol{g}_k^{(i)*}\right\|^2$ (i.e., noise gain/enhancement)

V-BLAST Algorithm

Basic idea: in each stage, minimize the detection error probability by choosing the stream k with the best post-detection signal to interference plus noise ratio

$$k^{(i)} = \underset{\boldsymbol{k}}{\operatorname{argmin}} \left\{ SINR_k^{(i)} \right\}$$

V-BLAST algorithm

- Set $\mathbf{y}^{(1)} = \mathbf{y}$, $\mathbf{H}^{(1)} = \mathbf{H}$
- For $i = 1$ to N_T
  - Compute the linear estimator for all streams $\mathbf{G}^{(i)} = \mathbf{H}^{(i)\dagger}$ with $\boldsymbol{g}_k^{(i)*} = \boldsymbol{e}_k \mathbf{G}^{(i)}$: rows of $\mathbf{G}$
  - Find the row of $\mathbf{H}^{(i)\dagger}$ with the smallest norm $k^{(i)} = \underset{\boldsymbol{k}}{\operatorname{argmin}} \left\{ \left\| \boldsymbol{g}_k^{(i)*} \right\|^2 \right\}$
  - Symbol detection of the $k^{(i)}$th stream $\hat{s}_i = \mathrm{Q}\left(\boldsymbol{g}_{k^{(i)}}^{(i)*} \mathbf{y}^{(i)} \right)$
  - Interference cancellation: $\mathbf{y}^{(i+1)} = \mathbf{y}^{(i)} - \mathbf{h}_i \hat{s}_i$
  - Remove $k^{(i)}$th column from $\mathbf{H}^{(i)} \rightarrow \mathbf{H}^{(i+1)}$ $\mathbf{H}^{(i+1)} = \left[\mathbf{h}_1^{(i)} \cdots \mathbf{h}_{k^{(i)}-1}^{(i)} \mathbf{h}_{k^{(i)}+1}^{(i)} \cdots \mathbf{h}_{N_T}^{(i)} \right]$
- End loop
- Reorder $\hat{s}_i$ according to $k^{(i)}$

MMSE V-BLAST

MMSE V-BLAST follows the same strategy

- Linear detection in each stage based on MMSE filter

$$\mathbf{H}^{(i)\dagger} \rightarrow \mathbf{G}^{(i)} = \left(\mathbf{H}^{(\boldsymbol{i})H}\mathbf{H}^{(\boldsymbol{i})} + N_T \frac{\sigma^2}{P_S}\mathbf{I}\right)^{-\mathbf{1}} \mathbf{H}^{(\boldsymbol{i})H}$$

- Calculate the SINR for each stream with linear MMSE filter

$$MMSE - SNR_k^{(i)} = \frac{P_S/N_T}{\sigma^2[(\mathbf{H}^{(\boldsymbol{i})H}\mathbf{H}^{(\boldsymbol{i})} + \mathbf{I}\sigma^2 N_T/P_S)^{-1}]_{kk}}$$

  and select the best stream for detection

- Detect the symbol on the selected stream

- Remove the impact of the detected stream from the received vector, the corresponding channel vector from the channel matrix

- Start from the beginning with the reduced system

Efficient SIC with QR Decomposition

- Drawback of SIC: each of the N_T stages requires a matrix inversion to compute the linear estimator
- More efficient implementation exists based on QR decomposition

QR-decomposition based SIC

- Given $\mathbf{H} = \mathbf{QR}$ of the channel
- Since $\mathbf{Q}$ is unitary, we can always consider $\tilde{\mathbf{y}} = \mathbf{Q}^{\mathrm{H}}\mathbf{y} = \mathbf{Rs} + \tilde{\boldsymbol{n}}$
- Start from the last stream and work backward to the first performing backsubstitution using quantized symbols

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{bmatrix}$$

- For $i = N_T$ downto 1
  - $\hat{s}_i = \mathrm{Q}(\tilde{y}_i / r_{ii})$
  - $\tilde{\mathbf{y}} \leftarrow \tilde{\mathbf{y}} - \mathbf{r}_i \hat{s}_i$
- End

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} r_{11} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} s_1 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{bmatrix}$$

Ordered SIC based on QR Decomposition

Problem: Backsubstitution determines detection order

- Nevertheless, we can start by reordering the channel matrix prior to QR decomposition
- **Unfortunately**, at this stage, the post-detection SNR is unknown!

Goal: find an alternative ordering strategy/criterion

- Consider the SINR before the slicer when detection is performed with backsubstitution

$$\tilde{s}_i = \frac{\tilde{y}_i}{r_{ii}} = s_i + \tilde{n}_i / r_{ii} \quad \Rightarrow \quad |\tilde{s}_i - s_i|^2 = |n_i|^2 / |r_{ii}|^2$$

- Large diagonal elements of $\mathbf{R}$ result in better SNR after the slicer

- Ordering strategy: Reorder the streams such that $r_{ii} > r_{kk}$ for $i > k$

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QR Decomposition

Objective

- Derive an orthonormal basis $\mathbf{Q}$ for $\mathbf{H}$ such that $\boldsymbol{q}_i \perp \mathbf{h}_j$, j<l

Gram Schmidt algorithm

- Set $\mathbf{Q} = \mathbf{H}$ will update $\boldsymbol{Q}$ later
- For $i = 1$ to N_T
  - $\boldsymbol{p} = \boldsymbol{q}_i$
  - For $k = i + 1$ to N_T
    - $r_{ik} = \boldsymbol{q}_k^H \boldsymbol{p}$
    - $\boldsymbol{q}_k = \boldsymbol{q}_k - r_{ik}\boldsymbol{q}_i$
  - End
  - $r_{ii} = ||\boldsymbol{p}||$
  - $\boldsymbol{q_i} = \boldsymbol{p}/r_{ii}$
- End

Orthogonalize $\mathbf{q}_{i+1} \quad \cdots \quad \mathbf{q}_{N_T}$

$$\boldsymbol{q}_k = \boldsymbol{q}_k - \sum_{k=1}^{i-1} \mathbf{q}_i \langle \mathbf{q}_k, \mathbf{q}_i \rangle$$

Normalize

Sorted QR Decomposition

- Unfortunately, r_{ii} are not known prior to performing QR decomposition

Solution: perform ordering within the QR decomposition (*Sorted QR decomposition*) [Wuebben, 2001]

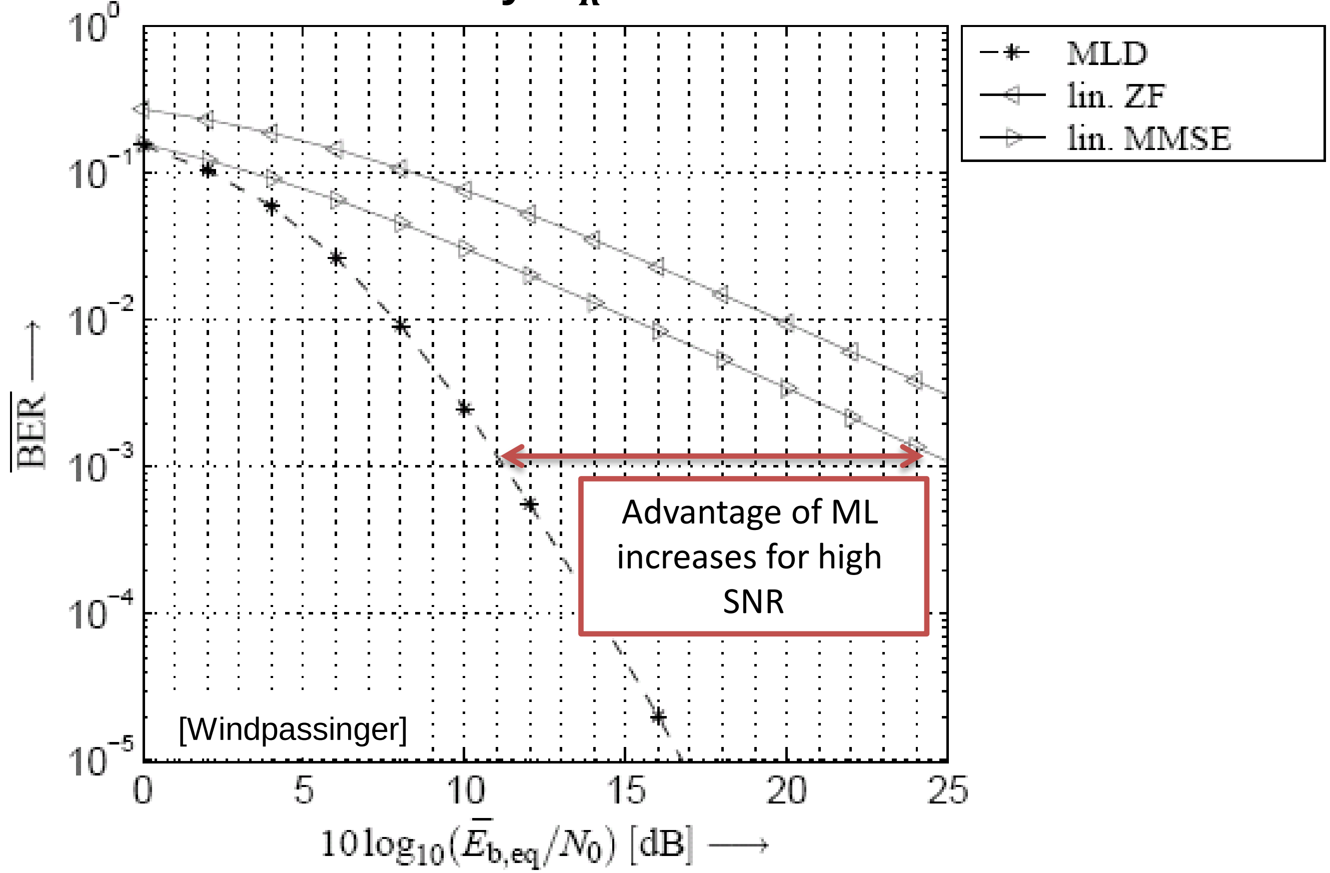
- Set $\mathbf{Q} = \mathbf{H}$
- For $i = 1$ to N_T
  - Among $\boldsymbol{q}_i \quad \cdots \quad \boldsymbol{q}_{N_T}$ find the one with the smallest norm ➔ m
  - Exchange $\boldsymbol{q}_i$ with $\boldsymbol{q}_m$
  - $\boldsymbol{p} = \boldsymbol{q}_i$
  - $r_{ii} = \|\boldsymbol{p}\|$
  - For $k = i + 1$ to N_T
    - $r_{ik} = \boldsymbol{q}_k^H \boldsymbol{p}$
    - $\boldsymbol{q}_k = \boldsymbol{q}_k - r_{ik}\boldsymbol{q}_i$
  - End
  - $\boldsymbol{q}_i = \boldsymbol{p}/r_{ii}$
- End

Greedy strategy: in each step choose among the remaining columns the one that yields the smallest r_{ii}
➔ r_{ii} in (approximately!) increasing order

Comparison of Basic MIMO Receivers with Fading for $N_T = N_R$

Observations

- Linear (ZF and MMSE) detection has diversity order of 1
- MMSE provides a gain in SNR (shift) over ZF
- **ML detection achieves full diversity N_R**



Comparison of Basic MIMO Receivers with Fading

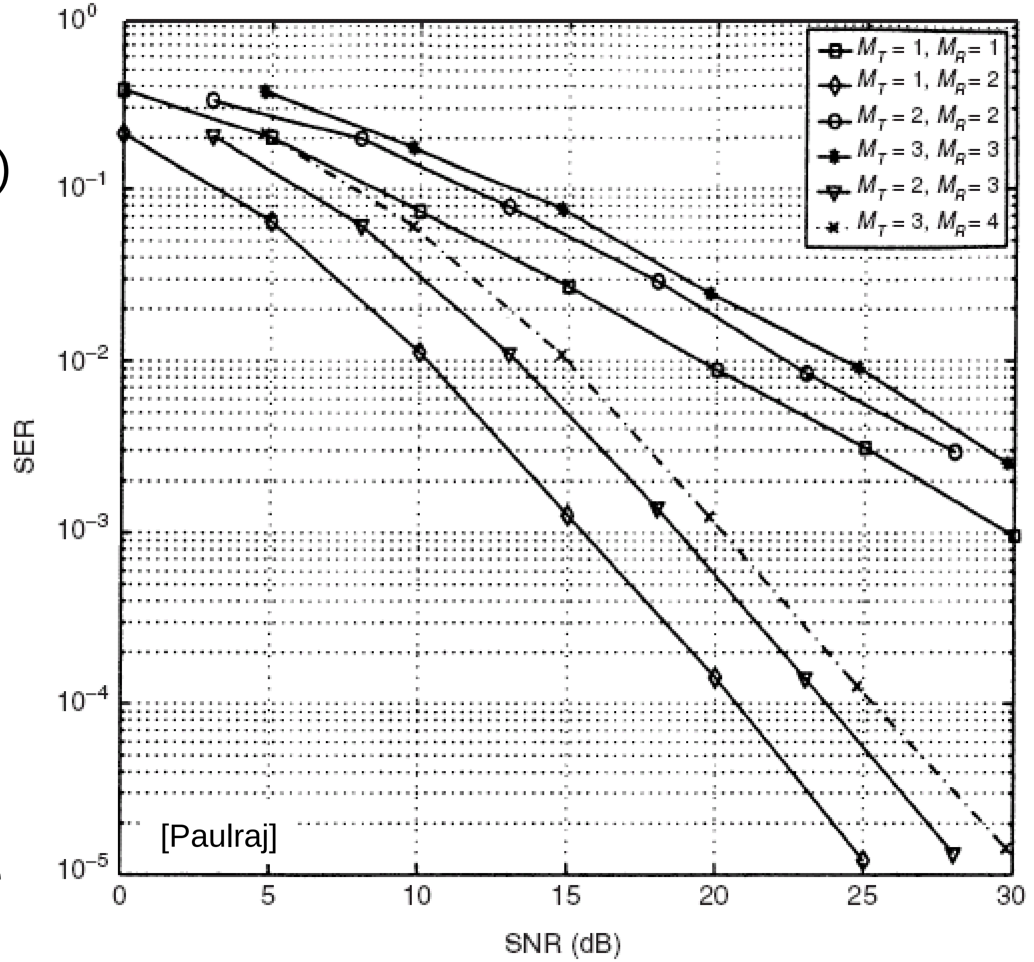
Observations: Increasing the number of receive antennas

- Improves diversity
- Provides a coding gain (SNR shift)

Diversity of linear detectors

$$N_R - N_T + 1$$

- The detailed proof is complicated



Intuition:

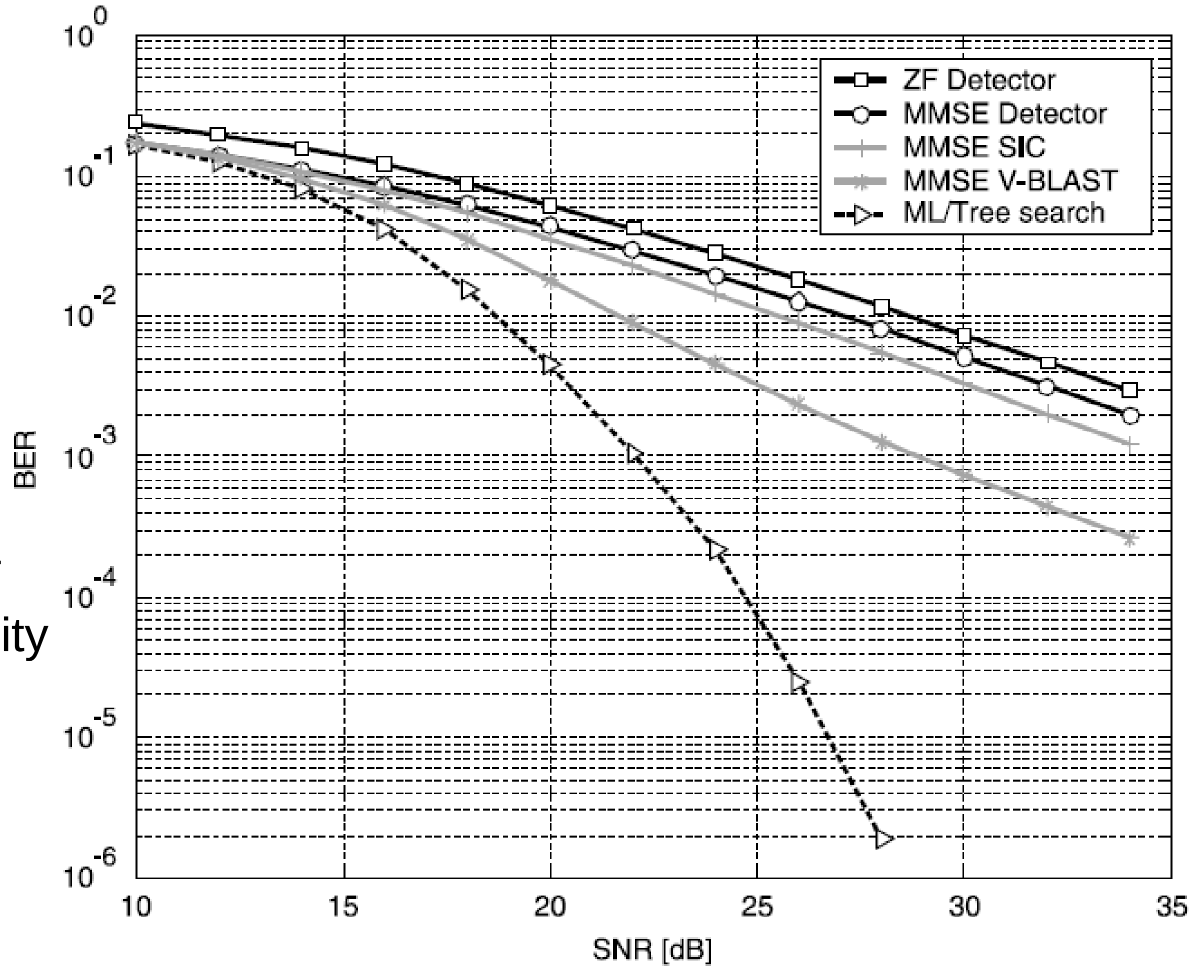
- N_R dimensional received vector $\mathbf{y}$
- Interference space: $N_T - 1$ dim.
  $\left[\mathbf{h}_1^{(i)} \cdots \mathbf{h}_{k^{(i)}-1}^{(i)} \mathbf{h}_{k^{(i)}+1}^{(i)} \cdots \mathbf{h}_{N_T}^{(i)}\right]$
- Project $\mathbf{y}$ into the null-space of the interference: $N_R + (N_T - 1)$ dim.
- Decision statistics has $N_R - N_T + 1$ degrees of freedom

Performance of SIC

- Symmetric 4x4 system

Observations:

- SIC yields an improvement over linear detection
- The real gain of SIC is only visible with ordering
- Unfortunetely, even V-BLAST achieves only 1st order diversity



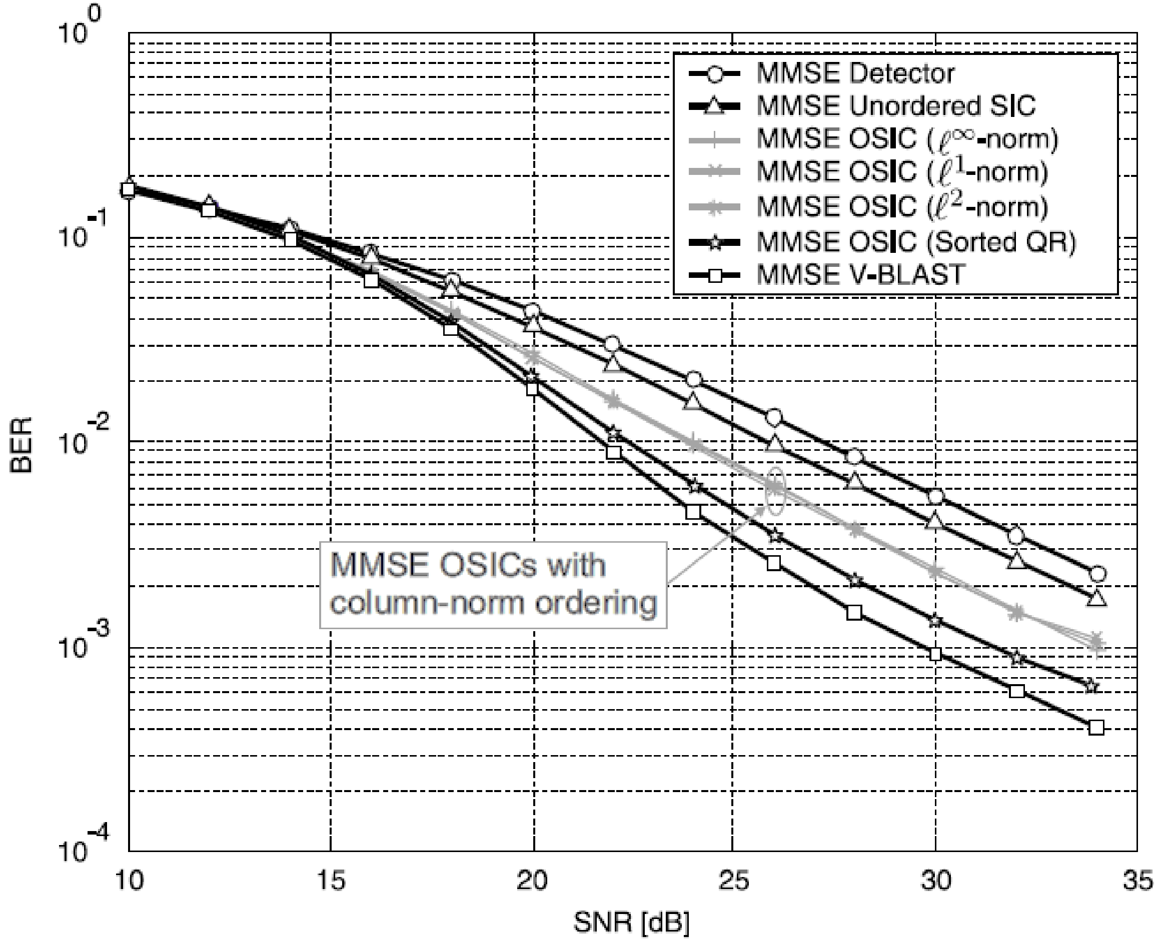
Performance of SIC with Sorted QR

Observation

- Sorted QR performs very close to V-BLAST, but is NOT the same

Note

- There is also a way to do optimum V-BLAST ordering with QR decomposition



Performance of SIC

The full performance analysis of SIC is difficult, though possible, but some intuitive arguments help to understand the behavior

- Given: N_R dimensional received vector $\mathbf{y}$

- Detecting the first stream requires projection onto the null-space of the remaining $N_R - 1$ interfering streams
- The corresponding projection is an $N_R - N_T + 1$ dimensional decision statistics

➢ **Diversity order of the first stream**: $N_R - N_T + 1$

- After cancellation of streams $1 - i$ the received vector $\mathbf{y}^{(i)}$ still has N_R dimensions
- But, (assuming no errors) only $N_T - i$ interferers are left ➔ null-space of the interference has only $N_T - i$ dimensions.

➢ **Diversity order of the ith stream**: $N_R - N_T + i$

➢ The subsequently detected streams have diversity order $N_R - N_T + i$

➢ Overall diversity is determined by the first stream. Hence still $N_R - N_T + 1$, BUT there is some performance improvement with appropriate ordering

Architecture for MIMO Detection

MIMO detection algorithms contain two types of operations

- Operations that depend only on the channel matrix (channel estimate)
- Operations that depend on the received vector

- Normally the channel (channel estimate) remain constant for many received vectors (keep training overhead low)

Receiver structure

- Preprocessing: pre-compute and store results that depend only on the channel
- Detection: processing for each individual received vector