Algorithms for Wireless Communications

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Lecture: Non-coherent modulation and Diversity

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Reference:
David Tse, Pramod Viswanath,
„Fundamentals of Wireless Communication“,
Cambridge Press 2005
http://www.eecs.berkeley.edu/~dtse/book.html

- **Detection in a Rayleigh fading channel (Chapter 3.1)**

Non Coherent Detection (chapter 3.1)

$$y(m) = h(m)x(m) + w(m)$$

$$h(m) \sim \mathcal{CN}(0,1)$$

$$w(m) \sim \mathcal{CN}(0,N_0)$$

- **Orthogonal sequence**

$$\mathbf{x}_A = \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \;;\mathbf{x}_B = \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$$

  - Note: x(0) = ±a does not work. The phase of y(m) is uniformly distributed [0, 2π] regardless whether ±a is transmitted. The received amplitude is independent of the transmitted symbol

- **ML-Detection**

Note for x_A: $y[0] \sim CN(0, a^2 + N_0)$ and $y[1] \sim CN(0, N_0)$

$$\Lambda(\mathbf{y}) = \ln\left\{\frac{f(\mathbf{y}|x_A)}{f(\mathbf{y}|x_B)}\right\} \underset{x_B}{\overset{x_A}{\gtrless}} 0$$

It can be solved

$$\Lambda(\mathbf{y}) = \ln\left\{\frac{(|y(0)|^2 - |y(1)|^2)a^2}{(a^2+N_0)N_0}\right\}$$

Non Coherent Detection

- **Error probability**

$$p_e = Pr\left[|y(1)|^2 > |y(0)|^2 \middle| x_A \right] = [2 + \tfrac{a^2}{N_0}]^{-1}$$

- **SNR definition [Tse]**

$$SNR = \frac{\text{average received signal energy per complex symbol time}}{\text{noise energy per (complex) symbol time}}$$

- **For the orthogonal modulation scheme we have**

$$\frac{a^2}{2} : \text{average received signal energy}$$

$$N_0 : \text{Noise energy}$$

$$SNR = \frac{a^2}{N_0}$$

and

$$\boxed{p_e = \frac{1}{2(1 + SNR)}}$$

- **Very discouraging result!**

To get $p_e = 10^{-3}$ requires SNR≈500 (27dB)

- **BER for transmission over AWGN**

$$p_e = Q\left(\frac{a}{\sqrt{N_0/2}}\right) = Q(\sqrt{2SNR}) \quad \text{with} \quad SNR = \frac{a^2}{N_0}$$

- **Approximation of**

$$Q(x) > \frac{1}{\sqrt{2\pi x}}(1 - \frac{1}{x^2})e^{-x^2/2}, \; x > 1$$

We see that p_e decreases exponentially in SNR while it decreases with only 1/SNR in the fading channel

- **If the channel h(m) is known, we achieve**

$$p_e = Q(\sqrt{2|h|^2 SNR})$$

Bit Error Rate in Rayleigh Fading Channels

- **For a given channel, the BER can be computed easily (QPSK)**
- $P(e|h) = Q\left(\sqrt{2|h|^2 SNR}\right)$
- **We are interested in the average bit error rate over many channel realizations**

$$\left.\begin{array}{l} \mathcal{E}_h\{P(e|h)\} = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right) \\ \sqrt{\frac{SNR}{1+SNR}} = 1 - \frac{1}{2SNR} + O\left(\frac{1}{4SNR^2}\right) \end{array}\right\} P(e) \sim \frac{1}{SNR}$$

  - Error probability decays only very slowly with increasing SNR

- **What is the reason for this poor performance?**
  - $|h|^2$SNR is the instantaneous SNR. Under typical channel conditions $|h|^2$SNR>>1 and the probability of error is very small.
  - If $|h|^2$SNR~1, then the error probability becomes significant. This event is called a *deep fade*
  - We are interested in

$$P(|h|^2 SNR < 1)$$

- **The pdf of $|h|^2$ decides how likely it is to end up in a deep fade**
- **If h is circular symmetric complex Gaussian random variable, then**

$$P(|h|^2 SNR < 1) = \int_0^{1/SNR} e^{-x} dx = \frac{1}{SNR} + O\left(\frac{1}{SNR^2}\right)$$

Performance of BPSK over Rayleigh Fading Channel

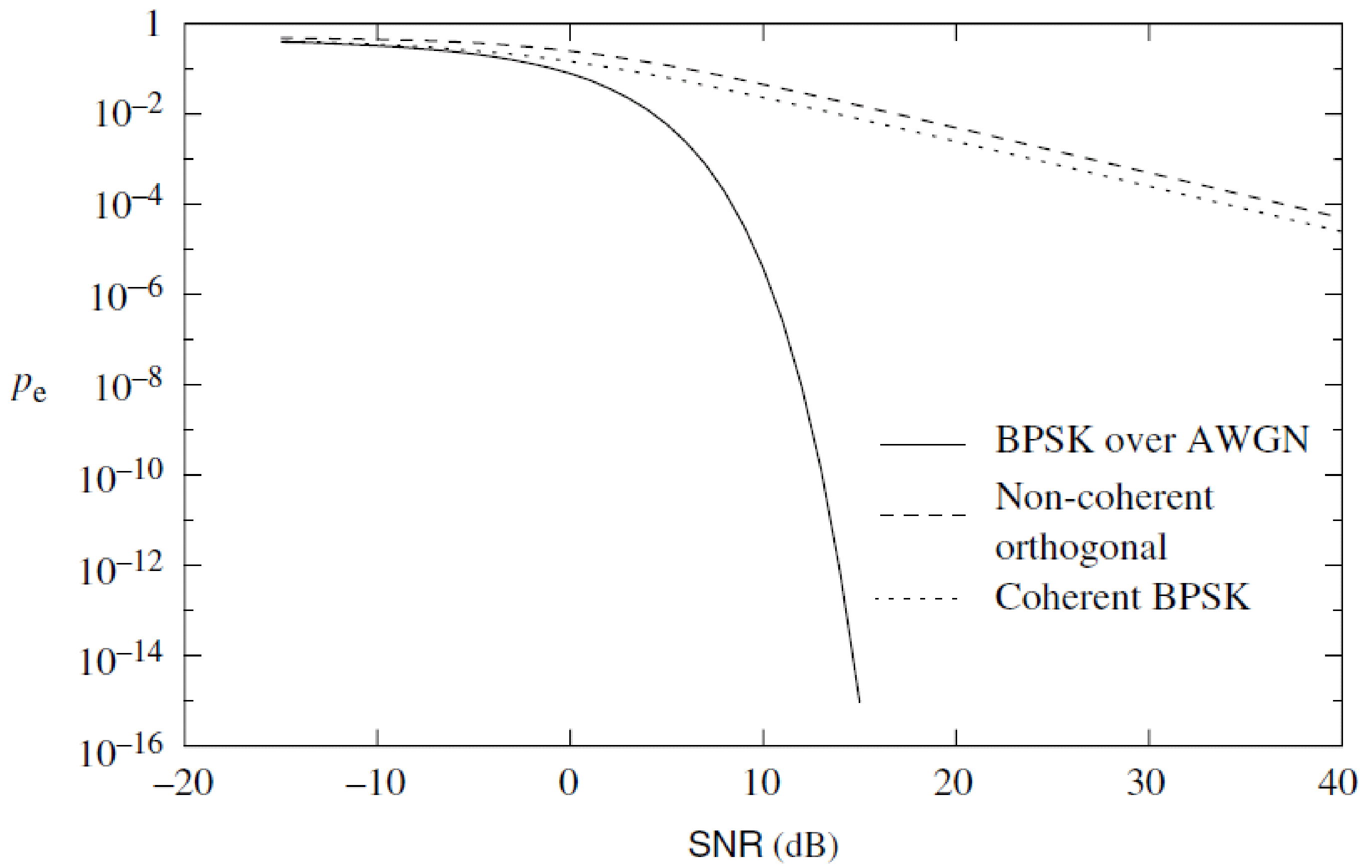


Figure 3.2 Performance of coherent BPSK vs. non-coherent orthogonal signaling over Rayleigh fading channel vs. BPSK over AWGN schannel.

- **Diversity (Chapter 3.1.4)**

The concept of diversity is to stabilize the link by providing multiple signal paths from transmitter to the receiver, so that when one path fails, other paths are still likely to provide a “good connection”

Idea:

Transmit a codeword $\mathbf{x} = [x_1, \ldots x_L]^T$ over L (nearly) independent fading gains by interleaving

$$y_l = h_l x_l + w_l, \qquad l = 1, \ldots, L$$

L : number of *diversity branches*

Simplest code: Repetition code

$$x_l = x_1, \quad \text{for } l = 1, \ldots, L$$

$$\mathbf{y} = \mathbf{h} x_1 + \mathbf{w}$$

where

$$\mathbf{y} = [y_1, \ldots, y_L]^T, \mathbf{h} = [h_1, \ldots, h_L]^T, \mathbf{w} = [w_1, \ldots, w_L]^T$$

This is a vector Gaussian detection problem.

We assume perfect knowledge of $\mathbf{h}$. $\Rightarrow$ *coherent detection*

Time diversity

The scalar

$$\frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{y} = \|\mathbf{h}\|x_1 + \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{w}$$

with

$$\frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{w} \sim \mathcal{CN}(0, N_0)$$

is sufficient statistics to decide for x_1 .

Receiver structure is a *matched filter* also called a *maximum ratio combiner*.

$$\|\mathbf{h}\|^2 = \sum_{l=1}^{L} |h_l|^2$$

weighs received signal in each branch and aligns phase

We get an equivalent channel gain $\|\mathbf{h}\|^2$.
The fluctuation of $\|\mathbf{h}\|^2$ becomes smaller because we average over *L*- independent value $|h_l|^2$.

Given $\|\mathbf{h}\|^2$, the error probability for a given channel equals (AWGN)

$$P_e(\|\mathbf{h}\|^2) = Q\left(\sqrt{2\|\mathbf{h}\|^2 SNR}\right)$$

Probability for a deep fade

- **We need the distribution of $\|\mathbf{h}\|^2$, since we are interested in the average BER over all possible channel realizations obtain p_e**

$$\chi^2_{2L}(\|\mathbf{h}\|^2) = \frac{1}{(L-1)!}\|\mathbf{h}\|^{2^{L-1}} e^{-\|\mathbf{h}\|^2}$$

$$P\left(\|\mathbf{h}\|^2 < \epsilon\right) \approx \frac{1}{L!}\epsilon^L$$

$$p_e \approx P\left(\|\mathbf{h}\|^2 < \frac{1}{\mathsf{SNR}}\right) \approx \frac{1}{L!}\frac{1}{\mathsf{SNR}^L}$$

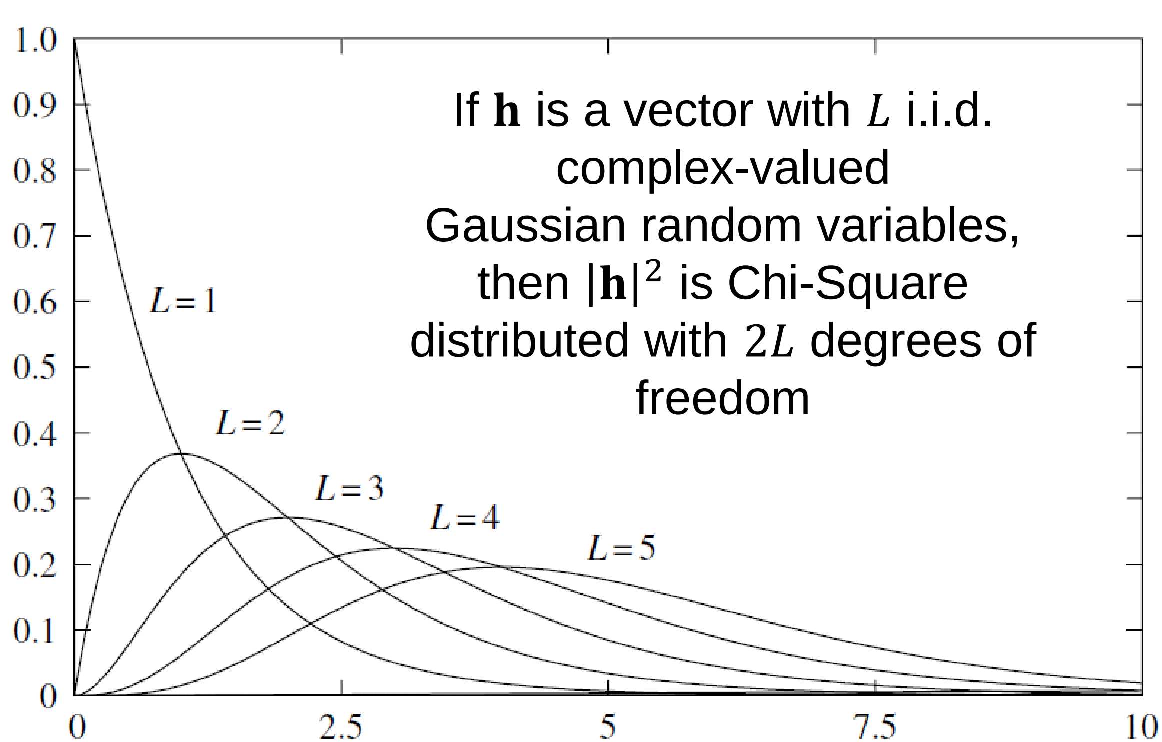


Figure 3.7 The probability density function of $\|\mathbf{h}\|^2$ for different values of L. The larger the L, the faster the probability density function drops off around 0.

The factor of 2 in the degrees of freedom of the Chi-Square distribution comes from the complex channel coefficients, which all comprise a real- and an imaginary part that are both independent (circular symmetric) Gaussians

Error Probability for BPSK

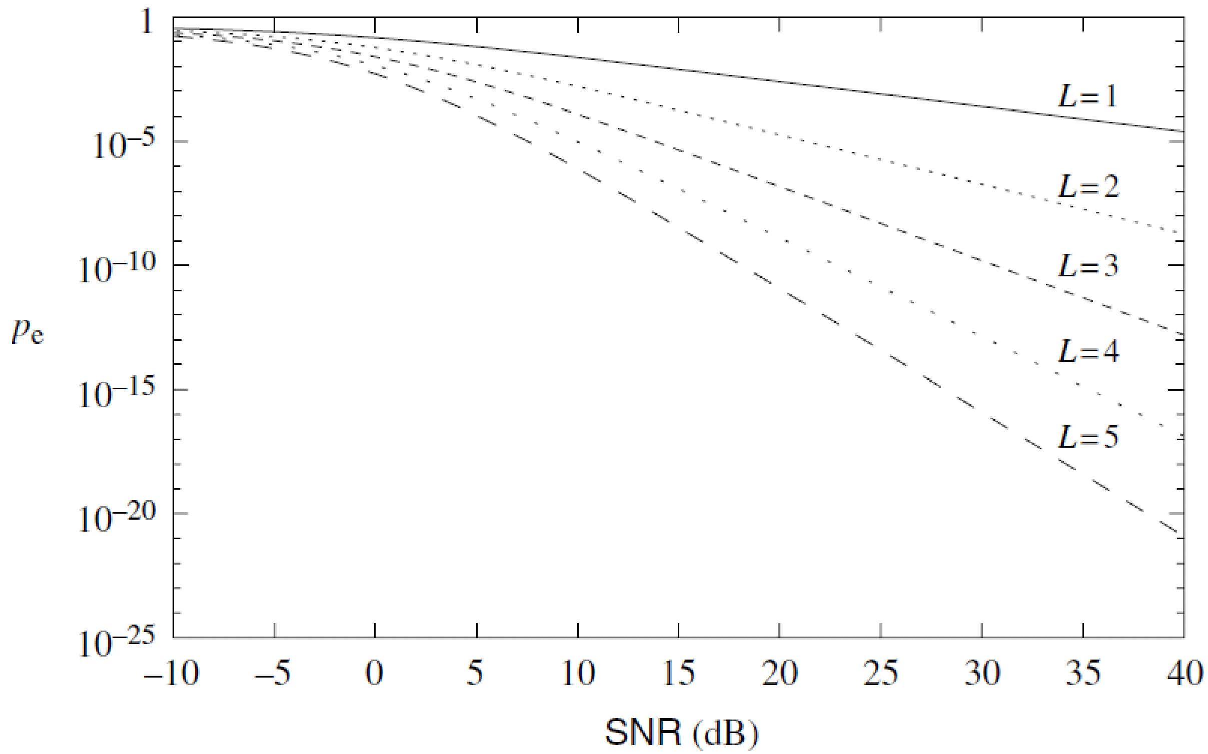


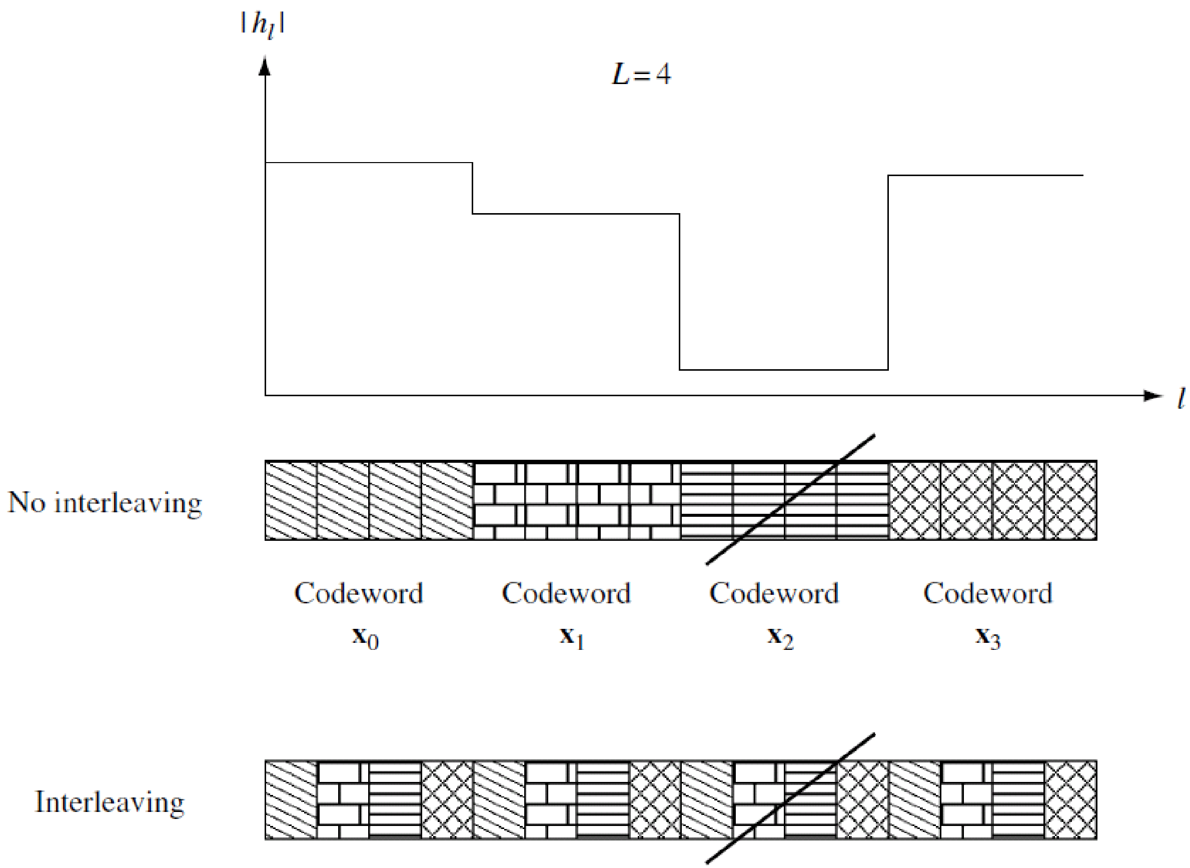
Figure 3.6 Error probability as a function of SNR for different numbers of diversity branches L.

P_e **decreases rapidly with increasing** L

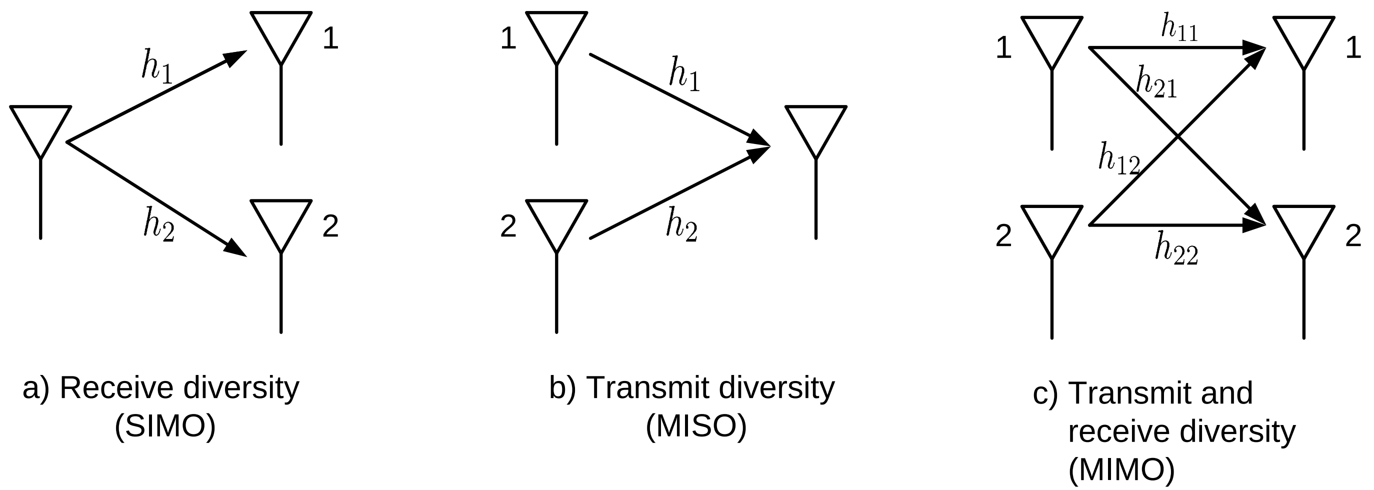
L: is called the *diversity gain*

Interleaving

Figure 3.5 The codewords are transmitted over consecutive symbols (top) and interleaved (bottom). A deep fade will wipe out the entire codeword in the former case but only one coded symbol from each codeword in the latter. In the latter case, each codeword can still be recovered from the other three unfaded symbols.



We consider transmission using several antennas at the transmitter or/and at the receiver.



If the antennas are placed sufficiently apart, the channel gains between different antenna pairs fade more or less independently

and create independent signal paths.

SIMO: Receive Diversity

- **Channel Model**

$$y_l(m) = h_l(m)x(m) + w_l(m), \quad l = 1, \ldots, L$$

Exactly the same model as for time diversity. The L-diversity branches are now over space instead over time

- **The error probability for BPSK equals**

$$Q\left(\sqrt{2||\mathbf{h}||^2 SNR}\right)$$

We can break up the total SNR conditioned on $||\boldsymbol{h}||^2$

$$||\mathbf{h}||^2 \cdot SNR = L \cdot SNR \cdot \frac{||\mathbf{h}||^2}{L}$$

$L \cdot SNR$: power gain (array gain)

$\frac{||\mathbf{h}||^2}{L}$: diversity gain $\rightarrow$ affects the exponent of $\left(\frac{1}{SNR}\right)^L$

for large $L \rightarrow$ the diversity gain tends to 1

- **Important case for small mobile devices when only one antenna can be placed**
- **We achieve the same diversity gain as for receive diversity using a simple repetition code**
  - At any time only *one* antenna transmits the symbol
  - For L antennas we have L independent signal paths
- **Disadvantage: we make poor use of the channel bandwidth since we transmit only one symbol over l-time slots**

- **Consider a channel with L transmit antennas and *one* receive antenna (<u>h</u>, <u>x</u> are Lx1-vectors) and fixed channel vector $\underline{h}^H$:**

$$y(k) = \underline{h}^H \underline{x}(k) + w(k)$$

- **There is a transmit power constraint: $\|\underline{x}\|^2 = \underline{x}^H \underline{x} = P'$.**
- **We get the maximum received signal power when $\underline{h}^H \underline{x}$ has maximum magnitude. Since this term is the projection of <u>x</u> onto <u>h</u>, which is maximum if <u>h</u> and <u>x</u> are parallel, we get the maximum received signal power for**

$$\underline{x}(k) = \frac{\underline{h}}{\|\underline{h}\|} \tilde{x}(k) \quad \text{with} \quad \|\underline{x}(k)\|^2 = |\tilde{x}(k)|^2 = P'$$

- **The resulting received signal is**

$$y(k) = \frac{\underline{h}^H \underline{h}}{\|\underline{h}\|} \tilde{x}(k) + w(k) = \|\underline{h}\| \tilde{x}(k) + w(k)$$

- **The previous two scenarios represent receive and transmit diversity.**
- **In both cases we get an equivalent channel gain**

$$h = \left\|\underline{h}\right\| = \sqrt{\sum_{\ell=0}^{L-1} \left|h_{\ell}\right|^{2}}$$

corresponding to a SNR gain $||\underline{h}||^2$

- **The probability of deep fades has decreased now as it would require all subchannels to be in deep fades at the same time.**

- **An issue with the MISO scenario is that it requires knowledge of the channel in the transmitter prior to transmission. It would be desirable to achieve the gain without this knowledge. Space-time coding targets this goal. Here we discuss the simplest and yet most elegant space-time code: the Alamouti scheme. It is designed for two transmit antennas: generalization to more than two antennas is possible, to some extent.**

Alamouti Space-Time Code (3.3, p. 73-74)

- **Assume two transmit antennas**

$$y(k) = \begin{bmatrix} h_1(k) & h_2(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + w(k)$$

- **The Alamouti scheme transmits two complex symbols u_1 and u_2 over two symbol periods:**
  - at time k: $x_1(k)=u_{1,k}$ and $x_2(k)=u_{2,k}$
  - at time k+1: $x_1(k+1)=-u^*_{2,k}$ and $x_2(k+1)=u^*_{1,k}$

- **Writing the scheme in matrix form**

$$\begin{bmatrix} y(k) & y(k+1) \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} u_{1,k} & -u^*_{2,k} \\ u_{2,k} & u^*_{1,k} \end{bmatrix} + \begin{bmatrix} w(k) & w(k+1) \end{bmatrix}$$

- **Since we are interested in $u_{1,k}$ and $u_{2,k}$ we rewrite this equation**

$$\begin{bmatrix} y(k) \\ y^*(k+1) \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h^*_2 & -h^*_1 \end{bmatrix} \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} + \begin{bmatrix} w(k) \\ w*(k+1) \end{bmatrix}$$

- **The two subsequent receive signals now are**

$$\begin{bmatrix} y(k) \\ y^*(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}}_{H} \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} + \begin{bmatrix} w(k) \\ w^*(k+1) \end{bmatrix}$$

- **Multiplying from the left with H^H yields**

$$\begin{bmatrix} \tilde{y}(k) \\ \tilde{y}(k+1) \end{bmatrix} = H^H \begin{bmatrix} y(k) \\ y^*(k+1) \end{bmatrix} = \begin{bmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{bmatrix} \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} + H^H \begin{bmatrix} w(k) \\ w^*(k+1) \end{bmatrix}$$

$$\begin{bmatrix} \tilde{y}(k) \\ \tilde{y}(k+1) \end{bmatrix} = \begin{bmatrix} \|h\|^2 & 0 \\ 0 & \|h\|^2 \end{bmatrix} \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} + \begin{bmatrix} v(k) \\ v(k+1) \end{bmatrix}$$

- **Thus, we have finally separated the transmitted symbols u_1, u_2**

$$\begin{bmatrix} \tilde{y}(k) \\ \tilde{y}(k+1) \end{bmatrix} = \|h\|^2 \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} + \begin{bmatrix} v(k) \\ v(k+1) \end{bmatrix}$$

Alamouti Space-Time Code (continued)

- **Since we transmit u_1 and u_2 (or their complex conjugates, respectively) at time k and at time k+1, the transmit power in each time instant is $P' = |u_1|^2 + |u_2|^2$. Thus, $|u_i|^2 = P'/2$, i.e. we have a loss of 3 dB compared to the optimum transmit diversity as shown before in the SIMO and MISO scenarios.**

Frequency Diversity

- **With OFDM we get parallel flat fading channels. Exceeding the coherence bandwidth of the channel, we get approximately independent channels. Thus, we could apply the concepts for time diversity also in the frequency direction, i.e. use frequency diversity.**