Algorithms for Wireless Communications II

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OFDM Channel Estimation

Notation Conventions

Type	Notation	Example
Scalars	Lower case, italic	a, n, b, x, y
Column vectors	Lower case, bold	$\mathbf{t}, \boldsymbol{\mathscr{t}}$
Matrices	Upper case, bold	$\mathbf{T}, \boldsymbol{\mathcal{T}}$
Time domain signals	Regular font	$\mathbf{H}, H, \mathbf{h}, h$
Frequency domain signals	Script font	$\boldsymbol{\mathscr{h}}, \mathscr{h}, \boldsymbol{\mathcal{H}}, \mathcal{H}, \boldsymbol{\mathcal{T}}, \mathcal{T}, \boldsymbol{\mathscr{t}}, \mathscr{t}$

We distinguish between

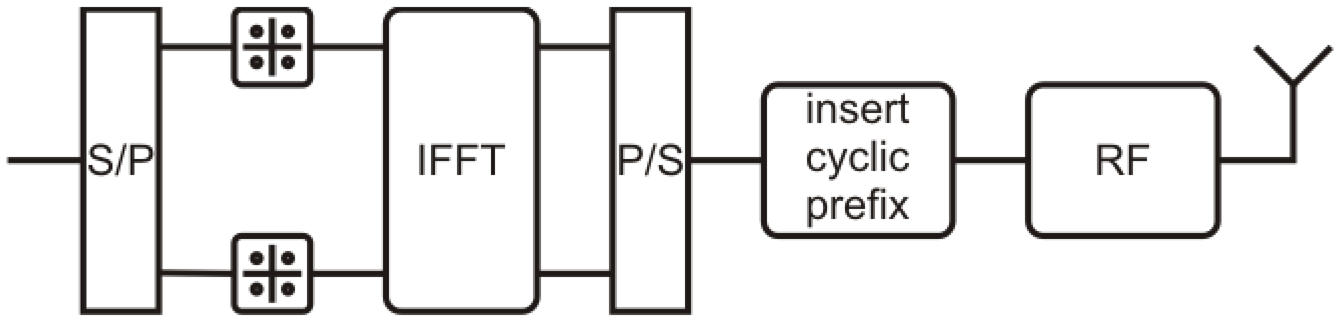
- Scalars and matrices or vectors by using regular and bold upper/lower case letters
- Time and frequency domain representation using regular and script font

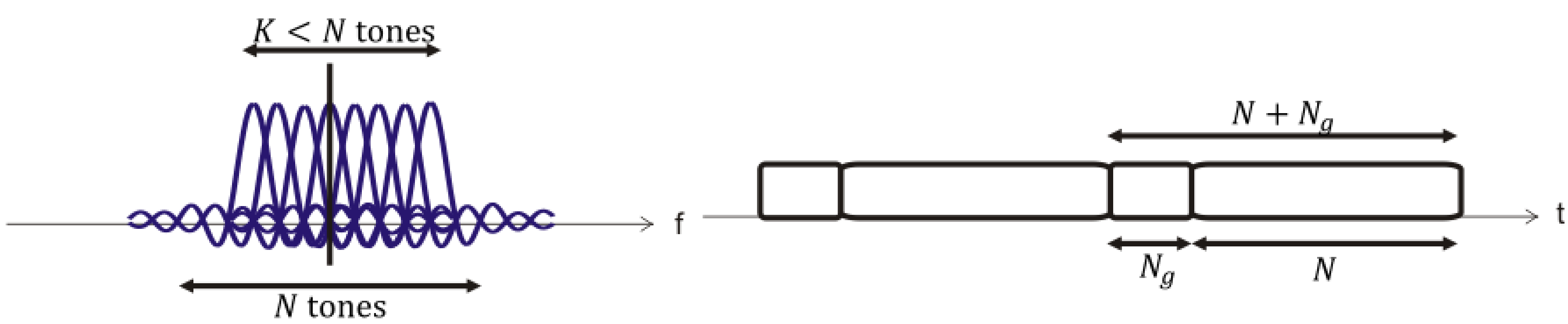
We carefully distinguish between a variable $\mathscr{h}$ and its estimate $\hat{\mathscr{h}}$ (denoted by adding a *hat*)

OFDM Reminder (1/2)

Wideband transmission using many (orthogonal) narrow-band channels

- Transmitted signal is divided into OFDM symbols, each comprising N tones
- Each tone is modulated separately with any modulation scheme
- IDFT or better IFFT generates time-domain signal
- A cyclic prefix is prepended to each OFDM symbol





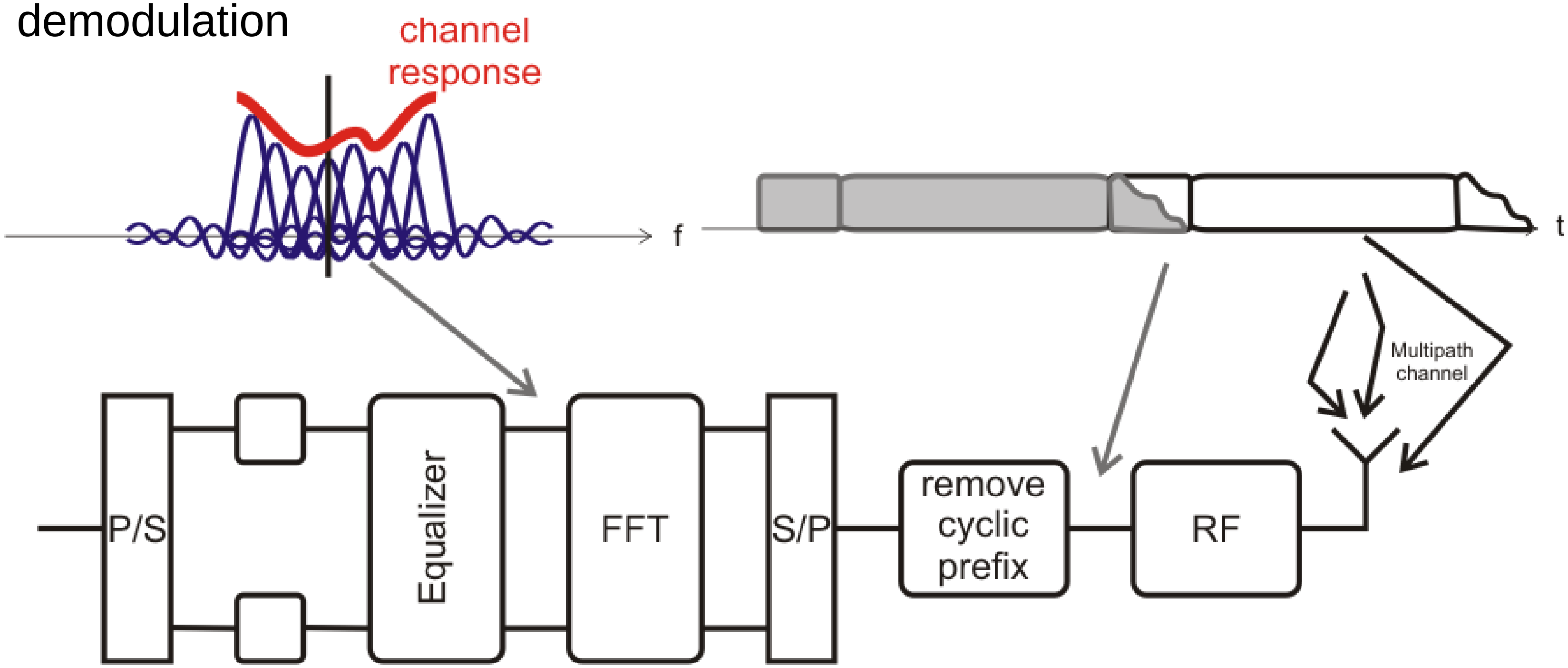
OFDM Reminder (2/2)

Transmission over multipath (frequency selective) channel

- Each tone experiences a different phase and attenuation
- Cyclic prefix captures ISI from previous symbol

Receiver

- Removes the cyclic prefix and transforms back into frequency domain (DFT or FFT)
- Per-tone equalization to undo the impact of the channel on each tone before demodulation



OFDM System Model

Recall the OFDM system model

$$\boldsymbol{y} = \mathbf{F}\left(\mathbf{H}\left(\mathbf{F}^{-1}\boldsymbol{s}\right) + \mathbf{n}\right) = \underbrace{\mathbf{F}\mathbf{H}\mathbf{F}^{-1}}_{diag(\mathbf{Fh})=\boldsymbol{h}} \boldsymbol{s} + \mathbf{F}\mathbf{n}$$

$\mathbf{F}^{-1}\boldsymbol{s}$: Frequency domain modulated data signal after IFFT

$\mathbf{H}$: Circulant matrix with time-domain channel coefficients

$\mathbf{F}$: FFT matrix

$\mathbf{n}$: i.i.d. Gaussian noise samples $\mathbb{E}\{\mathbf{n}\mathbf{n}^H\} = \mathbf{I}\sigma_n^2$

Frequency domain representation

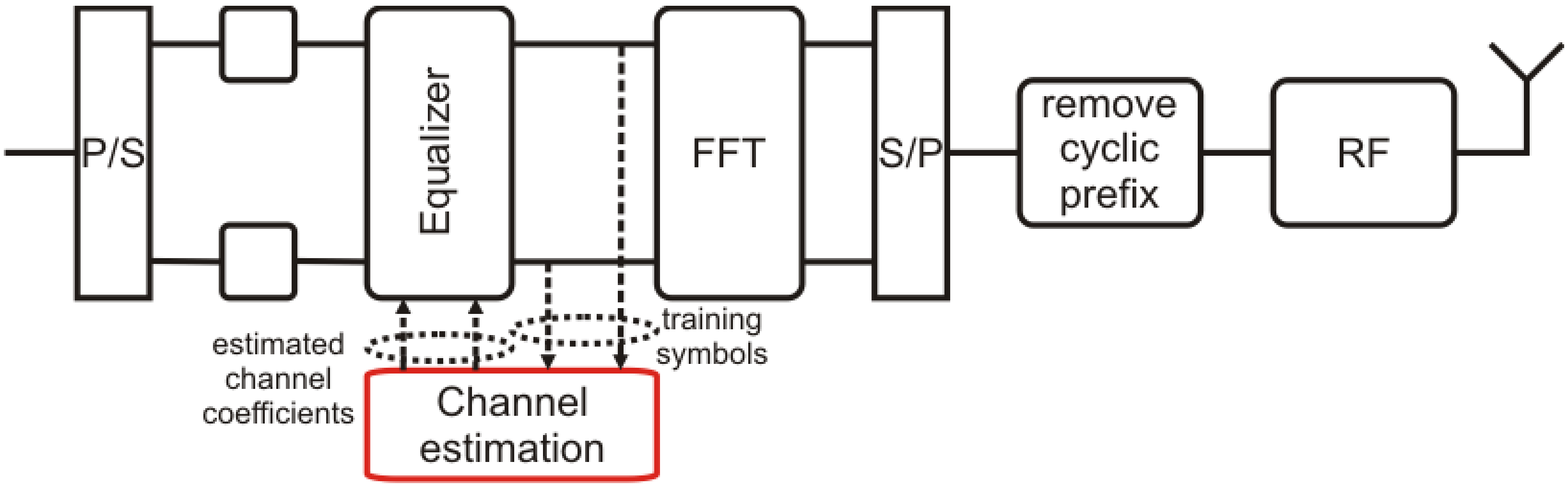
$$\boldsymbol{y} = \boldsymbol{\mathcal{H}}\boldsymbol{s} + \boldsymbol{n} \quad \text{with} \quad \mathcal{H} = \begin{bmatrix} h_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & h_{N-1} \end{bmatrix} = diag(\mathbf{Fh})$$

$$y_f = h_f s_f + n_f$$

- Note: noise properties remain unchanged $\mathbb{E}\{\boldsymbol{n}\boldsymbol{n}^H\} = \sigma_n^2\mathbf{I}$ since $\mathbf{F}$ is an orthonormal transform

Channel Estimation

Task: obtain knowledge (estimate) of the channel coefficients in the frequency domain h_f based on the received signal y_f



Channel Estimation

Estimation strategies

- Training/Pilot based: Transmitter sends known data (training/pilot symbols) so that h_f and n_f are the only unknown parameters
  - Overhead: training/pilot symbols do not carry data!!

- Data aided: channel estimation based on feedback of detected data

- Blind: Estimation only based on statistical properties

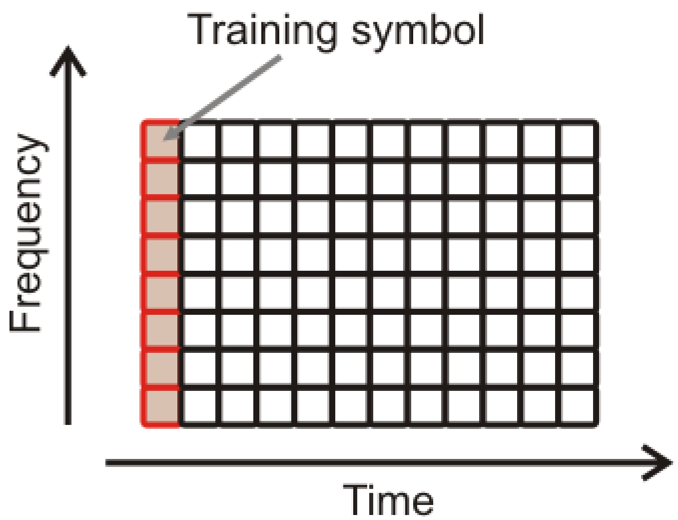
Here, we consider only training/pilot based channel estimation

- Most frequently used approach (very robust)
- Overhead is usually low when channels do not change to quickly

Common Training/Pilot Patterns

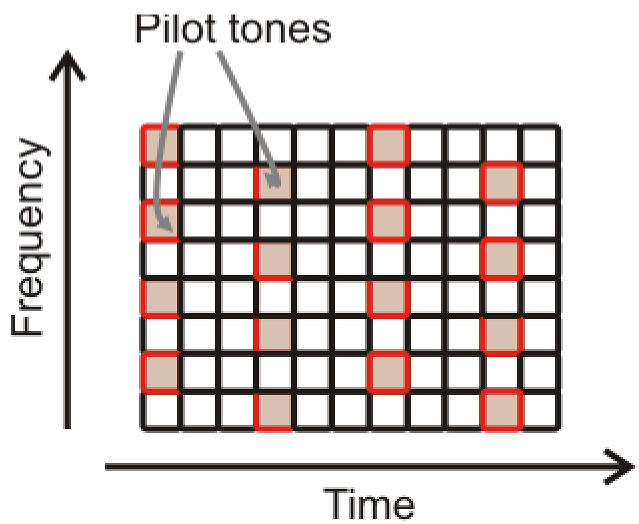
OFDM-training structures differ by how training/pilot symbols are distributed across time and frequency

Block type training (e.g., 802.11a/g/n)



- Dedicated OFDM training symbol
- Used in systems with relatively static channel conditions
- Packet based systems (burst traffic)

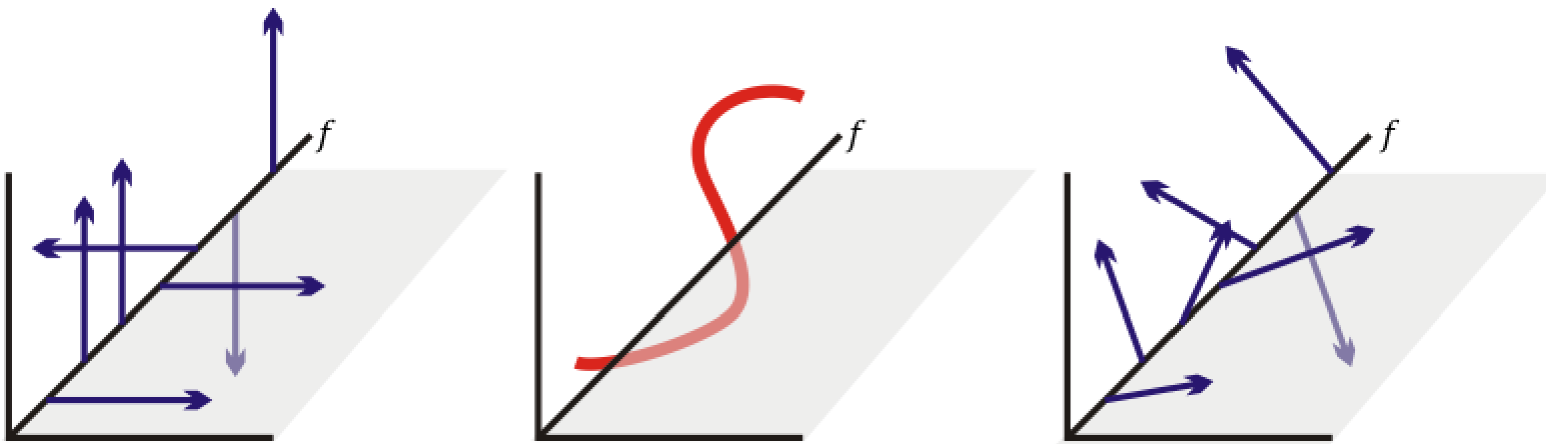
Comb type training (e.g., DVB-T, 3GPP LTE)



- Pilot tones interleaved with data
- Pilot location varies to cover all tones
- Used in rapidly time-varying channels
- Streaming (continuous transmission)

Input Output Relationship for Channel Estimation

Consider the transmission of a single training symbol t



Training symbol t_f

Channel frequency response h_f

Received symbol $y_f = h_f t_f + n_f$

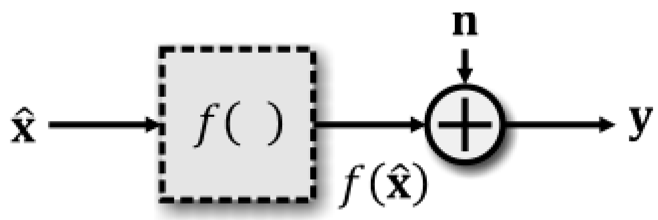
- Two options to formulate the input-output relationship

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & h_{N-1} \end{bmatrix} \begin{bmatrix} t_0 \\ \vdots \\ t_{N-1} \end{bmatrix} + \begin{bmatrix} n_0 \\ \vdots \\ n_{N-1} \end{bmatrix} = \boxed{\begin{bmatrix} t_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_{N-1} \end{bmatrix} \begin{bmatrix} h_0 \\ \vdots \\ h_{N-1} \end{bmatrix} + \begin{bmatrix} n_0 \\ \vdots \\ n_{N-1} \end{bmatrix}}$$

For our purposes (estimate t) the 2nd expression is more appropriate

Estimation Theory Basics

Problem statement: consider the following simplified system model



- Observation $\boldsymbol{y}$ is some function of an unknown parameter $\hat{\mathrm{x}}$ distorted by noise
  - Often, $f(\mathbf{x})$ is a linear function of $\mathbf{x}$
  - Often, the observation is distorted by additive Gaussian noise

- Objective: estimate $\mathbf{x}$ from $\mathbf{y}$ using some cost metric

- We may consider different types of prior knowledge on the variable to be estimated
  - A priori probabilities
  - Knowledge on 1st and 2nd order statistics
- OFDM channel estimation:
  - Observation: received training symbol/tones
  - $f(\quad)$: Impact of the channel on the (known) transmitted training symbol
  - Noise: thermal noise, additive Gaussian
  - A priori knowledge: **what can we say about the channel ??**

Estimation Theory Basics

Maximum likelihood estimation: Maximize likelihood of the observation under hypothesis $\hat{x}$

$$\hat{\mathbf{x}} = \underset{\hat{\mathbf{x}}}{\operatorname{argmax}}\ p(\mathbf{y}|\hat{\mathbf{x}})$$

Least squares estimation: Minimize distance between observation and noise free observation under hypothesis $\hat{x}$

$$\hat{\mathbf{x}} = \underset{\hat{\mathbf{x}}}{\operatorname{argmin}}\ |\mathbf{y} - f(\hat{\mathbf{x}})|^2$$

- Linear least squares approach: $\hat{\mathbf{x}} = \mathbf{W}_{LS}\mathbf{y}$
- No assumptions on 2nd order statistics
- For additive Gaussian noise: equivalent to ML estimation

Minimum mean squared error (MMSE) estimation

$$\hat{\mathbf{x}} = \underset{\hat{\mathbf{x}}}{\operatorname{argmin}}\, \mathbb{E}\{|\mathbf{x} - \hat{\mathbf{x}}|^2\}$$

- Linear least squares approach: $\hat{\mathbf{x}} = \mathbf{W}_{\mathrm{MMSE}}\mathbf{y}$

Per Tone (Linear) Least Squares Method

Linear least squares estimation: obtain the estimate $\hat{\boldsymbol{h}}$ of $\boldsymbol{h}$ according to

$$\hat{\boldsymbol{h}} = \underset{\hat{\boldsymbol{h}}}{\operatorname{argmin}} \left\{ \left| \boldsymbol{y} - \boldsymbol{\mathcal{T}} \hat{\boldsymbol{h}} \right|^2 \right\} \text{ with } \boldsymbol{\mathcal{T}} = \begin{bmatrix} t_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_{N-1} \end{bmatrix}$$

- Minimize Euclidean distance between $\boldsymbol{y}$ and $\boldsymbol{\mathcal{T}}\hat{\boldsymbol{h}}$
- Gaussian noise: least squares solution corresponds to the maximum likelihood solution

No assumptions are made on correlation of the individual tones:

- Treat all tones as if they had an independent channel coefficient
- Since $\boldsymbol{\mathcal{T}}$ is diagonal, tones can be estimated independently (locally optimal solutions leads to a global optimal solution)

$$\hat{\boldsymbol{h}} = \boldsymbol{\mathcal{T}}^{-\mathbf{1}} \boldsymbol{y} \text{ or } \hat{h}_f = \frac{1}{t_f} y_f$$

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Estimation Error for Per Tone Least Squares Estimation

- Define the per tone estimation error as

$$\Delta h_f = h_f - \hat{h}_f$$

We are interested in the mean squared error

- Mean: taken over many noise realizations (average error we expect over many independent channel estimations)

$$\sigma_{h_f}^{2} = \mathbb{E}\left\{\left|\Delta h_f\right|^{2}\right\} = \mathbb{E}\left\{\left|h_f - \frac{1}{t_f} y_f\right|^{2}\right\}$$

- Substitute the signal model $y_f = h_f t_f + n_f$

$$\sigma_{h_f}^{2} = \mathbb{E}\left\{\left|h_f - \frac{1}{t_f}\left(h_f t_f + n_f\right)\right|^{2}\right\} = \frac{\mathbb{E}\left\{\left|n_f\right|^{2}\right\}}{\left|t_f\right|^{2}} = \frac{\sigma_{n}^{2}}{\left|t_f\right|^{2}}$$

Per Tone Least Squares: remember from part 1 of the course

Channel Estimation

Quality of the channel estimate

$$\frac{r_k}{t} = h + w_k/t$$

Noise component on the channel est.

$$\frac{\mathcal{E}\{|h|^2\}}{\mathcal{E}\{|w_k/t|^2\}} = \frac{\mathcal{E}\{|h|^2\}}{\sigma_n^2}$$

Improving the channel estimate by averaging

- If the channel is static, we can use multiple training symbols and average

$$\hat{h} = \frac{1}{N}\sum_{k=1}^{N}\frac{r_k}{t} = \frac{1}{N}\sum_{k=1}^{N} h + \frac{1}{N}\sum_{k=1}^{N}\frac{w_k}{t}$$

N-fold averaging reduces noise on the channel estimate by a factor of N

If the channel is time varying, averaging is replaced by a low-pass filter

- Filter bandwidth determined by the Doppler frequency
- Remove the frequency components outside the Doppler bandwidth (noise!!)

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Design of the Training Symbol

Quality metric: average estimation MSE (across tones)

$$\sigma_{\boldsymbol{h}}^2 = \frac{\sigma_{\boldsymbol{n}}^2}{N} \sum \frac{1}{|t_f|^2}$$

- Average power constraint: $\sum |t_f|^2 = N P_t$ with P_t: transmit power during training

Distribution of power across tones has an impact on the average channel estimation error

Optimum power allocation

- Distribute power equally to all tones of the training symbol

$$|t_f|^2 \equiv P_t$$

Channel estimation MSE: $\sigma_{\boldsymbol{h}}^2 = \sigma_{h_f}^2 = \sigma_{\boldsymbol{n}}^2 / P_t$

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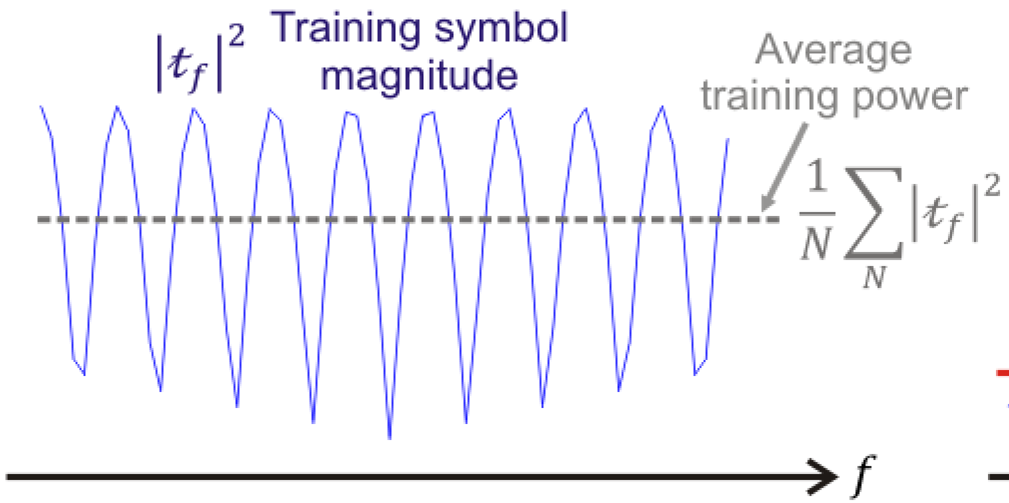
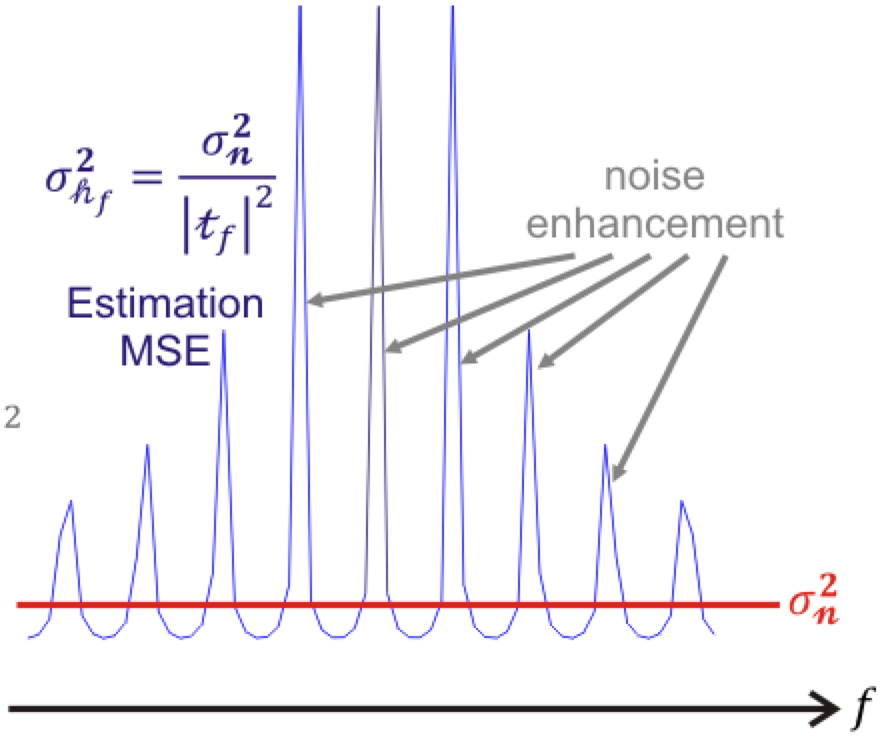
Implication for the Design of the Training Sequence

Example: training sequence with uneven power allocation across tones

- System constraints limit only the average (across tones) transmit power
- **Example**: training sequence designed in time domain (e.g., for good PAPR)



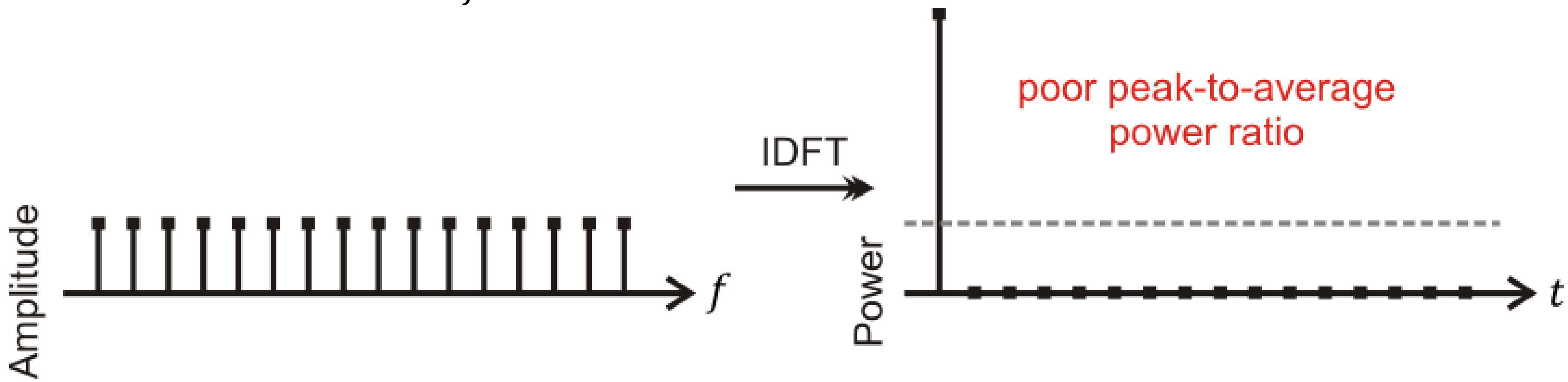


- MSE depends on the power allocated to each pilot tone
- Unequal power allocation -> unequal distribution of the MSE across tones

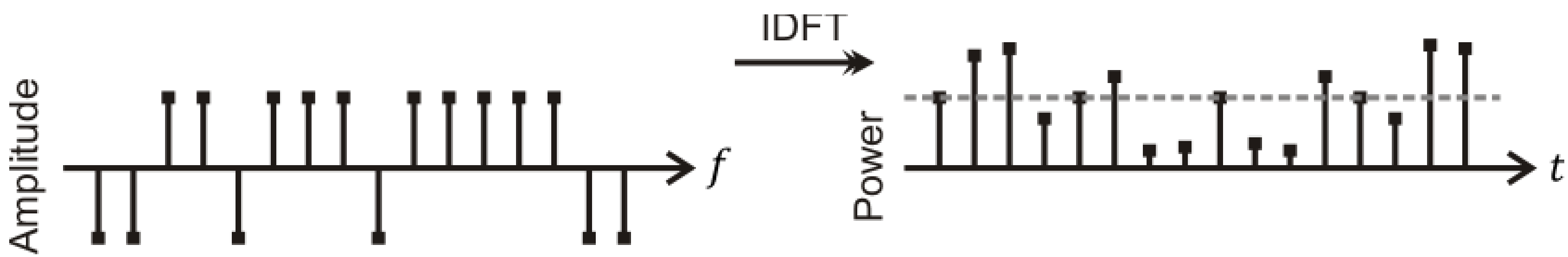
Distribute power equally across tones!

Design of the Training Symbol

Straightforward design: $t_f \equiv 1$



Solution: choose the phase of t_f to achieve an approximately constant envelope (PAPR$\approx$ 1)



- Real-world example: IEEE 802.11a long training symbol (tones -26 to +26)

t_f =++--++-+-++++++--++-+-++++0+--++-+-+-----++--+-+-++++

Impact of Channel Estimation Error on BER

- Channel estimation error $\Delta h_f = h_f - \hat{h}_f$
- Input-output relationship $y_f = h_f s_f + n_f$ (data symbols)

Express the real channel h_f as function of its estimate $\hat{h}_f$ and the error Δh_f

$$h_f = \hat{h}_f + \Delta h_f$$

$$y_f = \left(\hat{h}_f + \Delta h_f\right) s_f + n_f$$

Treat all unknown terms as noise

$$\breve{n}_f = \Delta h_f s_f + n_f$$

$$\sigma_{\breve{n}}^2 = \mathbb{E}\left\{\left|\breve{n}_f\right|^2\right\} = \underbrace{\mathbb{E}\left\{\left|\Delta h_f\right|^2\right\}}_{\sigma_n^2/P_t} \underbrace{\mathbb{E}\left\{\left|s_f\right|^2\right\}}_{P_s} + \underbrace{\mathbb{E}\left\{\left|n_f\right|^2\right\}}_{\sigma_n^2} = \left(\frac{P_s}{P_t} + 1\right) \sigma_n^2$$

Impact of Channel Estimation Error on BER

- Assume training and data have equal power (usually the case) : $P_s = P_t$

$$\sigma_{\breve{n}}^2 = 2\sigma_n^2$$

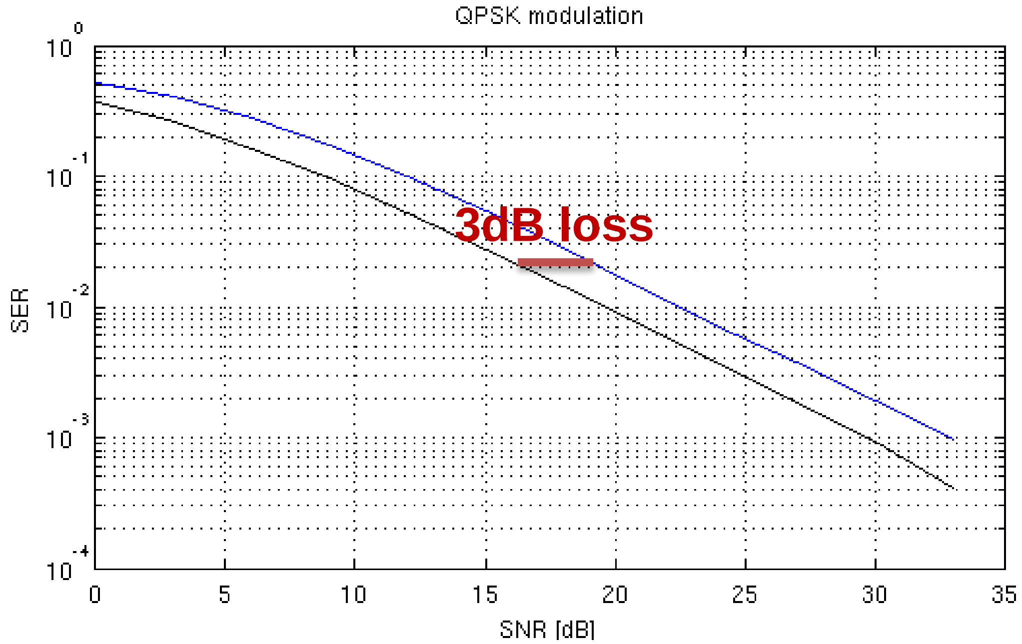
Channel estimation doubles the noise on the signal

Impact on bit error rate

- Approximation: ignore the fact that $\Delta h_f s_f$ is *data dependent* and *not necessarily, but often Gaussian*

Example

- *QPSK modulation*
- *With perfect channel state information (CSI): blue*
- *With estimated channel: black*



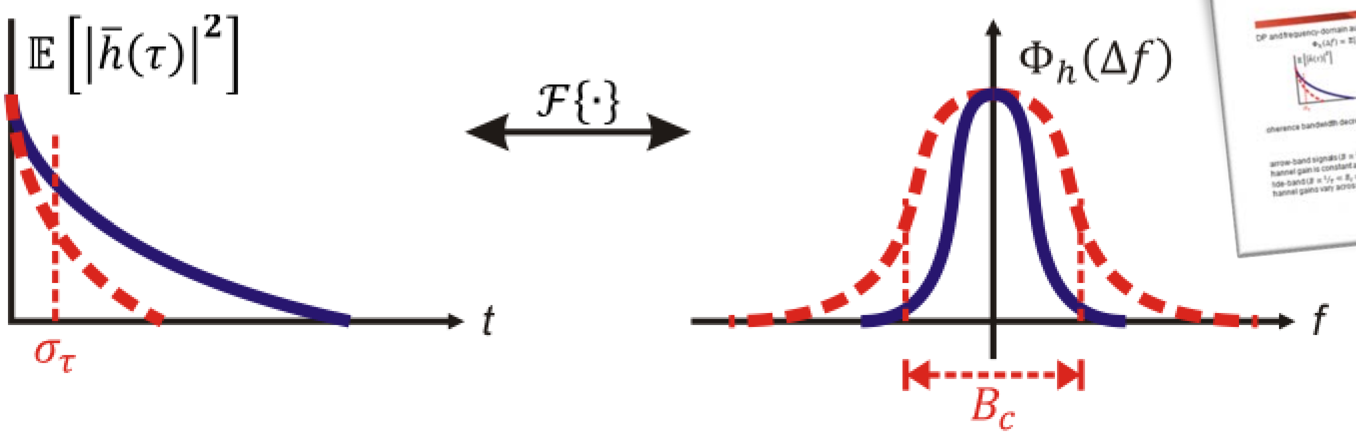
Suboptimality of Per Tone Channel Estimation

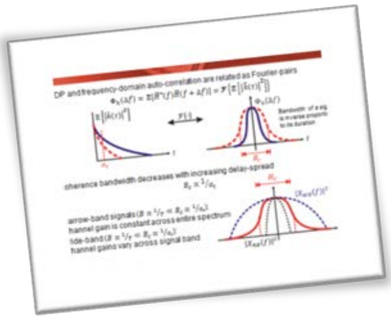
Assumes independence between channel coefficients across all tones

Remember:

- Correlation across tones measured by the **coherence bandwidth** and quantified by

$$\Phi_h(\Delta f) = \mathbb{E}[\bar{H}^*(f)\bar{H}(f+\Delta f)] = \boldsymbol{\mathcal{F}}\left\{\mathbb{E}\left[\left|\bar{h}(\tau)\right|^2\right]\right\}$$





- Coherence bandwidth is inverse proportional to the channel delay spread

- **OFDM** (by design): coherence bandwidth >> carrier spacing

Independence assumption is wrong (->pessimistic)

Least Squares Estimation (Freq. Domain View)

- Vector of channel coefficients spans an N-dimensional space

$$\boldsymbol{h} = [h_0 \quad h_1 \quad \cdots \quad h_{N-1}]^T$$

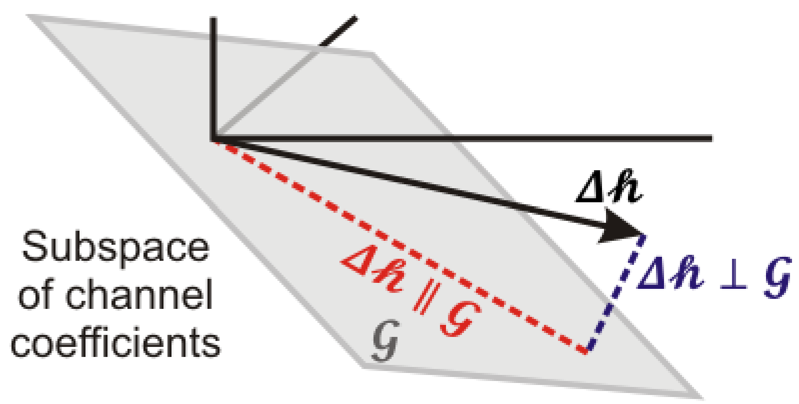
Per-tone channel estimation has two main signal components

$$\hat{\boldsymbol{h}} = \boldsymbol{h} + \boldsymbol{\Delta h}$$

- Uncorrelated (=independent) across tones noise $\boldsymbol{\Delta h}$
- *Correlated* channel across tones channel coefficients $\boldsymbol{h}$

The desired signal $\boldsymbol{h}$ lies in a lower dimensional subspace $\mathcal{G}$ (support)

- Two noise components
  - Orthogonal to the channel subspace $\boldsymbol{\Delta h} \perp \mathcal{G}$
  - Within the channel subspace $\boldsymbol{\Delta h} \parallel \mathcal{G}$
- Per tone estimation: no noise filtering
- But, if we know $\mathbf{G}$, we can remove $\boldsymbol{\Delta h} \perp \mathcal{G}$ completely => better estimate

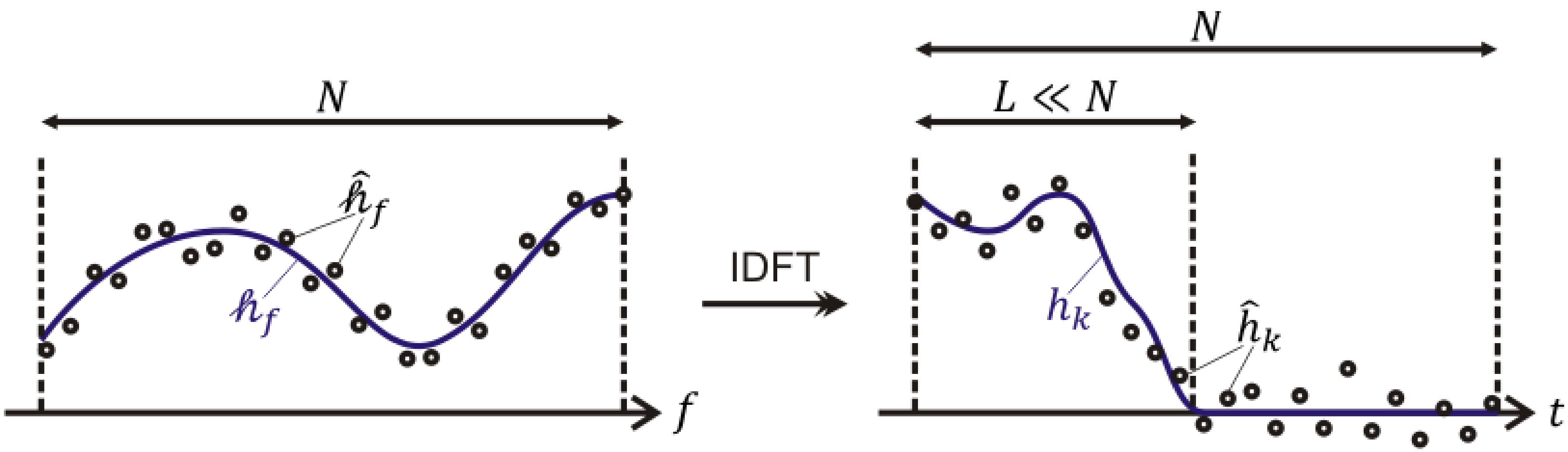


Least Squares Estimation (Time Domain View)

Frequency domain: subspace of the channel is not immediately obvious

- **OFDM** (by design): delay spread < guard interval L << number of carriers N

Consider the time domain representation



$$\mathbf{h} = [h_0 \quad \cdots \quad h_{L-1} \quad 0 \quad \cdots \quad 0]^T$$

$$\mathcal{G} = \mathbf{F}\boldsymbol{G} \xleftarrow{\text{DFT}} \boldsymbol{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

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Least Squares Estimation (Time Domain View)

- Write $\boldsymbol{\mathcal{h}}$ in terms of its (non-zero) time-domain representation $\mathbf{h}_L$

$$\widehat{\boldsymbol{\mathcal{h}}} = \mathbf{F}\boldsymbol{G}\hat{\mathbf{h}}_L = \mathbf{F}_{NxL}\hat{\mathbf{h}}_L$$

$\hat{\mathbf{h}}_L$ = $[\hat{h}_0 \quad \cdots \quad \hat{\mathrm{h}}_{L-1}]^T \quad (\hat{\mathbf{h}} = \boldsymbol{G}\hat{\mathbf{h}}_L)$

$\mathbf{F}_{NxL}$: Truncated Fourier matrix (only first L columns)

$$\boldsymbol{\mathcal{y}} = \boldsymbol{\mathcal{T}}\mathbf{F}_{NxL}\mathbf{h}_L + \mathbf{n}$$

Overdetermined system!!

Least squares solution

$$\hat{\mathbf{h}}_L = \underset{\hat{\mathbf{h}}_L}{\operatorname{argmin}}\left\{\left|\boldsymbol{\mathcal{y}} - \boldsymbol{\mathcal{T}}\mathbf{F}_{NxL}\hat{\mathbf{h}}_L\right|^2\right\}$$

- Pseudo inverse yields impulse response

$$\hat{\mathbf{h}}_L = (\mathbf{F}_{NxL}^H\boldsymbol{\mathcal{T}}^H\boldsymbol{\mathcal{T}}\mathbf{F}_{NxL})^{-1}\mathbf{F}_{NxL}^H\boldsymbol{\mathcal{T}}^H\boldsymbol{\mathcal{y}}$$

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Least Squares Estimation (Time Domain View)

Back to frequency domain ($\widehat{\boldsymbol{h}} = \mathbf{F}_{NxL}\hat{\mathbf{h}}_L$)

$$\widehat{\boldsymbol{h}} = \mathbf{F}_{NxL}(\mathbf{F}_{NxL}^H \boldsymbol{\mathcal{T}}^H \boldsymbol{\mathcal{T}} \mathbf{F}_{NxL})^{-1} \mathbf{F}_{NxL}^H \boldsymbol{\mathcal{T}}^H \boldsymbol{y}$$

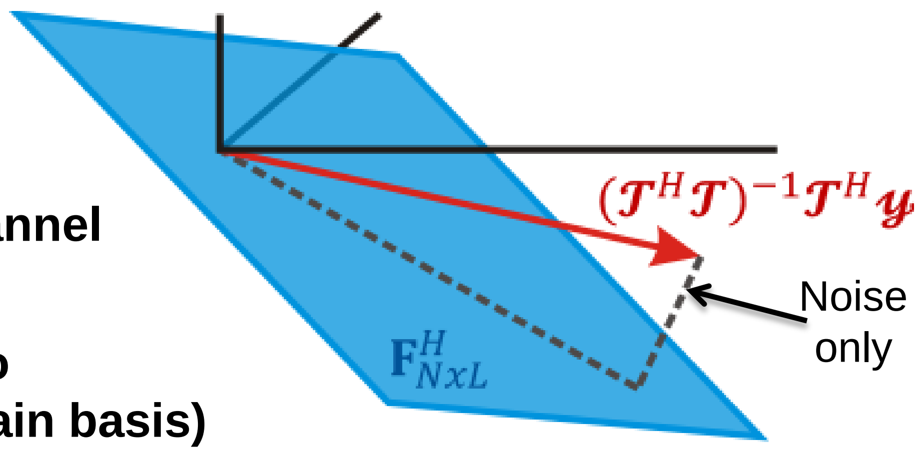
- $\boldsymbol{\mathcal{T}}^H \boldsymbol{\mathcal{T}} = \mathrm{P}_t \boldsymbol{I}$ is a scaled identity matrix
- $\mathbf{F}_{NxL}^H \mathbf{F}_{NxL} = \boldsymbol{I}_L$

support of the channel

$$\widehat{\boldsymbol{h}} = \mathbf{F}_{NxL} \mathbf{F}_{NxL}^H \underbrace{(\boldsymbol{\mathcal{T}}^H \boldsymbol{\mathcal{T}})^{-1} \boldsymbol{\mathcal{T}}^H \boldsymbol{y}}_{\text{per-tone estimate}}$$

Assume $\mathrm{P}_t = 1$

1. **Per tone estimate**
2. **Projection onto the support $\mathbf{F}_{NxL}^H$ of the channel**
3. **Transformation back to the original (freq. domain basis)**



Interpretation in the Frequency Domain

$$\widehat{\boldsymbol{h}} = \mathbf{F}_{NxL}\mathbf{F}_{NxL}^{H}(\boldsymbol{\mathcal{T}}^{H}\boldsymbol{\mathcal{T}})^{-1}\boldsymbol{\mathcal{T}}^{H}\boldsymbol{y}$$

- $(\boldsymbol{\mathcal{T}}^{H}\boldsymbol{\mathcal{T}})^{-1}\boldsymbol{\mathcal{T}}^{H}\boldsymbol{y}$: Per-tone channel estimation
- $\mathbf{F}_{NxL}\mathbf{F}_{NxL}^{H}$: Low-pass filter in frequency domain

$$\mathbf{F}_{NxL}\mathbf{F}_{NxL}^{H} = \mathbf{F}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}\mathbf{F}^{H} = \mathbf{F}\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}}_{diag\{[\underbrace{1 \ \cdots \ 1}_{L} \ \underbrace{0 \ \cdots \ 0}_{N}]\}}\mathbf{F}^{H}$$

- $\mathbf{FDF}^{H}$: Circulant matrix
- First row given by $\mathrm{DFT}\{diag\{\mathbf{D}\}\}$
- Multiplication with a circulant matrix = cyclic convolution with its first row

Multiplication of the per-tone estimate with $\mathbf{F}_{NxL}\mathbf{F}_{NxL}^{H}$ corresponds to a cyclic convolution in frequency domain

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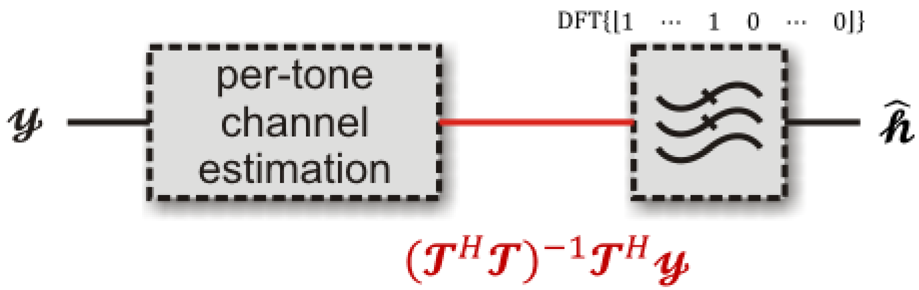
Interpretation in the Frequency Domain

Multiplication of the per-tome estimate with $\mathbf{F}_{NxL}\mathbf{F}_{NxL}^{H}$ corresponds to a cyclic convolution in frequency domain

Convolution with $\mathrm{DFT}\{[1 \quad \cdots \quad 1 \quad 0 \quad \cdots \quad 0]\}$ of the per-tone estimates corresponds to low-pass filter in the frequency domain



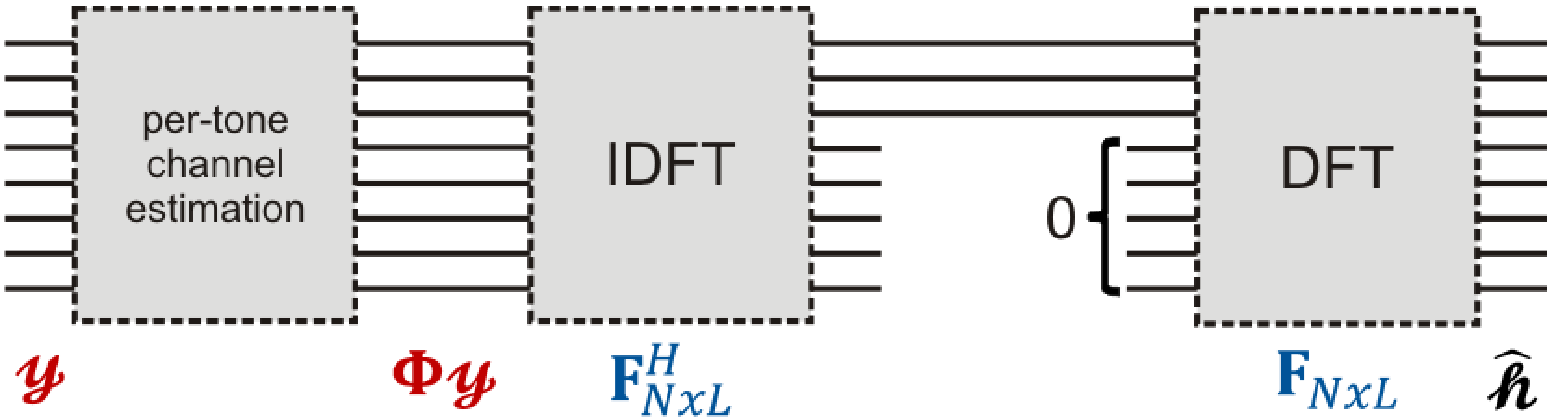
- Computational complexity (multiplications) $\propto N^2$

Low Complexity Implementation

More economic implementation

$$\hat{\boldsymbol{h}} = \mathbf{F}_{NxL}\mathbf{F}_{NxL}^{H}\underbrace{(\boldsymbol{\mathcal{T}}^{H}\boldsymbol{\mathcal{T}})^{-1}\boldsymbol{\mathcal{T}}^{H}}_{\boldsymbol{\Phi}}\boldsymbol{y}$$

- $\mathbf{F}_{NxL}^{H}$: Truncated IFFT (keep only L of N outputs)
- $\mathbf{F}_{NxL}$: Zero-padded FFT (Only first L coefficients are non-zero)



- Computational complexity (multiplications) $\propto N \log N$ (complexity of an FFT/IFFT)

Impact on Channel Estimation Error

Reminder: total channel estimation error with per-tone estimation

$$(\boldsymbol{\mathcal{T}}^H\boldsymbol{\mathcal{T}})^{-1}\boldsymbol{\mathcal{T}}^H\boldsymbol{\mathcal{y}} = \boldsymbol{\mathcal{h}} + \boldsymbol{\Delta\mathcal{h}}$$

$$\mathbb{E}\{|\boldsymbol{\Delta\mathcal{h}}|^2\} = N\sigma_{\boldsymbol{\mathcal{k}}}^{\boldsymbol{2}} = N\sigma_{\mathcal{h}_f}^{\boldsymbol{2}} = N\sigma_{\boldsymbol{n}}^{\boldsymbol{2}}/P_t$$

Time-domain least squares $\widehat{\boldsymbol{\mathcal{h}}} = \mathbf{F}_{NxL}\mathbf{F}_{NxL}^H(\boldsymbol{\mathcal{h}} + \boldsymbol{\Delta\mathcal{h}})$

- Consider the noise $\boldsymbol{\Delta\mathcal{k}}$ after projection onto the support of the channel $\mathbf{F}_{NxL}^H$

$$\mathbb{E}\left\{\left|\mathbf{F}_{NxL}^H\boldsymbol{\Delta\mathcal{h}}\right|^{\boldsymbol{2}}\right\} = \mathbb{E}\{|\boldsymbol{\Delta}\mathbf{h}|^2\} \text{ with } \boldsymbol{\Delta}\mathbf{h} = \mathbf{F}_{NxL}^H\boldsymbol{\Delta\mathcal{h}}$$

- Since $\mathbf{F}_{NxL}^H$ is an orthonormal transform: $\sigma_{h_k}^{\boldsymbol{2}} = \sigma_{\mathcal{h}_f}^{\boldsymbol{2}} = \sigma_{\boldsymbol{n}}^{\boldsymbol{2}}/P_t$ and all Δh_k are independent

$$\mathbb{E}\{|\boldsymbol{\Delta}\mathbf{h}|^2\} = \sum_{k=1}^{L}\mathbb{E}\{|\Delta h_k|^2\} = L\sigma_{\boldsymbol{n}}^{\boldsymbol{2}}/P_t$$

- Transformation back to frequency domain only re-distributes the noise across tones

$$\mathbb{E}\left\{\left|\widehat{\boldsymbol{\mathcal{h}}} - \boldsymbol{\mathcal{h}}\right|^2\right\} = \frac{L\sigma_{\boldsymbol{n}}^{\boldsymbol{2}}}{P_t} = \frac{L}{N}\mathbb{E}\{|\boldsymbol{\Delta\mathcal{h}}|^2\}$$

Improvement compared to per-tone estimation

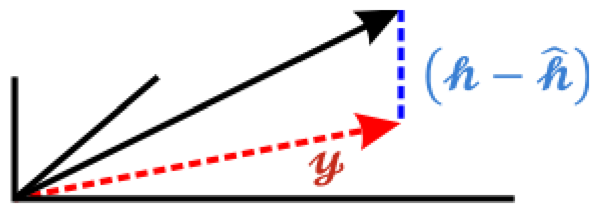
Linear MMSE Channel Estimation (Frequ. Domain View)

$$\hat{\boldsymbol{h}} = \underset{\hat{\boldsymbol{h}}}{\operatorname{argmin}}\, \mathbb{E}\left\{\left|\boldsymbol{h} - \hat{\boldsymbol{h}}\right|^2\right\}$$

- Linear MMSE estimator : $\hat{\boldsymbol{h}} = \boldsymbol{W}\boldsymbol{y}$

Solving the MMSE estimation problem:

- **Orthogonality principle**: ***residual error must be orthogonal to the observation***



$$\mathbb{E}\left\{\left(\boldsymbol{h} - \hat{\boldsymbol{h}}\right)\boldsymbol{y}^{\mathrm{H}}\right\} = \mathbf{0}$$

- Substitute $\boldsymbol{y}$ and $\hat{\boldsymbol{h}} = \boldsymbol{W}\boldsymbol{y}$, then take the expectation

$$\mathbb{E}\left\{\left(\boldsymbol{h} - \hat{\boldsymbol{h}}\right)\boldsymbol{y}^{\mathrm{H}}\right\} = \mathbb{E}\{(\boldsymbol{h} - \boldsymbol{W}\boldsymbol{y})\boldsymbol{y}^{\mathrm{H}}\} = 0 \qquad \overset{\boldsymbol{\Phi}_{\boldsymbol{y}\boldsymbol{h}}}{\mathbb{E}\{\boldsymbol{h}\boldsymbol{y}^{\mathrm{H}}\}} = \boldsymbol{W}\overset{\boldsymbol{\Phi}_{\boldsymbol{y}\boldsymbol{y}}}{\mathbb{E}\{\boldsymbol{y}\boldsymbol{y}^{\mathrm{H}}\}}$$

$$\boldsymbol{W} = \boldsymbol{\Phi}_{\boldsymbol{h}\boldsymbol{h}}\boldsymbol{\mathcal{T}}^{\mathrm{H}}\left(\boldsymbol{\mathcal{T}}\boldsymbol{\Phi}_{\boldsymbol{h}\boldsymbol{h}}\boldsymbol{\mathcal{T}}^{\mathrm{H}} + \boldsymbol{\Phi}_{\boldsymbol{n}\boldsymbol{n}}\right)^{-1}$$

$$\boldsymbol{W} = \boldsymbol{\Phi}_{\boldsymbol{y}\boldsymbol{h}}\boldsymbol{\Phi}_{\boldsymbol{y}\boldsymbol{y}}^{-1}$$

Covariance matrix of the channel coeffs

$$\boldsymbol{\Phi}_{\boldsymbol{h}\boldsymbol{h}} = \mathbb{E}\{\boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}\} = ?$$

Covariance matrix of the thermal noise

$$\boldsymbol{\Phi}_{\boldsymbol{n}\boldsymbol{n}} = \mathbb{E}\{\boldsymbol{n}\boldsymbol{n}^{\mathrm{H}}\} = \sigma_n^2\mathbf{I}$$

Linear MMSE Channel Estimation

How do we know the frequency domain covariance matrix of the channel?

- Start from the time-domain representation of the channel and substitute

$$\boldsymbol{\hbar} = \mathbf{F}\mathbf{h}$$
$$\boldsymbol{\Phi}_{\boldsymbol{\hbar}\boldsymbol{\hbar}} = \mathbf{F}\mathbb{E}\{\mathbf{h}\mathbf{h}^{\mathrm{H}}\}\mathbf{F}^{\mathrm{H}} = \mathbf{F}\boldsymbol{\Phi}_{\mathbf{hh}}\mathbf{F}^{\mathrm{H}}$$

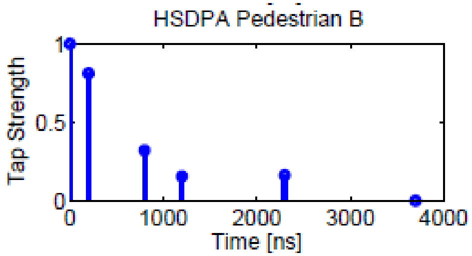
We can write the frequency domain channel covariance matrix in terms of the covariance matrix of the channels time-domain coefficients

Remember from discussion of channel models:

- In many models, individual taps are *uncorrelated* => $\boldsymbol{\Phi}_{\mathbf{hh}}$: diagonal
- Power-delay-profile specifies variance of the individual taps

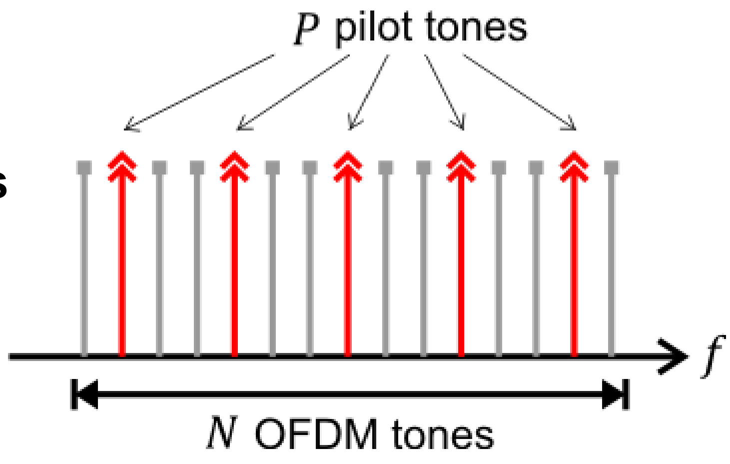
$$\boldsymbol{\Phi}_{\mathbf{hh}} = \mathrm{diag}\{PDP\}$$

- Length of the PDP (including trailing zeros is given by the OFDM symbol duration!!
- $\boldsymbol{\Phi}_{\boldsymbol{\hbar}\boldsymbol{\hbar}}$ is a circulant matrix by construction



Comb-Type Training

Only P out of N tones are used for training



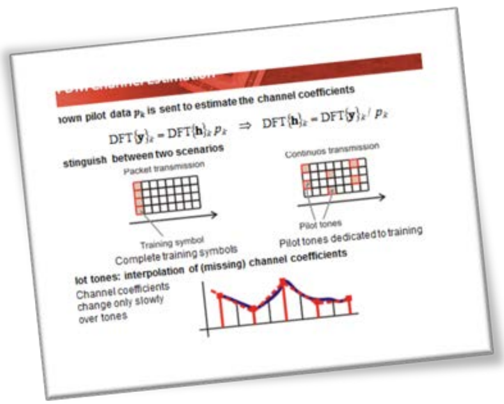
Per tone channel estimation only for P pilot tones

- Set of pilot tone indices $\mathcal{P} = \{p_0, p_1, \dots, p_{P-1}\}$

$$y_{p_k} = h_{p_k} t_k + n_{p_k}$$

- $\boldsymbol{h}_{\mathcal{P}} = [h_{p_0} \quad h_{p_1} \quad \cdots \quad h_{p_{P-1}}]^T$
- $\boldsymbol{y}_{\mathcal{P}} = [y_{p_0} \quad y_{p_1} \quad \cdots \quad y_{p_{P-1}}]^T$

$$\boldsymbol{T}_{\mathcal{P}} = \begin{bmatrix} t_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_{P-1} \end{bmatrix}$$



$$\boldsymbol{y}_{\mathcal{P}} = \boldsymbol{T}_{\mathcal{P}} \boldsymbol{h}_{\mathcal{P}} + \boldsymbol{n}_{\mathcal{P}}$$

How can we estimate the channel for all tones?

Time Domain Channel Estimation for Comb-Training

- Write channel frequency response in terms of the channels impulse response
- Consider only the tones occupied by pilot symbols
- Example $N = 8, L = 4$

$$
\begin{bmatrix} \mathscr{h}_0 \\ \mathscr{h}_{p_0} \\ \mathscr{h}_2 \\ \mathscr{h}_{p_1} \\ \mathscr{h}_4 \\ \mathscr{h}_{p_2} \\ \mathscr{h}_6 \\ \mathscr{h}_{p_3} \end{bmatrix}
=
\begin{bmatrix}
f_{0,0} & f_{0,1} & f_{0,2} & f_{0,3} & f_{0,4} & f_{0,5} & f_{0,6} & f_{0,7} \\
f_{1,0} & & & f_{1,3} & f_{1,4} & & & \\
f_{2,0} & & & f_{2,3} & f_{2,4} & & & \\
f_{3,0} & & & f_{3,3} & f_{3,4} & & & \\
f_{4,0} & & & f_{4,3} & f_{4,4} & & & \\
f_{5,0} & & & f_{5,3} & f_{5,4} & & & \\
f_{6,0} & & & f_{6,3} & f_{6,4} & & & \\
f_{7,0} & & & f_{7,3} & f_{7,4} & & &
\end{bmatrix}
\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

- $\mathbf{F}_{\mathcal{P}xL}$: Truncated Fourier matrix (only first L columns) keeping only the rows with indices in $\mathcal{P}$ (only pilot tones)

Time Domain Channel Estimation for Comb-Training

- System model for pilot subcarriers based on channel impulse response of length L

$$\boldsymbol{y}_{\mathcal{P}} = \boldsymbol{\mathcal{T}}_{\mathcal{P}}\mathbf{F}_{\mathcal{P}xL}\mathbf{h}_L + \boldsymbol{n}_{\mathcal{P}}$$

Solve for the time domain impulse response

$$\widehat{\mathbf{h}_L} = \left(\mathbf{F}_{\mathcal{P}xL}^H\boldsymbol{\mathcal{T}}_{\mathcal{P}}^H\boldsymbol{\mathcal{T}}_{\mathcal{P}}\mathbf{F}_{\mathcal{P}xL}\right)^{-1}\mathbf{F}_{\mathcal{P}xL}^H\boldsymbol{\mathcal{T}}_{\mathcal{P}}^H\boldsymbol{y}$$

- Necessary condition: $\left(\mathbf{F}_{\mathcal{P}xL}^H\boldsymbol{\mathcal{T}}^H\boldsymbol{\mathcal{T}}\mathbf{F}_{\mathcal{P}xL}\right)$ must be invertible
- **Note:** condition of $\left(\mathbf{F}_{\mathcal{P}xL}^H\boldsymbol{\mathcal{T}}^H\boldsymbol{\mathcal{T}}\mathbf{F}_{\mathcal{P}xL}\right)$ depends on $\mathcal{P}$ => improper selection of tones leads to ill conditioned estimator (noise enhancement)

- With $\widehat{\mathbf{h}_L}$ we can easily find $\widehat{\boldsymbol{h}}$ according to

$$\widehat{\boldsymbol{h}} = \mathbf{F}_{NxL}\widehat{\boldsymbol{h}}$$

Time Domain Channel Estimation for Comb-Training

Consider a frequently used special case: $P > L$ training tones are distributed equally spaced in frequency domain

- In this case: $\mathbf{F}_{\mathcal{P}xL}^{H}\mathbf{F}_{\mathcal{P}xL} = \boldsymbol{I}_P$

$$\hat{\boldsymbol{h}} = \mathbf{F}_{NxL}\left(\mathbf{F}_{\mathcal{P}xL}^{H}\boldsymbol{\mathcal{T}}_{\mathcal{P}}^{H}\boldsymbol{\mathcal{T}}_{\mathcal{P}}\mathbf{F}_{\mathcal{P}xL}\right)^{-1}\mathbf{F}_{\mathcal{P}xL}^{H}\boldsymbol{\mathcal{T}}_{\mathcal{P}}^{H}\boldsymbol{y} = \mathbf{F}_{NxL}\mathbf{F}_{\mathcal{P}xL}^{H}\underbrace{\left(\boldsymbol{\mathcal{T}}_{\mathcal{P}}^{H}\boldsymbol{\mathcal{T}}_{\mathcal{P}}\right)^{-1}\boldsymbol{\mathcal{T}}_{\mathcal{P}}^{H}\boldsymbol{y}}_{\text{per-tone estimate}}$$

- $\mathbf{F}_{\mathcal{P}xL}^{H}$: P-point DFT results in a P-periodic impulse response, truncated to the first $L < P$ coefficients

- $\mathbf{F}_{NxL}\mathbf{F}_{\mathcal{P}xL}^{H}$ is a low-pass (interpolation) filter in the frequency domain (cf. Slide 26)

For equally spaced pilot tones, channel estimation corresponds to interpolation of channel coefficients