On the competitiveness of memoryless strategies for the k-Canadian Traveller Problem

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Abstract

The k-Canadian Traveller Problem, defined and proven PSPACE-complete by Papadimitriou and Yannakakis, is a generalization of the Shortest Path Problem which admits blocked edges. Its objective is to determine the strategy that makes the traveller traverse graph G between two given nodes s and t with the minimal distance, knowing that at most k edges are blocked. The traveller discovers that an edge is blocked when arriving at its endpoint.

We study the competitiveness of randomized memoryless strategies for the k-CTP. Memoryless strategies are more attractive in practice as a decision taken by the strategy for a traveller in node v of G does not depend on its anterior moves. We establish that the competitive ratio of any randomized memoryless strategy cannot be better than $2k + O(1)$. This means that randomized memoryless strategies are asymptotically as competitive as deterministic strategies which achieve a ratio $2k + 1$ at best.

1. Introduction

The *Canadian Traveller Problem* (CTP), a generalization of the *Shortest Path Problem*, was introduced in [6]. Given an undirected weighted graph $G = (V, E, \omega)$ and two nodes $s, t \in V$, the objective is to design a strategy to make a traveller walk from s to t through G on the shortest path possible. Its particularity is that some edges of G are potentially blocked. The traveller does not know, however, which edges are blocked. This implies that we solve the CTP with online algorithms, called strategies. He discovers blocked edges, also called blockages, when arriving to its endpoints. The *k-Canadian Traveller Problem* (k-CTP) is the parameterized variant of CTP, where an upper bound k for the number of blocked edges is given. Both CTP and k-CTP are PSPACE-complete [2, 6].

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State-of-the-art. Strategies for the k-CTP are studied through the competitive analysis, which evaluates their quality [4]. The competitive ratio of a strategy is the maximum, over every satisfiable instance, of the ratio of the distance traversed by the traveller following the strategy and the *optimal offline cost*, which is the distance he traverses if he knows blocked edges from the beginning.

There are two classes of strategies: deterministic and randomized. Westphal [7] proved that there is no deterministic strategy that achieves a competitive ratio better than $2k + 1$. This ratio is reached by REPOSITION and COMPARISON strategies [7, 8]. The REPOSITION strategy repeats an attempt to reach t through the shortest (s, t)-path going back to s after the discovery of an obstacle. As in practical cases such as urban networks, it does not seem realistic, Xu *et al.* [8] introduced the GREEDY algorithm. For grids, it achieves an $\mathcal{O}(1)$ ratio, regardless of k. However, for any graph, this ratio is $\mathcal{O}\left(2^k\right)$.

We evaluate the competitiveness of the randomized strategies by calculating the maximal ratio of the mean distance traversed by the traveller following the strategy by the optimal offline cost. Westphal [7] proved that there is no randomized algorithm that can attain a ratio smaller than $k+1$. However, unlike the deterministic case, no $(\alpha k + 1)$-competitive randomized strategy, $\alpha < 2$, was identified, excepted for very particular cases. Two randomized strategies have been proposed. Demaine *et al.* [5] designed a strategy with a ratio $\left(1 + \frac{\sqrt{2}}{2}\right) k + 1$, executed in time of $\mathcal{O}\left(k\mu^2 \left|E\right|^2\right)$ where parameter μ may be exponential. It is dedicated to graphs that can be transformed into apex trees. Bender *et al.* studied in [3] a restriction of k-CTP for graphs composed of node-disjoint (s, t)-paths and proposed a polynomial-time strategy with ratio $(k + 1)$.

Contributions and paper plan. We study the competitiveness of memoryless strategies [1, 4]. The choice the traveller makes at node v (to decide where to go next) is independent of his travel before reaching node v. Memoryless strategies are worth of interest because they are easy to be implemented as they do not memorize the edges already visited by the traveller to make a decision. The only information they use is the graph $G \backslash E'_*$, which is the graph G deprived of the blocked edges discovered E'_*. Given that deterministic memoryless strategies cannot achieve a ratio better than $2k + 1$, our goal is to prove that randomized memoryless strategies are not more competitive asymptotically. To do this, we identify a lower bound $c_k = 2k + O(1)$ of the competitiveness of randomized memoryless strategies for the k-CTP.

We remind, in Section 2, the definitions of k-CTP, memoryless strategies and the competitive ratio. In Section 3, we present sets $\mathcal{R}_k$ (called road atlases) of *road maps*, *i.e.* pairs (G, E_*) with graph G and blocked edges E_*. These road maps, together with associated binary trees, are a means to study the performance of memoryless strategies. We prove in Section 4 that randomized memoryless strategies cannot drop below a ratio $c_k = 2k + O(1)$ on road maps in $\mathcal{R}_k$, where expression $O(1)$ is clarified. Eventually, we draw conclusions and highlight the future work in Section 5.

2. Definitions

We start by introducing the notation. For any graph $G = (V, E, \omega)$, let $G \backslash E'$ denotes its subgraph $(V, E \backslash E', \omega)$.

2.1. Memoryless Strategies for the k-CTP

We remind the definition of CTP. Let $G = (V, E, \omega)$ be an undirected graph with positive weights. The objective is to make a traveller traverse the graph from a source node s to a target one t, with $s, t \in V$. There is a set $E_* \subsetneq E$ of blocked edges. The traveller does not know a priori which edges are blocked. He discovers a blocked edge only when arriving to one of its endpoints. For example, if (v, w) is a blockage he will discover it when arriving to v (or w). The goal is to design the strategy A which minimizes the competitive ratio.

We focus on *memoryless strategies* (MS). Concretely, we suppose that the traveller remembers the blocked edges he has discovered but forgets the nodes which he has already visited. In other words, a decision of an MS is independent of the nodes already visited. In the literature, the term *memoryless* was used in the context of online algorithms (e.g. PAGING PROBLEM [4], LIST UPDATE PROBLEM [1]) which take decisions according to the current state, ignoring past events. An MS can be either deterministic or randomized.

Definition 1 (Memoryless Strategies for the k-CTP). *A deterministic strategy A is an MS if and only if (iff) the next node w the traveller visits depends on graph G deprived of blocked edges already discovered E'_* and the current traveller position v: $w = A(G \backslash E'_*, v)$. Similarly, a randomized strategy A is an MS iff node w is the realization of a discrete random variable $X = A(G \backslash E'_*, v)$.*

For example, the GREEDY strategy [8] is a deterministic MS. It consists in choosing at each step the first edge of the shortest path between the current node v and the target t. In contrast, the REPOSITION strategy [7] is not an MS as any decision refers to the past moves of the traveller. The polynomial-time strategies proposed in the literature do not use much memory information in the decision-taking process. Either they are memoryless or they use a weak quantity of memory. For example, REPOSITION (deterministic [7] or randomized [3]) can be implemented with a one bit memory given that the only information to retain is whether the traveller tries to reach t or to return to s.

The following process allows to identify whether a strategy A is a deterministic MS. Let us suppose that a traveller T_1 executes strategy A: he has already visited certain nodes of the graph, he is currently at node v but he has not reached target t yet. Let us imagine a second traveller T_2 who is airdropped on node v and starts applying strategy A. If the traveller T_2 always follows the same path as T_1 until reaching t, A is a deterministic MS. If T_1 and T_2 may follow different paths, then A is not an MS. Formally, prooving that a strategy is a MS consists in finding the function which transforms the pair $(G \backslash E_*, v)$ into node $w = A(G \backslash E'_*, v)$.

2.2. Competitive ratio

Let (G, E_*) be a *road map*, *i.e.* a pair with graph $G = (V, E, \omega)$ and blocked edges $E_* \subsetneq E$, such that there is an (s,t)-path in graph $G \backslash E_*$ (nodes s and t remain in the same connected component when all blocked edges are discovered). We note $\omega_A (G, E_*)$ the distance traversed by the traveller reaching t with strategy A on graph G with blocked edges E_* and $\omega_{\min} (G, E_*)$ the cost of the shortest (s,t)-path in graph $G \backslash E_*$.

The ratio $\omega_A (G, E_*) / \omega_{\min} (G, E_*)$ is abbreviated as $c_A (G, E_*)$. A strategy A is c_A-competitive [4, 8] iff for any $(G, E_*), \omega_A (G, E_*) \leq c_A \omega_{\min} (G, E_*)$. Otherwise stated, for any (G, E_*), $c_A (G, E_*) \leq c_A$. If strategy A is randomized, $\omega_A (G, E_*)$ is replaced by $\mathbb{E} (\omega_A (G, E_*))$ which is the expected distance traversed by the traveller to reach t with strategy A. The competitive ratio can also be evaluated on a family $\mathcal{R}$ of road maps, put formally,

$$c_{A,\mathcal{R}} = \max_{(G,E_*) \in \mathcal{R}} c_A (G, E_*) \tag{1}$$

This "local" competitive ratio fulfils $c_{A,\mathcal{R}} \leq c_A$. The definition of the competitive ratio can also be extended to families of strategies. We note c_{MS} the competitive ratio of MSes, which is the minimum over competitive ratios of any MSes: $c_{\text{MS}} = \min_{A \text{ MS}} c_A$.

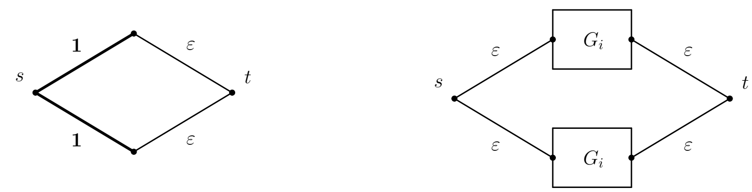
3. Road atlas used to study randomized MSes

We specify *road atlases* $\mathcal{R}_k$, *i.e.* sets of road maps, which will be used to evaluate the competitiveness of randomized MSes. We introduce a family of graphs G_i such that graphs composing these road atlases are subgraphs of G_i.

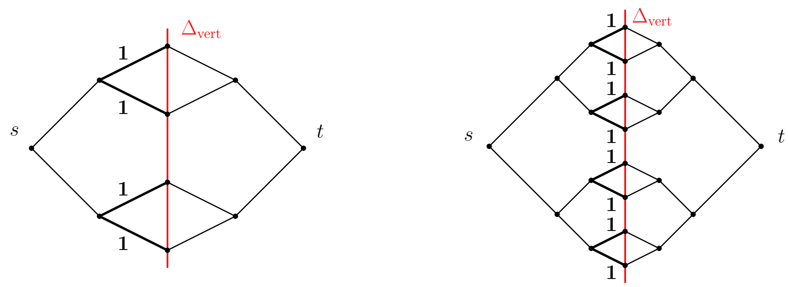
We define recursively a sequence of graphs G_i for $i \geq 1$ with weights from $\{1, \varepsilon\}$, $0 < \varepsilon \ll 1$. Graphs G_1 and G_{i+1} are represented in Figures 1a and 1b, respectively. Graphs G_2 and G_3 are shown in Figure 1c and 1d. Edges with weight 1 are thicker than edges with weight ε. On any graph G_i, axis Δ_{vert} is the vertical axis of symmetry (see Figures 1c and 1d).

We focus on the behavior of a traveller traversing road maps (G_k, E_*) composed of graph G_k and at most k blocked edges with an MS. Blocked edges in E_* are at the right of axis Δ_{vert}. Indeed, blocking edges left to Δ_{vert} affects negligibly the total distance traversed by a traveller. Let us suppose that he traverses graph G_k and has already discovered some blocked edges $E' \subseteq E_*$. He tries now to reach t on graph $G_k \backslash E'$, ignoring that the undiscovered blocked edges are $E'_* = E_* \backslash E'$. From the memoryless point of view, the graph considered at this step is $G_k \backslash E'$, without taking past moves into account. Even if road map (G_k, E_*) is the instance, an MS might take decisions in function of graphs $G_k \backslash E'$, where $E' \subseteq E_*$. We note $\mathcal{G}$ the set of all the subgraphs of G_k, *i.e.* graphs $G_k \backslash E'$ with at most k edges E' right to Δ_{vert}, for any integer $k \geq 1$. We call them *diamond graphs* because of their appearance, diamonds joined together. Formally, we set $\mathcal{G} = \bigcup_{k=1}^{+\infty} \{G_k \backslash E' : |E'| \leq k\}$. For any graph $G \in \mathcal{G}$, we partition its edges, noted E_G, in two sets $E_{G,\text{left}}$ (left to axis Δ_{vert})

and $E_{G,\text{right}}$ (right to axis Δ_{vert}). Road maps (G, E_*) composing road atlases $\mathcal{R}_k$ contain graphs in $\mathcal{G}$.



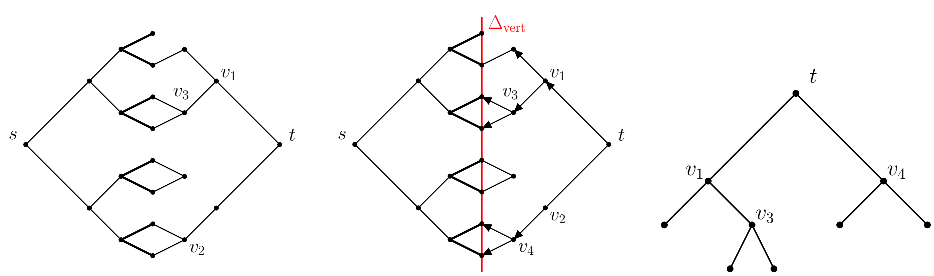
(a) Graph G_1 (b) Graph G_{i+1}



(c) Graph G_2 and axis Δ_{vert} (d) Graph G_3 and axis Δ_{vert}

Figure 1: Recursive construction of graphs G_i. Weights ε are not indicated in Figures 1c and 1d to keep them readable.

To any diamond graph G of $\mathcal{G}$, we associate a *diamond binary tree* (DBT) noted T_G which represents all *branching nodes* from t in graph G. The DBT is obtained thanks to a breadth-first search (BFS) from t in direction of nodes on axis Δ_{vert}. Node t is the root of the DBT. Graphically, each node from which there is a *branching node*, *i.e.* there are two possible directions for the next move of the BFS, provides two extra edges in the tree (see nodes t, v_1, v_2 and v_3 in Figure 2b). We note $T_\varnothing$ the empty tree. Any nonempty tree is a triplet (v, T_a, T_b) with a root $v \in V$ and two subtrees T_a and T_b. Figures 2a, 2b and 2c illustrate the construction of the DBT.



(a) Graph $G \in \mathcal{G}_3$ (b) Branching nodes in G (c) The DBT T_G

Figure 2: An example of graph $G \in \mathcal{G}_3$ and its DBT T_G.

To put the definition of DBTs formally, let $\textsc{bfs}\,(v)$ denote the set of nodes visited just after node v in the BFS starting from t. For example, $\textsc{bfs}\,(t) = \{v_1, v_2\}$, $v_3 \in \textsc{bfs}\,(v_1)$ and $\textsc{bfs}\,(v_2) = \{v_4\}$. We force this BFS to stop on axis Δ_{vert} such that $v \in \Delta_{\text{vert}} \Rightarrow \textsc{bfs}\,(v) = \emptyset$. We define below function BIN-TREE which gives the construction of tree T_G fulfilling $\text{BIN-TREE}\,(t) = T_G$.

$$\text{BIN-TREE}\,(v) = \begin{cases} T_\varnothing \text{ if } \textsc{bfs}(v) = \emptyset, \\ \text{BIN-TREE}\,(v_{\text{next}}) \text{ if } \textsc{bfs}(v) = \{v_{\text{next}}\}, \\ (v, \text{BIN-TREE}(v_{\text{up}}), \text{BIN-TREE}(v_{\text{down}})) \text{ if } \textsc{bfs}(v) = \{v_{\text{up}}, v_{\text{down}}\}. \end{cases}$$

We introduce notations which characterize a DBT. For any edge $e = (u, v)$ in DBT T where the depth of u is larger than those of v, we note $P(e) = (v, w)$ the "mother" edge of e where the depth of v is larger than the depth of w. We note $D_{\min}(T)$ the minimum depth among the leaves of T. For example, for DBT T_G of Figure 2c, $D_{\min}(T_G) = 2$ as the depth of leaves in T_G is 2 or 3. Finally, we set road atlases $\mathcal{R}_k$ in Definition 2.

Definition 2 (Road atlas $\mathcal{R}_k$ based on $\mathcal{D}_k$). *Road atlas $\mathcal{R}_k$ is composed of road maps (G, E_*) where $|E_*| = k$ and $G \in \mathcal{G}$ with a DBT fulfulling $D_{min}\,(T_G) \geq k$. Formally, we define set $\mathcal{D}_k = \{G \in \mathcal{G} : D_{min}(T_G) \geq k\}$ and road atlas $\mathcal{R}_k$ on which we study the competitiveness of MSes,*

$$\mathcal{R}_k = \{(G, E_*) : G \in \mathcal{D}_k, |E_*| = k\}.$$

Let $U(e)$ denote the "aunt" of edge e, which has the same parent than $P(e)$. Put formally, $U(e) = \{e' \in T : P(e') = P\,(P(e)) = P^2(e)\}$. This notation is used in Section 4.

4. Competitiveness of randomized MSes

We study the competitiveness of a randomized MS for each road atlas $\mathcal{R}_k$. We establish properties on DBTs T_G to understand the behavior of MSes on graphs $G \in \mathcal{G}$. Theorem 1 states that cutting one edge from $G \in \mathcal{D}_k$ results on a graph in $G \in \mathcal{D}_{k-1}$.

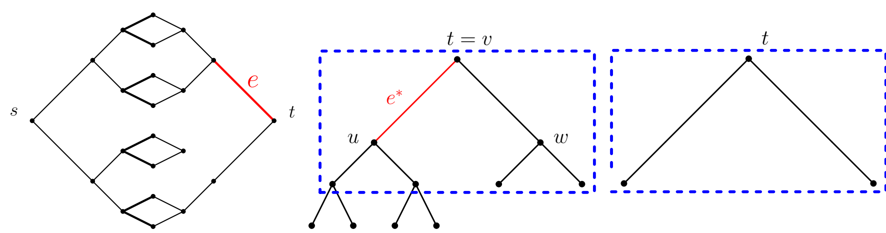
Theorem 1. *For any graph $G \in \mathcal{D}_k$ and edge $e \in E_{G,\text{right}}$, $G \setminus \{e\} \in \mathcal{D}_{k-1}$.*

Proof. Let $G \in \mathcal{D}_k$ and e be an edge of this graph. Removing e from $E_{G,\text{right}}$ makes at least one edge of the DBT T_G disappear and we note e^* the edge among them which is the nearest of t (said differently the edge with lowest depth). Let v be the "root" of edge e^*, *i.e.* node v such that $e^* = (u, v)$ and $v = P(u)$. We distinguish two cases:

- **The depth of node u is greater or equal to $D_{\min}(T_G)$.** If u is the unique leaf of depth $D_{\min}\,(T_G)$, the depth of leaves of the DBT $T_{G\setminus\{e\}}$ is $D_{\min}\,(T_G) - 1 \geq k - 1$. Otherwise, in DBT $T_{G\setminus\{e\}}$, the depth of leaves are still equal to $D_{\min}\,(T_G) \geq k$. In both cases, $G \setminus \{e\} \in \mathcal{D}_{k-1}$.

- **The depth of node u is strictly inferior to $D_{\min}(T_G)$.** Let T_v be the subtree of T_G with root v. We note w the sibling of u, said differently the other son of node v (see Figures 3a and 3b). When edge e is removed from G, edge e^* and its descendants are withdrawn in the DBT, so T_v becomes T_w, the subtree of root w. All the leaves of T_w have initially a depth greater than k so by removing e from G, all the leaves of T_w have a depth greater than $k-1$. All other leaves, which do not belong to T_v, keep the same depth which is greater than k. So all leaves of $T_{G\setminus\{e\}}$ have a depth greater than $k-1$.

In these two cases, we conclude that $G\setminus\{e\}$ belongs to $\mathcal{D}_{k-1}$. ■



(a) Example of graph $G \in \mathcal{D}_2$. Edge e (red) is about to be removed from this graph.

(b) The DBT T_G. Blue frame covers nodes of depth 0 to 2. Two leaves are at depth 2.

(c) The DBT $T_{G\setminus\{e\}}$. Nodes of depth 0 to 1 are framed. Graph $G\setminus\{e\}$ belongs to $\mathcal{D}_1$.

Figure 3: Illustrating the proof of Theorem 1 on a subgraph of G_3.

For any graph $G \in \mathcal{G}$, we note $c_{\mathrm{MS}}(G)$ the maximum over values $c_{\mathrm{MS}}(G, E_*)$ where $(G, E_*) \in \mathcal{R}_k$. This value represents the competitive ratio of MSes on road maps containing graph G. Formally,

$$c_{\mathrm{MS}}(G) = \min_{A \text{ MS}} \max_{(G,E_*)\in\mathcal{R}} c_A(G, E_*) \tag{2}$$

Please note that, even if the competitive ratio $c_{\mathrm{MS}}(G)$ is defined formally for a traveller starting from source s, it is still valid for a traveller starting his walk at any node left to axis Δ_{vert} as weights ε can be neglected. Theorem 2 states that value $c_{\mathrm{MS}}(G)$ is the same for any graph of $\mathcal{D}_k$. In other words, graphs in $\mathcal{D}_k$ are all similar regarding the competitiveness of MSes over them. Let c_k denotes the competitive ratio of MSes over atlas $\mathcal{R}_k$. The proof of Theorem 2 provides a recursive formula of ratio c_k.

Theorem 2. *For any graph $G, G' \in \mathcal{D}_k$, $c_{\mathrm{MS}}(G) = c_{\mathrm{MS}}(G') = c_k$.*

Proof. We prove it by induction. If $k = 0$, then for every graph of $\mathcal{D}_0$, $c_{\mathrm{MS}}(G) = c_0 = 1$ because any path the traveler takes is open. We assume the theorem holds for graphs which belong to $\mathcal{D}_{k-1}$. For any $G \in \mathcal{D}_k$, all leaves of T_G are at depth greater or equal to k. So, T_G is complete until depth k and has 2^k nodes at depth k. We suppose that the traveller, guided by a the most competitive MS A over atlas $\mathcal{R}_k$, is standing at source s and starts his walk on a road map $(G, E_*) \in \mathcal{R}_k$. We study his behavior until he discovers a blockage or reaches t without being blocked.

As strategy A is the most competitive, it makes the traveller traverse exactly a distance 1 (weights ε are neglected) before reaching t directly or meeting a blocked edge. Either the traveller reaches t directly with distance 1 or he meets a blocked edge and traverses a total distance $2 + c_{k-1}$: distance 1 to reach the blockage, distance 1 to go back left to Δ_{vert} and distance c_{k-1} to reach t on graph G deprived of the blocked edges, which belongs to $\mathcal{D}_{k-1}$ (see Theorem 1).

For any $1 \leq j \leq 2^k$, let p_j^A denote the probability that the traveler visits the j^{th} node at depth k, noted v_j (indexing nodes from left to right in the DBT representation). We have $\sum_{j=1}^{2^k} p_j^A = 1$. Let P_j be the simple (s,t)-path of length 1 which traverses node v_j. Given a set of E_* of blocked edges, we note $J(E_*)$ the set of node v_j at depth k where P_j is not blocked. For any node v_j of depth k, we can construct a set of blocked edges such that $J(E_*) = \{v_j\}$. Indeed, let e be the edge linking the sibling of v_j and $P(v_j)$. By blocking e, $U(e)$, $U(U(e))$ ... $U^{k-1}(e)$, we block $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$ paths $P_{j'}$, $j' \neq j$. Finally, only P_j is open. For this set of blocked edges E_*, if $v_j \in J(E_*)$, then the distance traversed is 1 by passing through v_j. If $v_j \notin J(E_*)$, the distance traversed is $2 + c_{k-1}$,

$$c_A(G, E_*) = \sum_{v_j \in J(E_*)} p_j^A + (2 + c_{k-1}) \sum_{v_j \notin J(E_*)} p_j^A$$

We note j^* the index of the node v_{j^*} in $J(E_*)$ which minimizes $p_{j^*}^A$. As $1 < 2 + c_{k-1}$, the greater $\sum_{v_j \in J(E_*)} p_j^A$ is, the lower $c_A(G, E_*)$ is, therefore

$$c_A(G, E_*) \leq p_{j^*}^A + (2 + c_{k-1}) \underbrace{\sum_{j \neq j^*} p_j^A}_{1 - p_{j^*}^A} .$$

The competitive ratio $c_A(G, E_*)$ attains this upper bound when blocked edges of road map (G, E_*) fulfils $J(E_*) = \{v_{j^*}\}$. Considering that any P_j, $1 \leq j \leq 2^k$ can be the single open path of road map (G, E_*), we obtain the competitive ratio of strategy A over road atlas $\mathcal{R}_k$,

$$\begin{aligned} c_{A,\mathcal{R}_k} &= \max_{1 \leq j \leq 2^k} p_j^A + \left(1 - p_j^A\right)(2 + c_{k-1}) , \\ &\geq \frac{1}{2^k} + \left(1 - \frac{1}{2^k}\right)(2 + c_{k-1}) . \end{aligned}$$

The MS with the weakest value $c_{A,\mathcal{R}_k}$ fulfils $p_j^A = \frac{1}{2^k}$ for any $1 \leq j \leq 2^k$. This value depends on k and c_{k-1}, not on the graph itself. For $G, G' \in \mathcal{D}_k$, $c_{\text{MS}}(G) = c_{\text{MS}}(G') = c_k = \min_A c_{A,\mathcal{R}_k} = \frac{1}{2^k} + \left(1 - \frac{1}{2^k}\right)(2 + c_{k-1})$. ■

The consequence of Theorem 2 and its proof is that, for k blockages, the competitive ratio of MSes on road atlas $\mathcal{R}_k$ is given by sequence c_k. Consequently, $c_{\text{MS}} \leq c_k$. We remark that $c_k - c_{k-1} = 2 - \frac{1}{2^k} - \frac{c_{k-1}}{2^k}$. By summing this expression from $j = 1$ to k, we obtain :

$$c_k = 2k + 1 - \sum_{j=0}^{k-1} \frac{c_j + 1}{2^{j+1}}$$

As $\sum_{j=0}^{+\infty} \dfrac{c_j + 1}{2^{j+1}}$ is finite, $c_k = 2k + O(1)$. We compute the value $\sum_{j=0}^{10^4} \dfrac{c_j + 1}{2^{j+1}} = 3.213$, so the competitive ratio of MSes is larger than $2k - 2.22$. As c_k represents the competitive ratio of the most competitive MSes over road atlas $\mathcal{R}_k$, no MS can go below $2k + O\left(1\right)$ in terms of competitiveness.

5. Conclusion and further work

We studied the competitiveness of the MSes for the k-CTP. An MS is a strategy which does not make decisions referring to the anterior moves of the traveller, in other words, the nodes the traveller visited until his current position.

Then, we constructed a series of k-CTP instances, called road atlases and noted $\mathcal{R}_k$. We foremost concluded that a randomized MS cannot reach a ratio more competitive than $2k + \mathcal{O}\left(1\right)$ on road atlas $\mathcal{R}_k$. That is to say that we identified an upper bound on the competitive ratio of randomized MSes which is significantly higher than the existing one $k + 1$. In future research, if we aim at designing a strategy with competitive ratio $\alpha k + O\left(1\right)$, $\alpha < 2$, we shall focus on strategies which not only are randomized but also use memory.

References

[1] S. Albers. Online algorithms: a survey. *Math. Program.*, pages 3–26, 2003.

[2] A. Bar-Noy and B. Schieber. The Canadian Traveller Problem. In *Proc. of ACM/SIAM SODA*, pages 261–270, 1991.

[3] M. Bender and St. Westphal. An optimal randomized online algorithm for the k-Canadian Traveller Problem on node-disjoint paths. *J. Comb. Optim.*, 30(1):87–96, 2015.

[4] A. Borodin and R. El-Yaniv. *Online computation and competitive analysis.* Cambridge University Press, 1998.

[5] E. D. Demaine, Y. Huang, C.-S. Liao, and K. Sadakane. Canadians Should Travel Randomly. *Proc. of ICALP*, pages 380–391, 2014.

[6] Ch. Papadimitriou and M. Yannakakis. Shortest paths without a map. *Theor. Comput. Sci.*, 84(1):127–150, 1991.

[7] St. Westphal. A note on the k-Canadian Traveller Problem. *Inform. Proces. Lett.*, 106(3):87–89, 2008.

[8] Y. Xu, M. Hu, B. Su, B. Zhu, and Z. Zhu. The Canadian traveller problem and its competitive analysis. *J. Comb. Optim.*, 18(2):195–205, 2009.