On the competitiveness of the memoryless strategies for the Canadian traveller

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Abstract

The k-Canadian Traveller Problem, defined and proven PSPACE-complete by Papadimitriou and Yannakakis, is a generalization of the Shortest Path Problem which admits blocked edges. Its objective is to determine the strategy that makes the traveller traverse graph G between two given nodes s and t with the minimal distance, knowing that at most k edges are blocked. The traveller discovers that an edge is blocked when arriving at its endpoint. Westphal showed that the competitive ratio, which is an indicator of the online algorithm quality, of any randomized strategy is not less than $k+1$.

We study the competitiveness of randomized memoryless strategies for the k-CTP. In this context, a decision taken by the traveler in node v of G does not depend on its anterior moves. We improve Westphal's result by establishing that the competitive ratio of any randomized memoryless strategy is not less than $\left(3-\sqrt{3}\right)k+1 \approx 1.268k+1$. This means that a randomized strategy achieving the competitive ratio below this bound incorporates information about the previous moves of the traveller. The primordial consequence of this result is that, if we aim at designing a strategy such that its competitive ratio is smaller than $1.268k+1$, we shall focus on strategies which are not memoryless.

1 Introduction

The *Canadian Traveller Problem* (CTP), a generalization of the *Shortest Path Problem*, was introduced in [6]. Given an undirected weighted graph $G=(V,E,\omega)$ and two nodes $s,t \in V$, the objective is to design a strategy to make a traveller walk from s to t through G on the shortest path possible. Its particularity is that some edges of G are potentially blocked. The traveller does not know, however, which edges are blocked. This implies that we solve the CTP with online algorithms, called strategies. He discovers blocked edges, also called blockages, when arriving to its endpoints. The k-*Canadian Traveller Problem* (k-CTP) is the parameterized variant of CTP, where an upper bound k for the number of blocked edges is given. Both CTP and k-CTP are PSPACE-complete [2, 6].

1.1 State-of-the-art

Strategies for the k-CTP are studied through the competitive analysis, which evaluates their quality [4]. The competitive ratio of a strategy is the maximum, over every satisfiable instance, of

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the ratio of the distance traversed by the traveller following the strategy and the *optimal offline cost*, which is the distance he traverses if he knows blocked edges from the beginning.

There are two classes of strategies: deterministic and randomized. Regarding the deterministic strategies, Westphal [7] proved that there is no algorithm that achieves a competitive ratio better than $2k+1$. This ratio is reached by REPOSITION and COMPARISON strategies [7, 8]. The REPOSITION strategy consists in traversing the shortest (s,t)-path on the current graph. If the traveller discovers that an edge e^* of this path is blocked, he goes back to node s and restarts the process on graph G deprived of the edge e^*. This algorithm is executed in polynomial time. However, considering specific practical cases such as urban networks, returning to node s every time the traveller is blocked does not seem realistic. This is why Xu *et al.* [8] introduced the GREEDY algorithm. For grids, it achieves an $\mathcal{O}(1)$ ratio, regardless of k. However, for any graph, this ratio is $\mathcal{O}\left(2^k\right)$.

We evaluate the competitiveness of the randomized strategies by calculating the maximal ratio of the mean distance traversed by the traveller following the strategy by the optimal offline cost. Westphal [7] proved that there is no randomized algorithm that can attain a ratio smaller than $k+1$. However, unlike the deterministic case, no $(k+1)$-competitive randomized strategy was identified, excepted for very particular cases. Recently, two randomized algorithms have been proposed. Demaine *et al.* [5] designed a strategy with a ratio $\left(1+\frac{\sqrt{2}}{2}\right)k+1$. It is executed in time of $\mathcal{O}\left(k\mu^2\left|E\right|^2\right)$, where μ is a parameter which may be exponential. It is dedicated to graphs that can be transformed into apex trees. It provides a $(k+1)$-competitive strategy for graphs G which are apex trees with equal-weight (s,t)-paths. Bender *et al.* studied in [3] a restriction of k-CTP for graphs composed of node-disjoint (s,t)-paths and proposed a polynomial-time strategy with ratio $(k+1)$.

1.2 Contributions and paper plan

We study the competitiveness of memoryless strategies [1, 4] for the k-CTP. The choice the traveller makes at node v (to decide where to go next) is independent of his travel before reaching node v. Our goal is to prove that, for randomized strategies, the $k+1$ competitive ratio cannot be attained. We list below our contributions.

- Proof, in Section 2.2, that one of the optimal deterministic strategies for the CTP, in terms of competitive analysis, is a memoryless strategy.
- Proof, in Section 3, that there is no randomized memoryless strategy which has a ratio $f(k) \leq k+1$ for the k-CTP.
- Establishing, in Section 4, a bound on the competitive ratio of memoryless strategies. There is no memoryless strategy that solves the k-CTP with a ratio less than $k\beta+1$ with $\beta = 3-\sqrt{3} \approx 1.268$. This implies that, if a randomized strategy has a ratio smaller than $k\beta+1$ for any graph, it is not memoryless.

We introduce, in Section 2.1, the definitions we use. Eventually, we draw conclusions and highlight the future work.

2 Definitions

We start by introducing the notation. For any graph $G=(V,E,\omega)$, let $G\backslash E'$ denotes its subgraph $(V,E\backslash E',\omega)$. If P is a path, we note its cost as $\omega\left(P\right)=\sum_{e\in P}\omega(e)$.

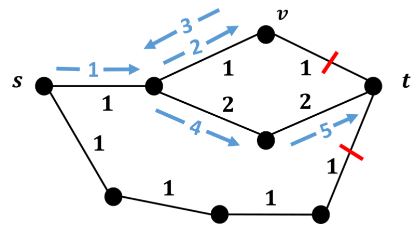


Figure 1: Illustration of the GREEDY strategy on a graph with 2 blocked edges

2.1 Memoryless Strategies for the k-CTP

We remind the definition of CTP. Let $G = (V, E, \omega)$ be an undirected graph with positive weights. The objective is to make a traveller traverse the graph from a source node s to a target one t, with $s, t \in V$. There is a set $E_* \subsetneq E$ of blocked edges. The traveller does not know a priori which edges are blocked. He discovers a blocked edge only when arriving to one of its endpoints. For example, if (v, w) is a blockage he will discover it when arriving to v (or w). The goal is to design the strategy A with the minimal competitive ratio.

We focus on *memoryless strategies* (MS). Concretely, we suppose that the traveller remembers the blocked edges he has discovered but forgets the nodes which he has already visited. In other words, a decision of an MS is independent of the nodes already visited. In the literature, the term *memoryless* was used in the context of online algorithms (e.g. PAGING PROBLEM [4], LIST UPDATE PROBLEM [1]) which take decisions according to the current state, ignoring past events. An MS can be either deterministic or randomized.

Definition 1 (Memoryless Strategies for the k-CTP) *A deterministic strategy A is an MS if and only if (iff) the next node w the traveller visits depends on graph G deprived of blocked edges already discovered E'_* and the current traveller position v: $w = A\left(G \backslash E'_*, v\right)$. Similarly, a randomized strategy A is an MS iff node w is the realization of a discrete random variable $X = A\left(G \backslash E'_*, v\right)$.*

MSes are easy to be implemented because they do not use past moves to take a decision. For the same reason, they do not need any extra memory either.

For example, the GREEDY strategy [8] is an MS. It consists in choosing at each step the first edge of the shortest path between the current node v and the target t. This strategy, illustrated in Figure 1, does not refer to anterior moves. In contrast, the REPOSITION strategy [7] is not an MS as any decision refers to the past moves of the traveller.

We propose the following process to identify whether a strategy A is a deterministic MS. Let us suppose that a traveller T_1 executes strategy A: he has already visited certain nodes of the graph, he is currently at node v but he has not reached target t yet. Let us imagine a second traveller T_2 who is airdropped on node v and starts applying strategy A. If the traveller T_2 always follows the same path as T_1 until reaching t, A is a deterministic MS. If T_1 and T_2 may follow different paths, then A is not an MS. Formally, prooving that a strategy is an MS consists in finding the function which transforms the pair $(G \backslash E_*, v)$ into node $w = A\left(G \backslash E'_*, v\right)$.

2.2 Competitive ratio

Let (G, E_*) be a *road map*, *i.e.* a pair with graph $G = (V, E, \omega)$ and blocked edges $E_* \subsetneq E$, such that there is an (s, t)-path in graph $G \backslash E_*$ (nodes s and t always remain in the same connected component). We note $\omega_A\left(G, E_*\right)$ the distance traversed by the traveller reaching t with strategy

(a) Graph G_1 (b) Graph G_{i+1}

Figure 2: Recursive construction of graphs G_i

A on graph G with blocked edges E_* and $\omega_{\min}(G, E_*)$ the cost of the shortest (s,t)-path in graph $G \backslash E_*$.

The ratio $\omega_A(G, E_*) / \omega_{\min}(G, E_*)$ is abbreviated as $c_A(G, E_*)$. A strategy A is c_A-competitive [4, 8] iff for any $(G, E_*), \omega_A(G, E_*) \leq c_A \omega_{\min}(G, E_*)$. Otherwise stated, for any (G, E_*), $c_A(G, E_*) \leq c_A$. If strategy A is randomized, $\omega_A(G, E_*)$ is replaced by $\mathbb{E}(\omega_A(G, E_*))$ which is the expected distance traversed by the traveller to reach t with strategy A.

Theorem 1 *The* COMPARISON *strategy [8] is an optimal deterministic MS.*

Proof. According to Lemma 2.1. in [7], there is no deterministic strategy with $c_A < 2k+1$. Xu *et al.* proved in [8] that the COMPARISON strategy is a deterministic strategy with $c_A = 2k+1$. We thus need to prove that the COMPARISON strategy is memoryless.

Let $P_{x,y}$ be the shortest path between nodes x and y. We assume, as before, that the traveler is at node v. The execution of COMPARISON may be summarized as:

- If $\omega(P_{v,t}) \leq \omega(P_{s,t})$ then go to the node following v in path $P_{v,t}$.
- If $\omega(P_{v,t}) > \omega(P_{s,t})$ then go to the node following v in path $P_{v,s}$.

The past traveller moves do not intervene in the decision process of the procedure. ■

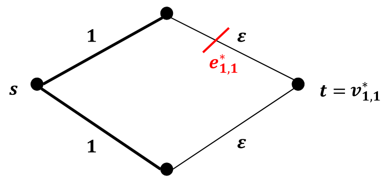
The consequence of another Westphal's result (Lemma 2.2 in [7]) is that there is no randomized MS with $c_A < k+1$. As a randomized MS reaching the bound $k+1$ on the competitive ratio is unknown, our goal is to identify a greater bound $k\beta + 1, \beta > 1$, such that there is no randomized MS with $c_A < k\beta + 1$.

2.3 Graphs G_k

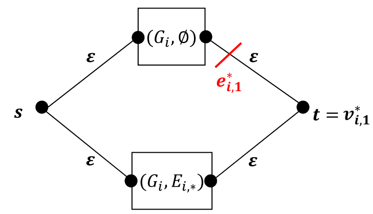
We construct a set $\mathcal{S}_k$ (also called *road atlas*) of road maps (G, E_*) which contain k blockages: $|E_*| = k$. In Sections 3 and 4, we will study randomized MSes on road maps of $\mathcal{S}_k$ to establish the bound $k\beta + 1$ on their competitive ratio.

We define recursively a sequence of graphs G_i for $i \geq 1$ with weights from $\{1, \varepsilon\}$, $0 < \varepsilon \ll 1$. Graphs G_1 and G_{i+1} are represented in Figures 2a and 2b, respectively, edges with weight 1 are thicker than edges with weight ε.

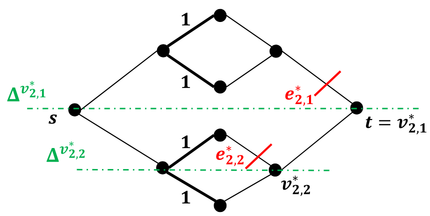
We associate with any graph G_i a set $E_{i,*}$ of blocked edges to obtain a road map $(G_i, E_{i,*})$. The recursive definition of set $E_{i,*}$ is illustrated in Figures 3a and 3b. We define $E_{i,*} = \left\{e^*_{i,1}, \ldots, e^*_{i,i}\right\}$,



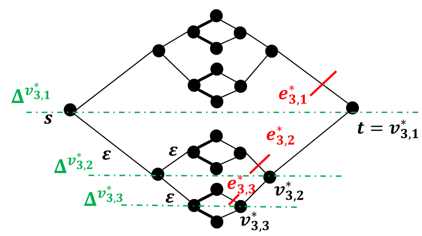
(a) Road map $(G_1, E_{1,*})$ with $E_{1,*} = \{e_{1,*}\}$



(b) Road map $(G_{i+1}, E_{i+1,*})$



(c) Road map $(G_2, E_{2,*})$



(d) Road map $(G_3, E_{3,*})$

Figure 3: Definition of road maps $(G_i, E_{i,*})$

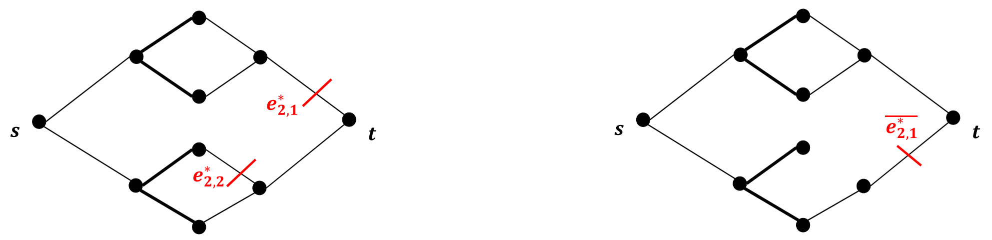
so $|E_{i,*}| = i$. We provide, as examples, road maps $(G_2, E_{2,*})$ and $(G_3, E_{3,*})$ in Figures 3c and 3d. Let $v^*_{i,j}$ be the endpoint on the right side of the blocked edge $e^*_{i,j} \in E_{i,*}$ (see Figures 3a and 3b).

We note $\Delta^{(v)}$ the horizontal axis of symmetry passing through a node v of graph G_i. The interest of defining horizontal axes is, given an edge e, to deal with the edge which is the symmetric to e with respect to a certain axis $\Delta^{(v)}$. Given an edge $e^*_{i,j}$, we are particularly interested in edge $\overline{e^*_{i,j}}$ which is a mirror reflexion of $e^*_{i,j}$ with respect to $\Delta^{(v^*_{j,i})}$. Axes $\Delta^{(v^*_{j,i})}$ are illustrated in Figures 3c and 3d.

Finally, we construct the road atlas $\mathcal{S}_k$. We insert $(G_k, E_{k,*})$ into $\mathcal{S}_k$. If it was the only map in atlas $\mathcal{S}_k$, MSes that encourage the traveller to visit nodes at the bottom of graph G_k would be very competitive. We add new maps which make the competitive ratio of such strategies increase significantly, so that the most competitive strategy on road atlas $\mathcal{S}_k$ shall reach a compromise. This is why we insert maps with the blocked edge $\overline{e^*_{k,1}}$, which is symmetric of the edge $e^*_{k,1}$ with respect to the horizontal axis $\Delta^{(t)}$, crossing node t. For any nonempty subset $E' \subsetneq E_{k,*}$ such that $e^*_{k,1} \notin E'$, we also insert $\left(G_k \backslash E', \left\{\overline{e^*_{k,1}}\right\}\right)$ into $\mathcal{S}_k$. In summary, road atlas $\mathcal{S}_k$ contains one map $(G_k, E_{k,*})$ with k blockages and other ones with only one blockage. Road atlas $\mathcal{S}_2$ is illustrated in Figure 4.

3 Memoryless Strategies are not $(k+1)$-competitive

Let A be an optimal (in terms of the minimal competitive ratio) randomized MS. We aim at establishing properties of strategy A on road maps of $\mathcal{S}_k$. The study of road atlas $\mathcal{S}_2$ will allow us to conclude that there is a lower bound, greater than $k+1$, of the competitive ratio of A. This implies that it is a lower bound of the competitiveness of any MS.



(a) $(G_2, E_{2,*})$ (b) $\left(G_2 \backslash \left\{e^*_{2,2}\right\}, \left\{\overline{e^*_{2,1}}\right\}\right)$

Figure 4: Road atlas $\mathcal{S}_2 = \left\{(G_2, E_{2,*}), \left(G_2 \backslash \left\{e^*_{2,2}\right\}, \left\{\overline{e^*_{2,1}}\right\}\right)\right\}$

3.1 Properties of MSes on road atlas $\mathcal{S}_k$

Definition 2 (Levels of graph G_k) *For any $0 \leq i \leq 2k$, let $L_k^{(i)}$ denotes the set of nodes v such that the shortest (v,t)-path is composed of exactly i edges. Consequently, $L_k^{(2k)} = \{s\}$. By convention, $L_k^{(0)} = \{t\}$.*

Figure 5a illustrates levels of graph G_2. We note Δ_{vert} the vertical axis of symmetry of graphs G_k and $W_G = \bigcup_{k<i\leq 2k} L_k^{(i)}$. Graphically, set W_G represents the nodes which are on the left of axis Δ_{vert}. The notion of levels can be extended to any subgraph of G_k. Let $\bar{v}$ denote the node symmetric to v with respect to axis Δ_{vert}. For any subgraph G of G_k, we note $G^{(v)}$ the subgraph of G such that $v \in G^{(v)}$, $\bar{v} \in G^{(v)}$ and any other node w (respectively edge e) is in graph $G^{(v)}$ iff there exists a shortest $(v,\bar{v})$-path which contains node w (respectively edge e). Figure 5b represents the axis Δ_{vert} and a graph $G^{(v)}$ on G_2.

A traveller at node $v \in L_k^{(i)}, i \neq k$, may choose between the three options:

- *up* move: going to the neighbor of v in $L_k^{(i-1)}$ which is above axis $\Delta^{(v)}$,
- *down* move: going to the neighbor of v in $L_k^{(i-1)}$ which is below axis $\Delta^{(v)}$,
- *back* move: going to the neighbor of v in $L_k^{(i+1)}$.

Directions *up* and *down* split graph $G^{(v)}$ into two subgraphs $G^{(v)}_{\text{up}}$ and $G^{(v)}_{\text{down}}$ (see Figure 5b). Put formally,

- Nodes v and $\bar{v}$ are both in $G^{(v)}_{\text{up}}$ and $G^{(v)}_{\text{down}}$ (they are egress nodes of these graphs) .
- A node w is in $G^{(v)}_{\text{up}}$ iff there is a shortest $(v,\bar{v})$-path which contains w and that takes direction *up* at start, in node v.
- A node w is in $G^{(v)}_{\text{down}}$ iff there is a shortest $(v,\bar{v})$-path which contains w and that takes direction *down* at start, in node v.

On any road map of road atlas $\mathcal{S}_k$, the traveller does not have any interest in moving on a *blockage-free cycle* , *i.e.* in making a round-trip (going back to a starting node) without discovering any blocked edge. A traveller moving on a blockage-free cycle does not gather new information about the blockages and comes back to his initial position which is absurd. The only benefit from a round-trip is the discovery of a blockage which prevents the traveller from reaching the destination t. As our objective is to identify a lower bound of ratio c_A, we consider that MS A does not make the traveller walk on blockage-free cycles.

Let us suppose that the traveller is standing at node $v \in W_G$. We say that directions *up* and *down* are *indistinguishable* iff graphs $G_{\text{up}}^{(v)}$ and $G_{\text{down}}^{(v)}$ are isomorphic. Theorem 2 says that, if the traveller is facing two indistinguishable directions, the probability to choose one is $\frac{1}{2}$. We give the intuition of its proof. For example, we consider that the traveller starts his walk at s on graph G_2 (see Figure 5). He does not know which edges are in E_*. He must choose between directions *up* and *down* which are indistinguishable. Perhaps one is more advantageous than another: this depends on set E_*. The traveller, however, does not have any idea of which direction should be taken. If the strategy favours one of the directions (for example, if the probability to go *up* is 0.6 instead of 0.5), it takes the risk to select the costliest direction. It degrades the competitive ratio which is calculated for every possible road map (G, E_*). This is why the uniform draw is the most appropriate solution.

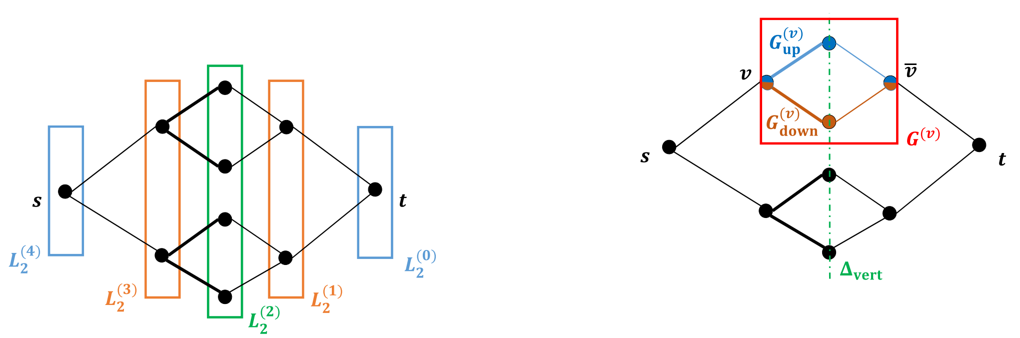
Theorem 2 *We suppose that the traveller, guided by strategy A, is standing at node $v \in L_k^{(i+1)}$ and goes to node $w \in L_k^{(i)}$ with $k < i < 2k$. At this very moment, if directions* up *and* down *are indistinguishable, the* up *move and the* down *move will be equiprobable.*

Proof. Let $p_\eta^{(w)}$ denotes the probability to choose direction $\eta \in \{up, down, back\}$, and $c_\eta(G, E_*, w)$ denotes the mean distance traversed by the traveller to reach t when he chooses direction η at node w. As $w \in W_G$, the construction of atlas $\mathcal{S}_k$ ensures that node w is not the endpoint of a blocked edge. Strategy A does not admit blockage-free cycles: $p_{back}^{(w)} = 0$. We thus can express the mean distance traversed to reach t from node w according to strategy A, $c_A(G, E_*, w)$ as:

$$c_A(G, E_*, w) = p_{\text{up}}^{(w)} c_{\text{up}}(G, E_*, w) + p_{\text{down}}^{(w)} c_{\text{down}}(G, E_*, w).$$

Let us have a look at the map which is complementary to (G, E_*) with respect to axis $\Delta^{(w)}$. Let $E_{w,*}$ be the blocked edges in subgraph $G^{(w)}$. We note $\overline{E_{w,*}}$ the set of edges which are the symmetric to edges in $E_{w,*}$ with respect to axis $\Delta^{(w)}$. We open edges from $E_{w,*}$ and block edges from $\overline{E_{w,*}}$ instead: we note $\overline{E_*}$ this new set. The ratio c_A for map $\left(G, \overline{E_*}\right)$ is:

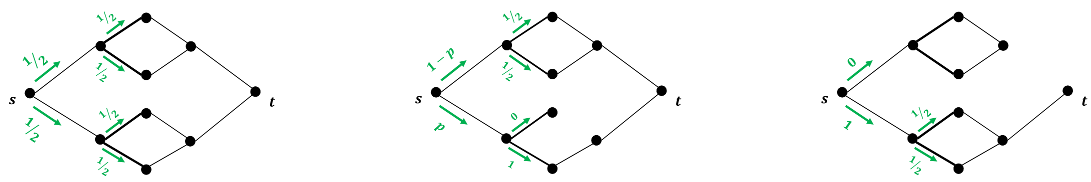
$$c_A\left(G, \overline{E_*}, w\right) = p_{\text{up}}^{(w)} c_{\text{down}}(G, E_*, w) + p_{\text{down}}^{(w)} c_{\text{up}}(G, E_*, w).$$



(a) Levels in graph G_2

(b) Axis Δ_{vert} and example of subgraphs $G^{(v)}$, $G_{\text{up}}^{(v)}$ and $G_{\text{down}}^{(v)}$ on graph G_2

Figure 5: Levels and graphs $G^{(v)}$, $G_{\text{up}}^{(v)}$, $G_{\text{down}}^{(v)}$ for $G = G_2$.



(a) Graph G_2 (b) Graph $G_2 \backslash \left\{e^*_{2,2}\right\}$ (c) Graph $G_2 \backslash \left\{e^*_{2,1}\right\}$

Figure 6: Probabilities of choosing directions on graphs G_2, $G_2 \backslash \left\{e^*_{2,2}\right\}$ and $G_2 \backslash \left\{e^*_{2,1}\right\}$

Strategy A must minimize the maximum in set $\left\{c_A\left(G, E_*, w\right), c_A\left(G, \overline{E_*}, w\right)\right\}$, otherwise this would imply that there was a strategy which obtains a competitive ratio better than A. We necessarily have $p^{(w)}_{\text{up}} = p^{(w)}_{\text{down}} = \frac{1}{2}$. ■

3.2 Optimal MS for $k = 2$

The objective is now to evaluate the competitive ratio of the optimal MS A on road atlas $\mathcal{S}_2$ when ε tends to 0 (in the remainder of this paper, we neglect any move of distance ε when computing the path cost). As shown above, strategy A does not make the traveller traverse blockage-free cycles and it imposes uniform probabilities of choosing indistinguishable directions.

Let us focus on road map $(G_2, E_{2,*}) \in \mathcal{S}_2$. The traveller is standing at source s. According to Theorem 2, the probabilities to go *up* and to go *down* are both equal to $\frac{1}{2}$. We apply again this result when the traveller is on level $L^{(3)}_k$ (see Figure 6a). We distinguish five cases:

- Case 1: the traveller does not discover any blocked edge and reaches t with distance 1. As illustrated in Figure 6a, the probability to reach node t with distance 1 is $\frac{1}{4}$.
- Case 2: the traveller is blocked by edge $e^*_{2,2}$ (probability $\frac{1}{4}$) and, then, reaches t without being blocked by edge $e^*_{2,1}$. Let p denotes the probability for the traveller to reach t without being blocked by edge $e^*_{2,1}$ after discovering blocked edge $e^*_{2,2}$. The cost of this scenario is 3 and its probability is $\frac{p}{4}$.
- Case 3: the traveller is blocked by edge $e^*_{2,1}$ (probability $\frac{1}{2}$) and, then, reaches t without being blocked by edge $e^*_{2,2}$ (probability $\frac{1}{2}$). The cost is 3 and its probability is $\frac{1}{4}$.
- Case 4: the traveller is blocked by edge $e^*_{2,2}$ (probability $\frac{1}{4}$) and, then, blocked by edge $e^*_{2,1}$ (probability $1-p$). The cost of this scenario is 5 and its probability is $\frac{1-p}{4}$.
- Case 5: the traveller is blocked by edge $e^*_{2,1}$ (probability $\frac{1}{2}$) and, then, blocked by edge $e^*_{2,2}$ (probability $\frac{1}{2}$). The cost of this scenario is 5 and its probability is $\frac{1}{4}$.

We determine the competitive ratio $c_A\left(G_2, E_{2,*}\right)$ of strategy A on road map $(G_2, E_{2,*})$:

$$c_A\left(G_2, E_{2,*}\right) = \frac{1}{4} + 3\left(\frac{p}{4} + \frac{1}{4}\right) + 5\left(\frac{1-p}{4} + \frac{1}{4}\right) = \frac{7}{2} - \frac{1}{2}p.$$

We focus now on road map $\left(G_2 \backslash \left\{e^*_{2,2}\right\}, \overline{e^*_{2,1}}\right)$. The probability to reach t with distance 1, without being blocked, is $1-p$. Otherwise, the traveller is blocked by edge $\overline{e^*_{2,1}}$ and traverses the graph with distance 3. Therefore, the competitive ratio of strategy A on map $\left(G_2\backslash\left\{e^*_{2,2}\right\}, \overline{e^*_{2,1}}\right)$ is $(1-p)+3p = 1+2p$.

Theorem 3 *There is no randomized MS with a competitive ratio better than* $\frac{10}{9}k+1$.

Proof. We identify the smallest value β_2 such that the competitive ratio of strategy A on road atlas $\mathcal{S}_2$ is larger than $k\beta_2+1$. Put formally,

$$\left\{\begin{array}{lcl} c_A\left(G_2, E_{2,*}\right) & \leq & 2\beta_2+1 \\ 1+2p & \leq & \beta_2+1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} \frac{7}{2}-\frac{1}{2}p & \leq & 2\beta_2+1 \\ 2p & \leq & \beta_2 \end{array}\right.$$

The solution of this inequation system which minimizes β_2 is $p=\frac{5}{9}$ and $\beta_2=\frac{10}{9}$ ■

4 Memoryless Strategies are at best $\left(\left(3-\sqrt{3}\right)k+1\right)$-competitive

We generalize the reasoning presented in Section 3.2 for $\mathcal{S}_2$ to any road atlas $\mathcal{S}_k$. The objective is to identify a bound β such that the competitive ratio of MSes is greater than $k\beta+1$.

Let $G = G_j\backslash E'$ with $E' \subsetneq E_{j,*}$, $e^*_{j,1} \notin E'$ and $1 \leq j \leq k$. We suppose that the traveller is at node $v \in W_G$. Let us remark that there is no difference between standing at $v \in W_G$ or $w \in W_G$, $w \neq v$ because the cost of the shortest (v,w)-path is negligible. Indeed, by construction, edges of significant cost are located only on the rightmost part of set W_G.

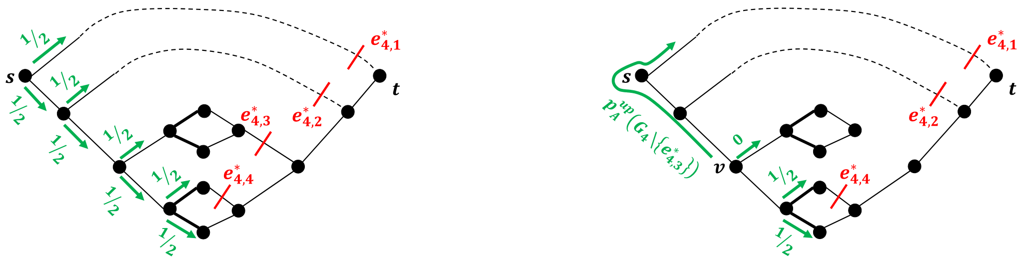
We remind that level $L^{(j)}_j$ of graph G is made up of nodes which are crossed by vertical axis Δ_{vert}: nodes which are graphically in the middle (see Figure 5a and 5b). We define functions p^{up}_A and p^{down}_A which give information about decisions taken by strategy A. Value $p^{\text{down}}_A(G)$ is the probability that the first node of level $L^{(j)}_j$ visited by the traveller on graph G is below axis $\Delta^{(t)}$, which is the horizontal axis passing through the destination t (there is necessarily one because he must pass through level $L^{(j)}_j$ to reach t). In other words, this is the probability that the traveller goes down below axis $\Delta^{(t)}$. Then, $p^{\text{up}}_A(G) = 1 - p^{\text{down}}_A(G)$ is the probability that the traveller climbs above axis $\Delta^{(t)}$, *i.e.* is blocked by edge $e^*_{j,1}$.

Figure 7a illustrates an outcome of strategy A on road map $(G_4, E_{4,*})$. If the traveller is standing at any node $v \in W_{G_4}$, directions *up* and *down* are indistinguishable. In Figure 7b, we suppose that the traveller discovered the blocked edge $e^*_{4,3}$. He necessarily goes back to node v to leave a dead end. He will discover $e^*_{4,1}$ if he goes up, which happens with the probability $p^{\text{up}}_A\left(G_4\backslash\left\{e^*_{4,3}\right\}\right)$.

Let $c^{\text{down}}_A(G_k, E_{k,*})$ denotes the mean distance traversed by the traveller on map $(G_k, E_{k,*})$ if he decides to go *down* starting his walk at source s. The probability to reach t without being blocked is $\left(\frac{1}{2}\right)^{k-1}$. The probability to be blocked by edge $e^*_{k,i}$ is $\left(\frac{1}{2}\right)^{i-1}$, *i.e.* probability $\frac{1}{2}$ to be blocked by edge $e^*_{k,2}$, probability $\frac{1}{4}$ to be blocked by edge $e^*_{k,3}$, etc. When the traveller is blocked, he traverses twice an edge of significant cost: once before being blocked and once after being blocked when he goes back. Then, he traverses the mean distance with strategy A on map $(G_k\backslash\{e_{k,i}\}, E_{k,*}\backslash\{e_{k,i}\})$. This gives:

$$c^{\text{down}}_A(G_k, E_{k,*}) = \left(\frac{1}{2}\right)^{k-1} + \sum_{i=2}^{k}\left(\frac{1}{2}\right)^{i-1}\left(2 + c_A\left(G_k\backslash\{e_{k,i}\}, E_{k,*}\backslash\{e_{k,i}\}\right)\right). \quad (1)$$

With probability $\frac{1}{2}$, the traveller goes *up* on map $(G_k, E_{k,*})$. He discovers blocked edge $e^*_{k,1}$. He goes back to source s and he never comes back to the part of the graph G_k which is above



(a) Probabilities of strategy A for graph G_4 when the traveller is on source s

(b) Probabilities of strategy A for graph $G_4 \backslash \{e^*_{4,3}\}$ when the traveller is on node v

Figure 7: Probabilities of strategy A with two situations obtained from road map $(G_4, E_{4,*})$ partially drawn

axis $\Delta^{(t)}$. Then, the mean distance traversed by the traveller is exactly the competitive ratio of strategy A on graph G_{k-1} with blocked edges $E_{k-1,*}$. In summary, if he goes *up*, he traverses a distance $c_A^{\text{up}}(G_k, E_{k,*}) = 2 + c_A(G_{k-1}, E_{k-1,*})$. On the contrary, if the traveller goes *down*, he does it with probability $\frac{1}{2}$ and distance $c_A^{\text{down}}(G_k, E_{k,*})$ defined previously and determined in Eq. (1). The mean distance traversed over the entire map $(G_k, E_{k,*})$, noted $c_A(G_k, E_{k,*})$, is thus:

$$c_A(G_k, E_{k,*}) = \frac{1}{2}\left(2 + c_A(G_{k-1}, E_{k-1,*})\right) + \frac{1}{2} c_A^{\text{down}}(G_k, E_{k,*}). \tag{2}$$

Let $E' \subsetneq E_{k,*}$ be a nonempty set and $e^*_{k,1} \notin E'$. It contains the edges which have already been discovered. To keep the notation short, we put $G'_k = G_k \backslash E'$. Let $c_A^{\text{down}}(G'_k, E_{k,*}\backslash E')$ denote the mean distance traversed by the traveller on map $(G'_k, E_{k,*}\backslash E')$ if he goes *down*, *i.e.* he cannot be blocked by edge $e^*_{k,1}$. The competitive ratio of A on map $(G'_k, E_{k,*}\backslash E')$ is:

$$c_A\left(G'_k, E_{k,*}\backslash E'\right) = p_A^{\text{up}}\left(G'_k\right)\left(2 + c_A\left(G'_{k-1}, E_{k-1,*}\backslash E'\right)\right) + p_A^{\text{down}}\left(G'_k\right) c_A^{\text{down}}\left(G'_k, E_{k,*}\backslash E'\right). \tag{3}$$

Our goal is to identify value β^*_k such that, for strategy A:

- the mean distance traversed on the road map $(G_k, E_{k,*}) \in \mathcal{S}_k$ is less than $k\beta^*_k + 1$,
- the mean distance traversed on all other maps of $\mathcal{S}_k$ is smaller than $\beta^*_k + 1$.

We know that, for any nonempty set $E' \subseteq E_{k,*}$, $e^*_{k,1} \notin E'$, road map $\left(G'_k, \left\{\overline{e^*_{k,1}}\right\}\right)$ is in atlas $\mathcal{S}_k$. On this map, going *up* makes the traveller reach the target t without being blocked with probability $p_A^{\text{up}}(G'_k)$. If he goes *down*, he is blocked by the unique blocked edge $\overline{e^*_{k,1}}$ and reaches t with a total distance 3 and probability $p_A^{\text{down}}(G'_k)$. So, the competitive ratio of strategy A on this road map is: $c_A\left(G'_k, \left\{\overline{e^*_{k,1}}\right\}\right) = 1 + 2p_A^{\text{down}}(G'_k)$.

Definition 3 (System of inequalities Π_k) *Coefficient β^*_k is the minimal value such that the system Π_k, introduced below, is solved for any nonempty $E' \subseteq E_{k,*}$, $e^*_{k,1} \notin E'$:*

$$\left\{\begin{array}{lcr} c_A(G_k, E_{k,*}) & \leq & k\beta^*_k + 1 \\ 1 + 2p_A^{down}\left(G'_j\right) & \leq & \beta^*_k + 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcr} c_A(G_k, E_{k,*}) & \leq & k\beta^*_k + 1 \\ 2p_A^{down}\left(G'_j\right) & \leq & \beta^*_k \end{array}\right.$$

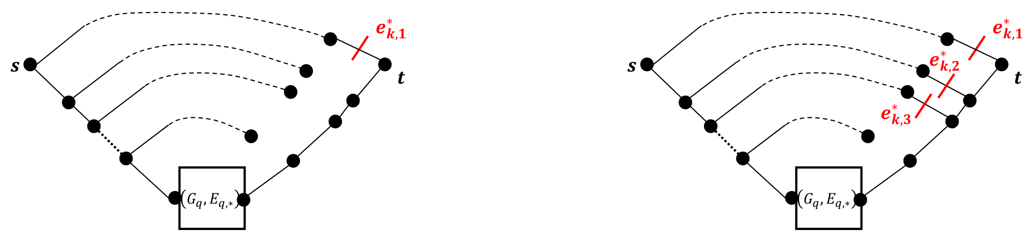
Theorem 4 *There is a solution of system Π_k, i.e. probabilities p_A^{down} and value β_k^*, such that, for any $G' = G_j \backslash E'$ with $j \leq k$ and $E' \subseteq E_{j,*}$, $p_A^{down}(G) = p_A^{down} \in [0,1]$.*

Proof. See Appendix A. ■

According to Theorem 4, $p_A^{\text{down}}(G) = p_A^{\text{down}}$ as this probability does not depend on graph G. The solution of the system Π_k is a pair (p_k^*, β_k^*).

Theorem 5 *There exists a sequence $(\beta_k)_{k \geq 2}$ such that $\beta_k \leq \beta_k^*$, β_k is growing and tends to $3 - \sqrt{3}$ when k tends to infinity.*

Proof. We note $r_k(E')$ the largest index of a blocked edge which has been already discovered: $r_k(E') = \max\left\{j : e_{k,j}^* \in E'\right\}$. The optimal offline path is the shortest (s,t)-path the traveller traverses when he knows which edges are blocked before starting his walk. Graphically, this path is composed of the nodes which are at the very bottom of the graph. By construction, value $q_k(E') = k - r_k(E')$ gives the largest index q such that there is a node v which is on the optimal offline path and $G_k^{\prime(v)} = G_q$. Figures 8a and 8b illustrate maps such that $q_k(E') = q$.



(a) $E' = \left\{e_{k,2}^*, \ldots, e_{k,r}^*\right\}$ (b) $E' \subsetneq \left\{e_{k,2}^*, \ldots, e_{k,r}^*\right\}$

Figure 8: Road map $(G', E_{k,*} \backslash E')$ with $q = k - r_k(E')$. If $E' = \left\{e_{k,2}^*, \ldots, e_{k,r}^*\right\}$ and if the traveller discovers edge $e_{k,1}^*$ next, the remaining map is equivalent to $(G_q, E_{q,*})$ as the traveller avoid traversing blockage-free cycles. If $E' \subsetneq \left\{e_{k,2}^*, \ldots, e_{k,r}^*\right\}$, the remaining map is equivalent to $(G_q, E_{q,*})$ with extra paths blocked by edges $e_{k,j}^*$ with $j < r$ and $e_{k,j}^* \notin E'$. This map is worse, in terms of competitiveness for strategy A, than map $(G_q, E_{q,*})$.

Now, we identify a lower bound of $c_A(G_k', E_{k,*} \backslash E')$ which is a function of $q = q_k(E')$ and probability p, for this reason it is noted $h(q,p)$. If the traveller goes *up* in graph G_k' (with probability $1-p$), he discovers edge $e_{k,1}^*$, traverses a distance 2 and comes back to s. Then, the map $\left(G_k' \backslash \left\{e_{k,1}^*\right\}, E_{k-1,*} \backslash E'\right)$, as it is illustrated in Figures 8a and 8b, is such that the distance traversed on it, is necessarily larger than the distance traversed on map $(G_q, E_{q,*})$. Moreover, as $c_A(G_k, E_{k,*})$ depends on $c_A(G_k', E_{k,*} \backslash E')$ thanks to Eq. (2), we obtain a lower bound of value $c_A(G_k, E_{k,*})$ which is noted $H(k,p)$. Based on Eqs. (2) and (3), we have:

$$H(k,p) = \frac{1}{2}(2 + H(k-1,p)) + \left(\frac{1}{2}\right)^k + \sum_{l=0}^{k-2} \left(\frac{1}{2}\right)^{k-l} (2 + h(l,p)), \tag{4}$$

$$h(q,p) = (1-p)(2 + H(q,p)) + p\left(\frac{1}{2}\right)^{q-1} + p\sum_{l=1}^{q-1} \left(\frac{1}{2}\right)^l (2 + h(q-l,p)). \tag{5}$$

After setting $H(1,p) = 2$ which is the competitive ratio of strategy A for road map $(G_1, E_{1,*})$, we obtain, out of Eqs. (4) and (5), a polynomial $H(k,p) = a_{0,k} + \sum_{i=1}^{k-1} a_{i,k} p^i$ with $a_{0,k} = \frac{3k+1}{2}$ and $a_{i,k} = -\frac{k+1-i}{2^{i+1}}$. We replace $c_A(G_k, E_{k,*})$ in system Π_k by the lower bound $H(k,p)$ and identify the minimal β_k which solves Π_k with the bound inserted. By this way, β_k is a lower bound of β_k^*.

$$\begin{cases} H(k,p) & \leq & k\beta_k + 1 \\ 2p & \leq & \beta_k \end{cases} \Leftrightarrow \begin{cases} \frac{H(k,p)-1}{k} & \leq & \beta_k \\ 2p & \leq & \beta_k \end{cases}$$

As function $f_k : p \rightarrow \frac{H(k,p)-1}{k}$ decreases on interval $[0;1]$, the solution β_k is obtained at the intersection of lines $y = 2x$ and $y = f_k(p)$, which corresponds to $2p_k = \beta_k$ and $H(k, p_k) - 2kp_k - 1 = 0$. So, p_k is the unique zero of polynomial $Q_k(p) = H(k,p) - 2kp - 1$ in the interval $[0,1]$. The proof that sequence $(p_k)_{k\geq 2}$ grows and tends to $\frac{3-\sqrt{3}}{2}$ is in Appendix B. ■

Now, we are ready to formulate our main result:

Corollary 1 *There is no randomized MS with a competitive ratio smaller than $k\beta + 1$ with $\beta = 3 - \sqrt{3}$.*

Proof. Let us suppose that there is a MS A' with a competitive ratio $c_{A'} = k\alpha + 1$ with $\alpha < \beta$. We proved with Theorem 4 that, to minimize the competitive ratio of A on road atlas $\mathcal{S}_k$, probability p_A^{down} is optimal when it is constant. The competitive ratio of A on road map $(G_k, E_{k,*})$ is greater than $k\beta_k + 1$. According to Theorem 5, sequence $(\beta_k)_{k\geq 2}$ grows and tends to β. So, there exists an index $l \geq 1$ such that, for any $k \geq l$, $\beta_k > \alpha$. As $c_A \geq k\beta_k + 1 > c_{A'}$ for $k \geq l$, A' is more competitive than A which is a contradiction. ■

5 Conclusion and further work

We studied the competitiveness of the MSes for the k-CTP. In this context, an MS is a strategy which does not make decisions referring to the anterior moves of the traveller, in other words, the nodes the traveller visited until his current position. We proved that the COMPARISON strategy, defined in [8], is a MS. This implies that the competitive ratio of the optimal deterministic MS is $2k + 1$.

Then, we constructed a series of k-CTP instances, called road atlases and noted $\mathcal{S}_k$. Identifying an efficient randomized MSes becomes harder when k tends to infinity. We foremost concluded that a randomized MS cannot reach a competitive ratio smaller than $\frac{10}{9}k + 1 = 3.22(2)$ on road atlas $\mathcal{S}_2$. We extended our reasoning to larger atlases and established that the competitive ratio of randomized MSes on maps with k blockages is larger than $\left(3 - \sqrt{3}\right) k + 1 \approx 1.268k + 1$. That is to say that we identified an upper bound on the competitive ratio of randomized MSes which is significantly higher than the existing one $k + 1$. Now, if the objective is to find a strategy with a competitive ratio smaller than $k\beta + 1$, one shall study strategies which are not memoryless.

The main issue that, in our opinion, needs further investigation, is the improvement of the new lower bound. There are, potentially, two ways to do this. The first one is the modification of road atlases $\mathcal{S}_k$ to make MSes be less efficient on them. The second possibility consists in increasing the lower bound of the competitive ratio by introducing new properties of MSes, more advantageous for bounding, on the road atlas $\mathcal{S}_k$. Other possible future work could be to design a randomized strategy with ratio less than $1.268k + 1$ and which is not memoryless, or to design a family of maps such that there exists a randomized strategy which, applied on it, has a ratio $k + 1$.

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A Proof of Theorem 4

We present a transitional result which allows to introduce next the proof of Theorem 4.

Lemma 1 *The relationship existing between the mean distance $c_A^{down}(G'_k, E_{k,*}\backslash E')$ when the traveller when he goes* down *on map $(G'_k, E_{k,*}\backslash E')$ and the mean distance over the entire map $(G'_{k-1}, E^*_{k-1}\backslash E')$ is:*

$$c_A^{down}\left(G'_k, E_{k,*}\backslash E'\right) < 2 + c_A\left(G'_{k-1}, E^*_{k-1}\backslash E'\right)$$

Proof. We suppose that this result is false. Value $c_A^{\text{down}}(G'_k, E_{k,*}\backslash E')$ gives the mean distance traversed by the traveller if he goes *down* on road map $(G'_k, E_{k,*}\backslash E')$. Even if the probability to reach the target t without being blocked is positive, we suppose that the traveller goes down and is blocked by edge $e^*_{k,j}$ with $j > 1$. The road map is now:

$$\left(G'_k\backslash\left\{e^*_{k,j}\right\}, E_{k,*}\backslash\left(E'\cup\left\{e^*_{k,j}\right\}\right)\right).$$

Let us imagine for a moment that we transform A into another strategy A'. We impose that there is a bijection $\rho : E^*_{k-1}\backslash E' \to E_{k,*}\backslash\left(E'\cup\left\{e^*_{k,j}\right\}\right)$ such that the probability to discover the l^{th} blocked edge e^* in the ascending order of indices in road map $\left(G'_{k-1}, E^*_{k-1}\backslash E'\right)$ is equal to the probability to discover the l^{th} edge $\rho(e^*)$ in the ascending order of indices in road map $\left(G'_k\backslash\left\{e^*_{k,j}\right\}, E_{k,*}\backslash\left(E'\cup\left\{e^*_{k,j}\right\}\right)\right)$. With strategy A', there are two possibilities when the traveller goes *down*: either he reaches t with distance 1, or he is blocked by edge $e^*_{k,j}$ and traverses a distance $2+c_A\left(G'_{k-1}, E^*_{k-1}\backslash E'\right)$. Bijection ρ makes two scenarios be equivalent: going *up* and being blocked, going *down* and being blocked. But, when he goes *down*, the traveller has a chance of reaching t without being blocked. As a consequence, $c_{A'}^{\text{down}}(G'_k, E_{k,*}\backslash E') < 2 + c_A\left(G'_{k-1}, E^*_{k-1}\backslash E'\right)$. So, $c_{A'}^{\text{down}}(G'_k, E_{k,*}\backslash E') < c_A^{\text{down}}(G'_k, E_{k,*}\backslash E')$ which makes strategy A' be more competitive than A. ■

Theorem 6 *There is a solution of system Π_k, i.e. probabilities p_A^{down} and value β_k^*, such that, for any $G' = G_j\backslash E'$ with $j \leq k$ and $E' \subseteq E_{j,*}$, $p_A^{down}(G') = p_A^{down} \in [0;1]$.*

Proof. Our first objective is to prove that, for any graph $G' = G_j\backslash E'$, $\frac{\partial c_A\left(G_k, E_{k,*}\right)}{\partial p_A^{\text{down}}(G')} \leq 0$. This result means that if the probability to go *down* increases, the competitive ratio decreases and we have more chance to reach t with a smaller distance. From Equation 3 we obtain:

$$\frac{\partial c_A\left(G'_k, E_{k,*}\backslash E'\right)}{\partial p_A^{\text{down}}(G')} = -2 - c_A\left(G_{k-1}\backslash E', E_{k-1,*}\backslash E'\right) + c_A^{\text{down}}\left(G', E_{k,*}\backslash E'\right)$$

According to Theorem 3, initially, $\frac{\partial c_A(G_2, E_{2,*})}{\partial p} = -\frac{1}{2}$. By induction, we suppose that $\frac{\partial c_A\left(G_{k-1}, E_{k-1,*}\right)}{\partial p_A^{\text{down}}(G')} \leq 0$. According to Lemma 1, we obtain $\frac{\partial c_A\left(G', E_{k,*}\backslash E'\right)}{\partial p_A^{\text{down}}(G')} \leq 0$ and it implies that $\frac{\partial c_A^{\text{down}}\left(G_k, E_{k,*}\right)}{\partial p_A^{\text{down}}(G')} \leq 0$. Referring to Equation 2, we get our result.

Now, fixing k, we note $p_A = \max_{G'=G_j\backslash E'} p_A^{\text{down}}(G')$. Let us propose another strategy A' such that, for any G', $p_A^{\text{down}}(G') = p_A^{\text{down}}$. Inequality $2p_{A'}^{\text{down}}(G') \leq \beta_k$ is still valid. Moreover, as $\frac{\partial c_A\left(G_k, E_{k,*}\right)}{\partial p_A^{\text{down}}(G')} \leq 0$, value $c_{A'}(G_k, E_{k,*})$ is smaller than with strategy A, as $p_{A'}^{\text{down}}(G')$ increases. Formula $c_{A'}(G_k, E_{k,*}) \leq k\beta_k + 1$ still holds. Finally, there exists a solution such that $p_A^{\text{down}}(G') = p_A^{\text{down}} \in [0;1]$ and which does not increase value β_k. ■

B Analysis of sequence $(p_k)_{k\geq 2}$

Polynomials Q_k and Q_{k+1} are decreasing in the interval $[0,1]$. Furthermore, $Q_k(0) = \frac{3k-1}{2} > 0$ whereas $Q_k(1) < \frac{3k-1}{2} - 2k \leq 0$. Value p_k is the unique zero of Q_k in the interval $[0,1]$. We introduce $R_k(p)$:

$$R_k(p) = Q_{k+1}(p) - Q_k(p) = \frac{3}{2} - 2p - \sum_{i=1}^{k-1} \frac{p^i}{2^{i+1}} - \frac{p^k}{2^k},$$

and observe a recursive relation between $R_{k+1}(p)$ and $R_k(p)$:

$$R_{k+1}(p) = R_k(p) + (1-p)\frac{p^k}{2^{k+1}}. \tag{6}$$

The formula above implies that $R_{k+1}(p) \geq R_k(p)$. We show by induction that $p_{k+1} \geq p_k$. This is true for $k = 1$: $Q_1(p) = 1 - 2p$ which leads to $p_1 = \frac{1}{2}$ and $Q_2(p) = \frac{5}{2} - \frac{9}{2}p$ which leads to $p_2 = \frac{5}{9}$.

We suppose that $p_{k+1} \geq p_k$. Our goal is to prove that $p_{k+2} \geq p_{k+1}$. We note y_k the value in the interval $[0,1]$ such that $Q_{k+1}(y_k) = Q_k(y_k)$, which is equivalent to $R_k(y_k) = 0$. According to Eq. (6), $R_{k+1}(y_k) \geq 0$ which means that $Q_{k+2}(y_k) \geq Q_{k+1}(y_k) = Q_k(y_k)$. Furthermore, for any $0 \leq y \leq y_k$, $R_{k+1}(y) \geq R_k(y) \geq 0$. As a consequence, we obtain that, for any $0 \leq y \leq y_k$, $Q_{k+2}(y) \geq Q_{k+1}(y)$ holds. Obviously, the zero of function Q_{k+2} is greater than the zero of function Q_{k+1}, so $p_{k+2} \geq p_{k+1}$.

Sequence $(p_k)_{k\geq 2}$ is bounded and grows: it converges to a value p^*. When k tends to infinity, $Q_k(p^*) \to 0$ and $R_k(p^*) \to 0$. For any $0 \leq p \leq 1$,

$$R_k(p) \to 2 - 2p - \frac{1}{2-p} = \frac{2p^2 - 6x + 3}{2-p}.$$

Eventually, $R_k(p^*) \to 0$ iff $p^* = \frac{3-\sqrt{3}}{2}$.