Canadians should use memory

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Abstract

We establish that the competitive ratio of any randomized memoryless strategy is not less than $2k+1$. Only randomized strategies using memory potentially overpass this ratio now.

1 Definitions

We start by introducing the notation. For any graph $G = (V, E, \omega)$, let $G \backslash E'$ denotes its subgraph $(V, E \backslash E', \omega)$. If P is a path, we note its cost as $\omega(P) = \sum_{e \in P} \omega(e)$.

1.1 Memoryless Strategies for the k-CTP

We remind the definition of CTP. Let $G = (V, E, \omega)$ be an undirected graph with positive weights. The objective is to make a traveller traverse the graph from a source node s to a target one t, with $s, t \in V$. There is a set $E_* \subsetneq E$ of blocked edges. The traveller does not know a priori which edges are blocked. He discovers a blocked edge only when arriving to one of its endpoints. For example, if (v, w) is a blockage he will discover it when arriving to v (or w). The goal is to design the strategy A with the minimal competitive ratio.

We focus on *memoryless strategies* (MS). Concretely, we suppose that the traveller remembers the blocked edges he has discovered but forgets the nodes which he has already visited. In other words, a decision of an MS is independent of the nodes already visited. In the literature, the term *memoryless* was used in the context of online algorithms (e.g. PAGING PROBLEM [**?**], LIST UPDATE PROBLEM [**?**]) which take decisions according to the current state, ignoring past events. An MS can be either deterministic or randomized.

Definition 1 (Memoryless Strategies for the k-CTP) *A deterministic strategy A is an MS if and only if (iff) the next node w the traveller visits depends on graph G deprived of blocked edges already discovered E'_* and the current traveller position v: $w = A(G \backslash E'_*, v)$. Similarly, a randomized strategy A is an MS iff node w is the realization of a discrete random variable $X = A(G \backslash E'_*, v)$.*

MSes are easy to be implemented because they do not use past moves to take a decision. For the same reason, they do not need any extra memory either.

For example, the GREEDY strategy [**?**] is an MS. It consists in choosing at each step the first edge of the shortest path between the current node v and the target t. This strategy, illustrated

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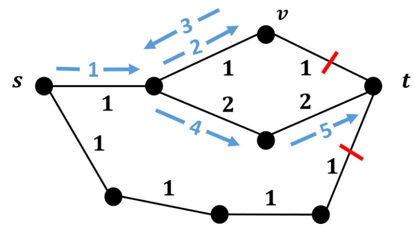


Figure 1: Illustration of the GREEDY strategy on a graph with 2 blocked edges

in Figure 1, does not refer to anterior moves. In contrast, the REPOSITION strategy [?] is not an MS as any decision refers to the past moves of the traveller.

We propose the following process to identify whether a strategy A is a deterministic MS. Let us suppose that a traveller T_1 executes strategy A: he has already visited certain nodes of the graph, he is currently at node v but he has not reached target t yet. Let us imagine a second traveller T_2 who is airdropped on node v and starts applying strategy A. If the traveller T_2 always follows the same path as T_1 until reaching t, A is a deterministic MS. If T_1 and T_2 may follow different paths, then A is not an MS. Formally, prooving that a strategy is an MS consists in finding the function which transforms the pair $(G \backslash E_*, v)$ into node $w = A\left(G \backslash E'_*, v\right)$.

1.2 Competitive ratio

Let (G, E_*) be a *road map*, *i.e.* a pair with graph $G = (V, E, \omega)$ and blocked edges $E_* \subsetneq E$, such that there is an (s,t)-path in graph $G \backslash E_*$ (nodes s and t remain in the same connected component when all blocked edges are discovered). We note $\omega_A\left(G, E_*\right)$ the distance traversed by the traveller reaching t with strategy A on graph G with blocked edges E_* and $\omega_{\min}\left(G, E_*\right)$ the cost of the shortest (s,t)-path in graph $G \backslash E_*$.

The ratio $\omega_A\left(G, E_*\right) / \omega_{\min}\left(G, E_*\right)$ is abbreviated as $c_A\left(G, E_*\right)$. A strategy A is c_A-competitive [?, ?] iff for any $\left(G, E_*\right), \omega_A\left(G, E_*\right) \leq c_A \omega_{\min}\left(G, E_*\right)$. Otherwise stated, for any (G, E_*), $c_A\left(G, E_*\right) \leq c_A$. If strategy A is randomized, $\omega_A\left(G, E_*\right)$ is replaced by $\mathbb{E}\left(\omega_A\left(G, E_*\right)\right)$ which is the expected distance traversed by the traveller to reach t with strategy A. The competitive ratio can also be evaluated on a family $\mathcal{R}$ of road maps, put formally,

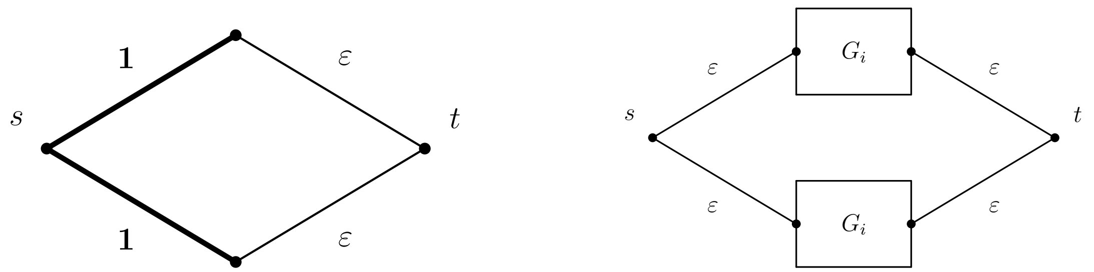
$$c_{A,\mathcal{R}} = \max_{(G,E_*) \in \mathcal{R}} c_A\left(G, E_*\right) \tag{1}$$

This "local" competitive ratio fulfils $c_{A,\mathcal{R}} \leq c_A$. The competitive ratio can also be extended to families of strategies. We note c_{MS} the competitive ratio of MSes, which is the minimum over competitive ratios of any MSes: $c_{\text{MS}} = \min_{A \text{ MS}} c_A$. In the remainder, we identify a set of road maps $\mathcal{R}$ such that the competitive ratio of MSes on it is $2k+1$, *i.e.* $c_{\text{MS},\mathcal{R}} \geq 2k+1$.

1.3 Graphs G_k

We define recursively a sequence of graphs G_i for $i \geq 1$ with weights from $\{1, \varepsilon\}$, $0 < \varepsilon \ll 1$. Graphs G_1 and G_{i+1} are represented in Figures 2a and 2b, respectively. Edges with weight 1 are thicker than edges with weight ε. For every graph G_k, we will only consider the road maps where the blocked edges are at the right of the graph, the left edges affecting negligibly the total cost of a traveler. We note E_{right} the set of all the right edges.

When the traveller discover blocked edges in G_k, parts of the graph become *dead ends*. If a traveller visits a dead end, the only chance for him to reach t is to leave it anyway. Dead ends



(a) Graph G_1 (b) Graph G_{i+1}

Figure 2: Recursive construction of graphs G_i

only provide to the traveller additional costs which make the competitive ratio increase. As a consequence, the most competitive strategies will not visit dead ends. From now on, as soon as a dead end appear when the traveller discovers a blockage, we make it disappear from the graph: graph $G \backslash E'$ becomes graph G deprived of both edges E' and dead ends appeared when edges E' are deleted. Let $\mathcal{G}_k$ denote the set of all sub-graphs of G_k where at most k edges have been removed. Formally, $\mathcal{G}_k = \{G_k \backslash E' : |E'| \leq k\}$.

1.4 Equivalent binary trees

We define for each graph of $\mathcal{G}_k$ an equivalent binary tree, which will represent all changes of direction from t. We note $T_\varnothing$ the empty tree, and (v, T_a, T_b) a non empty tree with $v \in V$, T_a the binary tree above and T_b the binary tree below.

We obtain the equivalent binary tree by applying the BIN-TREE algorithm. This algorithm take as an entry a directed tree obtained by first removing the left part of the graph and then directed all edges from t. Then, the BIN-TREE algorithm will construct a binary tree according to the outdegree of each node.

Algorithm 1: BIN-TREE algorithm

Data: a directed tree T and a node v
Result: binary tree
if $deg^-(v) = 0$ **then return** $T_\varnothing$;
if $deg^-(v) = 1$ **then return** BIN-TREE(T, v_{next}) ;
if $deg^-(v) = 2$ **then return** (v, BIN-TREE(T, v_{up}), BIN-TREE(T, v_{down}) ;

Given an edge in a binary tree, we will note $A(e)$ the child edge from above and $B(e)$ the child edge from below. We also note $C(e)$ the set of all the descendants of e. Formally, $C(e) = C(A(e)) \cup C(B(e)) \cup \{A(e), B(e)\}$. Finally, we note $P(e)$ the parent edge of e : $P(e) = e' \Leftrightarrow e = A(e')$ or $e = B(e')$.

We define recursively a set of edges E_k, E_0 being the edges connected to the root and $E_k = \{e : P(e) \in E_{k-1}\}$. E_k represents the edges which are at distance k from the root.

When an edge is cut from the original graph, the equivalent tree can be easily obtain by replacing the tree (v, T_a, T_b) by T_a (resp. T_b) if the cut edge is before T_b (resp. T_a)

Lemma 1 *For any road map $(G, E_*) \in \mathcal{R}_k$, $E_{T_G}(j-1) = 2^j$ with $j = |E_*|$*

Proof. If $G = G_k$, then $|E_*| = k$ and the binary tree is complete, so $E_{T_{G_k}}(k-1) = 2^k$.

We suppose now for any road map with $|E_*| = j$ that $E_{T_G}(j-1) = 2^j$. Let (G, E_*) a road map with $|E_*| = j-1$ Then it exists a road map $(G', E_* \cup \{e\})$ with G obtained by removing e to G'.

First, we remark that for any binary tree $T = (v, T_a, T_b)$ and i, if the number of edge of depth i is 2^{i+1}, the tree is complete until depth i, so T_a and T_b are complete until depth $i-1$ and each have 2^i edges at depth $i-1$: $|E_T(i)| = 2^{i+1} \implies |E_{T_a}(i-1)| = |E_{T_b}(i-1)| = 2^i$.

Second, we remark that for any binary tree T and i, if the number of edge of depth i is 2^{i+1}, then the number of edge for the depth $i-1$ is 2^i : $|E_T(i)| = 2^{i+1} \implies |E_T(i-1)| = 2^i$.

Let $T' = (v, T_a, T_b)$ the sub-tree of T'_G where v is the root of the edge e (we will suppose e is on the T_a side). Then if e is at depth i then the remarks and the hypothesis give $|E_T(j-i-1)| = 2^{j-i}$ and $|E_T(j-i-2)| = 2^{j-i-1} = |E_{T_a}(j-i-2)|$. By cutting e, T become in G the tree T_a. We find that the number of edge at the depth $j-i-2$ remains. So the rest of the tree being unchanged by cutting e, the number of edges of T_G for depth $j-2$ is unchanged from T'_G and is equal to 2^{j-1} : $|E_{T_G}(j-2)| = 2^{j-1}$. ∎