Implementing a checker for propositional logic in Circom

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Overview of the talk

- Arithmetic circuits and R1CS constraints
- Designing R1CS constraint systems using Circom
- Randomized (non-deterministic) programming
- Current implementation of Propositional Logic in Circom
    - Hilbert proofs
    - Arithmetization of terms/patterns/formulæ as Reed-Solomon polynomials
    - Well-formedness checking
    - Modus-ponens
    - Axioms and instantiation of (meta)variables

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Rank-1 Constraint Systems

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R1CS Formulations (programmer style)

- Constant values: only a constant value is allowed.
- Linear expression: an expression where only addition is used.
  - It can also be written using multiplication of variables by constants.
  - $2 * x + 3 * y + 2$ is allowed, as it is equivalent to $x + x + y + y + y + 2$.
- Quadratic expression: multiplication between two linear expressions and addition of a linear expression:
  - $A * B - C$, where A, B and C are linear expressions.
  - For instance, $(2 * x + 3 * y + 2) * (x + y) + 6 * x + y\breve{}2$.
- Quadratic constraint: $A * B - C = 0$, where A, B and C are linear expressions.

Source: (Documentation 2024)

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R1CS Formulations (mathematical style)

A **system of rank-1 quadratic equations** over $\mathbb{F}$ is a tuple $\mathcal{S} = \left((a_j, b_j, c_j)_{j \in \overline{1, N_g}}, n\right)$ where $a_j, b_j, c_j \in \mathbb{F}^{1+N_w}$ and $n \leq N_w$

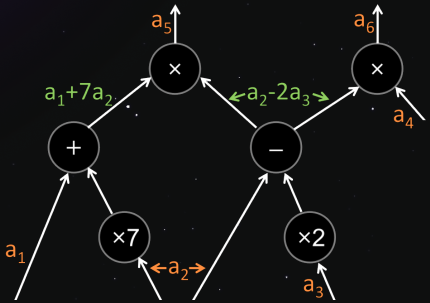
- N_g - the number of constraints
- N_w - the number of variables
- n - the input size

$\mathcal{S}$ is satisfiable with an input $x \in \mathbb{F}^n$ if there is a witness $w \in \mathbb{F}^{N_w}$ such that:

1. $x = (w_1, \ldots, w_n)$, and
2. $\langle a_j, (1, w)\rangle \cdot \langle b_j, (1, w)\rangle = \langle c_j, (1, w)\rangle forall j \in \overline{1, N_g}$.

Source (Ben-Sasson et al. 2013; Gennaro et al. 2012)

Arithmetic circuits to R1CS



An arithmetic circuit.
The value at each output wire of a multiplication gate is expressed in terms of the values of output wires of lower multiplication gates (or of input wires) (Gennaro et al. 2012).

$$(a_1 + 7a_2) \cdot (a_2 - 2a_3) = a_5$$
$$(a_2 - 2a_3) \cdot a_4 = a_6$$

R1CS: $N_g = 2, N_w = 6, n = 4$

$$(0, 1, 7, 0, 0, 0, 0), (0, 0, 1, -2, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0)$$
$$(0, 0, 1, -2, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 0, 0, 1)$$

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Programming in Circom

Circom

- A JavaScript-like language with extra support for signals and constraints
- The compiler checks that signals are used and constrained appropriately
  - no non-quadratic constraints
  - no conditionals based on signals
  - no unbounded loops for signals
- One can use the regular language to compute the witness
  - conditional and unbounded loops are allowed
  - one should constrain/check the witness using R1CS constraints

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Signals

- The “circuit” variables of the language
- Correspond to output wires of arithmetic circuit gates
- When assigned to, one should also generate the corresponding constraint(s)

  E.g., `a6 <== (a2 - 2 * a3) * a4`
- However, we can also assign them directly and specify the constraint(s) separately
  - especially if we want to assign them to non-quadratic expressions

```
b <-- 1/a;
a * b === 1;
```

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Compositionality via templates (Iden3 2024)

```
template AND() {
    signal input a;
    signal input b;
    signal output out;

    out <== a*b;
}
template OR() {
    signal input a;
    signal input b;
    signal output out;

    out <== a + b - a*b;
}
```

```
template IsZero() {
    signal input in_;
    signal output out;
    signal inv;

    inv <-- in_!=0 ? 1/in_ : 0;
    out <== -in_*inv +1;
    in_*out === 0;
}
```

- if `in_` is `0`, then `out=1` uniquely satisfies both equations regardless of `inv`
- if `in_` is not `0` then there exists `inv` s.t. `in_ * inv = 1` and `out=0` uniquely satisfies both equations

Template Recursion

```
template MultiAND(n) {
    signal input in[n];
    signal output out;
    component and1;
    component and2;
    component ands[2];
    if (n==1) {
        out <== in[0];
    } else if (n==2) {
        and1 = AND();
        and1.a <== in[0];
        and1.b <== in[1];
        out <== and1.out;
    } else {
        and2 = AND();
        var n1 = n\2;
        var n2 = n-n\2;
        ands[0] = MultiAND(n1);
        ands[1] = MultiAND(n2);
        var i;
        for (i=0; i<n1; i++)
          ands[0].in[i] <== in[i];
        for (i=0; i<n2; i++)
          ands[1].in[i] <== in[n1+i];
        and2.a <== ands[0].out;
        and2.b <== ands[1].out;
        out <== and2.out;
    }
}
```

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Conditional

```
template If() {
    signal input condition;
    signal input t;
    signal input e;
    signal els <== (1 - condition) * e;
    signal output result <== condition * t + els;
}
```

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Conversion to bits

```
// Converts felt into N-digit binary representation
// Fails if N is too small
template Num2Bits(N) {
    signal input in;
    signal output out[N];
    var lc1=0;

    var e2=1;
    for (var i = 0; i<N; i++) {
        out[i] <-- (in >> i) & 1;
        out[i] * (out[i] -1 ) === 0;
        lc1 += out[i] * e2;
        e2 = e2+e2
    }
    lc1 === in;
}
```

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Other functions: Less than and exponent

```
template LessThan(n) {
  assert(n <= 252);
  signal input in[2];
  signal output out;

  component n2b = Num2Bits(n+1);
  n2b.in <== in[0]+ (1<<n) - in[1];
  out <== 1-n2b.out[n];
  // checks that no borrowing
  // is needed for subtraction
}
```

```
// multiplies the elements in vector in selected
// by bits which should be a vector of 0/1
template ConditionalProduct(w) {
  signal input in[w];
  signal input bits[w];
  signal output out;
  signal aux[w];
  var lc = 1;
  for (var i=0; i<w; i++) {
    aux[i] <== (1 - bits[i]) + bits[i] * in[i];
  }
  out <== MultiAND(w)(aux);
}
```

Exponentiation

```
// Compute b ^ e in O(N = log e) gates
template BinaryExp(N) {
    signal input b;
    signal input e;
    signal pb[N];
    signal bits[N] <== Num2Bits(N)(e);
    signal output res;

    for (var i = 0; i < N; i++) {
        pb[i] <== i == 0 ? b : pb[i - 1] * pb[i - 1];
    }

    res <== ConditionalProduct(N)(pb, bits);
}
```

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Maximum value in an array

```
function array_max(N, A) {
    // returns the maximum value
    // in the array A of size N
    var m = 0;
    for (var i = 0; i < N; i ++) {
        if (A[i] > m) m = A[i];
    }
    return m;
}
```

- used to guess the maximum value as a witness
- no signals/constraints involved in `array_max`

```
template ArrayMax(N) {
    signal input A[N];
    signal output m <-- array_max(N, A);
    var m1 = m + 1;
    signal checks[N+1], diffs[N+1];
    diffs[0] <== 1;
    for (var i = 0; i < N; i ++) {
      checks[i] <== LessThan(64)([A[i], m1]);
      diffs[i+1] <== diffs[i] * (A[i] - m);
    }
    checks[N] <==  IsZero()(diffs[N]);
    signal ok <== MultiAND(N + 1)(checks);
    ok === 1;
}
```

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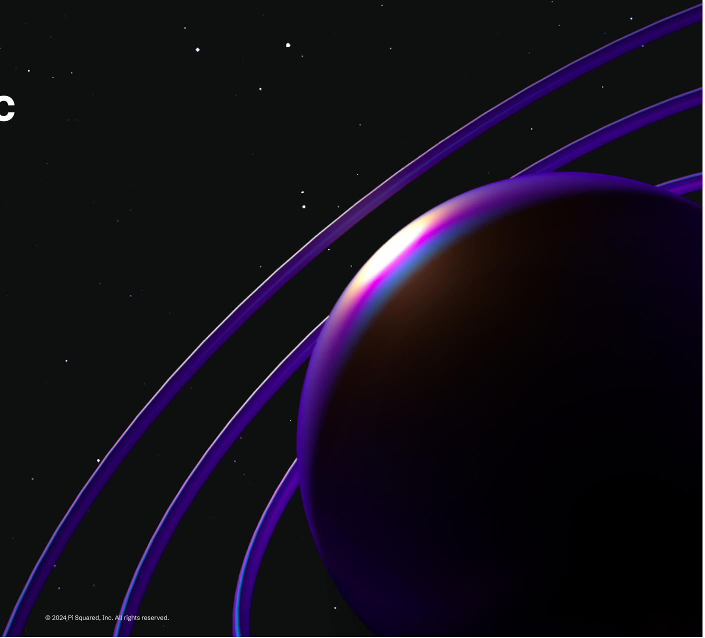
Propositional Logic



Syntax

Formulæ

- $\varphi ::= x \mid \bot \mid \varphi \to \varphi$
- $x ::= \phi_0 \mid \phi_1 \mid \phi_2 \mid \ldots$

Axioms (Łukasiewicz's third axiom system)

- ***Prop1***: $\phi_0 \to (\phi_1 \to \phi_0)$
- ***Prop2***: $(\phi_0 \to (\phi_1 \to \phi_2)) \to ((\phi_0 \to \phi_1) \to (\phi_0 \to \phi_2))$
- ***Prop3***: $((\phi_0 \to \bot) \to (\phi_1 \to \bot)) \to (\phi_1 \to \phi_0)$

Deduction

- ***ModusPonens***: $\dfrac{\varphi, \varphi \to \psi}{\psi}$

Instantiation

- ***Instantiate***: Substituting variables with formulæ

Notations

- $\neg\varphi ::= \varphi \to \bot$
- $\varphi \wedge \psi ::= \neg(\varphi \to \neg\psi)$
- $\varphi \vee \psi ::= \neg\varphi \to \psi$

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Hilbert Proofs for $\phi_0 \to \phi_0$

Proofs of pattern well-formedness:

1. ϕ_0 by HMetaVar(0)
2. $(\phi_0 \to \phi_0)$ by HImplies(1, 1)

Propositional Logic Proof:

1. $((\phi_0 \to ((\phi_0 \to \phi_0) \to \phi_0)) \to ((\phi_0 \to (\phi_0 \to \phi_0)) \to (\phi_0 \to \phi_0)))$
   by Instantiate(Prop2, $[\phi_1 \mapsto (\phi_0 \to \phi_0), \phi_2 \mapsto \phi_0, \phi_0 \mapsto \phi_0]$)
2. $(\phi_0 \to ((\phi_0 \to \phi_0) \to \phi_0))$ by Instantiate(Prop1, $[\phi_1 \mapsto (\phi_0 \to \phi_0), \phi_0 \mapsto \phi_0, \phi_2 \mapsto \phi_2]$)
3. $((\phi_0 \to (\phi_0 \to \phi_0)) \to (\phi_0 \to \phi_0))$ by ModusPonens(1, 2)
4. $(\phi_0 \to (\phi_0 \to \phi_0))$ by Instantiate(Prop1, $[\phi_1 \mapsto \phi_0, \phi_0 \mapsto \phi_0, \phi_2 \mapsto \phi_2]$)
5. $(\phi_0 \to \phi_0)$ by ModusPonens(3, 4)

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Arithmetization of formulæ

- Running example: $\phi_0 \vee \neg\phi_0$
- Desugar notations: $(\phi_0 \to \bot) \to (\phi_0 \to \bot)$
- Consider the prefix-form of the formula $\to\to \phi_0\bot \to \phi_0\bot$
- Associate numbers to each token
  - say $\to$ is 1
  - say $\bot$ is 2
  - say variables are numbered consecutively from 3, i.e. ϕ_0 is 3

Then we can associate to the formula the following vector of size 7:

$[1, 1, 3, 2, 1, 3, 2]$

Reed-Solomon fingerprint

Given a vector $(v_i)_{i=\overline{1,M}}$, the Reed-Solomon polynomial associated to v is $p(x) = \sum_{i=1}^{M} x^{i-1} * v_i$.

The Reed-Solomon fingerprint of v is the value of p at a random (but fixed) point r.

```
template Fingerprint(M) {
    signal input r;
    signal input pattern[M];
    signal r_pow[M];
    r_pow[0] <== 1;
    for (var i = 1; i < M; i++) {
        r_pow[i] <== r_pow[i - 1] * r;
    }
    signal output fingerprint <== EscalarProduct(M)(r_pow, pattern);
}
```

Arithmetization of formulæ (continued)

- We fix a randomly chosen number r
- We encode a formula φ as a pair $FP(\varphi) ::= (n, N)$ where:
  - n is the length of φ in its prefix representation
  - N is the fingerprint obtained by evaluating at point r the Reed-Solomon polynomial associated to the vector corresponding to φ
- E.g., assuming $r = 10$ and a large enough prime-field, $FP(\phi_0 \vee \neg\phi_0) = (7, 1132132)$

Properties

- Equality checking: $\varphi = \psi$ can be replaced by $FP(\varphi) = FP(\psi)$
- Concatenation
  - Assume $FP(\varphi) = (n, N)$ and $FP(\psi) = (m, M)$
  - Then $FP(\varphi \to \psi) = (n + m + 1, 1 + r * (N + r^n * M))$

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Checking Well-Formedness

As we saw before, we can consider a deduction system for well-formedness, where:

- **Bottom** is an axiom
- **MetaVar** is an axiom (parameterized by the index of the variable)
- **Implies** is a deduction rule $\dfrac{\varphi, \psi}{\varphi \to \psi}$

In a Hilbert proof using this system:

- we can consider that each step outputs a well-formed pattern
- We can thus consider Implies to (also) be parameterized by the previous indices (in the proof) where its hypotheses were established.

Proofs of pattern well-formedness for $\phi_0 \to \phi_0$:

1. ϕ_0 by HMetaVar(0)
2. $(\phi_0 \to \phi_0)$ by HImplies(1, 1)

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Arithmetization of Well-Formedness proofs

- To each proof step we associate
  - an identifier of the axiom/rule used
  - a list of parameters (padded with zeros)
- **Bottom**: identifier 1, no parameters
- **MetaVar**: identifier 2, index of the variable
- **Implies**: identifier 3, (n, N, i), (m, M, j) where:
  - (n, N) is the fingerprint of the formula at index i in the proof
  - (m, M) is the fingerprint of the formula at index j in the proof
- *Note:* the conclusion of each step will be computed during checking

Arithmetization of the proof of well-formedness for $\phi_0 \to \phi_0$:

1. ϕ_0 by HMetaVar(0)	1. (2, 0, 0, 0, 0, 0, 0)
2. $(\phi_0 \to \phi_0)$ by HImplies(1, 1)	2. (3, 1, 3, 1, 1, 3, 1)

The checking process

Given a row (at index i) corresponding to arithmetization of a proof step:

- use first argument to determine which rule was used
- use the parameters to compute the FP (n, N) of the conclusion according to this rule
- emit (n, N, i) as a proved statement of well-formedness
- if rule has hypotheses (Implies)
  - emit (n_h, N_h, i_h) as a hypothesis of well-formedness
  - check that $i_h < i$ (to avoid cyclic proofs)

Finally, check that all emitted hypotheses are included in all emitted proved statements.

Example

1. (2, 0, 0, 0, 0, 0, 0) — a MetaVar rule with variable of index 0
   - emit $(1, 3, 1)$ as proved statement
2. (3, 1, 3, 1, 1, 3, 1) — an Implies rule with arguments $(1, 3, 1), (1, 3, 1)$
   - emit $(3, 133, 2)$ as proved statement; emit $(1, 3, 1)$ twice as hypotheses

Finally, check that $[(1, 3, 1), (1, 3, 1)]$ is included in $[(1, 3, 1), (3, 133, 2)]$

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Well-formedness of implication

```
template WfBinaryConstructor(LogN, ConstructorId) {
    signal input index, h_a, len_a, idx_a, h_b, len_b, idx_b;
    signal input r; // evaluation point
    signal r_pow_len_a <== BinaryExp(LogN)(r, len_a);
    signal r_pow_len_b <== BinaryExp(LogN)(r, len_b);
    var check_idx_a = CheckIndex(LogN)(idx_a, index);
    var check_idx_b = CheckIndex(LogN)(idx_b, index);
    signal ab <== h_a + r_pow_len_a * h_b;
    signal h_ab <== ConstructorId + r * ab;
    signal len_ab <== len_a + len_b + 1;
    signal output o1 <== PatternAtIndexRelFp()(h_a, len_a, idx_a);
    signal output o2 <== PatternAtIndexRelFp()(h_b, len_b, idx_b);
    signal output p <== PatternAtIndexRelFp()(h_ab, len_ab, index);
    signal output correctness <== AND()(check_idx_a, check_idx_b);
}
```

Subset argument

There are various subset arguments. Here we describe one based on logarithmic derivatives (Haböck 2022).

Input two vectors $A = (a_i)_{i \in \overline{1,n}}$, $B = (b_j)_{j \in \overline{1,m}}$
Simplifying assumption: A is a set (no repetitions).

Output whether all elements of B are also in A

- Let $M = (m_i)_{i=\overline{1,n}}$ be the multiplicities of A in B $m_i = |\{j \in \overline{1,m} \mid b_j = a_i\}|$.
- Let $p_A(X) = \sum_{i=1}^{n} \frac{m_i}{X - a_i}$
- Let $p_B(X) = \sum_{j=1}^{m} \frac{1}{X - b_i}$
- Output whether $p_A = p_B$
  - by evaluating p_A and p_B at a random point.

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Checking a Propositional Logic Proof

- **ModusPonens** is a deduction rule $\dfrac{\varphi, \varphi \to \psi}{\psi}$
- **Instantiate** allows substituting variables with formulæ in already proved terms
- we assume `1. Prop1; 2. Prop2; 3. Prop3` and shift the proof by the number of axioms.

In a Hilbert proof using this system:

- we can consider that each step outputs a well-formed pattern
- We can consider ModusPonens to (also) be parameterized by the previous indices (in the proof) where its hypotheses were established.

Proofs of pattern well-formedness for $\phi_0 \to \phi_0$:

4. Instantiate(2, $[\phi_1 \mapsto (\phi_0 \to \phi_0), \phi_2 \mapsto \phi_0]$)
5. Instantiate(1, $[\phi_1 \mapsto (\phi_0 \to \phi_0)]$)
6. ModusPonens(1, 2)
7. Instantiate(1, $[\phi_1 \mapsto \phi_0]$)
8. ModusPonens(3, 4)

Arithmetization of proofs

- Similar to well-formedness proofs
- **ModusPonens**: identifier 1, (n, N, i), (m, M, j) where:
  - (n, N) is the fingerprint of formula φ at index i in the proof
  - (m, M) is the fingerprint of a formula ψ
  - Formula $\varphi \to \psi$ is at index i in the proof
- **Instantiate**: identifier 2, (n, N, i), $(m, M, 0)$, P
  - (n, N) is the fingerprint of the base pattern, at index i in the proof
  - (m, M) is the fingerprint of the result pattern to be outputed
  - P is a hash of the fingerprints of the instantiated formulæ

Arithmetization of the PL proof for $\phi_0 \to \phi_0$:

4. Instantiate(2, $[\phi_1 \mapsto (\phi_0 \to \phi_0), \phi_2 \mapsto \phi_0]$)
5. Instantiate(1, $[\phi_1 \mapsto (\phi_0 \to \phi_0)]$)
6. ModusPonens(1, 2)
7. Instantiate(1, $[\phi_1 \mapsto \phi_0]$)
8. ModusPonens(3, 4)

4. (2, 13, 11314..., 2, 17, 11311..., 0, H1)
5. (2, 5, 13143, 1, 7, 1311333, 0, H2)
6. (1, 7, 1311333, 5, 9, 113133133, 4, 0)
7. (2, 5, 13143, 1, 5, 13133, 0, H3)
8. (1, 5, 13133, 7, 3, 133, 6, 0)

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The checking process

Similar (largely) to checking the well-formedness proof:

ModusPonens we emit *FP*(φ) with provided index as hypothesis
we compute *FP*($\varphi \to \psi$) and emit it with provided index as hypothesis
we emit *FP*(ψ) with current index as proved statement
we check that provided indices are smaller than current index

Instantiate we emit base pattern with provided index as hypothesis
we emit result pattern with current index as proved statement
we emit an “instantiation hypothesis”
we check that base pattern index is smaller than current index

Finally - check that all emitted hypotheses are included in all emitted proved statements. - axioms are pre-emitted - check that all emitted instantiate hypotheses are included in all emitted instantiate statements. - instantiate proofs will be detailed later.

The checking process: Example

4. (2, 13, 113145113413, 2, 17, 1131133311313133, 0, H1) - Instantiate, $2 < 4$
    - emit (13, 113145113413, 2) as hypothesis
    - emit (17, 1131133311313133, 4) as proved statement
5. (2, 5, 13143, 1, 7, 1311333, 0, H2) - Instantiate, $1 < 5$
    - emit (5, 13143, 1) as hypothesis
    - emit (7, 1311333, 5) as proved statement
6. (1, 7, 1311333, 5, 9, 113133133, 4, 0) - ModusPonens, $5 < 6, 4 < 6$
    - emit (7, 1311333, 5) and (17, 1131133311313133, 4) as hypotheses
    - emit (9, 113133133, 6) as proved statement
7. (2, 5, 13143, 1, 5, 13133, 0, H3) - Instantiate, $1 < 7$
    - emit (5, 13143, 1) as hypothesis
    - emit (5, 13133, 7) as proved statement
8. (1, 5, 13133, 7, 3, 133, 6, 0) - ModusPonens, $7 < 8, 6 < 8$
    - emit (5, 13133, 7) and (9, 113133133, 6) as hypotheses
    - emit (3, 133, 8) as proved statement

For Instantiate we also emit a hash of all parameters (no indices) as instantiation hypothesis

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Well-formedness of implication

```
template ModusPonens(LogN) {
    signal input h_a, len_a, idx_a, h_b, len_b, idx_ab;
    signal input index,  r;
    signal r_pow_len_a <== BinaryExp(LogN)(r, len_a);
    signal r_pow_len_b <== BinaryExp(LogN)(r, len_b);
    var check_idx_a = CheckIndex(LogN)(idx_a, index);
    var check_idx_ab = CheckIndex(LogN)(idx_ab, index);
    signal ab <== h_a + r_pow_len_a * h_b;
    signal h_ab <== implies_id() + r * ab;
    signal len_ab <== len_a + len_b + 1;
    signal output obligation1 <== PatternAtIndexRelFp()(h_a, len_a, idx_a);
    signal output obligation2 <== PatternAtIndexRelFp()(h_ab, len_ab, idx_ab);
    signal output proof <== PatternAtIndexRelFp()(h_b, len_b, index);
    signal output correctness <== AND()(check_idx_a, check_idx_ab);
}
```

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