

SELF-ADAPTIVE DENSITY ESTIMATION OF PARTICLE DATA

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Abstract. We present a study of density estimation, the conversion of discrete particle positions to a continuous field of particle density defined over a 3D Cartesian grid. The study features a methodology for evaluating the accuracy and performance of various density estimation methods, results of that evaluation for four density estimators, and a large-scale parallel algorithm for a self-adaptive method that computes a Voronoi tessellation as an intermediate step. We demonstrate the performance and scalability of our parallel algorithm on a supercomputer when estimating the density of 100 million particles over 500 billion grid points.

Key words. Density estimation, cloud in cell, smoothed particle hydrodynamics, Voronoi tessellation, nearest grid point, triangular shaped clouds

AMS subject classifications. 51-04, 68-04, 70-04, 85-04.

1. Introduction. In the context of scientific data analysis, density estimation is a transformation from discrete particle data to a continuous density function defined over a 3D field. This field can be interpolated, differentiated, and integrated: operations not possible in the particles’ original discrete form. Moreover, the density field is discretized by a regular grid, which offers several advantages. (1) It is compact, in that x, y, z coordinates of the grid points are defined implicitly and need not be stored. (2) Applications using a regular grid are readily parallelizable because subdomain decomposition and processor assignment are also regular and implicit. (3) A regular grid is the most common data model for scientific data analysis and visualization algorithms. For example, most of the volume rendering literature in the past 25 years targets regular grids [9, 24, 37]; far less exists for adaptively refined grids [23] and unstructured meshes [30].

Density estimation is a fundamental step needed whenever a discrete particle dataset is sampled over a continuous field. Our research is motivated by cosmology and astrophysics, but many other applications exist, for example estimating population density in geospatial applications or electron charge density in computational chemistry. Density estimation is also a key visualization step when an algorithm calls for a regular scalar field but particles are provided instead. For example, atom positions from molecular dynamics simulations may be converted to a grid before rendering an isosurface of atomic density.

One of the earliest and arguably most popular density estimators, cloud in cell (CIC), was introduced 45 years ago [7]. Since then, smoothed particle hydrodynamics (SPH) and tessellation (TESS) methods have emerged. Combinations and variations also exist: for example, characteristics such as the window shape of CIC and the adaptivity of SPH can be combined. Nevertheless, two questions remain unanswered: Which density estimators are appropriate for a given problem domain and computational budget, and how can the estimator of choice be scaled to today’s problem sizes? In this paper, we examine the first question by designing an evaluation methodology and showing how several estimators can be compared using it. Then, we show how a parallel algorithm can be constructed for one of those estimators.

We begin by explaining the common ideas in estimation methods such as window shape and weighting function; and we briefly explain the operation of four estimation methods, noting their strengths and weaknesses. After the background and related work, we present

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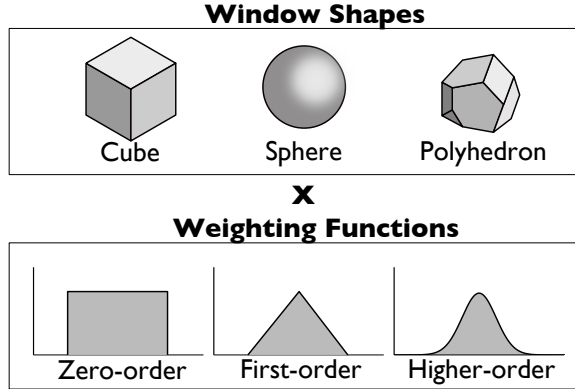


FIG. 1. Common window shapes and weighting functions in density estimation.

a methodology for designing synthetic datasets that mimic characteristics of actual data in a given application area. Our choice of design parameters is motivated by an application of density estimation in gravitational lensing to model the refraction of light rays caused by galaxies. The input data for lensing calculations comes from computational cosmology simulations. Using the datasets and test methodology we developed, we compare four density estimators with each other and with known analytical functions that underlie our particle data. We also derive the asymptotic time complexity of each of the four methods.

For our particular target of gravitational lensing, the TESS method turns out to be the best density estimator because it is relatively stable under small perturbations in particle sampling and therefore best captures the underlying density function. Because TESS needs no external parameters regarding window size or shape, it is entirely self-adaptive: the window over which a particle is deposited onto grid points adapts automatically to the size and shape of the Voronoi cell containing the particle. The paper concludes by describing a parallel algorithm for TESS that scales to high-performance computing (HPC) architectures; timing and scalability performance results are included.

Our work demonstrates that the choice of density estimator depends on characteristics of the target problem such as dynamic range, discontinuities, overall data size, and computational budget. Despite the importance of selecting an appropriate density estimator to capture the underlying density function accurately and efficiently, after four decades CIC still remains the de facto technique used in simulations and visualization tools. In fact, the only density estimator available in VTK [42], ParaView [1], and VisIt [10] is a serial CIC filter. Our study shows that CIC, while inexpensive to compute, is inaccurate for datasets such as ours with high dynamic range.

The key contributions of our work are threefold: (1) we describe a methodology for designing synthetic data and comparing estimators quantitatively and qualitatively; (2) we present the results of that comparison for synthetic data representative of N-body cosmology codes; and (3) we present a large-scale parallel algorithm for estimating the density field of 3D particle data using a 3D Voronoi tessellation to automatically adapt to the particle distribution.

2. Background and Related Work. In this paper, we refer to the input dataset of 3D points as *particles* and to the locations on the regular grid where the output density values are computed as *grid points*. It does not matter whether density values are vertex-centered (at the grid lines) or cell-centered (at the grid spaces); by convention, our terminology is vertex-

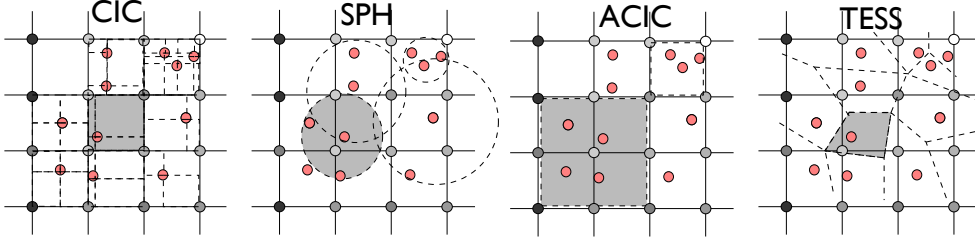


FIG. 2. Density estimation methods studied in this paper. In cloud in cell (CIC), particles are distributed to a fixed number of grid points, usually 8 in 3D, using first-order weighting. Smoothed particle hydrodynamics (SPH) use fixed-shape spherical windows whose size also adapts to contain the same number of particles; weighting is higher-order, usually Gaussian. In adaptive CIC (ACIC), the number of grid points adapts to encompass a fixed number of particles, and the weighting may be zero-, first-, or higher-order. In tessellation (TESS) methods, particles are distributed in windows of variable size and shape using zero-order weighting.

centered. We define *density estimation* as the process of distributing some property of the particles (we use particle mass in this paper, but the density of any property can be distributed) within an *estimation window* weighted according to a *weighting function*. Irrespective of the window and weighting, any estimation method must obey the conservation of physical properties; in our case, total particle mass is conserved.

2.1. Estimation Windows and Weighting Functions. We categorize different density estimators according to their estimation windows and weighting functions. The estimation window associated with a given particle is the region of grid points over which the particle’s mass is distributed. The weighting function determines the fraction of the particle mass deposited on each grid point in the window. Figure 1 shows three window shapes and three weighting functions. The window shape (cube, sphere, or polyhedron) may be either fixed or adaptive size. In this paper, the weighting function (zero-, first-, or higher-order) is a uniform distribution, linear interpolation, or Gaussian distribution, respectively. Window shapes and weighting functions can be combined in various ways; in this paper we examine four density estimation methods with various window shapes and weighting functions. Listed in Table 1, they are cloud in cell (CIC), smoothed particle hydrodynamics (SPH), adaptive cloud in cell (ACIC), and tessellation (TESS).

2.2. Density Estimators. Figure 2 shows four classes of **density estimators** studied in this paper; each class is identified by its window size and shape and weighting function as explained above. In addition to the four methods we studied, two others are mentioned at the end of this section, NGP and TSC.

2.2.1. Cloud in Cell Methods. CIC [7] is a first-order method that linearly interpolates the particle’s mass to neighboring grid points. CIC has a fixed size and shape window that encompasses eight grid points surrounding the particle. The weighting function distributes the mass of one particle to the eight adjacent grid points according to the distance from the particle to the grid points. Specifically, in 3D the mass of point P is distributed among the surrounding grid points G_0 through G_7 . If the volume of the grid cell with corners G_0, G_7 is $v(G_0, G_7) = 1.0$, then the mass assigned to grid point G_i is $m(G_i) = 1.0 - v(G_i, P)$. CIC is used in many simulations; astrophysics [21] and plasma fusion [32] are two examples. CIC is easy to compute, but undersampling in low-density regions with insufficient numbers of particles in grid cells can produce noisy results. Despite its shortcomings, CIC remains the most popular density estimator for visualization and visual analytics. Pang and Albrecht [33],

TABLE 1
Density Estimator Characteristics

| Method | Window Shape | Window Size | Weighting |
|--------|----------------------|----------------------------|--------------|
| CIC | Fixed: cube | Fixed: 8 grid points | First-order |
| SPH | Fixed: sphere | Adaptive: min. # particles | Higher-order |
| ACIC | Fixed: cube | Adaptive: min. # particles | Higher-order |
| TESS | Adaptive: polyhedron | Adaptive: Voronoi cell | Zero-order |

for example, sample air traffic density onto a regular grid with CIC prior to volume rendering.

2.2.2. Smoothed Particle Hydrodynamics Methods. In order to overcome the limitations of CIC, the size of the window can adapt to cover some prescribed number of particles. SPH [29] has a spherical window with variable size; the window radius r is large enough to encompass T particles, where T is a parameter provided by the user. The window is centered on the input particle. The weighting function $W(r)$ also must be specified; often a Gaussian is used. The main advantages of SPH are its adaptivity in window size and its ability to simultaneously act as a smoothing filter through its weighting function. Its disadvantage is the reliance on and sensitivity to the parameter T , the desired number of particles in a window. SPH is widely used in astrophysics fluid dynamical computations [16] to estimate mass distribution for cosmic structure formation [6] and in other areas of science and engineering.

2.2.3. Adaptive Cloud in Cell Methods. Characteristics of CIC and SPH can also be combined. One such combination that we call ACIC (adaptive CIC) features a cube-shaped window as in CIC whose size adapts to cover a given number of particles as in SPH. Birdsall and Fuss [8] first mentioned variable-size particle clouds. Many adaptive mesh refinement (AMR) simulations, for example *Nyx* [2], deposit density on a grid using ACIC windows, where the window is prescribed by the AMR grid level of the solver. In statistics, adaptive kernel density estimators (AKDE) [39,43,45] for computing a 1D probability density function (PDF) are a closely related topic. In both AKDE and ACIC, the kernel width adapts to a given number of input points, but the ACIC kernel is a 3D cube whose density does not integrate to 1, whereas AKDE has a 1D kernel over which a single or multivariate PDF is estimated. Lukasik [26] parallelized the (fixed-size) KDE using MPI, and Michailidis et al. [28] accelerated KDE for GPUs, but we have not found any parallel algorithms for (variable-size) AKDE in the literature. Our implementation of ACIC uses a cube-shaped window aligned with and centered on the grid cells, and we adapt the window symmetrically in three dimensions until it covers the desired number of particles. As in SPH, the advantage of ACIC is its adaptive size, and its drawback is selecting T , the number of input particles in the window.

2.2.4. Tessellation Methods. Both the window size and shape vary in TESS, which is self-adaptive and driven entirely by a Voronoi tessellation. The advantage of TESS is its self-adaptivity without reliance on any parameters. The estimation window is the Voronoi cell, and the weighting function distributes particle mass over the grid points in the window. The most expensive parts of the algorithm are computing the tessellation and finding the intersection of grid points with Voronoi cells. Our TESS method resembles the DTFE of Schaap and Van de Weygaert [40,41]. The main three differences between our work and theirs are: the DTFE used a Delaunay tessellation with first-order interpolation, whereas we use a Voronoi tessellation with zero-order weighting instead; they compared resulting density images visually, whereas a main focus of our work is developing a systematic method for designing test datasets and quantifying the differences with several metrics; and we develop

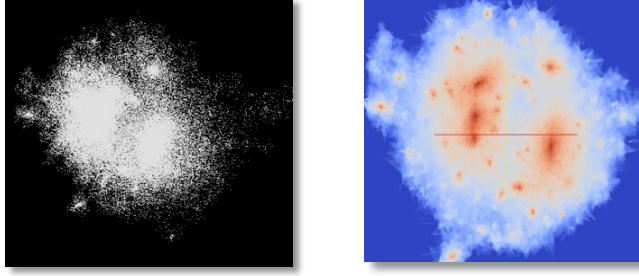


FIG. 3. *Density estimation of one cosmological halo for gravitational lensing. Left: original raw particle data. Right: 2D density field.*

an efficient parallel algorithm for the density estimation and evaluate its scalability.

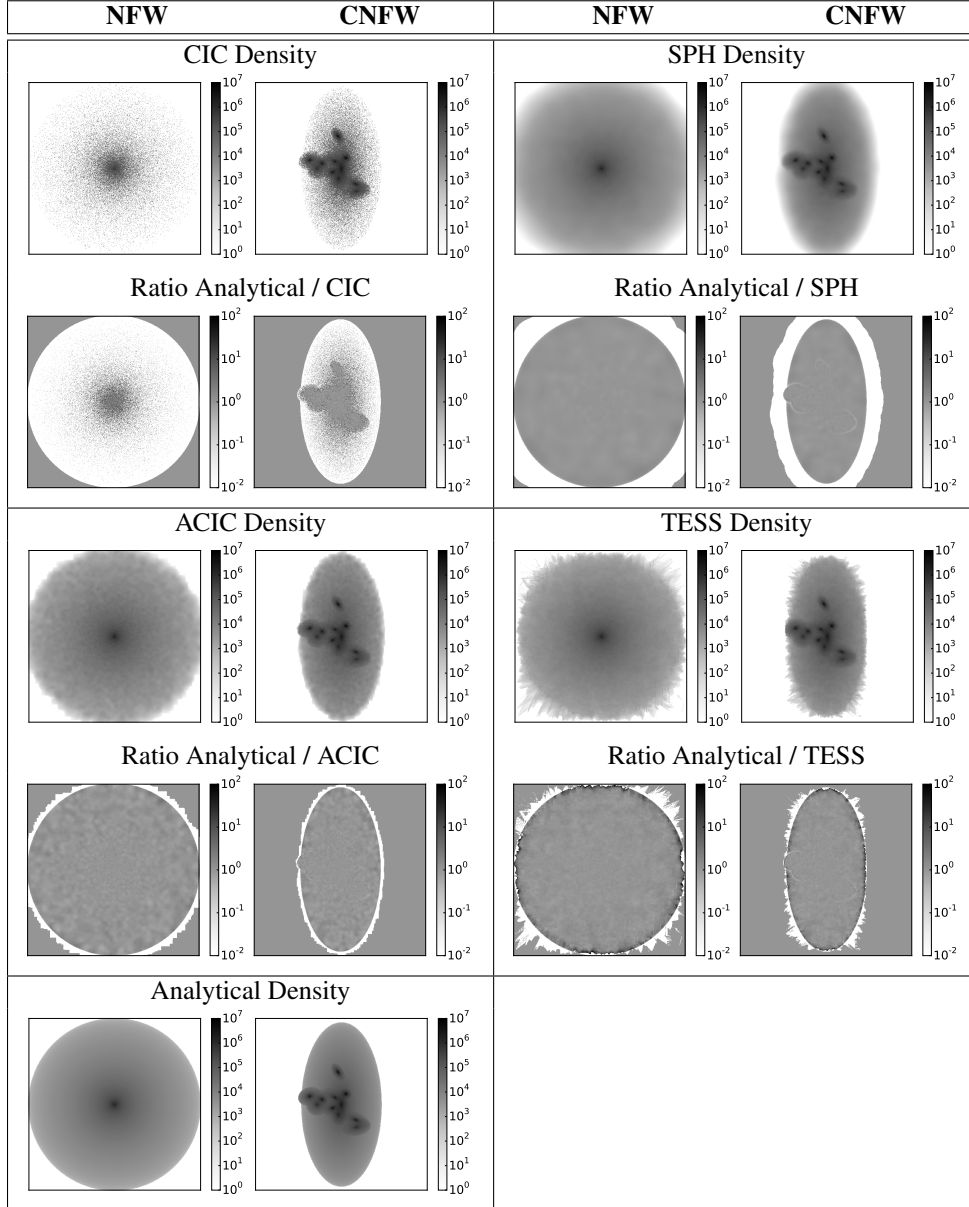
2.2.5. Other Methods. For completeness, we mention two other methods often found in particle simulations [18]. The nearest grid point (NGP) method deposits the entire mass of the particle to the nearest grid point. It is a zero-order estimation method with a window size of one grid space, centered on each grid point. Not every grid point is necessarily covered by a window, making the density field discontinuous (noisy) in low-density regions where grid cells are devoid of particles. One may consider NGP to be a degenerate version of CIC, where the window covers only one grid point instead of eight. On the other hand, the triangular-shaped cloud (TSC) method may be viewed as an expanded CIC, where the window covers a fixed size of 27 ($3 \times 3 \times 3$) grid points. Each particle is deposited onto its nearest 27 grid points using first-order weighting.

3. Synthetic Data Design. The regular grid estimation of particle density has many uses in science and engineering; we begin by describing one such application, gravitational lensing. Next, we describe a parameterized method for constructing synthetic input particle datasets with known underlying analytical density functions. The parameters of the datasets are derived from the characteristics (density range, substructure, anisotropy) of the target application. In this way, we can represent actual data while having a ground truth for comparison. We then describe ways to carry out this comparison.

3.1. Motivation: Gravitational Lensing. The deflection of light when passing through a gravitational potential is called gravitational lensing [5, 20]. Optical distortion caused by lensing is observed at all scales in the universe, making gravitational lensing a powerful tool for investigating dark matter and dark energy [27]. Gravitational lensing is simulated by computing the gravitational potential from the continuous density field of dark matter particles modeled by an N-body cosmological simulation [38]. To compute an accurate lensing simulation, one must have a density field that is low in noise, that reflects the underlying continuous density function, and that scales to the size of the simulation.

Figure 3 shows one cluster of dark matter particles whose density is estimated onto a 2D grid. For lensing calculations, the lens is modeled as a 2D density field located far away from the observer. Because the particles are in 3D, however, the density is estimated in 3D first before projecting to 2D. The input particles are produced by computational cosmology codes such as HACC (Hardware/Hybrid Accelerated Cosmology Code), a Gordon Bell finalist N-body code that exceeded 10 petaflops [17]. In the left image of Figure 3, we extracted one cluster consisting of approximately 100,000 particles from the simulation output. The right image shows the estimated density field.

TABLE 2
2D Density Images of NFW and CNFW



3.2. Analytical Functions. One may define analytical synthetic density functions to produce desired data characteristics. For example, molecular dynamics [22, 44] and cosmology simulations [2, 38], while both particle methods, have widely different behavior. We begin with a simple symmetrical density profile whose parameters are chosen to mimic the density profile of a given problem. Next, we construct more complex analytical functions that superimpose several instances of the original function. In so doing, we apply different parameters to each component and produce more complex features in the synthetic data, all

the while maintaining a known analytical solution.

Our first analytical function is a spherical density profile derived from an NFW (Navarro-Frenk-White) function [31]:

$$\rho(r) = \frac{k}{r(r+1)^2},$$

where ρ is the density; r is the radius, and k is a constant. The NFW function is an ideal testbed because it can model a high density range ($\rho = \infty$ at $r = 0$, and $\rho = 0$ at $r = \infty$). For our tests, we truncate the domain of $r \in [-1.5, 1.5]$; and we cap the range to a maximum $\rho = 10^6$, which approximates the density range of dark matter tracer particles in computational cosmology.

In addition to high dynamic range, dark matter forms clusters of large-scale structures and substructures that are asymmetrical. To capture this behavior in our synthetic data, we compose several NFW models in a complex NFW model that we call CNFW. In our tests, the CNFW has eleven NFW components scaled with different values of k . Furthermore, we asymmetrically truncate the domain to create ellipsoidal structures. We also translate each component before composing them. Therefore, the CNFW allows us to test a dataset that is anisotropic, contains substructure, and more closely resembles the actual data in N-body simulations. Because the model is composed of analytical functions, however, we still know the ground truth density at any grid point, and we can compare the estimated and known density.

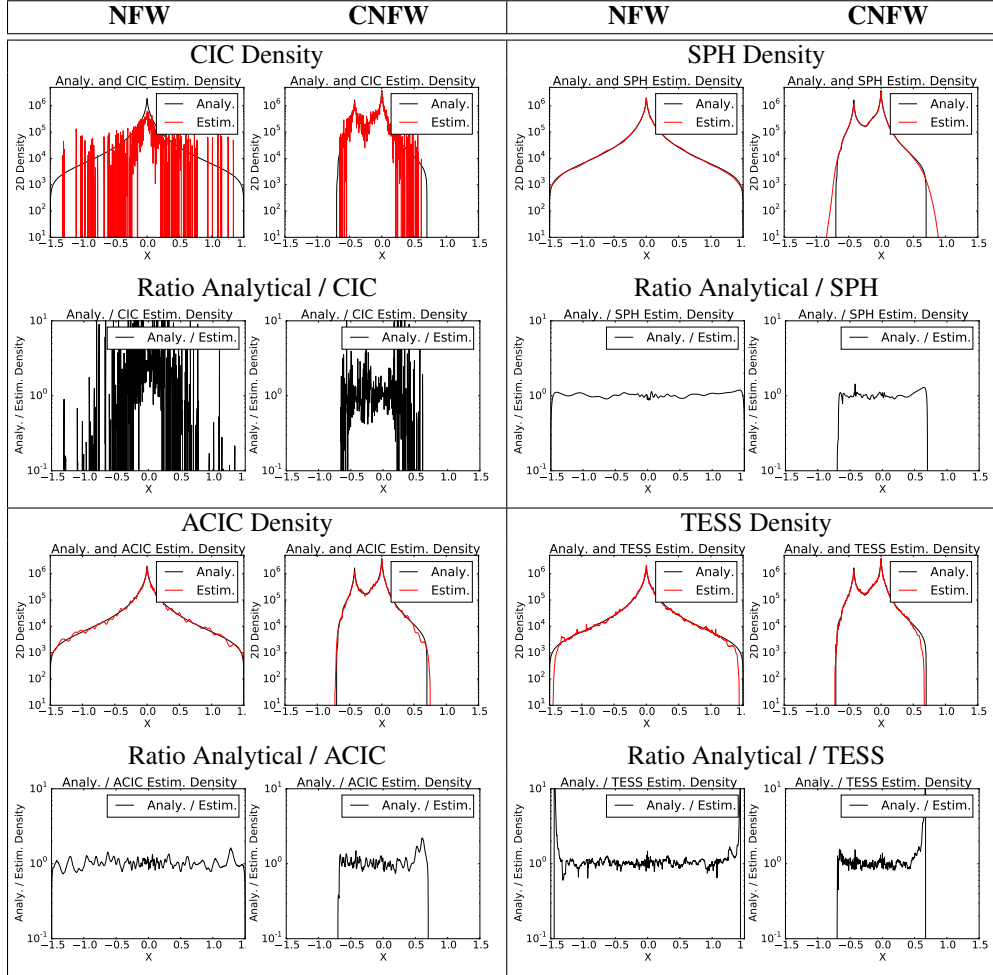
3.3. Input Particles. Monte Carlo sampling is used to produce discrete particle datasets from the analytical function. A random number is generated from a uniform distribution in $[0, 1]$; the inverse of the cumulative distribution function of the NFW determines r , which is then converted to (x, y, z) coordinates of a particle by choosing the spherical angle coordinates from a uniform distribution. A single instance of a Monte Carlo particle set may be used to compare estimators with each other and with the specific instance of the input data; performing the Monte Carlo sample multiple times (producing multiple particle datasets) can be used to better understand how well a density estimator captures the true underlying density that is represented by the particles.

3.4. Testing Methods. Each density estimator is then applied to the input datasets, producing a regular grid density field. All our tests are projected from a 3D density volume to a 2D density image, as explained in Section 5.1, to simplify visualizing and comparing results. In addition to visual inspection of the density images, images of the ratio of the analytical density divided by the estimated density are evaluated. Furthermore, cross sections through these images are taken to produce line plots. The power spectra of the density images are also analyzed to explain the estimation behavior in terms of spatial frequency.

4. Evaluation Results. Next, we demonstrate how to apply our design methodology in practice. Having constructed the NFW and CNFW analytical functions based on a particular science domain, we compare four density estimators as described above. Then we can decide which estimator is best suited for our particular science problem. In Section 4.1, we evaluate the accuracy of four density estimators. The results for a single Monte Carlo particle dataset appear in Tables 2 and 3; Figure 4 shows results for an ensemble of 20 Monte Carlo datasets. Section 4.2 compares the estimators in terms of their asymptotic computational complexity.

4.1. Accuracy. Table 2 shows the 2D density images for the four density estimators and the NFW and CNFW analytical functions. Each of the four estimations includes the density image with the image of the ratio between the analytical and estimated density below. A

TABLE 3
Cross Section of NFW and CNFW Density



ratio of 1.0 (no error) corresponds to the color of the background in the corners of the image. Hence, one can easily see the error in the foreground by comparing its color with that of the background. CIC suffers from noise caused by undersampling over much of the image except the high-density center region. Low-density regions in the input data combined with a high-resolution grid can generate grid cells with zero particles; this situation is the primary drawback of CIC.

The other estimators do better except for the area near the boundary of the sphere or ellipsoid. This edge effect is caused by the an incomplete estimation window at the boundary. For example, the tessellation has incomplete Voronoi cells there; similarly, ACIC and SPH cannot compute windows with the desired number of particles at the edges. Such edge effects are inherent in these methods, and density values near the edge of a boundary must be discarded.

For a more quantitative comparison, Table 3 is a cross section through the $y = 0$ plane of the data in Table 2. Included are plots showing the cross section of estimated density superimposed on the analytical density, and additional plots show the cross section of the ratio

between analytical and estimated density. SPH, ACIC, and TESS all do better than CIC, but there are subtle differences visible upon zooming in on the CNFW slices of SPH, ACIC, and TESS in Table 3. SPH is the smoothest, but it misses sharp discontinuities; in particular, SPH underestimates the maximum density at the two peaks of the CNFW. ACIC has more oscillations about the analytical curve than does SPH and also underestimates the left-hand peak. TESS reproduces the two peaks of the CNFW function slightly better than SPH and ACIC do, even though it is zero-order weighting. The reason is that the highest-density regions have very small Voronoi cells, where the weighting function is immaterial. (Voronoi cells smaller than one grid space default to CIC in our method.) TESS does, however, introduce more overall high-frequency noise; Table 3 shows higher-frequency oscillations in TESS than in SPH and ACIC.

For CNFW, we further quantify the deviation from the ground truth as a function of spatial frequency by taking the fast Fourier transform (FFT) of the analytical and estimated density and plotting the power spectrum in Figure 4. The power $P(k)$, or the square of the FFT magnitude, is plotted in log-log scale as a function of the spatial frequency k . For this test, we consider a subset of the density image in the interior of the CNFW, in effect cropping away the edge effects. For computing the FFT, we use an isolated boundary with zero-padding. The y axis of Figure 4 is normalized by the highest value of the power.

In Figure 4, 20 sample datasets were drawn from the CNFW function. The left and center panels show the median power spectrum of those 20 instances, and the right panel shows the range. The center and right panels are magnified views of the high-frequency region only. The units of k are $distance^{-1}$, although for our synthetic data the particle positions are dimensionless. The analytical curve is not perfectly smooth because the CNFW function is discretized onto a regular grid.

The center panel of Figure 4 demonstrates that density estimation is a delicate balance of high-pass and low-pass filtering. We define noise in the power spectrum as a deviation above the analytical curve caused by excessive high-frequency components. Similarly, we define (over)smoothing as a deviation below the analytical curve caused by excessive low-frequency components. For example, CIC deviates earliest from the analytical function. The upward direction of the error is caused by the sparse density values in low-density regions. The TESS curve overlays the analytical curve better than CIC does, until eventually the density discontinuities at Voronoi cell boundaries push the curve above the analytical. Of the median curves in the center panel, ACIC is the closest to the analytical function, and SPH is too smooth.

One should not conclude from the median curves of the center panel that ACIC is better than SPH and TESS. Rather, the large spread in the right panel for both ACIC and SPH indicates that these algorithms' use of the minimum number of particles (T) parameter is sensitive to variability of the Monte Carlo sampling. (The color-coded vertical bars at the right side of the plot indicate the maximum spread for each method.) ACIC and SPH have more spread than TESS does because ACIC and SPH have fixed-shape windows, and their decisions regarding window size are based on discrete particle counts. On the other hand, TESS adapts both its window size and shape automatically to the spatial particle distribution. Hence, it is more robust to Monte Carlo sampling variations in the input dataset, and the error compared with the underlying smooth density function is more consistent than with ACIC or SPH.

4.2. Computational Complexity. One naturally expects the quality of a method to come at a cost. In this section, we outline each method in pseudocode in order to compute the computational complexity in terms of number of input particles (P), number of output grid points (G^3), and number of particles in a window (T). In the following pseudocode, we

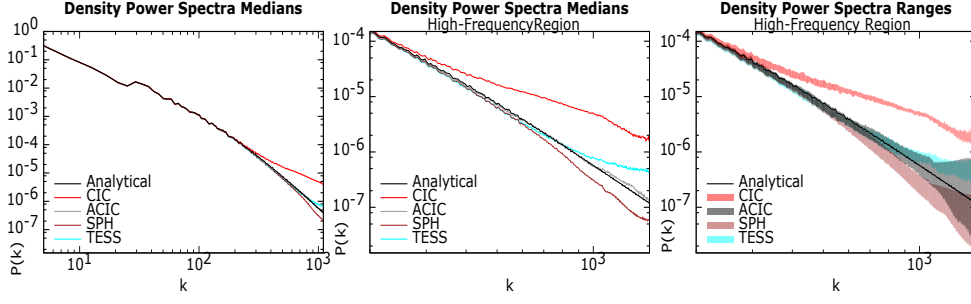


FIG. 4. Left: Power spectrum of CNFW estimated density. Center: Magnified high-frequency region. Right: Magnified high-frequency region showing the range of output density for 20 Monte Carlo samplings of input data.

assume that the volume of one grid cell is unity; hence, mass and density appear together in arithmetic expressions without dividing by volume. The four estimation methods are presented in order of increasing computational complexity.

4.2.1. CIC. The algorithm for CIC is straightforward and appears in Algorithm 1.

Algorithm 1 CIC

- 1: $D(\dots) \leftarrow 0$
 - 2: **for all** particles p **do**
 - 3: window $w \leftarrow$ grid cell containing p
 - 4: **for all** 8 points $g \in w$ **do**
 - 5: $D(g) \leftarrow D(g) + \text{mass_fraction}(w, p, g)$
-

The complexity of Algorithm 1 can be computed as follows.

$$\begin{aligned}
 C &= \sum_p \text{num_grid_pts}(w) \\
 &= 8 \times P
 \end{aligned}$$

The overall complexity is therefore $O(P)$.

4.2.2. SPH. Our implementation of SPH is divided into two parts: building a tree to find all the windows and distributing each particle's mass within a window. The first step consists of sorting the particles into a tree and traversing the tree to build a list of windows at least T particles in size. The complexity of the first step is $O(T \times P \log P)$.

The pseudocode for the second step, distributing the mass within a window, appears in Algorithm 2.

Algorithm 2 SPH

- 1: $D(\dots) \leftarrow 0$
 - 2: **for all** particles p **do**
 - 3: window $w_p \leftarrow$ sphere centered on p
 - 4: **for all** grid points $g \in w_p$ **do**
 - 5: $D(g) \leftarrow D(g) + \text{mass_fraction}(w_p, p, g)$
-

The complexity of Algorithm 2 can be computed as follows.

$$\begin{aligned} C &= \sum_p (\max(1, \text{num_grid_pts}(w_p))) \\ &\leq P + T \times G^3 \end{aligned}$$

The overall complexity of both steps, dropping the low-order P term, is therefore $O(T \times P \log P + T \times G^3)$.

4.2.3. ACIC. Our implementation of ACIC is divided into two parts: building a kd-tree of windows and distributing each particle's mass within a window. Each window is aligned with the grid and either is reduced to a single grid point or contains at most T particles. The first step, building the tree, is accomplished during the sorting of the particles in each of three dimensions and is bounded by the sorting time, $O(P \log P)$.

The pseudocode for the second step, distributing the mass within a window, appears in Algorithm 3.

Algorithm 3 ACIC

```

1:  $D(\dots) \leftarrow 0$ 
2: for all windows  $w \in \text{tree}$  do
3:   for all particles  $p \in w$  do
4:     window  $w_p \leftarrow w$  recentered on  $p$ 
5:     for all grid points  $g \in w_p$  do
6:        $D(g) \leftarrow D(g) + \text{mass\_fraction}(w_p, p, g)$ 
```

The complexity of Algorithm 3 can be computed as follows.

$$\begin{aligned} C &= \sum_w \sum_{p \in w_p} \max(1, \text{num_grid_pts}(w_p)) \\ &\leq P + T \times G^3 \end{aligned}$$

The overall complexity for both steps, dropping the low-order P term, is therefore $O(P \log P + T \times G^3)$.

4.2.4. TESS. TESS is also a two-part algorithm: computing the Voronoi tessellation and distributing each particle's mass within the window of the Voronoi cell associated with that particle. Computational geometry algorithms generate a Voronoi tessellation in dimension d by computing the convex hull in dimension $d + 1$. One such an algorithm [13] has a complexity of $O(P \log P + P^{\lceil d/2 \rceil})$. For $d = 3$ and keeping only the high-order polynomial term, the complexity in 3D is $O(P^2)$.

The pseudocode for the second step, distributing the mass within the Voronoi cell, appears in Algorithm 4.

Algorithm 4 TESS

```

1:  $D(\dots) \leftarrow 0$ 
2: for all windows  $w$ , i.e., Voronoi cells do
3:   for all grid points  $g \in w$  do
4:      $D(g) \leftarrow D(g) + \text{mass\_fraction}(w, g)$ 
```

The complexity of Algorithm 4 can be computed as follows.

$$\begin{aligned} C &= \sum_w \max(1, \text{num_grid_pts}(w)) \\ &\leq P + G^3 \end{aligned}$$

since the Voronoi cells are a tessellation of the entire grid. The overall complexity for both steps, dropping the low-order P term, is therefore $O(P^2 + G^3)$.

5. Parallel Algorithm for TESS Density Estimator. TESS’s appeal is its self-adaptivity without the need to tune input parameters such as window size, shape, or weighting function. TESS is also the most expensive estimator, motivating the need for a parallel algorithm. Our work is further motivated by large-scale science applications running on HPC architectures. Distributed-memory parallel algorithms for density estimation are needed in order to keep up with the increasing scale of particle-based simulations. One such simulation, HACC [17], already produces terabytes of data at each time step and runs on some of the largest supercomputers in the world.

As in the pseudocode above, our parallel algorithm is a two-step process—computing the Voronoi tessellation, followed by estimating density at grid points covered by Voronoi cells—except that now both stages are performed in parallel. The first stage, tessellating the input particles, is accomplished with a parallel library [34] that can tessellate up to 2048^3 particles using 128K MPI processes at up to 90% efficiency. This method is built on the Qhull implementation of the Quickhull algorithm [4], and alternatively on the Computational Geometry Algorithms Library (CGAL) [14]. We suggest Edelsbrunner’s text [12] for an overview of Voronoi tessellations.

Our density estimator can be either loosely or tightly coupled to the tessellation. In the first instance, the tessellation is written to disk in pNetCDF format [25] and is read back later by the density estimator. In the second instance, the same main program calls the tessellation and the density estimator without saving the tessellation to disk. Our performance results in Section 5.3 are measured by using this second, tightly coupled mode.

Both the tessellation and density estimator are implemented with a distributed-memory analysis infrastructure called DIY [35, 36]. DIY is a data-parallel programming library, built atop MPI [3, 15], that provides configurable data partitioning and communication. In this paper we define a *block*, the fundamental unit of domain decomposition and local work, as a hexahedral region of space containing a subset of the input particles, their associated Voronoi cells, and the regular grid points onto which the particles are deposited. An MPI process may own more than one block. A *neighborhood* is the union of a block with its immediate spatially adjacent blocks that share a face, edge, or corner. Neighborhood exchange is the flow of information between the blocks in some subset of a neighborhood.

We use a zero-order interpolation (left side of Figure 5) as follows. For Voronoi cells that cover one or more grid points (the cell associated with P_0 in the figure), we distribute the mass of the particle associated with the cell uniformly over the grid points in the interior of the cell. In the figure, each of the eight gray-colored grid points in the interior of the cell is assigned $1/8$ the mass of P_0 . For small cells that do not have any grid points in their interior (the cell associated with P_1 in the figure), we distribute the particle’s mass among the 8 nearest grid points using CIC. While the diagram is drawn in 2D for simplicity, the above steps are performed in 3D: cells are Voronoi polyhedra, and grid points lie on a 3D lattice.

After the particle mass is distributed among the grid points in each Voronoi cell, the density at each grid point is computed as a last step. For a given set of 3D input particles, our algorithm can produce either a 3D volume or a 2D image of density values as follows.

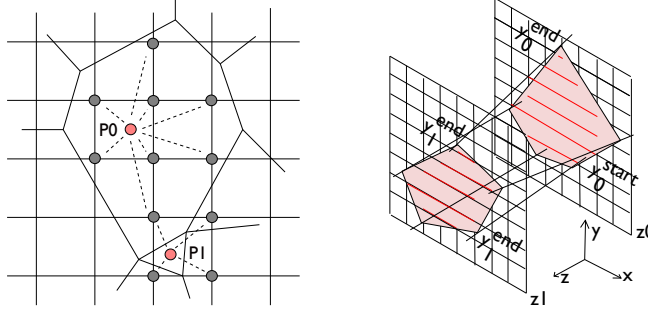


FIG. 5. Left: Voronoi tessellation density estimation. P_0 is a particle whose Voronoi cell covers several grid points. Its mass is uniformly distributed (zero-order weighting) to those grid points. P_1 has a small cell that covers no grid points. Its mass is distributed among the 8 nearest grid points using CIC. Right: Coherence in polyhedron scan conversion. The scans in the x direction are reduced by finding cell boundary crossings using the previous y points as starting points for next y line. The y range of scan lines can be further reduced from one z plane to the next by using the y limits at the previous z plane as starting y coordinates for the current z plane.

5.1. 3D Density Volume and 2D Density Image. For the 3D grid the result is volumetric density, or *mass/volume*, whereas in the 2D method the resulting grid is the surface density or *mass/area*. The 2D result is desired for certain applications, for example, gravitational lensing (Section 3.1). In all the subsequent performance tests, we enable the projection to a 2D density as we did in the preceding accuracy evaluation. Irrespective of the dimensionality of the output, the basic algorithm remains the same because we estimate the distribution of particles in 3D first before optionally choosing to project the result to 2D as a last step.

One may ask whether for the 2D output it suffices to compute a less expensive 2D Voronoi tessellation on the 2D projection of the input particles, in other words, to project early rather than late in the algorithm. Our early experiments showed that projecting 3D input particles first to a 2D plane introduces errors in the final density image because the 2D Voronoi cells associated with projected particles are different than those in 3D, which in turn affects the mass distribution.

Hence, we use a full 3D computation even when the resulting grid will be 2D. One need not, however, store the full 3D volumetric grid before projecting it to 2D. In order to save memory space, once a 3D grid point has been assigned a mass fraction (computed in 3D), its mass can be immediately accrued to its projected 2D grid point. In this way, we maintain the full accuracy of the 3D method with the memory footprint of a 2D grid. Currently, we have implemented projection only in the z direction to the xy plane, but we plan to support arbitrary projection planes in the future.

Once the mass of all particles has been distributed and accumulated in 3D or in 2D as described above, the density is computed. For 3D density, the accumulated mass at each 3D grid point is divided by the volume of a grid cell (one voxel). For 2D density, the accumulated mass at each 2D grid point is divided by the area of a grid cell (one pixel).

5.2. Algorithm. With the tessellation as an input, Algorithm 5 enumerates the main steps of finding which grid points lie inside a Voronoi cell, distributing the particle mass over those grid points, and optionally projecting the result to 2D.

Shared- and Distributed-Memory Parallelism Algorithm 5 is a hybrid combination of shared- and distributed-memory parallelism. First, lines 1 - 9 are parallelized by using OpenMP. The Voronoi cells are independent of each other, and up to the number of available threads can be scan-converted concurrently. The OpenMP parallel block terminates after line 9 so that subsequent distributed memory communication occurs only in the parent thread.

Algorithm 5 TESS Density Estimator

```

1: for all Voronoi cells do
2:   compute cell bounding box
3:   compute Voronoi cell interior grid points
   from cell bounding box
4:   mass fraction  $\leftarrow$  particle mass / number of
   cell interior grid points
5:   for all cell interior grid points do
6:     if grid point is in bounds of local block then
7:       grid point mass  $\leftarrow$  grid point mass + mass
       fraction
8:     else
9:       enqueue mass fraction for sending to neigh-
       boring block
10: Send enqueued mass fractions to their respec-
    tive neighbor blocks
11: At neighbor blocks, receive mass fractions
    and add to appropriate grid points
12: for all grid points in a block do
13:   if 3D then
14:     accumulated density  $\leftarrow$  accumulated mass /
     voxel volume
15:   if 2D then
16:     Send accumulated mass to block containing
     2D grid point that is the target of the projec-
     tion
17: if 2D then
18:   Receive accumulated masses
19:   for all grid points in a block do
20:     accumulated density  $\leftarrow$  accumulated mass /
     pixel area

```

The distributed-memory decomposition is a Cartesian lattice of regular-sized blocks that partition the 3D space: a block contains some subset of the output grid points and a subset of the total Voronoi cells in the tessellation. The left image of Figure 6 shows that while the output grid points align with block boundaries, the Voronoi cells do not. Hence, some of the grid points interior to a Voronoi cell of one block may belong to a neighbor block. Lines 10 - 11 of Algorithm 5 send messages over distributed memory using a neighbor exchange algorithm that we implemented over MPI so that mass contributions belonging to grid points in a neighboring block can be accumulated there.

When projecting the output to a 2D density image, additional distributed-memory communication is needed to aggregate the mass contribution of 3D grid points along the lines of projection to their corresponding 2D targets. The right side of Figure 6 shows this situation. The blocks containing the 2D grid (the bottom blocks in the diagram) are a subset of the blocks containing the 3D grid, and each of the blocks in a vertical column (assuming this is the direction of projection) must send its grid points' mass to the block at the bottom of the column. These communications are done by using MPI point-to-point communication. The receiving blocks at the bottom of each column accumulate the mass, convert to density, and write the density image to storage in parallel using MPI-IO [11].

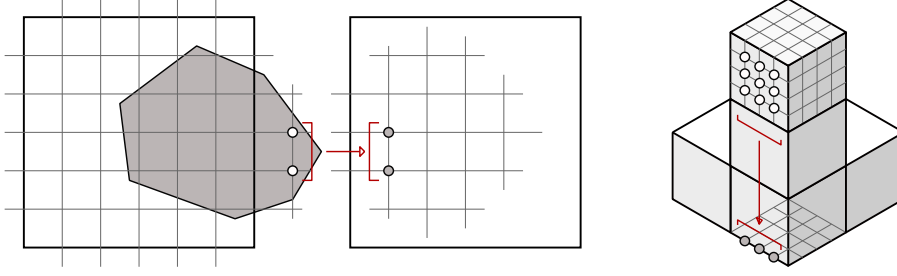


FIG. 6. *Communication in the distributed-memory parallel algorithm. Left: A Voronoi cell may cover grid points outside the local block, in which case the mass contributions of those grid points are sent to the neighboring block that owns them. Right: In the case of 2D output, the mass contributions of 3D grid points in blocks along the line of projection are sent to the block with the 2D output plane.*

Voronoi Cell Scan Conversion In line 3 of Algorithm 5, the 3D grid points in the bounding box of the Voronoi cell are checked whether they are interior to the polyhedron. In 2D, polygon scan line conversion is a textbook computer graphics algorithm [19]. The standard implementation of this algorithm includes edge coherence and an active edge table. While our situation is slightly simplified in that we have convex polytopes that have one entry and one exit per scan line, the extension from 2D to 3D is more complicated. The traditional scan line algorithm would require sorting polyhedral faces in x , y , and z ; and even so, we cannot guarantee that we know when we are leaving one face and entering another. While we borrowed some ideas such as edge coherence from 2D scan line conversion, we did not attempt to extend the algorithm per se to 3D. Instead, the following overview describes our implementation of scan conversion of Voronoi polyhedra in 3D. (Our software design is also modular, making the following algorithm easy to replace with another.)

Testing whether a 3D point lies inside a convex polyhedron (a 3D Voronoi cell) has linear complexity in the number of faces of the cell and is computed as follows. The dot product of the grid point with the normal vector of the face is computed for each face in the cell. If the dot product reverses sign, then the grid point is outside the cell. Conversely, the grid point is inside the cell if the dot product retains the same sign for all the cell faces. If the dot product changes sign, the test can terminate early; but on average, half the faces will need to be checked. We can further limit the number of grid points to test using edge and face coherence as follows.

Algorithm 6 lists the pseudocode for our Voronoi scan conversion. In the following discussion, we refer to the right side of Figure 5, which shows two slices of a Voronoi cell between two adjacent z planes in the output grid at $z = z_0$ and $z = z_1$. Because the polyhedron is convex, any y line will intersect the cell in at most two x points; and we need to find only those two x points to know the interior/exterior labeling of all x points on the y line. Furthermore, the two x points from the previous y line can be used to seed the search in the next y line to usually find the next cell intersection points within a few grid steps of the previous ones. Just as we constrained the search of x points from one y line to the next, the same idea can be applied to limit the search of y lines in successive z planes by starting the search at y_0^{start} of the prior z plane. Assuming that at z_0 the range of y lines intersecting the cell was $[y_0^{start}, y_0^{end}]$, we expect the new y range at z_1 , $[y_1^{start}, y_1^{end}]$, to be similar.

5.3. Performance. The following tests were conducted on up to 8K MPI processes of the IBM Blue Gene/Q *Vesta* machine at the Argonne Leadership Computing Facility at Argonne National Laboratory. *Vesta* is a testing and development platform consisting of 2K nodes, each node with 16 cores (PowerPC A2 1.6 GHz), 16 GB RAM, and 64 hardware

Algorithm 6 Cell Scan Conversion

```

1:  $x^{start}, x^{end}, y^{start} \leftarrow$  cell bounding box
2: for all  $z$  planes in cell bounding box do
3:    $x \leftarrow (x^{start} + x^{end})/2$ 
4:   adjust  $y^{start}$  up and down for inside/outside change
5:    $x^{start}, x^{end} \leftarrow x$ 
6:   for all  $y$  lines in  $y^{start}$  to cell bounding box do
7:     for all  $x$  points from  $x^{start}$  do
8:       if grid point remains inside the cell then
9:          $x \leftarrow x - 1$ 
10:      else if grid point remains outside the cell then
11:         $x \leftarrow x + 1$ 
12:      if inside/outside changes then
13:        mark  $(x, y, z)$  grid point as border
14:         $x^{start} \leftarrow x$ 
15:      for all  $x$  points from  $x^{end}$  do
16:        if grid point remains inside the cell then
17:           $x \leftarrow x + 1$ 
18:        else if grid point remains outside the cell then
19:           $x \leftarrow x - 1$ 
20:        if inside/outside changes then
21:          mark  $(x, y, z)$  grid point as border
22:           $x^{end} \leftarrow x$ 
23:      if no  $x$  intersection points found then
24:        terminate  $y$  lines loop

```

threads. We ran 8 MPI processes per compute node and 8 OpenMP threads per MPI process. The GNU GCC compiler, version 4.4.7, was used to compile the code with -O3 optimization.

We studied strong scaling and weak scaling for a synthetic test of 512^3 particles whose density is estimated onto 3D grids ranging from 1024^3 to 8192^3 and further projected to 2D density images from 1024^2 to 8192^2 . Weak scaling is in terms of grid size, with a fixed number of particles. While the theoretical complexity in number of particles was derived in Section 4.2.4, the number of particles was not varied in our performance evaluation. Varying the number of particles was studied in an earlier paper on the scalability of computing the Voronoi tessellation [34] instead. Our primary objective in the following performance evaluation is to study the scalability of the parallel density estimation algorithm presented in Section 5.2 as a function of grid size and number of processes.

The left plot in Figure 7 shows strong scaling in log-log scale. The plot shows end-to-end time that includes computing the tessellation, estimating the density, projecting the 3D density to 2D, and writing the final density image to storage. For the smallest grid size, 1024^3 , the strong-scaling efficiency of the end-to-end time is approximately 56% compared with the downward dashed line indicating 100% strong scaling. For the largest output, 8192^3 , the strong-scaling efficiency is 80%.

Of the components described above, two dominate the run time: computing the tessellation and estimating the 3D density. The center plot of Figure 7, in semilog scale, separates the run time into these two components. Each group of vertical bars shows the performance at a certain number of processes, and the columns within a group represent different output grid sizes (for example, “1K” is a 1024^3 grid). Within a group, the tessellation time is constant,

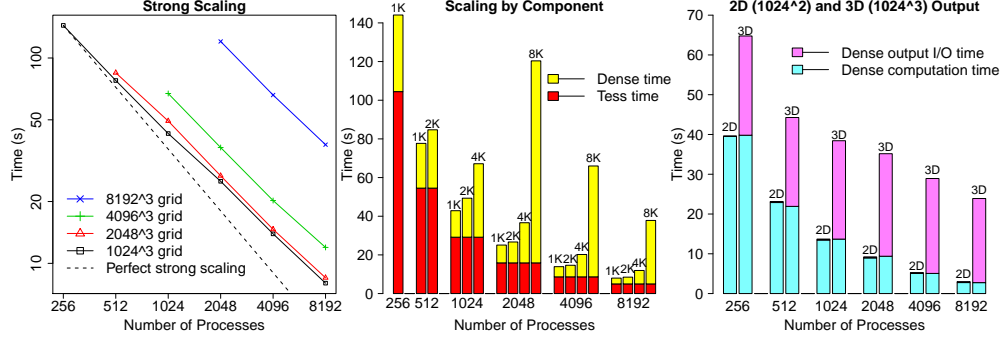


FIG. 7. Left: Strong scaling for 512^3 synthetic particles and grid sizes ranging from 1024^3 to 8192^3 . Center: Scaling of individual tessellation and density estimation components. Right: Comparison of 2D and 3D output for 512^3 synthetic particles and 1024^3 grid.

irrespective of output grid size, because it depends only on the number of input particles. The strong scaling of the tessellation time with process count is documented in [34]. The scaling of the density estimation component is evident in the upper portions of the bars, both within and across different numbers of process groups. The scaling efficiency of only the density estimation component is 80% for the 8192^3 grid size.

The decision whether to produce the final result in 2D or 3D is determined by the application. Nevertheless, it is instructive to compare the relative cost of these two output modes. In the 2D case, the algorithm needs to perform an additional projection compared with the 3D case, and this projection requires remote communication as explained in Section 5.2. The output file size, however, is smaller by several orders of magnitude when projecting to 2D. The right plot of Figure 7 shows this comparison in semilog scale for the 1024^3 and 1024^2 output grid size. In this case, the 3D output file is 1024 times the size of the 2D output file, evident by the substantial output I/O component in the figure for the 3D bars. The I/O component for the 2D bars is too small to be visible, taking approximately 0.25 seconds. The similar height of the lower section of each pair of bars also shows that the cost of the extra projection for 2D is negligible.

6. Conclusion. We presented a study of density estimation: the recovery of the underlying density function represented by the positions of a set of particles and the discretization of that density function onto a regular 3D or 2D grid. Even though density estimation is not a new topic, this paper presented a novel approach to constructing synthetic data and evaluating different density estimator methods. We also presented a parallel large-scale algorithm for one method using a Voronoi tessellation as an intermediate form. Although TESS is the most expensive of the algorithms we studied, it produced the highest quality density estimate for our data. We certainly do not advocate this algorithm for all uses, and the first half of the paper showed how to design experiments to determine what is the required level of accuracy and cost for a given problem domain. The use case we featured showed that for gravitational lensing problems, the high quality is in fact necessary. This motivated the development of a parallel algorithm in the second half of the paper.

To generate synthetic data, we built a parametric analytical model called CNFW based on the composition of NFW density profiles. The parameters of that model are the number of individual NFW components and the position, anisotropic scale, and density cutoffs of each component. In this way, the CNFW can mimic the structure and substructure of actual

experimental or simulation data while retaining the ability to compare with ground truth. We evaluated four density estimators—CIC, SPH, ACIC, and TESS—in a battery of tests, including visual comparison of 2D density images, 2D error images, and 1D line plots derived from cross sections of the 2D images. We also transformed the results to the frequency domain and evaluated the power spectra of each method. Furthermore, we presented the asymptotic computational complexity of each method.

All density estimators generate some error in attempting to recover an unknown density function and discretize it onto a regular grid. Nonetheless, given a particular application and computing budget, some estimators make better choices than others. For our particular set of CNFW parameters, we found ACIC, SPH, and TESS to be better than CIC. When we Monte Carlo sampled the CNFW function multiple times, the median value of ACIC was closer to the analytical power spectrum than was SPH or TESS. TESS, however, was less sensitive to the particular sample of input data points than ACIC and SPH were. The reason is TESS’s auto-adaptivity: both window size and shape adjust to the particle spacing automatically without requiring any input parameters. Not surprising, it is computationally the most expensive because it requires computing the Voronoi tessellation.

We presented a parallel algorithm for TESS-based density estimation using a hybrid shared- and distributed-memory parallel programming model and showed that our parallel algorithm scales to thousands of processes on a modern supercomputer architecture at grid sizes up to 0.5 trillion 3D grid points. To the best of our knowledge, no other parallel algorithm for density estimation exists at this scale. We applied our method to one science domain—gravitational lensing from the dark matter distribution in the universe—and we plan to explore other applications where the additional cost of a tessellation-based estimator is warranted.

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