Translations on graphs with neighborhood preservation

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Translations on graphs with neighborhood preservation

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***Abstract*—In the field of graph signal processing, defining translation operators is crucial to allow certain tasks, including moving a filter to a specific location or tracking objects. In order to successfully generalize translation-based tools existing in the time domain, graph based translations should offer multiple properties: a) the translation of a localized kernel should be localized, b) in regular cases, translating a signal to a vertex should have similar effect to moving the observer's point of view to this same vertex. In previous work several definitions have been proposed, but none of them satisfy both a) and b). In this paper we propose to define translations based on neighborhood preservation properties. We show that in the case of a grid graph obtained from regularly sampling a vector space, our proposed definition matches the underlying geometrical translation. We point out that identification of these graph-based translations is NP-complete and propose a relaxed problem as a proxy to find some of them. Our results are illustrated on highly regular graphs on which we can obtain closed form for the proposed translations, as well as on noisy versions of such graphs, emphasizing robustness of the proposed method with respect to small edge variations. Finally, we discuss the identification of translations on randomly generated graph.**

I. Introduction

Graph signal processing is a generalization of classical signal processing that arose a few years ago. The field developed around the obervations that eigenvectors of a particular matrix — the Laplacian matrix — associated with a ring graph as depicted in Figure 1 correspond to the Fourier modes. In more details, the *graph Fourier basis* associated with a graph of N vertices is the basis defined by these eigenvectors, and a spectral representation of any signal on a graph — a vector in $\mathbb{R}^N$ — can be obtained by projecting it into this particular basis, thus providing a *spectral representation* for the signal.

The correspondence between eigenvectors of the Laplacian matrix and the Fourier basis has been extended to any graph on which signals can be observed. Researchers have then successfully been able to find tools such as convolution, filtering, or modulation of signals on graphs (see [1] for an overview of such tools).

Among these tools, one of paramount importance is the translation operator, that allows one to move a signal on the graph. While understanding translation of temporal signals or images is straightforward due to the underlying vector space, it is not the case for graphs in general, since such objects only consist of vertices and edges linking them, without any underlying vector space. Multiple definitions of translations for signals on graphs have been proposed in the literature, but none has the property that adjacent signal entries necessarily remain adjacent after translation. This has the effect to deform the signal as it is translated, either by breaking neighborhoods or by changing the signal energy.

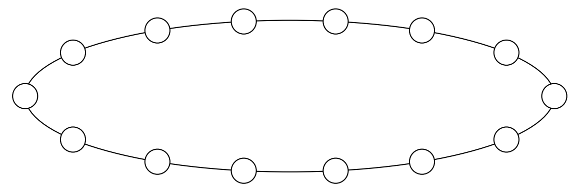

Figure 1: Example of a ring graph of 14 vertices. Eigenvectors of the Laplacian matrix associated with the smallest eigenvalues correspond to the Fourier modes associated with the lowest frequencies in classical Fourier analysis. The correspondence also holds as the eigenvalues increase.

In this article, we propose a framework to find translations of signals on graphs that preserve adjacency, while being compliant with the underlying graph. Translations as we define them can be seen as an orientation of a subset of edges in the graph, with some neighboring preservation constraints. In our approach, we do not modify the signal entries, with the exception of some special cases when we accept to lose some signal components. Such cases correspond to examples such as translation of an image *to the right*, in which case the last column of the image is lost if the Euclidean space is not toric.

Finding translations on graphs is a first step of a more global objective of extending convolutional neural networks to graph signals, thus making it possible to identify an object at different locations in a graph [2]. Applications include identification of moving patterns in brain imaging to obtain better models for causal connectivity [3].

The present article is organized as follows. First, Section II recalls existing definitions for translations defined on graphs. Then, in Section III, we introduce numerous defintions, and show some interesting properties for translations on graphs as we define them. Section IV then introduces results on the translations defined on graphs, such as the NP-completeness of the problem that consists in identifying them. To cope with this complexity issue, a relaxation of the problem is proposed in the form of an optimization problem, and an algorithm to identify some translations is provided. Finally, Section V studies the translations that are found by this algorithm, first on grid graphs, with and without noise, and then on graphs following a random model.

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II. Related work

A. The graph shift approach

Studying the case of the ring graph, Püschel and Moura [4], followed by Sandrihaila and Moura [5], propose a notion of *graph shift* as the adjacency matrix of the graph on which signals are defined. In particular, when considering the directed ring graph — *i.e.*, the orientation of all edges of the ring graph in the same direction — as a graph shift, multiplication of a signal by this shift has the effect to *advance* it in time. In the general case, considering an adjacency matrix as a translation operator has the effect to *diffuse* a signal as it is translated. Note that this is also the case where the adjacency matrix is normalized by its eigenvalue with the highest magnitude [6], in which case the signal energy only decreases as it is translated.

For this reason, translation of signals with this approach cannot match our objective of conserving the signal entries during translation. However, our approach is similar to this one in the sense that we identify translations by finding a subset of non-null entries of the adjacency matrix, which can be interpreted as an orientation of a subset of edges in the graph. In particular, the directed ring graph is a valid translation on the ring graph according to our definitions.

B. The convolutive approach

In the context of applying wavelets to graph signals, Hammond *et al.* [7] propose to define translation as a localization function to move a wavelet at a particular location of the graph. This is done by applying the wavelet to an *impulse*, *i.e.*, a signal that has all its energy concentrated at a single vertex.

The same approach is taken by Shuman *et al.* [8], [1], who propose a definition of translation of a signal *to a vertex* v, by convolution of this signal with an impulse located on v. This is done with analogy to the classical result in Fourier analysis that states that convolution in the time domain (in our case the graph) is equivalent to multiplication in the frequency domain (in our case the spectral domain of the graph).

With this approach, the signal is moved to a particular location rather than by a certain quantity. The convolution operation does not take the neighborhood in consideration, and allows modification of the signal when translating it.

C. The isometric approach

Girault *et al.* [9], [10] propose a translation operator for graphs that is isometric with respect to the ℓ_2 norm, *i.e.*, that does not change the signal energy as it is translated. Their approach consists in changing the phase of the signal in the spectral domain to move it in the graph domain. Additionally to keeping the signal norm unchanged, this operator has the property to preserve the signal localization, *i.e.*, to have its energy located around a target vertex [11].

This approach can also be considered as convolutive, since the translation is performed by convolving the signal with complex exponentials. Therefore, it suffers from the same drawback as the method introduced before, and can transform the signal while translating it.

Gavili and Zhang [12] take a similar direction, and also propose a phase change for translation. Contrary to the approach of Girault *et al.*, their solution does not take the graph spectrum into consideration. Again, this method does not have any neighboring preservation property.

D. Neighborhood-preserving translations

This article is an extended version of [13]. In this work, we explored translations on grid and torus graphs, and showed that Euclidean translations of images are equivalent to neighborhood preserving properties on these graphs.

In the present article, we first reformulate the resuts in [13] to make them more general and comprehensive. Additionally, we provide properties of the translations we propose, and show that identifying them is an NP-complete problem. Then, we propose a relaxation of this problem to identify *pseudo-translations*, and illustrate our results on grid graphs as well as on graphs following a random model.

III. Definitions

This section presents the notions that are needed for a full understanding of our work. After introducing some particular graphs, namely the grid graph and the torus graph, we propose some definitions of transformation and translation of a signal on a graph. Connections to intuitive translations on an Euclidean space are made in Section IV.

A. Some families of graphs

Definition 1 (Graph)**.** A *graph* is a tuple $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E} \subset \binom{\mathcal{V}}{2}$ is the set of edges[1]. Graphs defined this way are by construction *simple* (*i.e.*, $\forall v \in \mathcal{V} : \{v, v\} \notin \mathcal{E}$) and *symmetric* (*i.e.*, $\forall v_1, v_2 \in \mathcal{V} : \{v_1, v_2\} \in \mathcal{E} \Leftrightarrow \{v_2, v_1\} \in \mathcal{E}$).

Definition 2 (Digraph)**.** A *digraph*, or *directed graph*, is a tuple $\overrightarrow{\mathcal{G}} = \langle \mathcal{V}, \overrightarrow{\mathcal{E}} \rangle$, where $\mathcal{V}$ is the set of vertices and $\overrightarrow{\mathcal{E}} \subset \mathcal{V} \times \mathcal{V}$ is the set of directed edges, or *diedges*. Contrary to (undirected) graphs, the order of the vertices in $\overrightarrow{\mathcal{E}}$ matters. Therefore, digraphs can be *asymmetric*. In this article, we consider *simple* digraphs only.

A digraph can be seen as a graph, from which some edges have been oriented to a particular direction:

Definition 3 (Orientation of a graph)**.** Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$. An orientation of $\mathcal{G}$ is a digraph $\overrightarrow{\mathcal{G}} = \langle \mathcal{V}, \overrightarrow{\mathcal{E}} \rangle$ such that $\forall (v_1, v_2) \in \overrightarrow{\mathcal{E}} : \{v_1, v_2\} \in \mathcal{E}$.

It is often preferred to index vertices from 1 to $N = |\mathcal{V}|$, with $|\cdot|$ being the cardinality operator. We note $[\![C_1, C_2]\!]$ the set of all integers between C_1 and C_2, both included. Using indexation of vertices, we consider that $\mathcal{V} = [\![1, N]\!]$. This allows us to define the adjacency matrix of a graph:

Definition 4 (Adjacency matrix)**.** A *(binary) adjacency matrix* $\mathbf{A}$ for a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a $N \times N$ square matrix with

$$\forall v_1, v_2 \in \mathcal{V} : \mathbf{A}[v_1, v_2] = \left\{ \begin{array}{ll} 1 & \text{if } \{v_1, v_2\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{array} \right. .$$

[1] $\binom{\mathcal{V}}{2}$ denotes the set of unordered pairs of distinct elements in $\mathcal{V}$.

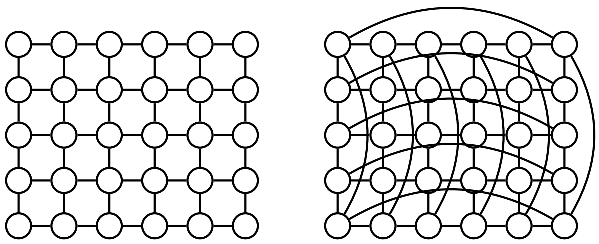


Figure 2: Example of the grid graph (left) and torus graph (right), both with dimensions $\mathbf{d} = \begin{bmatrix}6\\5\end{bmatrix}$.

In this article, we note $\mathbf{M}[i,j]$ the entry of matrix $\mathbf{M}$ at row i and column j. Additionally, we use the notations $\mathbf{M}[i,:]$ and $\mathbf{M}[:,j]$ for the i^{th} row and j^{th} column of $\mathbf{M}$, respectively.

Note that Definition 4 holds for digraphs, but in that case the adjacency matrix is not symmetric. This adjacency matrix provides a convenient way to represent adjacency between vertices. We note $\mathcal{N}(v_1) \subset \mathcal{V}$ the *neighborhood* of a vertex $v_1 \in \mathcal{V}$, *i.e.*, the set $\{v_2 \in \mathcal{V} \mid \mathbf{A}[v_1, v_2] = 1\}$.

In this article, we are interested in some families of graphs:

Definition 5 (Complete graph)**.** The *complete graph* $\mathcal{G}_c = \langle \mathcal{V}_c, \mathcal{E}_c \rangle$ of order N is the graph such that:

$$\forall v_1, v_2 \in \mathcal{V}_c, v_1 \neq v_2 : \{v_1, v_2\} \in \mathcal{E}_c \ .$$

Definition 6 (Grid graph)**.** Let $\mathbf{d} \in {\mathbb{N}^*}^D$. The *grid graph* $\mathcal{G}_g = \langle \mathcal{V}_g, \mathcal{E}_g \rangle$ yielded by the dimensions vector $\mathbf{d}$ is the graph such that:

- $\mathcal{V}_g = [\![1, \mathbf{d}[1]]\!] \times [\![1, \mathbf{d}[2]]\!] \times \cdots \times [\![1, \mathbf{d}[D]]\!]$;
- $\forall \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}_g : (\{\mathbf{v}_1, \mathbf{v}_2\} \in \mathcal{E}_g) \Leftrightarrow (\exists i \in [\![1, D]\!] : (|\mathbf{v}_1[i] - \mathbf{v}_2[i]| = 1) \wedge (\forall j \in [\![1, D]\!], j \neq i : \mathbf{v}_1[j] = \mathbf{v}_2[j]))$.

The torus graph of dimensions $\mathbf{d}$ can be defined just as the grid graph, considering operations to be performed over $\mathbb{Z}/\mathbf{d}[i]\mathbb{Z}$ for the i^{th} coordinate:

Definition 7 (Torus graph)**.** Let $\mathbf{d} \in {\mathbb{N}^*}^D$. The *torus graph* $\mathcal{G}_t = \langle \mathcal{V}_t, \mathcal{E}_t \rangle$ yielded by the dimensions vector $\mathbf{d}$ is the graph such that:

- $\mathcal{V}_t = [\![1, \mathbf{d}[1]]\!] \times [\![1, \mathbf{d}[2]]\!] \times \cdots \times [\![1, \mathbf{d}[D]]\!]$;
- $\forall \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}_t : (\{\mathbf{v}_1, \mathbf{v}_2\} \in \mathcal{E}_t) \Leftrightarrow ((|\mathbf{v}_1[i] - \mathbf{v}_2[i]| \in \{1, \mathbf{d}[D] - 1\}) \wedge (\forall j \in [\![1, D]\!], j \neq i : \mathbf{v}_1[j] = \mathbf{v}_2[j]))$.

As an example, Figure 2 provides a visual representation of the grid graph and the torus graph that are yielded by the dimensions vector $\mathbf{d} = \begin{bmatrix}6\\5\end{bmatrix}$. Vertices are placed according to the coordinates associated with vertices in $\mathcal{V}_g$ and $\mathcal{V}_t$.

It is important to notice that the grid graph and the torus graph are defined by associating coordinates with their vertices corresponding to a regular sampling of the Euclidean space. Therefore, vertices of $\mathcal{V}_g$ and $\mathcal{V}_t$ are by construction vectors, and defining a notion of translation is natural. However, when considering graphs in the general case, no underlying geometry is available, and only the existence of connections among vertices is observed. Thus, there is no notion of translation to a particular direction due to the underlying Euclidean space, and one can only rely on the neighborhood of the vertices. For this reason, we propose in the following subsections some definitions for transformations and translations on graphs that only rely on neighborhood properties.

Finally, in Section V, we study random graphs as follows:

Definition 8 (Watts-Strogatz graph)**.** A graph following a Watts-Strogatz model [14] of parameters P and K is built from the following graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$:

- $\mathcal{V} = [\![1, N]\!]$;
- $\forall v \in \mathcal{V}, \forall k \in [\![1, K]\!] : \{v, v+k\} \in \mathcal{E}$, where $+$ is the addition operator on $\mathbb{Z}/N\mathbb{Z}$.

Then, a *Watts-Strogatz graph* $\mathcal{G}_{ws} = \langle \mathcal{V}_{ws}, \mathcal{E}_{ws} \rangle$ is built by replacing with probability P one side of each edge from $\mathcal{E}$ with a randomly selected vertex, and avoiding duplicates:

- $\mathcal{V}_{ws} = \mathcal{V}$;
- $\forall \{v_1, v_2\} \in \mathcal{E}$:
$$\begin{cases} \{v_1, v_2\} \in \mathcal{E}_{ws} \quad \text{with probability } 1 - P \\ (\{v_1, v_3\} \in \mathcal{E}_{ws}) \wedge (v_1 \neq v_3) \wedge (\{v_1, v_3\} \notin \mathcal{E}_{ws}) \\ \hfill \text{with probability } P \end{cases} .$$

In this model, K controls the original neighborhood of every vertex, and P controls the quantity of *disorder* in the graph. In particular, for $K = 1$ and $P = 0$, the graph is a one-dimensional torus, and for $P = 1$, it is completely randomized.

B. Transformations and translations on graphs

Let us consider a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$. Additionally, let us introduce an element $\perp$ such that $\perp \notin \mathcal{V}$.

Definition 9 (Transformation)**.** A *transformation* on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a function $\phi : \mathcal{V} \to \mathcal{V} \cup \{\perp\}$ such that

$$\forall v_1, v_2 \in \mathcal{V} : (\phi(v_1) = \phi(v_2) \neq \perp) \Rightarrow (v_1 = v_2) \ .$$

We denote the set of transformations on $\mathcal{G}$ by $\Phi_{\mathcal{G}}$.

Informally, a transformation $\phi \in \Phi_{\mathcal{G}}$ on a graph is a function that is injective for every vertex whose image is not $\perp$.

Definition 10 (Loss of a transformation)**.** We call *loss* of a transformation the quantity $|\{v \in \mathcal{V} \mid \phi(v) = \perp\}|$, noted $loss(\phi)$. In the case where $loss(\phi) = 0$, we say that ϕ is *lossless*, and we note it ϕ^*. We denote the set of lossless transformations on $\mathcal{G}$ by $\Phi^*_{\mathcal{G}}$.

In the case of lossless transformations, every vertex has an image in $\mathcal{V}$. Therefore, they are bijective from $\mathcal{V}$ to $\mathcal{V}$.

It is also interesting to notice that every graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ admits a transformation of loss N:

$$\phi_\perp : \left\{ \begin{array}{ccc} \mathcal{V} & \to & \mathcal{V} \cup \{\perp\} \\ v & \mapsto & \perp \end{array} \right. . \quad (1)$$

Note that transformations do not take into consideration the edges of the graph. To add the constraint that vertices should be mapped to vertices in their neighborhood, we introduce edge-constrained transformations:

Definition 11 (Edge-constrained (EC) transformation)**.** A transformation on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is said to be *edge-constrained* if it is a function $\phi : \mathcal{V} \rightarrow \mathcal{V} \cup \{\perp\}$ such that:

$$(\{v, \phi(v)\} \in \mathcal{E}) \vee (\phi(v) = \perp) \ .$$

We denote the set of EC transformations on $\mathcal{G}$ by EC$(\Phi_{\mathcal{G}})$, and the set of lossless EC transformations on $\mathcal{G}$ by EC$\big(\Phi^*_{\mathcal{G}}\big)$.

Proposition 1. An EC transformation $\phi \in$ EC$(\Phi_{\mathcal{G}})$ on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ injectively defines an orientation of $\mathcal{G}$, with the possible exclusion of vertices of which image is $\perp$.

Proof. Let $\mathbf{A}$ be the adjacency matrix of $\mathcal{G}$, and let $\mathbf{A}_\phi$ be a $N \times N$ matrix defined as follows:

$$\forall v_1, v_2 \in \mathcal{V} : \mathbf{A}_\phi[v_1, v_2] = \begin{cases} 1 & \text{if } v_2 = \phi(v_1) \\ 0 & \text{otherwise} \end{cases} \ .$$

Remind that $\mathbf{A}$ and $\mathbf{A}_\phi$ take their values in $\{0, 1\}$. First, we show that $\forall v_1, v_2 \in \mathcal{V} : \mathbf{A}_\phi[v_1, v_2] \leq \mathbf{A}[v_1, v_2]$. Let us consider a vertex $v_1 \in \mathcal{V}$. There are three possible cases:

1) $\phi(v_1) = \perp$. In that case, $\forall v_2 \in \mathcal{V} : \mathbf{A}_\phi[v_1, v_2] = 0$.
2) $\exists v_2 \in \mathcal{N}(v_1) : (v_2 = \phi(v_1)) \wedge (v_1 = \phi(v_2))$. In that case, $\mathbf{A}_\phi[v_1, v_2] = \mathbf{A}_\phi[v_2, v_1] = \mathbf{A}[v_1, v_2] = 1$, and $\forall v_3 \in \mathcal{V}, v_3 \neq v_2 : \mathbf{A}_\phi[v_1, v_3] = \mathbf{A}_\phi[v_3, v_1] = 0$ (due to the injectivity of ϕ).
3) $\exists v_2 \in \mathcal{N}(v_1) : (v_2 = \phi(v_1)) \wedge (v_1 \neq \phi(v_2))$. In that case, $\mathbf{A}_\phi[v_1, v_2] = \mathbf{A}[v_1, v_2] = 1$, and $\mathbf{A}_\phi[v_2, v_1] < \mathbf{A}[v_2, v_1]$, and $\forall v_3 \in \mathcal{V}, v_3 \neq v_2 : \mathbf{A}_\phi[v_1, v_3] = \mathbf{A}_\phi[v_3, v_1] = 0$ (due to the injectivity of ϕ).

In all cases, entries of $\mathbf{A}_\phi$ are lower or equal than those of $\mathbf{A}$. Additionally, due to case 3), there may exist $v_1, v_2 \in \mathcal{V} : \mathbf{A}_\phi[v_1, v_2] < \mathbf{A}[v_1, v_2]$. Therefore, $\mathbf{A}_\phi$ is not necessarily symmetric, and corresponds to a digraph in which every diedge contains elements that form an edge in $\mathcal{E}$.

Additionally, if $\mathbf{A}_{\phi_1} = \mathbf{A}_{\phi_2}$, then $\forall v \in \mathcal{V} : \phi_1(v) = \phi_2(v)$, *i.e.*, $\phi_1 = \phi_2$. So the mapping $\phi \mapsto \mathbf{A}_\phi$ is injective. □

The proof of Proposition 1 shows that it is possible to represent an EC transformation ϕ on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ by a digraph $\overrightarrow{\mathcal{G}}^\phi = \langle \mathcal{V}, \overrightarrow{\mathcal{E}}^\phi \rangle$, with

$$\forall v \in \mathcal{V} : (\phi(v) \neq \perp) \Leftrightarrow \left((v, \phi(v)) \in \overrightarrow{\mathcal{E}}^\phi \right) \ .$$

This allows a visual representation of EC transformations on a graph, where edges of $\mathcal{E}$ are depicted with dotted lines, on top of which edges of $\overrightarrow{\mathcal{E}}^\phi$ are drawn with plain arrows. Additionally, we mark the vertices that have their image being $\perp$ by coloring them in black. Figure 3 depicts an EC transformation on an example graph.

Using this correspondence with a digraph, we can reformulate the loss of an EC transformation as follows:

Proposition 2. Let $\phi \in$ EC$(\Phi_{\mathcal{G}})$ be an EC transformation on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$, with associated digraph $\overrightarrow{\mathcal{G}}^\phi = \langle \mathcal{V}, \overrightarrow{\mathcal{E}}^\phi \rangle$. Let $\mathbf{A}_\phi$ be the adjacency matrix associated with $\overrightarrow{\mathcal{G}}^\phi$, then:

$$loss(\phi) = N - \left| \left\{ \{v_1, v_2\} \in \binom{\mathcal{V}}{2} \;\middle|\; \mathbf{A}_\phi[v_1, v_2] = 1 \right\} \right| \ .$$

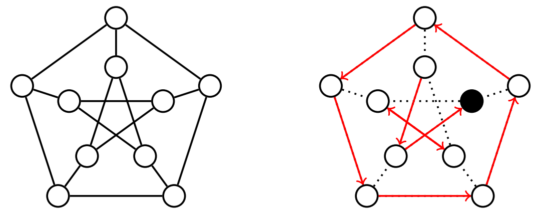


Figure 3: Example of a simple, symmetric graph (left) and an associated EC transformation with loss 1 (right)

Proof. Let $v_1 \in \mathcal{V}$. If $\phi(v_1) \in \mathcal{V}$, then due to injectivity, there is a unique $v_2 \in \mathcal{V}$ such that $\mathbf{A}_\phi[v_1, v_2] = 1$. If $\phi(v_1) = \perp$, then $\forall v_2 \in \mathcal{V} : \mathbf{A}_\phi[v_1, v_2] = 0$. Therefore, $loss(\phi)$ is N minus the number of vertices that have an image in $\mathcal{V}$. □

Not all graphs admit lossless EC transformations. Indeed, we can derive a few sufficient properties as well as necessary ones for an EC transformation to be lossless:

Proposition 3. Consider a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$. In order to have EC$\big(\Phi^*_{\mathcal{G}}\big) \neq \emptyset$, we have the following properties:

1) (Necessary): $\forall v \in \mathcal{V} : |\mathcal{N}(v)| > 0$;
2) (Necessary): No vertex is the unique neighbor for two other vertices;
3) (Sufficient): There exists an Hamiltonian cycle in $\mathcal{G}$, *i.e.*, a cycle that contains every vertex of $\mathcal{V}$ exactly once;
4) (Sufficient): There exists a perfect matching between all vertices in $\mathcal{V}$, *i.e.*, there is a subset $\mathcal{E}'$ of $\mathcal{E}$ such that every vertex appears exactly once in the edges of $\mathcal{E}'$.

Proof. Let $\phi \in$ EC$(\Phi_{\mathcal{G}})$ be an EC transformation. Let us consider the properties in the same order as above:

1) Let $v \in \mathcal{V}$. If $|\mathcal{N}(v)| = 0$, then the case $\{v, \phi(v)\} \in \mathcal{E}$ of Definition 11 is never matched, therefore $\phi(v) = \perp$.
2) Let $v_1, v_2, v_3 \in \mathcal{V}$, with $\mathcal{N}(v_1) = \{v_3\}$ and $\mathcal{N}(v_2) = \{v_3\}$. To avoid the case where a vertex has its image equal to $\perp$, we must have $\phi(v_1) = v_3$ and $\phi(v_2) = v_3$. However, this contradicts injectivity of transformations.
3) Let $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_N \rightarrow v_1$ be a Hamiltonian cycle. The transformation that associates every vertex with its sucessor in the cycle is EC, and lossless.
4) If a perfect matching exists, we can determine $\mathcal{E}' \subset \mathcal{E}$ with $\left|\mathcal{E}'\right| = \frac{N}{2}$ such that $\forall v_1 \in \mathcal{V} : \exists v_2 \in \mathcal{V} : \{v_1, v_2\} \in \mathcal{E}'$. In this case, the transformation that associates with v_1 its neighbor v_2 is EC and lossless.

□

Among all transformations, we are in particular interested in translations. Since their definition is not straightforward, let us first introduce the following properties for transformations:

Definition 12 (Weakly neighborhood-preserving (WNP) transformation)**.** We say that a transformation $\phi \in \Phi_{\mathcal{G}}$ on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is *weakly neighborhood-preserving* if

$$\forall \{v_1, v_2\} \in \mathcal{E} : (\{\phi(v_1), \phi(v_2)\} \in \mathcal{E}) \vee (\phi(v_1) = \perp) \vee (\phi(v_2) = \perp) \ .$$

We note WNP$(\Phi_{\mathcal{G}})$ the set of WNP transformations on $\mathcal{G}$, and WNP$\big(\Phi^*_{\mathcal{G}}\big)$ the set of lossless WNP transformations on $\mathcal{G}$.

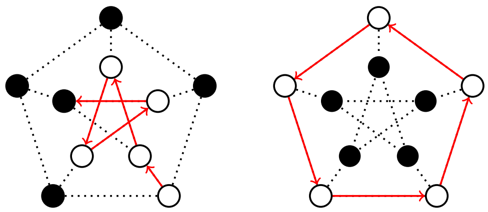


Figure 4: Examples of transformations that are translations on the Petersen graph introduced in Figure 3.

Informally, WNP transformations conserve existing neighborhoods. However, note that two vertices that are not neighbors may be associated with neighboring vertices through a WNP transformation. Transformations that do not create additional neighborhoods are characterized as follows:

Definition 13 (Strongly neighborhood-preserving (SNP) transformation)**.** We say that a transformation $\phi \in \Phi_{\mathcal{G}}$ on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is *strongly neighborhood-preserving* if

$$\forall v_1, v_2 \in \mathcal{V} : (\{v_1, v_2\} \in \mathcal{E} \Leftrightarrow \{\phi(v_1), \phi(v_2)\} \in \mathcal{E}) \\ \vee (\phi(v_1) = \perp) \vee (\phi(v_2) = \perp) \ .$$

We denote the set of SNP transformations on $\mathcal{G}$ by $\mathrm{SNP}(\Phi_{\mathcal{G}})$, and the set of lossless SNP transformations on $\mathcal{G}$ by $\mathrm{SNP}\left(\Phi^*_{\mathcal{G}}\right)$.

We can now define translations on graphs as follows:

Definition 14 (Translation on a graph)**.** A *translation* $\psi \in \Phi_{\mathcal{G}}$ on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is an EC and SNP transformation. We denote the set of translations on $\mathcal{G}$ by $\Psi_{\mathcal{G}}$, and the set of lossless translations on $\mathcal{G}$ by $\Psi^*_{\mathcal{G}}$.

Figure 4 depicts two examples of translations on a graph. Again, note that the function $\phi_\perp$ introduced in (1) is a translation for any graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$. Additionally, we observe the following property:

Proposition 4. Let $\psi \in \Psi_{\mathcal{G}}$ be a translation on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$, with associated digraph $\overrightarrow{\mathcal{G}}^{\psi} = \langle \mathcal{V}, \overrightarrow{\mathcal{E}}^{\psi} \rangle$. Edges of $\overrightarrow{\mathcal{E}}^{\psi}$ can be partitioned into directed cycles, and directed paths that have one vertex for which the image is $\perp$.

Proof. By injectivity of transformations, any vertex $v_1 \in \mathcal{V}$ has an image by ψ which is either a vertex $v_2 \in \mathcal{V}$ with no other inverse image, or $\perp$. Therefore, every vertex belongs either to a path $v_1 \to v_2 \to \cdots \to \perp$, or a cycle $v_1 \to v_2 \to \cdots \to v_1$. Additionally, the associated digraph restricts the existence of these paths and cycles to paths and cycles that exist in $\mathcal{E}$. □

It is also interesting to notice that every translation admits an inverse translation with the same loss:

Proposition 5. Let $\psi \in \Psi_{\mathcal{G}}$ be a translation on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$, with associated digraph $\overrightarrow{\mathcal{G}}^{\psi} = \langle \mathcal{V}, \overrightarrow{\mathcal{E}}^{\psi} \rangle$ of adjacency matrix $\mathbf{A}_\psi$. Let us call ψ^{-1} the inverse translation associated with the digraph $\overrightarrow{\mathcal{G}}^{\psi^{-1}} = \langle \mathcal{V}, \overrightarrow{\mathcal{E}}^{\psi^{-1}} \rangle$, with $(v_1, v_2) \in \overrightarrow{\mathcal{E}}^{\psi^{-1}} \Leftrightarrow (v_2, v_1) \in \overrightarrow{\mathcal{E}}^{\psi}$. We have the relation $loss(\psi) = loss(\psi^{-1})$.

Proof. First, let us notice that $\overrightarrow{\mathcal{E}}^{\psi^{-1}}$ is the exact same set of edges as in $\overrightarrow{\mathcal{E}}^{\psi}$, but with reverse direction. Therefore, since ψ is EC, it is also the case for ψ^{-1}. Additionally, since ψ is SNP, it preserves the existing neighborhoods and does not create additional ones. Therefore, it is also the case for the converse, and ψ^{-1} is thus SNP. From Definition 14, ψ^{-1} is therefore a translation. Finally, from Proposition 2, and noticing that the adjacency matrix associated with $\overrightarrow{\mathcal{G}}^{\psi^{-1}}$ is the transpose of $\mathbf{A}_\psi$, we conclude. □

Translations can be given a well-founded relation $\prec$:

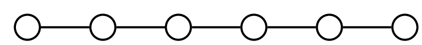
$$\forall \psi_1, \psi_2 \in \Psi_{\mathcal{G}} : (\psi_1 \prec \psi_2) \Leftrightarrow (loss(\psi_1) > loss(\psi_2)) \\ \wedge (\exists v \in \mathcal{V} : \psi_1(v) = \psi_2(v)) \ .$$

Proposition 6. Let $\psi_1, \psi_2, \psi_3 \in \Psi_{\mathcal{G}}$. The relation $\prec$ has the following properties:

1) It is *irreflexive*, *i.e.*, $\psi_1 \prec \psi_1$ is not true.
2) It is *antisymmetric*, *i.e.*, it is not possible to have both $\psi_1 \prec \psi_2$ and $\psi_2 \prec \psi_1$.
3) It is *intransitive*, *i.e.*, it is not true that $((\psi_1 \prec \psi_2) \wedge (\psi_2 \prec \psi_3)) \Rightarrow (\psi_1 \prec \psi_3)$.

Proof. Let us consider the three properties separately:

1) By definition, ψ_1 is comparable to itself, since there exists at least one edge in common. Comparison is then made using $<$, which is an irreflexive order on $\mathbb{Z}$.
2) In the case where $\nexists v \in \mathcal{V} : \psi_1(v) = \psi_2(v)$, then ψ_1 and ψ_2 are not comparable (noted $\psi_1 \sim \psi_2$). In the case where such an edge exists, $<$ is an antisymmetric order on $\mathbb{Z}$.
3) Let us consider the following graph:

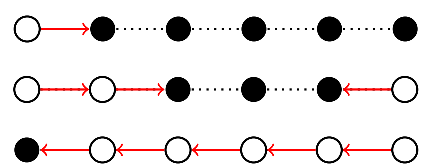


Let ψ_1, ψ_2, ψ_3 be the following translations:

ψ_1 :

ψ_2 :

ψ_3 :



In this example, $\psi_1 \prec \psi_2$ and $\psi_2 \prec \psi_3$. However, ψ_1 and ψ_3 have no edge in common, thus $\psi_1 \sim \psi_3$. Still, note that ψ_3^{-1}, the inverse translation of ψ_3, is comparable with ψ_1 and ψ_2. Therefore, $\prec$ is not an *antitransitive* relation.

□

Using this relation, we define minimal translations:

Definition 15 (Minimal translation)**.** A translation $\psi_1 \in \Psi_{\mathcal{G}}$ is *minimal* if there is no $\psi_2 \in \Psi_{\mathcal{G}}$ such that $\psi_1 \prec \psi_2$, *i.e.*, if it minimizes the loss.

Indeed, lossless translations $\psi^* \in \Psi^*_{\mathcal{G}}$ are necessarily minimal. Additionally, we define pseudo-minimal translations:

Definition 16 (Pseudo-minimal translation)**.** *Pseudo-minimal* translations are defined inductively. A translation $\psi_1 \in \Psi_{\mathcal{G}}$ is pseudo-minimal if one of the following holds:

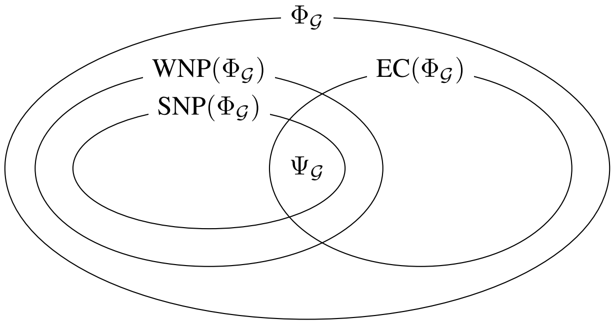


Figure 5: Venn diagram summarizing the various types of transformations introduced in this section.

1) ψ_1 is minimal;
2) Any translation $\psi_2 \in \Psi_\mathcal{G}$ such that $\psi_1 \prec \psi_2$ is not pseudo-minimal.

Proposition 7. Any graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ admits at least one minimal translation $\psi \in \Psi_\mathcal{G}$. Also, ψ^{-1} is minimal.

Proof. In order to show that a minimal translation exists, let us study the following cases:

1) In the case where $|\mathcal{E}| = 0$, the only possible transformation is $\phi_\perp$ (1), which is thus minimal;
2) In the more general case, let us consider an edge $\{v_1, v_2\} \in \mathcal{E}$. The function ψ_1 such that $\psi_1(v_1) = v_2$, and $\forall v_3 \neq v_1 : \psi_1(v_3) = \perp$ is obviously a translation. Now, consider a maximal sequence $(\psi_i)_i$ of translations of which first element is ψ_1 and such that $\forall i : \psi_i \prec \psi_{i+1}$. This sequence is necessarily finite, since $loss(\psi_i)$ decreases and $<$ is a well-founded order on $\mathbb{Z}$. By definition, the last element ψ_j of this sequence is a minimal translation.
Now, let us show that ψ_j^{-1} — the inverse translation as defined in Proposition 5 — is also minimal. From Proposition 5, we have $loss(\psi_j) = loss(\psi_j^{-1})$. Now, let us imagine that there exists ψ_k such that $\psi_j^{-1} \prec \psi_k$. Using the same reasoning as above, and noticing that ψ_j^{-1} and ψ_k share at least one edge, we obtain that $\psi_j \prec \psi_k^{-1}$. Since ψ_j is minimal, we reach a contradiction. As a consequence, both ψ_j and ψ_j^{-1} are minimal translations. It is interesting to notice that a special case occurs when ψ_j is a perfect matching between all vertices in $\mathcal{V}$. In this situation, we have $\psi_j = \psi_j^{-1}$.

□

To sum up the various sets we introduced in this section, Figure 5 presents the corresponding Venn diagram.

C. Isometries on graphs

Translations on Euclidean spaces are isometries. When it comes to graphs, we also want the translations to be distance-preserving functions. However, since in the general case there is no Euclidean space associated with the graph, we consider here the geodesic distance d.

Definition 17 (Isometry on a graph)**.** A transformation $\phi \in \Phi_\mathcal{G}$ on a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is an *isometry* if

$$\begin{aligned}\forall v_1, v_2 \in \mathcal{V} : (d(v_1, v_2) = d(\phi(v_1), \phi(v_2)))\\ \vee(\phi(v_1) = \perp) \vee (\phi(v_2) = \perp)\ .\end{aligned}$$

We denote the set of isometries on $\mathcal{G}$ by ISO($\Phi_\mathcal{G}$), and the set of lossless isometries on $\mathcal{G}$ by $\mathrm{ISO}\big(\Phi^*_\mathcal{G}\big)$.

Examples of isometries are the translations presented in Figure 4.

Proposition 8. Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a graph. We have that $\mathrm{SNP}\big(\Phi^*_\mathcal{G}\big) \subset \mathrm{ISO}\big(\Phi^*_\mathcal{G}\big)$.

Proof. Let $v_1, v_2 \in \mathcal{V}$. For a lossless SNP transformation $\phi^* \in \mathrm{SNP}\big(\Phi^*_\mathcal{G}\big)$, we distinguish the following cases:

1) There is no path between vertices v_1 and v_2, *i.e.*, $d(v_1, v_2)$ is infinite. This corresponds to the case where they belong to different connected components. By contradiction, let $d(\phi^*(v_1), \phi^*(v_2))$ be finite. This implies that there exists a path $\phi^*(v_1) \to \phi^*(v_{i_1}) \to \phi^*(v_{i_2}) \to \dots \to \phi^*(v_{i_k}) \to \phi^*(v_2)$ in the graph. Since ϕ^* is SNP, we have that $v_1 \to v_{i_1} \to v_{i_2} \to \dots \to v_{i_k} \to v_2$ is also a path in the graph, therefore we reach a contradiction. As a consequence, $d(v_1, v_2)$ and $d(\phi^*(v_1), \phi^*(v_2))$ are both infinite.
2) A shortest path $v_1 \to v_{i_1} \to v_{i_2} \to \dots \to v_{i_k} \to v_2$ exists. Since ϕ^* is lossless, $\phi^*(v_1) \to \phi^*(v_{i_1}) \to \phi^*(v_{i_2}) \to \dots \to \phi^*(v_{i_k}) \to \phi^*(v_2)$ is also a path (no intermediar vertex has its image equal to $\perp$). Additionally, ϕ^* being SNP, it does not create nor remove neighborhoods, so $\phi^*(v_1) \to \phi^*(v_{i_1}) \to \phi^*(v_{i_2}) \to \dots \to \phi^*(v_{i_k}) \to \phi^*(v_2)$ is also a shortest path from $\phi^*(v_1)$ to $\phi^*(v_2)$. Therefore, we have $d(v_1, v_2) = d(\phi^*(v_1), \phi^*(v_2))$.

□

Corrolary 1. Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a graph. $\Psi^*_\mathcal{G} \subset \mathrm{ISO}\big(\Phi^*_\mathcal{G}\big)$.

Proof. $\Psi^*_\mathcal{G} \subset \mathrm{SNP}\big(\Phi^*_\mathcal{G}\big) \subset \mathrm{ISO}\big(\Phi^*_\mathcal{G}\big)$. □

Note that Proposition 8 holds for lossless SNP transformations only. In the more general case of SNP transformations $\phi \in$ SNP($\Phi_\mathcal{G}$), having a vertex that has its image equal to $\perp$ may cause $d(\phi(v_1), \phi(v_2))$ and $d(v_1, v_2)$ to be different. As an example, Figure 6 depicts a SNP transformation $\phi \in$ SNP($\Phi_\mathcal{G}$) for which $d(v_1, v_2) < d(\phi(v_1), \phi(v_2))$. When considering the inverse transformation ϕ^{-1}, we have $d(v_1, v_2) > d(\phi^{-1}(v_1), \phi^{-1}(v_2))$.

Still, it is interesting to note that some transformations with a non-zero loss are isometries. Examples of such are depicted in Figure 4.

IV. Results and proofs

In this section, we are interested in identifying translations on arbitrary graphs. After introducing some generic results, we study the case of the torus graph, on which defining a notion of translation is intuitive. This gives us intuition on how to find these translations without considering the underlying Euclidean space. Then, we show that we are able to find the

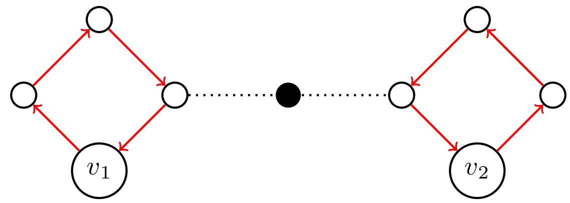


Figure 6: Example of a SNP transformation $\phi \in \mathrm{SNP}(\Phi_{\mathcal{G}})$ with a non-zero loss that is not an isometry. In this example, $d(v_1, v_2) = 4$, $d(\phi(v_1), \phi(v_2)) = 6$ and $d(\phi^{-1}(v_1), \phi^{-1}(v_2)) = 2$.

translations on a grid graph, and extend our results to families of graphs that are more generic.

A. Results on generic graphs

As stated before, the function $\phi_\perp$ introduced in (1) is a translation for any graph. However, it is not very interesting, since it destroys all signal information when translating it. Therefore, we need to identify more complex translations that keep most of the signal entries. As a consequence, we are particularly interested in minimal translations.

Before trying to identify translations, let us provide bounds on the number of translations:

Proposition 9. A graph with order N cannot admit more than

$$\sum_{k=0}^{N} \frac{1}{(N-k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (N-j)!$$

translations. This number is reached for the complete graph $\mathcal{G}_c = \langle \mathcal{V}_c, \mathcal{E}_c \rangle$.

Proof. Every EC transformation on $\mathcal{G}_c$ is necessary SNP, since all vertices are pairwise linked and therefore share the same neighborhood. However, not every transformation on such graph is EC, since transformations can map vertices to themselves, and we consider simple graphs only. Therefore, for a fixed loss $N-k$ ($k \leq N$), the set of translations of loss $N-k$ is exactly the set of injective functions that have no fixed points of k elements to N. The cardinal of such a set is given by the solution of the (N,k)-*matching problem* in [15] as follows:

$$\frac{1}{(N-k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (N-j)! \,. \tag{2}$$

By summing for every possible value of k, corresponding to the number of vertices that have an image different to $\perp$, we obtain the number of translations on $\mathcal{G}_c$. Then, note that any graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ of order N has its edges $\mathcal{E} \subset \mathcal{E}_c$. As a consequence, since any EC transformation is a translation on $\mathcal{G}_c$, it follows that any translation on $\mathcal{G}$ is also a translation on $\mathcal{G}_c$. Therefore, a graph of order N cannot admit more translations than the complete graph. □

This characterization of the number of translations gives us the following result on the number of minimal translations:

Proposition 10. A graph of order N cannot admit more than

$$N! \sum_{j=0}^{N} \frac{(-1)^j}{j!}$$

minimal translations. This number is reached for the complete graph $\mathcal{G}_c = \langle \mathcal{V}_c, \mathcal{E}_c \rangle$.

Proof. Minimal translations on $\mathcal{G}_c$ are necessarily lossless, since any translation shares at least a diedge with an Hamiltonian cycle on this graph, which is lossless. By particularizing (2) for $k = N$, we obtain the number of lossless translations on $\mathcal{G}_c$, which is exactly the number of derangements of a set of N elements, *i.e.*, the number of permutations of N elements with no fixed points [16].

Using the same reasoning as in the proof of Proposition 9, any minimal translation on a graph $\mathcal{G}$ is included in a minimal translation on $\mathcal{G}_c$. Therefore, a graph of order N cannot admit more minimal translations than the complete graph. □

The number of translations on graphs is therefore exponential in the general case. Additionally, we prove that identifying translations is a complex problem:

Proposition 11. The problem of deciding, for an input graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ and two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{V}$, if there is a translation for which the image is exactly $\mathcal{V}_2$ and with inverse images only in $\mathcal{V}_1$ is NP-complete.

Proof. To prove this result, we first prove that the problem is NP, and then that it is NP-hard.

All possible transformations with inverse image set $\mathcal{V}_1$ and image set $\mathcal{V}_2$ can be generated non-deterministically by induction as follows:

1) $\phi_\perp \in \Phi_{\mathcal{G}}$;
2) If we have a transformation $\phi_1 \in \Phi_{\mathcal{G}}$, and two vertices $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ such that $\phi_1(v_1) = \perp$ and $\nexists v_3 \in \mathcal{V}_1 : \phi_1(v_3) = v_2$, then define $\phi_2 \in \Phi_{\mathcal{G}}$ as follows:
   - $\phi_2(v_1) = v_2$;
   - $\forall v_3 \in \mathcal{V}_1, v_3 \neq v_1 : \phi_2(v_3) = \phi_1(v_3)$.

Furthermore, determining whether any such transformation is a translation or not can be done by checking EC and SNP constraints, which can be done in polynomial time. So the problem is NP.

Then, we prove the problem is NP-hard by reduction from the subgraph isomorphism problem. Consider two graphs $\mathcal{G}_1 = \langle \mathcal{V}_1, \mathcal{E}_1 \rangle$ and $\mathcal{G}_2 = \langle \mathcal{V}_2, \mathcal{E}_2 \rangle$. Without loss of generality, we consider that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. The more general case where $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \emptyset$ can be included in the proof by duplication of the vertices in the intersection.

From these graphs, we build the graph $\mathcal{G}_3 = \langle \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}_3 \rangle$, where $\mathcal{E}_3 = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{\{v_1, v_2\}, v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\}$. Note that this construction is at most quadratic in the order of $\mathcal{G}$. Then, we show that answering our problem on $\mathcal{G}_3$ solves the problem of subgraph isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$.

To this end, consider the following two properties, that we prove to be equivalent:

1) There is a translation of which image set is $\mathcal{V}_2$ and inverse images are in $\mathcal{V}_1$;
2) There is a subgraph of $\mathcal{G}_1$ isomorph to $\mathcal{G}_2$.

We prove this in two steps. First, consider there exists such a translation. Then, since it is SNP, the subgraph corresponding to the inverse images of vertices in $\mathcal{V}_2$ is isomorph to $\mathcal{G}_2$.

Conversely, consider there exists an isomorphism, then the transformation that associates each vertex in $\mathcal{V}_2$ with its corresponding vertex in $\mathcal{V}_1$ is a translation. indeed, it is EC because of the complete bipartite subgraph connecting vertices in $\mathcal{V}_2$ to vertices in $\mathcal{V}_1$, and it is SNP as a particularization of the isomorphism property.

As a consequence, the problem of deciding, for an input graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ and two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{V}$, if there is a translation for which the image is exactly $\mathcal{V}_2$ and with inverse images only in $\mathcal{V}_1$ is at least as difficult as the subgraph isomorphism problem. Since it is also NP, it is NP-complete. $\square$

These results tell us that finding the translations on a graph is a hard problem. Therefore, one may need to establish approximate methods to identify interesting translations on a given graph. In the following subsections, we focus on the particular case of highly regular graphs, namely the torus graph and the grid graph. We develop generic results on these graphs and extend them to any class of graphs.

B. Results on the torus graph

Let us first consider the case of the torus graph $\mathcal{G}_t = \langle \mathcal{V}_t, \mathcal{E}_t \rangle$ yielded by a dimensions vector $\mathbf{d} \in \mathbb{N}^{*D}$. Such graphs are highly regular, and are often used to model classical domains, such as the periodical time with a 1-dimensional torus graph, or the pixels of a periodical image with a 2-dimensional torus graph. Additionally, what makes these graphs interesting is the fact that they are constructed using an Euclidean space (see Definition 7). For this reason, we use the notation $v \in \mathcal{V}_t$ when referring to the index of vertex v, and $\mathbf{v} \in \mathcal{V}_t$ when referring to its Euclidean coordinates in $\mathbf{d}$.

In this section, we aim to find a relation between translations defined on an Euclidean space, and those defined on the graph, with no reference to the underlying metrics. To do so, let us first formalize the notion of Euclidean translation on $\mathcal{G}_t$.

Definition 18 (Euclidean translation on the torus graph)**.** An *Euclidean translation* ψ_t on the torus graph is such that:

$$\exists \boldsymbol{\delta} \in \mathbb{N}^D : \forall \mathbf{v} \in \mathcal{V}_t : \psi_t(\mathbf{v}) = \mathbf{v} + \boldsymbol{\delta} \ .$$

Definition 19 (Dirac vector for dimension i)**.** Let $\mathbf{e}_i$ be the vector in $\{0,1\}^D$ with a single non-null entry $i \in [\![1, D]\!]$, defined as follows:

$$\forall j \in [\![1, D]\!] : \mathbf{e}_i[j] = \left\{ \begin{array}{ll} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{array} \right. .$$

Remember from Definition 7 that coordinates of the torus graph are defined in $\mathbb{Z}/\mathbf{d}[i]\mathbb{Z}$ (for the i^{th} coordinate). Therefore, addition and subtraction take into account the modulo. By construction of the torus graph, we have that $\forall \mathbf{v} \in \mathcal{V}_t, \forall i \in [\![1, D]\!] : \{\mathbf{v}, \mathbf{v} + \mathbf{e}_i\} \in \mathcal{E}_t$.

Note that this is also true for the inverse Dirac vectors containing a single non-null entry i being -1. Consequently, the following results also apply using such vectors.

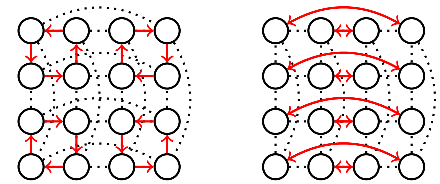


Figure 7: Examples of lossless translations on a torus of dimensions $\mathbf{d} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ that are not translations by a Dirac vector.

Lemma 1 (Contamination lemma on the torus graph)**.** Let $\psi \in \Psi^*_{\mathcal{G}_t}$ be a lossless translation on the torus graph, with $\forall i \in [\![1, D]\!] : \mathbf{d}[i] \geq 5$. Let $\mathbf{v}_1 \in \mathcal{V}_t$. Let us consider the Dirac vector $\mathbf{e}_j = \psi(\mathbf{v}_1) - \mathbf{v}_1$. Then, $\forall \mathbf{v}_2 \in \mathcal{V}_t : \psi(\mathbf{v}_2) = \mathbf{v}_2 + \mathbf{e}_j$.

Proof. We proceed in two steps:

1) First, let us show that $\psi(\mathbf{v}_1 - \mathbf{e}_j) = \mathbf{v}_1$. By construction of the torus graph in Definition 7, we have that $\{\mathbf{v}_1 - \mathbf{e}_j, \mathbf{v}_1\} \in \mathcal{E}_t$. Since ψ is EC, we must have $\psi(\mathbf{v}_1 - \mathbf{e}_j) \in \mathcal{N}(\mathbf{v}_1 - \mathbf{e}_j)$. Also, since ψ is SNP, we must have $\psi(\mathbf{v}_1 - \mathbf{e}_j) \in \mathcal{N}(\psi(\mathbf{v}_1))$. As a consequence, $\psi(\mathbf{v}_1 - \mathbf{e}_j) \in \mathcal{N}(\mathbf{v}_1 - \mathbf{e}_j) \cap \mathcal{N}(\psi(\mathbf{v}_1))$. The neighborhood of $\psi(\mathbf{v}_1)$ is $\mathcal{N}(\psi(\mathbf{v}_1)) = \mathcal{N}(\mathbf{v}_1 + \mathbf{e}_j) = \{\mathbf{v}_1 + \mathbf{e}_j + \mathbf{e}_1, \mathbf{v}_1 + \mathbf{e}_j - \mathbf{e}_1, \mathbf{v}_1 + \mathbf{e}_j + \mathbf{e}_2, \mathbf{v}_1 + \mathbf{e}_j - \mathbf{e}_2, \ldots, \mathbf{v}_1 + 2\mathbf{e}_j, \mathbf{v}_1, \ldots, \mathbf{v}_1 + \mathbf{e}_j + \mathbf{e}_D, \mathbf{v}_1 + \mathbf{e}_j - \mathbf{e}_D\}$. Similarly, the neighborhood of $\mathbf{v}_1 - \mathbf{e}_j$ is $\mathcal{N}(\mathbf{v}_1 - \mathbf{e}_j) = \{\mathbf{v}_1 - \mathbf{e}_j + \mathbf{e}_1, \mathbf{v}_1 - \mathbf{e}_j - \mathbf{e}_1, \mathbf{v}_1 - \mathbf{e}_j + \mathbf{e}_2, \mathbf{v}_1 - \mathbf{e}_j - \mathbf{e}_2, \ldots, \mathbf{v}_1, \mathbf{v}_1 - 2\mathbf{e}_j, \ldots, \mathbf{v}_1 - \mathbf{e}_j + \mathbf{e}_D, \mathbf{v}_1 - \mathbf{e}_j - \mathbf{e}_D\}$. Since $\forall i \in [\![1, D]\!] : \mathbf{d}[i] \geq 3$, we have $\mathbf{v}_1 + \mathbf{e}_j + \mathbf{e}_k \neq \mathbf{v}_1 - \mathbf{e}_j + \mathbf{e}_k$ $(j \neq k)$. Therefore, vertices with coordinates that differ by an entry in dimension $k \neq j$ cannot belong to the intersection by construction of the torus graph. Similarly, because $\forall i \in [\![1, D]\!] : \mathbf{d}[i] \geq 5$, we obtain that $\mathbf{v}_1 + 2\mathbf{e}_j$ cannot be the same vertex as $\mathbf{v}_1 - 2\mathbf{e}_j$, since they differ by $\alpha \mathbf{e}_j$ $(\alpha > 1)$. As a consequence, $\mathcal{N}(\mathbf{v}_1 - \mathbf{e}_j) \cap \mathcal{N}(\psi(\mathbf{v}_1)) = \{\mathbf{v}_1\}$, and thus $\psi(\mathbf{v}_1 - \mathbf{e}_j) = \mathbf{v}_1$.
2) Now, let us consider a vertex $\mathbf{v}_2 \in \mathcal{N}(v_1) \setminus \{v_1 - \mathbf{e}_j, v_1 + \mathbf{e}_j\}$. Let us show that $\psi(\mathbf{v}_2) = \mathbf{v}_2 + \mathbf{e}_j$. As in the step 1), comparing the neighborhoods of $\mathbf{v}_2$ and $\psi(\mathbf{v}_1)$ gives us $\mathcal{N}(\mathbf{v}_2) \cap \mathcal{N}(\psi(\mathbf{v}_1)) = \{\mathbf{v}_1, \mathbf{v}_2 + \mathbf{e}_j\}$. However, step 1) gives us that $\mathbf{v}_1$ is necessarily the image of $\mathbf{v}_1 - \mathbf{e}_j$. Since ψ is injective, it follows that $\psi(\mathbf{v}_2) = \mathbf{v}_2 + \mathbf{e}_j$.

By induction, we conclude that for every vertex $\mathbf{v}_2 \in \mathcal{V}_t$: $\psi(\mathbf{v}_2) = \mathbf{v}_2 + \mathbf{e}_j$. $\square$

The constraint of having all dimensions in $\mathbf{d}$ being larger than 5 allows any lossless translation to verify $\forall \mathbf{v} \in \mathcal{V}_t$: $\psi(\mathbf{v}) = \mathbf{v} + \mathbf{e}_j$. For smaller graphs, lossless translations can be found for which this property is not true. As an example, Figure 7 depicts lossless translations on a torus of dimensions $\mathbf{d} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ that are not translations by a Dirac vector. Still, note that any translation ψ such that $\forall \mathbf{v} \in \mathcal{V}_t : \psi(\mathbf{v}) = \mathbf{v} + \mathbf{e}_j$ is lossless even for smaller grid graphs.

Note that a direct consequence of Lemma 1 is that there are as many lossless translations as there are neighbors for a given vertex. By composing the lossless translations on the torus graph, we obtain more complex functions, that induce the following monoid:

Definition 20 (Monoid induced by $\Psi^*_{\mathcal{G}_t}$)**.** We call *monoid induced by* $\Psi^*_{\mathcal{G}_t}$ the minimum monoid containing $\Psi^*_{\mathcal{G}_t}$ with the composition of fonctions as inner law.

Proposition 12. For torus graphs with $\forall i \in [\![1, D]\!] : \mathbf{d}[i] \geq 5$, the monoid induced by $\Psi^*_{\mathcal{G}_t}$ is exactly the set of Euclidean translations on the torus graph.

Proof. A direct consequence of Lemma 1 is that lossless translations $\psi \in \Psi^*_{\mathcal{G}_t}$ on the torus graph can be obtained by choosing a Dirac vector for a dimension $i \in [\![1, D]\!]$ and applying the contamination. Therefore, $\forall i \in [\![1, D]\!] : \exists! \psi \in \Psi^*_{\mathcal{G}_t} : \forall \mathbf{v} \in \mathcal{V}_t : \psi(\mathbf{v}) = \mathbf{v} + \mathbf{e}_i$. We obtain that $\boldsymbol{\delta}$ in Definition 18 is a linear combination of vectors in $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_D\}$. As a consequence, any Euclidean translation on the torus graph can be written as a composition of lossless translations on the torus graph, which are elements of the monoid induced by $\Psi^*_{\mathcal{G}_t}$. □

C. Results on the grid graph

Let us now proceed with grid graphs $\mathcal{G}_g = \langle \mathcal{V}_g, \mathcal{E}_g \rangle$ yielded by a dimensions vector $\mathbf{d} \in {\mathbb{N}^*}^D$. We can adapt the definition of Euclidean translation on the torus graph in Definition 18 to grid graphs as follows:

Definition 21 (Euclidean translation on the grid graph)**.** An *Euclidean translation* ψ_g on the grid graph $\mathcal{G}_g = \langle \mathcal{V}_g, \mathcal{E}_g \rangle$ of dimensions $\mathbf{d} \in {\mathbb{N}^*}^D$ is such that:

$$\exists \boldsymbol{\delta} \in \mathbb{N}^D : \forall \mathbf{v} \in \mathcal{V}_g : \psi(\mathbf{v}) = \begin{cases} \mathbf{v} + \boldsymbol{\delta} & \text{if } \mathbf{v} + \boldsymbol{\delta} \in \mathcal{V}_g \\ \perp & \text{otherwise} \end{cases} .$$

Proposition 13. Let us consider the translation $\psi \in \Psi_{\mathcal{G}_g}$ such that

$$\forall \mathbf{v} \in \mathcal{V}_g : \psi(\mathbf{v}) = \begin{cases} \mathbf{v} + \mathbf{e}_i & \text{if } \mathbf{v} + \mathbf{e}_i \in \mathcal{V}_g \\ \perp & \text{otherwise} \end{cases} ,$$

for $\mathbf{e}_i$ the Dirac vector for dimension i. We have $loss(\psi) = \prod_{j \in [\![1,\, D]\!], j \neq i} \mathbf{d}[j]$.

Proof. By construction of the grid graph, two vertices are neighbors if their coordinates differ by 1 or -1 along a single dimension. In particular, it is true for dimension i. Therefore, any vertex $\mathbf{v}$ such that $\mathbf{v} + \mathbf{e}_i \notin \mathcal{V}_g$ is such that $\mathbf{v}[i] = \mathbf{d}[i]$. The product of dimensions that are different from i gives us the number of such vertices, hence the loss of ψ. □

Remark 1. As for torus graphs, note that all the results in this section also apply when considering inverse Dirac vectors containing a single non-null entry i being -1.

As for torus graphs, we can introduce the monoid induced by the translations on the grid graph as follows:

Definition 22 (Monoid induced by $\Psi_{\mathcal{G}_g}$)**.** We call *monoid induced by* $\Psi_{\mathcal{G}_g}$ the minimum monoid containing $\Psi_{\mathcal{G}_g}$ with the composition of fonctions as inner law.

Proposition 14. The monoid induced by $\Psi_{\mathcal{G}_g}$ includes the set of Euclidean translations on the grid graph.

Proof. Translations by Dirac vectors introduced in Proposition 13 exist for every dimension i. It follows that $\boldsymbol{\delta}$ in Definition 21 is a linear combination of vectors in $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_D\}$. As a consequence, any Euclidean translation on the grid graph can be written as a composition of translations on the grid graph, which are elements of the monoid induced by $\Psi_{\mathcal{G}_g}$.

However, contrary to the case of torus graphs in Proposition 12, translations introduced in Proposition 13 are only a subset of $\Psi_{\mathcal{G}_g}$. Therefore, Euclidean translations are included in the monoid induced by $\Psi_{\mathcal{G}_g}$, but the converse is not true.

As an counterexample, for the Dirac vector $\mathbf{e}_1$ and a grid graph such that $\forall i \in [\![1, D]\!] : \mathbf{d}[i] \geq 3$, the translation $\psi \in \Psi_{\mathcal{G}_g}$ such that

$$\forall \mathbf{v} \in \mathcal{V}_g : \psi(\mathbf{v}) = \begin{cases} \mathbf{v} + \mathbf{e}_1 & \text{if } (\mathbf{v} + \mathbf{e}_1 \in \mathcal{V}_g) \wedge \left(\mathbf{v} \neq \begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix} \right) \\ \perp & \text{otherwise} \end{cases}$$

is not an Euclidean translation on the grid graph. □

Now, we are interested in showing that Euclidean translations by $\mathbf{e}_i$ (or $-\mathbf{e}_i$) are pseudo-minimal on the grid graph. We restrict our study to a subclass of grid graphs such that each dimension is large compared to the following ones, *i.e.*,

$$\mathbf{d}[D] \geq 3 \wedge \forall i \in [\![1, D-1]\!] : \mathbf{d}[i] \geq 2 + 2 \prod_{j=i+1}^{D} \mathbf{d}[j] . \quad (3)$$

This hypothesis is necessary for the subsequent proofs. However, we conjecture the following result:

Conjecture 1. The forthcoming results apply for grid graphs such that $\forall i \in [\![1, D]\!] : \mathbf{d}[i] \geq 6$.

To ease exposition of the following results, we introduce the notion of *slice of a grid graph* as follows:

Definition 23 (Grid graph slice)**.** We call *slice of a grid graph*, noted $\mathcal{V}_g^{i,j}$ the subset of vertices $\mathcal{V}_g$ such that they have their i^{th} coordinate equal to j, *i.e.*,

$$\mathcal{V}_g^{i,j} = \{\mathbf{v} \in \mathcal{V}_g \mid \mathbf{v}[i] = j\} .$$

Lemma 2. Let $\mathcal{G}_g$ be a grid graph respecting assumption (3). If $\psi \in \Psi_{\mathcal{G}_g}$ is a minimal translation, then:

$$\exists i : \forall \mathbf{v} \in \mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1} : \psi(\mathbf{v}) \neq \perp .$$

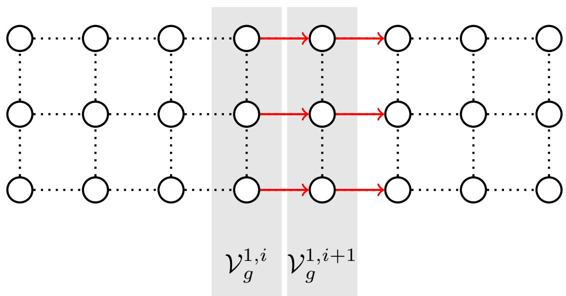
Proof. By Proposition 13, we have an upper bound on the loss of minimal translations when translating vertices along a single dimension. When considering dimension 1, any minimal translation has therefore at most $\prod_{j \in [\![2,\, D]\!]} \mathbf{d}[j]$ vertices that have their image through ψ being $\perp$. From assumption (3), we have that $\mathbf{d}[1] \geq 2 + 2 \prod_{j \in [\![2,\, D]\!]} \mathbf{d}[j]$. Since $\mathbf{d}[1] - 2 \prod_{j \in [\![2,\, D]\!]} \mathbf{d}[j] + 1 > 1$, there cannot be a strict alternance of vertices with image in $\mathcal{V}_g$, and vertices with image equal to $\perp$, as it would violate the upper bound on the loss. Therefore, there exist two slices $\mathcal{V}_g^{1,i}$ and $\mathcal{V}_g^{1,i+1}$ that contain no vertex $\mathbf{v}$ such that $\psi(\mathbf{v}) = \perp$. □

Lemma 3. Let $\mathcal{G}_g$ be a grid graph respecting assumption (3), and let $\psi \in \Psi_{\mathcal{G}_g}$ be a minimal translation. If $\mathcal{V}_g^{1,i}$ and $\mathcal{V}_g^{1,i+1}$ are two slices containing no vertex of which image by ψ is $\perp$, then:

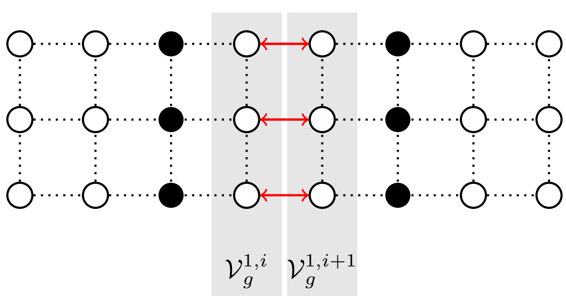
$$\psi(\mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}) \not\subset \mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1} .$$

Proof. Let us consider a vertex $\mathbf{v}_1 \in \mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$. Proposition 4 tells us that there are two cases to consider:

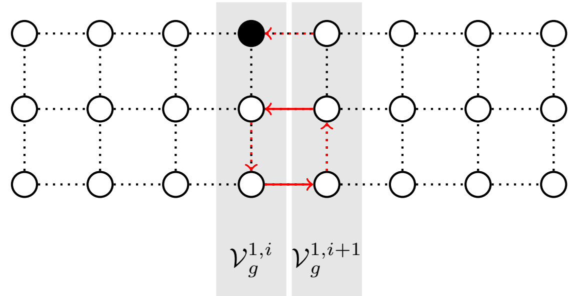
1) $\exists n : \psi^n(\mathbf{v}_1) = \perp$. Since no vertex in $\mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$ has its image being $\perp$, the sequence $(\psi^n(\mathbf{v}_1))_n$ necessarily contains a vertex $\mathbf{v}_2 \notin \mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$.
2) $\exists n : \psi^n(\mathbf{v}_1) = \mathbf{v}_1$. In this case, we distinguish the following situations, illustrated on a $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ grid graph:
   a) Every vertex from $\mathcal{V}_g^{1,i}$ is sent to its neighbor in $\mathcal{V}_g^{1,i+1}$, and every vertex from $\mathcal{V}_g^{1,i+1}$ is sent to its neighbor in $\mathcal{V}_g^{1,i+2}$.



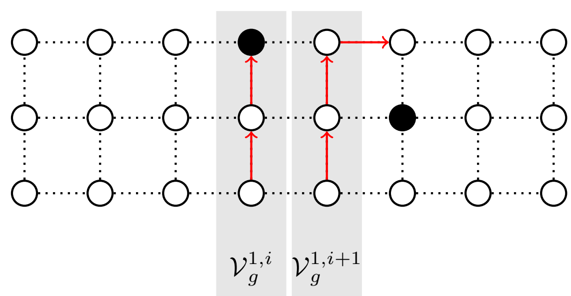
   In this situation, there cannot exist a cycle such that $\psi^n(\mathbf{v}_1) = \mathbf{v}_1$ due to injectivity of ψ. Note that this situation also applies in the case where every vertex from $\mathcal{V}_g^{1,i}$ is sent to its neighbor in $\mathcal{V}_g^{1,i-1}$, and every vertex from $\mathcal{V}_g^{1,i+1}$ is sent to its neighbor in $\mathcal{V}_g^{1,i}$.
   b) Every vertex from $\mathcal{V}_g^{1,i}$ is sent to its neighbor in $\mathcal{V}_g^{1,i+1}$, and every vertex from $\mathcal{V}_g^{1,i+1}$ is sent to its neighbor in $\mathcal{V}_g^{1,i}$.

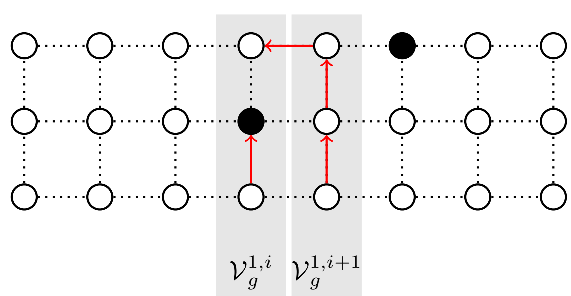


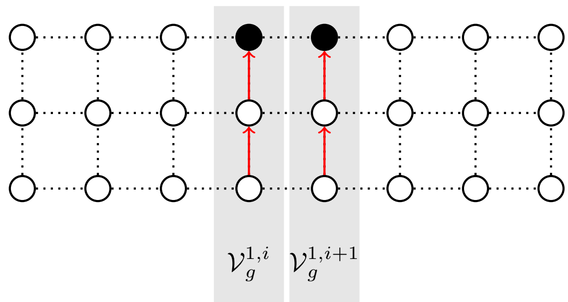
   This causes all vertices in $\mathcal{V}_g^{1,i-1} \cup \mathcal{V}_g^{1,i+2}$ to be sent to $\perp$, leading to a loss twice higher than the upper bound for minimal translations given in Proposition 13. Therefore we reach a contradiction.
   c) There exists a vertex $\mathbf{v}_2 \in \mathcal{V}_g^{1,i}$ such that $\psi(v_2) \in \mathcal{V}_g^{1,i+1}$, and a vertex There exists a vertex $\mathbf{v}_3 \in \mathcal{V}_g^{1,i+1} \cap \mathcal{N}(\psi(v_2))$ such that $\psi(v_3) \in \mathcal{V}_g^{1,i}$. Necessarily, $\psi^2(\mathbf{v}_2) = \mathbf{v}_3$ and $\psi^2(\mathbf{v}_3) = \mathbf{v}_2$, causing apparition of a cycle of 4 vertices.



   Then, since all dimensions are larger than 3, at least one of the vertices from this cycle has a neighbor $\mathbf{v}_4 \in \mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$ for which neighborhood cannot be preserved. As a consequence, there exists a vertex in $\mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$ that has its image equal to $\perp$, and we reach a contradiction.
   Additionally, note that in the case where the diedges of opposite directions are not adjacent, there is necessarily at least a vertex of which image is $\perp$ between them.
   d) Every vertex from $\mathcal{V}_g^{1,i}$ (resp. $\mathcal{V}_g^{1,i+1}$) is sent to a neighbor in $\mathcal{V}_g^{1,i}$ (resp. $\mathcal{V}_g^{1,i+1}$). If the corresponding diedges are of opposite directions, this situation eventually leads to a *turn*, in this case situation c) concludes. If they take the same direction, then due to border effects, at least a vertex in $\mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$ has its image equal to $\perp$, leading to a contradiction.







Note that all these situations lead to a contradiction. Therefore, a minimal translation ψ cannot lead to the creation of a cycle such that $\psi^n(\mathbf{v}_1) = \mathbf{v}_1$. As a consequence, only case 1) applies, and $\psi(\mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}) \not\subset \mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$.

$\square$

Corrolary 2. Let $\mathcal{G}_g$ be a grid graph respecting assumption (3), and let $\psi \in \Psi_{\mathcal{G}_g}$ be a minimal translation. Let $\mathcal{V}_g^{1,i}$ and $\mathcal{V}_g^{1,i+1}$ be two slices containing no vertex of which image through ψ is $\perp$. Every vertex from $\mathcal{V}_g^{1,i}$ is sent to its neighbor in $\mathcal{V}_g^{1,i+1}$, and every vertex from $\mathcal{V}_g^{1,i+1}$ is sent to its neighbor in $\mathcal{V}_g^{1,i+2}$.

Proof. This is a direct consequence of the proof of Lemma 3. Any other case corresponds to the situations described by cases 2b), 2c) and 2d) of the proof of Lemma 3, leading to existence of vertices of which image through ψ is $\perp$ in $\mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$. $\square$

Lemma 4. Let $\mathcal{G}_g$ be a grid graph respecting assumption (3), and let $\psi \in \Psi_{\mathcal{G}_g}$ be a minimal translation. Let $\mathcal{V}_g^{1,i}$ and $\mathcal{V}_g^{1,i+1}$ be two slices containing no vertex of which image by ψ is $\perp$:

$$\forall j \in [\![1, i+1]\!] : \forall \mathbf{v} \in \mathcal{V}_g^{1,j} : \psi(\mathbf{v}) \neq \perp .$$

Proof. From the proof of Lemma 3, there cannot exist any cycle including vertices in $\mathcal{V}_g^{1,i} \cup \mathcal{V}_g^{1,i+1}$. As a consequence, for every vertex $\mathbf{v}_1 \in \mathcal{V}_g^{1,i}$, the sequence $(\psi^n(\mathbf{v}_1))_n$ eventually leads to $\perp$. Since the cardinal of $\mathcal{V}_g^{1,i}$ is $\prod_{j \in [\![2,\,D]\!]} \mathbf{d}[j]$, there cannot exist a vertex $\mathbf{v}_2$ in slices $\mathcal{V}_g^{1,j}$ $(j < i)$ such that $\psi(\mathbf{v}_2) = \perp$, since ψ would not be minimal. $\square$

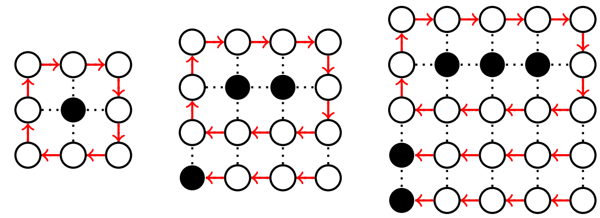
Proposition 15. Let $\psi_1 \in \Psi_{\mathcal{G}_g}$ be the Euclidean translation by $\mathbf{e}_1$ as introduced in Proposition 13 on a grid graph $\mathcal{G}_g$ respecting assumption (3). We have that ψ_1 is minimal.

Proof. Let $\psi_2 \in \Psi_{\mathcal{G}_g}$ be a minimal translation on $\mathcal{G}_g$. Let $\mathcal{V}_g^{1,i}$ and $\mathcal{V}_g^{1,i+1}$ be two slices containing no vertex of which image through ψ_2 is $\perp$. Lemma 4 tells us that no vertex $\mathbf{v}$ in slices $\mathcal{V}_g^{1,j}$ $(j \leq i+1)$ has its image equal to $\perp$. Additionally Lemma 3 and Corrolary 2 indicate that for these vertices, $\psi_2(\mathbf{v}) = \mathbf{v} + \mathbf{e}_1$.

Now, let us consider the minimum $k > i+1$ such that $\exists \mathbf{v}_1 \in \mathcal{V}_g^{1,k} : \psi_2(\mathbf{v}_1) = \perp$. Corrolary 2 tells us that every vertex from $\mathcal{V}_g^{1,k-1}$ has its image through ψ_2 in $\mathcal{V}_g^{1,k}$. Now, let us distinguish two cases:

1) If $k = \mathbf{d}[1]$, then $\psi_2 = \psi_1$, for which the loss is equal to the upper bound on losses for minimal translations;
2) If $k < \mathbf{d}[1]$, then we proceed by contradiction. By Corrolary 2, we have that $\psi_2(\mathbf{v}_1 - \mathbf{e}_1) = \mathbf{v}_1$. Since $\mathcal{N}(\mathbf{v}_1 - \mathbf{e}_1) \cap \mathcal{N}(\mathbf{v}_1 + \mathbf{e}_1) = \{\mathbf{v}_1\}$, and since $\psi_2(\mathbf{v}_1) = \perp$, we obtain that $\mathbf{v}_1 + \mathbf{e}_1$ cannot be the image of any vertex. As a consequence, we have a sequence $(\psi_2^n(\mathbf{v}_1 + \mathbf{e}_1))_n$ that ends with $\perp$. Therefore, the loss of ψ_2 is at least $1 + \prod_{j \in [\![2,\,D]\!]} \mathbf{d}[j]$, and ψ_2 is not minimal.

$\square$



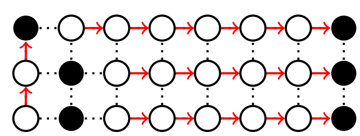
(a) (b) (c)

Figure 8: Counterexamples for grid graphs of dimensions $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ (a), $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ (b) and $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ (c). For such graphs, the translations that are depicted are minimal, while not being Euclidean translations by $\mathbf{e}_i$ as introduced in Proposition 13.

Corrolary 3. Let $\mathcal{G}_g$ be a grid graph respecting assumption (3). Euclidean translations by Dirac vectors $\mathbf{e}_i$ $(i \in [\![1, D]\!])$ are pseudo-minimal.

Proof. Let us denote by ψ_i the translation of every vertex by $\mathbf{e}_i$ as introduced in Proposition 13. Proposition 15 shows that ψ_1 is minimal, hence pseudo-minimal.

Now, let us consider a translation $\psi \in \Psi_{\mathcal{G}_g}$ such that $\forall \mathbf{v} \in \mathcal{V}_g : \psi(\mathbf{v}) \neq \mathbf{v} + \mathbf{e}_1$. For such translations, we can perform the same reasoning as above, leading to ψ_2 being pseudo-minimal. However, in the case where such a vertex exists, we can have the situation where $\psi_2 \prec \psi$. The following illustration depicts such a possible ψ:



However, in this situation, ψ is not pseudo-minimal since $\psi \prec \psi_1$. It follows that ψ_2 is pseudo-minimal. The same reasoning can be made for all higher dimensions. $\square$

Finally, recall from Conjecture 1 that we believe all the results in this section apply for grid graphs such that $\forall i \in [\![1, D]\!] : \mathbf{d}[i] \geq 6$. Interestingly, for smaller dimensions, counterexamples as depicted in Figure 8 can be found. The depicted translations are minimal, while not being Euclidean translations by $\mathbf{e}_i$ as introduced in Proposition 13.

D. Extension to generic graphs

In Proposition 11, we have shown that finding translations is an NP-complete problem. However, Section IV-B and Section IV-C have shown that in some particular cases, identifying the minimal or pseudo-minimal translations is possible in a minimum amount of time.

In this section, we are interested in generalizing the results we obtained for grid graphs to generic graphs. To do so, let us first show the following result:

Proposition 16. Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a graph of adjacency matrix $\mathbf{A}$, and let $\psi \in \Psi_{\mathcal{G}}$ be a translation on $\mathcal{G}$. Let $\overrightarrow{\mathcal{G}}^{\psi} = \langle \mathcal{V}, \overrightarrow{\mathcal{E}}^{\psi} \rangle$ be the digraph associated with ψ, of adjacency matrix $\mathbf{A}_\psi$. The following propositions are equivalent:

1) For every connected component of vertices $\mathcal{V}_1 \subseteq \mathcal{V}$ in $\mathcal{G}$, we have either $\forall v \in \mathcal{V}_1 : \psi(v) = \bot$, or $\forall v \in \mathcal{V}_1 : \psi(v) \neq \bot$;
2) $\mathbf{A}\mathbf{A}_\psi = \mathbf{A}_\psi \mathbf{A}$.

Proof. Without loss of generality, let us consider that $\mathcal{G}$ is connected, *i.e.*, has only one connected component.
$1) \Rightarrow 2)$:

- If $\psi = \phi_\bot$, then $\mathbf{A}_\psi$ is a matrix full of zeros, and the equality in 2) holds.
- If ψ is lossless, let us consider a vertex $v_1 \in \mathcal{V}$. The equality in 2) is the matrix representation of $\mathcal{N}(\psi(v_1)) = \{\psi(v_2) \mid v_2 \in \mathcal{N}(v_1)\}$. Since translations are SNP, necessarily we have that $\{v_1, v_2\} \in \mathcal{E} \Leftrightarrow \{\psi(v_1), \psi(v_2)\} \in \mathcal{E}$, unless $\psi(v_1) = \bot$ or $\psi(v_2) = \bot$. Since we consider here lossless translations, we have the equivalence. As a conclusion, for any vertex $v_1 \in \mathcal{V}$, the translation of neighbors of v_1 (which in matrix notation translates to $\mathbf{A}_\psi \mathbf{A}$) are the neighbors of the translation of v_1 (which in matrix notation translates to $\mathbf{A}\mathbf{A}_\psi$).

$2) \Rightarrow 1)$: If ψ is not lossless, there exists v such that $\psi(v) = \bot$. To verify equality 2), all neighbors of v must have their image equal to $\bot$. By contamination, $\psi = \phi_\bot$. ☐

Following Proposition 16, and except for translation $\phi_\bot$, it is interesting to note that the number of non-null entries of $\mathbf{A}\mathbf{A}_\psi - \mathbf{A}_\psi \mathbf{A}$ is low when the loss of ψ is low. In particular, we remark the following properties:

1) For every vertex $v_1 \in \mathcal{V}$ such that $\nexists v_2 \in \mathcal{V} : \psi(v_2) = v_1$, we have $\forall v_3 \in \mathcal{V} : (\mathbf{A}\mathbf{A}_\psi)[v_3, v_1] = 0$.
2) For every vertex $v_1 \in \mathcal{V}$ such that $\psi(v_1) = \bot$, we have $\forall v_2 \in \mathcal{V} : (\mathbf{A}_\psi \mathbf{A})[v_1, v_2] = 0$.

Basically, $\mathbf{A}\mathbf{A}_\psi$ is a reorganization of the columns of $\mathbf{A}$, except for vertices that start a path, for which the associated column becomes null (item 1). Similarly, $\mathbf{A}_\psi \mathbf{A}$ is a reorganization of the rows of $\mathbf{A}$, except for vertices that have their image equal to $\bot$, for which the associated row becomes null (item 2). It follows that there are a number of non-null entries in $\mathbf{A}\mathbf{A}_\psi - \mathbf{A}_\psi \mathbf{A}$ proportional to the number of neighborhoods that are lost (due to a neighbor having its image equal to $\bot$) or created (due to a neighbor having no inverse image by ψ).

From these remarks, we can derive a method to estimate translations that minimize the loss, through the following optimization problem, where $\mathbf{A}^*_\psi$ is the adjacency matrix of the digraph associated with the target translation, C is a constant, and $\|\cdot\|_1$ is the entrywise ℓ_1 norm for matrices:

$$\mathbf{A}^*_\psi = \underset{\mathbf{A}_\psi \in \{0,1\}^{N \times N}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{A}_\psi - \mathbf{A}_\psi \mathbf{A}\|_1$$

$$\textit{s.t.} \begin{cases} \forall i, j \in [\![1, N]\!] : \mathbf{A}_\psi[i,j] \leq \mathbf{A}[i,j] \\ \forall j \in [\![1, N]\!] : \sum_{i=1}^{N} \mathbf{A}_\psi[i,j] \leq 1 \\ \forall i \in [\![1, N]\!] : \sum_{j=1}^{N} \mathbf{A}_\psi[i,j] \leq 1 \\ \|\mathbf{A}_\psi\|_1 = C \end{cases} . \tag{4}$$

The first constraint imposes non-null entries of $\mathbf{A}_\psi$ to be a subset of non-null entries of $\mathbf{A}$, thus enforcing the EC property of translations. Additionally, the constraints enforcing rows and columns to sum to at most 1 force injectivity of the solution. Finally, the fourth constraint is necessary to avoid the trivial solution where $\mathbf{A}_\psi$ is a matrix full of zeros. Since ψ is injective, we have $C \in [\![0, N]\!]$.

Note that no constraints enforce the SNP property of the solution, which can thus be a transformation that is not a translation. Still, minimizing the sacrifice of this property is the objective of the problem, as explained above. The solution of (4) is therefore an EC transformation that is as close as possible to being SNP. We will thus call solutions to (4) *pseudo-translations*.

Due to the sacrifice of SNP, and because it is necessary to fix a norm for $\mathbf{A}_\psi$, we need a measure to describe which value of C is the correct one. In practice, we solve (4) for every possible value of C in $[\![0, N]\!]$, and keep the associated pseudo-translation ψ_C that minimizes the following quantity:

$$|\mathcal{E}_{\text{lost}} \cup \mathcal{E}_{\text{created}}| , \tag{5}$$

with

$$\mathcal{E}_{\text{lost}} = \{\{v_1, v_2\} \in \mathcal{E} \mid (\psi_C(v_1) = \bot) \vee (\psi_C(v_2) = \bot) \\ \vee (\{\psi_C(v_1), \psi_C(v_2)\} \notin \mathcal{E})\} ,$$

and

$$\mathcal{E}_{\text{created}} = \{\{v_1, v_2\} \notin \mathcal{E} \mid (\psi_C(v_1) \neq \bot) \wedge (\psi_C(v_2) \neq \bot) \\ \wedge (\{\psi_C(v_1), \psi_C(v_2)\} \in \mathcal{E})\} .$$

The quantity measured by (5) is the exact error of the pseudo-translation ψ_C, *i.e.*, it is the total number of neighborhoods that are lost or created by ψ_C, and should not be. Note that this quantity does not depend on C. Therefore, it provides a way of selecting the best pseudo-translation found, witout a bias related to the imposed norm in (4).

Once a pseudo-translations ψ_1 is found by solving (4) with the value of C minimizing (5), we add additional constraints to (4) in order to prevent subsequent solutions to share edges with ψ_1. While this prevents some pseudo-minimal translations to be found in the general case — consider for instance the case of the complete graph — this gives us a greedy and efficient algorithm to find a set of pseudo-translations, by increasing progressively the set of constraints to restrict the possible neighborhood of previously found pseudo-translations. In more details, let $\{\psi_1, \ldots, \psi_i\}$ be the

i first translations found by solving (4). For an optimization variable $\mathbf{A}_\psi$, such constraints are as follows:

$$\forall \psi \in \{\psi_1, \dots, \psi_i\}, \forall v \in \mathcal{V} : \mathbf{A}_\psi[v, \psi(v)] = 0\ . \quad (6)$$

Algorithm 1 summarizes the whole process to identify pseudo-translations on a graph.

Algorithm 1: $findPseudoTranslations\ (\mathcal{G})$

```
result := {};
do
    ψ_best := φ_⊥;
    error_best := ∞;
    foreach C ∈ ⟦0, N⟧ do
        ψ := solve (4) with ‖A_ψ‖_1 = C;
        error := compute (5) for ψ and C;
        if error ≤ error_best then
            ψ_best := ψ;
            error_best := error;
    result := result ∪ {ψ_best};
    Update (4) with constraints (6) using result;
while ψ_best ≠ φ_⊥;
return result;
```

Although we will not study the following, it is interesting to remark that the constraints in (4) can be relaxed more to sacrifice the EC property in some extent. This would allow pseudo-translations to artificially *create* edges in the graph in order to minimize the overall error. In such case, the corresponding optimization problem can be written as follows:

$$\mathbf{A}^*_\psi = \underset{\mathbf{A}_\psi \in \{0,1\}^{N \times N}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{A}_\psi - \mathbf{A}_\psi\mathbf{A}\|_1 + \alpha\|\mathbf{A} - \mathbf{A}_\psi\|_1$$

$$\textit{s.t.} \begin{cases} \forall j \in [\![1, N]\!] : \sum_{i=1}^{N} \mathbf{A}_\psi[i,j] \leq 1 \\ \forall i \in [\![1, N]\!] : \sum_{j=1}^{N} \mathbf{A}_\psi[i,j] \leq 1 \\ \|\mathbf{A}_\psi\|_1 = C \end{cases} ,$$

where α is a regularization parameter, and $\|\mathbf{A} - \mathbf{A}_\psi\|_1$ replaces the constraint enforcing non-null entries of $\mathbf{A}_\psi$ to be a subset of those of $\mathbf{A}$, by encouraging it in the objective function. Note that it also requires to update (5) to take the number of violations of the EC property into consideration.

V. Experiments

In this section, we first evaluate the results obtained with Algorithm 1 for various families of graphs. We illustrate that it finds the pseudo-minimal translations on a grid graph, and then we study the impact of small transformations of such graph on the translations that are found. Finally, we apply Algorithm 1 to identify translations on randomly generated graphs following a Watts-Strogatz model. In the following experiments, Algorithm 1 is implemented using CVX [17] package for MATLAB [18].

A. Experiments on the grid graph

First, we consider a grid graph of dimensions $\mathbf{d} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$ respecting assumptions (3). Figure 9 depicts the errors (5) obtained for each value of C on this graph, as well as the pseudo-translations that correspond to the norm minimizing this error. As expected, the algorithm finds all the pseudo-minimal translations on the grid graph. Similar results have been observed for grid graphs of different dimensions, provided that all dimensions are larger than 6, corresponding to Conjecture 1.

To evaluate whether the algorithm is robust to small variations of the graph, we consider a deformation of the grid graph $\mathcal{G}_g = \langle \mathcal{V}_g, \mathcal{E}_g \rangle$ of dimensions $\mathbf{d} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. More precisely, we build a deformed grid graph $\mathcal{G}_{g\sigma} = \langle \mathcal{V}_{g\sigma}, \mathcal{E}_{g\sigma} \rangle$ as follows:

- $\forall \mathbf{v} \in \mathcal{V}_g : \mathbf{v} + \boldsymbol{\epsilon} \in \mathcal{V}_{g\sigma}$, where $\boldsymbol{\epsilon} \sim \mathcal{N}\left(0, \sigma^2 \mathbf{I}_D\right)$;
- $\forall \{\mathbf{v}_1, \mathbf{v}_2\} \in \binom{\mathcal{V}_{g\sigma}}{2} : (\{\mathbf{v}_1, \mathbf{v}_2\} \in \mathcal{E}_{g\sigma}) \Leftrightarrow \left(d(\mathbf{v}_1, \mathbf{v}_2) < \frac{\sqrt{2}+1}{2}\right)$, where $d(\mathbf{v}_1, \mathbf{v}_2)$ is the Euclidean norm between vertices $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then, we solve (4) once on this deformed grid to find a pseudo-translation ψ, of associated matrix $\mathbf{A}_\psi$. Let ψ_i be a translation by $\mathbf{e}_i$ on a non-deformed grid graph as introduced in Proposition 13, of associated matrix $\mathbf{A}_{\psi_i}$. The following quantity measures the difference between ψ and the closest translation ψ_i (or ψ_i^{-1}):

$$\min_{i \in [\![1, D]\!], j \in \{-1, 1\}} \left\|\mathbf{A}_{\psi_i^j} - \mathbf{A}_\psi\right\|_1 \quad (7)$$

where j allows the consideration of inverse Dirac translations.

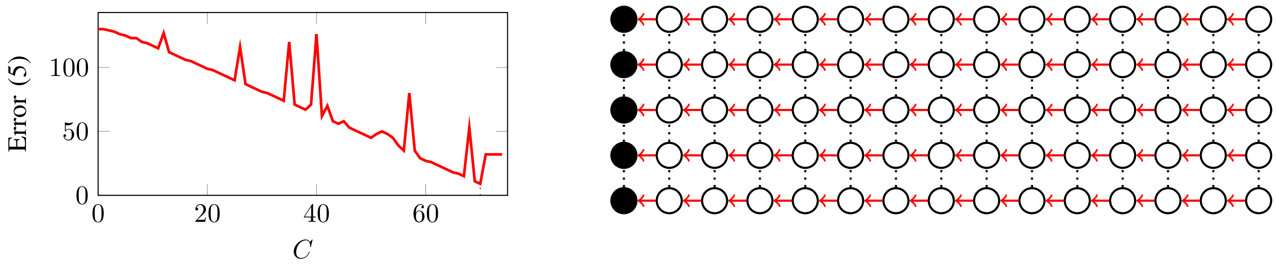
Figure 10 depicts the difference (7) between ψ and the closest translation by a Dirac, as a function of the standard deviation σ controlling the deformation of the grid. For every value of $\sigma \in [0.03, 0.12]$ with a range of 0.005, we compute the mean difference for 100 randomly deformed grid graphs.

From Figure 10, we can see that small modifications on the grid do not strongly impact the translations that are found on it. In particular, around locations in the graph where an edge has been suppressed, we have observed that the solution of (4) tends to favor sending close vertices to $\perp$ rather than modifying the whole translation. However, when the modifications become too strong, the best solution becomes a translation that is not related to Dirac translations, and we observe a strong increase of the difference (7).

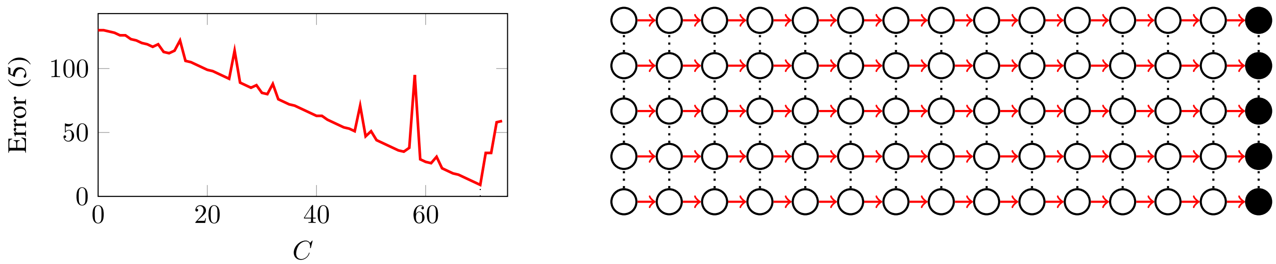
In our observations, we have noticed that addition of edges has a smaller impact on the difference than suppression. This can easily be explained by the fact that relaxation of the problem by solving (4) sacrifices the SNP property, but still enforces pseudo-translations that are found to be EC. Therefore, when an edge is added, it results in a slightly larger loss in neighborhood, that may not change the overall result. On the contrary, when an edge is removed, as the EC property must be met, this implies sending vertices to $\perp$ or changing the whole solution.

B. Translations on random graphs

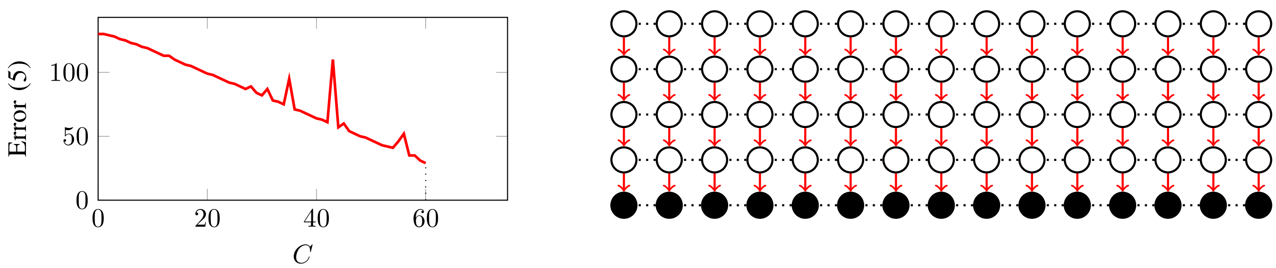
In the previous experiments, we considered grid graphs, or small variations of it. In this section, we apply Algorithm 1 to



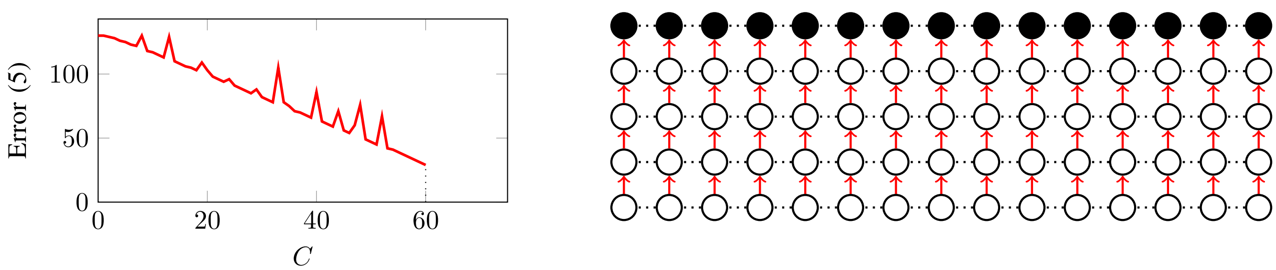
(a) First pseudo-translation found by Algorithm 1 for $C = 70$: ψ_1.



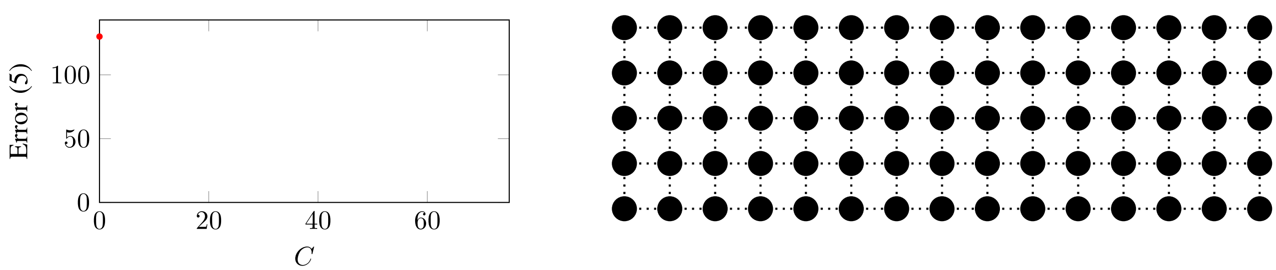
(b) Second pseudo-translation found by Algorithm 1 for $C = 70$: ψ_1^{-1}.



(c) Third pseudo-translation found by Algorithm 1 for $C = 60$: ψ_2.



(d) Fourth pseudo-translation found by Algorithm 1 for $C = 60$: ψ_2^{-1}.



(e) Fifth pseudo-translation found by Algorithm 1 for any C: $\phi_\perp$.

Figure 9: Pseudo-translations found by Algorithm 1 on a grid graph of dimensions $\mathbf{d} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$. The left column depicts the error in (5) as a function of C. Values of (5) for which no solution exists are not depicted on the curve. The right column represents the translation associated with the value of C minimizing this quantity.

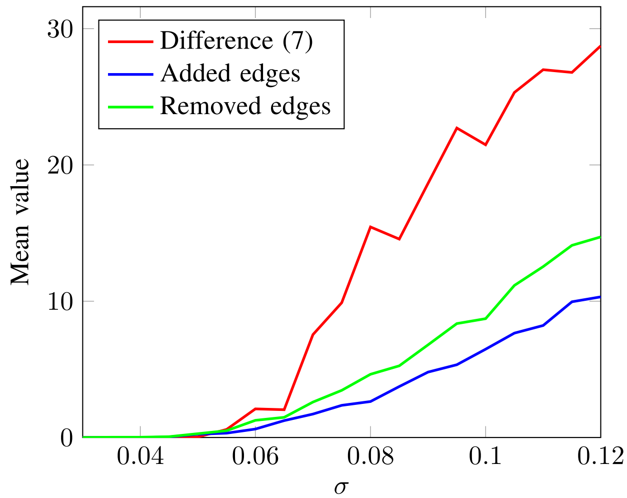


Figure 10: Mean difference between the first solution of (4) and a translation by a Dirac vector, as a function of the deformation of a grid of dimensions $\mathbf{d} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$.

randomly generated graphs in order to find pseudo-translations on different structures.

More particularly, we study the Watts-Strogatz model (see Definition 8). Figure 11 depicts the pseudo-translations that are found for a Watts-Strogatz graph of $N = 20$ vertices with $K = 2$ and $P = 0.1$.

It appears that Algorithm 1 first finds pseudo-translations that are very significative. In particular, since P is low, the graph is close to a ring graph, and the first pseudo-translations that are found tend to follow edges along the border of the graph. After the first few, pseudo-translations that are found appear to be *residuals*, as they can only use the edges that do not appear in the previously found ones, due to the greedy strategy of Algorithm 1. Still, these residual pseudo-translations favor edges linking vertices that belong to subsets of higher connectivity when it is possible, resulting in appearance of small cycles or small paths.

Figure 12 depicts the results we obtain for Watts-Strogatz graphs associated with larger probabilities $P \in \{0.3, 0.5, 0.7, 0.9\}$. Obviously, the results are not as good as for low values of P, as it can be seen from the curves in Figure 12. However, it is interesting to notice that the pseudo-translations continue to consist of paths in most cases and therefore make sense, as vertices sharing common neighbors tend to be sent from one to another.

VI. Conclusions

In this article, we have introduced a novel definition for translations on graphs. By observing that translations on a toric Euclidean space are exactly lossless translations on a torus graph, we have been able to propose a general notion of translation on graph that preserves neighboring properties. Our translations have the property to follow the edges of the graph and, when lossless, they guarantee that two neighboring signal entries become located on neighboring vertices after translation of the whole signal. Additionally, lossless translations do not change any entry in the signal that is translated, contrary to most existing approaches. When lossless translations do not exist due to the graph being irregular, we have proposed a method to identify translations that allow some vertices to lose the signal entries they carry. On a grid graph, translations admitting a loss are exactly those we expect to find when shifting an image on a non-toric Euclidean space. We have shown that identification of these translations — lossless or not — is an NP-complete problem, and have proposed a greedy algorithm based on the resolution of an optimization problem to approximate a subset of these translations. Our experiments have demonstrated that our algorithm is able to correctly identify translations on a grid graph — even with small modifications — and provides interesting translations on any graph.

We believe this work to have applications in tasks such as classification. Translations on arbitrary graphs could be used to shift a local filter on the graph, allowing the definition of adapted kernels for convolutional neural networks. This would generalize such networks to signals evolving on irregular structures, and could possibly allow detection of patterns in signals, independently from their localizations on the graph.

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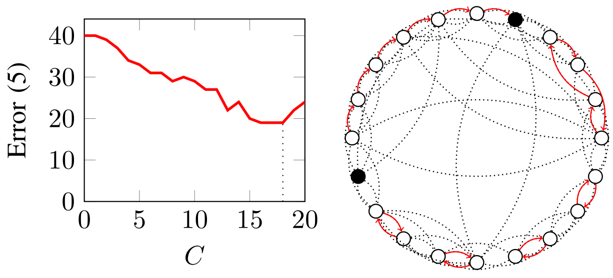
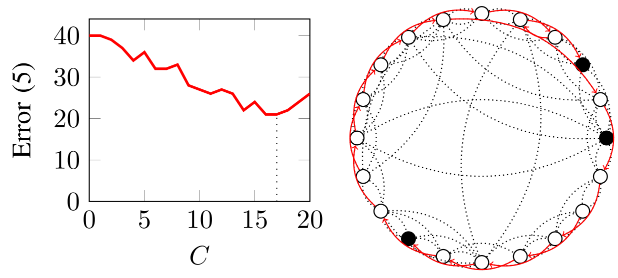
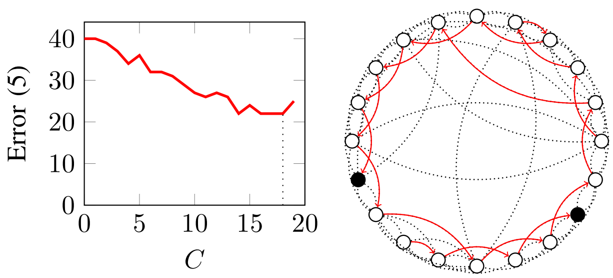


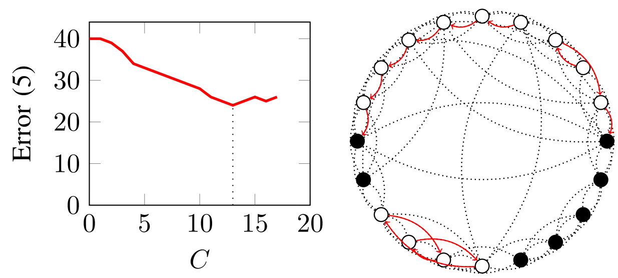
(a) $P = 0.1$, first pseudo-translation found for $C = 18$.



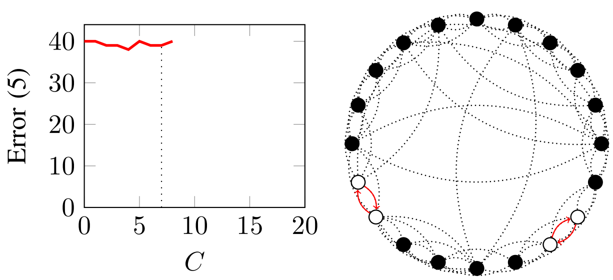
(b) $P = 0.1$, second pseudo-translation found for $C = 17$.



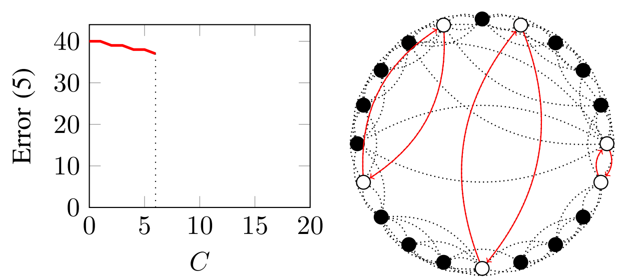
(c) $P = 0.1$, third pseudo-translation found for $C = 18$.



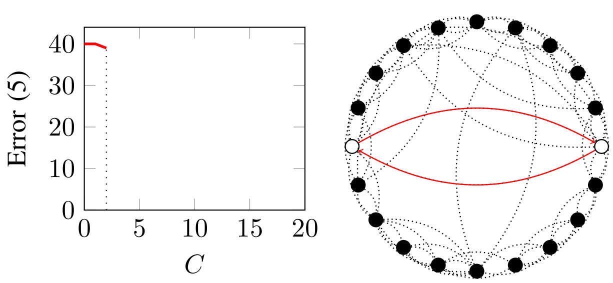
(d) $P = 0.1$, fourth pseudo-translation found for $C = 13$.



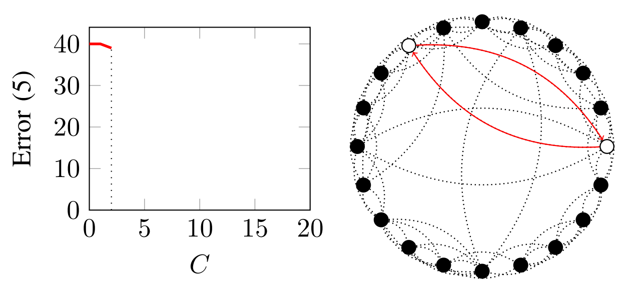
(e) $P = 0.1$, fifth pseudo-translation found for $C = 4$.



(f) $P = 0.1$, sixth pseudo-translation found for $C = 6$.



(g) $P = 0.1$, seventh pseudo-translation found for $C = 2$.



(h) $P = 0.1$, eighth pseudo-translation found for $C = 2$.

Figure 11: Pseudo-translations found for random graphs following a Watts-Strogatz model with parameters $K = 2$ and $P = 0.1$. The translation that sends every vertex to $\perp$ is not depicted.

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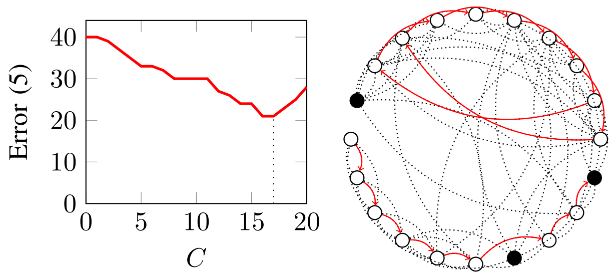
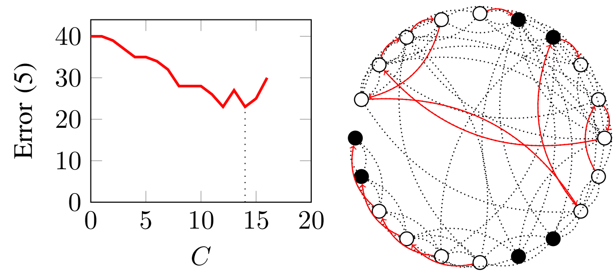
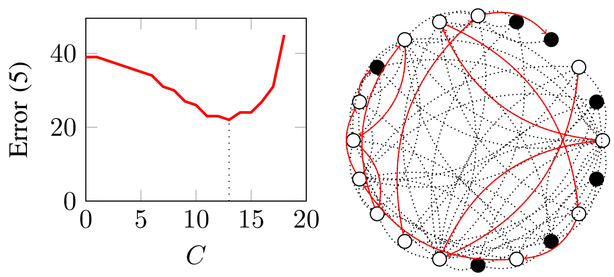


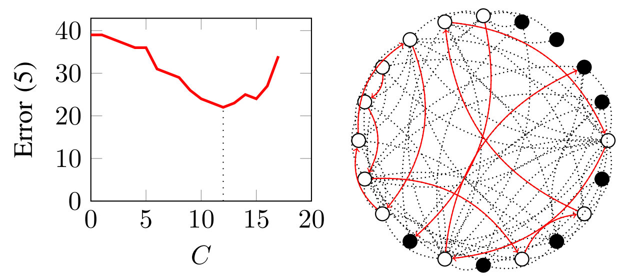
(a) $P = 0.3$, first pseudo-translation found for $C = 17$.



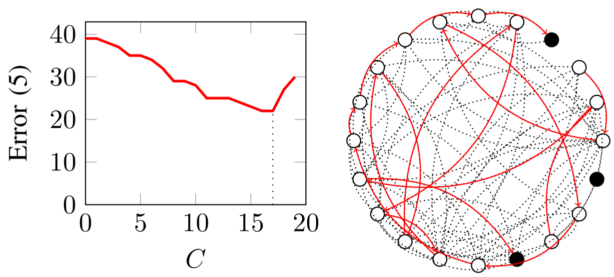
(b) $P = 0.3$, third pseudo-translation found for $C = 14$. The second pseudo-translation found was the inverse of the first.



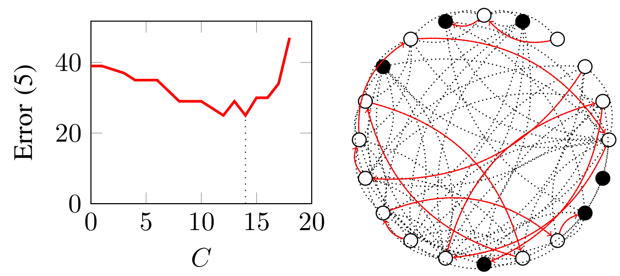
(c) $P = 0.5$, first pseudo-translation found for $C = 13$.



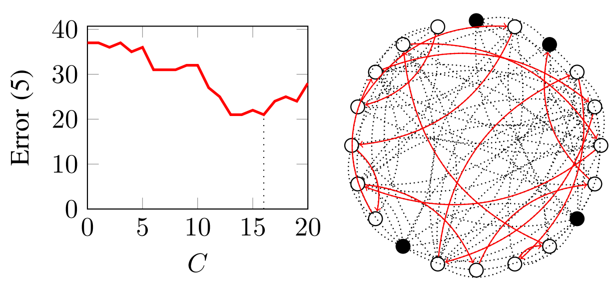
(d) $P = 0.5$, second pseudo-translation found for $C = 12$.



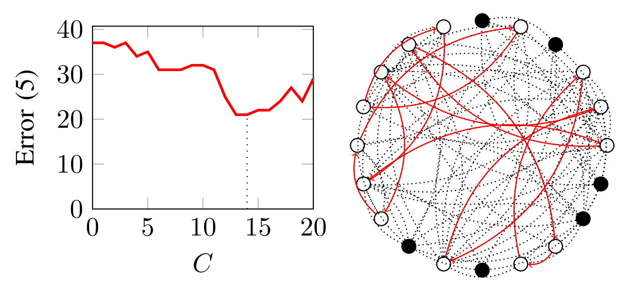
(e) $P = 0.7$, first pseudo-translation found for $C = 17$.



(f) $P = 0.7$, third pseudo-translation found for $C = 14$. The second pseudo-translation found was the inverse of the first.



(g) $P = 0.9$, first pseudo-translation found for $C = 16$.



(h) $P = 0.9$, second pseudo-translation found for $C = 14$.

Figure 12: Pseudo-translations found for random graphs following a Watts-Strogatz model with parameters $K = 2$ and $P \in \{0.3, 0.5, 0.7, 0.9\}$. For each value of P, only the two first distinct pseudo-translations that are found are depicted.