Finals -Lecture Notes #1 – Introduction to Graph Theory 04.19.2023

Your mindset has a direct impact
on how you perform in life
as well as the quality of
the results you get.

Ian Wallace Harper

REMINDER!

1) ALWAYS FOLLOW MY CLASSROOM PROCEDURES.

2) ACTIVITIES ARE ALWAYS UPDATED IN GOOGLE CLASSROOM, MYOPENMATH

3) ALWAYS SUBMIT YOUR WORK ON TIME. IF YOU MISSED SOME ACTIVITIES, BE RESPONSIBLE ENOUGH TO DO THEM.

4) THE WEEKLY HOMEWORK IS ALWAYS DUE EVERY MONDAY.

5) CHECK THE UPDATES ON GOOGLE CLASSROOM AND DO YOUR MISSING ASSIGNMENTS as the GRADES are always UPDATED.

Activities for today
MyOpenMath
   1) **Bellwork#1 (Finals) – Let Us Check Your Knowledge 04.19.23**
   2) **Activity # 1 (Finals) – All About Graph Theory 04.19.23**
   3) **Finals - Weekly Homework #1 – Always Remember These 04.19-26.23**
   4) **Finals – Lecture Notes #1** Introduction to Graph Theory 04.19.2023

At the end of the lesson, students will be able to
   1) identify graph theory.
   2) problems associated with the graph theory.

Graph Theory

In the domain of mathematics and computer science, ***graph theory is the study of graphs that concerns with the relationship among edges and vertices.*** It is a popular subject having its applications in computer science, information technology, biosciences, mathematics, and linguistics to name a few.

What is a Graph?

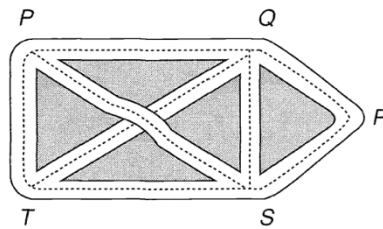We begin by considering Figs. 1.1 and 1.2, which depict part of a road map and part of an electrical network.
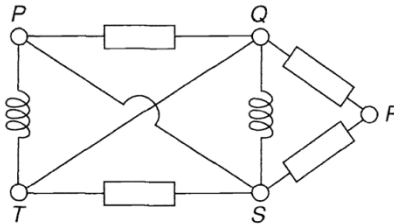

Fig. 1.1

Fig. 1.2

Either of these situations can be represented diagrammatically by means of points and lines, as in Fig. 1.3. The points P, Q, R, S and T are called ***vertices***, the lines are called ***edges***, and the whole diagram is called a graph. Note that the ***intersection*** of the lines ***PS and QT*** is not a vertex since it does not correspond to cross-roads or to the meeting of two wires. The ***degree of a vertex*** is the number of edges with that vertex as an endpoint; it corresponds in Fig. 1.1 to the number of roads at an intersection. For example, **the degree** of the vertex Q is 4.

Key Idea. A graph is a pictorial representation of a set of objects where some pairs of objects are connected by links. The interconnected objects are represented by points termed as ***vertices***, and the links that connect the vertices are called ***edges***.

Formally, a graph is a pair of sets **(V, E)**, where *V* is the set of vertices and *E* is the set of edges, ***connecting the pairs of vertices***. Take a look at the following graph.
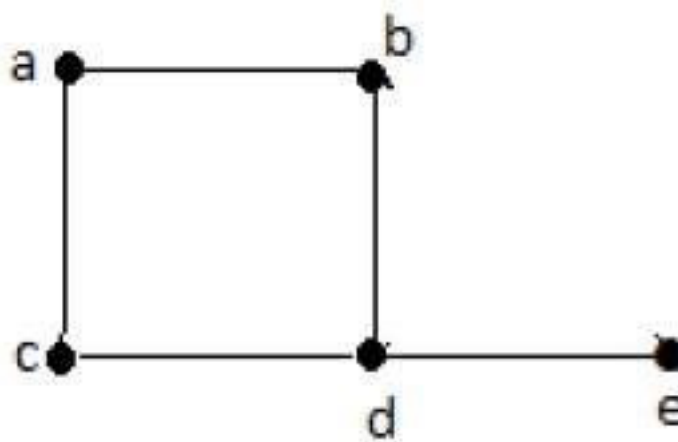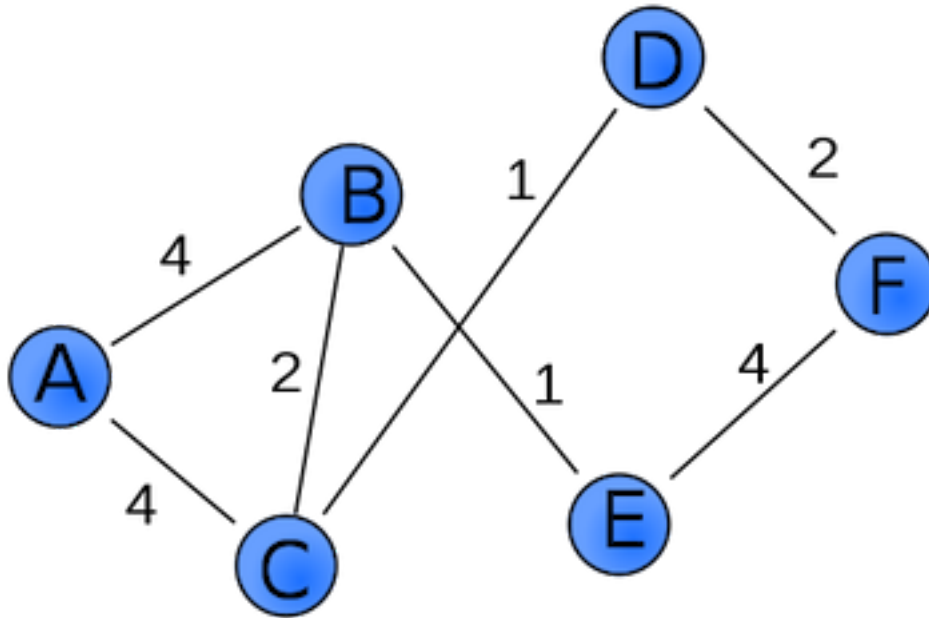

In the above graph,

Vertices, V = {a, b, c, d, e}

Edges, E = {ab, ac, bd, cd, de}

A **weighted graph** is a graph with weighted edges. The weights may represent factors like cost, or the distance required to travel between nodes.

Example of a weighted graph

Applications of Graph Theory

Graph theory has its applications in diverse fields of engineering −

1) Electrical Engineering − The concepts of graph theory is used extensively in designing circuit connections. The types or organization of connections are named as topologies. Some examples for topologies are star, bridge, series, and parallel topologies.

2) Computer Science − Graph theory is used for the study of algorithms. For example,

- Kruskal's Algorithm
- Prim's Algorithm
- Dijkstra's Algorithm

3) Computer Network − The relationships among interconnected computers in the network follows the principles of graph theory.

4) Science − The molecular structure and chemical structure of a substance, the DNA structure of an organism, etc., are represented by graphs.

5) Linguistics − The parsing tree of a language and grammar of a language uses graphs.

6) General − Routes between the cities can be represented using graphs. Depicting hierarchical ordered information such as family tree can be used as a special type of graph called tree.

Now I know… **A graph is a diagram of points and lines connected to the points.** It has at least one line joining a set of two vertices with no vertex connecting itself. The concept of graphs in graph theory stands up on some basic terms such as *point, line, vertex, edge, degree of vertices, properties of graphs, etc.*

Common Graph Terms
1) Point. It is particular position in a *one-dimensional, two-dimensional, or three-dimensional space.* It can be denoted by an alphabet. It can be represented with a dot.

Example

• a

2) Line is a connection between two points. It can be represented with a solid line.

Example
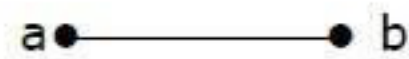
a•————————• b   Here, 'a' and 'b' are the points. The link between these two points is called a line.

3) Vertex. A vertex is a point where multiple lines meet. It is also called a *node*. Similar to points, a vertex is also denoted by an alphabet.

Example

• a   Here, the vertex is named with an alphabet 'a'.

4) Edge. An *edge* is the mathematical term for a line that connects two vertices. Many edges can be formed from a single vertex. Without a vertex, an edge cannot be formed. There must be a starting vertex and an ending vertex for an edge.
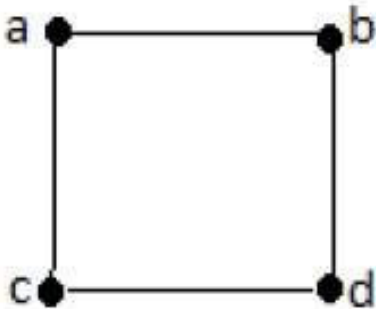
Example

a •――――――• b   Here, 'a' and 'b' are the two vertices and the link between them is called an edge.

5) Graph. A graph 'G' is defined as G = (V, E) Where V is a set of all vertices and E is a set of all edges in the graph.

Example 1

In this example, *ab, ac, cd, and bd* are the edges of the graph. Similarly, *a, b, c, and d* are the vertices of the graph.

Example 2
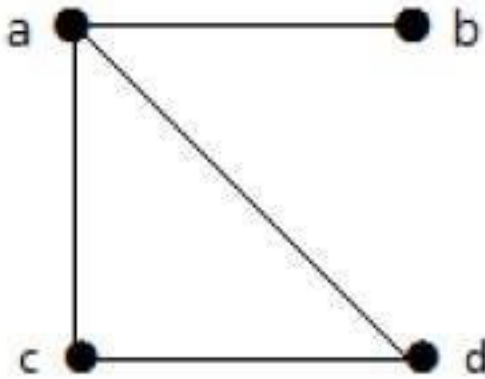
In this graph, there are four vertices *a, b, c, and d,* and four edges *ab, ac, ad, and cd.*

6) Loop. In a graph, if an edge is drawn from vertex to itself, it is *called a loop.*

Example 1

In the given example, *V* is a vertex for which it has an *edge (V, V)* forming a loop.

Example 2
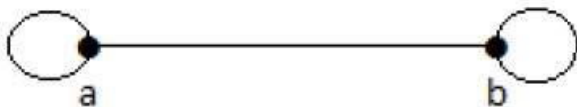
In this graph, there are two loops which are formed at vertex a, and vertex b.

Degree of Vertex

It is the number of vertices adjacent to a vertex V.

Notation − deg(V).

The symbol ∀ means "**for all" or "for any"**.

In a simple graph with n number of vertices, the degree of any vertices is −

$$\deg(v) \leq n - 1 \ \forall \ v \in G$$

Key Idea. A vertex can form an edge with all other vertices except by itself. So, the *degree of a vertex* will be up to the *number of vertices in the graph minus 1.* This 1 is for the self-vertex as it cannot form a loop by itself. If there is a loop at any of the vertices, then it is not a *Simple Graph.*

Key Idea. The number of edges of a given tree is equal to number of vertices minus 1 (n of edges =n of vertices – 1)

Question 1: What is the number of edges in a tree with 11 vertices?

Question 2: What is the number of edges in a tree with 15 vertices?

Degree of vertex can be considered under two cases of graphs −

- Undirected Graph
- Directed Graph

Degree of Vertex in an Undirected Graph

Undirected graph definition:

- An undirected graph is a set of nodes and a set of links between the nodes.
- Each node is called a **vertex**, each link is called an **edge**, and each edge connects two vertices.
- The order of the two connected vertices is unimportant.
- An undirected graph is a finite set of vertices together with a finite set of edges. Both sets might be empty, which is called the empty graph.

An *undirected graph* has no directed edges. Consider the following examples.

A Motivating Example: Coin Game (Undirected State Graphs)

Start of the game   Goal of the game

Rules:

1. You may flip the middle coin whenever you want to.
2. You may flip one of the end coin only if the other two coins are the same as each other.

Example 1
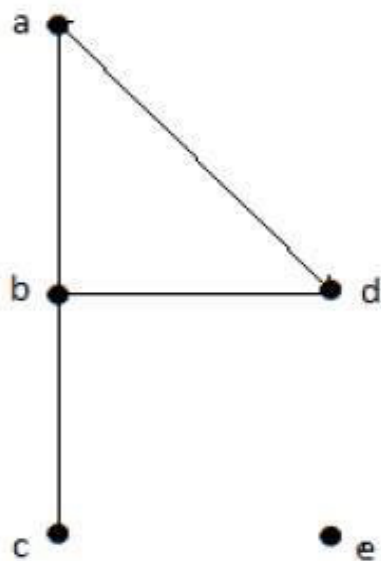
In the given undirected graph,

- ***deg(a)*** = 2, as there are 2 edges meeting at vertex 'a'.
- ***deg(b)*** = 3, as there are 3 edges meeting at vertex 'b'.
- ***deg(c)*** = 1, as there is 1 edge formed at vertex 'c', so it is a ***pendent vertex.***
- ***deg(d)*** = 2, as there are 2 edges meeting at vertex 'd'.
- ***deg(e)*** = 0, as there are 0 edges formed at vertex 'e', so 'e' is an isolated vertex.

Example 2

Take a look at the following graph −
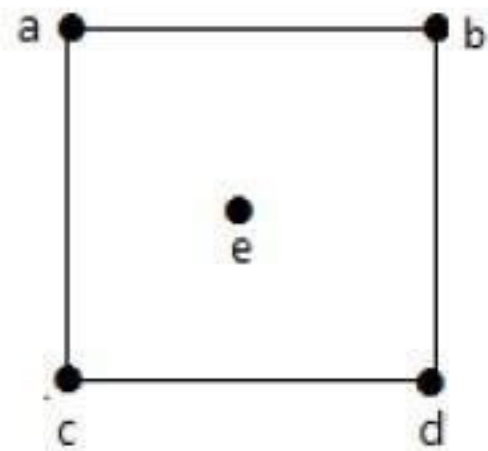
In the given graph,

deg(a) = 2, deg(b) = 2, deg(c) = 2, deg(d) = 2, and deg(e) = 0.

The vertex 'e' is an ***isolated vertex***. The graph does not have any pendent vertex.

CHECK FOR UNDERSTANDING 1 – (UNDIRECTED GRAPH)

Identify the degree of Vertex in the given undirected Graph
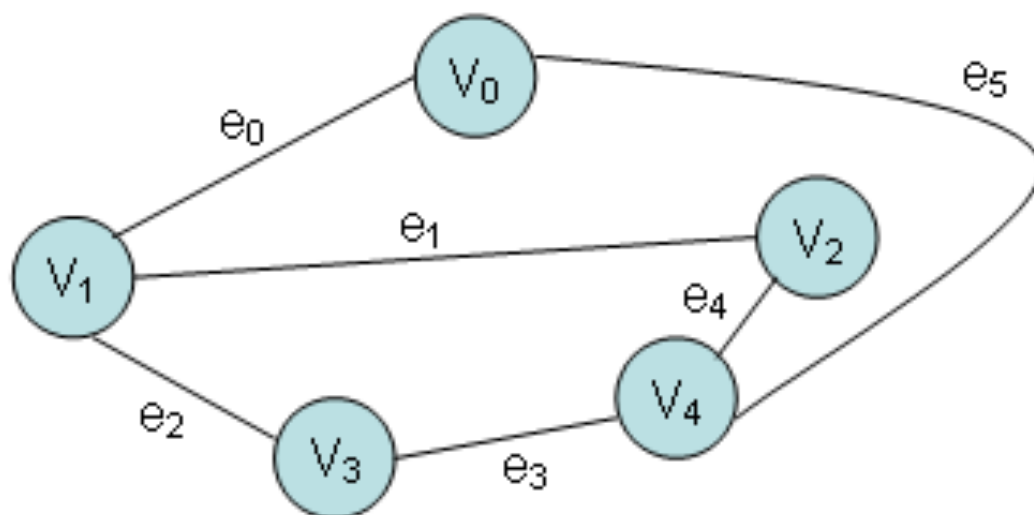
Deg (V_0) =   Deg (V_3) =

Deg (V_1) =   Deg (V_4) =

Deg (V_2) =

Degree of Vertex in a Directed Graph

Directed graph definition

- A directed graph is a finite set of vertices together with a finite set of edges. Both sets might be empty, which is called the **empty graph.**
- Each edge is associated with two vertices, called its **source** and **target** vertices.
- We say that the edge connects its source to its target.
- The **order** of the two connected vertices is important
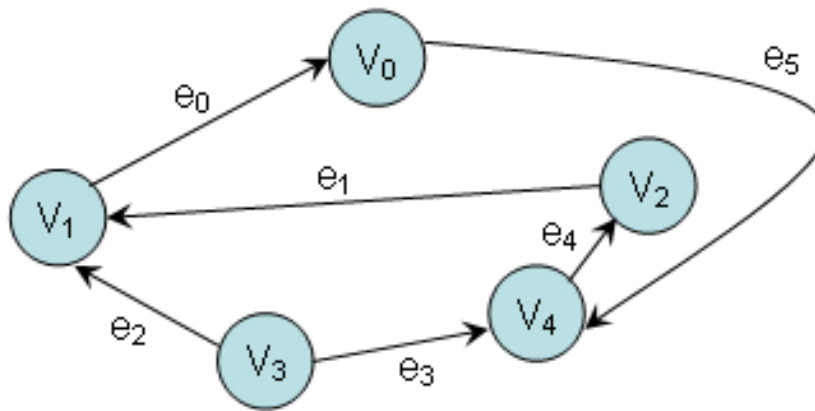
One application of directed graphs is a state graph for a game where reversing a move is sometimes forbidden. For example, tic-tac-toe.
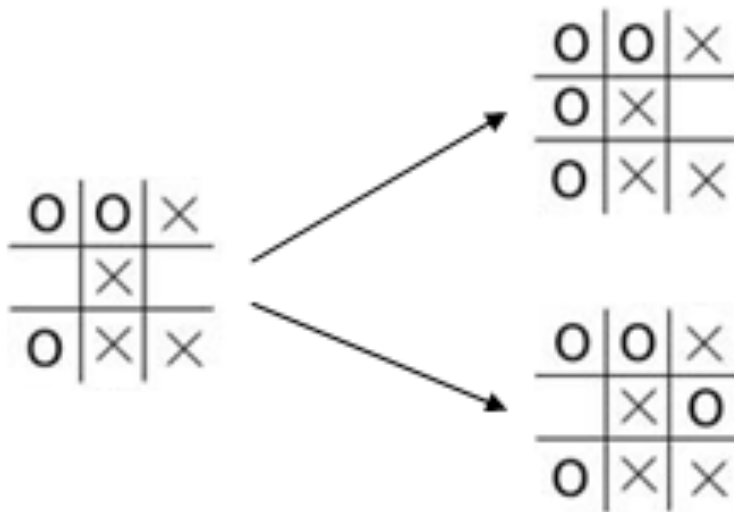
In a directed graph, each vertex has an *indegree* and an *outdegree.*

Indegree of a Graph
- Indegree of vertex V is the number of edges which are coming into the vertex V.
- **Notation** − deg−(V).

Outdegree of a Graph
- Outdegree of vertex V is the number of edges which are going out from the vertex V.
- **Notation** − deg+(V).

Example 1

Take a look at the following directed graph. Vertex 'a' has two edges, 'ad' and 'ab', which are going outwards. Hence its outdegree is 2. Similarly, there is an edge 'ga', coming towards vertex 'a'. Hence the indegree of 'a' is 1.
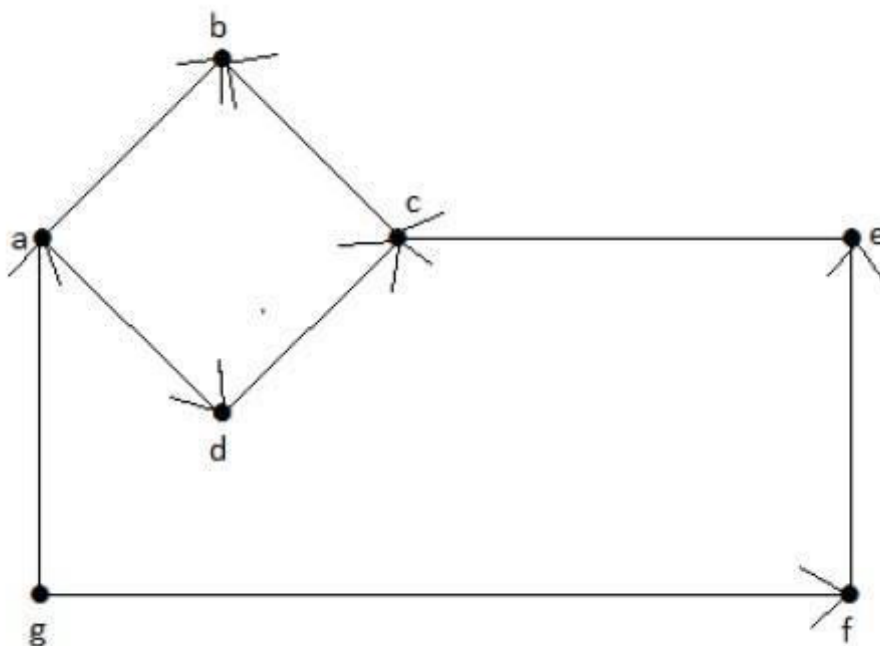
The indegree and outdegree of the given vertices are shown in the following table.

Vertex	Indegree	Outdegree
a	1	2
b	2	0
c	2	1
d	1	1
e	1	1
f	1	1
g	0	2

CHECK FOR UNDERSTANDING 2 – (DIRECTED GRAPH)
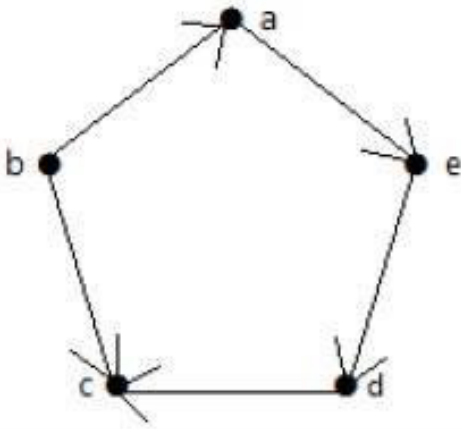
List down the indegree and outdegree of the given vertices

Vertex	Indegree	Outdegree
a		
b		
c		
d		
e		

Additional Graph Terminologies:

Loop: an edge that connects a vertex to itself.

Path: a sequence of vertices, $p_0, p_1, ..., p_m$, such that each adjacent pair of vertices p_i and p_{i+1} are connected by an edge.

Cycle: a simple path with no repeated vertices or edges other than the starting and ending vertices. A cycle in a directed graph is called a directed cycle.

Multiple edges: in principle, a graph can have two or more edges connecting the same two vertices in the same direction.

Simple graphs: the graphs that have no loops and no multiple edges. In fact, many applications require only simple directed graphs or even simple undirected graphs.

Adjacency

Here are the norms of adjacency –

- In a graph, **two vertices** are said to be **adjacent** if there is an edge between the two vertices. Here, the adjacency of vertices is maintained by the single edge that is connecting those two vertices.

- In a graph, **two edges** are said to be **adjacent** if there is a common vertex between the two edges. Here, the adjacency of edges is maintained by the single vertex that is connecting two edges.

Example 1
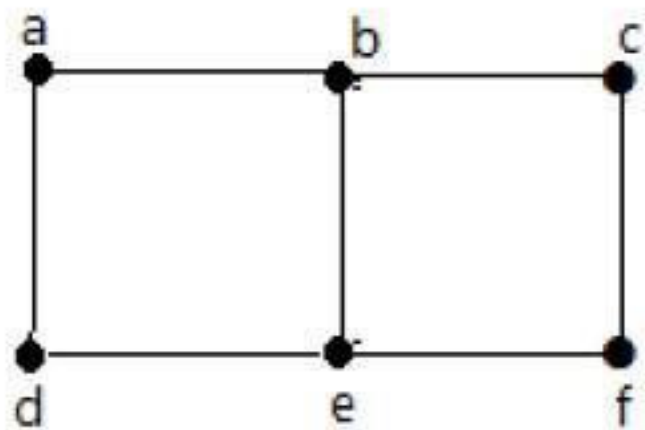
In the above graph −

- 'a' and 'b' are the **_adjacent vertices_**, as there is a **_common edge_** 'ab' between them.
- 'a' and 'd' are the **_adjacent vertices,_** as there is a **_common edge_** 'ad' between them.
- ab' and 'be' are the **_adjacent edges_**, as there is a **_common vertex_** 'b' between them.
- be' and 'de' are the **_adjacent edges,_** as there is a **_common vertex_** **'e' between them.**

CHECK FOR UNDERSTANDING 3 – (ADJACENCY)

Which vertices are **adjacent** to vertex C?
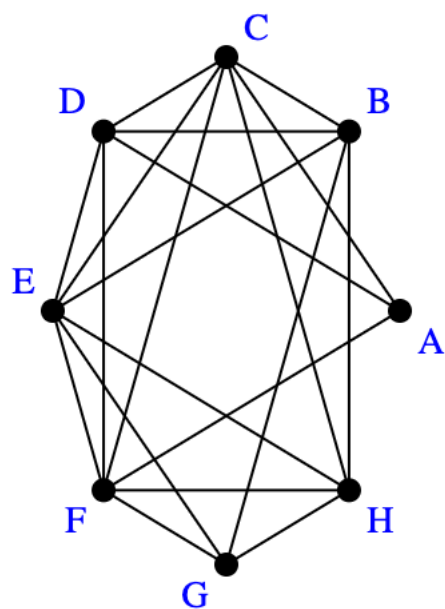
CHECK FOR UNDERSTANDING 4 – (ADJACENCY)

a. List down at least two adjacent vertices and adjacent edges of the given graph below.

b. What is the adjacency matrix for this graph (with the given the labeling on the vertices)?

Key: Adjacency Matrix (b)

	a	b	c	d	e
a	0	0	0	1	0
b	0	0	1	0	0
c	0	1	0	0	1
d	1	0	0	0	1
e	0	0	1	1	0

c. What is the adjacency matrix for this graph (with the given the labeling on the vertices)?
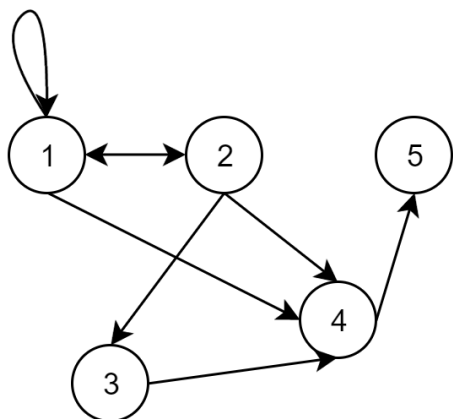
What is the adjacency matrix for this graph (given the natural ordering on vertices)?

	⚷ 1		⚷ 1		⚷ 0		⚷ 1		⚷ 0
	⚷ 1		⚷ 0		⚷ 1		⚷ 1		⚷ 0
	⚷ 0		⚷ 0		⚷ 0		⚷ 1		⚷ 0
	⚷ 0		⚷ 0		⚷ 0		⚷ 0		⚷ 1
	⚷ 0		⚷ 0		⚷ 0		⚷ 0		⚷ 0

Parallel Edges

In a graph, if a pair of vertices is connected by more than one edge, then those edges are called *parallel edges.*
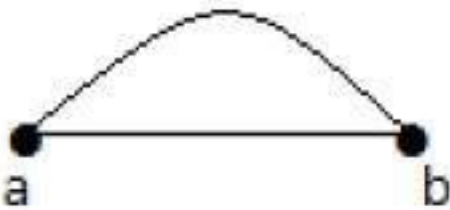
In the above graph, 'a' and 'b' are the two vertices which are connected by two edges 'ab' and 'ab' between them. So it is called as a *parallel edge.*

Multi Graph

A graph having parallel edges is known as a *Multigraph.*

Example 1
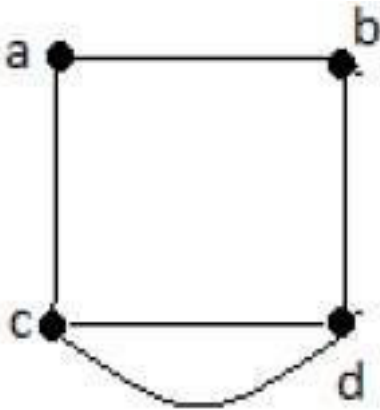
In the given graph, there are five edges 'ab', 'ac', 'cd', 'cd', and 'bd'. Since 'c' and 'd' have two parallel edges between them, it a *Multigraph.*

Example 2
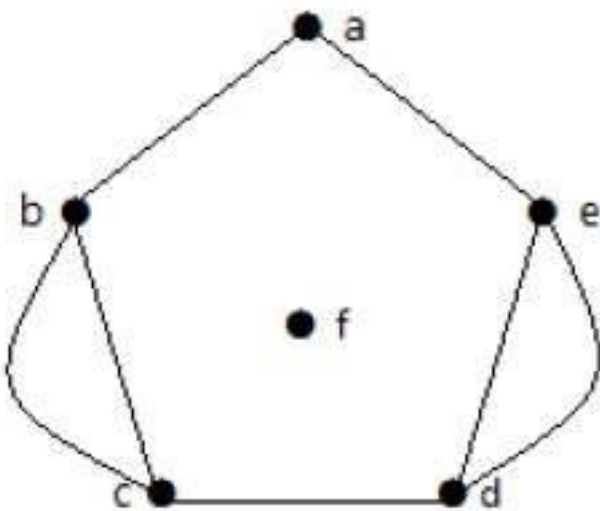
In the above graph, the vertices 'b' and 'c' have two edges. The vertices 'e' and 'd' also have two edges between them. Hence it is a Multigraph.

Degree Sequence of a Graph

If the degrees of all vertices in a graph are arranged in descending or ascending order, then the sequence obtained is known as the *degree sequence of the graph.*

Example 1
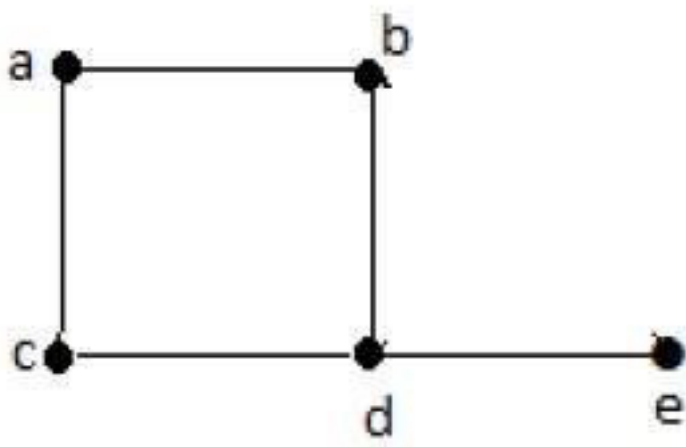
Vertex	a	b	c	d	e
Connecting to	b, c	a, d	a, d	c, b, e	d
Degree	2	2	2	3	1

In the above graph, for the vertices {d, a, b, c, e}, the degree sequence in descending order is {3, 2, 2, 2, 1}.

In the above graph, for the vertices {e, a, b, c, d}, the degree sequence in ascending order is {1, 2, 2, 2, 3}.

CHECK FOR UNDERSTANDING 5 (DEGREE SEQUENCE)

Fill out the table by identifying the vertices connected to a given vertex and identify the degree. Also, arrange the degree sequence in both ascending and descending order.
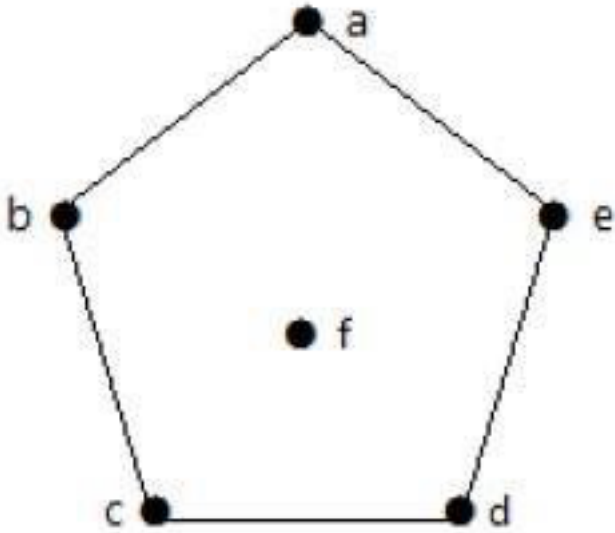
Vertex	a	b	c	d	e	f
Connecting to						
Degree						

Activity #2 – All About Graph Theories

Airline Routing Example: Crocodile Airlines Routes
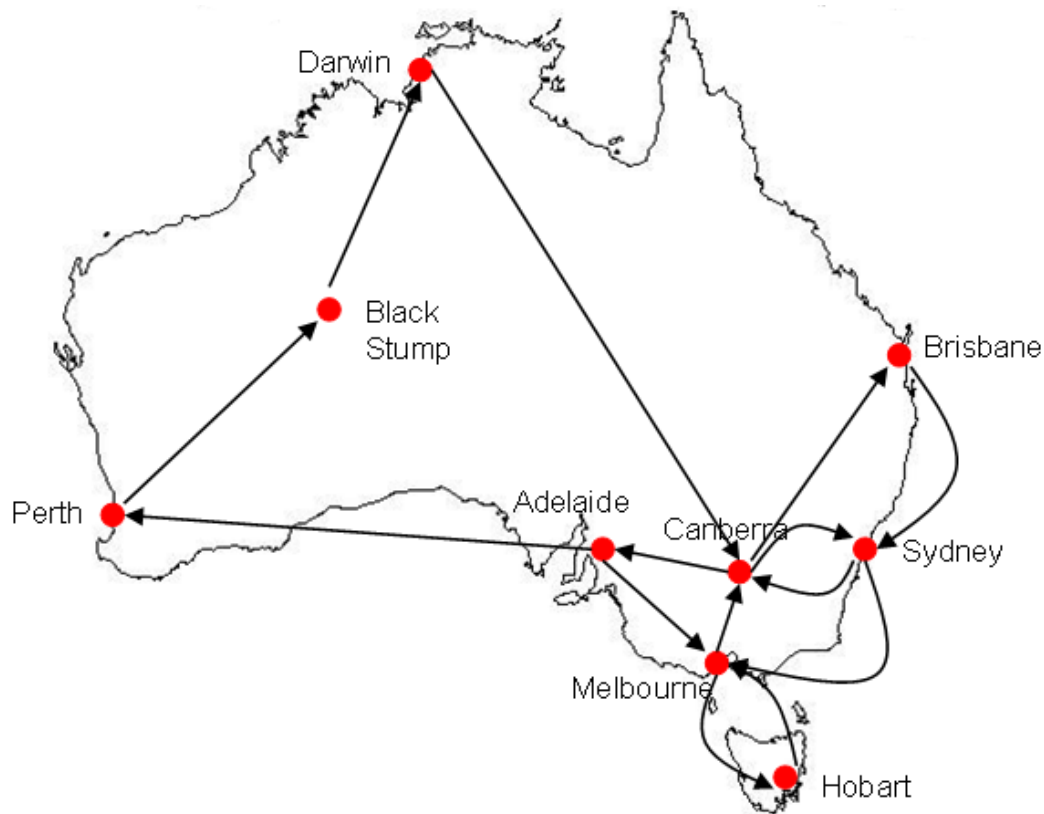
Questions:

1. **How many vertices and edges does the graph have? How many loops?**

2. **Is it a simple graph? Why or why not?**

3. **What is the shortest path from "Black Stump" to "Melbourne"? -- The shortest path problem.**

Pendent Vertex

By using degree of a vertex, we have a two special types of vertices. A vertex with degree one is called a ***pendent vertex.***

Example

Here, in this example, vertex 'a' and vertex 'b' have a connected edge 'ab'. So with respect to the vertex 'a', there is only one edge towards vertex 'b' and similarly with respect to the vertex 'b', there is only one edge towards vertex 'a'. Finally, vertex 'a' and vertex 'b' has degree as one which are also called as the pendent vertex.

Isolated Vertex

A vertex with degree zero is called an isolated vertex.

Example

Here, the vertex 'a' and vertex 'b' has a no connectivity between each other and also to any other vertices. So, the degree of both the vertices 'a' and 'b' are zero. These are also called as ***isolated vertices.***

Equivalent Graphs

Graphs are **equivalent** if they have the same number of vertices and the same edge connections. The vertices do *not* need to have the same labels, and they do not have to be drawn in the same positions.

Select all the graphs that are equivalent.
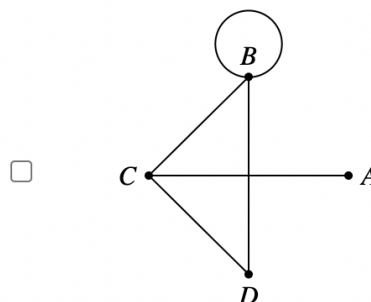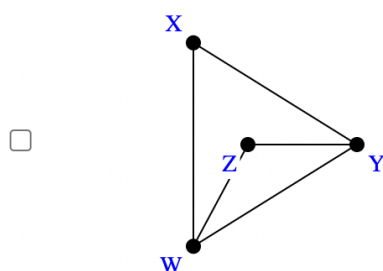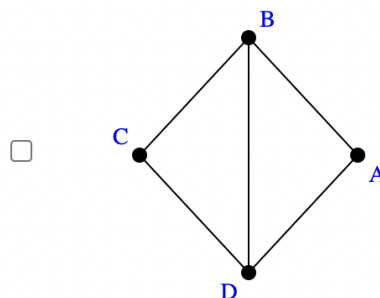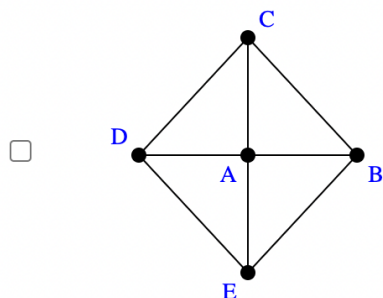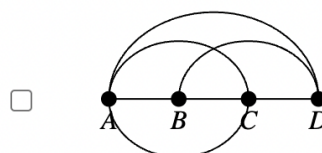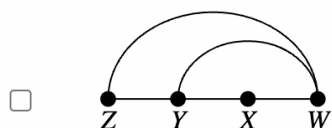
EVEN & ODD VERTICES
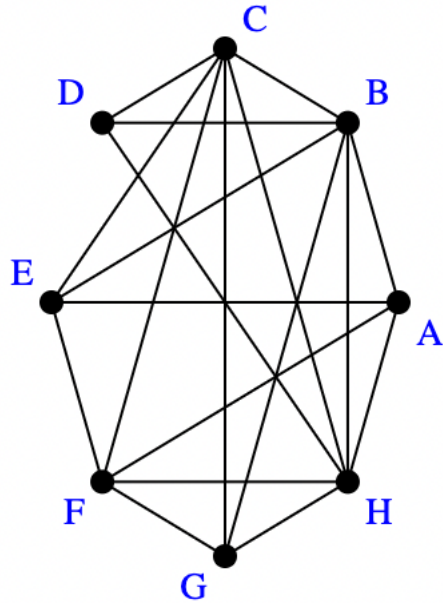
Which vertices are **even**?

☐ A   ☐ F

☐ B   ☐ G

☐ C   ☐ H

☐ D   ☐ None are even

☐ E

Which vertices are **odd**?

☐ A   ☐ F

☐ B   ☐ G

☐ C   ☐ H

☐ D   ☐ None are odd

☐ E

Sources

https://www.cpp.edu/~ftang/courses/CS241/notes/graph.htm

https://www.tutorialspoint.com/graph_theory/graph_theory_fundamentals.htm

Quick Review

Graphs come with various properties which are used for characterization of graphs depending on their structures. These properties are defined in specific terms pertaining to the domain of graph theory. In this chapter, we will discuss a few basic properties that are common in all graphs.

A *graph* is defined as set of points known as 'Vertices' and line joining these points is known as 'Edges'. It is a set consisting of where 'V' is vertices and 'E' is edge.

Vertices: {A, B, C, D, E, F}
Edges: {{A, B}, {A, D}, {A, E}, {B, C}, {C, E}, {C, F}, {D, E}, {E, F}}
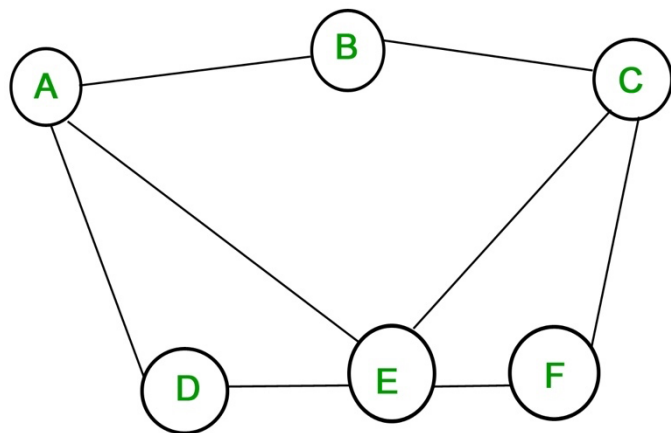
Graph measurements: length, distance, diameter, eccentricity, radius, center

1) Distance between Two Vertices

It is number of edges in a shortest path between Vertex U and Vertex V. If there are multiple paths connecting two vertices, then the shortest path is considered as the distance between the two vertices.

Notation − d (U, V)

There can be any number of paths present from one vertex to other. Among those, you need to choose only the shortest one.

Key Idea. The distance between two vertices in a graph is the number of edges in a shortest or minimal path. It gives the available minimum distance between two edges. There can exist more than one shortest path between two vertices.

Example 1)

Take a look at the following graph −
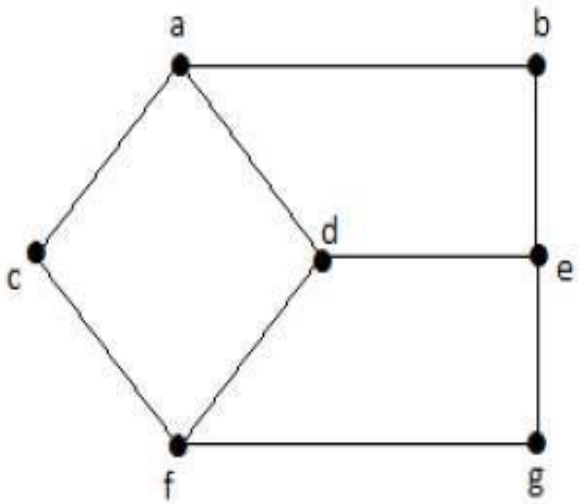
Here, the distance from vertex 'd' to vertex 'e' or simply 'de' is 1 as there is one edge between them. There are many paths from vertex 'd' to vertex 'e' −
- da, ab, be
- df, fg, ge
- de (It is considered for distance between the vertices)
- df, fc, ca, ab, be
- da, ac, cf, fg, ge

Example 2)
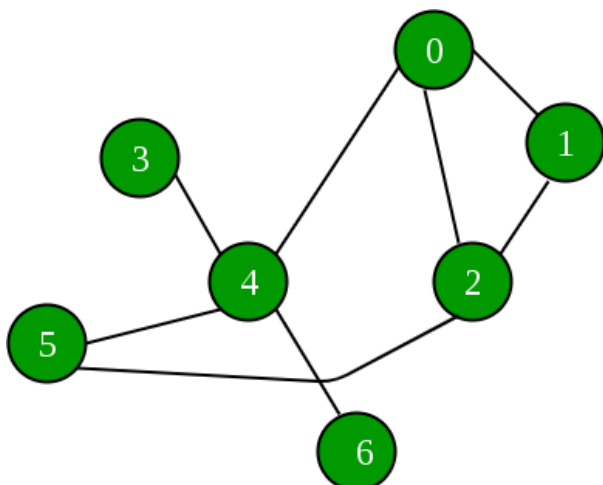
Shortest Distance between 1 - 5 is **2**
$1 \rightarrow 2 \rightarrow 5$

SHORTEST PATH PROBLEM

Find the length of the shortest path from vertex A to vertex L.

```
 C          F            I          L
  •——20——•——24——•——13——•
  |           |            |           |
  6          17           3           1
  |           |            |           |
  •——26——•——18——•——4——•
 B          E            H          K
  |           |            |           |
 19         10           22          16
  |           |            |           |
  •——8———•——21——•——7——•
 A          D            G          J
```
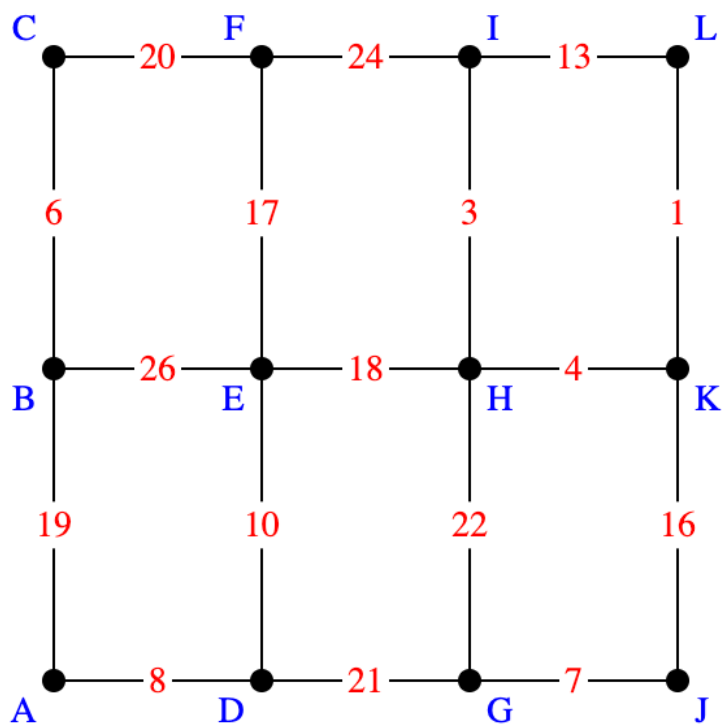
2) Length – Length of the graph is defined as the number of edges contained in the graph.
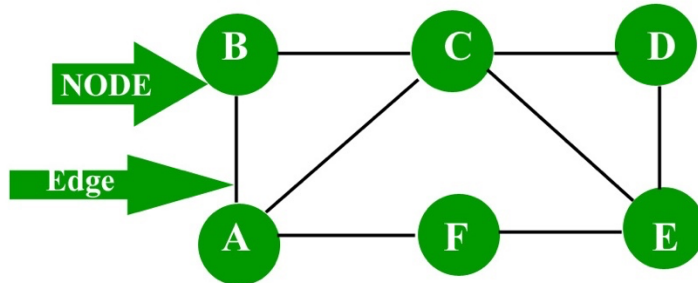
Length of the graph: 8

AB, BC, CD, DE, EF, FA, AC, CE

3) Diameter of graph – The diameter of graph is the maximum distance between the pair of vertices. It can also be defined as the maximal distance between the pair of vertices. Way to solve it is to find all the paths and then find the maximum of all.
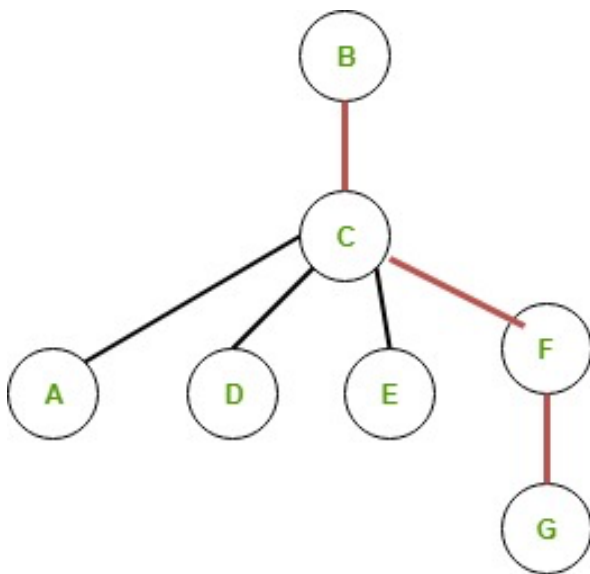
Diameter: **3**

BC → CF → FG

4) Radius of graph –A radius of the graph exists only if it has the diameter. The minimum among all the maximum distances between a vertex to all other vertices is considered as the radius of the Graph G. It is denoted as r(G).
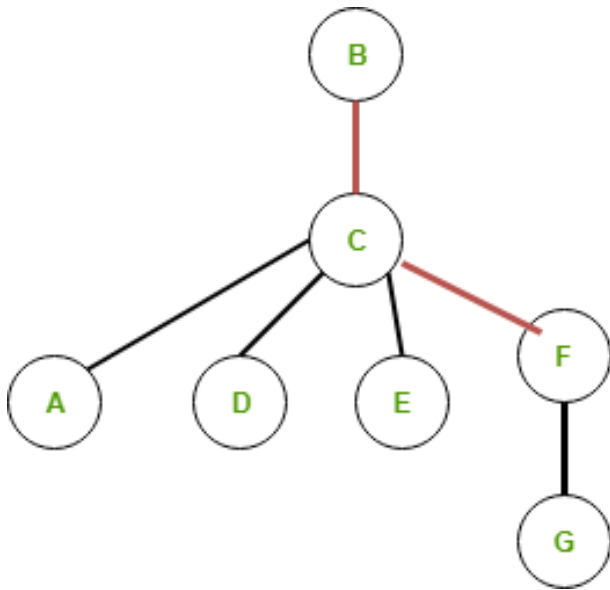
Radius: 2
All available minimum radius:
BC → CF,
BC → CE,
BC → CD,
BC → CA

5) Centre of graph – It consists of all the vertices whose eccentricity is minimum. Here the eccentricity is equal to the radius. For example, if the school is at the centre of town, it will reduce the distance buses has to travel.
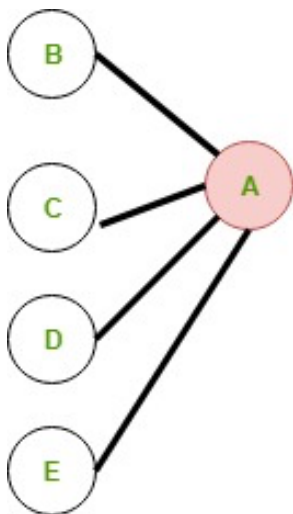
Centre: **A**

Key idea. The set of all central points of 'G' is called the centre of the Graph.

Example
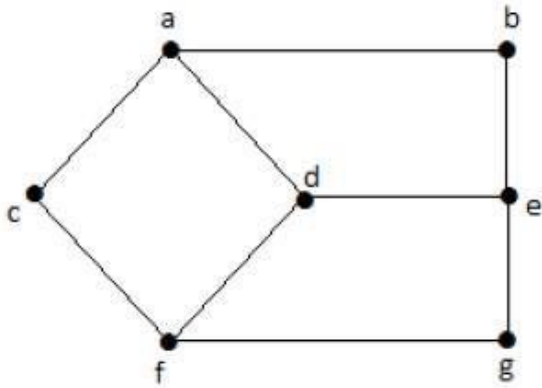
In the example graph, {'d'} is the centre of the Graph.

6) **Eccentricity of graph** – It is defined as the maximum distance of one vertex from other vertex. The maximum distance between a vertex to all other vertices is considered as the eccentricity of the vertex. It is denoted by e(V).
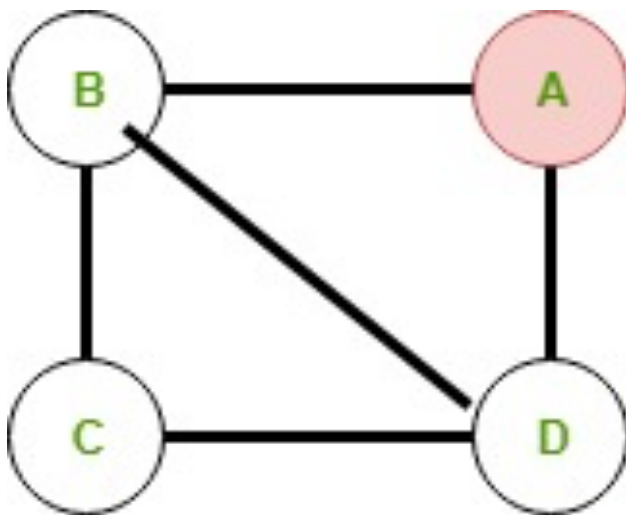

```
Eccentricity from:
(A, A) = 0
(A, B) = 1
(A, C) = 2
(A, D) = 1
Maximum value is 2, So
Eccentricity is 2
```

Key Idea. The maximum distance between a vertex to all other vertices is considered as the eccentricity of vertex. The distance from a particular vertex to all other vertices in the graph is taken and among those distances, the eccentricity is the highest of distances.

Example

In the given graph, the eccentricity of 'a' is 3.
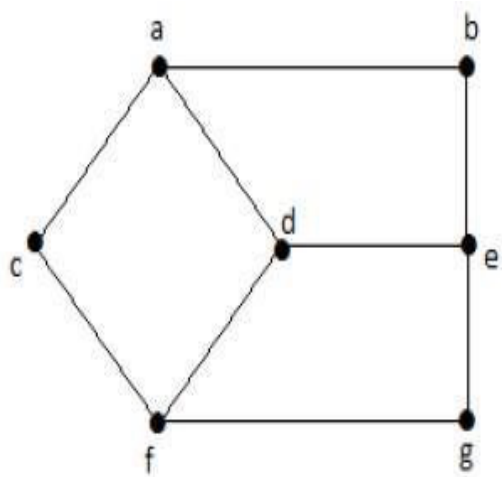
The distance from 'a' to 'b' is 1 ('ab'),

from 'a' to 'c' is 1 ('ac'),

from 'a' to 'd' is 1 ('ad'),

from 'a' to 'e' is 2 ('ab'-'be') or ('ad'-'de'),

from 'a' to 'f' is 2 ('ac'-'cf') or ('ad'-'df'),

from 'a' to 'g' is 3 ('ac'-'cf'-'fg') or ('ad'-'df'-'fg').

So, the eccentricity is 3, which is a maximum from vertex 'a' from the distance between 'ag' which is maximum.

In other words,

e(b) = 3

e(c) = 3

e(d) = 2

e(e) = 3

e(f) = 3

e(g) = 3

CHECK FOR UNDERSTANDING 6 (3 minutes)

1) It is number of edges in a shortest path between Vertex U and Vertex V.

A) Distance
B) Length
C) Radius
D) Diameter

2) It is defined as the maximum distance of one vertex from other vertex.

A) Diameter of a graph
B) Radius of a graph
C) Center of a graph
D) Eccentricity of graph

Types of Graphs

There are various types of graphs depending upon the number of vertices, number of edges, interconnectivity, and their overall structure.

1) Null Graph. A graph having no edges is called a Null Graph.

Key Idea. A graph whose edge set is empty is called as a null graph. In other words, a null graph does not contain any edges in it.

Example 1

a

b   c
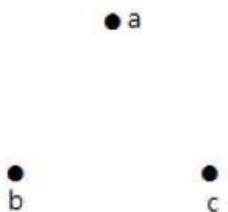
In the above graph, there are three vertices named 'a', 'b', and 'c', but there are no edges among them. Hence it is a Null Graph.

Example 2
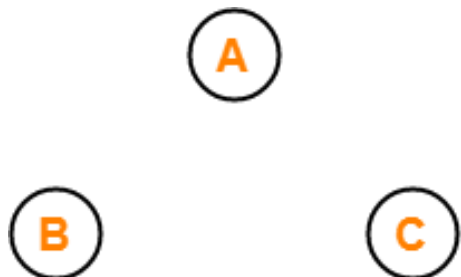
Example of Null Graph

Here,

- This graph consists only of the vertices and there are no edges in it.
- Since the edge set is empty, therefore it is a null graph.

2) Trivial Graph. A **graph with only one vertex** is called a Trivial Graph.

Key Idea. A graph having only one vertex in it is called as a trivial graph. It is the smallest possible graph.

Example 1

In the above shown graph, there is only one vertex 'a' with no other edges. Hence it is a Trivial graph.

Example 2

Example of Trivial Graph

Here,

- This graph consists of only one vertex and there are no edges in it.
- Since only one vertex is present, therefore it is a trivial graph.

3) Non-Directed Graph. A non-directed graph contains edges, but the edges are not directed ones.

Key Idea. A graph in which all the edges are undirected is called as a non-directed graph. In other words, edges of an undirected graph do not contain any direction.

Example 1
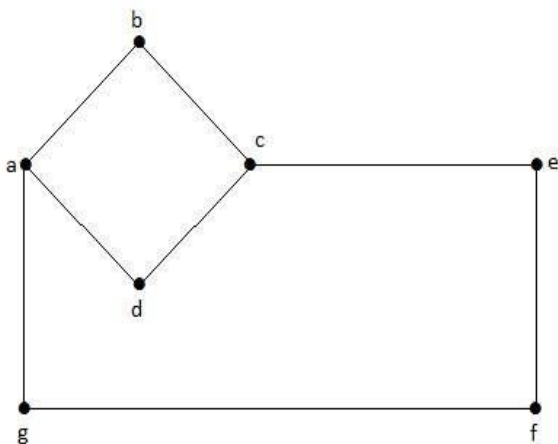
In this graph, 'a', 'b', 'c', 'd', 'e', 'f', 'g' are the vertices, and 'ab', 'bc', 'cd', 'da', 'ag', 'gf', 'ef' are the edges of the graph. Since it is a non-directed graph, the edges 'ab' and 'ba' are same. Similarly other edges also considered in the same way.

Example 2
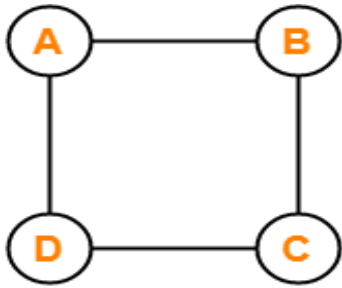
Example of Non-Directed Graph

Here,

- This graph consists of four vertices and four undirected edges.
- Since all the edges are undirected, therefore it is a non-directed graph.

4) Directed Graph. In a directed graph, each edge has a direction.

Key idea. A graph in which all the edges are directed is called as a directed graph. In other words, all the edges of a directed graph contain some direction. Directed graphs are also called as **digraphs**.

Example 1
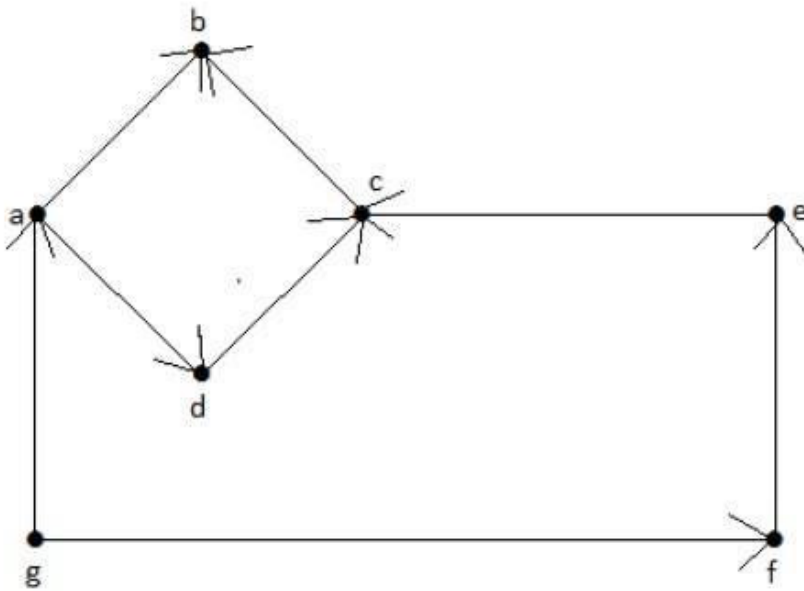
In the given graph, we have seven vertices 'a', 'b', 'c', 'd', 'e', 'f', and 'g', and eight edges 'ab', 'cb', 'dc', 'ad', 'ec', 'fe', 'gf', and 'ga'. As it is a directed graph, each edge bears an arrow mark that shows its direction. Note that in a directed graph, 'ab' is different from 'ba'.

Example 2
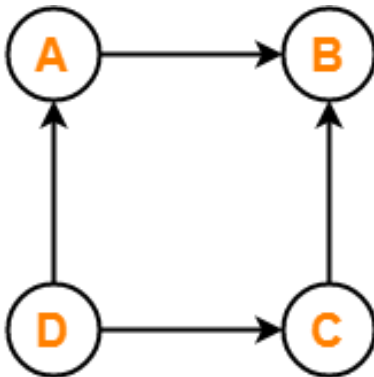
Example of Directed Graph

Here,

- This graph consists of four vertices and four directed edges.
- Since all the edges are directed, therefore it is a directed graph.

5) Connected Graph. A graph *G* is said to be connected **if there exists a path between every pair of vertices**. There should be at least one edge for every vertex in the graph. So that we can say that it is connected to some other vertex at the other side of the edge.

Key Idea. A graph in which we can visit from any one vertex to any other vertex is called as a connected graph. In connected graph, at least one path exists between every pair of vertices.

Example 1

In the following graph, each vertex has its own edge connected to other edge. Hence it is a connected graph.
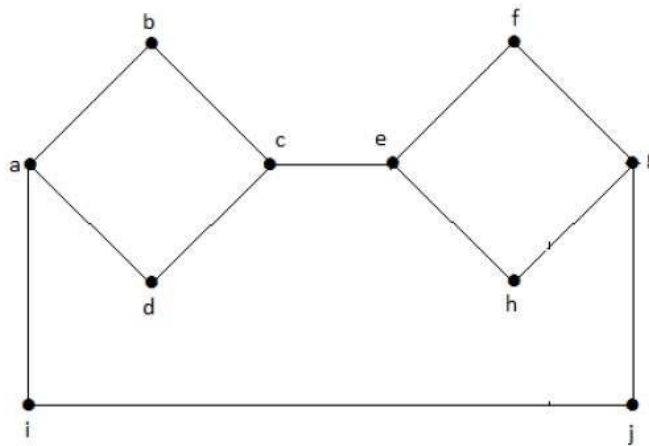
Example 2
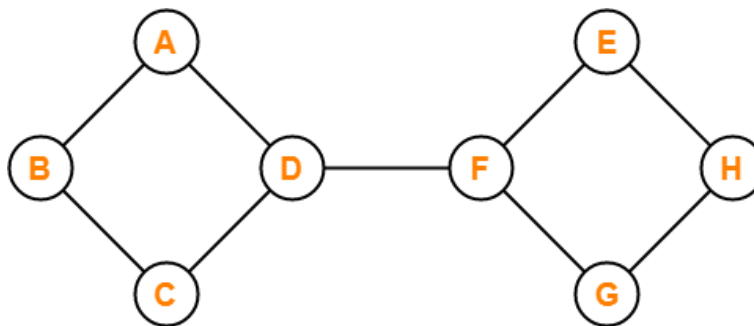
Example of Connected Graph

Here,

- In this graph, we can visit from any one vertex to any other vertex.
- There exists at least one path between every pair of vertices.
- Therefore, it is a connected graph.

6) **Disconnected Graph.** A graph G is disconnected, if it does not contain at least two connected vertices.

Example 1

The following graph is an example of a Disconnected Graph, where there are two components, one with 'a', 'b', 'c', 'd' vertices and another with 'e', 'f', 'g', 'h' vertices.
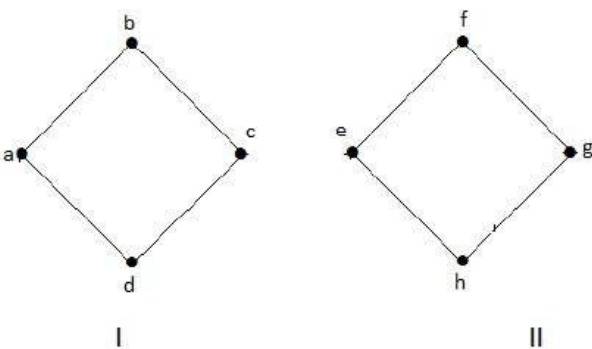
The two components are independent and not connected to each other. Hence it is called disconnected graph.
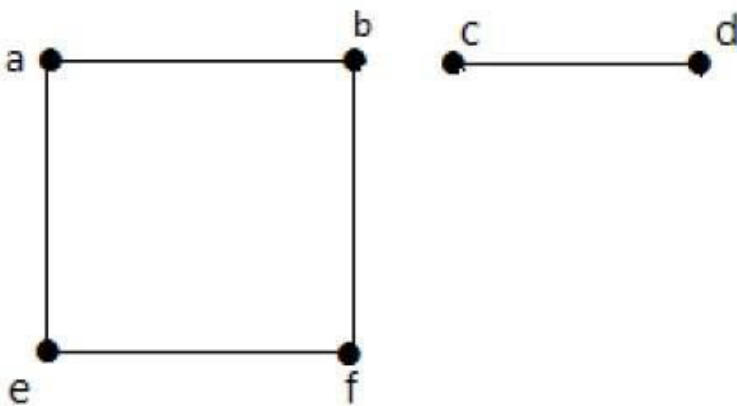
Example 2

In this example, there are two independent components, a-b-f-e and c-d, which are not connected to each other. Hence this is a disconnected graph.

Example 3
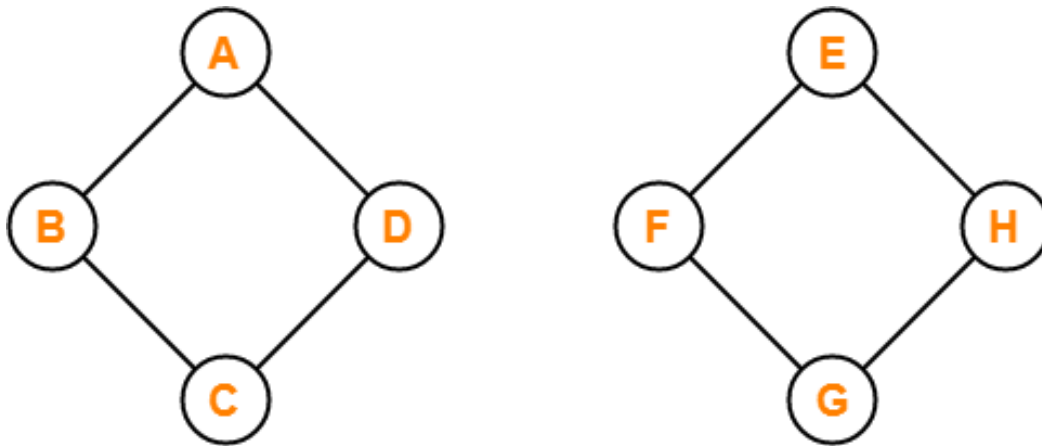
Example of Disconnected Graph

Here,

- This graph consists of two independent components which are disconnected.
- It is not possible to visit from the vertices of one component to the vertices of other component.
- Therefore, it is a disconnected graph.

7) Regular Graph. A graph G is said to be regular, **if all its vertices have the same degree**. In a graph, if the degree of each vertex is 'k', then the graph is called a 'k-regular graph'.

Key Idea. A graph in which degree of all the vertices is same is called as a regular graph. If all the vertices in a graph are of degree 'k', then it is called as a "**k-regular graph**".

Example 1

In the following graphs, all the vertices have the same degree. So these graphs are called regular graphs.
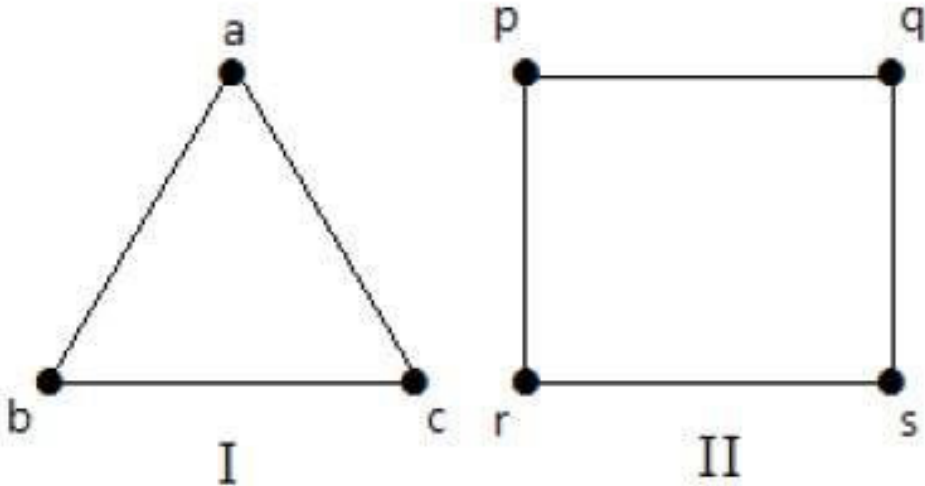
In both the graphs, all the vertices have degree 2. They are called 2-Regular Graphs.

Example 2
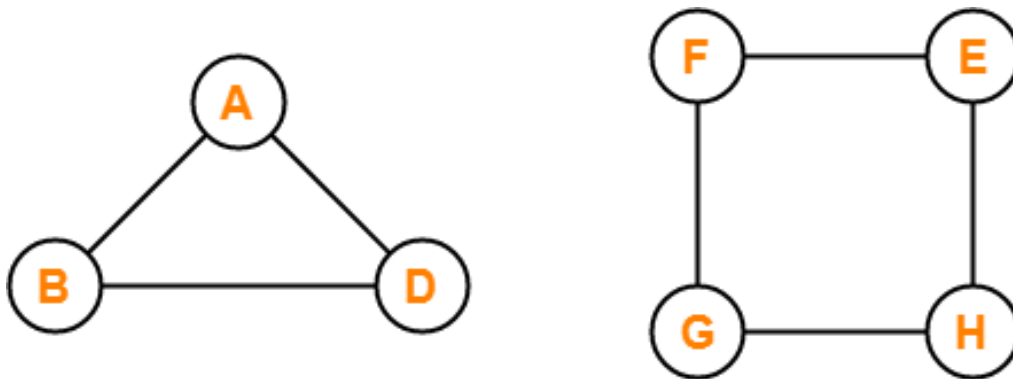
Examples of Regular Graph

In these graphs,

- All the vertices have degree-2.
- Therefore, they are 2-Regular graphs.

8) **Complete Graph.** A simple graph with 'n' mutual vertices is called a complete graph and it is **denoted by 'K_n'**. In the graph, **a vertex should have edges with all other vertices**, then it called a complete graph.

In other words, if a vertex is connected to all other vertices in a graph, then it is called a complete graph.

Key idea. A graph in which exactly one edge is present between every pair of vertices is called as a complete graph. A complete graph of 'n' vertices contains exactly $^{n}C_2$ edges. A complete graph of 'n' vertices is represented as K_n.

Example 1

In the following graphs, each vertex in the graph is connected with all the remaining vertices in the graph except by itself.
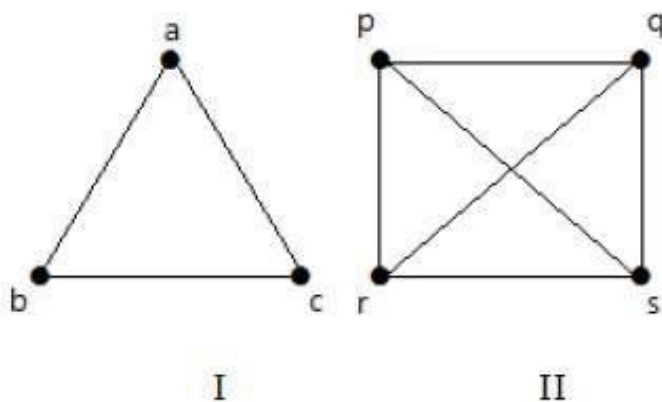
In graph I,

	a	b	c
a	Not Connected	Connected	Connected
b	Connected	Not Connected	Connected
c	Connected	Connected	Not Connected

In graph II,

	p	q	r	s
p	Not Connected	Connected	Connected	Connected
q	Connected	Not Connected	Connected	Connected
r	Connected	Connected	Not Connected	Connected
s	Connected	Connected	Connected	Not Connected

Example 2
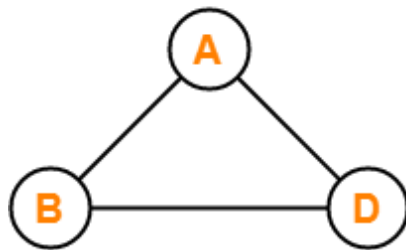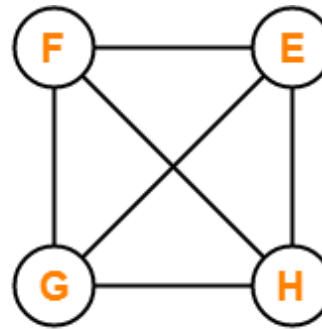
Examples of Complete Graph

In these graphs,

- Each vertex is connected with all the remaining vertices through exactly one edge.
- Therefore, they are complete graphs.

9) Cycle Graph. A simple graph with 'n' vertices (n >= 3) and 'n' edges is called a cycle graph if all its edges form a cycle of length 'n'.

If the degree of each vertex in the graph is two, then it is called a Cycle Graph. **Notation** − C_n

Key Idea. A simple graph of 'n' vertices (n>=3) and n edges forming a cycle of length 'n' is called as a cycle graph. In a cycle graph, all the vertices are of degree 2.

Example 1

Take a look at the following graphs −

- Graph I has 3 vertices with 3 edges which is forming a cycle 'ab-bc-ca'.
- Graph II has 4 vertices with 4 edges which is forming a cycle 'pq-qs-sr-rp'.
- Graph III has 5 vertices with 5 edges which is forming a cycle 'ik-km-ml-lj-ji'.
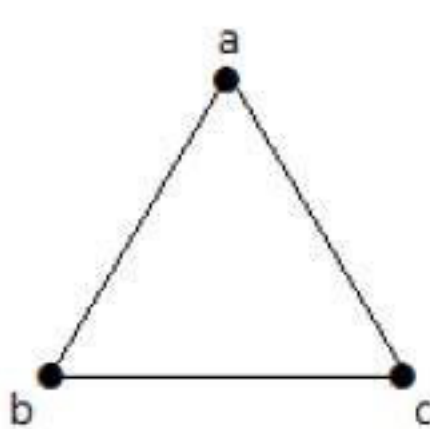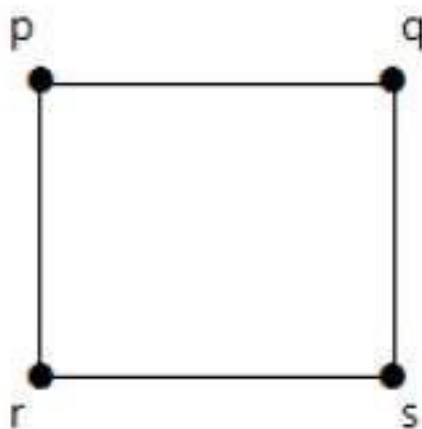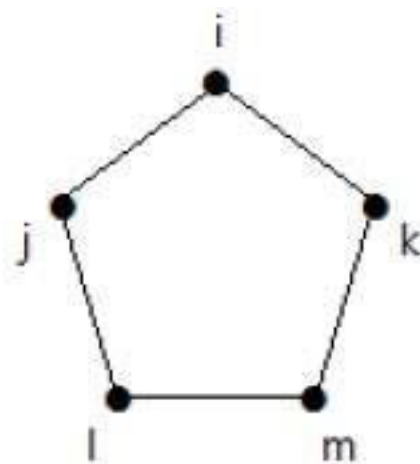
Hence all the given graphs are cycle graphs.

Example 2
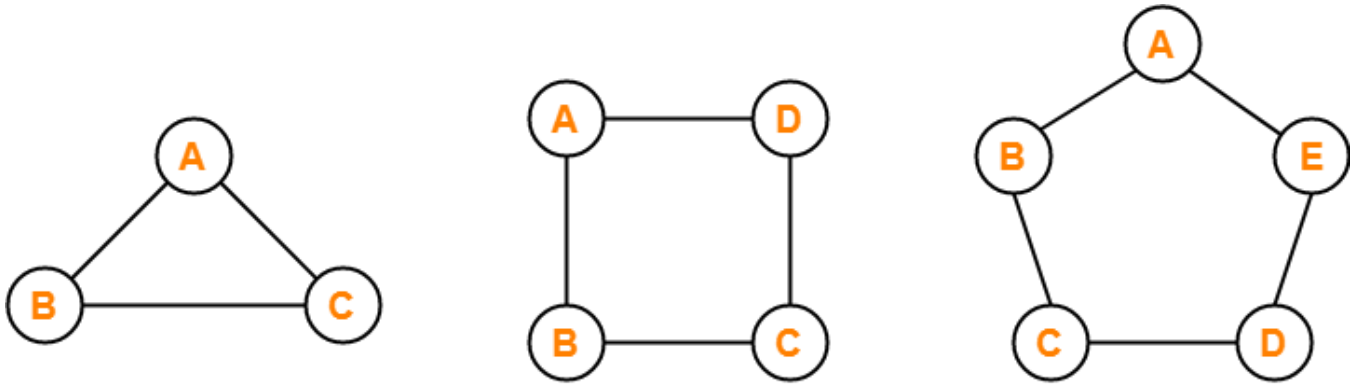
Examples of Cycle Graph

In these graphs,

- Each vertex is having degree 2.
- Therefore, they are cycle graphs.

10) Cyclic Graph. A graph **with at least one** cycle is called a cyclic graph.

Key Idea. A graph containing at least one cycle in it is called as a cyclic graph.

Example 1
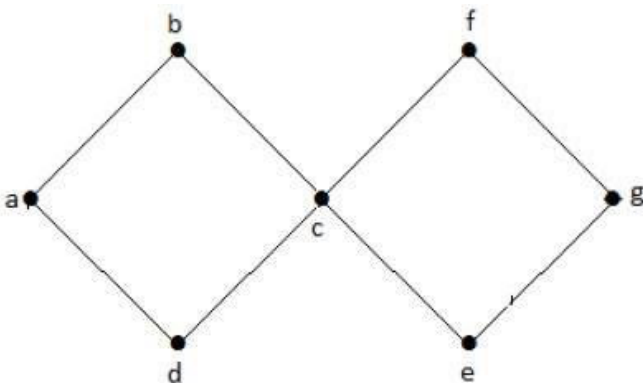
In the above example graph, we have two cycles a-b-c-d-a and c-f-g-e-c. Hence it is called a *cyclic graph.*

Example 2
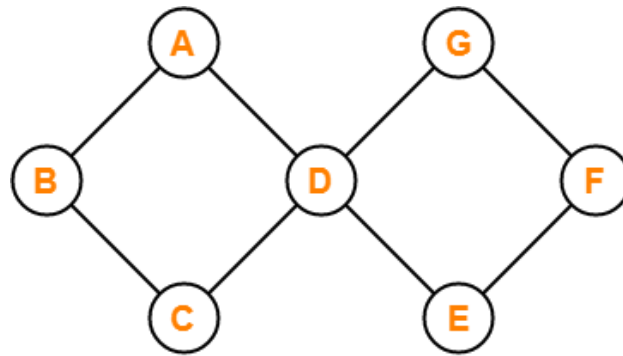
Example of Cyclic Graph

Here,

- This graph contains two cycles in it.
- Therefore, it is a cyclic graph.

11) Acyclic Graph. A graph **with no cycles** is called an acyclic graph.
Key Idea. A graph not containing any cycle in it is called as an acyclic graph.

Example 1
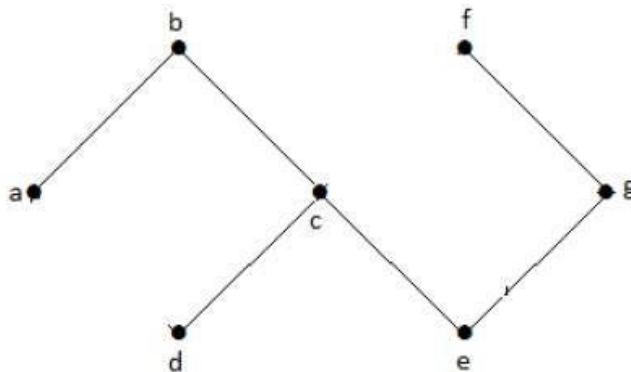
In the above example graph, we do not have any cycles. Hence it is a non-cyclic graph.

Example 2
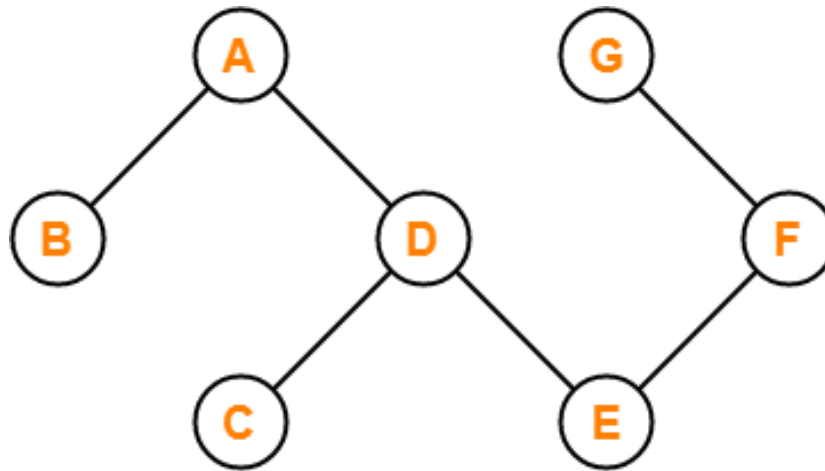
Example of Acyclic Graph

Here,

- This graph do not contain any cycle in it.
- Therefore, it is an acyclic graph.

12) Finite Graph. A graph consisting of finite number of vertices and edges is called as a finite graph.

Example 1
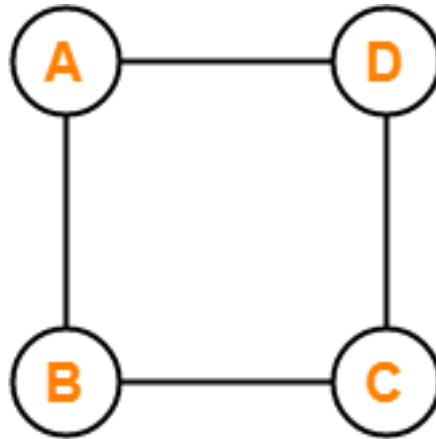
Example of Finite Graph

Here,

- This graph consists of finite number of vertices and edges.
- Therefore, it is a finite graph.

13) Infinite Graph. A graph consisting of infinite number of vertices and edges is called as an infinite graph.
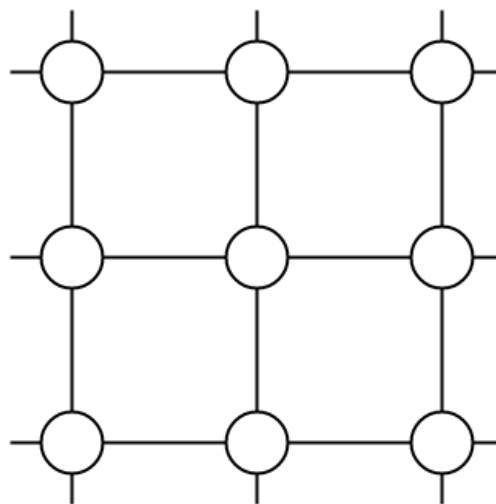
Example of Infinite Graph

Here,

- This graph consists of infinite number of vertices and edges.
- Therefore, it is an infinite graph.

14) Bipartite Graph. A simple graph G = (V, E) with vertex partition V = {V_1, V_2} is called a bipartite graph **if every edge of E joins a vertex in V1 to a vertex in V_2.**

In general, a Bipartite graph has two sets of vertices, let us say, V1 and V2, and if an edge is drawn, it should connect any vertex in set V_1 to any vertex in set V_2.

Key Idea. A bipartite graph is a graph where- Vertices can be divided into two sets X and Y. The vertices of set X only join with the vertices of set Y. None of the vertices belonging to the same set join each other.

Example 1
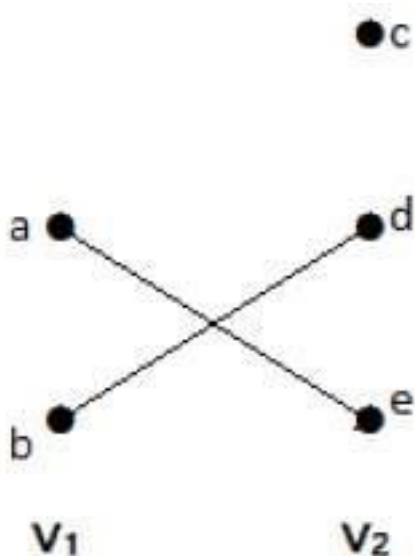
In this graph, you can observe two sets of vertices − V_1 and V_2. Here, two edges named 'ae' and 'bd' are connecting the vertices of two sets V_1 and V_2.

Example 1
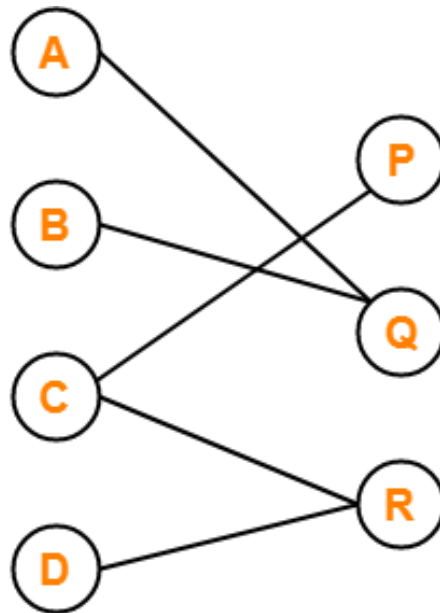
Example of Bipartite Graph

15) Planar Graph. A planar graph is a graph that we can draw in a plane such that no two edges of it cross each other.
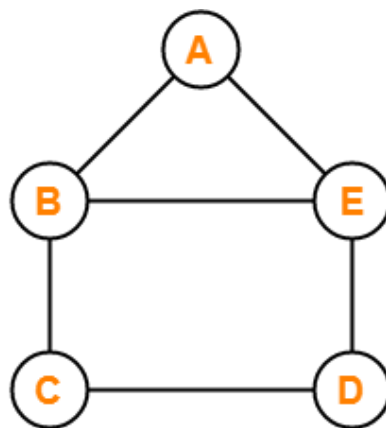
Example of Planar Graph

Here,

- This graph can be drawn in a plane without crossing any edges.
- Therefore, it is a planar graph.

16) **Simple Graph.** A graph **with no loops** and no **parallel edges** is called a simple graph.

- The maximum number of edges possible in a single graph with 'n' vertices is $^nC_2$ where $^nC_2 = n(n-1)/2$.
- The number of simple graphs possible with 'n' vertices = $2^{nc_2} = 2^{n(n-1)/2}$.

Key Idea. A graph having no self-loops and no parallel edges in it is called as a simple graph.

Example 1

In the following graph, there are 3 vertices with 3 edges which is maximum excluding the parallel edges and loops. This can be proved by using the above formulae.
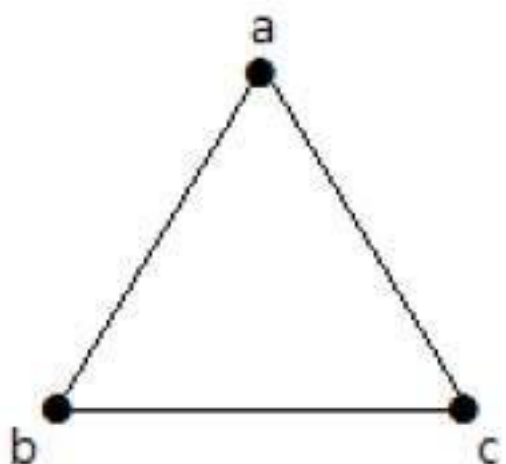
The maximum number of edges with n=3 vertices –

$$^nC_2 = n(n-1)/2$$
$$= 3(3-1)/2$$
$$= 6/2$$
$$= 3 \text{ edges}$$

The maximum number of simple graphs with n=3 vertices –

$$2^nC_2 = 2^{n(n-1)/2}$$
$$= 2^{3(3-1)/2}$$
$$= 2^3$$
$$8$$

These 8 graphs are as shown below −
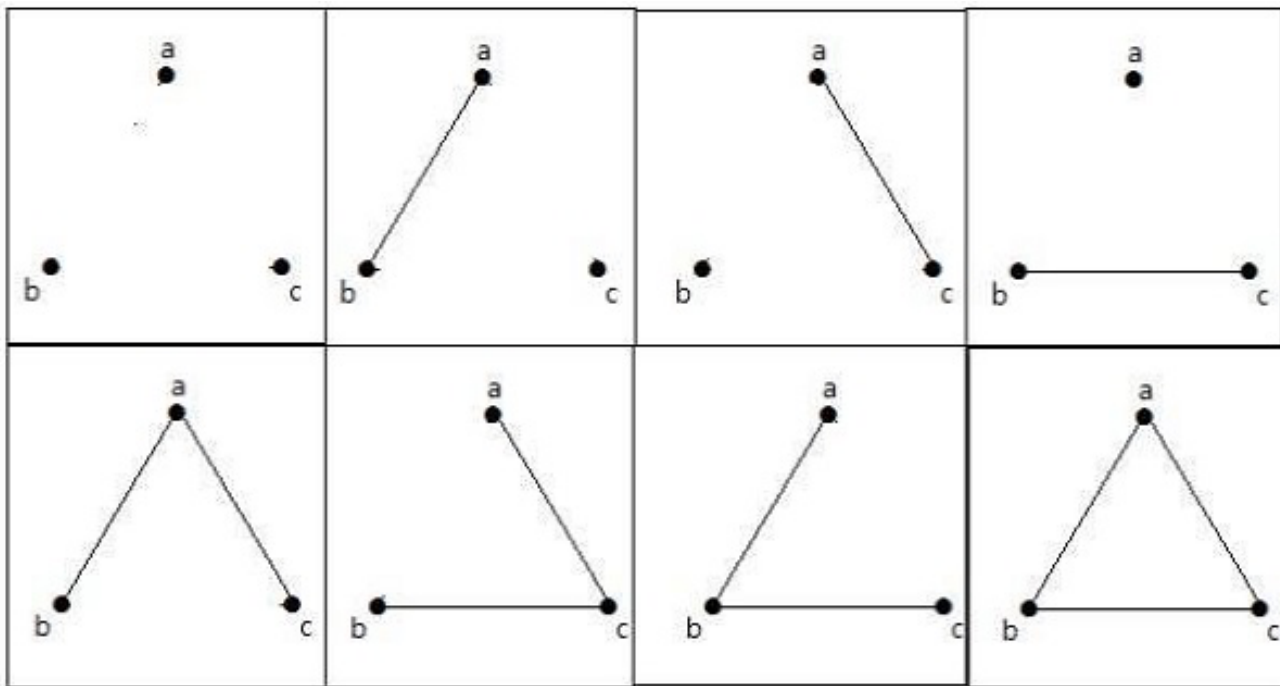
Example 2
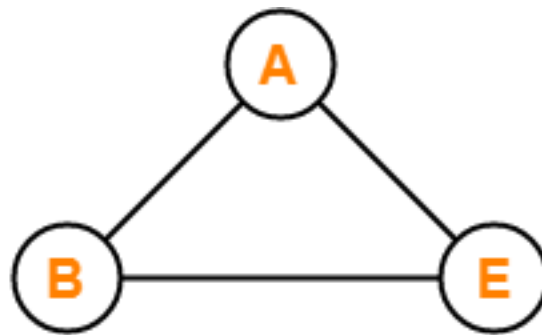
Example of Simple Graph

Here,

- This graph consists of three vertices and three edges.
- There are neither self-loops nor parallel edges.
- Therefore, it is a simple graph.

16) **Multi Graph.** A graph having no self-loops but having parallel edge(s) in it is called as a multi graph.
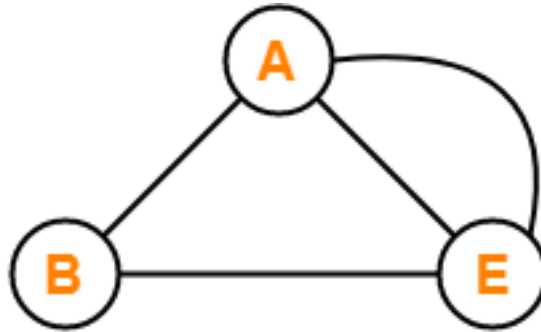
Example of Multi Graph

Here,

- This graph consists of three vertices and four edges out of which one edge is a parallel edge.
- There are no self-loops, but a parallel edge is present.
- Therefore, it is a multi-graph.

16) **Pseudo Graph.** A graph having no parallel edges but having self loop(s) in it is called as a pseudo graph.
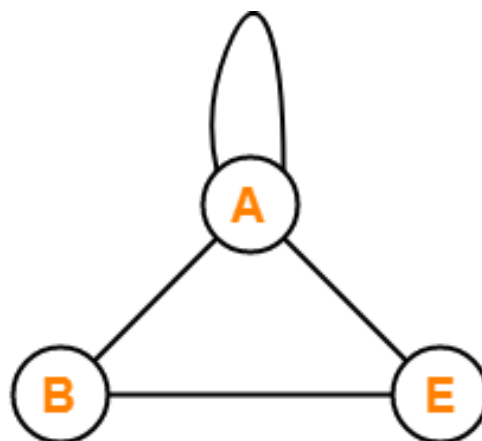
Example of Pseudo Graph

Here,

- This graph consists of three vertices and four edges out of which one edge is a self-loop.
- There are no parallel edges but a self-loop is present.
- Therefore, it is a pseudo graph.

17) Euler Graph. It is a connected graph in which all the vertices are even degree.
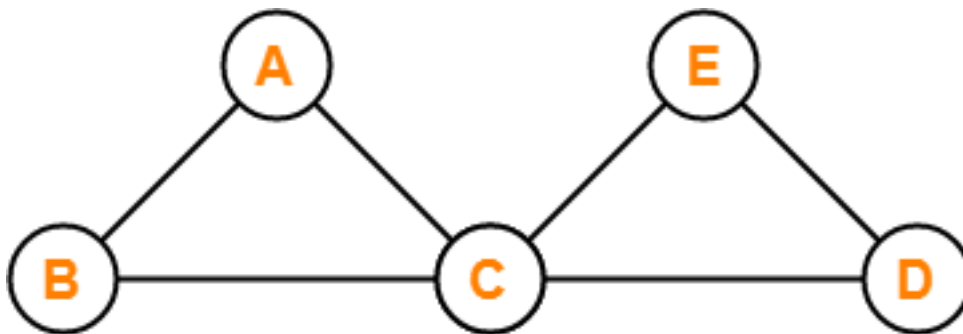
Example of Euler Graph

Here,

- This graph is a connected graph.
- The degree of all the vertices is even.
- Therefore, it is a Euler graph.

18) *Hamiltonian Graph.* If there exists a closed walk in the connected graph that visits every vertex of the graph exactly once (except starting vertex) without repeating the edges, then such a graph is called as a Hamiltonian graph.

Example 1
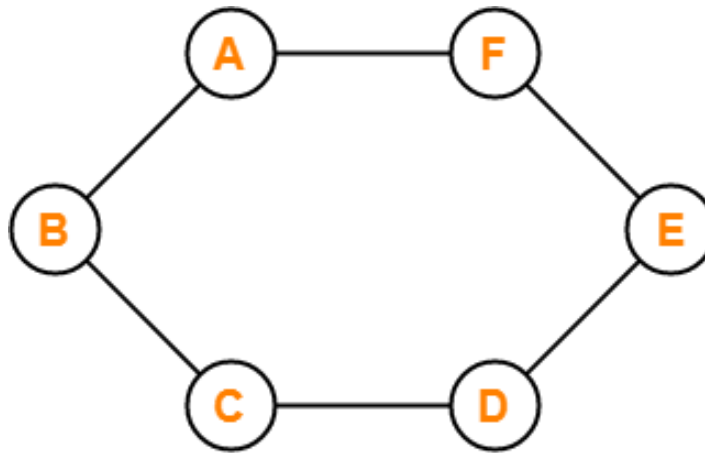
Example of Hamiltonian Graph

Here,

- This graph contains a closed walk ABCDEFG that visits all the vertices (except starting vertex) exactly once.
- All the vertices are visited without repeating the edges.
- Therefore, it is a Hamiltonian Graph.

19) Wheel Graph. A wheel graph is obtained from a cycle graph C_{n-1} by adding a new vertex. That new vertex is called a *Hub* which is connected to all the vertices of C_n.

Notation – W_n

No. of edges in W_n = No. of edges from hub to all other vertices +

> No. of edges from all other nodes in cycle graph without a hub.
>
> $= (n–1) + (n–1)$
>
> $= 2(n–1)$

Example 1

Take a look at the following graphs. They are all wheel graphs.

I(W_4)   II(W_5)   III(W_7)
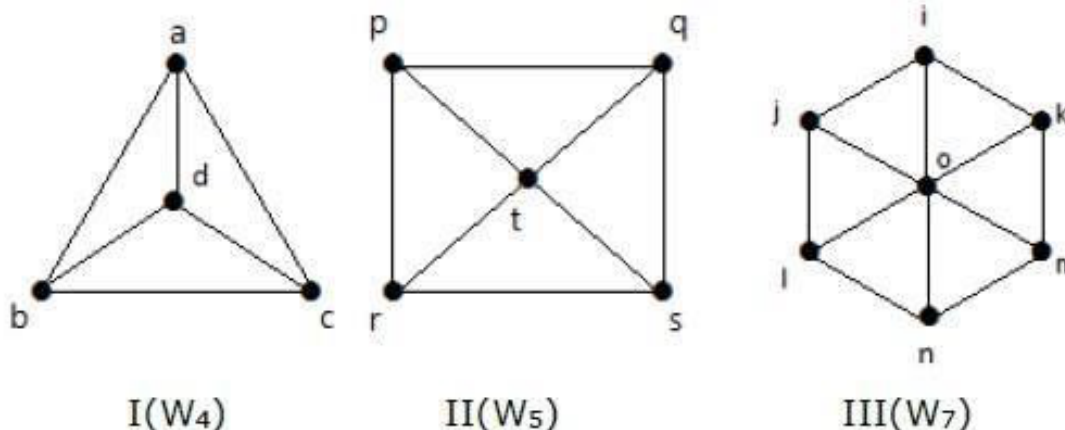
In graph I, it is obtained from C_3 by adding an vertex at the middle named as 'd'. It is denoted as W_4.

Number of edges in W4 = 2(n-1) = 2(3) = 6

In graph II, it is obtained from C4 by adding a vertex at the middle named as 't'. It is denoted as W_5.

Number of edges in W5 = 2(n-1) = 2(4) = 8

In graph III, it is obtained from C_6 by adding a vertex at the middle named as 'o'. It is denoted as W_7.

Number of edges in W4 = 2(n-1) = 2(6) = 12

20) ***Complete Bipartite Graph.*** A bipartite graph 'G', G = (V, E) with partition V = {V_1, V_2} is said to be a complete bipartite graph if every vertex in V_1 is connected to every vertex of V_2.

In general, a complete bipartite graph connects each vertex from set V_1 to each vertex from set V_2.

Example 1

The following graph is a complete bipartite graph because it has edges connecting each vertex from set V_1 to each vertex from set V_2.
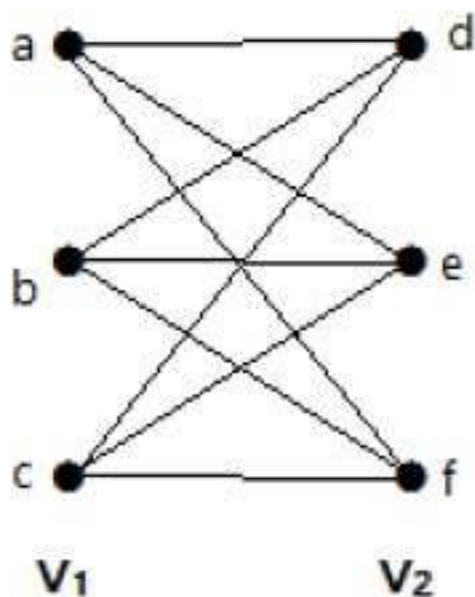
If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m, n}$.

- $K_{m, n}$ has (m + n) vertices and (mn) edges.
- $K_{m,n}$ is a regular graph if m=n.

In general, a **complete bipartite graph is not a complete graph**.

$K_{m, n}$ is a complete graph if m=n=1.

The maximum number of edges in a bipartite graph with n vertices is −

$[n^2/4]$

If n=10, k5, 5= $[n2/4]$ = $[10^2/4]$ = 25.

Similarly, K6, 4=24

K7, 3=21

K8, 2=16

K9, 1=9

If n=9, k5, 4 = $[n2/4]$ = $[92/4]$ = 20

Similarly, K6, 3=18

K7, 2=14

K8, 1=8

'G' is a bipartite graph if 'G' has no cycles of odd length. A special case of bipartite graph is a star graph.

21) *Star Graph.* A complete bipartite graph of the form K^1, n^{-1} is a star graph with n-vertices. A star graph is a complete bipartite graph if a single vertex belongs to one set and all the remaining vertices belong to the other set.

Example
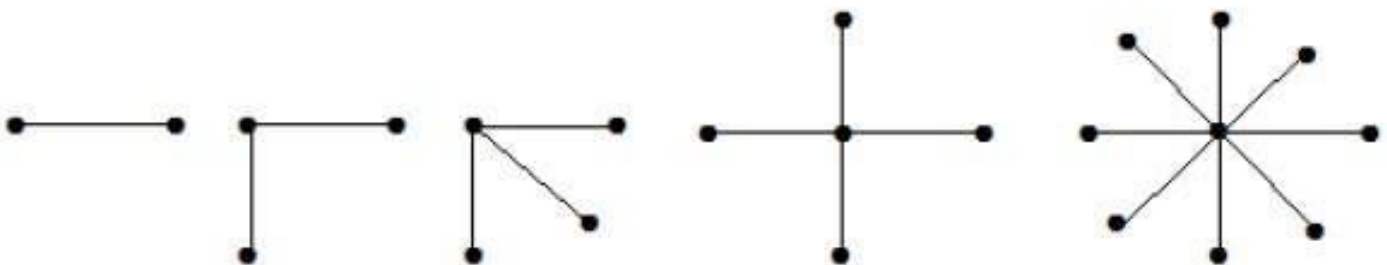
In the above graphs, out of 'n' vertices, all the 'n–1' vertices are connected to a single vertex. Hence it is in the form of K^1, $^{n-1}$ which are star graphs.

22) Complement of a Graph. Let 'G−' be a simple graph with some vertices as that of 'G' and an edge {U, V} is present in 'G−', if the edge is not present in G. It means, two vertices are adjacent in 'G−' if the two vertices are not adjacent in G.
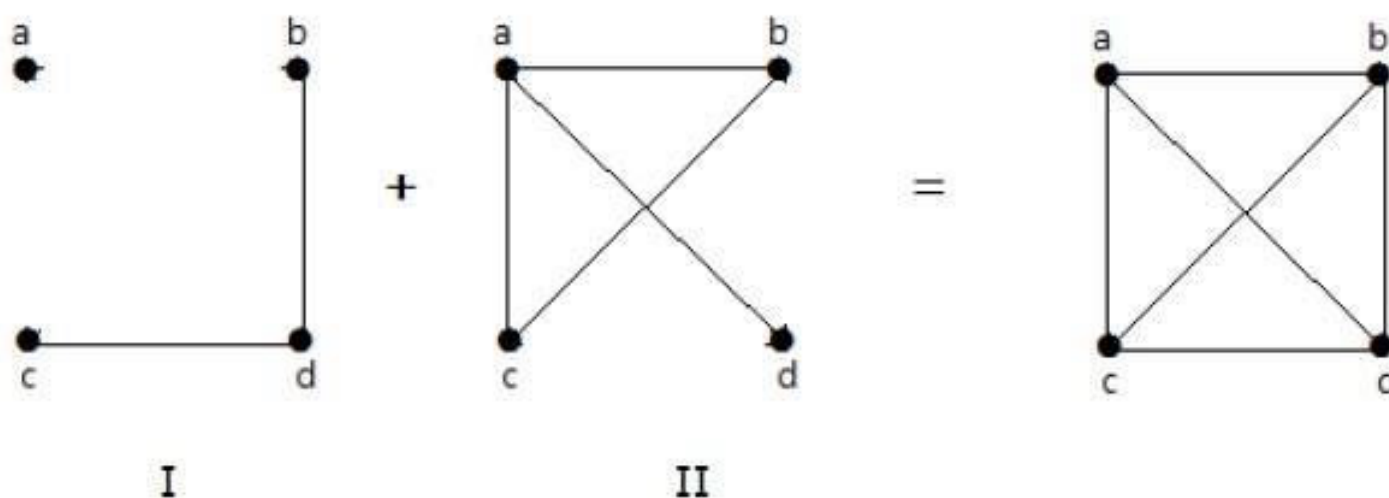
If the edges that exist in graph I are absent in another graph II, and if both graph I and graph II are combined together to form a complete graph, then graph I and graph II are called complements of each other.

Example

In the following example, graph-I has two edges 'cd' and 'bd'. Its complement graph-II has four edges.

Note that the edges in graph-I are not present in graph-II and vice versa. Hence, the combination of both the graphs gives a complete graph of 'n' vertices.

Note − A combination of two complementary graphs gives a complete graph.

If 'G' is any simple graph, then

$|E(G)| + |E('G-')| = |E(K_n)|$, where n = number of vertices in the graph.

Example

Let 'G' be a simple graph with nine vertices and twelve edges, find the number of edges in 'G-'.

You have, $|E(G)| + |E('G-')| = |E(Kn)|$

$12 + |E('G-')| =$

$9(9-1) / 2 = {}^9C_2$

$12 + |E('G-')| = 36$

$|E('G-')| = 24$

'G' is a simple graph with 40 edges and its complement 'G−' has 38 edges. Find the number of vertices in the graph G or 'G−'.

Let the number of vertices in the graph be 'n'.

We have, $|E(G)| + |E('G-')| = |E(Kn)|$

$40 + 38 = n(n-1)/2$

$156 = n(n-1)$

$13(12) = n(n-1)$

$n = 13$

Important Points-

- Edge set of a graph can be empty, but vertex set of a graph cannot be empty.

- Every polygon is a 2-Regular Graph.

- Every complete graph of 'n' vertices is a (n-1)-regular graph.

- Every regular graph need not be a complete graph.

Remember-

The following table is useful to remember different types of graphs-

	Self-Loop(s)	Parallel Edge(s)
Graph	Yes	Yes
Simple Graph	No	No
Multi Graph	No	Yes
Pseudo Graph	Yes	No

Sources
https://www.geeksforgeeks.org/graph-measurements-length-distance-diameter-eccentricity-radius-center/

https://www.gatevidyalay.com/tag/cycle-graph-definition/

Thank you for listening!
Do the assigned activities!