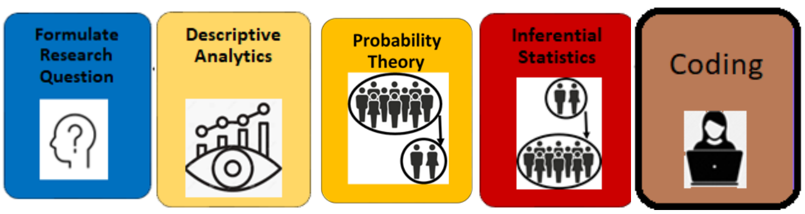
<u>Unit 7</u>: Standard Errors and Confidence Intervals

<u>Case Studies</u>:

- *Testing for association between childhood <u>lead exposure levels (<40 vs. ≥40 μg/mL)</u> and <u>IQ score.</u>*
- *Estimating a plausible range of values for the <u>average age of an adult living in the U.S.</u>*
- *Estimating a plausible range of values for the <u>proportion of adults living in the U.S. that hold a certain opinion.</u>*



<u>Summary of Concepts:</u>

- <u>Research Question:</u>
  - Set up an analysis to answer the research question: is there an **association** between a **categorical variable (with 2 levels)** and a **numerical variable** *in a **population?***
  - Give a **range of plausible values** for a **population mean** *(dealing with a single numerical variable).*
  - Give a **range of plausible values** for a **population proportion** *(dealing with a single categorical variable).*

- <u>Descriptive Analytics</u>
  - Some **summary statistics and visualizations** you can give to answer the question:
    - Is there an **association** between a **<u>categorical</u> variable (with 2 levels)** and a **<u>numerical</u> variable** *in the **collected sample?***
    - Describe a **single <u>numerical</u> variable** *in a collected sample.*
    - Describe a **single <u>categorical</u> variable** *in a collected sample.*

- <u>Probability Theory (Inference Building Blocks)</u>
  - Calculate **expected value** and **variance** for combinations and functions of random variables.
  - Introduce a new random variable: **binomial random variable.**

- <u>Central Limit Theorem:</u>
  - When is the **random variable** $\overline{X}$ approximately a **normal random variable?**
  - When is the **random variable** $\hat{p}$ approximately a **normal random variable?**

- <u>Conducting Inference</u>
  - Give a **range of plausible values** for a **population mean** *(dealing with a single numerical variable).*
  - Give a **range of plausible values** for a **population proportion** *(dealing with a single categorical variable).*

[Formulating a Research Question]: About *a Population* and a **Categorical Variable (with 2 levels)** and a **Numerical Variable**

Example: Lead Exposure Study

As an example, consider a study of lead exposure in 124 children living near a lead-emitting smelter in El Paso, Texas (Landrigan et al. 1975). Blood levels of lead were measured and collected along with information on age, gender, and full scale IQ ('fulliq' in the data set). Full IQ is a measure of intellectual function scaled to have a mean of 100 and standard deviation of 15 in the general population. In the collected data low and high exposure groups are identified as the groups with 'lead < 40' (less than 40 μg/mL) and 'lead >= 40' (at least 40 μ g /mL. The data are in the comma separated file, 'lead.csv'.

In [1]:
```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns; sns.set()
```

```
Bad key "text.kerning_factor" on line 4 in
C:\Users\vme3\AppData\Local\Continuum\anaconda3\lib\site-packages\matplotlib\mpl-data\stylelib\_classic_test_patch.mplstyle.
You probably need to get an updated matplotlibrc file from
https://github.com/matplotlib/matplotlib/blob/v3.1.3/matplotlibrc.template
or from the matplotlib source distribution
```

In [2]:
```python
df = pd.read_csv('lead.csv')
df.head()
```

Out[2]:

	id	age	sex	status	verbiq	perfiq	fulliq	iqtype	totyrs	hyperact	tapping	group
0	101	11.083333	M	77	61	85	70	WISC	11	NaN	72.0	lead < 40
1	102	9.416667	M	77	82	90	85	WISC	6	0.0	61.0	lead < 40
2	103	11.083333	M	30	70	107	86	WISC	5	NaN	49.0	lead < 40
3	104	6.916667	M	77	72	85	76	WISC	5	2.0	48.0	lead < 40
4	105	11.250000	M	62	72	100	84	WISC	11	NaN	51.0	lead < 40

Q: Assuming this sample was collected randomly and representative of all children, what types of research questions can we ask with this data?

a) Does higher exposure to lead causes a decrease in childhood IQ?

b) Did higher exposure to lead causes a decrease in childhood IQ for the children studied in this analysis?

c) Is there an association between lead exposure levels and childhood IQ score for the children studied in this analysis?

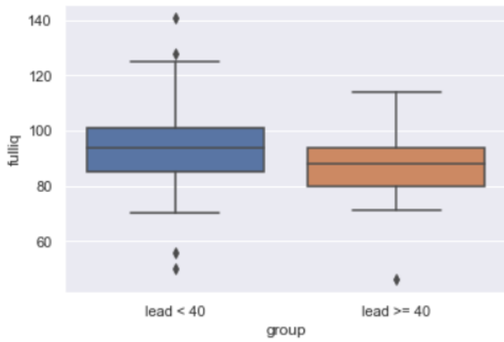
d) Is there an association between lead exposure levels and childhood IQ score?

Descriptive Analytics: Is there an **association** between a **categorical variable (with 2 levels)** and a **numerical variable** *in the **collected sample**?*

*Using **summary statistics** and **visualizations** to answer.*

Let's compare boxplots and sample means of 'fulliq' for the two exposure levels.

In [3]:
```
sns.boxplot(x='group', y='fulliq', data=df)
plt.show()
```



In [4]:
```
mean_lo = df['fulliq'][df['group']=='lead < 40'].mean()
mean_hi =  df['fulliq'][df['group']=='lead >= 40'].mean()
print('mean difference (low exposure - high exposure) = ',
      np.round(mean_lo - mean_hi, 4))
```

```
mean difference (low exposure - high exposure) =  4.8629
```

In [5]:
```
n_lo = df['fulliq'][df['group']=='lead < 40'].count()
n_hi =  df['fulliq'][df['group']=='lead >= 40'].count()

print('Sample Size n_lo: ',n_lo)
print('Sample Size n_hi: ',n_hi)
```

```
Sample Size n_lo:  78
Sample Size n_hi:  46
```

Research Question about the Population: How do these sample findings relate back to our question about an association in the larger population of children?

Question: Could the mean difference in 'fulliq' between the low and high exposure groups be due to chance variation or some other factors than exposure? To answer this kind question we need more background development.

To assess if this difference is just due to chance we need to be able to define and calculate the following things.

1. How can we define the **process of randomly selecting** a single child from the population of LOW-lead exposed children and **measuring** their IQ score?

2. How can we define and calculate what we would **expect** [the IQ of a **single child** from the **population of LOW-lead exposed]** to be? *(Aka: if we sampled many, many children from the population of LOW-lead exposed children, what would the average IQ score be?)*

3. How can we define and calculate what we would **expect the variance of** [the IQ of a **single child** from the **population of LOW-lead exposed]** to be? *(Aka: if we sampled many, many children from the population of LOW-lead exposed children, what would the variance of these IQ scores be?)*

4. How can we define the **process of randomly selecting** a random sample of size n_lo=78 from the population of LOW-lead exposed children and **measuring** the average IQ score of the sample?

5. How can we define and calculate what we would **expect** [the average IQ score of **a random sample of size n_lo=78** from the **population of LOW-lead exposed]** to be? *(Aka: if we collected many, many random samples of size 78 from the population of LOW-lead exposed children and took the average IQ score what would the average of these averages to be?)*

6. How can we define and calculate what we would **expect the variance of the** [the average IQ score of **a random sample of size n_lo=78** from the **population of LOW-lead exposed]** to be? *(Aka: if we collected many, many random samples of size 78 from the population of LOW-lead exposed children and took the average IQ score what would the variance of these averages to be?)*

7. How can we define the **process of randomly selecting** a single child from the population of HIGH-lead exposed children and **measuring** their IQ score?

8. How can we define and calculate what we would **expect** [the IQ of a **single child** from the **population of HIGH-lead exposed]** to be? *(Aka: if we sampled many, many children from the population of HIGH-lead exposed children, what would the average IQ score be?)*

9. How can we define and calculate what we would **expect the variance of** [the IQ of a **single child** from the **population of HIGH-lead exposed]** to be? *(Aka: if we sampled many, many children from the population of HIGH-lead exposed children, what would the variance of these IQ scores be?)*

10. How can we define the **process of randomly selecting** a random sample of size n_hi=46 from the population of HIGH-lead exposed children and **measuring** the average IQ score of the sample?

11. How can we define and calculate what we would **expect** [the average IQ score of **a random sample of size n_hi=46** from the **population of HIGH-lead exposed]** to be? *(Aka: if we collected many, many random samples of size 46 from the population of LOW-lead exposed children and took the average IQ score what would the average of these averages to be?)*

12. How can we define and calculate what we would **expect the variance of the** [the average IQ score of **a random sample of size n_hi=46** from the **population of HIGH-lead exposed]** to be? *(Aka: if we collected many, many random samples of size 46 from the population of HIGH-lead exposed children and took the average IQ score what would the variance of these averages to be?)*

Research Question about the Population: How do these definitions relate back to our question about whether noticing a sample mean difference of $\bar{x}_{lo} - \bar{x}_{hi} = 4.8629$ is unlikely or not?

Probability Theory: Calculating the summary statistics (ie. **mean**, **variance**, and **standard deviation**) for:

1.) a **random variable**

2.) a **function** of a random variable

3.) a **linear combination** of random variables

Distributions of sums and averages of random variables: expectations and variance

Our previous simulation studies demonstrated that sample means from random samples have expectations equal to the population mean, and their populatoin standard deviations equal to the population standard deviation divided square root of sample size, assuming sampling with replacement. What is the theory behind this?

Recall that the population mean of a random variable X, often called the expectation, has the following form:

$$E[X] = \begin{cases} \sum_x xp(x), & \text{discrete } X \\ \int_{-\infty}^{\infty} xf(x)dx, & \text{continuous } X \end{cases}$$

where $\sum_x p(x) = 1$ or $\int f(x)dx = 1$ depending on whether X is deiscrete or continuous.

A function of X, say $Y = g(X)$, is a random variable as well. Its expectation can be computed similarly:

$$E[g(X)] = \begin{cases} \sum_x g(x)p(x), & \text{discrete } X \\ \int_{-\infty}^{\infty} g(x)f(x)dx, & \text{continuous } X \end{cases}$$

For example, let $\mu = E[X]$. Setting $g(x) = (x - \mu)^2$ in the formula shows how to compute the variance, $Var[X] = E[(X - \mu)^2)]$.

Remark: For future reference, it is common in many areas of application to have a mixture of discrete and continuous distributions, for example, nonnegative insurance claims data where many individuals have zero claims, but some have positive claims. That will not affect the basic ideas here.

Expectation of a weighted sum

Now suppose we have two random variables X and Y and consider the combined random variable $aX + bY$, where a and b are constants. With more advanced techniques it can be shown that for any random variables X and Y:

$$E[aX + bY] = aE[X] + bE[Y]$$

Application to sums and averages: Suppose we have a random sample $X_1, X_2, \dots, X_n$ where each observation has mean μ and standard deviation σ. Define the sample mean $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Then:

$$E[X_1 + X_2 + \cdots + X_n] = n\mu$$

$$E[\bar{X}] = E[n^{-1}(X_1 + X_2 + \cdots + X_n)] = \mu$$

Covariance between two random variables

Returning to our friends X and Y, we define the **covariance** between them as:

$$Cov[X, Y] = E[(X - E(X))(Y - E(Y)] = E[XY] - E[X]E[Y]$$

and the **correlation**

$$\rho[X, Y] = \frac{Cov[X, Y]}{SD[X]SD[Y]}$$

where $SD[X] = \sqrt{Var[X]}$ and $SD[Y] = \sqrt{Var[Y]}$. The correlation measures linear dependence between X and Y. It is always between -1 and 1. Values close to 1 indicate strong postive linear dependence. Values close to -1 indicate strong negative linear dependence. Correlation = 0 means there is no linear dependence.

Variances for sums and averages random variables

Next consider the variance of the weighted sum $aX + bY$. Whether they are correlated or not we have:

$$Var[aX + bY] = E[a(X - \mu_X) + b(Y - \mu_Y))^2]$$

$$= a^2 E[(X - \mu_X)^2] + b^2 E[(Y - \mu_Y)^2] + 2abE[(X - \mu_x)(Y - \mu_Y)]$$

$$= a^2 Var[X] + b^2 Var[Y] + 2abCov[X, Y]$$

Only if X and Y are uncorrelated (i.e. $Cov[X, Y] = 0$), we have the simplified formula:

$$Var[aX + bY] = a^2 Var[X] + b^2 Var[Y].$$

Independent random samples and the square root rule

For an independent random sample, $X_1, X_2, \ldots, X_n$, suppose each observation has the same mean μ and standard deviation σ.

The square root rule says that the sample mean, $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$, has a populatoin standard deviation proportional to $\frac{1}{\sqrt{n}}$. Why is this? Observe that

$$Var(X_1 + X_2 + \cdots + X_n) = n\sigma^2$$

and

$$SD(X_1 + X_2 + \cdots + X_n) = \sqrt{n}\sigma.$$

Therefore,

$$Var(\bar{X}) = Var[n^{-1}(X_1 + X_2 + \cdots + X_n)] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

and

$$SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

This is the origin of the square root rule for sample means and proportions.

Formulating a Research Question: What are a range of plausible values for a population mean μ (single numerical variable)?

What is a range of plausible values for the average age of an adult living in the U.S.?

Data Collection and Descriptive Analytics: Calculate summary statistics from a random sample.

Example: Pew Research Survey Age Distribution

The February 2017 Pew Research Center random phone number dialing survey had 1,503 respondents in total. How close is the mean age of respondents in the survey to the mean age for the population frame of the survey? Let's compute the sample mean and consider how to estimate the standard error.

In [6]:
```
# Read in the data, extracting only the 'age' column
import zipfile as zp
zf = zp.ZipFile('Feb17-public.zip')
missing_values = ["NaN", "nan", "Don't know/Refused (VOL.)"]
df_pew_age = pd.read_csv(zf.open('Feb17public.csv'),
                   na_values=missing_values)['age']
```

In [7]:
```
df_pew_age
```

Out[7]:
```
0       80.0
1       70.0
2       69.0
3       50.0
4       70.0
        ...
1498    37.0
1499    30.0
1500    72.0
1501    67.0
1502    35.0
Name: age, Length: 1503, dtype: float64
```

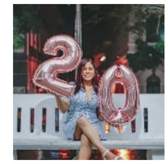
In [8]:
```
# compute and display sample statistics
age_mn = df_pew_age.mean()
age_std = df_pew_age.std()
age_n = df_pew_age.shape[0]
print('mean age=', round(age_mn, 2),
      ' std age=', round(age_std,2),
      ' n=', age_n)
```

```
mean age= 50.49  std age= 17.84  n= 1503
```

So our natural estimates of the population mean and standard deviation for age are $\bar{x} = 50.49$ and $s = 17.84$.

Inference Building Blocks
Population of a Single Numerical Variable

Unknown	Known
Population Data	Random Sample of Size n
Population Mean μ	Sample Mean $\bar{x}$
Population Standard Deviation σ	Sample Standard Deviation *s*



Ultimate Goal: Use the sample data to create a plausible range of values for μ.

Sub-Goal: *Estimate* the variability of the sample means of this size n.

Standard error for the mean

Assuming random sample from the population, we saw previously that the standard deviation of the sample mean is given by

$$SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

where σ is the standard deviation of the population distribution, and n is the sample size. Because σ is an unknown population parameter, we need to estimate it. Using the sample version standard deviation S, we obtain the **standard error:**

$$SE(\bar{X}) = \frac{S}{\sqrt{n}}$$

where S is the sample standard deviation. This depends only on sample statistics, so we are able to compute it from the data.

Example: Pew Research Survey Age Distribution

We can now compute the standard error for the mean age in the Pew survey.

In [9]:
```
# standard error calculation
age_se = age_std/np.sqrt(age_n)
print('mean age=', round(age_mn, 2),
      ' standard error=', round(age_se, 3))
```

```
mean age= 50.49  standard error= 0.46
```

Q: Suppose I asked a random sample of people in this Zoom room how many hours they slept last night. Which would be more unusual?

a) Randomly selecting one person in the class who had slept 4 hours last night

b) Randomly selecting 5 people in the class where the average number of hours they slept last night was 4 hours.

Remark: It is important to understand that the *standard error* for the mean is much smaller than the *standard deviation* for the sample. The standard deviation is an estimate of the spread of the values in the population, whereas the standard error is an estimate of the spread in the distribution of $\bar{X}$, which is much smaller due to the averaging effect captured by the square root rule.

Inference Building Blocks

Population of a Single Numerical Variable

Unknown	Known
Population Data	Random Sample of Size n
Population Mean μ	Sample Mean $\bar{x}$
Population Standard Deviation σ	Sample Standard Deviation s

Goal: Create a plausible range of values for which μ could be.

Sub-Goal: Define this range.

Confidence intervals

With an estimate of the sampling distribution of the sample mean we are prepared to develop a confidence interval for the population mean. In general, we define a **confidence interval** with confidence level $1 - \alpha$ as an interval of the form

$$[C_{lo}(\text{Data}),\ C_{hi}(\text{Data})]$$

with the property that

$$P\left(C_{lo}(\text{Data}) \leq \text{Parameter} \leq C_{hi}(\text{Data})\right) = 1 - \alpha$$

where 'Parameter' refers to the unknown value of the parameter that we are trying to estimate.

Sub-Goal: How to calculate this range *when* $\bar{X}$ is a normal random variable.

It is frequently the case that we have an estimate that is approximately normally distributed, due to the central limit theorem, and a standard error that is a good estimate of the standard deviation of that normal distribution. In these cases a natural confidence interval takes the form

$$(\text{Estimate} - z_{1-\frac{\alpha}{2}} * SE, \quad \text{Estimate} + z_{1-\frac{\alpha}{2}} * SE)$$

where z_q is shorthand for the qth quantile of the standard normal distribution. In this case the margin of error is the half width:

$$\text{margin-of-error} = z_{1-\frac{\alpha}{2}} * SE$$

Ex: Suppose we wanted to calculate a confidence interval (ie. a range of plausible values) for μ (the average age of ALL adults living in the U.S.) with confidence level 90%.

- We have a random sample of size n=1503 that has a mean age of 50.49 years and a standard deviation of 17.84 years.
- Assume we know that $\bar{X}$ (the random variable that represents our estimate for μ) is normally distributed.

What equation would we use to calculate this 90% confidence interval?

Ex: Suppose we wanted to calculate a confidence interval (ie. a range of plausible values) for μ (the average age of ALL adults living in the U.S.) with confidence level 97%.

- We have a random sample of size n=1503 that has a mean age of 50.49 years and a standard deviation of 17.84 years.
- Assume we know that $\bar{X}$ (the random variable that represents our estimate for μ) is normally distributed.

What equation would we use to calculate this 97% confidence interval?

Sub-Goal: How to determine when $\bar{X}$ is a normal random variable. (aka: **Introduce the Central Limit Theorem).**

Margin of error and confidence interval for the mean

Case 1: X is a normal random variable.

Normality of the sample mean in a normal sample

If $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution, then the sample mean, $\bar{X}$ is normally distributed as well, with the same mean, μ, as the individual observations, and with a reduced population standard deviation of $\sigma/\sqrt{n}$.

Case 2: X is not a normal random variable.

Approximate normality of sample means in general

Even if the original sample is not normally distributed, the **Central Limit Theorem** implies that the sample mean is approximately normal in large samples n.

A rule of thumb for saying a sample size is "large enough" is if $n > 30$.

We can therefore use the form a normal confidence interval even if the original sample is not precisely normally distributed.

In particular, we have the following equivalent results for the sample mean $\bar{X}$:

$$\bar{X} \text{ is approximately normal with mean } \mu \text{ and standard deviation } SD(\bar{X}) = \frac{\sigma}{\sqrt{n}};$$

$$Z_\sigma = \frac{\bar{X} - \mu}{SD(\bar{X})} \text{ is approximately normal with mean 0 and standard deviation 1.}$$

What to do when you don't know σ?

Plug-in estimate of standard deviation: When all we have is data we cannot use these expressions, because they depend on the unobservable value of σ for the population! However, in large samples we expect, with high probability,

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2} \quad \approx \quad \sigma.$$

"Plugging in" we have the following useful approximations:

$$\bar{X} \text{ is approximately normal with mean } \mu \text{ and standard deviation } SE(\bar{X}) = \frac{S}{\sqrt{n}};$$

$$Z = \frac{\bar{X} - \mu}{SE(\bar{X})} \text{ is approximately normal with mean 0 and standard deviation 1.}$$

Ex: Create a 90% confidence interval for the average age of an adult living in the U.S.

- We have a random sample of size n=1503 that has a mean age of 50.49 years and a standard deviation of 17.84 years.
- ~~Assume we know that $\bar{X}$ (the random variable that represents our estimate for μ) is normally distributed.~~ *(aka: don't assume that we know this).*

Based on these approximations we can compute confidence intervals with target levels of confidence. For example, suppose we want a 90% confidence interval for the population mean. Observe that because our sample size n=1503>30, according to the statements above

$$P\left(-z_{0.95} < \frac{\bar{X}-\mu}{SE(\bar{X})} < z_{0.95}\right) \approx 0.90,$$

which is the same as

$$P\left(\bar{X} - z_{0.95} * SE(\bar{X}) < \mu < \bar{X} + z_{0.95} * SE(\bar{X})\right) \approx 0.90.$$

Quick calculation:

In [10]:

```python
from scipy.stats import norm
norm.ppf(0.95)
```

Out[10]: 1.6448536269514722

Therefore, the **margin of error** for a 90% confidence level is

$$MOE = \pm 1.645 * SE(\bar{X})$$

and the 90% confidence interval has the form

$$(\bar{X} - 1.645 * SE(\bar{X}), \quad \bar{X} + 1.645 * SE(\bar{X}))$$

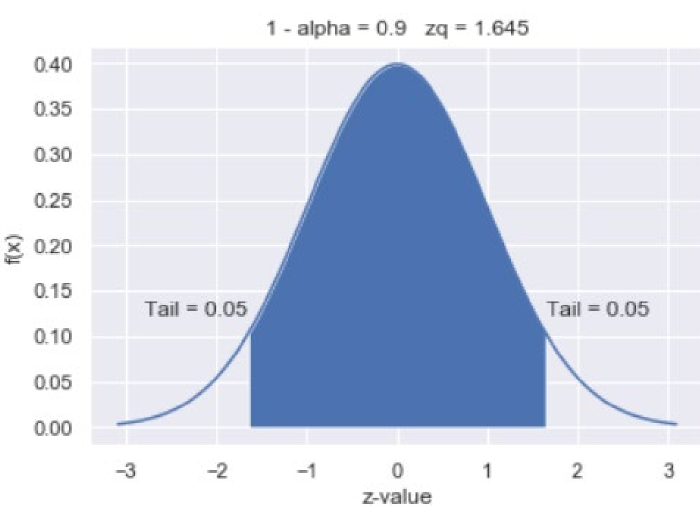
In general, if we desire a $100(1-\alpha)\%$ confidence interval, then we need to use the mutiplier

$$z_{1-\frac{\alpha}{2}} = 100(1-\frac{\alpha}{2}) \text{ percentile of the normal distribution.}$$

This ensures that the middle part of the distribution, excluding the tails includes $100(1-\alpha)$ percent as illustrated below.

In [11]:

```python
# alpha is the probability of being
# outside of the confidence interval
alpha = 0.1
zq = norm.ppf(1-(alpha/2))
eps = 0.02
# plotting grid
x = np.linspace(norm.ppf(0.001), norm.ppf(0.999), 100)
# density curve
plt.plot(x, norm.pdf(x))
plt.xlabel('z-value')
plt.ylabel('f(x)')
# adaptive title and labels
plt.title('1 - alpha = ' + str(1-alpha) + '   zq = ' + str(round(zq, 3)))
plt.text(zq, norm.pdf(zq) + eps, 'Tail = ' + str(alpha/2), horizontalalignment='left')
plt.text(-zq , norm.pdf(zq) + eps, 'Tail = ' + str(alpha/2), horizontalalignment='right')
# add shaded areas whose probability we need
xalpha = np.linspace(norm.ppf(alpha/2), norm.ppf(1-(alpha/2)), 100)
plt.fill_between(xalpha, 0, norm.pdf(xalpha))
plt.show()
```



<u>Ex:</u> Create a 95% confidence interval for μ, the average age of an adult living in the U.S.

- We have a random sample of size n=1503 that has a mean age of 50.49 years and a standard deviation of 17.84 years.
- ~~Assume we know that $\bar{X}$ (the random variable that represents our estimate for μ) is normally distributed.~~ *(aka: don't assume that we know this).*

Here are the calculations using Python:

```
In [29]:  age_mn = df_pew_age.mean()
          age_std = df_pew_age.std()
          age_n = df_pew_age.shape[0]
          age_se = age_std/np.sqrt(age_n)
          alpha = 0.01
          zq = norm.ppf(1-(alpha/2))
          print('mean age=', round(age_mn, 2), ' std age=',
                round(age_std,2), ' n=', age_n)
          print('Z-score for this 95% confidence interval:', zq)
          print('Std Error for mean:', round(age_se, 3))
          print('Confidence level:', 1-alpha)
          print('Confidence limits:', (round(age_mn - zq*age_se, 3),
                                       round(age_mn + zq*age_se, 3)))

          mean age= 50.49  std age= 17.84  n= 1503
          Z-score for this 95% confidence interval: 2.5758293035489004
          Std Error for mean: 0.46
          Confidence level: 0.99
          Confidence limits: (49.303, 51.674)
```

<u>General:</u> How to interpret a confidence interval

Once we've calculated your confidence interval (at confidence level $1-\alpha$ for your population parameter, we say:

"We are $(1-\alpha)\cdot 100\%$ confident that our <u>population parameter</u> is within <u>lower bound of confidence interval</u> and <u>upper bound of confidence interval.</u>"

<u>Ex:</u> Interpret your 95% confidence interval from above.

General: What does "XX% confident" mean in the interpretation of a confidence interval?

Suppose we have calculated a XX% confidence interval for a ______________________________ using a random sample of size __________.

Now suppose we do the following:

- Collect many, many random samples, each with a sample size of ____________.
- For each random sample we calculate ______________________.
- And then we construct a confidence interval centered around each _________________________.

When we say "XX% confident" in our original confidence interval interpretation, we are saying that we would expect ____________ of the confidence intervals that we would create, using the method described above would _______________________________.

Ex: So what does 95% confident mean in our interpretation above?

We are 95% confident that μ the average age of adults living in the U.S. is between 49.303 and 51.674.

Suppose we do the following:

- Collect many, many random samples, each with a sample size of ____________.
- For each random sample we calculate ______________________.
- And then we construct a confidence interval centered around each _________________________.

When we say "_____% confident" in our original confidence interval interpretation, we are saying that we would expect ____________ of the confidence intervals that we would create, using the method described above would ____.

Formulating a Research Question: What is a range of plausible values for a population proportion p (single categorical variable)?

What is a range of plausible values for the proportion *p* of adults living in the U.S. that "approve of the way Donald Trump is handling his job as president?"

Data Collection and Descriptive Analytics: Calculate summary statistics from a random sample.

Estimating population proportions with standard errors

Example: Pew Research Survey

In the February 2017 Pew Research Center random phone number dialing survey of 1,503 adults of voting age, the first question in the survey was:

- Q.1 Do you approve or disapprove of the way Donald Trump is handling his job as President?

The possible answers were:

- 1 Approve
- 2 Disapprove
- 9 Don't know/Refused (VOL.)

Let's extract the data for this question.

In [13]:
```
import zipfile as zp
zf = zp.ZipFile('Feb17-public.zip')
missing_values = ["NaN", "nan", "Don't know/Refused (VOL.)"]
dfpewq1 = pd.read_csv(zf.open('Feb17public.csv'),
                      na_values=missing_values)['q1']
```

In [14]:
```
dfpewq1.shape
```

Out[14]:
```
(1503,)
```

In [15]:
```
dfpewq1.head()
```

Out[15]:
```
0    Disapprove
1    Disapprove
2    Disapprove
3       Approve
4    Disapprove
Name: q1, dtype: object
```

In [16]:
```
approve = sum(dfpewq1=='Approve')
disapprove = sum(dfpewq1=='Disapprove')
n = dfpewq1.shape[0]
q1sum = pd.Series(
    data= [approve, disapprove, n-approve-disapprove, n],
    index= ['Approve', 'Disapprove', 'Other', 'Total']
)
q1sum
```

Out[16]:
```
Approve        572
Disapprove     861
Other           70
Total         1503
dtype: int64
```

Now we can compute the sample proportion of approval.

In [17]:
```
prop = q1sum['Approve']/q1sum['Total']
round(prop, 4)
```

Out[17]:
```
0.3806
```

<u>Probability Theory</u>: To answer our research question, we first need to learn about a new type of random variable, and calculate it's mean and variance.

1. How can we define the **process of randomly selecting** a <u>single U.S. adult</u> from the <u>population adults that live in the U.S.</u> and **measuring** whether they approve of Trump or not?

2. What type of **random variable** is this?

3. How can we define and calculate what we would **expect** [the opinion of the single adult living in the U.S.] to be? *(Aka: if we sampled many, many adults from the population, what would the "average" opinion value be (where 1 = approve, 0 = don't approve?)*

4. How can we define and calculate what we would **expect the variance of** [the opinion of the single adult living in the U.S.] to be? *(Aka: if we sampled many, many adults from the population, what would the "variance" opinion value be (where 1 = approve, 0 = don't approve?)*

A **Bernoulli** random variable is one that takes only two possible values. We'll assume those values are 0 and 1. Typically the "1" corresponds to an outcome we're counting, e.g., "heads" on a coin" or "Plans to vote yes" on a state proposition.

Bernoulli random variables have extremely simple distributions. For any probability value p between 0 and 1, let $p = P(X = 1) = 1 - P(X = 0)$. We've just defined the entire probability mass function for a Bernoulli random variable!

$$p(x) = \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

For this simple distribution we have $E(X) = p$ and $Var(X) = p(1 - p)$ (Exercise).

5. How can we define the **process of randomly selecting** a <u>random sample of n=1503 U.S. adult</u> from the <u>population adults that live in the U.S.</u> and **measuring** the number that approve of Trump?

6. What type of **random variable** is this?

Next suppose we have a random sample of these Bernoulli random variables, i.e., n draws at random without replacement from a population of 0/1 values. Denote them by

$$X_1, X_2, \ldots, X_n$$

where each of these random variables independently equals 1 or 0 with probabilities p and $1-p$ respectively. The total count is the sum

$$Y = \sum_{i=1}^{n} X_i = \text{Number of 1's in the sample}$$

This combined random variable Y has the **Binomial distribution** with parameters n = sample size or number of draws, and p = probability of a 1 for each draw.

Associated with this we define the Binomial proportion, $\hat{p}$ given by

$$\hat{p} = \frac{Y}{n} = \frac{\text{Number of 1's in the sample}}{n}.$$

Binomial probability mass function

The pmf of the Binomial distribution can be derived using combinatorial counting methods. It has the form:

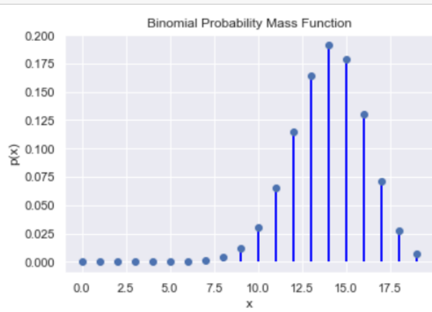
$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

where $\binom{n}{k} = k!(n-k)!/n!$ is the number of ways to choose k positions out of n for the 1's in a sequence of n 1's and 0's.

The Binomial distribution is included in SciPy as scipy.stat.binom with parameters n and p. The graph below shows the binomial pmf for n=20 and p=0.7.

In [18]:
```
from scipy.stats import binom
```

In [19]:
```
n, p = 20, 0.7
x = np.arange(0,20)
plt.plot(x, binom.pmf(x, n, p), 'bo')
plt.title("Binomial Probability Mass Function")
plt.vlines(x, 0, binom.pmf(x, n=n, p=p), lw=2, colors='blue')
plt.xlabel("x")
plt.ylabel("p(x)")
plt.show()
```



Ex: Suppose we collect a random sample of 5 adults living in the U.S. What is the probability that 2 approve of Donald Trump?

7. How can we define and calculate what we would **expect** [the *number* of adults in a random sample of size n that approve of Trump**]** to be? *(Aka: if we collected many, many random samples of size n from the population and found how many people in each sample approved of Trump, what would the average of this be?)*

8. How can we define and calculate what we would **expect the variance of** [the *number* of adults in a random sample of size n that approve of Trump**]** to be? *(Aka: if we collected many, many random samples of size n from the population and found how many people in each sample approved of Trump, what would the variance of this be?)*

Using our results so far, we can easily find the mean and standard deviation for Y and $\hat{p}$. Y is the sum of n independent Bernoulli random variables, each with mean $\mu = p$ and variance $\sigma^2 = p(1-p)$. Therefore the general results above give the following formulas for $Y \sim Binomial(n, p)$:

$$E(Y) = np, \quad Var(Y) = np(1-p), \quad SD(Y) = \sqrt{np(1-p)}$$

9. How can we define and calculate what we would **expect** [the *proportion* of adults in a random sample of size n that approve of Trump**]** to be? *(Aka: if we collected many, many random samples of size n from the population and found the proportion of adults in each sample that approved of Trump, what would the average of this be?)*

10. How can we define and calculate what we would **expect the variance of** [the *proportion* of adults in a random sample of size n that approve of Trump**]** to be? *(Aka: if we collected many, many random samples of size n from the population and found the proportion of adults in each sample that approved of Trump, what would the variance of this be?)*

Similarly, for the binomial proportion $\hat{p} = Y/n$:

$$E(\hat{p}) = p, \quad Var(\hat{p}) = \frac{p(1-p)}{n}, \quad SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}.$$

In [20]:

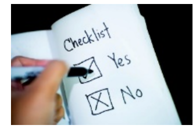
```
# Compare
n, p = 20, 0.7
print('Direct:', n*p, np.sqrt(n*p*(1-p)))
print('Via binom:', binom.mean(n, p), binom.std(n, p))
```

```
Direct: 14.0 2.04939015319192
Via binom: 14.0 2.04939015319192
```

Inference Building Blocks

Population of a Single Categorical Variable

Unknown	Known
Population Data	Random Sample of Size n
Population Proportion p	Sample Proportion $\hat{p}$



Ultimate Goal: Create a plausible range of values for *p*.

Sub-Goal:

*** *Estimate* the variability of the sample proportions of this size n (ie. standard error).**

*** *Estimate* the margin of error of our sample proportion.**

A special problem when calculating the standard error for a sample proportion!

In this survey the President's approval rate is 38.1% ($\hat{p} = 0.381$). What is the margin of error for this estimate?

First, we need a **standard error**, which is defined to be an estimate of the standard deviation of the estimate. If we knew the true proportion we could compute the standard deviation, but p is what we are trying to estimate! What are the alternatives?

Standard Error $SD(\hat{p}) = \sqrt{\dfrac{p(1-p)}{n}}$.

Margin of Error: $Z_{\alpha/2} \cdot SD(\hat{p})$

Method 1 (Upper bound method). The SD function for $\hat{p}$ is bounded:

$$SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} \le \sqrt{\frac{0.25}{n}} = \frac{0.5}{\sqrt{n}}$$

So we could report a **standard error** of *at most* this computed value:

In [21]:
```
se_bound = 0.5/np.sqrt(q1sum['Total'])
round(se_bound, 4)
```

Out[21]: 0.0129

This is likely to be reasonable for values of p in the range of 0.4 to 0.6 and too conservative for more extreme values of p. A more commonly used method is to insert $\hat{p}$ into the standard deviation formula.

Method 2 (Plug-in method). Inserting the estimate itself into the standrd deviation formula gives:

$$SE(\hat{p}) = \hat{SD}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

For the approval question in the Pew survey we obtain:

In [22]:
```
se = np.sqrt(prop*(1-prop)/n)
round(se, 4)
```

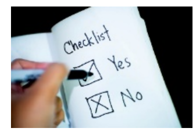
Out[22]: 0.1086

We see that the plug in method gives a smaller standard error. Is it accurate? The upper bound method already shows that the expected deviation of at most $\hat{p}$ from p is on the order of 1.3%, so plugging it into the SE formula is likely to be quite accurate for this large of a sample.

Inference Building Blocks

Population of a Single Categorical Variable

Unknown	Known
Population Data	Random Sample of Size n
Population Proportion p	Sample Proportion $\hat{p}$



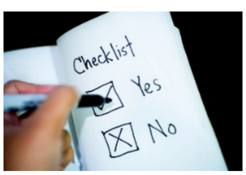
Ultimate Goal: Create a plausible range of values for *p*.

Sub-Goal:

**** Determine when we can use the confidence interval formula:***

$$\left(estimate - z_{1-\frac{\alpha}{2}}SE, estimate + z_{1-\frac{\alpha}{2}}SE\right)$$

**** Aka: Determine when $\hat{p}$ is a normal random variable.***



Probability Theory: To answer our research question, we first need to learn about a new type of random variable, and calculate it's mean and variance.

Normal Approximation for the Binomial via Central Limit Theorem

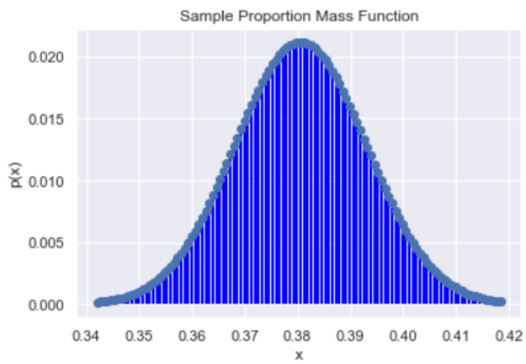
The sample proportion $\hat{p} = Y/n$ from a Binomial sample is just a rescaling of the the total count Y for the outcome of interest. We now know that Y follows a Binomial distribution with n draws and "succes" probability p for each draw. The probabilty mass function for $\hat{p}$ follows from that of Y as follows:

$$p_{\hat{p}}(k/n) = P\left(\hat{p} = \frac{k}{n}\right) = P(Y = k) = \binom{n}{k} p^k (1-p)^{n-p}, \quad k = 0, 1, 2, \ldots, n$$

Using the sample estimates in the Pew example, the estimated pmf looks like the figure below.

In [23]:

```
n, p = 1503, prop
x = np.arange(binom.ppf(0.001, n, p), binom.ppf(0.999, n, p))
plt.plot(x/n, binom.pmf(x, n, p), 'bo')
plt.title("Sample Proportion Mass Function")
plt.vlines(x/n, 0, binom.pmf(x, n, p), lw=2, colors='blue')
plt.xlabel("x")
plt.ylabel("p(x)")
plt.show()
```



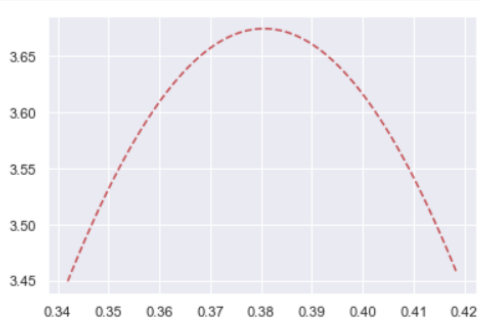
Compare the shape of the normal pdf with the same mean and standard deviation as the scaled binomial pmf.

In [24]:

```
from scipy.stats import norm
```

In [25]:

```
plt.plot(x/n, norm.pdf(x/n, loc=prop, scale=se), 'r--')
plt.show()
```



Normal Approximation for Binomial Sum

Both the binomial distribution for Y and the equivalent rescaled binomial distribution for $\hat{p}$ can be approximated by normal distributions with the same means and standard deviations if the sample size is large enough. This is a consequence of the central limit theorem for sample averages.

As an example, let's compare $P(X \le x)$ for Binomial(n=100, p=0.2) and Normal(loc=20, scale=4) for several values of x. For binomial counts it is more accurate to add or substract 0.5 to the interger count value depending on whether the probability includes or excludes that value.

In [26]:

```
n, p = 100, 0.2
loc, scale = n*p, np.sqrt(n*p*(1-p))
# values at which to compare
z = [15.5, 20.5, 25.5, 30.5]
# binomial cdf values
bprob = binom.cdf(z, n, p)
nprob = norm.cdf(z, loc, scale)
pd.DataFrame({
    'z': z,
    'binom.cdf(z, 100, 0.2)': bprob,
    'norm.cdf(z, 20, 4)': nprob
})
```

Out[26]:

	z	binom.cdf(z, 100, 0.2)	norm.cdf(z, 20, 4)
0	15.5	0.128506	0.130295
1	20.5	0.559462	0.549738
2	25.5	0.912525	0.915434
3	30.5	0.993941	0.995668

<u>Ex</u>: Create a 95% confidence interval for the *p* the proportion of adults living in the U.S. that approve of Donald Trump.

Margin of error for Binomial proportion

In the Pew survey example, let's compute the margin of error for a confidence level of 95%.

In [27]:

```
# Estimate
prop = q1sum['Approve']/q1sum['Total']
# compute the multiplier from N(0,1)
zq = norm.ppf(0.975)
# compute the standard error
SE = np.sqrt(prop*(1-prop)/q1sum['Total'])
# compute and display margin of error
print(' Estimate=', round(prop, 4))
print(' Standard Error=', round(SE, 4))
print(' SE multiplier=', round(zq, 4))
print(' Margin of error:', round(zq*SE, 4))
```

```
Estimate= 0.3806
Standard Error= 0.0125
SE multiplier= 1.96
Margin of error: 0.0245
```

Conclusion: We are 95% confident that the sample estimate of 38.1% within $\pm$2.45% of the population percentage for approval.

Equivantly, we compute a 95% confidence interval:

In [28]:

```
np.round([prop-zq*SE, prop+zq*SE], 3)
```

Out[28]:

```
array([0.356, 0.405])
```

Probability Theory: Comparing the way to calculate:

- Population Standard Deviation
- Biased Sample Standard Deviation
- Unbiased Sample Standard Deviation

Why are there two ways to calculate sample standard deviation?

Advanced topic: Why do we divide by n-1 in the sample standard deviation?

The short answer is that this makes S^2 **unbiased** as an estimator of the population variance σ^2. But this is worth delving into in a bit more detail, because it relates to a general idea that we will see repeatedly, that using the sample **training data** for estimating accuracy often makes results biased and we need to adjust for this in one way or another.

Framework: Consider a random sample $X_1, X_2, \ldots, X_n$ from a population or distribution with mean μ and standard deviation σ. Our standard estimators for (μ, σ^2) are $(\bar{X}, S^2)$ where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

What gives with the $n-1$??

Let's work out the expected value using the facts we studied earlier in this section. Recall that the expectation of a sum of random variables is the sum of the expectations. Using this fact we already figured out that

$$E(\bar{X}) = \mu \quad \text{and} \quad E((\bar{X}-\mu)^2) = \frac{\sigma^2}{n}.$$

Next, consider the sum-of-squares of deviations of X_i about the true, unknown μ:

$$SS(\mu) = \sum_{i=1}^{n}(X_i - \mu)^2.$$

Using our facts about expectations we have:

$$E(SS(\mu)) = \sum_{i=1}^{n} E((X_i - \mu)^2) = \sum_{i=1}^{n} \sigma^2 = n\sigma^2.$$

Therefore, if we knew μ, then we could use $SS(\mu)/n$ as an unbiased estimtator because

$$E\left(\frac{1}{n}SS(\mu)\right) = \sigma^2.$$

But we don't know μ. What is the impact of plugging in the estimate $\bar{X}$ into the sum-of-squares?

$$\begin{aligned}
SS(\bar{X}) &= \sum_{i=1}^{n}(X_i - \bar{X})^2 \\
&= \sum_{i=1}^{n}(X_i - \mu)^2 + \sum_{i=1}^{n}(\mu - \bar{X})^2 + 2\sum_{i=1}^{n}(X_i - \mu)(\mu - \bar{X}) \\
&= SS(\mu) + n(\bar{X} - \mu)^2 - 2n(\bar{X} - \mu)^2 \\
&= SS(\mu) - n(\bar{X} - \mu)^2.
\end{aligned}$$

$$\begin{aligned}SS(\bar{X}) &= \sum_{i=1}^{n}(X_i - \bar{X})^2 \\ &= \sum_{i=1}^{n}(X_i - \mu)^2 + \sum_{i=1}^{n}(\mu - \bar{X})^2 + 2\sum_{i=1}^{n}(X_i - \mu)(\mu - \bar{X}) \\ &= SS(\mu) + n(\bar{X} - \mu)^2 - 2n(\bar{X} - \mu)^2 \\ &= SS(\mu) - n(\bar{X} - \mu)^2.\end{aligned}$$

Taking expectations on both sides of the equation gives

$$E(SS(\bar{X}) = E(SS(\mu)) - n * Var(\bar{X}) = n\sigma^2 - \sigma^2 = (n-1)\sigma^2.$$

Therefore, we get an unbiased, sample-based estimator of σ^2 if we use S^2, because

$$E(S^2) = E\left(\frac{SS(\bar{X})}{n-1}\right) = \frac{(n-1)\sigma^2}{n-1} = \sigma^2.$$

Conclusion: plugging the sample estimate $\bar{X}$ into the expression for the sample variance (and standard deviation) underestimates the spread in the population distribution. We adjust for that by dividing by $n-1$ instead of n. This inflates the standard deviation estimate by the factor

$$\sqrt{\frac{n}{n-1}}.$$

In very large samples this makes very little difference, but it has a noticable impact in smaller samples.