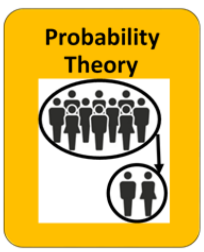
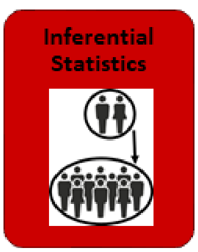
<u>Unit 5</u>: For Loops, Functions, and Monte Carlo Simulation

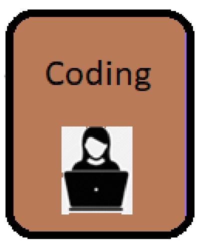
<u>Case Study</u>: UIUC Couse Data and Coin Flip Data

We will learn:

- Building blocks for conducting **<u>frequentist inference</u>**
- From a population of **<u>numerical data</u>** we can collect many random samples (with replacement) and calculate the **<u>mean</u> of each sample.** How will this **distribution of sample means behave?**
- From a population of **<u>categorical data</u>** we can collect many random samples (with replacement) and calculate the **<u>proportion</u> of each sample.** How will this **distribution of sample proportions behave?**







<u>Summary of Concepts:</u>

<u>ESTIMATING Population Means and Proportions:</u>

- From a **population** of **<u>numerical data</u>** we can collect a ***single*** **random sample**.
    - Will our ***sample*** **mean** be more likely to be close to our ***population*** **mean** if the sample was small or large?
- From a **population** of **<u>categorical data</u>** we can collect a ***single*** **random sample**.
    - Will our ***sample*** **proportion** be more likely to be close to our ***population*** **proportion** if the sample was small or large?

<u>Determining the NATURE of Sample Means and Sample Proportions</u>

- From a population of **<u>numerical data</u>** we can collect many random samples (with replacement) and calculate the **<u>mean</u> of each sample.** How will this **distribution of sample means behave?**
    - Mean
    - Standard Deviation
    - Shape of Distribution
- From a population of **<u>categorical data</u>** we can collect many random samples (with replacement) and calculate the **<u>proportion</u> of each sample.** How will this **distribution of sample proportions behave?**
    - Mean
    - Standard Deviation
    - Shape of Distribution

Populations vs. Samples

Populations are usually too large to collect completely, so the data and summary statistics we can collect about the populations is usually unknown. We can have different types of populations.

Population 1: Comprised of <u>Numerical</u> Data

	course	section	enrolled
0	adv307	A	37
1	badm210	A	215
2	badm210	B	178
3	badm210	C	197
4	cs105	A	345
5	cs105	B	201
6	stat107	A	197
7	stat207	A	53

What kind of **summary statistics (population parameters)** can we use to summarize this **numerical population** data?

If we take a **random sample of size *n*** what kind of summary statistic can we use to summarize this **numerical sample data?**

Population 2: Comprised of <u>Categorical</u> Data

	toss	value
0	heads	1
1	tails	0

What kind of **summary statistics (population parameters)** can we use to summarize this **categorical population** data?

If we take a **random sample of size *n*** what kind of summary statistic can we use to summarize this **categorical sample data?**

QUESTION OF THE LECTURE: How does the **size *n* of the random sample** effect the likelihood that the **sample statistic** will be *close* to the **population parameter?**

Sampling Distributions

If we collect many, many **random samples** from a **population of data** (each drawn

____________________________), where each sample is of the

________________________, then the _________________________ is the

distribution of ____________________________.

Ex: The ____________________________________ is a numerical distribution of

___________________________ of random samples drawn from a population of

numerical data with replacement.

Ex: The ____________________________________ is a numerical distribution of

___________________________ of random samples drawn from a population of

numerical data with replacement.

When we make an **inference** about an unknown **population parameter,** there are usually three things we are interested in knowing about the corresponding sampling distributions:

1. ______________________________________
2. ______________________________________
3. ______________________________________

Sampling Distributions of the Mean

How to generate a sampling distribution of the mean.

Population of Numerical Data

	course	section	enrolled
0	adv307	A	37
1	badm210	A	215
2	badm210	B	178
3	badm210	C	197
4	cs105	A	345
5	cs105	B	201
6	stat107	A	197
7	stat207	A	53

Collect Many Random Samples (all of size n=10) drawn with replacement.

Random Sample of n=10 Course Enrollments *(drawn with replacement from population)*	**Random Sample of n=10 Course Enrollments** *(drawn with replacement from population)*	**Random Sample of n=10 Course Enrollments** *(drawn with replacement from population)*	...	**Random Sample of n=10 Course Enrollments** *(drawn with replacement from population)*
197	215	53	...	215
37	215	53	...	197
345	53	197	...	37
201	201	53	...	215
178	53	197	...	197
37	345	197	...	197
53	197	178	...	345
201	37	215	...	345
201	201	197	...	197
197	37	197	...	201

Sampling Distribution

What's the:

- Mean,
- Standard deviation,
- Shape?

Sample Means
164.7
155.4
153.7
...
214.6

Sampling Distribution

How do these things change when *n* changes?

- Mean,
- Standard deviation,
- Shape?

For Loops, Functions and Monte Carlo Simulation

Flow control methods such as for loops allow us to automate repetitive operations. When running simulations this allows us to repeatedly sample from data either to test out a sampling model or to construct resampling based inferences such as the bootstrap, which will be discussed in later chapters.

We use Python functions constantly, and a valuable feature of Python and many programming languages is the ability to build our own functions to perform frequent specialized tasks.

In this section we first introduce how for loops work, then develop a small function for Monte Carlo sampling from a data frame. Then we use this to investigate the sample distributions of sample means and proportions for random samples of varying sample sizes, when sampling with replacement.

Using Monte Carlo simulation we demonstrate the square root rule for the standard deviation of the mean and the approximate normal distribution of the sample mean of a large sample. This latter approximation is consistent with the Central Limit Theorem from probability theory.

Flow control: for loop

In order to do simulations we use Python's flow control to allow us to repeatedly draw samples. The **for** loop is fundamental in many programming languages. Here's a simple version. Notice that for Python the **colon (:)** and **indentation** are important. The indentation needs to be 4 characters wide! The notebook formats this automatically.

In [1]:

```
for x in ["Fido", "Rex", "Mitzi", "Fluffy", "Mr. Lizard"]:
    print("Here ", x, "!", sep="")
```

```
Here Fido!
Here Rex!
Here Mitzi!
Here Fluffy!
Here Mr. Lizard!
```

Here's another example, with the same general principle that the for loop passes through all the values in the "in" list.

In [2]:

```
for i in range(5):
    print("Hello Fidotron-R", i, ", welcome!", sep="")
```

```
Hello Fidotron-R0, welcome!
Hello Fidotron-R1, welcome!
Hello Fidotron-R2, welcome!
Hello Fidotron-R3, welcome!
Hello Fidotron-R4, welcome!
```

The for loop allows us to do an operation repeatedly by stepping through a finite list. This is extremely useful for performing computer simulations in which we repeatedly draw samples and study the the effects of random variation on the statistics.

Sampling distribution of the mean

The sample mean is common statistic used to summarize the central tendancies of particular variables in the data. When the data are drawn from a larger population at random, the sample mean provides an estimate of the mean for the whole population. In a sample survey the sample that we get from the population is random, and woul be different if we were to repeat the sampling process. However, if the *sample* is large enough then sample statistics will tend to be close to the correpsonding population parameters, and the variation in the sample statistics is predictable.

Example: the proportion of voting age citizens who support a given policy in a population can be thought of as the mean of all the 0/1 indicators for whether each citizen supports the policy. 1 means they support it; 0 means they don't. The average of all these 0s and 1s is the proportion supporting the policy. If we draw a random sample from the population, the sample proportion is the sample mean of all the 0's and 1's selected for the sample. How much variation is there in this sample proportion due to the random sampling? We investigate questions like these through simulation. Later we will see that theory provides precise information about the variation in the sample proportion.

Before specifically addressing the special case of the sample proportion, let's consider the case of sample means for variables in data frames in general. We set up a Monte Carlo simulation scheme in which we repeatedly draw random samples and see how the resulting sample means vary.

In [3]:

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns; sns.set()
```

```
Bad key "text.kerning_factor" on line 4 in
C:\Users\vme3\AppData\Local\Continuum\anaconda3\lib\site-packages\matplotli
b\mpl-data\stylelib\_classic_test_patch.mplstyle.
You probably need to get an updated matplotlibrc file from
https://github.com/matplotlib/matplotlib/blob/v3.1.3/matplotlibrc.template
 (https://github.com/matplotlib/matplotlib/blob/v3.1.3/matplotlibrc.templat
e)
or from the matplotlib source distribution
```

In [4]:

```python
# create a data frame for illustration and testing
courses = ['adv307', 'badm210', 'badm210', 'badm210',
           'cs105', 'cs105', 'stat107', 'stat207']
sections = ['A', 'A', 'B', 'C', 'A', 'B', 'A', 'A']
enrollments = [37, 215, 178, 197, 345, 201, 197, 53]
sectdf = pd.DataFrame({'course': courses,
                       'section': sections,
                       'enrolled': enrollments})
sectdf
```

Out[4]:

	course	section	enrolled
0	adv307	A	37
1	badm210	A	215
2	badm210	B	178
3	badm210	C	197
4	cs105	A	345
5	cs105	B	201
6	stat107	A	197
7	stat207	A	53

In [5]:

```python
sectdf['enrolled'].mean()
```

Out[5]:

```
177.875
```

Let's generate 5 random samples form the data frame and calculate the corresponding sample means. This is a very simple example of a Monte Carlo simulation.

We use a for loop where we initialize an empty array, SampleMeans, and then iterate a sepcified number of times. To understand how each step works you might find it helpful to break out the individual steps and run them with different values of the iteration variable i.

In [6]:

```python
#What will sampling from this look like?
test_sample=sectdf['enrolled'].sample(10, replace=True)
print('Test Sample of Size n=10 (drawn with replacement):')
print(test_sample)
print('')
print('Mean of Sample', test_sample.mean())
```

```
Test Sample of Size n=10 (drawn with replacement):
0     37
1    215
4    345
1    215
2    178
2    178
4    345
6    197
1    215
3    197
Name: enrolled, dtype: int64

Mean of Sample 212.2
```

In [7]:

```python
# iterate 5 times
x = 'enrolled'
SampleMeans = []
for i in range(5):
    SampleMeans.append(sectdf[x].sample(10, replace=True).mean())
    print(SampleMeans)
print(pd.DataFrame({x: SampleMeans}))
```

```
[164.9]
[164.9, 178.3]
[164.9, 178.3, 165.0]
[164.9, 178.3, 165.0, 199.5]
[164.9, 178.3, 165.0, 199.5, 245.3]
   enrolled
0     164.9
1     178.3
2     165.0
3     199.5
4     245.3
```

In [8]:

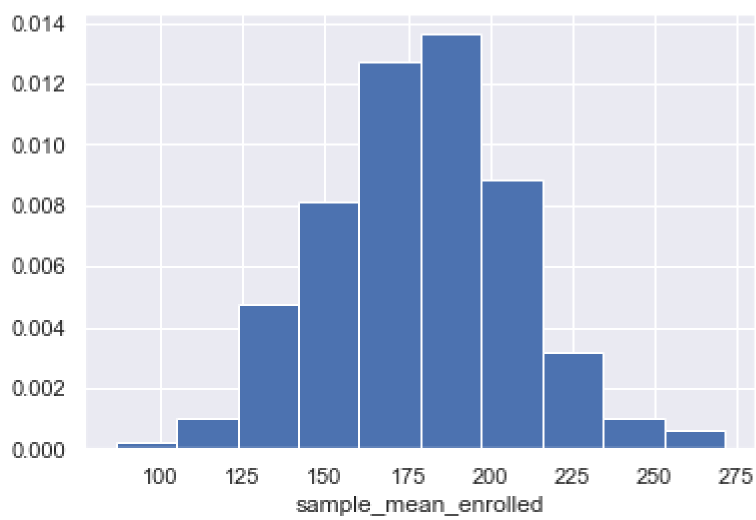
```python
# iterate 1000 times
# parametrize the sample size. number of random samples,
# and the variable name
x = 'enrolled'
n=10
M=1000
SampleMeans = []
for i in range(M):
    SampleMeans.append(sectdf[x].sample(n, replace=True).mean())
MonteCarlo = pd.DataFrame({'sample_mean_enrolled': SampleMeans})
```

In [9]:
```
MonteCarlo.shape
```
Out[9]:
```
(1000, 1)
```

In [10]:
```
MonteCarlo.head(10)
```
Out[10]:

	sample_mean_enrolled
0	183.2
1	197.0
2	151.5
3	194.6
4	165.5
5	245.7
6	202.2
7	184.2
8	134.3
9	166.5

In [11]:
```
MonteCarlo['sample_mean_enrolled'].hist(density=True)
plt.xlabel('sample_mean_enrolled')
plt.show()
```



Making our own functions

If we want to try lots of different values for n and M it gets tedious to keep copying the code and changing the parameters in multiple locations. Instead, we can make our own function to do this kind of task with varying inputs. This saves a lot of redundant effort. It also makes it easier to understand and debug the code.

Here's a function to do the Monte carlo simulation of the sample mean for different sample sizes and numbers of Monte Carlo samples. Notice that we input the data frame (or data series), variable name x as a text string, sample size n, and number of Monte Carlo samples M. Here again, the **colon (:)** and **indentation (4 characters)** are important to indicate that the ensuing lines of code are included in the function.

In [12]:
```
def MCmeans(df, x='', replace=True, n=1, M=1):
    # df is a data frame
    # x is a text-valued name for a variable in the data frame
    # replace = True or False depending on whether
    #    draws are with or without replacement
    # n = number of draws per sample
    # M = number of samples to draw
    MCstats = []
    for i in range(M):
        MCstats.append(df[x].sample(n, replace=replace).mean())
    return pd.DataFrame({'sample_mean_'+x: MCstats})
```

In [13]:
```
MCmeans(df=sectdf, x='enrolled')
```

Out[13]:

	sample_mean_enrolled
0	53.0

In [14]:
```
MCmeans(df=sectdf, x="enrolled", n=1, M=10)
```

Out[14]:

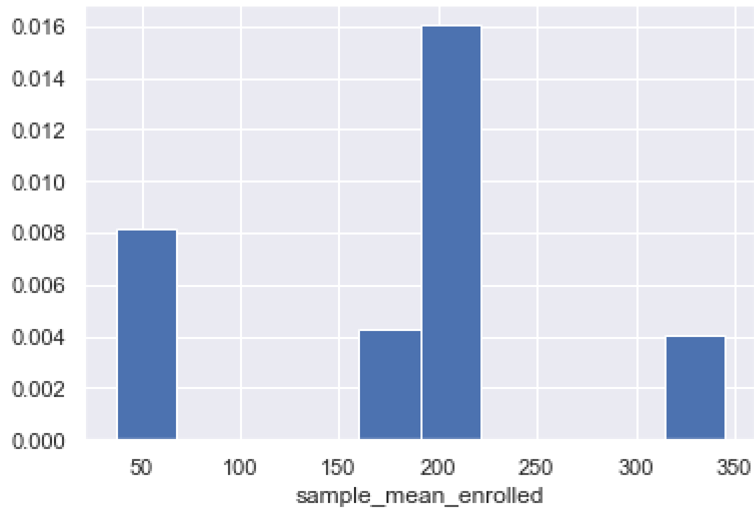
	sample_mean_enrolled
0	178.0
1	53.0
2	201.0
3	201.0
4	215.0
5	197.0
6	53.0
7	37.0
8	197.0
9	345.0

In [15]:
```
MonteCarlo = MCmeans(df=sectdf, x="enrolled", n=1, M=10000)
MonteCarlo.shape
```

Out[15]:
```
(10000, 1)
```

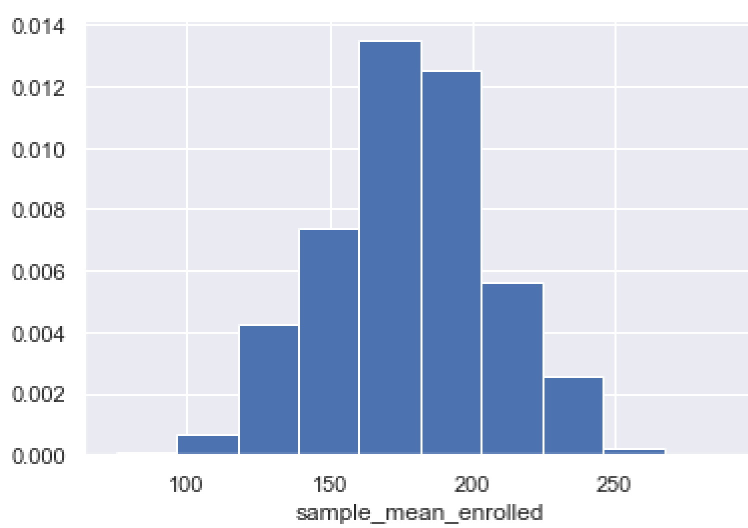
In [16]:

```
MonteCarlo['sample_mean_enrolled'].hist(density=True)
plt.xlabel('sample_mean_enrolled')
plt.show()
```



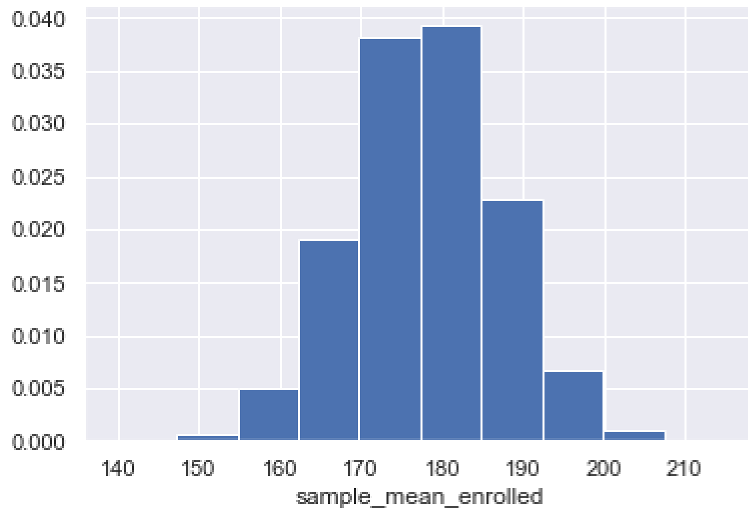
In [17]:

```
MonteCarlo = MCmeans(df=sectdf, x="enrolled", n=10, M=10000)
MonteCarlo['sample_mean_enrolled'].hist(density=True)
plt.xlabel('sample_mean_enrolled')
plt.show()
```



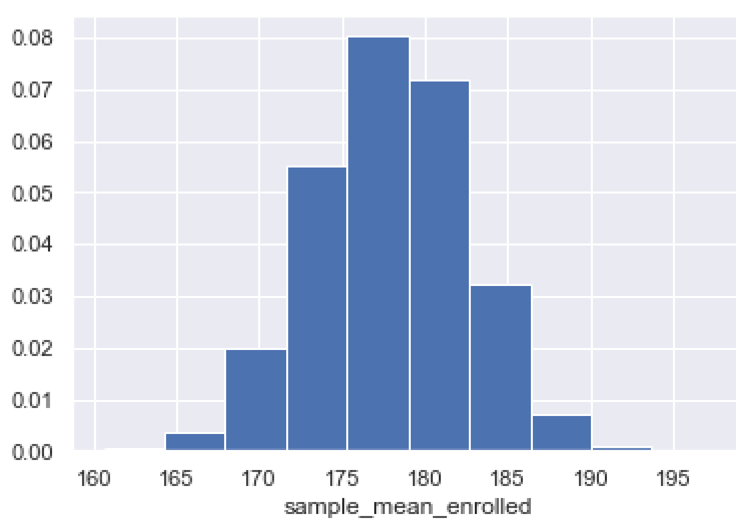
In [18]:

```
MonteCarlo = MCmeans(df=sectdf, x="enrolled", n=100, M=10000)
MonteCarlo['sample_mean_enrolled'].hist(density=True)
plt.xlabel('sample_mean_enrolled')
plt.show()
```



In [19]:

```
MonteCarlo = MCmeans(df=sectdf, x="enrolled", n=400, M=10000)
MonteCarlo['sample_mean_enrolled'].hist(density=True)
plt.xlabel('sample_mean_enrolled')
plt.show()
```



Sampling distribution of the proportion of "heads" in coin tossing

What did we <u>observe</u> about the sampling distribution of the mean as our sample size n changed?

In an upcoming unit, we will learn about the Central Limit Theorem, which will <u>prove</u> these observations.

Sampling Distributions of the Proportion

How to generate a sampling distribution of the proportion.

Population of Categorical Data

	toss	value
0	heads	1
1	tails	0

Collect Many Random Samples (all of size n=10) drawn with replacement.

Random Sample of n=10 Tosses *(drawn with replacement from population)*	Random Sample of n=10 Tosses *(drawn with replacement from population)*	Random Sample of n=10 Tosses *(drawn with replacement from population)*	...	Random Sample of n=10 Tosses *(drawn with replacement from population)*
1	1	1	...	0
0	1	0	...	0
1	1	1	...	0
0	1	0	...	0
1	1	0	...	1
0	1	0	...	0
0	1	1	...	0
1	1	1	...	0
0	1	0	...	1
1	0	0	...	1

Sampling Distribution

What's the:

- Mean,
- Standard deviation,
- Shape?

Sample Proportions
0.5
0.9
0.4
...
0.3

Sampling Distribution

How do these things change when *n* changes?

- Mean,
- Standard deviation,
- Shape?

With our function available we can simulate all kinds of things. Here is a "data frame" of the possible outcomes when flipping a coin.

In [20]:

```
df = pd.DataFrame({'toss':['heads','tails'], 'value': [1, 0]})
df
```

Out[20]:

	toss	value
0	heads	1
1	tails	0

If we make one draw, i.e., toss the coin once, then the uniform probability principle tells us the probability of a 1 is p = 1/2. What if we draw (flip) 10 times randomly and without replacement? What proportion $\hat{p}$ of "heads" do we expect? How much is it likely to vary from this expectation? What if we toss 100 times, or 400?

Let's consider a series of sample sizes and study how the sampling distribution is affected. We'll compute mean and standard deviation of the Monte Carlo values, and look at the histrogram as well to get a picture fo the sample distributions for different sample sizes.

In [21]:

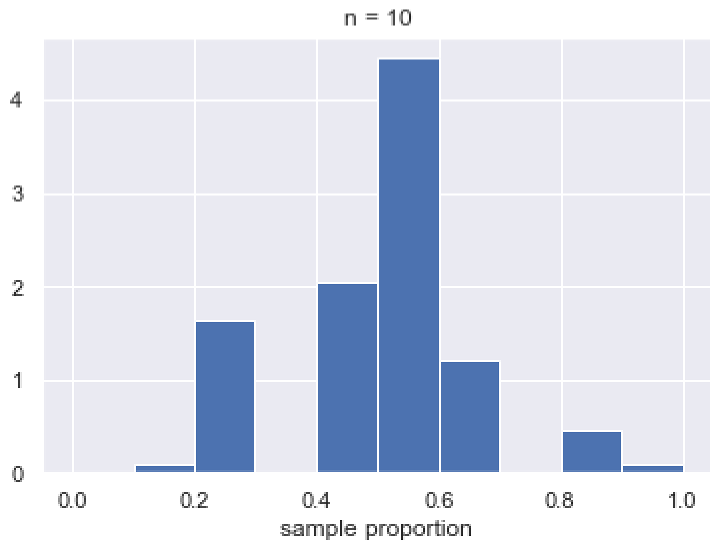
```
MonteCarlo = MCmeans(df=df, x="value", n=10, M=10000)
print('Sample size = ', 10)
print('Mean value for sample proportion =',
      np.round(MonteCarlo.mean().sample_mean_value, 5))
print('Standard Deviation for sample proportion =',
      np.round(MonteCarlo.std().sample_mean_value, 5))
```

```
Sample size =  10
Mean value for sample proportion = 0.49988
Standard Deviation for sample proportion = 0.1588
```

What does the sample distribution look like?

In [22]:

```
MonteCarlo['sample_mean_value'].hist(density=True)
plt.title('n = 10')
plt.xlabel('sample proportion')
plt.show()
```



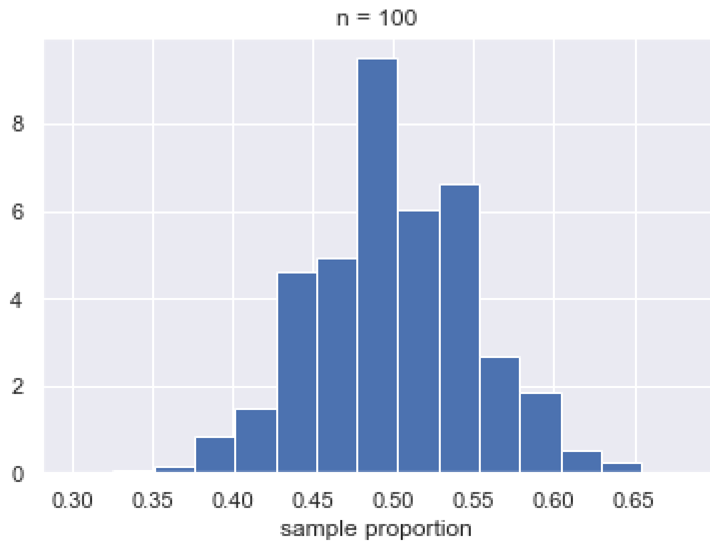
Let's compare the sample distribution of $\hat{p}$ for 100, 400 and 1600 tosses.

In [23]:

```
MonteCarlo = MCmeans(df=df, x="value", n=100, M=10000)
print('Sample size = ', 100)
print('Mean value for sample proportion =',
      np.round(MonteCarlo.mean().sample_mean_value, 5))
print('Standard Deviation for sample proportion =',
      np.round(MonteCarlo.std().sample_mean_value, 5))
```

```
Sample size =  100
Mean value for sample proportion = 0.49965
Standard Deviation for sample proportion = 0.04976
```

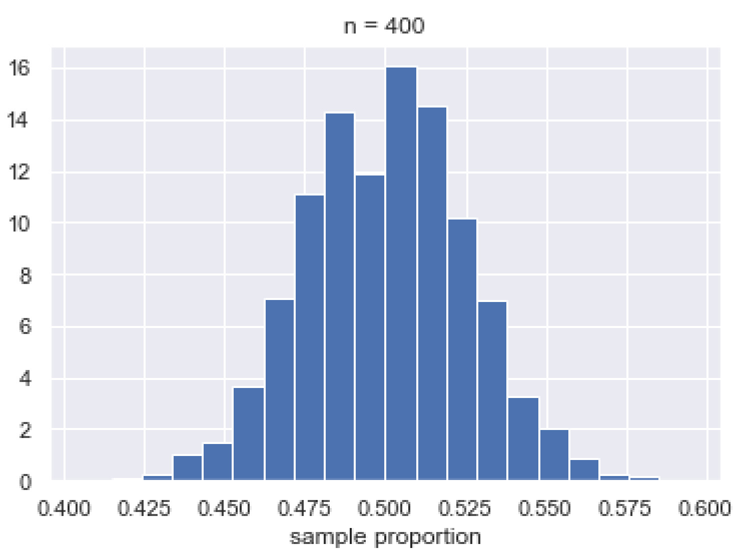
In [24]:
```
MonteCarlo['sample_mean_value'].hist(density=True, bins=15)
plt.title('n = 100')
plt.xlabel('sample proportion')
plt.show()
```



In [25]:
```
MonteCarlo = MCmeans(df=df, x="value", n=400, M=10000)
print('Sample size = ', 400)
print('Mean value for sample proportion =',
      np.round(MonteCarlo.mean().sample_mean_value, 5))
print('Standard Deviation for sample proportion =',
      np.round(MonteCarlo.std().sample_mean_value, 5))
```

```
Sample size =  400
Mean value for sample proportion = 0.49986
Standard Deviation for sample proportion = 0.0253
```

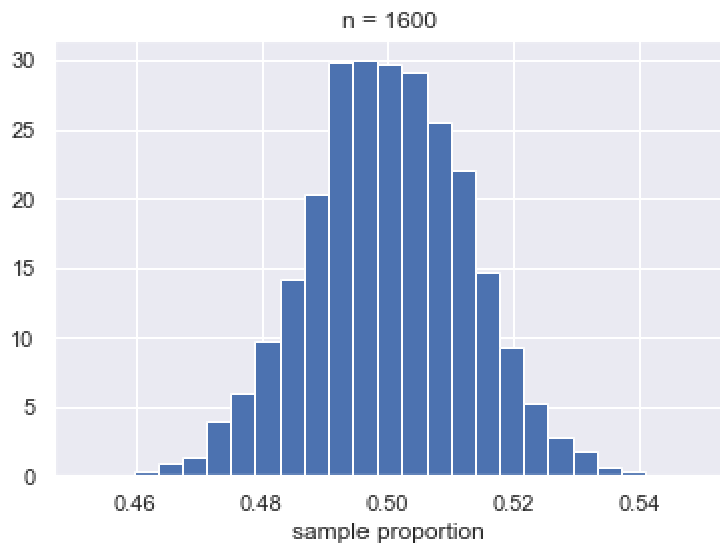
In [26]:
```
MonteCarlo['sample_mean_value'].hist(density=True, bins=20)
plt.title('n = 400')
plt.xlabel('sample proportion')
plt.show()
```



In [27]:
```
MonteCarlo = MCmeans(df=df, x="value", n=1600, M=10000)
print('Sample size = ', 1600)
print('Mean value for sample proportion =',
      np.round(MonteCarlo.mean().sample_mean_value, 5))
print('Standard Deviation for sample proportion =',
      np.round(MonteCarlo.std().sample_mean_value, 5))
```

```
Sample size =  1600
Mean value for sample proportion = 0.50001
Standard Deviation for sample proportion = 0.01261
```

In [28]:
```
MonteCarlo['sample_mean_value'].hist(density=True, bins=25)
plt.title('n = 1600')
plt.xlabel('sample proportion')
plt.show()
```



Effect of sample size on sampling distribution of $\hat{p}$

Comparing the histograms of the sample proportion for n=10, 100, 400 and 1600 above, we see that:

1. the sampling distribution is centered around the "population" proportion 0.50.
2. the sample distribution becomes more and more concentrated near 0.50 as we increase n, both in terms of the concentration of the histogram and the decrease in the standard deviation.
3. the sampling distribution becomes more like a "bell curve" as the sample size increases.

The **standard deviation** is a measure of the **spread** of the distribution. To a close approximation it is the square-root of mean-square deviation of a set of numbers from their mean.

Our simulations are suggestive concerning the behavior sample means and sample proportions. The following general result shows why these statistics become more and more precise as estimates, as the sample size is increased.

Mean and Standard Deviation of sample means and proportions: Square root rule

When a random sample is drawn from a population the sample mean and sample proportion have expected values equal to the population mean and proportion, respectively.

More is known: the standard deviation of a sample mean ($\bar{X}$) or sample proportion ($\hat{p}$) from a random sample decreases in proportion to the square root of the sample size n:

$$SD(\bar{X}) \propto \frac{1}{\sqrt{n}}$$

and

$$SD(\hat{p}) \propto \frac{1}{\sqrt{n}}$$

as the sample size n drawn from a given population is increased.

Monte Carlo investigation of the square root rule

Let's set up another simulation to test the square root rule. We'll use an asymmetrical distribution of 0s and 1s.

In [29]:
```
np.repeat(1,9)
```

Out[29]:
```
array([1, 1, 1, 1, 1, 1, 1, 1, 1])
```

In [30]:
```
df49 = pd.DataFrame({'binary': np.concatenate((np.repeat(1, 4),
                                               np.repeat(0, 9)))})
df49
```

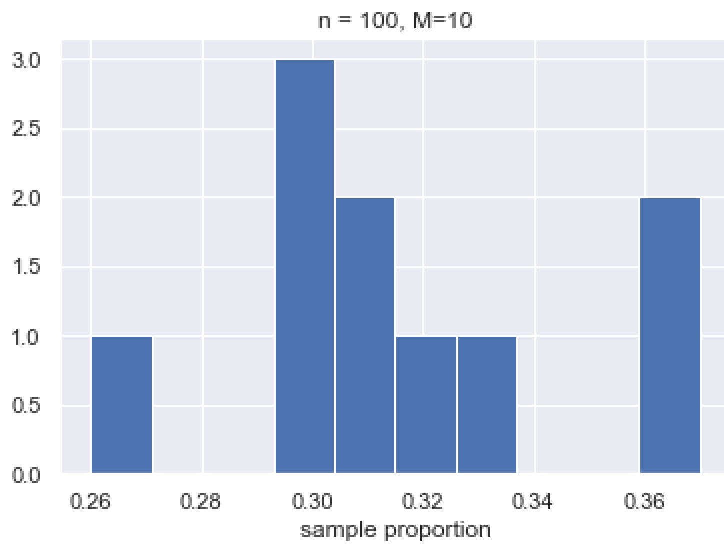
Out[30]:

	binary
0	1
1	1
2	1
3	1
4	0
5	0
6	0
7	0
8	0
9	0
10	0
11	0
12	0

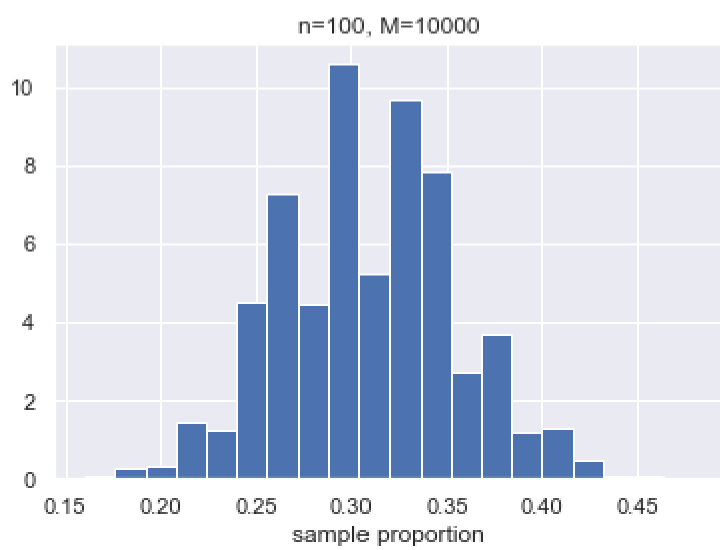
In [31]:
```
print('population proportion = ', df49['binary'].sum()/df49.shape[0])
```

```
population proportion =  0.3076923076923077
```

In [32]:
```
MonteCarlo = MCmeans(df=df49, x='binary', n=100, M=10)
MonteCarlo['sample_mean_binary'].hist()
plt.title("n = 100, M=10")
plt.xlabel('sample proportion')
plt.show()
```



In [33]:
```
MonteCarlo = MCmeans(df=df49, x='binary', n=100, M=10000)
MonteCarlo['sample_mean_binary'].hist(density=True, bins=20)
plt.title("n=100, M=10000")
plt.xlabel('sample proportion')
plt.show()
```



In [34]:

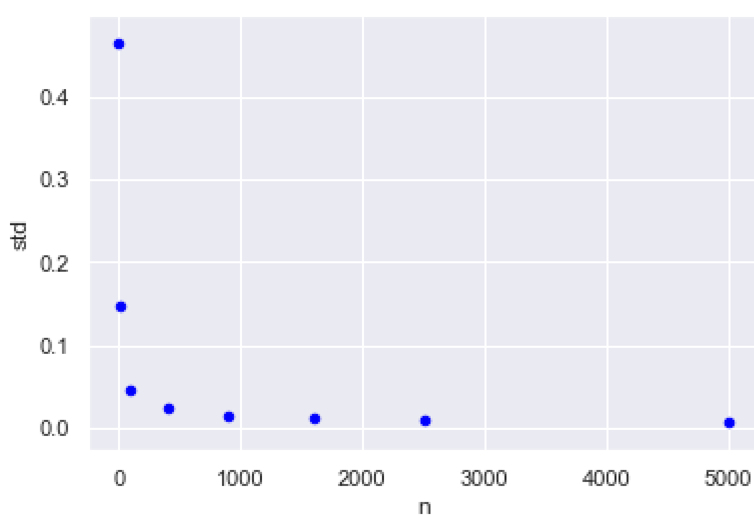
```python
# set sample sizes and initialize arrays of means and standard deviations
meanset = []
sdset = []
nset = [1,10,100,400,900,1600,2500,5000]
#nset = [1,10]    # test values while debugging
M=10000
df=df49
x='binary'
for n in nset:
    MC = MCmeans(df=df, x=x, n=n, M=M)
    # print(MC[x].mean())    # uncomment to see values as they're computed
    # print(MC[x].std())     # ditto
    meanset.append(MC['sample_mean_'+x].mean())
    sdset.append(MC['sample_mean_'+x].std())
summary = pd.DataFrame({'n': nset, 'mean': meanset, 'std': sdset })
summary
```

Out[34]:

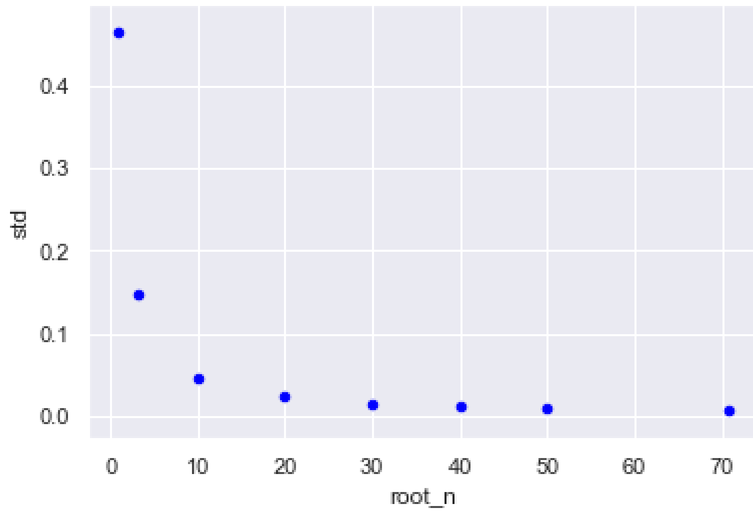
	n	mean	std
0	1	0.315500	0.464738
1	10	0.306740	0.147251
2	100	0.306676	0.046406
3	400	0.308058	0.023186
4	900	0.307742	0.015398
5	1600	0.307679	0.011519
6	2500	0.307673	0.009177
7	5000	0.307590	0.006572

In [35]:

```python
summary.plot.scatter(x='n', y='std', c='blue')
plt.xlabel('n')
plt.ylabel('std')
plt.show()
```

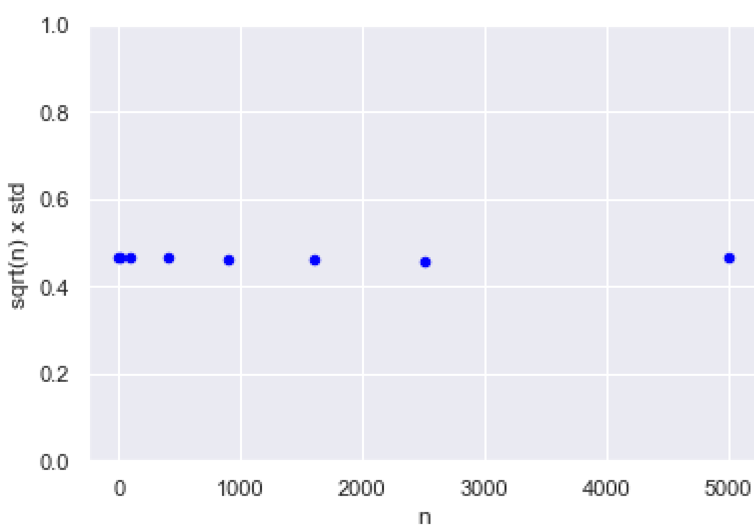


In [36]:
```
summary['root_n'] = np.sqrt(summary['n'])
summary.plot.scatter(x='root_n', y='std', c='blue')
plt.xlabel('root_n')
plt.ylabel('std')
plt.show()
```



Finally, to make it easier to see if the square root rate is the correct stabilization of the standard deviation, we graph $\sqrt{n} * STD$ versus n.

In [37]:
```
summary['root_n_std'] = summary['root_n']*summary['std']
summary.plot.scatter(x='n', y='root_n_std', c='blue')
plt.xlabel('n')
plt.ylabel('sqrt(n) x std')
plt.ylim([0,1])
plt.show()
```



What did we observe about the sampling distribution of the proportion as our sample size n changed?

In an upcoming unit, we will learn about the Central Limit Theorem, which will prove these observations.