Unit 6: Statistics, Parameters, and Random Variables

***Case Studies*: Weekly Hours Spent Watching Youtube Data and Coin Flip Data**

We will learn about:

- Random variables definition
- Types of random variables
- How they behave
- How to calculate probabilities that involve random variables.
- More general probabilities rules (for *any* type of event… not just ones that involve random variables).

Probability Theory

Coding

Summary of Concepts:

- **Needed concept for inference:**
  - Calculate the **probability** that a **(known) sample statistic** from a random sample is a *certain distance away* from the **(unknown) population parameter**.
- **Goal of this lecture:**
  - These probabilities involve **random variables**. How do we calculate probabilities that involve random variables?
- **Main Definitions**
  - **Random variable**
  - **Discrete** random variables
  - **Continuous** random variables
- **Discrete Random Variables**
  - More definitions
  - How to **calculate probabilities** involving discrete random variables *"from scratch."*
- **Bernoulli Random Variable *(special type of discrete random variable)***
  - How to **identify** if a random variable is a Bernoulli random variable.
  - How to **calculate probabilities** involving Bernoulli random variables.
  - How to calculate the **mean** and **variance** of a Bernoulli random variable.
- **Continuous Random Variables**
  - More definitions
  - How to **calculate probabilities** involving continuous random variables *given a pdf curve.*
- **Normal Random Variable *(special type of continuous random variable)***
  - How to **identify** if a random variable is a normal random variable.
  - How to **calculate probabilities** involving normal random variables.
  - How to calculate the **mean** and **variance** of a normal random variable.
- **Standard Normal Random Variable *(special type of continuous random variable)***
  - How to **identify** if a random variable is a standard normal random variable.

  - How to **calculate probabilities** involving standard normal random variables.
  - What is the **mean** and **variance** of a standard normal random variable.
- **Truncated Normal Random Variable *(special type of continuous random variable)***
  - How to **identify** if a random variable is a truncated normal random variable.
  - How to **calculate probabilities** involving truncated normal random variables.
  - How to calculate the **mean** and **variance** of a truncated normal random variable.
- **Exponential Random Variable *(special type of continuous random variable)***
  - How to **identify** if a random variable is a exponential random variable.
  - How to **calculate probabilities** involving exponential random variables.
  - How to calculate the **mean** and **variance** of an exponentialrandom variable.
- **Population Parameters**
  - How to calculate population parameters for a random variable
- **Sample Statistics**
  - How to calculate sample statistics using random variables.

- **More general probabilities rules *(for any type of event… not just events involving random variables).***

Needed concept for inference

Calculate the **probability** that a **(known) sample statistic** from a random sample is a *certain distance away* from the **(unknown) population parameter**.

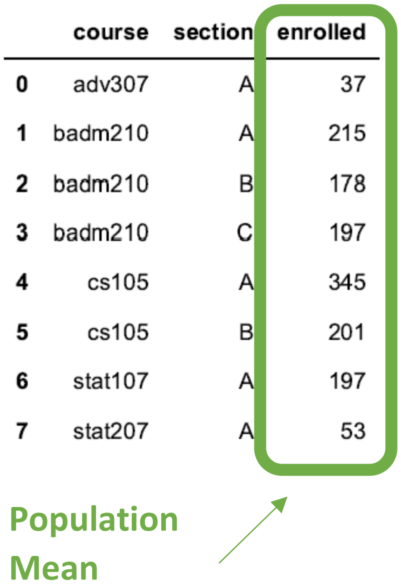
Intuition:

From our explorations from Unit 05, which do we expect to be smaller?

a) $P(|\bar{X} - \mu| \leq k)$ (where $\bar{X}$ is the mean of a random sample of size n=100 drawn from the population with replacement.)

b) $P(|\bar{X} - \mu| \leq k)$ (where $\bar{X}$ is the mean of a random sample of size n=1000 drawn from the population with replacement.)

Population



	course	section	enrolled
0	adv307	A	37
1	badm210	A	215
2	badm210	B	178
3	badm210	C	197
4	cs105	A	345
5	cs105	B	201
6	stat107	A	197
7	stat207	A	53

Population Mean

μ = 177

Goal of Lecture

$\bar{X}$ can be thought of as a **random variable**.

- What is a random variable?
- How do we calculate probabilities with a random variable?
- Building blocks for: "How do we calculate $P(|\bar{X} - \mu| \leq k)$?"

Main Definitions

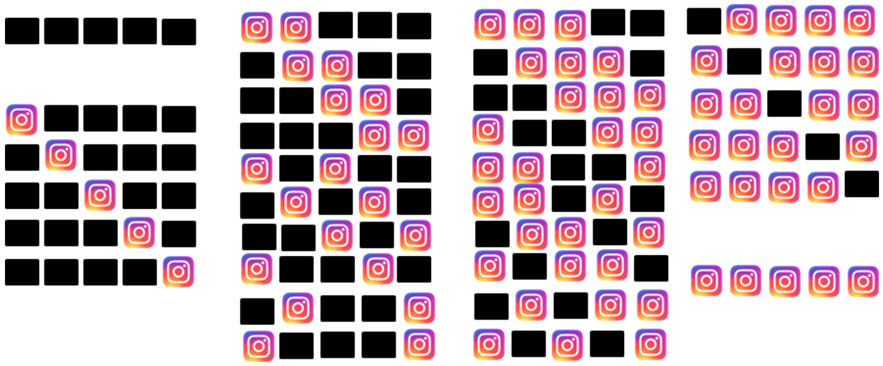
Random Variable:

a variable that assigns some ___________________ to each simple event in a sample space.

Example:

About 35% of American adults use Instagram. We decide to collect a random sample of 5 American adults and ask if they use Instagram or not.

Sample Space



Ex: X = # of randomly selected American adults (out of 5) that are Instagram users.

Ex: Come up with another random variable, call it Y, that involves this sample space.

Discrete vs. Continuous Random Variable

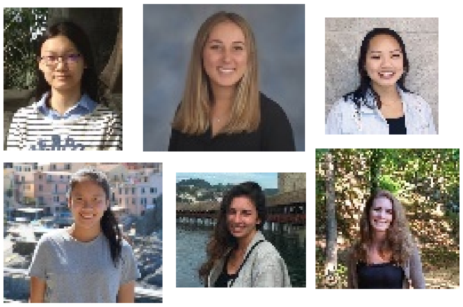
- For **discrete random variables** there exists a way to write out every value the random variable can take on. There exist “gaps” in between the values that they can take on.

- For **continuous random variables** there is no possible way to write out every possible value the random variable can take on.

Example of Continuous Random Variable

We decide to random select an adult female.

Sample Space



... many more

Ex: X = height of the random selected adult female

List of Events	Probabilities
X = 0"	0.00
...	...
X = 5'8"	
X = 5'9"	
...	...
...	...
...	...
...	...
X = 10"	

List of Events	Probabilities
X = 0"	0.00
...	...
X = 5'8"	
X = 5'8.5"	
X = 5'9"	
...	...
...	...
...	...
X = 10"	0.00

AGH!

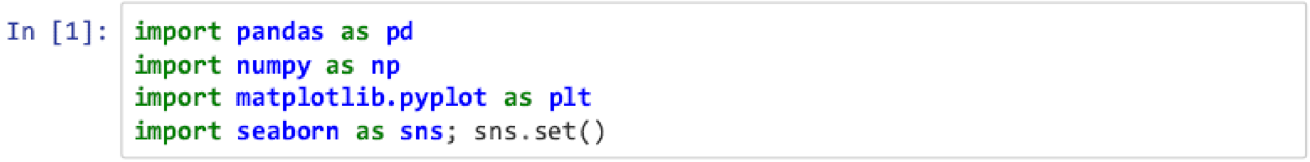
Set of events is continuous
.

Can't write them all out.

List of Events	Probabilities
X = 0"	0.00
...	...
X = 5'8"	
X = 5'8.53"	
X = 5'8.5"	
X = 5'9"	...
...	...
...	...
X = 10"	0.00

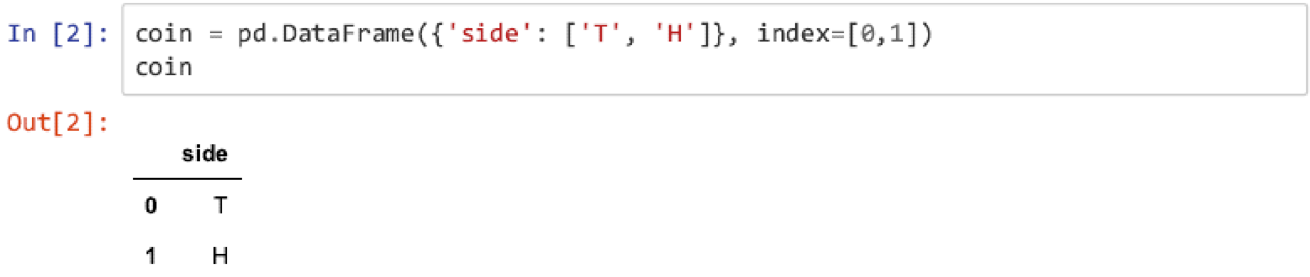
Discrete Random Variables

Example of how to calculate probabilities for a discrete random variable "from scratch"

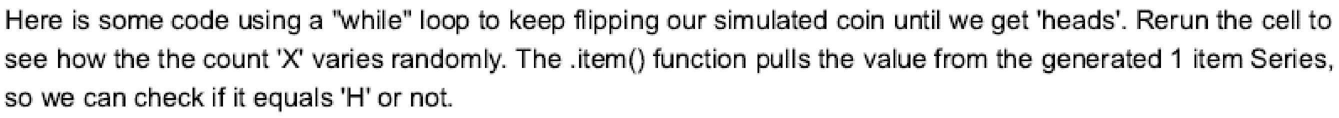


```
In [1]: import pandas as pd
        import numpy as np
        import matplotlib.pyplot as plt
        import seaborn as sns; sns.set()
```

Ex: Random Variable X = # of coin flips until you get a head

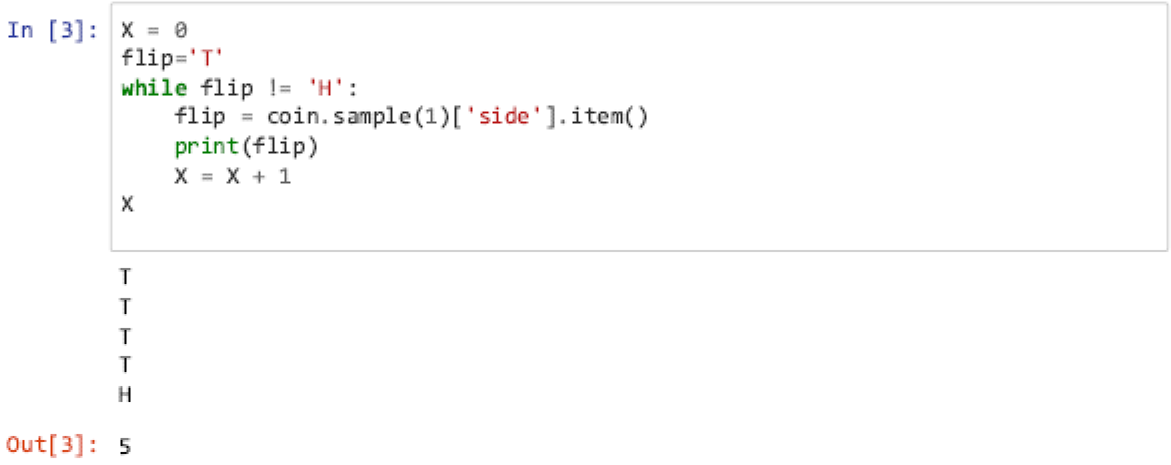


```
In [2]: coin = pd.DataFrame({'side': ['T', 'H']}, index=[0,1])
        coin

Out[2]:
           side
        0     T
        1     H
```



Here is some code using a "while" loop to keep flipping our simulated coin until we get 'heads'. Rerun the cell to see how the the count 'X' varies randomly. The .item() function pulls the value from the generated 1 item Series, so we can check if it equals 'H' or not.

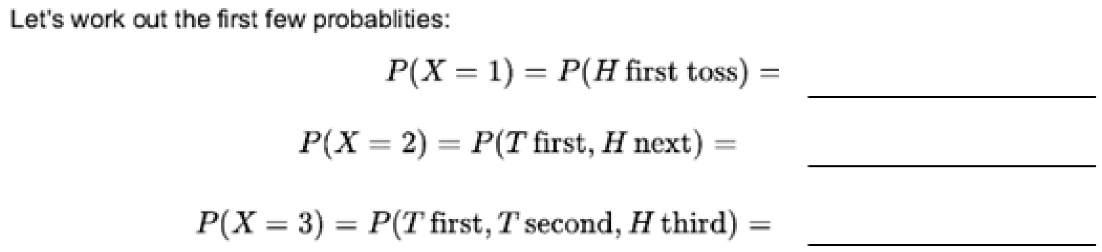
Ex: How to simulate random values for X.



```
In [3]: X = 0
        flip='T'
        while flip != 'H':
            flip = coin.sample(1)['side'].item()
            print(flip)
            X = X + 1
        X

        T
        T
        T
        T
        H
Out[3]: 5
```

Ex: Possible values for X

<u>Ex</u>: How to calculate the probability of each possible value of X

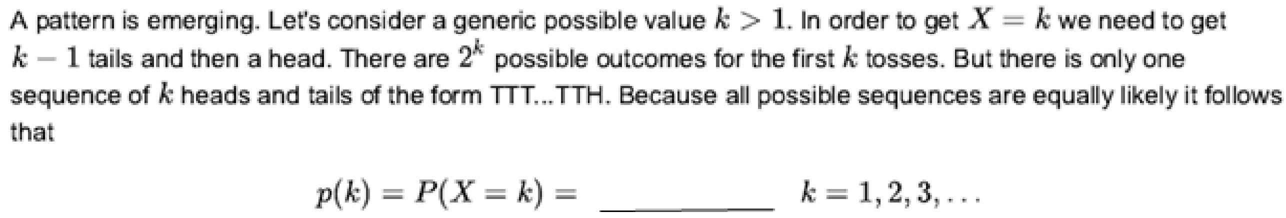


Let's work out the first few probablities:

$$P(X=1) = P(H\,\text{first toss}) = \underline{\qquad\qquad\qquad}$$

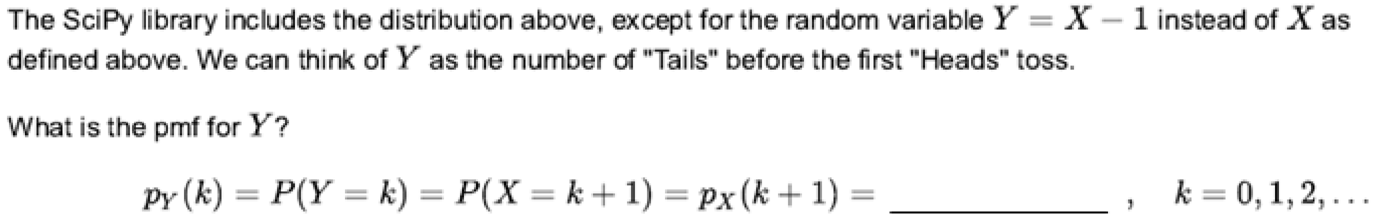
$$P(X=2) = P(T\,\text{first}, H\,\text{next}) = \underline{\qquad\qquad\qquad}$$

$$P(X=3) = P(T\,\text{first}, T\,\text{second}, H\,\text{third}) = \underline{\qquad\qquad\qquad}$$



A pattern is emerging. Let's consider a generic possible value $k > 1$. In order to get $X = k$ we need to get $k-1$ tails and then a head. There are 2^k possible outcomes for the first k tosses. But there is only one sequence of k heads and tails of the form TTT...TTH. Because all possible sequences are equally likely it follows that

$$p(k) = P(X=k) = \underline{\qquad\qquad} \qquad k = 1, 2, 3, \ldots$$



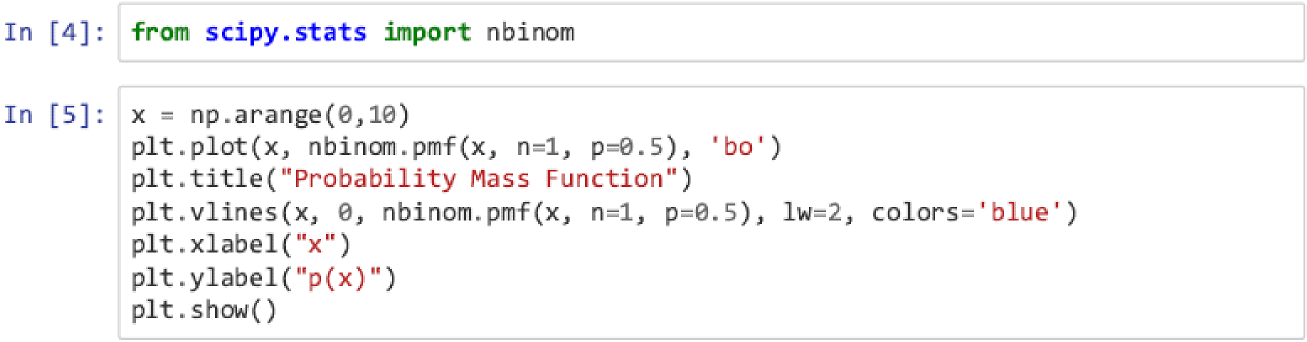
The SciPy library includes the distribution above, except for the random variable $Y = X - 1$ instead of X as defined above. We can think of Y as the number of "Tails" before the first "Heads" toss.

What is the pmf for Y?

$$p_Y(k) = P(Y=k) = P(X=k+1) = p_X(k+1) = \underline{\qquad\qquad\qquad}, \qquad k = 0, 1, 2, \ldots$$

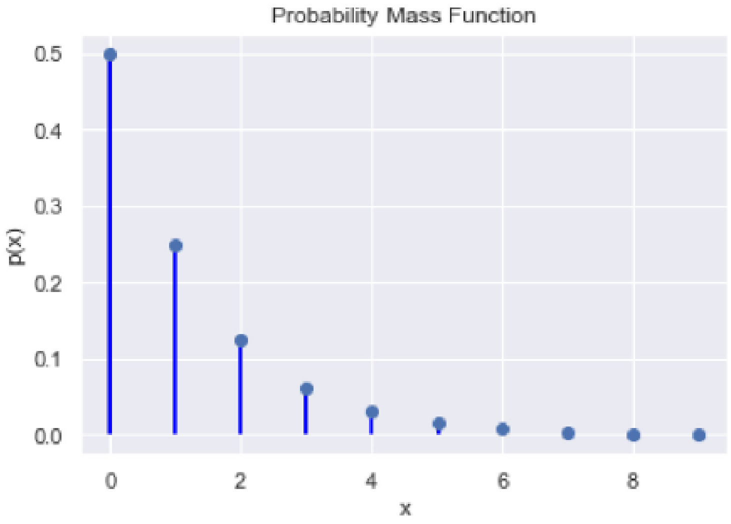
Definitions – for General Discrete Variables

Next we import a from a new library, scipy.stats, which includes many different probablity models for random variables and their distributions. In this case we import the negative binomial model package, 'nbinom'.



```
In [4]: from scipy.stats import nbinom
```

```
In [5]: x = np.arange(0,10)
        plt.plot(x, nbinom.pmf(x, n=1, p=0.5), 'bo')
        plt.title("Probability Mass Function")
        plt.vlines(x, 0, nbinom.pmf(x, n=1, p=0.5), lw=2, colors='blue')
        plt.xlabel("x")
        plt.ylabel("p(x)")
        plt.show()
```



For an arbitrary discrete distribution, let's denote the possible values in the population as

$$x_1, x_2, x_3, \ldots$$

These might be integers, or they could be some other discrete set such as a grid of numbers between 0 and 1.

The ____________________ is then given by

$$p(x) = P(X = x) \quad x = x_1, x_2, x_3, \ldots$$

and $p(x) = 0$ if x is not in the discrete set $\{x_1, x_2, x_3, \ldots\}$.

For discrete random variables we can define the **cumulative distribution function** in the same way as we do later for continuous random variables:

$$F(x) = P(X \leq x) \quad \text{for any value of } x$$

Example: For the coin tossing example, what is the probability of at most 2.7 tails before the first heads? This is

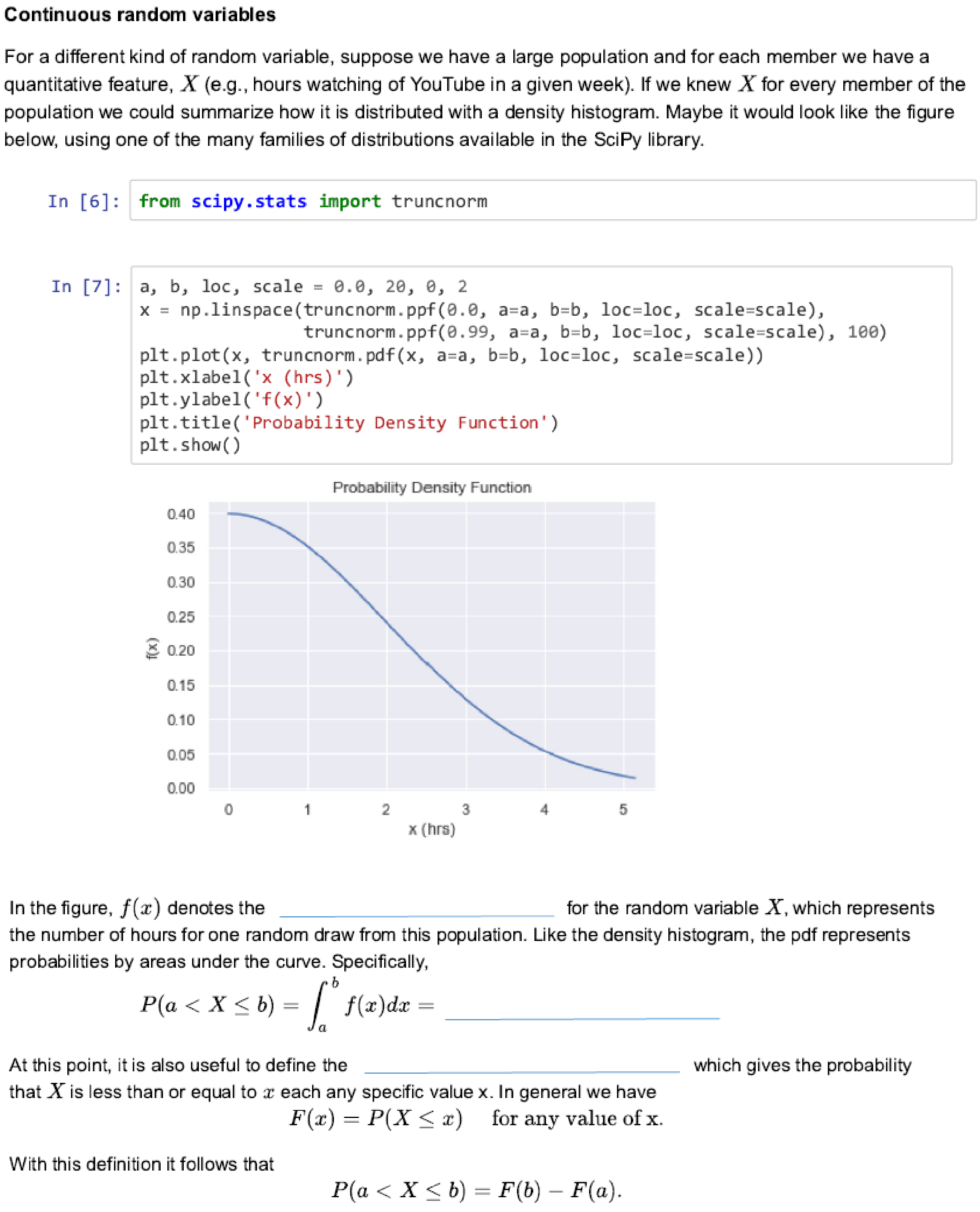
$$\begin{aligned} F(2.7) &= P(Y \leq 2.7) \\ &= \end{aligned}$$

Continuous Random Variables

Definitions

How to Calculate Probabilities of Continuous Random Variables

Ex: Random Variable X = # of hours watching Youtube in a given week.



Continuous random variables

For a different kind of random variable, suppose we have a large population and for each member we have a quantitative feature, X (e.g., hours watching of YouTube in a given week). If we knew X for every member of the population we could summarize how it is distributed with a density histogram. Maybe it would look like the figure below, using one of the many families of distributions available in the SciPy library.

```
In [6]: from scipy.stats import truncnorm
```

```
In [7]: a, b, loc, scale = 0.0, 20, 0, 2
        x = np.linspace(truncnorm.ppf(0.0, a=a, b=b, loc=loc, scale=scale),
                        truncnorm.ppf(0.99, a=a, b=b, loc=loc, scale=scale), 100)
        plt.plot(x, truncnorm.pdf(x, a=a, b=b, loc=loc, scale=scale))
        plt.xlabel('x (hrs)')
        plt.ylabel('f(x)')
        plt.title('Probability Density Function')
        plt.show()
```

In the figure, $f(x)$ denotes the ______________________ for the random variable X, which represents the number of hours for one random draw from this population. Like the density histogram, the pdf represents probabilities by areas under the curve. Specifically,

$$P(a < X \leq b) = \int_a^b f(x)dx = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

At this point, it is also useful to define the ______________________ which gives the probability that X is less than or equal to x each any specific value x. In general we have

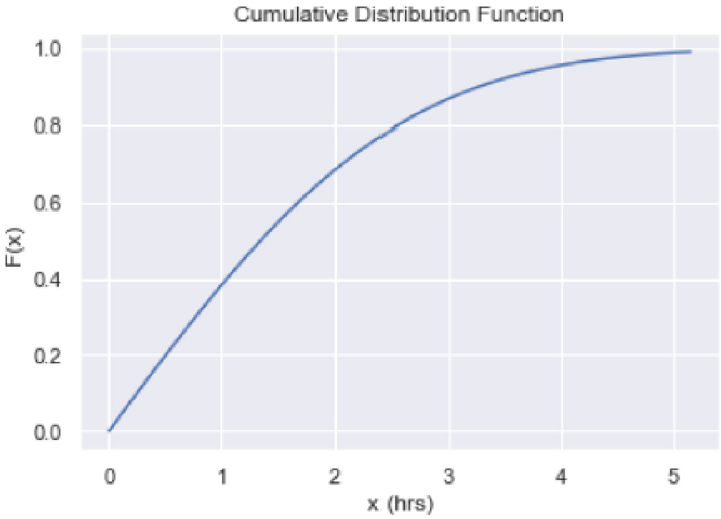
$$F(x) = P(X \leq x) \quad \text{for any value of x.}$$

With this definition it follows that

$$P(a < X \leq b) = F(b) - F(a).$$

In [8]:

```
plt.plot(x, truncnorm.cdf(x, a=a, b=b, loc=loc, scale=scale))
plt.xlabel('x (hrs)')
plt.ylabel('F(x)')
plt.title('Cumulative Distribution Function')
plt.show()
```



Because the cdf is *continuous* with no jumps in the function, we say that the random variable X is a *continuous* random variable, and the distribution is a *continuous* distribution.

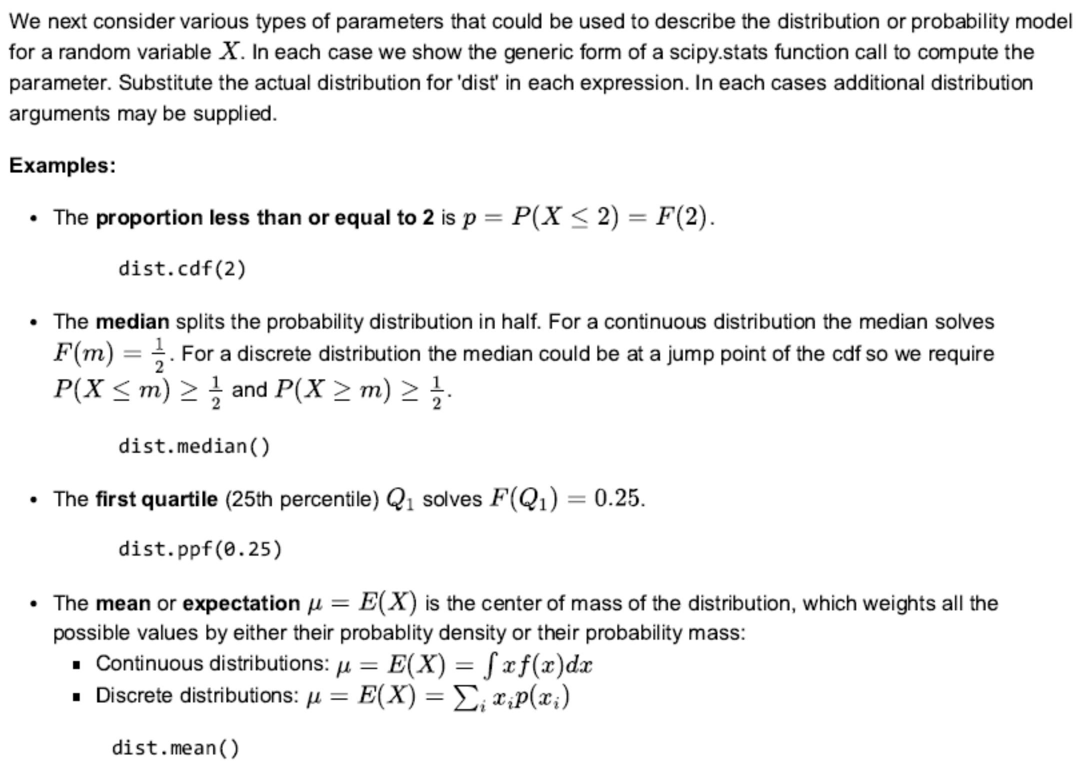
The cdf increases as a function of the potential value x. Why? For any numbers a and b with $a < b$ the event $\{X \le a\}$ implies the event $\{X \le b\}$ (why?). Therefore, for any such numbers we have:

$$a < b \quad \text{implies} \quad F(a) = P(X \le a) \le P(X \le b) = F(b).$$

The cdf curve is handy because the height of the curve at any point x gives the probability that X is less than or equal to that value x.

Population Parameters

How to calculate population parameters of a random variable.



We next consider various types of parameters that could be used to describe the distribution or probability model for a random variable X. In each case we show the generic form of a scipy.stats function call to compute the parameter. Substitute the actual distribution for 'dist' in each expression. In each cases additional distribution arguments may be supplied.

Examples:

- The **proportion less than or equal to 2** is $p = P(X \leq 2) = F(2)$.

```
dist.cdf(2)
```

- The **median** splits the probability distribution in half. For a continuous distribution the median solves $F(m) = \frac{1}{2}$. For a discrete distribution the median could be at a jump point of the cdf so we require $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$.

```
dist.median()
```

- The **first quartile** (25th percentile) Q_1 solves $F(Q_1) = 0.25$.

```
dist.ppf(0.25)
```

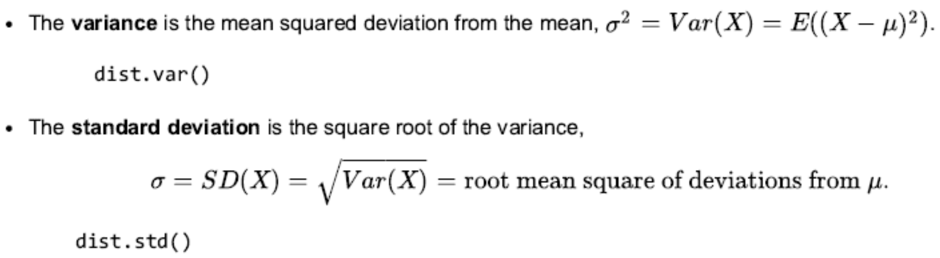
- The **mean** or **expectation** $\mu = E(X)$ is the center of mass of the distribution, which weights all the possible values by either their probablity density or their probability mass:
  - Continuous distributions: $\mu = E(X) = \int x f(x) dx$
  - Discrete distributions: $\mu = E(X) = \sum_i x_i p(x_i)$

```
dist.mean()
```

Ex: Let X = # of adults in a random sample of size 5 that use Instagram. Below is the probability table detailing the probability for each possible value for X. Find the mean of X.

Random Variable X	P(X = #)
X=0	0.12
X=1	0.31
X=2	0.34
X=3	0.18
X=4	0.05
X=5	0.01

Question: How is $\mu = E(X) = \sum_i x_i p(x_i)$ related to the "traditional way" of calculating a mean (ie. adding up all the values and dividing by the number of values)?



- The **variance** is the mean squared deviation from the mean, $\sigma^2 = Var(X) = E((X-\mu)^2)$.

```
dist.var()
```

- The **standard deviation** is the square root of the variance,

$$\sigma = SD(X) = \sqrt{Var(X)} = \text{root mean square of deviations from } \mu.$$

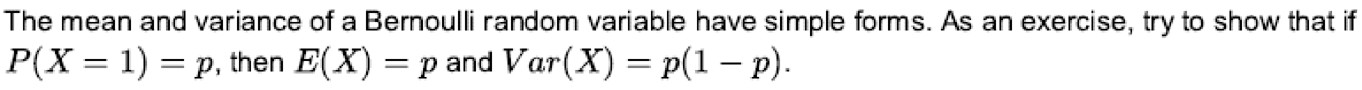
```
dist.std()
```

Bernoulli Random Variables: *(Special Type of Discrete Random Variable)*

How to identify if a random variable is a Bernoulli random variable.

How to calculate the probability of a Bernoulli random variable.

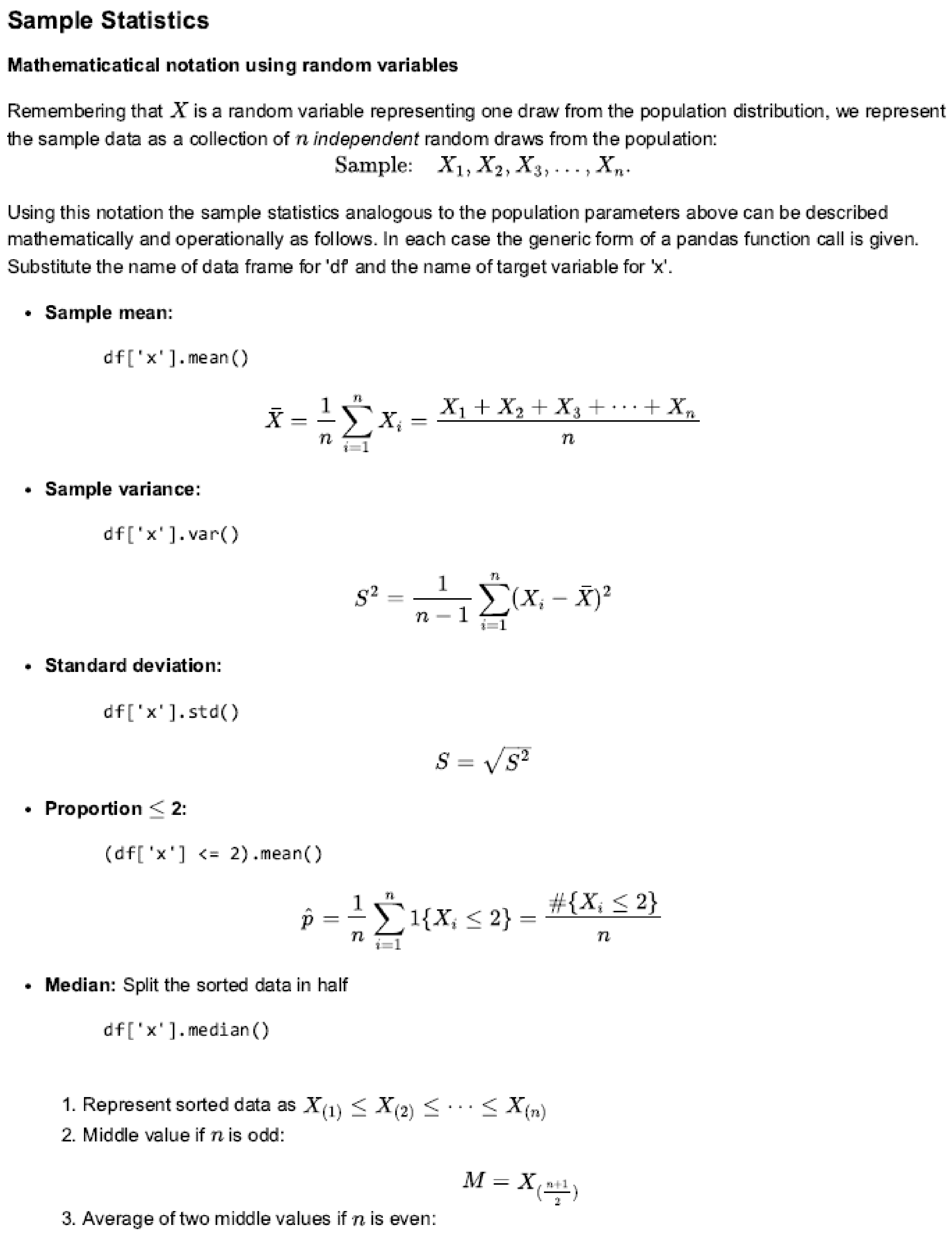
How to find the mean and variance of a Bernoulli random variable.



The mean and variance of a Bernoulli random variable have simple forms. As an exercise, try to show that if $P(X=1)=p$, then $E(X)=p$ and $Var(X)=p(1-p)$.

Sample Statistics

How to calculate sample statistics using random variables.



Sample Statistics

Mathematicatical notation using random variables

Remembering that X is a random variable representing one draw from the population distribution, we represent the sample data as a collection of n *independent* random draws from the population:

$$\text{Sample:} \quad X_1, X_2, X_3, \ldots, X_n.$$

Using this notation the sample statistics analogous to the population parameters above can be described mathematically and operationally as follows. In each case the generic form of a pandas function call is given. Substitute the name of data frame for 'df' and the name of target variable for 'x'.

- **Sample mean:**

```
df['x'].mean()
```

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$$

- **Sample variance:**

```
df['x'].var()
```

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- **Standard deviation:**

```
df['x'].std()
```

$$S = \sqrt{S^2}$$

- **Proportion $\leq$ 2:**

```
(df['x'] <= 2).mean()
```

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} 1\{X_i \leq 2\} = \frac{\#\{X_i \leq 2\}}{n}$$

- **Median:** Split the sorted data in half

```
df['x'].median()
```

1. Represent sorted data as $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$
2. Middle value if n is odd:

$$M = X_{(\frac{n+1}{2})}$$

3. Average of two middle values if n is even:

$$M = \frac{X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)}}{2}$$

- **First quartile (25th percentile):** Quick way for hand calculation is median of sorted observations below the median.

```
df['x'].quantile(q=0.25)
```

Additional information about the relationship between sample statistics and random variables.

Two important observations:

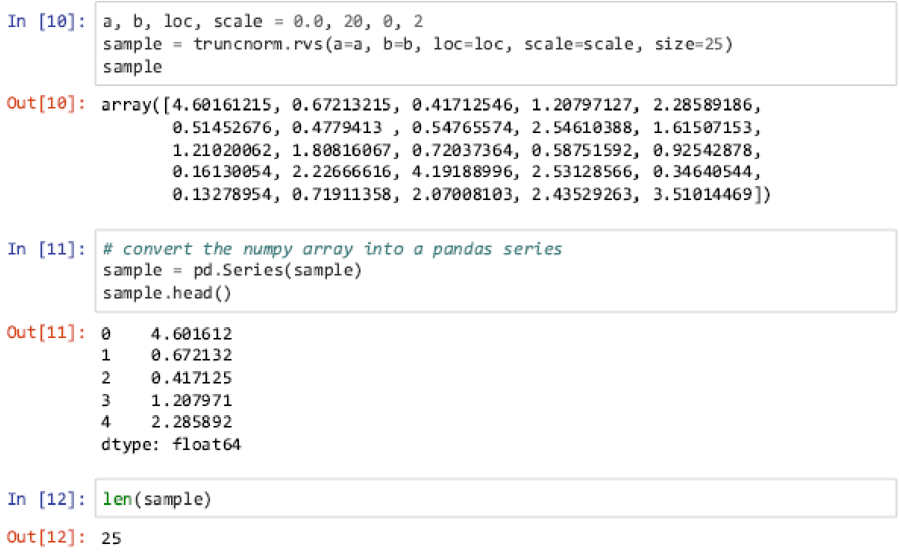
1. All of these sample statistics are themselves random variables. Getting their observed values amuonts to collecting the data and computing the statistics of interest.
2. Because the sample statistics are random varaibles they have distributions, means, standard deviations etc. that can aid us in determining their accuracy as population parameter estimates.

Comparing the Population Distribution to the Sample Distribution

- Let’s compare the following for a **population distribution** and a **random sample drawn from that population.**
    - Shape
    - Measures of Center
    - Measures of a spread

Example: Youtube viewing population with truncated normal distribution.

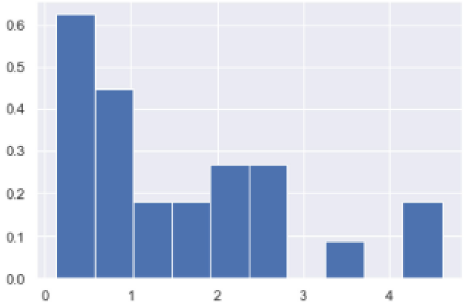
Let's draw a sample of 25 individuals (Sim people!) from the YouTube watching population described previously and see how their sample statistics compare to the population parameters.



```
In [10]: a, b, loc, scale = 0.0, 20, 0, 2
         sample = truncnorm.rvs(a=a, b=b, loc=loc, scale=scale, size=25)
         sample

Out[10]: array([4.60161215, 0.67213215, 0.41712546, 1.20797127, 2.28589186,
                0.51452676, 0.4779413 , 0.54765574, 2.54610388, 1.61507153,
                1.21020062, 1.80816067, 0.72037364, 0.58751592, 0.92542878,
                0.16130054, 2.22666616, 4.19188996, 2.53128566, 0.34640544,
                0.13278954, 0.71911358, 2.07008103, 2.43529263, 3.51014469])

In [11]: # convert the numpy array into a pandas series
         sample = pd.Series(sample)
         sample.head()

Out[11]: 0    4.601612
         1    0.672132
         2    0.417125
         3    1.207971
         4    2.285892
         dtype: float64

In [12]: len(sample)

Out[12]: 25
```

In [13]:

```
sample.hist(density=True)
plt.show()
```



How do the sample statistics compare to the population parameters?

In [14]:

```
# params = ['mean', 'median', 'std', 'prop <= 2'] # -- for reference, already
 defined
samp = [sample.mean(),
        sample.median(),
        sample.std(),
        (sample<=2).mean()]
pd.DataFrame({'population': pop, 'sample': samp}, index=params)
```

Out[14]:

	population	sample
mean	1.595769	1.538507
median	1.348980	1.207971
std	1.205621	1.255055
prop	0.682689	0.640000

The sample yields statistics that do not exactly equal the population parameters, due to sampling variation, but are reasonably close.

Normal Random Variables: *(Special Type of Continuous Random Variable)*

How to identify if a random variable is a normal random variable.

Properties of Normal Random Variables

Probability density function (pdf):

Shape:

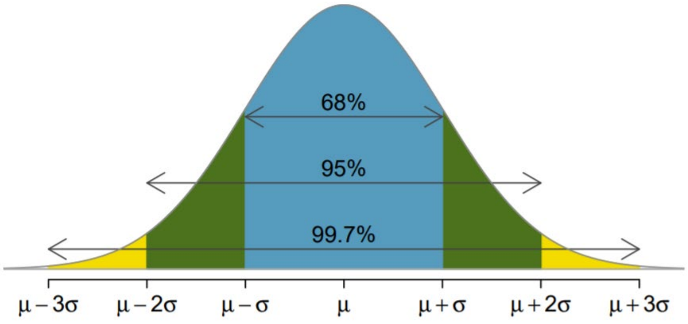
Mean:

Variance:

Standard Deviation:

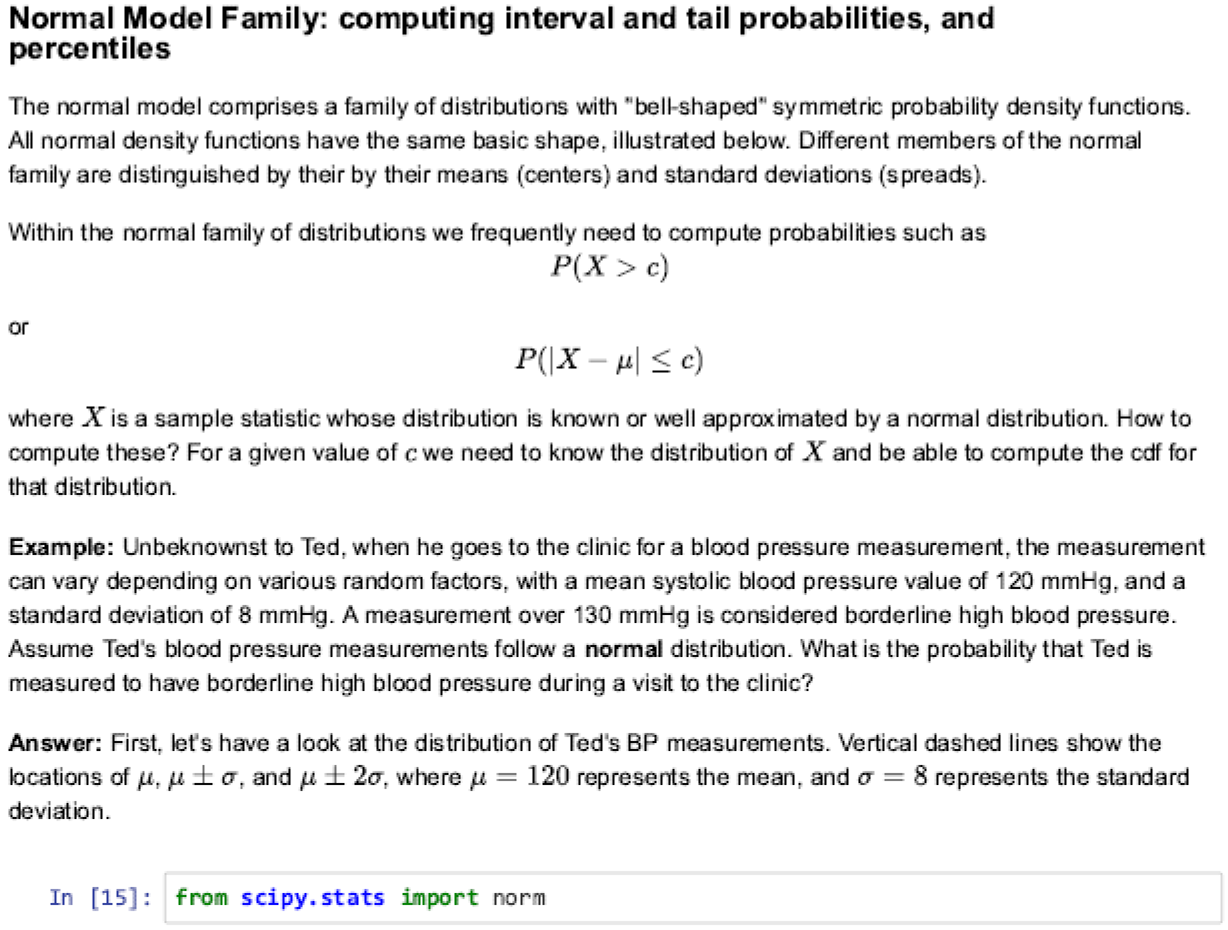
Other properties:

Normal Distribution Follows the **68-95-99.7 rule**



Ex: The average height of an adult female in the U.S. is about 64” with a standard deviation of 2.5”. What percent of adult females in the U.S are taller than 69”?

Calculating probabilities involving normal random variables *(with Python).*



Normal Model Family: computing interval and tail probabilities, and percentiles

The normal model comprises a family of distributions with "bell-shaped" symmetric probability density functions. All normal density functions have the same basic shape, illustrated below. Different members of the normal family are distinguished by their by their means (centers) and standard deviations (spreads).

Within the normal family of distributions we frequently need to compute probabilities such as

$$P(X > c)$$

or

$$P(|X - \mu| \leq c)$$

where X is a sample statistic whose distribution is known or well approximated by a normal distribution. How to compute these? For a given value of c we need to know the distribution of X and be able to compute the cdf for that distribution.

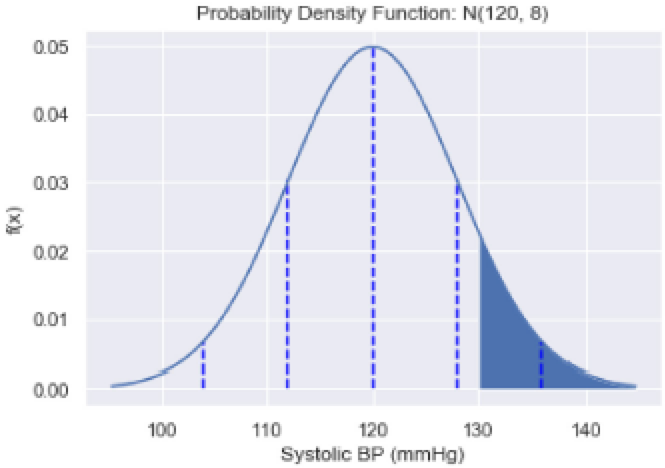
Example: Unbeknownst to Ted, when he goes to the clinic for a blood pressure measurement, the measurement can vary depending on various random factors, with a mean systolic blood pressure value of 120 mmHg, and a standard deviation of 8 mmHg. A measurement over 130 mmHg is considered borderline high blood pressure. Assume Ted's blood pressure measurements follow a **normal** distribution. What is the probability that Ted is measured to have borderline high blood pressure during a visit to the clinic?

Answer: First, let's have a look at the distribution of Ted's BP measurements. Vertical dashed lines show the locations of μ, $\mu \pm \sigma$, and $\mu \pm 2\sigma$, where $\mu = 120$ represents the mean, and $\sigma = 8$ represents the standard deviation.

In [15]:

```
from scipy.stats import norm
```

In [16]:
```
mu, sd = 120, 8
x = np.linspace(norm.ppf(0.001, loc=mu, scale=sd),
                norm.ppf(0.999, loc=mu, scale=sd), 100)
plt.plot(x, norm.pdf(x, loc=mu, scale=sd))
plt.xlabel('Systolic BP (mmHg)')
plt.ylabel('f(x)')
plt.title('Probability Density Function: N(120, 8)')
# add shaded areas whose probability we need
x130 = np.linspace(130, norm.ppf(0.999, loc=mu, scale=sd), 100)
plt.fill_between(x130, 0, norm.pdf(x130, loc=mu, scale=sd))
# add sd lines
xsd = np.array([mu-2*sd, mu-sd, mu, mu+sd, mu+2*sd])
plt.vlines(xsd, 0, norm.pdf(xsd, loc=mu, scale=sd), colors='blue', linestyle=
'--')
plt.show()
```



Let X denote Ted's systolic blood pressure measurement. We need

$$P(X > 130) = 1 - P(X \leq 130) = 1 - F_X(130)$$

We compute this probability using the 'norm.cdf' function in SciPy:

In [17]:
```
1-norm.cdf(130, loc=120, scale=8)
```

Out[17]:
```
0.10564977366685535
```

In [18]:
```
norm.cdf(1.25)
```

Out[18]:
```
0.8943502263331446
```

We see there is just over a 10% chance of a borderline high blood pressure measurement.

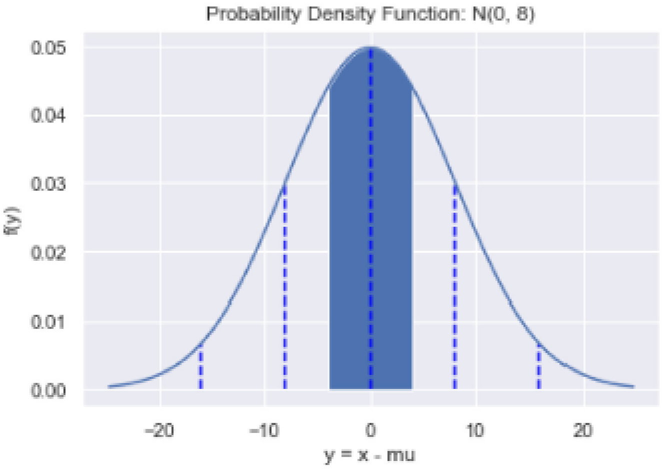
Example: The aforementioned Ted does not know his current blood pressure. In the past his measurements tended to jump around, with a standard deviation of 8 mmHg. When he gets a new measurement, what is the probability that it is within 4 mmHg of his true mean systolic blood pressure? Assume his blood pressure measurements are like random draws from a normal distribution with unknown mean μ.

Answer:

How can we do this if we don't know the mean? The key is that we can deduce the distribution of the **difference** between the measurement and the mean, and that is sufficient to answer the question. If X is normally distributed with mean μ, then $Y = X - \mu$ is normally distributed with mean 0 and the same standard deviation as X. Here is what the pdf looks like now:

In [19]:

```
x0 = np.linspace(norm.ppf(0.001, loc=0, scale=sd),
                 norm.ppf(0.999, loc=0, scale=sd), 100)
plt.plot(x0, norm.pdf(x0, loc=0, scale=sd))
plt.xlabel('y = x - mu')
plt.ylabel('f(y)')
plt.title('Probability Density Function: N(0, 8)')
# add shaded areas whose probability we need
xshade = np.arange(-4,4,0.01)
plt.fill_between(xshade, 0, norm.pdf(xshade, loc=0, scale=sd))
# add sd lines
xsd0 = np.array([0-2*sd, 0-sd, 0, 0+sd, 0+2*sd])
plt.vlines(xsd0, 0, norm.pdf(xsd0, loc=0, scale=sd), colors='blue', linestyle=
'--')
plt.show()
```



We need to compute

$$P(|X - \mu| \leq 4) = P(|Y| \leq 4) = P(-4 \leq Y \leq 4) = F_Y(4) - F_Y(-4).$$

Using norm.cdf() we compute:

```
In [20]: norm.cdf(4, loc=0, scale=8) - norm.cdf(-4, loc=0, scale=8)

Out[20]: 0.38292492254802624
```

Only 38%. Apparently ± 4 mmHg is too much precision to expect from one measurement of Ted's blood pressure!

Example: We're not done with Ted yet. Let's find a value c such that there is a 95% chance that Ted's blood pressure measurement is within c standard deviations of his true mean blood pressure. In this case we are assuming the standard deviation of the measurement is 8 mmHg, though it turns out the answer does not require that information. Thus, we want to find c so that

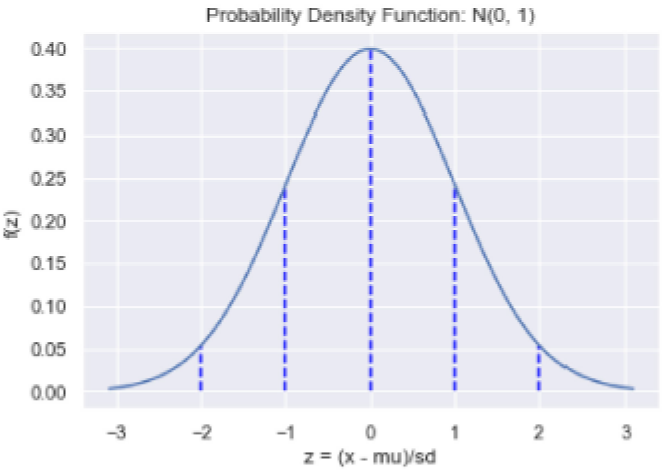
$$P(|X - \mu| \leq 8c) = 0.95.$$

This is the same as

$$P\left(\left|\frac{X-\mu}{8}\right| \leq c\right) = P(|Z| \leq c) = 0.95$$

where Z is normal with mean 0 and standard deviation 1. This is the **standard normal distribution**. Its pdf looks like this:

```
In [21]: x01 = np.linspace(norm.ppf(0.001),
                           norm.ppf(0.999), 100)
         plt.plot(x01, norm.pdf(x01))
         plt.xlabel('z = (x - mu)/sd')
         plt.ylabel('f(z)')
         plt.title('Probability Density Function: N(0, 1)')
         # add sd lines
         xsd0 = np.array([-2, -1, 0, 1, 2])
         plt.vlines(xsd0, 0, norm.pdf(xsd0), colors='blue', linestyle='--')
         plt.show()
```



So we see the same value of c works for any standard deviation. OK, so what is c? Solve

$$0.95 = P(Z \le c) - P(Z \le -c) = P(Z \le c) - (1 - P(Z \le c)) = 2P(Z \le c) - 1$$

because a normal distribution with mean zero is **symmetric**. This is the same as

$$P(Z \le c) = \frac{1 + 0.95}{2} = 0.975.$$

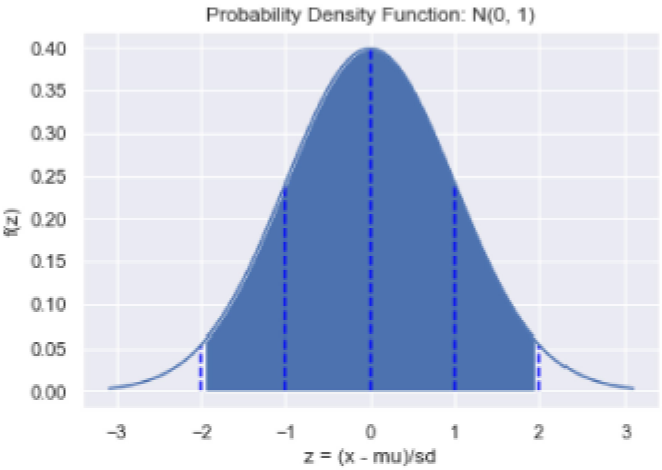
Therefore we need the 97.5th percentile of the **standard normal distribution** (loc=0, scale=1, the default values). This is given by the function norm.ppf (normal ppoint percentile function):

```
In [22]: print('c=', norm.ppf(0.975))
```

```
c= 1.959963984540054
```

We conclude that there is a 95% probability that a normally distributed random variable (such as Ted's blood pressure measurement) is within 1.96 standard devaitions of it's mean, whatever the mean and standard deviation might be!

```
In [23]: plt.plot(x01, norm.pdf(x01))
         plt.xlabel('z = (x - mu)/sd')
         plt.ylabel('f(z)')
         plt.title('Probability Density Function: N(0, 1)')
         # add shaded areas whose probability we need
         xshade = np.arange(norm.ppf(0.025), norm.ppf(0.975), 0.01)
         plt.fill_between(xshade, 0, norm.pdf(xshade))
         # add sd lines
         xsd0 = np.array([-2, -1, 0, 1, 2])
         plt.vlines(xsd0, 0, norm.pdf(xsd0), colors='blue', linestyle='--')
         plt.show()
```



How does c change if we need only 90% or 80% probability of being within c standard deviations of the true mean?

Standard Normal Random Variables:

**** Special Type of Continuous Random Variable***

**** Special Type of Normal Random Variable***

How to identify if a random variable is a standard normal random variable.

Properties of Standard Normal Random Variables

Calculating probabilities involving standard normal random variables *(with Python).*

Relationship between:

- **a normal random variable (X)**
- **the z-score of a random variable**
- **a standard normal random variable (Z)**

Definition of z-score

The **z-score** of a random variable X with mean μ and standard deviation σ is defined as:

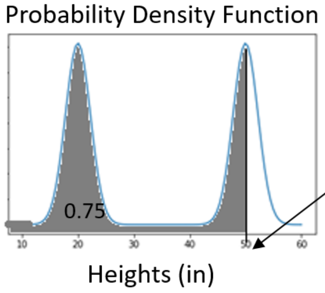
Relationship between:

- Random variables
- Z-scores of random variables
- And pdfs of the random variable and the z-score of the random variable

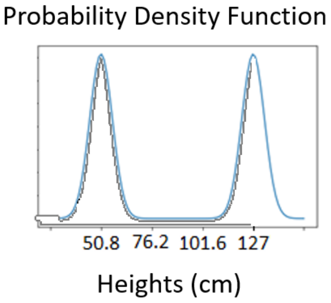
Example

Population: babies and teenagers
Distribution: heights (in)

- Mean = 35”
- Standard deviation = 5”



Joe is taller than 75% of the population when measured in inches. If we converted *all* heights to centimeters, would Joe still be taller than 75% of the population?



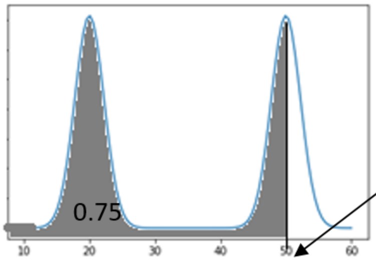
Example

<u>Population</u>: babies and teenagers
<u>Distribution</u>: heights (in)

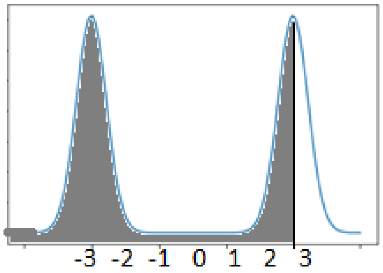
- Mean = 35”
- Standard deviation = 5”

Joe is taller than 75% of the population when measured in inches. If we converted *all* heights to z-scores, would Joe’s z-score be larger than the z-scores of 75% of the population?

Probability Density Function



Heights (in)



Z-scores of the heights

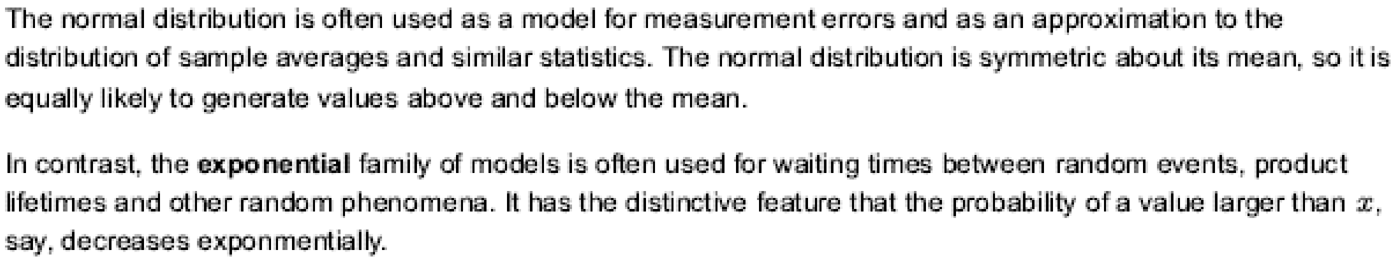
Properties:

- *<u>In general</u>, if X is a random variable, with mean μ and standard deviation σ, and* $Z = \frac{X-\mu}{\sigma}$*, then*
  - $P(X \leq k) = P(Z \leq \qquad\qquad)$

- *More specifically, if X is a ____________ random variable, with mean μ and standard deviation σ, and* $Z = \frac{X-\mu}{\sigma}$*, then*
  - $P(X \leq k) = P(Z \leq \qquad\qquad)$
  - and Z is a ____________________________ random variable.

Exponential Random Variables: *(Special Type of Continuous Random Variable)*

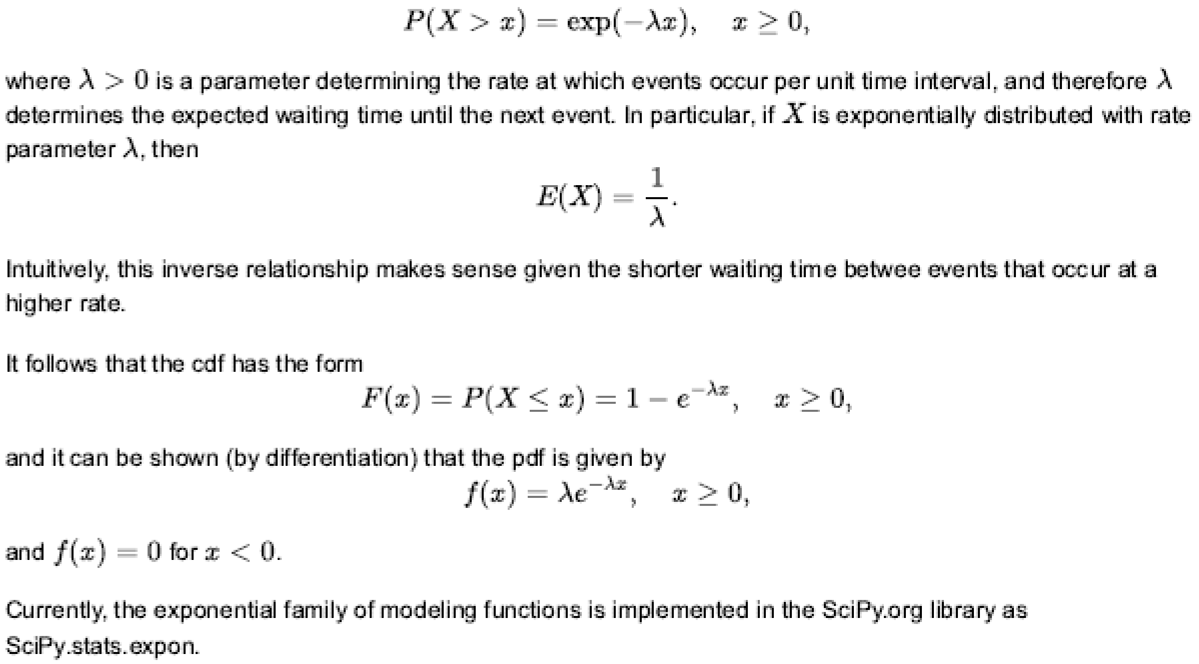
Examples of exponential random variables



The normal distribution is often used as a model for measurement errors and as an approximation to the distribution of sample averages and similar statistics. The normal distribution is symmetric about its mean, so it is equally likely to generate values above and below the mean.

In contrast, the **exponential** family of models is often used for waiting times between random events, product lifetimes and other random phenomena. It has the distinctive feature that the probability of a value larger than x, say, decreases exponmentially.

Properties of Exponential Random Variables



$$P(X > x) = \exp(-\lambda x), \quad x \geq 0,$$

where $\lambda > 0$ is a parameter determining the rate at which events occur per unit time interval, and therefore λ determines the expected waiting time until the next event. In particular, if X is exponentially distributed with rate parameter λ, then

$$E(X) = \frac{1}{\lambda}.$$

Intuitively, this inverse relationship makes sense given the shorter waiting time betwee events that occur at a higher rate.

It follows that the cdf has the form

$$F(x) = P(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0,$$

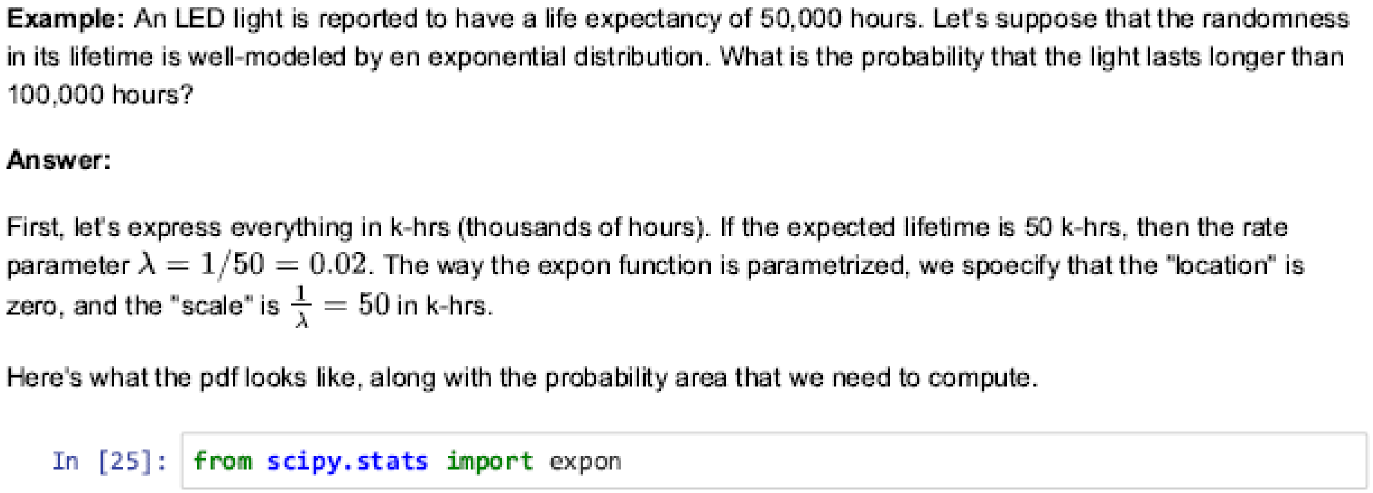
and it can be shown (by differentiation) that the pdf is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0,$$

and $f(x) = 0$ for $x < 0$.

Currently, the exponential family of modeling functions is implemented in the SciPy.org library as SciPy.stats.expon.

Calculating probabilities involving exponential random variables (with Python).



Example: An LED light is reported to have a life expectancy of 50,000 hours. Let's suppose that the randomness in its lifetime is well-modeled by en exponential distribution. What is the probability that the light lasts longer than 100,000 hours?

Answer:

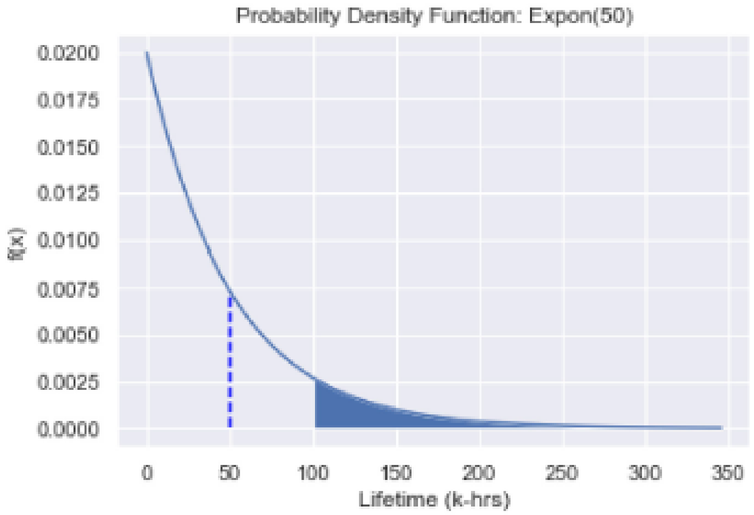
First, let's express everything in k-hrs (thousands of hours). If the expected lifetime is 50 k-hrs, then the rate parameter $\lambda = 1/50 = 0.02$. The way the expon function is parametrized, we spoecify that the "location" is zero, and the "scale" is $\frac{1}{\lambda} = 50$ in k-hrs.

Here's what the pdf looks like, along with the probability area that we need to compute.

In [25]:
```
from scipy.stats import expon
```

In [26]:

```
scale = 50
x = np.linspace(0,
                expon.ppf(0.999, loc=0, scale=scale), 100)
plt.plot(x, expon.pdf(x, loc=0, scale=scale))
plt.xlabel('Lifetime (k-hrs)')
plt.ylabel('f(x)')
plt.title('Probability Density Function: Expon(50)')
# add shaded areas whose probability we need
x100 = np.linspace(100, expon.ppf(0.999, loc=0, scale=scale), 100)
plt.fill_between(x100, 0, expon.pdf(x100, loc=0, scale=scale))
# add mean value
plt.vlines(50, 0, expon.pdf(50, loc=0, scale=scale), colors='blue', linestyle=
'--')
plt.show()
```



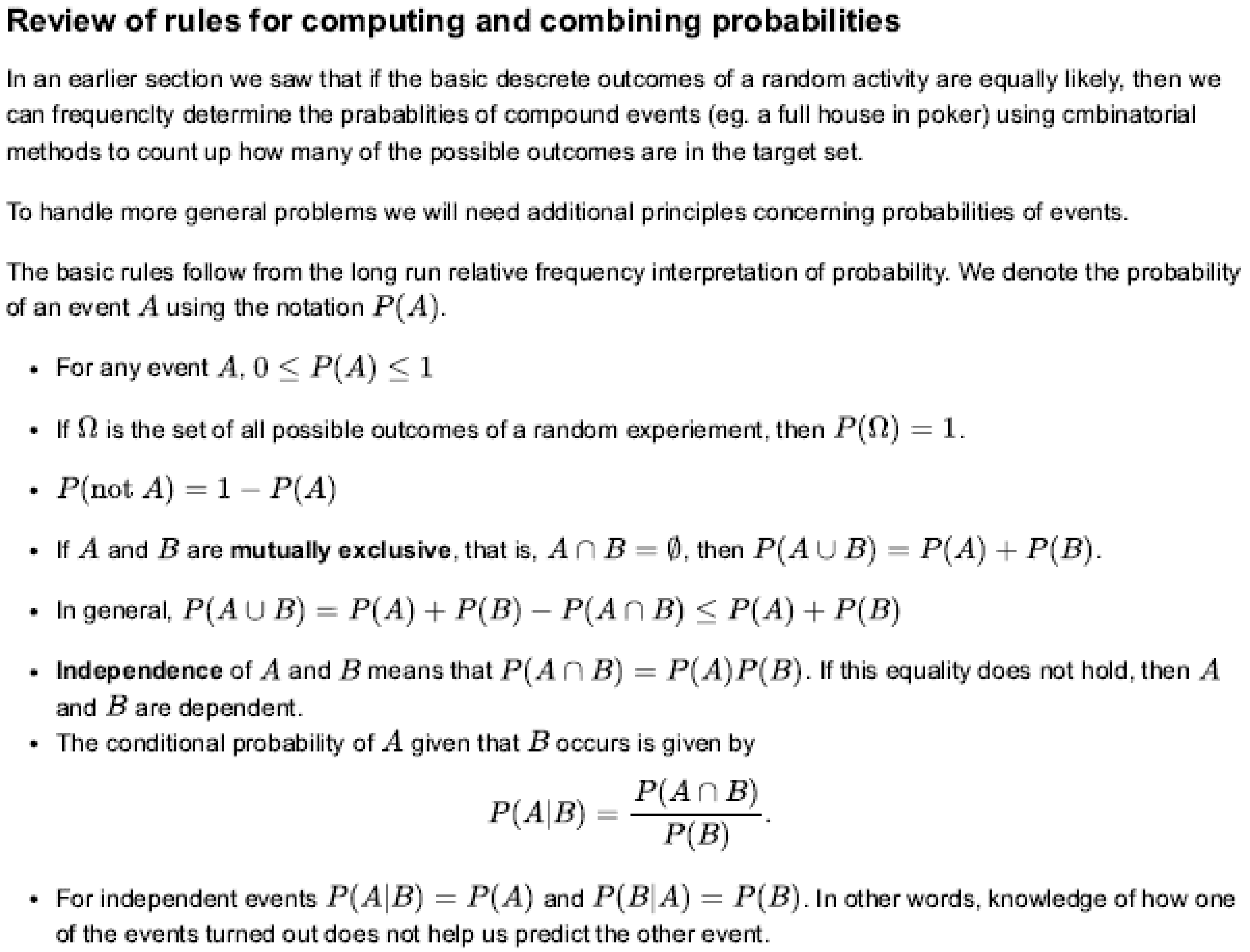
We need to compute the area of the shaded region. In this case, it is possible to do this directly using the formula. We can also do it using the expon.cdf() function. Try it!

In [ ]:

It is interesting to compare the mean lifetime (50 k-hrs) and the median. How can we compute the median lifetime for this distribution?

In [ ]:

<u>Rules that Help you Calculate Probabilities *of any event*</u> (not just ones that involve random variables)



Review of rules for computing and combining probabilities

In an earlier section we saw that if the basic descrete outcomes of a random activity are equally likely, then we can frequenclty determine the prabablities of compound events (eg. a full house in poker) using cmbinatorial methods to count up how many of the possible outcomes are in the target set.

To handle more general problems we will need additional principles concerning probabilities of events.

The basic rules follow from the long run relative frequency interpretation of probability. We denote the probability of an event A using the notation $P(A)$.

- For any event A, $0 \leq P(A) \leq 1$
- If Ω is the set of all possible outcomes of a random experiement, then $P(\Omega) = 1$.
- $P(\text{not } A) = 1 - P(A)$
- If A and B are **mutually exclusive**, that is, $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.
- In general, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$
- **Independence** of A and B means that $P(A \cap B) = P(A)P(B)$. If this equality does not hold, then A and B are dependent.
- The conditional probability of A given that B occurs is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- For independent events $P(A|B) = P(A)$ and $P(B|A) = P(B)$. In other words, knowledge of how one of the events turned out does not help us predict the other event.

Application to events involving random variables

We will often consider events involving random varaibles, such as the event that X lies in some interval $[a, b]$. Probabiliites of these types of events follow all of the above rules.

Example of mutually exclusive events: X is a random number between 0 and 1. Then, using the fact that $\{X < 0.2\}$ and $\{X > 0.8\}$ are mutually exclusive events,

$$P(X < 0.2 \text{ or } X > 0.8) = P(X < 0.2) + P(X > 0.8) = 0.2 + 0.2 = 0.4$$

Independent random variables:

Two random variables X and Y are **independent** if for any sets of values A for X and B for Y we have

$$P(\{X \in A\} \cap \{Y \in B\}) = P(X \in A)P(Y \in B).$$

Otherwise, X and Y are **dependent**.

Example: Dependent or Independent? A jar contains 5 red chips and 10 blue chips. We sample 2 chips at random without replacement. X is the number of red chips. Y is the number of blue chips. Are X and Y dependent or independent? Consider the probability that $X = 2$ and $Y = 2$. Does the probability that both of these things happen equal the product of the individual probabilities that each of them happens?

Example: Dependendent or Independent? Same jar. Now we draw two chips with replacement. X is 1 if the first chip is blue and 0 otherwise. Y is 1 if the second chip is blue, and zero otherwise. Are X and Y dependent or independent? Consider whether knowledge of the first draw tells us anything about what will happen on the second draw.