Partial Differential Equations Study Guide

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1 Solving the transport equation

1. Solve $u_x + 2xu_y = 0$

We use the method of characteristics to solve the transport equation. Note that we can rewrite the equation $u_x+2xu_y = 0$ as $\nabla u(1, 2x)$. This equation describes the directional derivative of u in the direction of $< 1, 2x >$. We can also describe this as having the scalar product of the gradient u with the vector $< 1, 2x >$. Thus, we can parametrize the transport equation with respect to such characteristic curves.

We parametrize with respect to a variable which we will call $x^{'}$. With our equation $u(x, y)$, we now have $u(x(x^{'}), y(x^{'}))$. Thus, we can differentiate and achieve,

$$\frac{du}{dx'} = \frac{\partial u}{\partial x}\frac{dx}{dx'} + \frac{\partial u}{\partial y}\frac{dy}{dx'}$$

$$u_x + 2xu_y \implies 1\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y}$$

Thus, it is clear that,

$$\frac{dx}{dx'} = 1 \ *, \frac{dy}{dx'} = 2x**$$

Knowing this, we can write $dx^{'} = dx$ from *. We can then replace $dx^{'}$ with dx in ** to achieve: $\frac{dy}{dx} = 2x$.

Integrating, we get $y = x^2 + c$. Thus, the solution to this transport equation can be described as:

$$f(y - x^2) = u(x, y)$$