1 Preliminaries

Gõ và trình bày lại văn bản sau, với các yêu cầu:

- Căn chỉnh lề trên dưới 2cm, trái phải 3,5cm
- Phông chữ Times new roman, cỡ chữ 11pt
- Chuẩn trình bày toán học: có tham chiếu, các dấu biến đổi thẳng nhau, ngắt đoạn phù hợp với công thức dài,...

2 Proof

Firstly, let $u \in \Omega_{\mathrm{SVIP}}$. Hence, $u \in \mathscr{S}_{(\mathscr{A},\mathscr{B})}$.

$$
\begin{aligned}
\|y^k - u\|^2 = \Big\| & \beta_k(x^k - u) \\
& + (1-\beta_k)\Big[P_{\mathscr{C}}\big(I^{\mathscr{H}_1} - \lambda\mathscr{A}\big)x^k - P_{\mathscr{C}}\big(I^{\mathscr{H}_1} - \lambda\mathscr{A}\big)u\Big]\Big\|^2 \\
\leq{} & \beta_k\|x^k - u\|^2 + (1-\beta_k)\|x^k - u\|^2 \\
& - \beta_k(1-\beta_k)\big\|x^k - P_{\mathscr{C}}\big(I^{\mathscr{H}_1} - \lambda\mathscr{A}\big)x^k\big\|^2 \\
={} & \|x^k - u\|^2 - \beta_k(1-\beta_k)\big\|x^k - P_{\mathscr{C}}\big(I^{\mathscr{H}_1} - \lambda\mathscr{A}\big)x^k\big\|^2 \qquad (2.1) \\
\leq{} & \|x^k - u\|^2. \qquad (2.2)
\end{aligned}
$$

It follows from Step 3 in Algorithm 1, the property of adjoint operator $\mathscr{F}^*$ that

$$
\begin{aligned}
\|v^k - u\|^2 &= \big\|y^k + \gamma_k\mathscr{F}^*\big(z^k - \mathscr{F}y^k\big) - u\big\|^2 \\
&= \|y^k - u\|^2 + \gamma_k^2\|\mathscr{F}^*\big(z^k - \mathscr{F}y^k\big)\|^2 \\
&\quad + \gamma_k\Big(\big\|z^k - \mathscr{F}u\big\|^2 - \big\|\mathscr{F}y^k - \mathscr{F}u\big\|^2 - \big\|z^k - \mathscr{F}y^k\big\|^2\Big).
\end{aligned}
$$

Since $u \in \Omega_{\text{SVIP}}$, $\mathscr{F} u \in \mathscr{S}_{(\mathscr{B},\mathscr{Q})}$.

$$\begin{aligned}
\|v^k - u\|^2 &= \|y^k - u\|^2 + \gamma_k^2 \big\|\mathscr{F}^*\big(z^k - \mathscr{F} y^k\big)\big\|^2 \\
&\quad + \gamma_k \Big(\big\|P_{\mathscr{Q}}\big(I^{\mathscr{H}_2} - \lambda \mathscr{B}\big)\mathscr{F} y^k - P_{\mathscr{Q}}\big(I^{\mathscr{H}_2} - \lambda \mathscr{B}\big)\mathscr{F} u\big\|^2 \\
&\quad - \big\|\mathscr{F} y^k - \mathscr{F} u\big\|^2 - \big\|z^k - \mathscr{F} y^k\big\|^2 \Big) \\
&\le \|y^k - u\|^2 + \gamma_k^2 \big\|\mathscr{F}^*\big(z^k - \mathscr{F} y^k\big)\big\|^2 \\
&\quad + \gamma_k \Big(\big\|\mathscr{F} y^k - \mathscr{F} u\big\|^2 - \big\|\mathscr{F} y^k - \mathscr{F} u\big\|^2 - \big\|z^k - \mathscr{F} y^k\big\|^2 \Big) \\
&= \|y^k - u\|^2 + \gamma_k^2 \big\|\mathscr{F}^*\big(z^k - \mathscr{F} y^k\big)\big\|^2 - \gamma_k \big\|z^k - \mathscr{F} y^k\big\|^2 \\
&\le \|y^k - u\|^2 + \rho_k^2 \frac{\big\|z^k - \mathscr{F} y^k\big\|^4}{\big(\big\|\mathscr{F}^*(z^k - \mathscr{F} y^k)\big\|^2 + \kappa_k\big)^2} \big(\big\|\mathscr{F}^*\big(z^k - \mathscr{F} y^k\big)\big\|^2 + \kappa_k\big) \\
&\quad - \rho_k \frac{\big\|z^k - \mathscr{F} y^k\big\|^4}{\big\|\mathscr{F}^*(z^k - \mathscr{F} y^k)\big\|^2 + \kappa_k} \\
&= \|y^k - u\|^2 - \rho_k(1-\rho_k) \frac{\big\|z^k - \mathscr{F} y^k\big\|^4}{\big\|\mathscr{F}^*(z^k - \mathscr{F} y^k)\big\|^2 + \kappa_k} \qquad (2.3) \\
&\le \|y^k - u\|^2. \qquad (2.4)
\end{aligned}$$

Now we claim that $\lim_{n\to\infty} \|x^k - u^*\| = 0$, where u^* is the unique solution of the VIP$(I^{\mathscr{H}_1} - \mathscr{T}, \Omega_{\text{SVIP}})$, that is, $u^* = P_{\Omega_{\text{SVIP}}} \mathscr{T} u^*$. Indeed, from the convexity of $\|.\|^2$, Step 4 in Algorithm 1, (2.1), (2.3) with u replaced by u^*, we get

$$\begin{aligned}
\|x^{k+1} - u^*\|^2 &= \big\|\alpha_k(\mathscr{T} x^k - u^*) + (1-\alpha_k)(v^k - u^*)\big\|^2 \\
&\le \alpha_k \big\|\mathscr{T} x^k - u^*\big\|^2 + \|x^k - u^*\|^2 - \rho_k(1-\rho_k) \frac{\big\|z^k - \mathscr{F} y^k\big\|^4}{\big\|\mathscr{F}^*(z^k - \mathscr{F} y^k)\big\|^2 + \kappa_k} \\
&\quad - \beta_k(1-\beta_k) \big\|x^k - P_{\mathscr{C}}\big(I^{\mathscr{H}_1} - \lambda \mathscr{A}\big)x^k\big\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}\rho_k(1-\rho_k)&\frac{\left\|z^k-\mathscr{F}y^k\right\|^4}{\left\|\mathscr{F}^*(z^k-\mathscr{F}y^k)\right\|^2+a_k}\\ &\quad+\beta_k(1-\beta_k)\left\|x^k-P_{\mathscr{C}}\left(I^{\mathscr{H}_1}-\lambda\mathscr{A}\right)x^k\right\|^2\\ &\leq\left(\|x^k-u^*\|^2-\|x^{k+1}-u^*\|^2\right)+\alpha_k\left\|\mathscr{T}x^k-u^*\right\|^2. \quad (2.5)\end{aligned}$$