Appendix A
Group Theory

This appendix is a survey of only those topics in group theory that are needed to understand the composition of symmetry transformations and its consequences for fundamental physics. It is intended to be self-contained and covers those topics that are needed to follow the main text. Although in the end this appendix became quite long, a thorough understanding of group theory is possible only by consulting the appropriate literature in addition to this appendix. In order that this book not become too lengthy, proofs of theorems were largely omitted; again I refer to other monographs. From its very title, the book by H. Georgi [206] is the most appropriate if particle physics is the primary focus of interest. The book by G. Costa and G. Fogli [100] is written in the same spirit. Both books also cover the necessary group theory for grand unification ideas. A very comprehensive but also rather dense treatment is given by [418]. Still a classic is [248]; it contains more about the treatment of dynamical symmetries in quantum mechanics.

A.1 Basics

A.1.1 Definitions: Algebraic Structures

From the structureless notion of a set, one can successively generate more and more algebraic structures. Those that play a prominent role in physics are defined in the following.

Group

A *group* $\mathbf{G}$ is a set with elements g_i and an operation $\circ$ (called group multiplication) with the properties that (i) the operation is closed: $g_i \circ g_j \in \mathbf{G}$, (ii) a neutral element $g_0 \in \mathbf{G}$ exists such that $g_i \circ g_0 = g_0 \circ g_i = g_i$, (iii) for every g_i exists an inverse element $g_i^{-1} \in \mathbf{G}$ such that $g_i \circ g_i^{-1} = g_0 = g_i^{-1} \circ g_i$, (iv) the operation is associative: $g_i \circ (g_j \circ g_k) = (g_i \circ g_j) \circ g_k$. The closure property can be expressed

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Table A.1 The group D_4

	1	i	-1	-i	C	Ci	-C	-Ci
1	1	i	-1	-i	C	Ci	-C	-Ci
i	i	-1	-i	1	-Ci	C	Ci	-C
-1	-1	-i	1	i	-C	-Ci	C	Ci
-i	-i	1	i	-1	Ci	-C	-Ci	C
C	C	Ci	-C	-Ci	1	i	-1	-i
Ci	Ci	-C	-Ci	C	-i	1	i	-1
-C	-C	-Ci	C	Ci	-1	-i	1	i
-Ci	-Ci	C	Ci	-C	i	-1	-i	1

by considering the operation as a mapping

$$\circ : G \times G \to G.$$

There is indeed only one neutral element and the inverse of a group element is unique, as easily can be shown.

A group is called *finite* if the number of its elements (also called the *order* of the group) is finite. An example of a finite group is the rotations of a square by $90°$ (in mathematics and crystallography called $\mathbf{C}_4$; a group of order 4). Examples of infinite groups are the integers with respect to addition (countably infinite) and the symmetry group of the circle, characterized by a parameter φ $[0 \leq \varphi < 2\pi]$ (continuous group).

The *center of a group* is defined as the set of those elements of $\mathbf{G}$ which commute with all other elements of the group:

$$\mathrm{Cent}(\mathbf{G}) = \{c \in \mathbf{G} \mid cg = gc \quad \text{for all} \quad g \in \mathbf{G}\}.$$

Example: From the multiplication table of $\mathbf{D}_4$ (see Table A.1; more about this group later), one can see that its center is given by the set $\{1, -1\}$.

A group is called *Abelian*[1] if the operation is commutative: $g_i \circ g_j = g_j \circ g_i$. Obviously the center of an Abelian group is the group itself.

The symmetry group $\mathbf{C_4}$ of the square is Abelian; the analogous group of the cube is an example of a non–Abelian group. In physics we often deal with groups (or subgroups of) $\mathbf{GL}(n, \mathbb{F})$. This is the set of non–singular $n \times n$ matrices with elements from the field $\mathbb{F}$ which are in the majority of cases real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. $\mathbf{GL}(n, \mathbb{F})$ is in general a non–Abelian group with matrix multiplication as the group operation. The restriction to non–singular matrices is necessary because otherwise the inverse would not be defined.

A subset of $\mathbf{G}$, which itself is a group, is called a *subgroup* of $\mathbf{G}$. By this definition, every group has two trivial subgroups, namely itself ($\mathbf{G} \subset \mathbf{G}$) and the set consisting of the neutral element ($\{g_0\} \subset \mathbf{G}$).

Although in the context of symmetries in fundamental physics, continuous group are the most important ones, many notions from group theory will subsequently

[1] This term derives from the name of the Norwegian mathematician Niels Henrik Abel (1802–1829). He is treated with some disrespect by the fact that contemporary theoreticians use his name in a negated form, by talking for instance of “non–Abelian” gauge theories.

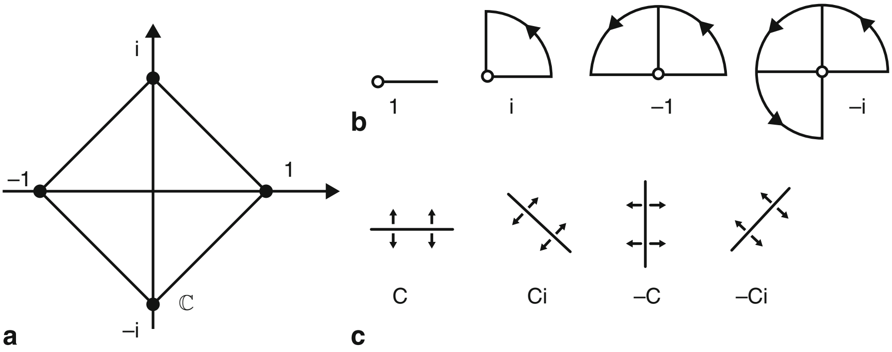

Fig. A.1 Symmetries of the square

be explained on a finite group. This is the group $\mathbf{D}_4$, defined by its multiplication table above. The notation of its elements is taken from [400]. $\mathbf{D}_4$ is a group of order 8, its neutral element is (1), and it is non–Abelian (for instance $(Ci)C = -i$, but $C(Ci) = i$). This abstract group is realized by 90°-rotations of the square and reflections along its diagonals; see Fig. A.1. Here the vertices of the square are represented by the points 1, i, –1, –i. The group $\mathbf{D}_4$ has various nontrivial subgroups; these are

- order 2: rotations by $180°$ realized by $\mathbf{C_2} = \{1, -1\}$, and four reflections realized by $\{1, C\}$, $\{1, Ci\}$, $\{1, -C\}$, $\{1, -Ci\}$.
- order 4: rotations by $90°$ realized by $\mathbf{C_4} = \{1, i, -1, -i\}$, reflections along diagonals, that is $\{1, -1, C, -C\}$, and reflections along the two parallels to the edges of the square: $\{1, -1, Ci, -Ci\}$.

Field

A *field* is a set $\mathbb{F}$ with elements f_i together with two operations

$$+ : \ \mathbb{F} \times \mathbb{F} \to \mathbb{F} \qquad\qquad * : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$

with the properties that (i) $\mathbb{F}$ is an Abelian group under the operation $+$ with f_0 as the identity, (ii) $\mathbb{F} \setminus \{f_0\}$ is a group under $*$, and (iii) there is a distributive law

$$f_i * (f_j + f_k) = f_i * f_j + f_i * f_k, \qquad\qquad (f_i + f_j) * f_k = f_i * f_k + f_j * f_k.$$

Prominent examples of fields are the real and the complex numbers under the usual addition and multiplication operations. Further fields are known, as for instance quaternions (building a 4–dimensional number system and having some link to Dirac matrices), octonions, sedenions, ... the latter bearing a relation to the exceptional Lie groups.

Vector Space

A (linear) *vector space* consists of a set V with elements $\mathbf{v}_i$ and a field $\mathbb{F}$ together with two operations

$$+: \qquad V \times V \to V \qquad\qquad \star : \mathbb{F} \times V \to V$$

with the properties that (i) $(V,+)$ is an Abelian group, (ii) the operation $\star$ fulfills

$$f_i \star (f_j \star \mathbf{v}_k) = (f_i * f_j) \star \mathbf{v}_k$$
$$1 \star \mathbf{v}_i = \mathbf{v}_i = \mathbf{v}_i \star 1$$
$$f_i \star (\mathbf{v}_j + \mathbf{v}_k) = f_i \star \mathbf{v}_j + f_i \star \mathbf{v}_k, \qquad (f_i + f_j) \star \mathbf{v}_k = f_i \star \mathbf{v}_k + f_j \star \mathbf{v}_k.$$

Further definitions:

- A set of vectors $\mathbf{v}_k$ is called *linearly dependent* iff $f_j \in \mathbb{F}$ exist such that $\sum\limits_k f_k \, \mathbf{v}_k \equiv 0$.
- The *dimension* D of a vector space is defined as the maximal number of linearly independent vectors.
- In every vector space of dimension D one can find a set of basis vectors $\mathbf{e}_k$ which are linearly independent and which make it possible to expand any vector as $\mathbf{v} = \sum\limits^{D} f_j \mathbf{e}_j$.

Prominent examples of vector spaces are the Euclidean space, the solution space of linear differential and integral operators, the sets of $N \times M$–matrices, and Hilbert spaces.

Algebra

A (linear) *algebra* A consists of a set V with elements $\mathbf{v}_i$, a field $\mathbb{F}$ together with three operations $+, \star, \Box$. The algebra is a vector space with respect to $(V, \mathbb{F}, +, \star)$. The $\Box$ is a mapping $\Box : V \times V \to V$ with

$$(\mathbf{v}_i + \mathbf{v}_j)\Box\mathbf{v}_k = \mathbf{v}_i\Box\mathbf{v}_k + \mathbf{v}_j\Box\mathbf{v}_k$$
$$\mathbf{v}_i\Box(\mathbf{v}_j + \mathbf{v}_k) = \mathbf{v}_i\Box\mathbf{v}_j + \mathbf{v}_i\Box\mathbf{v}_k.$$

Prominent examples are *Lie algebras*, which aside from the algebra properties additionally fulfill

$$\mathbf{v}_i\Box(a\mathbf{v}_j + b\mathbf{v}_k) = a\mathbf{v}_i\Box\mathbf{v}_j + b\mathbf{v}_i\Box\mathbf{v}_j, \qquad a, b \in \mathbb{F}$$
$$\mathbf{v}_i\Box\mathbf{v}_j = -\mathbf{v}_j\Box\mathbf{v}_i$$
$$\mathbf{v}_i\Box(\mathbf{v}_j\Box\mathbf{v}_k) = (\mathbf{v}_i\Box\mathbf{v}_j)\Box\mathbf{v}_k + \mathbf{v}_j\Box(\mathbf{v}_i\Box\mathbf{v}_k),$$

where the latter relation is the Jacobi identity, as it is known in physics for instance for the Poisson brackets or the generators of bosonic/Grassmann-even[2] symmetry transformations.

A.1.2 Mapping of Groups

Group Homomorphism

Let f be a group mapping $f : \mathbf{G} \to \hat{\mathbf{G}}$. If the mapping respects the group structure, that is if

$$f(g_1) \circ_{\hat{\mathbf{G}}} f(g_2) = f(g_1 \circ_{\mathbf{G}} g_2),$$

(where $\circ_{\mathbf{F}}$ denotes the operation in the respective group) it is called a (group) *homomorphism*. The definition of a homomorphism can be generalized to fields, vector spaces, and algebras as structure–preserving maps between two of these algebraic structures.

The *kernel* of the homomorphism f is defined as the subset of $\mathbf{G}$ for which each element is mapped to the neutral element in $\hat{\mathbf{G}}$:

$$\mathrm{Ker} f := \{g \in \mathbf{G} \mid f(g) = \hat{g}_0 \ \in \hat{\mathbf{G}}\}.$$

The definition of homomorphism does not exclude that $f(g_1) = f(g_2)$ for $g_1 \neq g_2$ (a so-called "many-to-one" relation). For instance, mapping all elements of an arbitrary group to the real number 1 is a homomorphism between the group and the group of real numbers. A "one-to-one" mapping is called an *isomorphism*. Isomorphic groups are not distinguishable in their mathematical structure; one denotes $\mathbf{G} \cong \hat{\mathbf{G}}$. If $\mathbf{G} = \hat{\mathbf{G}}$ the map f is called an *automorphism*. An *inner automorphism* is a mapping f_h such that $f_h \mapsto g^{-1}hg$.

For finite groups an isomorphism between two groups is easily detected in that both groups have identical multiplication tables. In the symmetry group of the square $\mathbf{D}_4$, for example, the group called $\mathbf{C_2}$ has the same multiplication table as all the other subgroups of order two (in the classification by crystallography these are called $\mathbf{D_1}$:

$$\mathbf{C}_2 \cong \mathbf{D}_1 \cong \{1, -1\} \cong \mathbf{Z}_2$$

which also defines the group $\mathbf{Z}_2$, as it occurs in the context of C-,P-, and T-transformations. In general, the groups $\mathbf{C_{2N}}$ are the cyclic groups: If their elements are written as $\{1, a, \ldots, a^{2N-1}\}$ their multiplication table can be generated compactly by the rule $a^n \circ a^m = a^{(n+m) \bmod 2N}$. Therefore, the group of rotations by $90°$ is isomorphic to $\mathbf{C_4}$ (take $a = i$). The two other order–4 subgroups are isomorphic

$$\mathbf{D}_2 \cong \{1, -1, C, -C\} \cong \{1, -1, Ci, -Ci\}.$$

[2] to be explained in B.2.1.

Indeed $\mathbf{C_2}$ is the only abstract group of order two, and $\mathbf{C_4}$ and $\mathbf{D_2}$ are the only groups of order four.

Representation of Groups

In fundamental physics, it is not the symmetry groups themselves that are of primary significance, but–for reasons arising from quantum theory–the "irreducible unitary representations of their central extensions". These terms are defined in Sect. 3 of this appendix, a section entirely devoted to representation theory. A (matrix)–*representation* D is a homomorphism

$$D : \mathbf{G} \to \mathbf{GL}(n, \mathbb{F}),$$

where n is called the *dimension* of the representation.

Example: Rotations of the square, now denoted by $\{r0, r90, r180, r270\}$

- 1-dimensional complex representation of $\mathbf{C}_4$:

$$r0 \ \mapsto 1, \qquad r90 \ \mapsto i, \qquad r180 \ \mapsto -1, \qquad r270 \ \mapsto -i.$$

- 2-dimensional real representation of $\mathbf{C}_4$: Since rotations in a plane can be written as $\vec{x}' = \mathrm{R}\,\vec{x}$ with a matrix

$$\mathrm{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

we get immediately the representation

$$r0 \ \mapsto \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad r90 \ \mapsto \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$r180 \ \mapsto \ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad r270 \ \mapsto \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A.1.3 Simple Groups

It is obvious to ask how many different (that is, not mutually isomorphic) groups there are. A first step towards an answer is the investigation of *simple* groups, where "simple" is to be understood in the sense of "prime": The idea is to factorize a group into its simple subgroups. Before a definition can be given, we need some further group–theoretical terms and definitions.

Equivalence Classes

An *equivalence relation* $\sim$ on a set $M = \{a, b, \ldots\}$ is defined by its properties: (i) $a \sim a$, (ii) $a \sim b \Leftrightarrow b \sim a$, (iii) $a \sim b, b \sim c \Rightarrow a \sim c$. Every equivalence relation on M allows to subdivide the set into *equivalence classes*

$$[a] := \{a \sim b_i \mid a, b_i \in M\},$$

i.e. the class $[a]$ contains all those elements, which are equivalent to a. Thus M is divided in disjunct classes:

$$M = \cup_i \, [a_i] \quad \text{with} \quad [a_i] \cap [a_j] = \emptyset \quad \text{if} \quad a_i \nsim a_j.$$

Cosets

If $\mathbf{H}$ is a subgroup of $\mathbf{G}$, the set

$$g\mathrm{H} := \{g \circ h_i \mid g \in \mathbf{G}, \; h_i \in \mathbf{H}\}$$

is called *left coset*. (A right coset $\mathrm{H}g$ can be defined analogously.) This gives rise to the introduction of an equivalence relation

$$g_i \sim g \;\; \Leftrightarrow g_i \;\; \in g\mathrm{H}.$$

Thus the group G can (as a set) be subdivided in disjunct cosets

$$\mathrm{G} = \cup_1^p \, g_i\mathrm{H} \qquad \text{with} \qquad g_i\mathrm{H} \cap g_j\mathrm{H} = \emptyset \quad \text{for} \quad g_i \neq g_j.$$

Example: Take $\mathbf{H} = \{1, Ci\}$ to be a subgroup of $\mathbf{D}_4$. Because of

$$\begin{aligned} 1\{1, Ci\} &= \{1, Ci\}, & i\{1, Ci\} &= \{i, C\}, \\ -1\{1, Ci\} &= \{-1, -Ci\}, & -i\{1, Ci\} &= \{-i, -C\} \end{aligned}$$

the full set G is representable as $\mathrm{G} = 1\mathrm{H} \;\; \cup i\mathrm{H} \;\; \cup (-1)\mathrm{H} \;\; \cup (-i)\mathrm{H}$.

The *quotient* of a group w.r.t. to one of its left cosets is defined as

$$\mathbf{G}/\mathbf{H} = \{g_1 H, g_2 H, \ldots, g_p H\}.$$

For finite groups, one can show, that if $\mathbf{H}$ is of order q, $\mathbf{G}$ is of order $p * q$. (In the example, the subgroup is of order 2, the number of independent cosets is 4, the full group is of order $8 = 2 * 4$.) Thus the order of a subgroup must be a divisor of the order of the full group, and therefore groups of prime order (2, 3, 5, 7, 11, ...) do not have non–trivial subgroups.

Conjugate Classes and Conjugate Groups

An element $\bar{g} \in \mathbf{G}$ is said to be conjugate to the element g if one can find an element $\tilde{g} \in \mathbf{G}$ such that

$$\tilde{g} g \tilde{g}^{-1} = \bar{g}.$$

This gives rise to another equivalence relation and to a separation of the group in a set of *conjugacy classes*. If the group is Abelian, each element forms a class by itself. The *conjugate group* $\mathbf{H}_{\tilde{g}}$ with respect to a subgroup $\mathbf{H} \subset \mathbf{G}$ and to a group element $\tilde{g} \in \mathbf{G}$ is formed by all elements $\tilde{g} H \tilde{g}^{-1}$.
In the example with $\mathbf{H} = \{1, Ci\}$, we find for instance $\mathbf{H}_{(i)} = \{1, -Ci\}$.

Invariant/Normal Subgroups

If it happens that the conjugate group with respect to $\mathbf{H}$ and all elements in $\mathbf{G}$ are identically the same, $\mathbf{H}$ is called an *invariant* or *self-conjugate* or *normal* subgroup. Definition: $\mathbf{N} \subseteq \mathbf{G}$ is an *invariant* subgroup iff $gN = Ng$ for all $g \in \mathbf{G}$. This can be interpreted as if $\mathbf{N}$ commutes with all $g \in \mathbf{G}$. By this definition (1) the trivial subgroups of $\mathbf{G}$ are invariant subgroups, (2) all subgroups of an Abelian group are invariant subgroups.
If $\mathbf{N}$ is an invariant subgroup, the quotient **G/N** is itself a group, as can be seen from

$$(g_1\mathrm{N}) \circ (g_2\mathrm{N}) = g_1\mathrm{N} \circ \mathrm{N}g_2 = g_1 \circ \mathrm{N}g_2 = (g1 \circ g_2)\mathrm{N}.$$

This group is called *factor group*.
Let us come back to the $\mathbf{D}_4$ example with the subgroup $\mathbf{H} = \{1, Ci\}$. This is not an invariant subgroup, since for instance

$$iH = \{i, iCi\} = \{i, -Ci^2\} = \{i, C\} \qquad \text{but} \qquad Hi = \{i, Ci^2\} = \{i, -C\}.$$

On the other hand $\mathbf{D}_2 = \{1, -1, C, -C\}$ is an invariant subgroup of $\mathbf{D}_4$. The full group can be written in terms of this subgroup as $\mathbf{D}_4 = 1\mathbf{D}_2 \cup i\mathbf{D}_2$. The factor group $\mathbf{D}_4/\mathbf{D}_2 = \{1\mathbf{D}_2, i\mathbf{D}_2\}$ is isomorphic to $\mathbf{Z}_2$ by the mappings $1\mathbf{D}_2 \mapsto 1, i\mathbf{D}_2 \mapsto -1$.

Simple and Semi-Simple Groups

With the previous introduction to subgroup structures we are finally able to define "simple":

(1) A group is *simple* if it has no non-trivial invariant subgroup.
(2) A group is *semi-simple* if it has no Abelian invariant subgroup. In other words: A semi-simple group may have a nontrivial invariant subgroup, but this is not allowed to be Abelian.

All simple finite groups have been known since 1982. All other finite groups are "products" of these in the following sense: A group $\mathbf{G}$ is a *direct product* of its subgroups $\mathbf{A}, \mathbf{B}$ written as $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ if

(1) for all $a_i \in \mathbf{A}, b_j \in \mathbf{B}$, it holds that $a_i \circ b_j = b_j \circ a_i$
(2) every element $g \in \mathrm{G}$ can be uniquely expressed as $g = a \circ b$.

Some consequences of this definition are

- $g_1 \circ g_2 = (a_1 \circ b_1) \circ (a_2 \circ b_2) = (a_1 \circ a_2) \circ (b_1 \circ b_2)$.
- Both $\mathbf{A}$ and $\mathbf{B}$ are invariant subgroups of $\mathbf{G}$, e.g. for $\mathbf{A}$

$$g_i A = (a_i \circ b_i)A = (b_i \circ a_i)A = b_i A \quad \text{and} \quad Ag_i = A(a_i \circ b_i) = Ab_i = b_i A.$$

- $\mathbf{G}$ can be written in terms of the subgroups $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{G} = \cup b_i A = \cup a_i B \qquad \text{suchthat} \qquad \mathbf{G/A} \cong \mathbf{B}, \qquad \mathbf{G/B} \cong \mathbf{A}.$$

- $\mathbf{G}$ is not simple.

For example $\mathbf{D}_2$ can be written as $\mathbf{D_2} = 1\mathbf{C_2} \cup C\ \mathbf{C_2}$, i.e. $\mathbf{D}_2/\mathbf{C}_2 = \mathbf{C}_2$, and also $\mathbf{D}_2 \cong \mathbf{C}_2 \times \mathbf{C}_2$.

Observe that in general a relation $\mathbf{G}_1 = \mathbf{G}_2/\mathbf{G}_3$ does not allow to conclude that $\mathbf{G}_2$ is a direct product of $\mathbf{G}_1$ and $\mathbf{G}_3$.

A.2 Lie Groups

A.2.1 Definitions and Examples

An *r-parameter Lie group*[3] $\mathbf{G}$ is a group which at the same time is an r–dimensional differentiable manifold[4]. This means that the group operations

$$C : \mathbf{G} \times \mathbf{G} \to \mathbf{G}, \qquad C(g, g') = g \circ g'$$

$$I : \mathbf{G} \to \mathbf{G} \qquad I(g) = g^{-1}$$

are differentiable maps. The group elements of a Lie-group have the generic form

$$\mathbf{G} \ni g(\xi^1, \ldots \xi^r) =: g(\xi),$$

where the $\{\xi^j\}$ are (local) coordinates in the group manifold. If the group multiplication is written as $g(\xi) \circ g(\xi') = g(\xi''(\xi, \xi'))$, the Lie group property implicates that the functions $\xi''(\xi, \xi')$ are differentiable[5] in the parameters/coordinates.

All notions and results of abstract group theory as derived in the previous section do hold for Lie groups except those in which reference is made to finiteness and to the order of the group. This is especially true for the notions subgroups, cosets, conjugacy classes, invariant subgroups, quotient groups, etc.

In the following, several examples of Lie groups will be given, many of them of utmost importance for describing symmetries in fundamental physics. Most of these groups are not just abstract structures, but can be interpreted/realized as transformations in geometric spaces; therefore being called transformation groups.

Transformations of the Straight Line

The *translation group* is realized by the set of all displacements

$$T_a : \quad x' = x + a \qquad a \in \mathbb{R}.$$

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[3] named after Sophus Lie (1842–1899), a Norwegian mathematician. Interesting enough, he arrived at these notions by investigating solutions of differential equations and using symmetry arguments in this context. These techniques were mentioned in Sect. 2.2.4.

[4] A Lie group is a subcategory of topological groups, namely as a topological space for which the group operation is a continuous mapping.

[5] Remarkably, due to the group structure the manifold is even analytic.

This is obviously an Abelian group with $T_b \circ T_a = T_{a+b}$. The *dilation group* is defined by its elements

$$D_A : \quad x' = Ax \qquad A \in \mathbb{R}, \qquad A \neq 0$$

again an Abelian group. The condition $A \neq 0$ is imposed to guarantee the invertibility. The group is a local symmetry group if $A \in \mathbb{R}^+$. Both the translations and the dilations may be combined to the one-dimensional coordinate transformations $x' = Ax + a$ constituting a two-parameter continuous group. In writing its elements as (A, a), the group multiplication becomes $(A, a) \circ (A', a') = (AA', a + Aa')$. The neutral element is $(1, 0)$, and the inverse is $(A, a)^{-1} := A^{-1}(1, -a)$. This group is a subgroup of the *linear fractional transformations*

$$x' = \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}} \qquad a_{11}\, a_{22} - a_{12}\, a_{21} \neq 0.$$

The identity transformation is recovered for $a_{12} = 0 = a_{21}$ and $a_{11} = a_{22}$ and we can write this three-parameter subgroup as

$$x' = \frac{a_1 x + a_2}{a_3 x + 1} \qquad a_1 \neq a_2 a_3.$$

This is the *projective group* of the straight line.

Groups in the Plane

The most general transformations mapping any straight line in a plane again into a straight line are

$$x' = \frac{Ax + By + C}{Lx + My + N} \qquad y' = \frac{Dx + Ey + F}{Lx + My + N}.$$

These transformations realize the projective group in the plane. The parameter are not completely independent, because one insists on invertibility. Further, the subgroup containing the identity transformation has $N \neq 0$, and thus there are eight essential parameter. This projective group has as important subgroups, for instance, translations $x' = x + a,\ y' = y + b,$ dilations $x' = e^{\kappa}x,\ y' = e^{\lambda}y,$ general linear transformations $x' = a_{11}x + a_{12}y,\ y' = a_{21}x + a_{22}y$, and among these rotations $x' = x\cos\alpha + y\sin\alpha,\ y' = y\cos\alpha - x\sin\alpha$, and special conformal transformations $x' = x(1 - ax)^{-1},\ y' = y(1 - ax)^{-1}$. A further prominent group of transformation (providing mappings between circles and straight lines) is given by the inversion with respect to a circle of radius R centered for example in the origin;

$$I_R : \qquad x' = R^2 \frac{x}{r^2} \qquad y' = R^2 \frac{y}{r^2} \qquad \text{with} \qquad r^2 = x^2 + y^2.$$

From inversions I_1 and translations T_a (along the x-direction, say) one can build conformal transformations:

$$C_a = I_1 T_a I_1 : \qquad x' = \frac{x + ar^2}{1 + 2ax + a^2r^2} \qquad y' = \frac{y}{1 + 2ax + a^2r^2}.$$

Matrix Groups

The Lie group $\mathbf{GL}(\mathbf{n},\mathbb{F})$ is given by the set of all nonsingular $(n\times n)$ matrices with values in the field $\mathbb{F}$

$$A=\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \cdot & & \cdot \\ a_{n1} & \dots & a_{n1} \end{pmatrix}, \qquad a_{ik}\in\mathbb{F}, \qquad \det A\neq 0,$$

and the group operation realized as matrix multiplication. If there are no further restrictions on the matrices, $\mathbf{GL}(n,\mathbb{R})$ is an n^2 parameter and $\mathbf{GL}(\mathbf{n},\mathbb{C})$ a $2n^2$ parameter group. These matrices describe linear transformations in an n-dimensional real or complex vector space as: $z_i' = \sum_k a_{ik}z_k$. The matrices with positive determinant constitute the Lie subgroup $\mathbf{SL}(\mathbf{n},\mathbb{F})$.

In fundamental physics, especially regarding symmetries, the matrix groups given below are of major relevance:

- *orthogonal groups*
  The group $\mathbf{O}(\mathbf{n})\subset\mathbf{GL}(\mathbf{n},\mathbb{R})$ contains all matrices for which
  $$AA^T=\mathbf{1}, \quad \text{i.e.} \quad A_{ik}A_{kj}=\delta_{ij}$$
  where $(A^T)_{ik}=A_{ki}$ is the transposed matrix of A. Because of this restriction, $\mathbf{O}(\mathbf{n})$ is an $\frac{n(n-1)}{2}$-parameter group. $\mathbf{O}(\mathbf{n})$ characterizes the symmetry of the sphere, with n real coordinates, in the sense that the "radius"
  $$x_1^2+x_2^2+\ldots+x_n^2$$
  is invariant with respect to linear transformations with matrices from $\mathbf{O}(\mathbf{n})$.
  A special case is the subgroup $\mathbf{SO}(\mathbf{n})$, which contains all matrices of $\mathbf{O}(\mathbf{n})$ with $\det A=1$. Since as a set, $\mathbf{O}(\mathbf{n})$ has elements with either $\det A=1$ or $\det A=-1$ (this follows from the defining condition) it can be understood as the quotient
  $$\mathbf{O}(\mathbf{n})=\mathbf{SO}(\mathbf{n})/\mathbf{Z}_2.$$
  As will be seen later, from the classification of groups (or rather, their associated algebras) one needs to distinguish even and odd n, because these differ in their spinor representations.
- *unitary groups*
  The group $\mathbf{U}(\mathbf{n})\subset\mathbf{GL}(\mathbf{n},\mathbb{C})$ contains all matrices for which
  $$AA^\dagger=\mathbf{1} \qquad \text{with} \qquad A^\dagger:=(A^T)^*.$$
  $\mathbf{U}(\mathbf{n})$ has n^2 parameter. It is the invariance group of the sphere with n complex coordinates. A special case is the (n^2-1)-parameter group $\mathbf{SU}(\mathbf{n})$, in which all matrices have determinants equal to 1. It leaves invariant the real quadratic form
  $$z_1z_1^*+z_2z_2^*+\ldots+z_nz_n^*$$
  where the z_i are complex variables.

- *pseudo-orthogonal groups*
  $\mathbf{O(n,m)}$ leaves invariant $\sum_{ij} \eta_{ij} x^i x^j$, with

$$\eta_{ij} = \delta_{ij} \quad \text{for } i = 1, \ldots, n \qquad\qquad \eta_{ij} = -\delta_{ij} \quad \text{for } i = n+1, \ldots, m.$$

  The further restriction on uni-modularity $(\det A = 1)$, leads to the $\mathbf{SO(n,m)}$ groups. Prominent examples in fundamental physics are the Lorentz group $\mathbf{SO(3,1)}$, the de Sitter group $\mathbf{SO(4,1)}$ and the conformal group $\mathbf{SO(4,2)}$.
- *symplectic groups*
  Matrices of the group $\mathbf{Sp(2\ m)}$ are those elements A of the general linear groups which obey the condition

$$A^T \Gamma A = \Gamma, \qquad \Gamma := \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}.$$

  The transformations generated by $\mathbf{Sp(2\ m)}$ leave the real antisymmetric quadratic form $\sum x_i \Gamma_{ij} y_j$ invariant. These groups typically arise in the phase-space formulation of physical systems, for instance in defining canonical transformations.

Inhomogeneous Linear Transformations

The transformation $x' = Ax + a$ is a mixture of a linear transformation (mediated by a real number A) and a translation (described by the parameter a) in one dimension. This can be generalized to higher dimensions: Let $\mathbf{G} = \mathbf{GL(n}, \mathbb{R}) \ni A$, and $\mathbb{R}^n \ni a$ be the n-dimensional real space with addition of vectors. Define

$$\mathbf{IG} := \{(A, a) |\ A \in \mathbf{G},\ a \in \mathbb{R}^n\}$$

with the group operation

$$(A, a) \circ (A', a') := (AA', a + Aa').$$

One discovers $\mathbf{IG} \subset \mathbf{GL(n+1}, \mathbb{R})$ by the identification

$$(A, a) \mapsto \begin{pmatrix} A & a \\ 0^T & 1 \end{pmatrix}.$$

Another notation is $\mathbf{IG} = \mathbf{G} \ltimes \mathbb{R}^n$ in terms of the *semi-direct product* $\ltimes$.

Groups of Motion

The *group of motion* of a geometry, also called *isometry group* is a transformation group that leaves distances invariant. In one dimension this is just the translation group. For the Euclidean plane it is the group of inhomogeneous linear transformations composed out of rotations and translations. It is the largest group that changes only the location and orientation of a figure (triangle, circle, etc.) but not its magnitude and shape. For $\mathbb{E}^3$ the transformations of its group of motion have the form $\vec{x}' = O(\vec{x}) + \vec{a}$ with an $\mathbf{SO(3)}$ matrix O. For Minkowski space it is the Poincaré group as the semi–direct product of the Lorentz group with the group of spacetime translations.

A.2.2 Generators of a Lie Group

The generators of a Lie group may be defined through the tangent space of the group manifold near the neutral element g_0. One can always choose a chart with coordinates $\{\xi^i\}$ such that $g(\xi = 0) = g_0$. A Taylor expansion for the group elements near the identity is of the form

$$g(\xi) = g_0 + i\xi^a X^a + \mathcal{O}(\xi^2) \qquad \xi \in \mathbb{R}.$$

The X^a are called the infinitesimal *generators* of the Lie group[6]. Clearly,

$$X^a = -i\frac{\partial g}{\partial \xi^a}\bigg|_{\xi=0}. \tag{A.1}$$

In some cases, it is possible to construct the complete group from its infinitesimal generators. Consider for instance a one-parameter Lie group. Because of the analyticity of $g(\xi)$, it is possible to choose a parametrization in which $\xi''(\xi, \xi') = \xi + \xi'$. We may now attempt to construct finite group elements by making an infinite series of infinitesimally small steps away from the group identity:

$$g(\xi) = g(\xi/n)^n = \lim_{n\to\infty}\left(1 + \frac{i\xi X}{n}\right)^n = \exp(i\xi X).$$

This can be extended to an r-parameter Lie group if it is Abelian

$$g(\xi^1, \ldots, \xi^r) = \exp(i\xi^a X^a). \tag{A.2}$$

In case of a non-Abelian Lie group the expression (A.2) is generically not valid, but is only valid near the identity; otherwise one needs to invoke the Campbell-Baker-Hausdorff formula, see (A.6) below.

As an example consider again the 2-parameter group of coordinate transformations $x \to (1 + \xi^1)x + \xi^2$. A function $f(x)$ changes into

$$f(x) \to f((1 + \xi^1)x + \xi^2) = f(x) + \xi^1 x\frac{d}{dx}f(x) + \xi^2\frac{d}{dx}f(x) + \mathcal{O}(\xi^2)$$

---

[6] Singling out a factor i here is merely a convention. This convention is chosen throughout this book, since from the point of view of quantum physics the generators X are to be interpreted as representations of Hermitean operators.

from which we read the two infinitesimal generators

$$X^1 = -ix\frac{d}{dx},\ X^2 = -i\frac{d}{dx}.$$

Taking the commutator of these generators, we calculate

$$(X^1X^2 - X^2X^1)\ f(x) = -(x\frac{d}{dx}\frac{d}{dx} - \frac{d}{dx}x\frac{d}{dx})f(x) = \frac{d}{dx}f(x) = iX^2\ f(x).$$

The commutator of the two infinitesimal generators is again an infinitesimal generator.

As a second example, consider rotations in a plane, characterized by an angle θ. In polar coordinates, a clockwise rotation amounts to

$$\begin{aligned} x = r\cos\theta \to &\quad r\cos(\theta-\xi) = r\cos\theta + \xi r\sin\theta + \mathcal{O}(\xi^2) = x + \xi y \\ y = r\sin\theta \to &\quad r\cos(\theta-\xi) = r\sin\theta - \xi r\cos\theta + \mathcal{O}(\xi^2) = y - \xi x. \end{aligned}$$

Thus the generator can be written as a 2×2 matrix

$$X = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.3}$$

with $X^2 = I$. The exponential map (A.2) gives

$$\begin{aligned} g(\xi) &= \sum_{n=0}^{\infty}\frac{1}{n!}\xi^n(iX)^n = \sum_{n=0}^{\infty}\left\{\frac{1}{2n!}(-\xi^2)^nI + \frac{1}{(2n+1)!}(-1)^n\xi^{2n+1}iX\right\} \\ &= \cos\xi I + i\sin\xi X = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \end{aligned}$$

which indeed is simply a two-dimensional rotation.

This can be extended to rotations in three dimensions: Think of the infinitesimal rotation (A.3) as being a rotation in the x-y plane around the z-axis with an infinitesimal angle $\xi^3 = \xi$. Any rotation in three dimensions can be constructed from rotations around the x-, y-, and z-axis with angles (ξ^1, ξ^2, ξ^3), according to

$$\vec{r} \to \vec{r} + \vec{r}\times\vec{\xi} + \mathcal{O}(|\xi|^2) = \vec{r} + i\xi^aX^a\vec{r}$$

$$X^1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} X^2 = -i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} X^3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.4}$$

A.2.3 Lie Algebra Associated to a Lie Group

The infinitesimal generators of a Lie group are local representatives of the group elements. So far, we have not investigated how the defining attributes of a group become manifest in features of the generators. Firstly, because of the closure property of a group, there must exist a relation

$$
\begin{aligned}
g(\xi^1, \ldots, \xi^r) \circ g(\bar{\xi}^1, \ldots, \bar{\xi}^r) &= \exp(i\xi^a X^a) \circ \exp(i\bar{\xi}^a X^a) \\
&= \exp(i\bar{\bar{\xi}}^a X^a) = g(\bar{\bar{\xi}}^1, \ldots, \bar{\bar{\xi}}^r).
\end{aligned} \tag{A.5}
$$

Now the product of the two exponentials can be expressed by the *Campbell-Baker-Hausdorff* formula:

$$
\begin{aligned}
e^A e^B &= e^{A+B+C(A,B)} \\
C(A,B) &= \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \ldots.,
\end{aligned} \tag{A.6}
$$

where the dots indicate the possible presence of higher-order commutators of A and B. Therefore, group closure (A.5) demands that the commutator of two matrices/generators is itself a generator:

$$
[X^a, X^b] = X^a X^b - X^b X^a = i f^{abc} X^c \qquad f^{abc} \in \mathbb{R}. \tag{A.7}
$$

The constants f^{abc} are called the *structure constants* of the group. Because of the antisymmetry of the commutator these are of course antisymmetric in their first two indices:

$$
f^{abc} = -f^{bac}.
$$

A further identity among the structure constants originates from the commutator identity

$$
[[A,B],C] + [[B,C],A] + [[C,A],B] \equiv 0
$$

resulting in the *Jacobi identity*

$$
\sum_e (f^{abe} f^{ecd} + f^{bce} f^{ead} + f^{cae} f^{ebd}) \equiv 0. \tag{A.8}
$$

Given these relations, one can easily see that the infinitesimal generators of a Lie group constitute a Lie algebra (in the sense introduced in Sect. A.1.1) if the $\Box$ operator is realized by the commutator, that is if $v_1 \Box v_2 := [v_1, v_2]$.

For an Abelian (sub)group all the structure coefficients vanish. Since for an invariant subgroup with generators Y^b, it holds that

$$
\exp(i\xi^a X^a) \circ \exp(i\xi^b Y^b) = \exp(i\xi^{b'} Y'^{b'}) \circ \exp(i\xi^a X^a)
$$

the Campbell–Baker–Hausdorff formula shows that

$$
[X^a, Y^b] = f^{abb'} Y^{b'}. \tag{A.9}
$$

The Lie algebra associated to a Lie group $\mathbf{G}$ is by convention denoted by $\mathfrak{g}$. Lie groups with the "same" Lie algebra are locally isomorphic, that is isomorphic near the neutral element:

$$\mathfrak{g} = \hat{\mathfrak{g}} \qquad \Leftrightarrow \qquad \mathbf{G} \cong \hat{\mathbf{G}} \mid_{g_0} .$$

Be aware that in giving meaning to "same", linear combinations $X'^a = \Gamma^{ab} X^b$ of generators are again generators. The algebra now reads

$$[X'^a, X'^b] = f'^{abc} X'^c,$$

where the new structure constants f'^{abc} are linear combinations of the old ones:

$$f'^{abc} = \Gamma^{ad} \Gamma^{be} \, f^{deg} \, (\Gamma^{-1})^{gc}. \tag{A.10}$$

This might mimic another algebra. The techniques to identify all non–isomorphic Lie algebras make use of 'root spaces' and 'Dynkin diagrams', techniques that will not be covered here.

Examples

- Lie algebra of $\mathbf{SO(3)}$
  The Lie algebra is directly derived from the explicit representation of the generators as given by (A.4):
  $$[X^a, X^b] = i\epsilon^{abc} X^c. \tag{A.11}$$
  This is the same as the algebra of the three angular momenta $\{J^1, J^2, J^3\}$, as we expect them to arise from rotational symmetry.
- Lie algebra of $\mathbf{SU(2)}$
  $\mathbf{SU(2)}$ is a three-dimensional group. The requirement that the group elements $g(\xi) = \exp(i\xi^a X^a)$ are unitary matrices with determinant $+1$ demands that
  $$X^a = (X^a)^\dagger \qquad \text{and} \qquad \mathrm{tr}(X^a) = 0.$$
  In the physical literature, it is common to choose $X^a = \frac{1}{2}\sigma^a$ with the *Pauli matrices*
  $$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.12}$$
  Since the Pauli matrices do have the property $\sigma^a \sigma^b = i\epsilon^{abc}\sigma^c + \delta^{ab} I$, the $\mathfrak{su}(2)$ commutators are
  $$[X^a, X^b] = i\epsilon^{abc} X^c.$$

This is identical to the $\mathfrak{so}(3)$ algebra (A.11). Therefore the Lie groups $\mathbf{SU(2)}$ and $\mathbf{SO(3)}$ are locally isomorphic. Globally, however, both groups differ, and in A.3.6 $\mathbf{SU(2)}$ will be identified as the 'universal covering group' of $\mathbf{SO(3)}$. This mathematical subtlety becomes physically manifest in the existence of half-integer spin particles, as explained in Sect. 5.3. Here, we keep in mind the representation of $\mathbf{SU(2)}$ group elements in terms of the Pauli matrices as

$$g(\vec{\alpha}) = \exp\left(\frac{i}{2}\vec{\sigma}\cdot\vec{\alpha}\right). \tag{A.13}$$

- Lie algebra of $\mathbf{SO(2,1)}$
  If one determines the $\mathfrak{so}(2,1)$ according to the previously specified recipe, one finds

$$[X^1, X^2] = iX^3, \qquad [X^2, X^3] = -iX^1, \qquad [X^3, X^1] = iX^2.$$

  There is no choice of linear combinations of these X^a by which one regains the algebra of the rotation group. Thus $\mathbf{SO(2,1)}$ and $\mathbf{SO(3)}$ are different as groups. But one would also find that $\mathfrak{so}(2,1) \equiv \mathfrak{so}(1,2)$ and thus–at least locally–$\mathbf{SO(2,1)} \cong \mathbf{SO(1,2)}$.
- Lie algebra of the 2-dimensional Poincaré group
  The group elements of the Poincaré group in one spatial and one temporal coordinate are parametrized through the rapidity η (characterizing the Lorentz boost, see Sect. 3.1.2) and the constant shifts (a,τ) in spacetime:

$$g(\eta, a, \tau) = \begin{pmatrix} \cosh\eta & \sinh\eta & a \\ \sinh\eta & \cosh\eta & \tau \\ 0 & 0 & 1 \end{pmatrix} \tag{A.14}$$

  from which by (A.1) we obtain the generators

$$X^\eta = -i\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^a = -i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^\tau = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

  These obey the $\mathfrak{iso}(2,1)$ algebra

$$[X^\eta, X^a] = iX^\tau, \qquad [X^\eta, X^\tau] = iX^a, \qquad [X^a, X^\tau] = -iX^a.$$

- $\mathfrak{so}(\mathfrak{n},\mathfrak{m})$
  The algebra of the pseudo-orthogonal group $\mathbf{SO(n,m)}$ with $(n+m=D)$ is spanned by $\frac{D(D-1)}{2}$ generators $M^{AB} = -M^{BA}$, where the indices A, B range from 1 to D. The Lie algebra $\mathfrak{so}(\mathfrak{n},\mathfrak{m})$ is

$$[M^{AB}, M^{CD}] = i(\eta^{BC}M^{AD} - \eta^{BD}M^{AC} + \eta^{AD}M^{BC} - \eta^{AC}M^{BD}). \tag{A.15}$$

Here $\eta^{AB} = \delta^{AB}$ for $(A = 1, \ldots, n)$, $\eta^{AB} = -\delta^{AB}$ for $(A = n+1, \ldots, m)$ (all others vanishing). In coordinate space (that is as pseudo-rotations) the generators can be realized in the form

$$M^{AB} = i(x^A \partial^B - x^C \partial^A). \tag{A.16}$$

Important examples (in D=4) are the Lorentz algebra $\mathfrak{so}(3,1)$, the de Sitter algebra $\mathfrak{so}(4,1)$, the anti-de Sitter algebra $\mathfrak{so}(3,2)$, and the conformal algebra $\mathfrak{so}(4,2)$.

A.2.4 Inönü–Wigner Contraction of Lie Groups

"The process of group contraction is a method for obtaining from a given Lie group another, non-isomorphic Lie group through a limit operation." (from [281]). This was originally investigated by E. Inönü and E. Wigner [282], motivated by the desire to establish a mathematically sound derivation of the Galilei algebra from the Poincaré algebra (see Sect. 3.4). They found that every Lie group can be contracted with respect to any of its Lie subgroups–and only with respect to these. Technically one can describe the construction of the contracted group in terms of local algebras.

Consider a Lie algebra L with a Lie subalgebra L'. Assume that $\{X_1, \ldots, X_n\}$ is a basis in L, and $\{X'_1, \ldots, X'_{n'}\}$ is a basis in L'. The Lie algebra is explicitly of the form[7]

$$\begin{aligned}
[X_a, X_b] &= i f_{ab}^{\ \ c} X_c \ + \ i f_{ab}^{\ \ c'} X'_{c'} \\
[X_a, X'_{b'}] &= i f_{ab'}^{\ \ c} X_c \ + \ i f_{ab'}^{\ \ c'} X'_{c'} \\
[X'_{a'}, X'_{b'}] &= i f_{a'b'}^{\ \ c'} X'_{c'}.
\end{aligned}$$

where the latter reflects that the X' span a subalgebra. Define rescaled generators as

$$Y_a := \epsilon X_a, \qquad Y'_{a'} := X'_{a'}$$

which fulfill the algebra

$$\begin{aligned}
[Y_a, Y_b] &= \epsilon^{-1} i f_{ab}^{\ \ c} Y_c \ + \ \epsilon^{-2}\, i f_{ab}^{\ \ c'} Y'_{c'} \\
[Y_a, Y'_{b'}] &= i f_{ab'}^{\ \ c} Y_c \ + \ \epsilon^{-1} i f_{ab'}^{\ \ c'} Y'_{c'} \\
[Y'_{a'}, Y'_{b'}] &= i f_{a'b'}^{\ \ c'} Y_{c'}.
\end{aligned}$$

[7] Here, I prefer to write the infinitesimal generators with lower indices and the structure constants with mixed indices. This is only to make the expressions more comprehensible.

Now the limit $\epsilon \to 0$ can be taken only if all $f_{ab}^{\ \ c}, f_{ab}^{\ \ c'}, f_{ab'}^{\ \ c'}$ vanish identically. This results in the contracted algebra L_c:

$$[Y_a, Y_b] = 0 \tag{A.17a}$$

$$[Y_a, Y'_{b'}] = i f_{ab'}^{\ \ c} Y_c \tag{A.17b}$$

$$[Y'_{a'}, Y'_{b'}] = i f_{a'b'}^{\ \ c'} Y'_{c'}. \tag{A.17c}$$

Looking at the associated Lie groups, we see that if a Lie group $\mathbf{G}$ is contracted with respect to one of its Lie subgroups $\mathbf{G}'$, one arrives at a contracted Lie group $\mathbf{G_c}$. This group has the same dimension as its "mother" group. $\mathbf{G_c}$ contains a subgroup which is isomorphic to $\mathbf{G}'$ and another Abelian subgroup $\bar{\mathbf{G}}$ spanned by the contracted generators X_a. $\bar{\mathbf{G}}$ is an invariant subgroup of $\mathbf{G}'$, and thus $\mathbf{G}' \sim \mathbf{G_c}/\bar{\mathbf{G}}$. Inönü and Wigner also showed that the necessary condition for a group $\mathbf{G_c}$ to be derivable from another group $\mathbf{G}$ by contraction is the existence of an Abelian invariant subgroup in $\mathbf{G_c}$ and the possibility of choosing from each of its cosets an element so that these form a subgroup.

Following [281], the technique and the consequences of group contractions shall be made explicit with two examples:

1. contraction of the 3-dim rotation group $\mathbf{SO(3)}$
   We start from the algebra L of the generators X^a which generate rotations around the x-, y-, and z-axis according to (A.11): $[X^a, X^b] = i\epsilon^{abc}X^c$. Take as L' the generator X^3 for the one-parameter subgroup of rotations around the z-axis. Then with the procedure explained before, that is defining $Y^3 = X^3$, $Y^1 = \epsilon X^1$, $Y^2 = \epsilon X^2$ and taking the limit $\epsilon \to 0$ we obtain the algebra L_c

$$[Y^1, Y^2] = 0 \tag{A.18a}$$

$$[Y^2, Y^3] = iY^1 \tag{A.18b}$$

$$[Y^3, Y^1] = iY^2. \tag{A.18c}$$

   This is isomorphic to the algebra of $\mathbf{E_2}$, the Euclidean group in two dimensions consisting of translations (generators Y^1 and Y^2) and rotations (generator Y^3) in the plane.
2. contraction of the 3-dim Lorentz-group
   In splitting the set of generators in $\mathfrak{so}(3,1)$ as M^{ij} and M^{0i} $(i = 1,2,3)$ one rewrites the algebra (A.15)

$$[M^{ij}, M^{kl}] = i(\eta^{jk}M^{il} - \eta^{jl}M^{ik} + \eta^{il}M^{jk} - \eta^{ik}M^{jl})$$
$$[M^{0j}, M^{kl}] = i(\eta^{jk}M^{0l} - \eta^{jl}M^{0k})$$
$$[M^{0i}, M^{0j}] = iM^{ij}.$$

   Taking for L' the subalgebra generated by the M^{ij} and defining

$$J^i = \frac{1}{2}\epsilon^{ikl}M^{kl}, \qquad G^i = \epsilon M^{0i}$$

Table A.2 Cartan classification

Class	Range	Dimension	Compact group
$\mathbf{A_n}$	$n \geqslant 1$	n(n+2)	$\mathbf{SU(n+1)}$
$\mathbf{B_n}$	$n \geqslant 2$	n(2n+1)	$\mathbf{SO(2n+1)}$
$\mathbf{C_n}$	$n \geqslant 3$	n(2n-1)	$\mathbf{Sp(2n)}$
$\mathbf{D_n}$	$n \geqslant 4$	n(2n-1)	$\mathbf{SO(2n)}$
$\mathbf{E_n}$	$n = 6, 7, 8$	78, 133, 248	$\mathbf{E_n}$
$\mathbf{F_4}$	$n = 4$	52	$\mathbf{F_4}$
$\mathbf{G_2}$	$n = 2$	14	$\mathbf{G_2}$

results in the algebra

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$
$$[J^i, G^j] = i\epsilon^{ijk} G^k$$
$$[G^i, G^j] = 0.$$

This is the algebra of the homogeneous Galilei group made up of three-dimensional rotations and Galilei boosts; compare (2.75).

A.2.5 Classification of Lie Groups

Strangely enough the classification of Lie groups (i.e. the reduction of an arbitrary Lie-group to its "prime" factors) was completed already in 1894 by Élie Cartan, and this was nearly a century earlier then the analogous classification for finite groups, which as mentioned above was completed only in 1982.

Cartan Classification

The classification of all simple Lie-groups by Cartan[8] is as follows: There are four infinite series of groups and five exceptional groups. The series are $\mathbf{A_n}$, $\mathbf{B_n}$, $\mathbf{C_n}$, $\mathbf{D_n}$. The subscript n denotes the *rank* of the group. The groups in the Cartan series are isomorphic to the groups introduced in section A.2.1 as shown in Table A.2. The value n may take on smaller values than the limits indicated in the table, but because of mutual (local) isomorphisms these are not independent of the others given. For example $\mathbf{B_1} \cong \mathbf{A_1}, \mathbf{D_2} \cong \mathbf{B_1} \oplus \mathbf{B_1}, \mathbf{D_3} \cong \mathbf{A_3}$. The exceptional groups are called $\mathbf{G_2}$, $\mathbf{F_4}$, $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$. The transformations generated by these groups typically leave higher than quadratic polynomials invariant. The exceptional groups are indeed every now and then considered in fundamental physics (for instance in "Grand Unified Theories" or in string models). We note that

$$\mathbf{E}_8 \supset \mathbf{E}_7 \supset \mathbf{E}_6 \supset \mathbf{SO(10)}.$$

[8] Actually, the classification was already published by W. Killing some years before. He however had a further exceptional group, which Cartan showed to be isomorphic to $\mathbf{G_2}$. Wilhelm Killing (1847–1923) was a professor of mathematics in Münster, a town in Lower Saxony of Germany. His name also continues to be used in the term 'Killing symmetries', which gives a precise meaning to symmetries of metrics independent of their coordinate expressions.

The classification of Cartan includes both compact and non-compact groups, a topic to be clarified next. Thus, for instance, $\mathbf{A_{n-1}}$ stands for the series of groups $\mathbf{SU(p, n-p)}$.

The problem of finding all possible Lie groups can be transferred to finding the classification of Lie algebras since all group elements that are continuosly connected to the identity can be generated by exponentiation.

Compact Lie Groups

The classification program of Lie groups is largely similar to the classification of finite groups, and it is just the same if the group in question is compact. A Lie group is *compact* if the space of parameters ξ^a is bounded ($u^a \leq \xi^a < o^a$) and closed, that is, the limit of every convergent sequence of parameters is itself a parameter value.

- The rotation group $\mathbf{SO(n)}$ is compact.
- The translation–group $\mathbb{R}^n$ is not compact, since it is not bounded.
- The Lorentz group is not compact, since it is not closed ($v = c$ is not a valid parameter).

Cartan-Killing Metric

An extremely useful concept in the classification and representation theory of Lie algebras is the Cartan-Killing metric, which can be expressed directly by the structure constants as

$$g^{ab} = \sum_{c,d} f^{acd} f^{bdc}. \tag{A.19}$$

The Cartan-Killing metric provides criteria for compactness and semi–simplicity of a Lie algebra:

- A Lie algebra is semi-simple if the metric is nonsingular ($\det g \neq 0$); otherwise the Lie algebra is not semi-simple.
- A Lie algebra is compact if the metric is negative definite; that is if for every $A = \alpha_a X^a$, we have $|A, A| = g^{ab}\, \alpha_a \alpha_b < 0$.

Let us understand these notions by resorting to examples of Lie groups and their Lie algebras which were already discussed before. These are the three-dimensional groups $\mathbf{SO(3)}$, $\mathbf{SO(2,1)}$, $\mathbf{E_2}$. The first two are simple, while $\mathbf{E_2}$ is not semi-simple. This can be seen from their respective Lie algebras. For an invariant subalgebra with generators X' (A.9) requires that its commutator with any other generator is contained in the subalgebra; in short $[X, X'] \Rightarrow X'$. There are Abelian one-parameter subgroups in $\mathbf{SO(3)}$ and in $\mathbf{SO(2,1)}$ generated by any of the X^a, but their commutators with any of the other generators are not elements of the respective subalgebras. Hence these two groups are simple. The situation is different for $\mathbf{E_2}$. Its Abelian

Table A.3 Groups and (nonvanishing) structure constants

Group	f^{123}	f^{231}	f^{312}
$\mathbf{SO(3)}$	1	1	1
$\mathbf{SO(2,1)}$	1	-1	1
$\mathbf{E_2}$	0	1	1

subalgebra, generating the translations Y^1, Y^2 (see (A.18)), is an invariant algebra; thus this Lie group is not semi-simple. The non-vanishing structure coefficients for the three groups are aggregated in Table A.3.

For the respective Cartan-Killing metric one finds

$$G_{\mathbf{SO(3)}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} G_{\mathbf{SO(2,1)}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} G_{\mathbf{E_2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Thus indeed the metrics are regular for the two simple groups, and singular for the non-semi-simple one. Furthermore, the metric is negative definite for the compact group $\mathbf{SO(3)}$.

For compact semi-simple Lie groups, one can always find a basis X^a in the algebra such that the structure constants in $[X^a, X^b] = i\, f^{abc} X^c$ are totally antisymmetric in their indices.

Casimir Operators

In quantum mechanics, use is made of the fact that $J^2 = (J_1)^2 + (J_2)^2 + (J_3)^2$ commutes with each of the $(J_i)^2$. In general terms, J^2 is a Casimir operator of the group $\mathbf{SO(3)}$. A Casimir operator generically is a polynomial in the generators that commutes with all generators. For a semi-simple algebra the number of independent Casimir operators is equal to the rank of the algebra. Notice that the Casimir operators are not part of the Lie algebra–in mathematical terms they belong to the enveloping algebra.

For semi-simple Lie groups, the Cartan-Killing metric can be used to calculate the quadratic Casimir operator of the group by

$$C := g_{ab} X^a X^b$$

since the Cartan-Killing metric has an inverse g_{ab}. Because this is a real symmetric matrix, one can always bring it to the form $C = X^a X^a$ by an orthonormal transformation of the generators.

A.2.6 *Infinite-Dimensional Lie Groups*

Kac-Moody Algebra

Kac–Moody algebras are a generalization of finite-dimensional semi-simple Lie algebras defined by

$$\left[X^a_m, X^b_n\right] = if^{abc}X^c_{m+n} + c\delta^{ab}\delta_{m,-n}.$$

Here the $f's$ are the structure constants of a finite-dimensional Lie-algebra and the constant c is a central term. Kac-Moody algebras typically arise in conformal field theories and in dimensional reductions of gravity and supergravity theories.

A related algebra is the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}.$$

This algebra generates the diffeomorphisms of the circle. It turns up in string theory together with further generators for fermionic degrees of freedom.

Diffeomorphism Group

The diffeomorphism group arises in all theories which are reparametrization invariant, or generally covariant. Examples are the relativistic point particle, the relativistic string and general relativity.

Diffeomorphisms are smooth, invertible mappings of a D-dimensional manifold M onto itself: $d: M \to M$. These form a group $\mathbf{Diff}(\mathbf{M})$. In a coordinate patch, the mapping is

$$d : x \to d(x) = x'(x) \qquad \text{orlocally} \qquad x^\mu \to x^\mu + \xi^\mu.$$

The Lie algebra associated to $\mathbf{Diff}(\mathbf{M})$ is spanned by generators

$$L_\xi(x) = \xi^\mu(x)\partial_\mu$$

that is by the set of all vector fields on the manifold M. On any tensor T, a diffeomorphism acts as the Lie derivative $\delta T = -\pounds_\xi T$. The "structure constants" of the diffeomorphism group are derived from the commutator of two Lie derivatives

$$[\pounds_X, \pounds_Y]T = \pounds_{[X,Y]}T$$

where $[X,Y] = \pounds_X Y = -\pounds_Y X$. Then (in the condensed DeWitt notation)

$$\int d^Dx' \int d^Dx'' C^\mu{}_{\nu'\rho''}X^{\nu'}Y^{\rho''} = -[X,Y]^\mu = X^\mu_{,\nu}Y^\nu - Y^\mu_{,\nu}X^\nu$$

from which

$$C^\mu{}_{\nu'\rho''} = \delta^\mu_{\nu',\sigma}\delta^\sigma_{\rho''} - \delta^\mu_{\rho'',\sigma}\delta^\sigma_{\nu'} = \delta^\mu_{\nu';\sigma}\delta^\sigma_{\rho''} - \delta^\mu_{\rho'';\sigma}\delta^\sigma_{\nu'} \tag{A.20}$$

with

$$\delta^\sigma_{\rho'} = \delta^\sigma_\rho\delta^D(x,x'), \qquad \delta^\mu_{\nu',\sigma} = \delta^\mu_\nu\partial_\sigma\delta^D(x,x').$$

A.3 Representation of Groups

As elaborated in the main text, quantum physics shifts the attention from groups to their representations (or even more precisely, to ray representations; see Sect. A 3.5).

A.3.1 Definitions and Examples

The n-dimensional *representation* $D^{(n)}$ of a group $\mathbf{G}$ is a homomorphism

$$D^{(n)} : \mathbf{G} \to \mathbf{L}(V^n)$$

where $\mathbf{L}(V^n)$ is a group of linear operators that act on an n-dimensional vector space V^n. In physical applications the vector space is mainly a Hilbert space. If the operators are realized as matrices the representation is a *matrix representation*. The mapping D obeys the postulates of a group homomorphism: If $D(g_i) \in \mathbf{GL}(\mathbf{n}, \mathbb{F})$ it fulfills

$$D(g_i) \star D(g_k) = D(g_i \circ g_k) \qquad D(g_i) \star D({g_i}^{-1}) = D(g_0) = I.$$

Here $\circ$ denotes the group multiplication in $\mathbf{G}$, and $\star$ is the group multiplication in $D(\mathbf{G}) \subset \mathbf{L}(V^n)$.

In order to motivate and illustrate notions introduced later, let us consider some real representations of the most rudimentary non-trivial group, written as $\mathbf{Z}_2 = \{e, a\}$:

$$e \circ a = a \circ e = a \qquad a \circ a^{-1} = e \qquad a \circ a = e \qquad \text{i.e.} \qquad a = a^{-1}.$$

- 1-dimensional representations:

$$D_1^{(1)}(e) = 1 \qquad D_1^{(1)}(a) = -1$$
$$D_2^{(1)}(e) = 1 \qquad D_2^{(1)}(a) = 1.$$

  The representation $D_1^{(1)}$ is an isomorphism (in the context of representations called a *faithful representation*).
- 2-dimensional representations:

$$D_1^{(2)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D_1^{(2)}(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  is the *trivial* representation,

$$D_2^{(2)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D_2^{(2)}(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  is a faithful representation.

$$D_3^{(2)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D_3^{(2)}(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This representation is related to the representation $D_2^{(2)}$ by a *similarity transformation*

$$D_3^{(2)}(Z_2) = S \star D^{(2)}(Z_2) \star S^{-1} \qquad \text{with} \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- 4-dimensional representations, for instance

$$D^{(4)}(e) = \begin{pmatrix} D_1^{(2)}(e) & 0 \\ 0 & D_2^{(2)}(e) \end{pmatrix} \qquad D^{(4)}(a) = \begin{pmatrix} D_1^{(2)}(a) & 0 \\ 0 & D_2^{(2)}(a) \end{pmatrix}.$$

This representation is–by construction–completely reducible into $D_1^{(2)}(Z_2)$ and $D_2^{(2)}(Z_2)$; one writes $D^{(4)} = D_1^{(2)} \oplus D_2^{(2)}$. Obviously, reducibility is equivalent to the fact that the action of $D^{(4)}$ on the 4-dimensional vector space leaves two 2-dimensional subspaces invariant.

A.3.2 Representations of Finite Groups

On the evidence of the previous example, there are many representations of a group. Some of them are related by similarity transformations, others are reducible to (dimensionally) smaller representations. Of course one is interested in the most basic (or "elementary", or "prime") ones. The theory of (finite) group representations was essentially developed by F.G. Frobenius and I. Schur in 1906. (Further refinements are due to W. Burnside and R. Brauer.) They found (1) a criterion for a representation to be irreducible, (2) the number and dimensions of inequivalent representations, (3) the important role played by the so-called character of the representation, and (4) how to construct the representation using the character table. These findings shall be outlined in the following

Some More Definitions...

- If $D(G)$ is a representation and if a matrix S exists such that $D'(G) = SD(G)S^{-1}$ is a representation, both representations are called *equivalent with similarity transformation* S. The further investigation can then be restricted to the equivalence classes of equivalent (sic!) representations.
- If a representation is equivalent to

$$D(g) = \begin{pmatrix} D_1(g) & A(g) \\ 0 & D_2(g) \end{pmatrix} \tag{A.21}$$

it is called *reducible*. Otherwise, if this triangle form cannot be attained, the representation is *irreducible*. If D_2 has dimension m, the representation $D(g)$ leaves an m-dimensional subspace invariant.

- If for a representation a similarity transformation exists such that in (A.21) $A(g) = 0$, the representation is *completely reducible*; one writes in this situation

$$D^{(n)} = \oplus\, m_j\, D^{(j)}; \tag{A.22}$$

the integers m_j indicating how often the respective representation appears in this decomposition.

... and Some Theorems

- Every representation of a finite group is equivalent to a unitary representation, i.e. $D^{(n)}(g) \in U(n)$.
- A unitary representation of a finite group is always completely reducible, and thus every representation of a finite group is completely reducible.
- Abelian finite groups have only 1-dimensional irreducible representations.
- Schur's two lemmata:
  (1) If $D(G)$ is a finite dimensional irreducible representation of a group, and if $D(g)A = AD(g)\ \forall g \in G$ then A is a multiple of the unit matrix.
  (2) If D_1 and D_2 are two different irreducible representations and if for a matrix $A,\ D_1(g)A = AD_2(g)\ \forall g \in G$ holds, then $A = 0$.

Character of a Representation

With the aid of the two lemmata named after I. Schur one can prove orthogonality properties of irreducible representations and of the character of a representation. The notion of the character of a representation has its origin in the fact that similarity equivalent representations of a group element, that is $D'(g) = SD(g)S^{-1}$, although realized by different matrices, do have a common index: Because of the cyclic property of the trace $\mathrm{tr}\ (ABC) = \mathrm{tr}\ (BCA) = \mathrm{tr}\ (CAB)$, we immediately see that the traces of $D(g)$ and $D'(g)$ are the same. And this is the character of a group element with respect to the representation D. The *character* of a representation (also called the character table) is defined as the set

$$\chi(D) := \{\chi_D(g_i)\} \qquad \text{where} \qquad \chi_D(g_i) := \mathrm{tr}\ D(g_i).$$

In the following, in line with the established group representation literature, the dependence of the characters on the representation will be denoted as $\chi^{(\mu)}$. It can be shown that

- For finite groups of order $[g]$ the characters of two irreducible representations $\chi^{(\mu)}$ und $\chi^{(\nu)}$ fulfill

$$\sum_g^{[g]} \chi^{(\mu)}(g)\chi^{*(\nu)}(g) = [g]\delta^{\mu\nu}. \tag{A.23}$$

- The characters are constant on conjugacy classes: Conjugate elements in the group do have the same character. If $\mathbf{G}$ has k conjugacy classes $C_1, \ldots, C_k$, each containing $h_1, \ldots, h_k$ elements, and denoting by $\chi_r (r = 1, \ldots, k)$ the character of the conjugacy classes, one can write (A.23) in the form

$$\sum_{r=1}^{k} h_r \chi_r^{(\mu)} \chi_r^{*(\nu)} = [g] \delta^{\mu\nu}.$$

  This entails specifically that the number of irreducible representations of a group $\mathbf{G}$ is the same as the number of its conjugacy classes.
- Any representation can be expanded as a direct sum of irreducible representations:

$$D(g_i) = \oplus_\nu \, m_\nu^D D^{(\nu)}(g_i) \qquad \text{with} \qquad m_\nu^D = \frac{1}{[g]} \sum_r h_r \, \chi_r^{*(\nu)} \chi_r^D .$$

- The criterion for irreducibility of a representation is $\sum_r h_r |\chi_r|^2 = [g]$.

Product Representations

Multiplying two representations results in another representation: The *direct product* of an $(n \times n)$-matrix A_{ij} and an $(m \times m)$ -matrix $B_{\alpha\beta}$ is an $(n \cdot m \times n \cdot m)$-matrix $(A \times B)$ defined by

$$(A \times B)_{i\alpha, j\beta} := A_{ij} B_{\alpha\beta}.$$

The representation matrices of a product representation are then given by the product of the single representations:

$$D_{i\alpha, j\beta}^{(n \times m)}(g) = D_{ij}^{(n)}(g) D_{\alpha\beta}^{(m)}(g).$$

The result can be decomposed into irreducible components. This process is familiar from quantum mechanics as the addition of angular momenta and the subsequent determination of the Clebsch-Gordan series.

$$D^{(n)} \otimes D^{(m)} = \oplus \, \Gamma^{nmk} \;\; D^{(k)}. \tag{A.24}$$

The coefficients Γ^{nmk} can be derived from the characters of the groups in question as

$$\Gamma^{nmk} = \frac{1}{[g]} \sum_g \chi^{(n)}(g) \, \chi^{(m)}(g) \, \chi^{(k)}(g^{-1}). \tag{A.25}$$

A.3.3 Representation of Continuous Groups

Many notions and results for the representation of finite groups are directly transferable to Lie groups. You saw that a central technical means was the character for distinguishing inequivalent representations. Together with a "group mean value"

$$M(f) = \frac{1}{[g]} \sum_{g} f(g)$$

of a function $f : G \to \mathbb{C}$ this gave clues on type and number of irreducible representations.

Invariant Integration

Only for those continuous groups for which a substitute of the group mean value with its specific properties can be defined do all results from the representation theory of finite groups hold. The specific properties of $M(f)$ are (1) linearity: $M(\alpha f_1 + \beta f_2) = \alpha M(f_1) + \beta M(f_2)$ (2) positiveness: from $f \geqslant 0$ follows $M(f) \geqslant 0,$ and $M(f) = 0$ only for $f = 0$ (3) normalization: if $f(g) = 1$ for all $g \in G,$ then $M(f) = 1$ (4) left and right invariance, e.g. left invariance:

$$M_{l_a}(f) = \frac{1}{[g]} \sum_{g} f(ag) = \frac{1}{[g]} \sum_{g} f(g) = M(f)$$

since for any fixed $a \in G$, we have $\{ag|g \in G\} = \{g|g \in G\}$. The same applies to right invariance.

For a continuous group, the mean group value defined in terms of a summation needs to be replaced by an integral. Then the question arises of whether one can find an (integration) measure. The answer is positive if the Lie group is compact. In this case

$$M(f) = \int_G d\mu(g) f(g)$$

where $d\mu$ is called the *Haar measure*. I refrain from further details and from deriving the Haar measure for those compact groups most important in the Standard Model, namely $\mathbf{U(1)}$, $\mathbf{SU(2)}$, and $\mathbf{SU(3)}$, because this would require several additional pages.

Some Results on Representations

Here I state without proof that

- All representations of a semi-simple group are fully reducible.
- Unitary representations are fully reducible.

- The finite-dimensional representations of compact Lie groups are all unitary (and the finite-dimensional representations of compact Lie-algebras are all Hermitean).
- Every representation of a compact group is equivalent to a unitary representation.
- A compact group has a countably infinite number of inequivalent, unitary, irreducible representations which are all finite-dimensional.
- Non-compact Lie groups have uncountable numbers of inequivalent, unitary irreducible representations which are all infinite dimensional. Their finite-dimensional representations are not unitary.

Adjoint Representation

For every Lie algebra, there exists a distinguished representation. This is the adjoint[9] representation defined by matrices $\hat{T}^a$:

$$(\hat{T}^a)^{bc} := -if^{bca}. \tag{A.26}$$

(Observe the different positions of the indices on both sides, and remember that the structure constants are antisymmetric in their first two indices.) By inserting these into the Jacobi–identity (A.8) one indeed finds $([\hat{T}^a, \hat{T}^b])^{de} = if^{abc}(\hat{T}^c)^{de}$ which assures that the matrices $(\hat{T}^a)$ are a representation of the algebra. The adjoint representation has–by construction–the same dimension as the group itself.

We saw that a linear transformation on the Lie group generators induces a linear transformation of the structure constants in the form (A.10). Now one can easily show that taking the transformed structure functions to define the adjoint representation is nothing but a similarity transformation on the matrices $(\hat{T}^a)$:

$$(\hat{T}^a) \quad \to \quad (\hat{T}'^a) = \Gamma^{ad}\, \Gamma(\hat{T}^d)\Gamma^{-1}.$$

This similarity transformation induces a change

$$\mathrm{tr}\,(\hat{T}^a\hat{T}^b) \quad \to \quad \mathrm{tr}\,(\hat{T}'^a\hat{T}'^b) = \Gamma^{ac}\Gamma^{bd}\,\mathrm{tr}\,(\hat{T}^c\hat{T}^d).$$

The trace can be diagonalized by an appropriate choice of the Γ-matrix, and thus we can assume that

$$\mathrm{tr}\,(\hat{T}^a\hat{T}^b) = C^a\delta^{ab} \qquad (\mathrm{nosum}).$$

By a suitable rescaling of the generators the C^a can be assumed to have values $\{+1, 0, -1\}$. As it turns out, the case with all C^a positive is characteristic for a compact Lie algebra: In a suitable basis $(\hat{T}^a\hat{T}^b) = \delta^{ab}$, multiply the commutator of the representation matrices with $\hat{T}^d$ and take the trace:

$$\mathrm{tr}\,([\hat{T}^a, \hat{T}^b]\hat{T}^d) = \mathrm{tr}\,(\hat{T}^a\hat{T}^b\hat{T}^d) - \mathrm{tr}\,(\hat{T}^b\hat{T}^a\hat{T}^d) = \mathrm{tr}(f^{abc}\hat{T}^c\hat{T}^d) = if^{abd}.$$

Because of the cyclic property of the trace, the previous relation reveals that the structure functions are completely antisymmetric in all indices. Therefore the $(\hat{T}^a)$ are Hermitean matrices: $((\hat{T}^a)^{bc})^\dagger = ((\hat{T}^a)^{cb})^* = if^{cba} = -if^{bca} = (\hat{T}^a)^{bc}$.

[9] A similar notion exists for finite groups where it is called the regular representation.

Defining/Fundamental Representation

If the group in question is already a matrix group $\mathbf{GL(n,F)}$, there obviously is a *defining representation* $D[\mathbf{GL}(\mathbf{n},\mathbb{F})] = \mathbf{GL}(\mathbf{n},\mathbb{F})$, that is a $\mathbb{F}$-valued n-representation. This is sometimes also called the *fundamental* representation. Other representations may be obtained by taking the repeated direct product of this representation (and/or its complex conjugate).

Complex and Real Representations

Let T^a be a representation of the generators X^a associated to the Lie group $\mathbf{G}$: $\left[T^a, T^b\right] = if^{abc}T^c$. Taking the complex conjugate of this expression, we see that $(-T^{a*})$ also forms a representation:

$$\left[(-T^{a*}), (-T^{b*})\right] = if^{abc}(-T^{c*}).$$

Thus every representation T has an associated *conjugate representation* $\bar{T}$ with $\bar{T}^a = -T^{a*}$. If a representation is equivalent to its conjugate, there is an unitary similarity transformation such that $\bar{T}^a = UT^aU^{-1}$. In this case the representation is real. Otherwise the representation is complex.

Casimir's and Dynkin's

Denote by T^a_ρ the matrices of an irreducible representation ρ with dimension $d(\rho)$. For every representation one can define characteristic indices: Assume that the representation matrices satisfy the normalization condition

$$\mathrm{Tr}\ (T^a_\rho T^b_\rho) = C(\rho)\delta^{ab}$$

(it can be shown that this is always possible to arrange for semi-simple algebras). The constant $C(\rho)$ is called the Dynkin index of the representation.
A further index relates to the quadratic Casimir operator $g_{ab}X^aX^b$ where g_{ab} is the Cartan–Killing metric. For any simple Lie algebra, one can choose a basis in which it has the form $X^2 = X^aX^a$. By definition, it commutes with all generators in the algebra. Thus, according to the Schur lemma, its representation takes on a constant value in each irreducible representation:

$$T^a_\rho T^a_\rho = C_2(\rho)I.$$

The constant $C_2(\rho)$ is called the quadratic Casimir index. In the adjoint representation it becomes

$$f^{cba}f^{cda} = C_2(\mathrm{adj})\delta^{bd}.$$

The quadratic Casimir index is related to the Dynkin index by

$$d(\rho)C_2(\rho) = d(\mathrm{adj})C(\rho). \tag{A.27}$$

These indices appear for instance in the beta functions of field theories.
Another index arises repeatedly in expressions for the chiral anomaly. Take

$$\mathrm{Tr}\ \left[T^a_\rho\{T^b_\rho T^c_\rho\}\right] = \mathcal{A}^{abc}(\rho).$$

Specifically for $\mathbf{SU(N)}$ one defines the anomaly coefficient $A(\rho)$ by

$$\mathcal{A}^{abc}(\rho) = \frac{1}{2}A(\rho)d^{abc}$$

where d^{abc} is defined from the anti-commutator of the representation matrices in the fundamental representation

$$\{T^a_f T^b_f\} = \frac{1}{N}\delta^{ab} + d^{abc}T^c_f.$$

A.3.4 Examples: Representations of SO(2), SO(3), SU(3)

SO(2)

The group **SO**(2) describes the 1-parameter Lie-group of rotations in the plane, parameterized by an angle $0 \leq \varphi < 2\pi$. The defining/fundamental representation is given by matrices

$$R = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{A.28}$$

This group is Abelian, therefore all irreducible representations are one dimensional. However, the defining representation is obviously not reducible in the domain of the real numbers but only in the domain of complex numbers:

$$\begin{pmatrix} x' + iy' \\ x' - iy' \end{pmatrix} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} x + iy \\ x - iy \end{pmatrix}$$

and $e^{i\varphi}$ is indeed a $\mathbf{U(1)}$-matrix. Hence $\mathbf{SO(2)}$ and $\mathbf{U(1)}$ are locally isomorphic. (As will be explained in A.3.6, $\mathbf{U(1)}$ is the universal covering group of $\mathbf{SO(2)}$.) All representations are classified by the integer (quantum) number m as

$$D^{[m]} = e^{im\varphi}.$$

SO(3)

This is the rotation group in three dimensions. The defining representation is given in form of real-valued (3×3)-matrices $R_n(\varphi)$ which, referring to Cartesian coordinates, implement the effect of a rotation around the axis $\vec{n}$ by an angle φ. If one denotes by $R_i(\varphi)$ the rotation around the coordinate axis x_i, one simply can enlarge the $SO(2)$ matrix (A.28) for each of the three axis, i.e.

$$R_3 = \begin{pmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}$$
$$R_2 = \begin{pmatrix} \cos\varphi_2 & 0 & -\sin\varphi_2 \\ 0 & 1 & 0 \\ \sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}.$$

The generators $X_k = -i\frac{dR_k}{d\varphi}|_{\varphi=0}$ were already calculated as the matrices in (A.4).

In order to exemplify how to construct irreducible representations of a group, we will recapitulate how one proceeds in the case of quantum mechanics. The generators X_k correspond up to a factor $\hbar$ (which in the subsequent calculation is set to 1 in any case) to the angular momentum operators J_k with

$$[J_x, J_y] = i\hbar J_z \qquad [J_y, J_z] = i\hbar J_x \qquad [J_z, J_x] = i\hbar J_y.$$

In the subsequent considerations, a prominent role is played by the quadratic Casimir operator

$$J^2 = J_x^2 + J_y^2 + J_z^2.$$

It follows indeed from the Cartan-Killing metric of $\mathfrak{so}(3)$ according to (A.19). Therefore–complying with the quantum mechanical interpretation– J^2 can be measured simultaneously with any one of the angular momentum operators. Customarily one chooses the z-component, for which the eigenvalues are already known from the $\mathbf{SO(2)}$ consideration in the previous subsection. Therefore we have the two eigenvalue equations

$$J^2|\alpha, m\rangle = \alpha|\alpha, m\rangle \qquad J_z|\alpha, m\rangle = m|\alpha, m\rangle$$

with real-valued α and integer m (and $\hbar = 1$). Because of

$$(J_x^2 + J_y^2)|\alpha, m\rangle = (J^2 - J_z^2)|\alpha, m\rangle = (\alpha - m^2)|\alpha, m\rangle$$

the eigenvalues α must obey the inequalities

$$\alpha \geq 0, \qquad \alpha - m^2 \geq 0. \tag{A.29}$$

Now introduce the ladder operators

$$J_\pm := J_x \pm iJ_y \qquad \text{with} \qquad [J_z, J_\pm] = \pm J_\pm.$$

Their name 'ladder operators' (or 'raising and lowering operators') is due to their property of changing the eigenvalue of J_z by ± 1:

$$J_{\pm}|\alpha, m\rangle = \sqrt{\alpha - m^2 \mp m}\, |\alpha, m \pm 1\rangle. \tag{A.30}$$

As a consequence of this and of the inequalities (A.29), one argues that the eigenvalues can be found only in the spectrum

$$\alpha = j(j+1); \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \qquad m = -j, -j+1, \ldots., j-1, j.$$

Therefore the eigenvalues of J_z are restricted to $(2j+1)$ values. The $(2j+1)$-dimensional irreducible representations can be enumerated by the quantum numbers (j, m) and the representation of the generators in terms of matrices are

$$\langle j, m'|J_z|j, m\rangle = m\delta_{m'm} \tag{A.31a}$$

$$\langle j, m'|J_+|j, m\rangle = [(j-m)(j+m+1)]^{1/2}\delta_{m'm+1} \tag{A.31b}$$

$$\langle j, m'|J_-|j, m\rangle = [(j+m)(j-m+1)]^{1/2}\delta_{m'm-1}, \tag{A.31c}$$

where (A.31b, A.31c) follow from (A.30). The first three lowest dimensional representations are therefore:

- one-dimensional representation $(j = 0)$
  $J_z = J_{\pm} = 0$ with the consequence that $R_k = \exp(i\varphi J_k) = 1$, and thus–as to be expected–this is simply the trivial representation.
- two-dimensional representation $(j = \frac{1}{2})$
  The representation matrices (A.31) are

  $$J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

  which amounts to $J_i = \frac{1}{2}\sigma_i$ with the Pauli matrices (A.12). This representation is called the spinor representation of the rotation group. Like all other half-integral representations, it cannot be derived or expressed in terms of tensor representations. The half-integral representations of $\mathbf{SO}(n, \mathbb{C})$ were discovered by E.Cartan in 1913. Later in the 20th they were rediscovered independently (by e.g. Dirac and Pauli) in quantum physics in the context of describing the spin of the electron. As shown in subsection A 2.4 the groups $\mathbf{SO(3)}$ and $\mathbf{SU(2)}$ have isomorphic Lie algebras. This comes about again since the Pauli matrices arising in the $j = 1/2$ representation of $\mathbf{SO(3)}$ constitute the defining representation of $\mathbf{SU(2)}$ in the form $U(\xi) = \exp(\frac{i}{2}\, \xi^a\sigma^a)$; see (A.13). In the next section, $\mathbf{SU(2)}$ is identified as the universal covering group of $\mathbf{SO(3)}$.
- three-dimensional representation $(j = 1)$
  For $j = 1$ the matrices (A.31) combine to

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

At first sight they seem to be unrelated to the three-dimensional matrices (A.4) from the defining representation of $\mathbf{SO(3)}$. But remember that irreducible representations are defined up to similarity transformations. In this case the unitary matrix

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

mediates the transformation to $X_k = S J_k S^\dagger$.

The coefficients of the Clebsch-Gordan series (A.24) for $\mathbf{SO(3)}$ can be determined from the character of the group and the orthogonality of its elements. According to (A.31a), the generator J_3 has in the $(2j+1)$-dimensional representation always the form

$$J_3 = \operatorname{diag}(j, j-1, \ldots, -j)$$

and thus

$$R_3(\varphi) = e^{iJ_3\varphi} = \operatorname{diag}(e^{ij\varphi}, e^{i(j-1)\varphi}, \ldots, e^{-ij\varphi}),$$

and the trace of this representation is calculated to be

$$\chi^{[j]} = e^{-ij\varphi} + e^{-i(j-1)\varphi} + \ldots + e^{ij\varphi} = \frac{\sin(j+\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi}.$$

In the case of $\mathbf{SO(3)}$ the "group mean value" derived from the Haar measure is

$$\frac{1}{[g]} \sum_g \Rightarrow \int_0^{2\pi} \frac{d\varphi}{2\pi}(1 - \cos\varphi),$$

and it can be deduced that the coefficients in (A.24) calculated from the character elements as in (A.25) yield

$$D^{[j_1]} \otimes D^{[j_2]} = \sum_{|j_1-j_2|}^{|j_1+j_2|} \oplus D^{[j]}.$$

You can find this result in books on quantum mechanics from the addition of angular momenta and its subsequent reduction in terms of multiplets. In introducing the notation $\mathbf{1}$ for a singlet $(j=0)$, $\mathbf{2}$ for a doublet $(j=1/2)$, $\mathbf{3}$ for a triplet $(j=1)$, etc. one arrives at expressions such as

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} &= \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \\ \mathbf{2} \otimes \mathbf{2} &= \mathbf{1} \oplus \mathbf{3} \end{aligned}$$

from which the dimension of the product representation and its reduction can be immediately recognized.

SU(3)

Historically, the group $\mathbf{SU(3)}$ arose in physics in the original (up, down, strange) quark model in explaining the composition of hadrons from constituent quarks. Today this is recognized as an approximately valid "flavor" $\mathbf{SU_F(3)}$), and it is the $\mathbf{SU_C(3)}$ with respect to the three color degrees of freedom that is considered to be the exact symmetry of quantum chromodynamics; see Sect. 6.2.

$\mathbf{SU(3)}$ is an eight-dimensional compact semi-simple group. Thus it is possible to find a basis F_a in its associated Lie algebra such that the structure constants in

$$[F_a, F_b] = i f_{abc} F_c \qquad (a, b, c = 1, \ldots 8)$$

are real and totally antisymmetric. The convention of choosing the non-vanishing structure constants as

$$f_{123} = 1 \qquad\qquad f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}$$

$$f_{156} = f_{367} = -\frac{1}{2} \qquad\qquad f_{458} = f_{678} = \frac{1}{2}\sqrt{3}.$$

is due to Gell-Mann. The defining three-dimensional matrix representation of the generators is provided by $F_a = \frac{1}{2}\lambda_a$, with the *Gell-Mann matrices*

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Observe that λ_1, λ_4 and λ_6 look like the Pauli matrix σ_1, aside from having an additional column and row with zeros. In the same manner, λ_2, λ_5 and λ_7 are related to σ_2, and λ_3 with σ_3 [10].

Other representations of $\mathbf{SU(3)}$ can be obtained in a similar manner to the procedure for $\mathbf{SO(3)}$: One has to identify a set of operators which can be diagonalized simultaneously. Again, this task is supported by means of Casimir operators. The quadratic Casimir operator derived from the Cartan-Killing metric is

$$F^2 = \sum_{a=1}^{8} F_i^2.$$

[10] This observation gives a hint for how to construct the defining representations of $\mathbf{SU(n)}$ for $n > 3$.

There are two further commuting generators, namely F_3 and F_8, or–following the denotation of quantum numbers in particle physics–the isospin and the hypercharge operator

$$I_3 = F_3, \qquad Y = \frac{2}{\sqrt{3}} F_8.$$

Quite in analogy with the further procedure in case of the rotation group, one defines linear combinations with the remaining operators:

$$I_\pm := F_1 \pm iF_2, \qquad U_\pm := F_6 \pm iF_7, \qquad V_\pm := F_4 \pm iF_5$$

which act as ladder operators with respect to the isospin (I-spin), the so-called U-spin and the V-spin. Indeed, accordingly there are three $\mathfrak{su}(2)$ -subalgebras of $\mathfrak{su}(3)$, spanned by the generators

$$\{I_\pm, I_3\}, \qquad \{U_\pm, U_3 = \frac{3}{4}Y - \frac{1}{2}I_3\}, \qquad \{V_\pm, V_3 = -\frac{3}{4}Y - \frac{1}{2}I_3\}.$$

Again in analogy with the ladder operators of the rotation group, whose action can be visualized on a line (stretched along the eigenvalue of I_3), it is immediately clear that in the present case, one can argue within a two-dimensional visualization in the (Y, I_3)-plane. The eigenvalues of I_3 are $\{0, \pm^1\!/_2, \pm 1, \pm^3\!/_2, \ldots\}$, and the eigenvalues of $Y = {}^2\!/_3(U_3\text{-}V_3)$ are $\{0, \pm^1\!/_3, \pm^2\!/_3, \pm 1, \pm^4\!/_3, \ldots\}$. Thus, all the states are localized on a hexagonal lattice in which, starting from an arbitrary point, all displaced states can be reached by ladder operations I_+, U_+, V_+ with $(I_3\text{+}^1\!/_2, Y)$, $(I_3\text{-}^1\!/_2, Y\text{+}^1\!/_3)$, and $(I_3\text{-}\frac{1}{2}, Y\text{-}^1\!/_3)$, respectively; see Fig. A.2.
The representations of $\mathbf{SU}(\mathbf{3})$ are therefore characterized by two numbers α and β which indicate by how many units the ladder operations act. A standard has been established (see [159], Chap. 11) by which parameters α and β are chosen such that

$$\Delta I_3 = \alpha \qquad \Delta Y = \frac{1}{3}(\alpha + 2\beta).$$

The dimension d of the representation $D^{[\alpha\beta]}$ is then

$$d(\alpha, \beta) = \frac{1}{2}(\alpha + 1)(\beta + 1)(\alpha + \beta + 2).$$

Accordingly $D^{[00]}$ is the singlet, $D^{[10]}$ the triplet, $D^{[01]}$ the anti-triplet, $D^{[20]}$ the sextet, $D^{[11]}$ the octet, and $D^{[30]}$ the decuplet representation.

The smallest non-trivial representations are the triplet and the anti-triplet. These are the defining representations of **SU(3)** in terms of the Gell-Mann λ-matrices or their Hermitean conjugates. The objects which transform under these representations are three-component $\mathbf{SU}(\mathbf{3})$-vectors ψ, i.e. $\psi' = e^{i(\alpha^a \lambda_a)}\psi$.

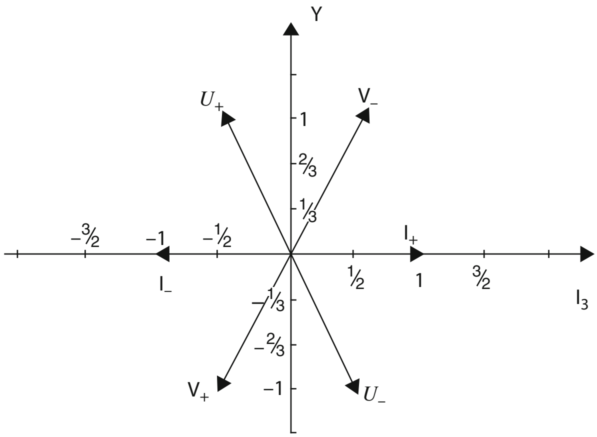

Fig. A.2 Ladder operations for SU(3)

Some of the lower-dimensional product representations of $\mathbf{SU(3)}$ are

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1},$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1},$$

$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}.$$

If one performs the reduction of $\mathbf{3} \otimes \overline{\mathbf{3}}$ with the $\mathbf{SU}(3)$-vectors ψ, one can by use of the identity

$$\psi_\alpha \bar{\psi}^\beta = \left(\psi_\alpha \bar{\psi}^\beta - \frac{1}{3} \delta_\alpha^\beta \psi_\gamma \bar{\psi}^\gamma \right) + \frac{1}{3} \delta_\alpha^\beta \psi_\gamma \bar{\psi}^\gamma, \tag{A.32}$$

directly read of the octet–and singlet-parts.

Finally, some remarks on the Dynkin and Casimir indices of $\mathbf{SU(N)}$ are appropriate. The fundamental representation of $\mathbf{SU(2)}$ is given in terms of Pauli matrices $T_2^a = \sigma^a/2$, which satisfy $\mathrm{Tr}\ \left[T_2^a T_2^b\right] = \frac{1}{2}\delta^{ab}$. As we saw for $\mathbf{SU(3)}$, three of the matrices of its fundamental representation contain the Pauli matrices, which here act on the first two components of an $\mathbf{SU(3)}$-vector. A quite similar choice of generators can be made for larger $\mathbf{SU(N)}$ algebras. Therefore

$$\mathrm{Tr}\ \left[T_N^a T_N^b\right] = \frac{1}{2}\delta^{ab}$$

and thus the Dynkin index is $C(N) = 1/2$ and the quadratic Casimir is found from (A.27). We note down that in the fundamental representation of SU(N)

$$C(N) = \frac{1}{2} \qquad C_2(N) = \frac{N^2 - 1}{2N}. \tag{A.33}$$

At various places (for instance in the generic expression for the β function), one also needs the Dynkin and Casimir in the adjoint representation of $\mathbf{SU(N)}$. This may be derived from the product of fundamental representations

$$\mathbf{N} \otimes \overline{\mathbf{N}} = (\mathbf{N^2} - \mathbf{1}) \oplus \mathbf{1}. \tag{A.34}$$

For a product representation $r_1 \times r_2 = \Gamma^i\ r_i$, it holds that

$$\big(C_2(r_1) + C_2(r_2)\big)\ d(r_1)d(r_2) = \sum \Gamma^i\ C_2(r_i)d(r_i).$$

In applying this to (A.34) by using (A.33), we derive from

$$\big(2 \cdot \frac{N^2-1}{2N}\big)N^2 = C_2(adj) \cdot (N^2-1)$$

the $\mathbf{SU(N)}$ quadratic Casimir and Dynkin indices for the adjoint representation:

$$C_2(adj) = C(adj) = N. \tag{A.35}$$

A.3.5 Projective Representations and Central Charges

As explained in Sect. 4.2., the fact that quantum physical states are not vectors but "rays" in a Hilbert space makes it necessary to consider ray representations (or projective representations) of symmetry groups.

Ray Representations

A *ray representation* of a group $\mathbf{G} = \{g, g', \ldots\}$ by unitary operators $\mathbf{U}$ in a Hilbert space is a map $\mathbf{G} \to \mathbf{U}$ i.e. $g \mapsto U_g$ obeying

$$U_{g'} \cdot U_g = e^{i\phi(g',g)} U_{g' \circ g} \tag{A.36}$$

with a phase $\phi(g', g)$. The phases in (A.36) are constrained because of the associativity of the group operation: From $U_{g''} \cdot (U_{g'} \cdot U_g) \stackrel{!}{=} (U_{g''} \cdot U_{g'}) \cdot U_g$, one immediately derives the condition

$$\phi(g', g) + \phi(g'', g' \circ g) \stackrel{!}{=} \phi(g'', g') + \phi(g'' \circ g', g) \tag{A.37a}$$

$$\phi(g', g_0) = \phi(g_0, g) = 1. \tag{A.37b}$$

The second line is due to the property of the phase if either g or g' is the neutral element g_0 in the group. We are still allowed to redefine the unitary operators as $\tilde{U}_g = U_g \exp(i\alpha(g))$. Then a simple calculation shows that the phase $\tilde{\phi}(g', g)$ in $\tilde{U}_{g'} \cdot \tilde{U}_g = e^{i\tilde{\phi}(g',g)}\ U_{g' \circ g}$ is related to $\phi(g', g)$ by

$$\tilde{\phi}(g', g) = \phi(g', g) - \alpha(g' \circ g) + \alpha(g') + \alpha(g). \tag{A.38}$$

and also obeys the conditions (A.37). The ray representations U and $\tilde{U}$ are said to be equivalent.

Curiosity provokes the question as to whether and when it is possible to find representations for which the phase $\phi(g', g)$ can be made to vanish. The answer was given by V. Bargmann in 1954 [26]:

(1) All central charges of the Lie algebra associated to the symmetry group vanish or can be transformed away.

(2) The group is simply connected.

Interestingly enough, the first condition refers to properties of the group on a small scale (near the identity) and the second one is related to properties on a large scale (i.e. the topology of the group manifold). These two points will be explained below.

Central Charges

There is a relation between the phases of a projective representation of a Lie group and central charges in its Lie algebra. This will be derived next by expanding the group elements and their representation operators (near the respective identities) in terms of the coordinates $\{\xi^a\}$. At first observe that if we write $g' \circ g = g(\xi')g(\xi) = g(\xi''(\xi', \xi))$, the dependence of the "composed" parameter set ξ'' takes on the form

$$\xi''^a = \xi'^a + \xi^a + \gamma^{abc}\xi'^b\xi^c + \dots$$

(with real coefficients γ^{abc}) because $\xi''^a(0, \xi) = \xi^a$ and $\xi''^a(\xi', 0) = \xi'^a$. Expanding the phase $\phi(g', g)$, we observe that because of (A.37)

$$\phi(g', g) = \phi(\xi', \xi) = \gamma^{bc}\, \xi'^b\xi^c + \dots \tag{A.39}$$

Finally, the representation matrices near the identity can be expanded as

$$U_{g(\xi)} = \mathbf{1} + i\xi^a T^a + \frac{1}{2}\xi^a\xi^b T^{ab} + \dots$$

with Hermitean matrices T^a and with $T^{ab} = T^{ba}$. Now expand (A.36) as

$$\begin{aligned}\Big(\mathbf{1} + i\xi'^a T^a + \frac{1}{2}\xi'^a\xi'^b T^{ab} + \dots\Big)&\Big(\mathbf{1} + i\xi^a T^a + \frac{1}{2}\xi^a\xi^b T^{ab} + \dots\Big)\\ &= \Big(1 + i\gamma^{bc}\xi'^b\xi^c + \dots\Big)\Big(\mathbf{1} + i(\xi'^a + \xi^a + \gamma^{abc}\xi'^b\xi^c + \dots)T^a\\ &\qquad + \frac{1}{2}(\xi'^a + \xi^a + \dots)(\xi'^b + \xi^b + \dots)T^{ab} + \dots\Big).\end{aligned}$$

Comparing both sides of this expression order by order in the ξ and the ξ', we observe that the terms of order $1, \xi, \xi', \xi^2, \xi'^2$ match. The terms proportional to $\xi'^c\xi^b$ disappear if

$$-T^c T^b = i\gamma^{cb}\mathbf{1} + i\gamma^{acb}T^a + T^{cb}.$$

By defining

$$f^{abc} := \gamma^{acb} - \gamma^{abc} \qquad f^{bc} := \gamma^{cb} - \gamma^{bc}$$

the previous condition becomes

$$[T^b, T^c] = i f^{abc} T^a + i f^{bc} \mathbf{1}. \tag{A.40}$$

Thus we recover in the first term on the right-hand side the structure coefficients f^{abc} of the Lie algebra. The further terms f^{bc} are the *central charges*. Due to (A.39) the central charges can be calculated from the phase as

$$f^{bc} = \left(\frac{\partial^2 \phi(\xi', \xi)}{\partial \xi^b \partial \xi'^c} - \frac{\partial^2 \phi(\xi', \xi)}{\partial \xi^c \partial \xi'^b} \right)_{|\xi=0=\xi'}.$$

Now the Jacobi identities in the Lie algebra impose conditions on the central charges in the form

$$f^{abd}\, f^{cd} + f^{bcd}\, f^{ad} + f^{cad}\, f^{bd} = 0. \tag{A.41}$$

In some cases, these conditions are so stringent that the only solution is $f^{ab} = 0$. In other cases one may be able to eliminate the central charges by redefining the Lie algebra generators as $X^a \mapsto \tilde{X}^a = X^a + C^a(f^{bc})$. This is possible specifically for solutions of (A.41) in the form $f^{ab} = f^{abd} C_d$ with an arbitrary set of real constants C_d. However, there are Lie algebras for which these are not the only solutions. In this case, not all central charges can be transformed away and one must concentrate one's considerations onto an algebra which is extended by these central charges. This extension amounts to augmenting the set of generators by additional ones being proportional to the non-vanishing central charges. A prominent example is given in Sect. 3.4 by the Galilei algebra centrally extended to the Bargmann algebra.

Global Properties of Lie Groups

As before, let $\mathbf{G}$ be a Lie group with elements $g, g', \ldots$. Two group elements g and g' are called *connected* if a continuous curve $c(\lambda)$ $(0 \leq \lambda \leq 1)$ exists in $\mathbf{G}$ with $c(0) = g$ and $c(1) = g'$. The set of all elements which are connected to the unit element is an invariant subgroup $\mathbf{G}_0$ of $\mathbf{G}$. A connected group $\mathbf{G}$ is called *simply-connected* if every loop in the group can be shrunk continuously to an element in $\mathbf{G}$.

Each connected Lie group $\mathbf{G}$ has a unique simply-connected *universal covering group* $\mathbf{G}^*$ with the properties (i) There is a homomorphic mapping ϱ: $\mathbf{G}^* \to \mathbf{G}$ (ii) The kernel of the homomorphism ϱ is an invariant discrete subgroup $\mathbf{N}$ of the center of $\mathbf{G}^*$.

Examples for physically important universal covering groups are

- $\mathbf{SU(2)}$ as universal cover of $\mathbf{SO(3)}$
The defining representation of $\mathbf{SU(2)}$ is provided by (A.13) as

$$U(\vec{\alpha}) = \exp(\frac{i}{2}\vec{\sigma}\cdot\vec{\alpha})$$

in terms of Pauli matrices σ. Introducing a normal vector $\vec{n}$ by $\vec{\alpha} = \alpha\vec{n}$ with $(\vec{\sigma}\cdot\vec{n})^2 = I$, one can write

$$U = \cos\frac{1}{2}\alpha + i\vec{\sigma}\cdot\vec{n}\sin\frac{1}{2}\theta.$$

Comparing this with the generic form of an $\mathbf{SO(3)}$ matrix

$$R(\theta) = \exp(i\vec{n}\cdot\vec{J}\theta),$$

where the J_k are the defining generators, we identify a homomorphism ϱ : $\mathbf{SU(2)} \to \mathbf{SO(3)}$. Since $U(2\pi) = -1$ and $R(2\pi) = +1 = R(0)$, two elements from $\mathbf{SU(2)}$, namely $U(0)$ and $U(2\pi)$, are mapped onto the neutral element of $\mathbf{SO(3)}$, the kernel of the homomorphism is

$$\mathrm{Ker}\ \varrho = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

This is the center of $\mathbf{SU(2)}$ and is non other than $\mathbf{Z_2}$. Therefore

$$\mathbf{SO(3)} \cong \mathbf{SU(2)}/\mathbf{Z_2},$$

and $\mathbf{SU(2)}$ is the (twofold) universal covering group of $\mathbf{SO(3)}$.
- $\mathbf{SL(2,\mathbb{C})}$ as universal cover of $\mathbf{SO(3,1)}$
First define from the Pauli matrices the four-vectors $\sigma^\mu := (1,\vec{\sigma})$, $\overline{\sigma}^\mu := (1,-\vec{\sigma})$. These obey

$$\mathrm{Tr}(\sigma^\mu\overline{\sigma}^\nu) = 2\eta^{\mu\nu} \qquad \sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu = 2\eta^{\mu\nu}.$$

Now construct a mapping from Minkowski space to the set of Hermitean complex 2×2 matrices by

$$x^\mu \mapsto X = x^\mu\sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

with an arbitrary Lorentz vector x^μ. Observe that $\det X = x^2$. The action of $\mathbf{SL(2,C)}$ on X, that is $X' = AXA^\dagger$ preserves $\det X = x^2$ and can be identified as a Lorentz transformation. Explicitly: To each transformation $A \in \mathbf{SL(2,C)}$ there corresponds a Lorentz transformation $\Lambda \in \mathbf{SO(3,1)}$ of x^μ:

$$x'^{\mu} = \Lambda^\mu{}_\nu x^\mu \qquad \Lambda^\mu{}_\nu(A) = \frac{1}{2}\mathrm{Tr}(\overline{\sigma}^\mu A\sigma_\nu A^\dagger).$$

The mapping $A \mapsto \Lambda(A)$ is a homomorphism of $\mathbf{SL(2, C)}$ to $\mathbf{SO(3, 1)}$ as

$$\Lambda^{\mu}{}_{\nu}(A) \in \mathbf{SO(3,1)}$$
$$\Lambda^{\mu}{}_{\nu}(A_1 A_2) = \Lambda^{\mu}{}_{\varrho}(A_1)\Lambda^{\varrho}{}_{\nu}(A_2).$$

But since $\Lambda(-A) = \Lambda(A)$, the inverse mapping is defined only up to a sign. Or–put another way–the elements $A = \pm 1$ of $\mathbf{SL(2, C)}$ are mapped onto the identity element of $\mathbf{SO(3, 1)}$. Thus, by definition, the kernel of the mapping is the discrete invariant subgroup $\mathbf{Z_2}$. In conclusion, $\mathbf{SL(2, C)}$ is the universal covering group of the Lorentz group

$$\mathbf{SO(3,1)} \cong \mathbf{SL(2,C)}/\mathbf{Z_2}.$$

A proof that $\mathbf{SL(2, C)}$ is simply connected can be found in [524], Sect. 2.7.

Representation Procedure

Based on the concepts and findings of the previous two sections, in this concluding section the generic procedure to find the irreducible (ray) representations of a symmetry group $\mathbf{S}$ in a Hilbert space is summarized:

1. The first question refers to the connection components of $\mathbf{S}$. If the symmetry group is not simply connected one must call on its universal covering group $\mathbf{S}'$.
2. Secondly, there is the question about the possible appearance of central charges in the algebra of generators, i.e. whether there are non-trivial solutions of (A.41).
3. If non-trivial solutions exist, one investigates whether by a re-definition of the generators one can transform the central charges away.
4. For the remaining central charges one enlarges the set of generators (and the group) and one applies the procedure on the central extension of $\mathbf{S}$.

Prominent examples in fundamental physics in which this procedure is brought to bear are the representation theory of the Galilei group (in which just one central charge, namely the mass, remains), and the representation theory of the Poincaré group, in which all central charges can be transformed away, but where one needs to regard $\mathbf{SL(2, C)}$ as the universal covering group of the Lorentz group.

Appendix B
Spinors, $\mathbb{Z}_2$-gradings, and Supergeometry

In this appendix I have collected all those formal techniques which are relevant to the fact that to the best of our knowledge, nature seems to prefer two rather distinct variants of fields, namely

- bosons $\triangleq$ commuting fields $\triangleq$ of integer spin
- fermions $\triangleq$ anti-commuting fields $\triangleq$ of half-integer spin.

B.1 Spinors

The following citation is from the preface of [401]: "To a very high degree of accuracy, the space-time we inhabit can be taken to be a smooth four-dimensional manifold, endowed with the smooth Lorentzian metric of Einstein's special or general relativity. The formalism most commonly used for the mathematical treatment of manifolds and their metric is, of course, the tensor calculus (... [1]). But in the specific case of four dimensions and Lorentzian metric there happens to exist–by accidence or providence–another formalism which is in many ways more appropriate, and that is the formalism of 2-spinors. Yet 2-spinor calculus is still comparatively unfamiliar even now–some seventy years[2] after Cartan first introduced the general spinor concept, and over fifty years since Dirac, in his equation for the electron, revealed a fundamentally important role for spinors in relativistic physics and van der Waerden provided the basic 2-spinor algebra and notation."

This already contains the central message about spinors. They are fundamental objects which can describe four-dimensional Minkowski spacetime. Their geometric origin was revealed in 1913 by E. Cartan. They were so to speak rediscovered in 1928 by Dirac in physics (however as 4-spinors), and gave rise to the concepts of particle spin and antiparticles. And the so-called spinor calculus and the "dot"-notation,

[1] Here, I have deleted a side remark from the original text.

[2] We have to add another 30 years considering the year of publication of the book by R. Penrose and W. Rindler.

K. Sundermeyer, *Symmetries in Fundamental Physics,*
Fundamental Theories of Physics 176, DOI 10.1007/978-94-007-7642-5,

which allows one to construct tensors and spinors of higher weight from the basic 2-spinors, is due to B. van der Waerden. The term “spinor” was coined by P. Ehrenfest, who is said to have encouraged van der Waerden to develop his calculus.

Interestingly enough, nature seems to prefer to be described by 2-spinors: For one thing, as outlined in some detail in Chap. 6, due to parity violation the Lagrangian of the Standard Model is formulated in terms of left-handed leptons and quarks. For another thing, in supersymmetric models the building blocks of matter are chiral supermultiplets, each of which contains a single Weyl fermion.

First of all, we need to fix some conventions on signs and factors i and (1/2), since–mostly for historical reasons–you find various slightly different notations in textbooks and in the research literature. Here, I follow essentially [45] and [134]. The latter authors state explicitly how things change in case of “the other” metric convention and how their definitions relate to those of the influential textbook [533].

B.1.1 Pauli and Dirac Matrices

Pauli Matrices and their Relatives

The three Pauli matrices are introduced in App. A.2.3 as the defining representation of $\mathfrak{su}(2)$. A widely used explicit representation is

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.1}$$

They have the property

$$\sigma^a \sigma^b = i\epsilon^{abc}\sigma^c + \delta^{ab} I. \tag{B.2}$$

The Pauli matrices are used to define the “four-vectors”

$$\sigma^\mu := (1, \vec{\sigma}), \qquad \overline{\sigma}^\mu := (1, -\vec{\sigma}). \tag{B.3}$$

From (B.2) one finds

$$\sigma^\mu \overline{\sigma}^\nu + \sigma^\nu \overline{\sigma}^\mu = 2\eta^{\mu\nu}.$$

Further definitions are

$$\sigma^{\mu\nu} := \frac{1}{4}\big(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\big) \qquad \bar{\sigma}^{\mu\nu} := \frac{1}{4}\big(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu\big). \tag{B.4}$$

These can be expressed in terms of the Pauli matrices themselves:

$$\sigma^{0i} = -\bar{\sigma}^{0i} = -\frac{1}{2}\sigma^i \qquad \sigma^{jk} = \bar{\sigma}^{jk} = -\frac{i}{2}\epsilon^{jkl}\sigma^l.$$

Furthermore, because of $\sigma_a^\dagger = \sigma_a$, one has $(\sigma^{\mu\nu})^\dagger = -\bar{\sigma}^{\mu\nu}$.

Dirac Matrices and their Relatives

The 4×4 Dirac matrices are defined by their algebraic relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \tag{B.5}$$

In the metric convention chosen in this book $(\gamma^0)^2 = +1$ and $(\gamma^k)^2 = -1$. Since the γ^μ anti-commute with each other, any product of gamma matrices can be reduced to products which contain at most four factors of gamma matrices. This gives rise to the definition of the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{B.6}$$

for which $(\gamma^5)^2 = 1$. The matrix γ^5 anti-commutes with all the other γ^μ:

$$\{\gamma^5, \gamma^\nu\} = 0.$$

Products of three different gamma matrices can be written in terms of $\gamma^\mu\gamma^5$. (For instance $\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^5$.) And finally, products of two different gamma matrices can be directly expressed via (B.5) as $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\gamma^{\mu\nu}$ where

$$\gamma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{B.7}$$

The set of the 16 matrices

$$\Gamma = \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\} = \{\Gamma^A\} \tag{B.8}$$

constitutes a basis of $GL(4, \mathbb{C})$, that is every complex 4×4 matrix can be written as a linear combination in this basis.

The Clifford condition (B.5) does not have a unique solution. If a certain set of matrices γ^μ is a solution, so is $\gamma'^\mu = S^{-1}\gamma^\mu S$. This means that the two sets of matrices are connected by a similarity transformation. Specifically by taking the Hermitean conjugate and the transpose of (B.5), we notice that $\gamma^{\mu\dagger}$ and -$\gamma^{\mu T}$ are representations of the Clifford algebra. This gives rise to the specific similarity transformations with matrices A and C

$$A\gamma^\mu A^{-1} = \gamma^{\mu\dagger}, \qquad C^{-1}\gamma^\mu C = -\gamma^{\mu T}$$

from which

$$(CA^T)^{-1}\gamma^\mu(CA^T) = -\gamma^{\mu *}. \tag{B.9}$$

Simply by algebraic manipulations, one can derive various properties of A and C, such as

$$A = A^\dagger, \qquad C = -C^T, \qquad C\gamma^5 = -\gamma^5\, C.$$

The previous relations are valid independently of the representation of the gamma matrices. They hold generally as long as the algebraic relation (B.5) is fulfilled. In terms of specific representations, there are two privileged ones. Their choice depends of whether one wants to work with four-component Dirac spinors (as appropriate to all parity–conserving elementary processes) or with two-component Weyl spinors (as appropriate to the left-right asymmetry in nature). In the *Dirac representation* the gamma matrices are represented as

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{B.10}$$

where of course the explicit form of γ^5 follows from its definition (B.6). In the *Weyl representation* (also called the *chiral representation* or the *spinor representation*),

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{B.11}$$

Thus in both the Dirac and the Weyl representations, the γ^i assume the same form. Furthermore, γ^0 and γ^5 change their role (up to a sign). In the Weyl basis one can write compactly

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{B.12}$$

by using the definition (B.3). Furthermore, (B.7) becomes

$$\gamma^{\mu\nu} = 2i \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}. \tag{B.13}$$

By the way, the terminology in this context is a little confusing. Here we referred to the Dirac and Weyl representations. In fact, there is another privileged one, namely the Majorana representation. On the other hand, in the following we will investigate Weyl, Dirac, and Majorana spinors. Their properties are completely unrelated to the representation issues of the Clifford algebra. And to make it even worse: Majorana spinors are specific Dirac spinors, but Dirac and Majorana fermions obey different field equations, characterized by Dirac and Majorana mass terms.

B.1.2 Weyl Spinors

Weyl spinors do have a genuine geometric and group–theoretical origin, like vectors. They are directly related to the smallest non-trivial representations $(\mathbf{2}) = (\frac{1}{2}, 0)$ and $(\mathbf{2}^*) = (0, \frac{1}{2})$ of the Lorentz group $\mathbf{SO(3,1)}$, or rather its double covering group $\mathbf{SL(2,C)}$. A left-handed Weyl spinor is a two-component complex objects transforming under $\mathbf{SL(2,C)}$ as

$$\xi'_\alpha = M_\alpha{}^\beta \xi_\beta \tag{B.14}$$

with a matrix M from $\mathbf{SL(2, C)}$. This Weyl spinor belongs to the representation $(\mathbf{2})$, as deduced in Sect. 5.3.3.

Introduce the matrix

$$(\varepsilon^{\alpha\beta}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2.$$

For an arbitrary 2×2 matrix A, it holds that $A^T \varepsilon A = (\det A)\,\varepsilon$. Now consider the expression bilinear in two left-handed Weyl spinors ξ and η

$$\varepsilon^{\alpha\beta}\xi_\alpha\eta_\beta = \xi_1\eta_2 - \xi_2\eta_1.$$

This expression is invariant, since

$$\varepsilon^{\alpha\beta}\xi'_\alpha\eta'_\beta = \varepsilon^{\alpha\beta}M_\alpha{}^\gamma M_\beta{}^\delta\xi_\gamma\eta_\delta = (\det M)\,\varepsilon^{\gamma\delta}\xi_\gamma\eta_\delta = \varepsilon^{\gamma\delta}\xi_\gamma\eta_\delta.$$

Therefore $\varepsilon^{\alpha\beta}$ can be considered as a metric on $\mathbf{SL(2, C)}$. Its inverse $\varepsilon_{\alpha\beta}$, defined of course by $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\beta$, becomes in components: $(\varepsilon_{\alpha\beta}) = -(\varepsilon^{\beta\alpha})$. Making use of the ε-matrix, define the dual of a left-handed spinor ξ_α as

$$\xi^\alpha := \varepsilon^{\alpha\beta}\xi_\beta.$$

(This construction resembles how one switches from covariant to contravariant indices for Lorentz vectors.) Component-wise, this is $\xi^1 = \xi_2$ and $\xi^2 = -\xi_1$, and–again in analogy with Lorentz vectors–the product of a lower-index spinor with an upper-index spinor transforms as a scalar. However there is a subtlety: Unlike in the case of Lorentz vectors, one must pay attention to the position of indices in the Einstein summation convention:

$$\xi\eta := \xi^\alpha\eta_\alpha = \varepsilon^{\alpha\beta}\xi_\beta\eta_\alpha = -\varepsilon^{\beta\alpha}\xi_\beta\eta_\alpha = -\xi_\beta\eta^\beta = \eta^\beta\xi_\beta = \eta\xi, \tag{B.15}$$

assuming that spinors anti-commute. Since for a matrix M with unit determinant, $\varepsilon M \varepsilon^{-1} = (M^{-1})^T$ holds, the transformation behavior of an upper-index spinor under Lorentz transformations is calculated as

$$\xi'^{\alpha} = \varepsilon^{\alpha\beta}\xi'_\beta = \varepsilon^{\alpha\beta}M_\beta{}^\gamma\xi_\gamma = \varepsilon^{\alpha\beta}M_\beta{}^\gamma\varepsilon_{\gamma\delta}\xi^\delta = (M^{-1})_\gamma{}^\alpha\xi^\gamma. \tag{B.16}$$

The representations defined by (B.14) and (B.16) are equivalent due to the similarity transformation $M = \varepsilon^{-1}(M^{-1})^T\varepsilon$.

The complex conjugate spinor ξ^*_α transforms as

$$\xi'^*_\alpha = M^*_\alpha{}^\beta\xi^*_\beta.$$

The notation with dotted indices, where by definition[3]

$$\bar{\xi}_{\dot\alpha} := \xi^*_\alpha.$$

[3] This is only a notational sophistry: complex conjugation puts on a dot and a bar. The bar notation stems from [533]; others, like [134] use a † instead.

is due to B.L. van der Waerden. Thus

$$\bar{\xi}'_{\dot{\alpha}} = M^{*}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\xi}_{\dot{\beta}}. \tag{B.17}$$

This representation is not equivalent to (B.14) since there is in general no matrix S which provides a similarity transformation $M = SM^* S^{-1}$. Spinors which transform as in (B.17) are called right-handed Weyl spinors. As shown in the main text, they constitute the $(\mathbf{2}^*) = (0, \frac{1}{2})$ representation of $\mathbf{SL}(\mathbf{2}, \mathbf{C})$. Of course, one can also define the dual partners to right-handed Weyl spinors by

$$\bar{\xi}^{\dot{\alpha}} := \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}} \qquad\qquad (\varepsilon^{\dot{\alpha}\dot{\beta}}) \equiv (\varepsilon^{\alpha\beta})$$

which transforms with $(M^{*-1\,T})$. And again, the bilinear form

$$\bar{\xi}\bar{\eta} := \bar{\xi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} = -\bar{\xi}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}} = \bar{\eta}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} = \bar{\eta}\bar{\xi} \tag{B.18}$$

is a Lorentz scalar. If the dotted and the undotted indices are not denoted explicitly, it is always understood that the contraction of right-handed spinors follows this "down-up rule", whereas the contraction of two left-handed spinors happens with the "up-down rule" (B.15). With this convention, we employ the notation

$$\xi^2 = \xi\xi = \xi^\alpha \xi_\alpha \qquad\qquad \bar{\xi}^2 = \bar{\xi}\bar{\xi} = \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}.$$

Furthermore,

$$(\xi\eta)^\dagger = (\xi^\alpha \eta_\alpha)^\dagger = \bar{\eta}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} = \bar{\eta}\bar{\xi} = \bar{\xi}\bar{\eta}. \tag{B.19}$$

Two-Component Fierz Identities

In supersymmetry calculations, one frequently needs the Fierz identities[4]. To formulate them, first adapt the matrices σ^μ and $\bar{\sigma}^\mu$ to the van der Waerden calculus. In the dot-notation σ^μ has the index structure $\sigma^\mu_{\alpha\dot{\alpha}}$ with

$$(\sigma^\mu_{\alpha\dot{\alpha}})^* = (\sigma^{\mu *})_{\dot{\alpha}\alpha} = (\sigma^{\mu\dagger})_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}}.$$

The ε-tensors relate this to $\bar{\sigma}^\mu$ via

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} (\sigma^\mu)_{\beta\dot{\beta}}.$$

The following relations hold:

$$\sigma^\mu_{\alpha\dot{\alpha}} \overline{\sigma}^{\nu\dot{\alpha}\beta} + \sigma^\nu_{\alpha\dot{\alpha}} \overline{\sigma}^{\mu\dot{\alpha}\beta} = 2\eta_{\mu\nu} \delta_\alpha{}^\beta \tag{B.20a}$$

$$\overline{\sigma}^{\mu\dot{\alpha}\alpha} \sigma^\nu_{\alpha\dot{\beta}} + \overline{\sigma}^{\nu\dot{\alpha}\alpha} \sigma^\mu_{\alpha\dot{\beta}} = 2\eta_{\mu\nu} \delta^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{B.20b}$$

[4] named after the Swiss physicist Markus Fierz (1912–2006) who derived them as identities for Dirac spinors.

from which

$$\mathrm{Tr}(\sigma_\mu \overline{\sigma}_\nu) = 2\eta_{\mu\nu}.$$

Furthermore,

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} \qquad \bar\sigma^{\mu\dot\alpha\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = 2\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} \qquad \sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = 2\delta_\alpha{}^\beta\delta^{\dot\beta}{}_{\dot\alpha}.$$

The generators of the Lorentz group are in the dot notation given by

$$\begin{aligned}
(\sigma^{\mu\nu})_\alpha{}^\beta &= \frac{1}{4}\Big((\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} - (\sigma^\nu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\Big) \\
(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} &= \frac{1}{4}\Big((\bar\sigma^\mu)^{\dot\alpha\alpha}(\sigma^\nu)_{\alpha\dot\beta} - (\bar\sigma^\nu)^{\dot\alpha\alpha}(\sigma^\mu)_{\alpha\dot\beta}\Big).
\end{aligned} \tag{B.21}$$

For two anti-commuting spinors, there are the following Fierz identities

$$\xi_\alpha\eta_\beta = -\frac{1}{2}\xi\eta\epsilon_{\alpha\beta} - \frac{1}{8}\xi\sigma^{\mu\nu}\eta(\sigma_{\mu\nu})_{\alpha\beta} \tag{B.22a}$$

$$\bar\xi_{\dot\alpha}\bar\eta_{\dot\beta} = -\frac{1}{2}\bar\xi\bar\eta\bar\epsilon_{\dot\alpha\dot\beta} - \frac{1}{8}\bar\xi\bar\sigma^{\mu\nu}\bar\eta(\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta} \tag{B.22b}$$

$$\xi_\alpha\bar\eta_{\dot\beta} = \frac{1}{2}\xi\sigma^\mu\bar\eta(\bar\sigma_\mu)_{\dot\beta\alpha}. \tag{B.22c}$$

These identities allow one to derive further specific identities by contracting indices and making use of the relations (B.20) among the Pauli matrices. Examples are

$$(\xi\sigma^\mu\bar\xi)(\xi\sigma^\nu\bar\xi) = \frac{1}{2}\xi^2\bar\xi^2\eta^{\mu\nu} \tag{B.23a}$$

$$\xi\psi\xi\sigma^\mu\bar\eta = -\frac{1}{2}\xi^2\psi\sigma^\mu\bar\eta \tag{B.23b}$$

$$\bar\xi\bar\psi\bar\xi\bar\sigma^\mu\eta = -\frac{1}{2}\bar\xi^2\bar\psi\bar\sigma^\mu\eta. \tag{B.23c}$$

B.1.3 Spinors and Tensors

We saw in the previous subsection that it is possible to form a Lorentz scalar $\xi\eta$ from two Weyl spinors. Lorentz vectors and higher-order tensors can also be built explicitly from these spinors. This is again due to the fact that $\mathbf{SL(2, C)}$ is the double universal covering group of $\mathbf{SO(3, 1)}$. The notations and techniques for the explicit construction were already prepared in Appendix A.3.4.

The σ^μ and $\bar\sigma^\mu$ matrices can be used to establish a one-to-one correspondence between any bi-spinor $V_{\alpha\dot\alpha}$ respectively $W_{\dot\alpha\alpha}$ and the Lorentz four vector V^μ or W^μ, respectively, by

$$V^\mu = \bar\sigma^{\mu\dot\alpha\alpha}V_{\alpha\dot\alpha} \leftrightarrow V_{\alpha\dot\alpha} = \frac{1}{2}V^\mu\sigma_{\mu\alpha\dot\alpha} \quad W^\mu = \frac{1}{2}\bar\sigma^{\mu\alpha\dot\alpha}W_{\dot\alpha\alpha} \leftrightarrow W_{\dot\alpha\alpha} = \frac{1}{2}W^\mu\sigma_{\mu\dot\alpha\alpha}. \tag{B.24}$$

One easily verifies that V^2 and W^2 are Lorentz scalars and that both V^μ and W^μ transform as Lorentz vectors, e.g.

$$V'^{\mu} = \bar{\sigma}^{\mu\dot{\alpha}\alpha} M_\alpha{}^\beta M^{*\dot{\gamma}}_{\dot{\alpha}} V_{\beta\dot{\gamma}} = (M^\dagger \bar{\sigma}^\mu M)^{\dot{\gamma}\beta} V_{\beta\dot{\gamma}} = \Lambda^\mu{}_\nu \bar{\sigma}^{\mu\dot{\gamma}\beta} V_{\beta\dot{\gamma}} = \Lambda^\mu{}_\nu V^\nu .$$

This construction of a vector from Weyl spinors as

$$V^\mu = \xi^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\eta}^{\dot{\beta}}$$

shows that one is dealing with a $(1/2, 1/2)$ representation.

Two spinors may also serve to construct objects

$$F_{\alpha\beta} = \frac{1}{2}(\xi_\alpha \chi_\beta + \xi_\beta \chi_\alpha), \qquad \bar{F}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\bar{\zeta}_{\dot{\alpha}} \bar{\eta}_{\dot{\beta}} + \bar{\zeta}_{\dot{\beta}} \bar{\eta}_{\dot{\alpha}})$$

and from these

$$F^+_{\mu\nu} = \frac{i}{4}(\epsilon \sigma_{\mu\nu})^{\alpha\beta} F_{\alpha\beta}, \qquad F^-_{\mu\nu} = -\frac{i}{4}(\bar{\sigma}_{\mu\nu} \epsilon^T)^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}}$$

which obey the duality relations

$$F^{\pm\mu\nu} = \pm \epsilon^{\mu\nu\rho\sigma} F^\pm_{\rho\sigma}.$$

Finally $F_{\mu\nu} = F^+_{\mu\nu} + F^-_{\mu\nu}$ is a real antisymmetric Lorentz tensor. Since $F^+_{\mu\nu}$ belongs to the $(1, 0)$-representation, and $F^-_{\mu\nu}$ to $(0, 1)$, $F_{\mu\nu}$ is associated to the representation $(1, 0) \oplus (0, 1)$.

B.1.4 Dirac and Majorana Spinors

A Dirac spinor can be built from two Weyl spinors in the form

$$\psi(x) := \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} \doteq \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix}. \tag{B.25}$$

It transforms under $\mathbf{SL(2, C)}$ as

$$\psi' = S(M)\psi = \begin{pmatrix} M\xi \\ M^{-1\dagger}\bar{\eta} \end{pmatrix}.$$

Explicitly (and infinitesimally) this is

$$\begin{aligned} \psi' = \begin{pmatrix} \xi'_\alpha \\ \bar{\eta}'^{\dot{\alpha}} \end{pmatrix} &= \begin{pmatrix} \left(1 + \frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta & 0 \\ 0 & \left(1 + \frac{1}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \begin{pmatrix} \xi_\beta \\ \bar{\eta}^{\dot{\beta}} \end{pmatrix} \\ &= \left(I + \frac{1}{2}\omega_{\mu\nu} \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \right) \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix} = \left(I - \frac{i}{4}\omega_{\mu\nu}\gamma^{\mu\nu}\right)\psi. \end{aligned} \tag{B.26}$$

where the expression for $\gamma^{\mu\nu}$ according to (B.13) was used. Therefore, the spin matrix for a Dirac field is

$$\Sigma_D^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}.$$

The Weyl components–also called chirality eigenstates–are recovered from the Dirac spinor by projection operators

$$P_L = \frac{1-\gamma^5}{2} \qquad P_R = \frac{1+\gamma^5}{2} \tag{B.27}$$

as

$$\psi_L = P_L\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix} \qquad \psi_R = P_R\psi = \begin{pmatrix} 0 \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}.$$

The matrices A and C, implicitly given by (B.9), serve to define the adjoint and the charge conjugate spinor as

$$\overline{\psi} := \psi^\dagger A \qquad \psi^C := C\overline{\psi}^T = CA^T\psi^*. \tag{B.28}$$

The Dirac bilinears built from the basis (B.8) as $\overline{\psi}_1\Gamma^A\psi_2$ can be shown to obey

$$(\overline{\psi}_1\Gamma^\alpha\psi_2)^\dagger = +\overline{\psi}_2\Gamma^\alpha\psi_1 \qquad \text{for} \qquad \Gamma^\alpha = (1, \gamma^\mu, \gamma^5) \tag{B.29a}$$

$$(\overline{\psi}_1\Gamma^\alpha\psi_2)^\dagger = -\overline{\psi}_2\Gamma^\alpha\psi_1 \qquad \text{for} \qquad \Gamma^\alpha = (\gamma^\mu\gamma^5, \gamma^{\mu\nu}). \tag{B.29b}$$

In terms of the Weyl components and in the chiral basis, the adjoint becomes explicitly

$$\overline{\psi} = (\bar{\xi}_{\dot{\alpha}}, \ \ \eta^\alpha)\begin{pmatrix} 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix} = (\eta^\beta, \ \ \bar{\xi}_{\dot{\beta}}). \tag{B.30}$$

According to (B.11), numerically $A = \gamma^0$, but using the gamma matrix here would spoil the spinor index structure. One verifies that

$$\overline{\psi}_1\psi_2 = \eta_1\xi_2 + \bar{\xi}_1\bar{\eta}_2,$$

revealing that $\bar{\psi}_1\psi_2$ is a Lorentz scalar. The bilinears

$$\overline{\psi}_1\gamma^5\psi_2, \qquad \overline{\psi}_1\gamma^\mu\psi_2, \qquad \overline{\psi}_1\gamma^\mu\gamma^5\psi_2, \qquad \overline{\psi}_1\gamma^{\mu\nu}\psi_2$$

transform as pseudo-scalars, vectors, axial-vectors and tensors with respect to parity transformations.

Charge conjugation is in the chiral representation mediated by

$$C = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} = i\gamma^2\gamma^0 \tag{B.31}$$

that is,

$$\psi^C := C\bar{\psi}^T = \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}. \tag{B.32}$$

As a consequence we derive the identity

$$\overline{\psi}^C \psi = \xi^2 + \bar{\eta}^2 = \overline{\psi}^C_R \psi_L + \overline{\psi}^C_L \psi_R. \tag{B.33}$$

The charge conjugate of a left-handed field is right-handed and vice versa. This is obvious in the Weyl notation: If for example $\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}$ then $\psi^C = \begin{pmatrix} 0 \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}$. Furthermore,

$$\begin{aligned}
\psi^C_L &= P_L \psi^C = P_L C \overline{\psi}^T = C(\overline{\psi} P_L)^T = C\overline{\psi}^T_R = C\gamma^0 \psi^*_R \\
\overline{\psi}^C_L &= \psi^{C\dagger}_L \gamma^0 = \psi^{*\dagger}_R \gamma^{0\ T\dagger} C^\dagger \gamma^0 = -\psi^T_R C^{-1} = \psi^T_R C.
\end{aligned}$$

Be aware that $(\psi^C)_L \neq (\psi_L)^C$ but $(\psi^C)_L = (\psi_R)^C$.

A *Majorana spinor* ψ_M is a Dirac spinor which is identical to its charge conjugate: $\psi^C_M = \psi_M$. Thus a Majorana spinor can be expressed in terms of a single Weyl spinor

$$\psi_M = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}. \tag{B.34}$$

It is possible to define two Majorana spinors from a Dirac spinor $\psi = \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}$ as

$$\chi_L = \psi_L + (\psi^C)_R = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \qquad \chi_R = \psi_R + (\psi^C)_L = \begin{pmatrix} \eta_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}. \tag{B.35}$$

Majorana spinors satisfy the identities

$$(\overline{\psi}_1 \Gamma \psi_2) = +\overline{\psi}_2 \Gamma \psi_1 \qquad \text{for} \qquad \Gamma = (1, \gamma^5, \gamma^\mu \gamma^5) \tag{B.36a}$$

$$(\overline{\psi}_1 \Gamma \psi_2) = -\overline{\psi}_2 \Gamma \psi_1 \qquad \text{for} \qquad \Gamma = (\gamma^\mu, \gamma^{\mu\nu}). \tag{B.36b}$$

B.2 *$\mathbb{Z}_2$ Gradings

By the term $\mathbb{Z}_2$ gradings I denote in the following those mathematical techniques which allow one to speak of bosonic and fermionic degrees of freedom even in an non-quantized theory.

B.2.1 Definitions

A $\mathbb{Z}_2$ graded vector space is a vector space which admits the direct sum decomposition

$$V = V^{(0)} \bigoplus V^{(1)}.$$

Hereafter, I will leave off the $\mathbb{Z}_2$, and also refer to $V^{(0)}$ and $V^{(1)}$ as the even and odd subspaces. An element $X \in V$ is called 'homogeneous' if it belongs either to $V^{(0)}$ or to $V^{(1)}$. Homogeneous elements are assigned a *Grassmann parity* $|X|$ by the convention that $|X| = i$ for $X \in V^{(i)}$. Thus, homogeneous elements are either even/bosonic or odd/fermionic.

A graded algebra is a graded vector space $A = A^{(0)} \bigoplus A^{(1)}$ which is also an algebra, that is a structure with a linear and associative mapping $\Box : A \times A \to A$; see A.1.1. A graded algebra is called supercommutative if all homogeneous elements $X, Y \in A$ satisfy $X \Box Y = (-1)^{|X||Y|} Y \Box X$.

The elementary supercommutative graded algebra is the *Grassmann algebra*[5]. It consists of the set of generators $\{\psi_a\}$ $(a = 1, \ldots, N)$ with $\psi_a \psi_b + \psi_b \psi_a = 0$. An arbitrary element in this algebra (also called a 'supernumber') has the form

$$\Omega(\psi) = \omega_0 + \omega^a \psi_a + \frac{1}{2} \omega^{ab} \psi_a \psi_b + \ldots + \frac{1}{N!} \omega^{1 \ldots N} \psi_1 \psi_2 \ldots \psi_N \tag{B.37}$$

with real or complex coefficients $\omega^{a_1, \cdots, a_k}$ which can be considered completely antisymmetric in their indices.

B.2.2 Supertrace and Superdeterminant

In the space of supernumbers, one can establish the notions of supermatrices: The object

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

built from submatrices A and D with even, and B and C with odd supernumbers constitutes a supermatrix. The standard matrix multiplication of two supermatrices is again a supermatrix. The supertrace of the supermatrix A is defined by

$$\text{str } M := \text{tr } A - \text{tr } D.$$

It obeys the usual relation for a trace of a matrix:

$$\text{str } (M_1 + M_2) = \text{str } M_1 + \text{str } M_2 \qquad \text{str } (M_1 M_2) = \text{str } (M_2 M_1).$$

---

[5] named after the German mathematician H.G. Grassmann (1809-1877).

The superdeterminant (sometimes called the *Berezinian* Ber) of a supermatrix is defined by

$$\mathrm{Ber} = \mathrm{sdet}\ M := \exp \mathrm{str}\ \ln M$$

with

$$\ln M = \ln(I + M - I) = \sum_{i \geqslant 1} (-1)^{i-1} \frac{1}{i} (M - I)^i.$$

For matrices M which differ infinitesimally from the unit matrix $(M = I + \epsilon)$:

$$\mathrm{sdet}\ M \approx 1 + \mathrm{str}\ \epsilon.$$

Furthermore,

$$\mathrm{sdet}\ M = \det A [\det(D - CA^{-1}B)]^{-1} = (\det D)^{-1} [\det(A - BD^{-1}C)],$$

from which one can see that M is non-invertible if either A or d are non-invertible. Therefore, $\mathrm{sdet}\ M$ is defined only for invertible supermatrices M. If it exists, the superdeterminant fulfills

$$\mathrm{sdet}\ (M_1 M_2) = (\mathrm{sdet}\ M_1)(\mathrm{sdet}\ M_2).$$

The (super)transpose of a supermatrix M is

$$M^T = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}$$

with

$$(M_1 M_2)^T = M_2^T M_1^T \qquad (M^{-1})^T = (M^T)^{-1}$$
$$\mathrm{str}\ M^T = \mathrm{str}\ M \qquad \mathrm{sdet}\ M^T = \mathrm{sdet}\ M.$$

B.2.3 Differentiation and Integration

It is possible to define derivatives of the multinomials (B.37). Consider the variations

$$\delta\Omega(\psi) = \delta\psi_a \frac{\partial^R \Omega}{\partial \psi_a} = \frac{\partial^L \Omega}{\partial \psi_a}_L \delta\psi_a$$

in which a R(ight)- and a L(eft)-derivative is defined. Only for even elements in the Grassmann algebra are they identical. In this text, if not stated otherwise, R-derivatives are always used. For these,

$$\frac{\partial}{\partial \psi_a}(\psi_{a_1}\dots\psi_{a_k}) = \delta_{aa_1}\psi_{a_2}\dots\psi_{a_k} - \delta_{aa_2}\psi_{a_1}\psi_{a_3}\dots\psi_{a_k} \\ +\dots(-1)^{k-1}\delta_{aa_k}\psi_{a_1}\dots\psi_{a_{k-1}}. \qquad \text{(B.38)}$$

As with ordinary differentiation, there is a product rule

$$\frac{\partial}{\partial \psi_\alpha}(\Omega_1\Omega_2) = \frac{\partial \Omega_1}{\partial \psi_\alpha}\Omega_2 + (-1)^{|\Omega_1|}\Omega_1\frac{\partial \Omega_2}{\partial \psi_\alpha}.$$

For $\psi'_a = A_{ab}\psi_b$, there also is a chain rule

$$\frac{\partial}{\partial \psi_a}\Omega(\psi'(\psi)) = \frac{\partial \Omega}{\partial \psi'_b}A_{ba}.$$

A more or less heuristic, but nevertheless very successful concept of integration within a Grassmann algebra[6] is due to F.A. Berezin. Berezin's recipe is motivated by requiring the definite integral over a Grassmann function to have properties of an ordinary integral in the limits $-\infty$ and $+\infty$, namely linearity and shift invariance. For instance, for one variable

$$\int_{-\infty}^{+\infty} dx\ cF(x) = c\int_{-\infty}^{+\infty} dx\ F(x) \qquad \int_{-\infty}^{+\infty} dx\ F(x+a) = \int_{-\infty}^{+\infty} dx\ F(x).$$

A Grassmann function in one odd variable necessarily has the form $f(\psi) = \alpha + \theta\beta$. Demanding that linearity and shift invariance hold for $\int d\psi\ f(\psi)$ leads to

$$\int d\psi\ f(\psi) = \beta \qquad \text{or} \qquad \int d\psi = 0, \qquad \int d\psi\ \psi = 1.$$

This is immediately extended to more than one Grassmann variable as

$$\int d\psi_a = 0 \qquad \int \psi_b d\psi_a = \delta_{ab} \qquad \text{(B.39)}$$

together with the linearity relation

$$\int (c_1\Omega_1 + c_2\Omega_2)d\psi_\alpha = c_1\int \Omega_1 d\psi_\alpha + c_2\int \Omega_2 d\psi_\alpha.$$

Observe a peculiarity arising in the change of variables: Take first the simple example of changing only one variable as in $\psi'_a = \lambda\psi_a$. Since

$$\int \psi'_a d\psi_a = \lambda\int \psi_a d\psi_a = \lambda,$$

consistency requires $d\psi'_a = \lambda^{-1}d\psi_a$. In general, for $\psi'_a = \lambda_{ab}\psi_b$:

$$\int \Omega\big(\psi(\psi')\big)d\psi'_N\dots d\psi'_1 = (\det\lambda)^{-1}\int \Omega(\psi)d\psi_N\dots d\psi_1.$$

[6] A sophisticated differential geometric meaning is elaborated in [404].

Thus changes in the integration of Grassmann odd variables transform with the inverse of the Jacobian, contrary to the case for "ordinary" even variables. If there are both odd and even variables $z^A = (x_k, \psi_a)$ one can show that

$$\int_{\mathcal{V}} F(z)dz = \int_{\mathcal{V}} F(x(x',\psi'),\psi(x',\psi'))\ \mathrm{sdet}\Big(\frac{\partial(x,\psi)}{\partial(x',\psi')}\Big).$$

This is immediately obvious for the case that even variables are transformed to even ones and odd variables to odd ones, where the super(Jacobian) factorizes as

$$\mathrm{sdet}\frac{\partial(x,\psi)}{\partial(x',\psi')} = \det\Big(\frac{\partial x}{\partial x'}\Big)\det{}^{-1}\Big(\frac{\partial\psi}{\partial\psi'}\Big).$$

And it can easily be proven for infinitesimal transformations.

The following calculation with Grassmann numbers seems to be a little preposterous, but the result is beneficial for formulating the functional-integral version of gauge theories:

$$I(A) = \int\prod_{k=1}^{N} d\psi_k d\tilde{\psi}_k \exp\Big(\sum_{i,j=1}^{N}\tilde{\psi}_i A_{ij}\psi_j\Big) = \det A. \tag{B.40}$$

In this expression $\psi_j, \tilde{\psi}_i$ are two distinct sets of Grassmann variables, and A is some real or complex matrix. This identity offers a seemingly strange way to write a determinant. The proof is straightforward: Think of expanding the exponential and integrating each summand. Due to the rules of Berezin integration (B.39) the only term that survives in this step is the Nth order term:

$$I(A) = \int\prod_{k=1}^{N} d\psi_k d\tilde{\psi}_k \frac{1}{N!}\Big(\sum_{i,j=1}^{N}\tilde{\psi}_i A_{ij}\psi_j\Big)^N.$$

But still by far not all terms contribute; instead,

$$I(A) = \Big(\int\prod_{k=1}^{N} d\psi_k d\tilde{\psi}_k\tilde{\psi}_k\psi_k\Big)\Big(\sum_{\mathrm{perm}}\epsilon^{i_1 i_2\ldots i_N}A_{1i_1}A_{2i_2}\ldots A_{Ni_N}\Big) = \det A.$$

By the shift invariance of the Berezin integration, we can immediately generalize (B.40) to

$$I(A,\eta,\tilde{\eta}) = \int\prod_{k=1}^{N} d\psi_k d\tilde{\psi}_k\ \exp(\tilde{\psi}A\psi + \tilde{\eta}\psi + \tilde{\psi}\eta) = \det A\ \exp(-\tilde{\eta}A^{-1}\eta). \tag{B.41}$$

Another generalization can be obtained by additionally considering the Gaussian integration (D.4) with respect to even variables:

$$\int d^N z \exp\Big(-\frac{1}{2}z^A M_{AB} z^B\Big) = (2\pi)^{N/2}(\mathrm{sdet}M)^{-1/2},$$

where $M = M^T$ is an $N \times N$ supermatrix.

B.2.4 Pseudo-Classical Mechanics

Based upon the notion of Grassmann variables it is possible to formulate dynamical theories with bosonic and fermionic degrees of freedom; see e.g. [78]. Assume an action

$$S = \int dt\ L(q, \dot{q}, \psi, \dot{\psi})$$

where the Lagrangian depends on even coordinates q_i and odd coordinates ψ_α. Define the momenta

$$p^i = \frac{\partial L}{\partial q_i} \qquad \pi^\alpha = \frac{\partial L}{\partial \psi_\alpha}.$$

Then from the Hamiltonian

$$H = \dot{q}_i p^i + \dot{\psi}_\alpha \pi^\alpha - L$$

the equations of motion are derived as

$$\dot{p}^i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p^i} \tag{B.42a}$$

$$\dot{\pi}^\alpha = -\frac{\partial H}{\partial \psi_\alpha} \qquad \dot{\psi}_\alpha = -\frac{\partial H}{\partial \pi^\alpha} \tag{B.42b}$$

if the derivatives from the right are used. (The minus sign in the last equation is not a mistake!) The Hamilton equations can be written in terms of generalized Poisson brackets or *Bose-Fermi brackets* $\{X, Y\}_{\mathrm{BF}}$:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}_{\mathrm{BF}}.$$

The Bose-Fermi brackets are most compactly defined in the following way–motivated by the symplectic structure of “ordinary” mechanics: Call $(Z^A) = \{q_i, \psi_\alpha, p^i, \pi^\alpha\}$, then

$$\{F, G\}_{\mathrm{BF}} := \frac{\partial^R F}{\partial Z^A} C^{AB} \frac{\partial^L G}{\partial Z^B} \tag{B.43}$$

with $C^{AB} = \{Z^A, Z^B\}_{BF}$, or explicitly

$$\{q_i, p^j\}_{BF} = -\{p^j, q_i\}_{BF} = \delta_i^j, \qquad \{\psi_\alpha, \pi^\beta\}_{BF} = \{\pi^\beta, \psi_\alpha\}_{BF} = -\delta_\alpha^\beta, \tag{B.44}$$

all others vanishing. Thus we not only get back the Poisson brackets for bosonic phase space variables, but also brackets for the fermionic phase space variables that prepare the commutator relations for fermionic quantum operators or–conversely–are the classical limit for a fermionic system. This of course is taken to be only formal, but it is a successful calculational device. In the following–and in all those cases where use is made of the Bose-Fermi bracket–the index BF on the bracket will be omitted.

The Bose-Fermi brackets obey $\{F,G\} = -(-1)^{|F||G|}\{F,G\}$ and therefore they qualify as a supercommutative graded algebra with the identification $F \,\square\, G = \{F,G\}$. Furthermore

$$\begin{aligned}
&\{A, B+C\} = \{A,B\} + \{A,C\} \\
&\{A, BC\} = (-1)^{|A||B|}B\{A,C\} + \{A,B\}C \\
&(-1)^{|A||C|}\{\{A,B\},C\} + (-1)^{|B||A|}\{\{B,C\},A\} + (-1)^{|C||B|}\{\{C,A\},B\} = 0,
\end{aligned}$$

which makes it a graded Lie algebra[7].

Every Lie group has an associated Lie algebra. In extending the group concept to graded Lie groups there is also the notion of a graded Lie algebra. This is spanned by a set of even and odd generators $X = \{X^A\}$ with a mapping $X \times X \to X$:

$$\{X^A, X^B\} := X^A X^B - (-1)^{|A||B|} X^B X^A = \sum_C \gamma^{ABC} X^C \tag{B.45}$$

with the shorthand notation $|A| = |X_A|$, i.e. $+1(-1)$ if $|X_A|$ is bosonic(fermionic). The graded Lie algebra properties request that the structure functions γ^{ABC} must satisfy the conditions

$$\begin{aligned}
&\gamma^{BAC} + (-1)^{|A||B|}\gamma^{ABC} = 0 \\
&(-1)^{|A||C|}\gamma^{ABD}\gamma^{DCE} + (-1)^{|B||A|}\gamma^{BCD}\gamma^{DAE} + (-1)^{|C||B|}\gamma^{CAD}\gamma DBE = 0.
\end{aligned}$$

B.3 *Supergeometry

The supermathematics behind local and global supersymmetry became quite advanced from the 1980s on. If you are interested in this aspect of supersymmetry you might like to refer to [60, 119].

B.3.1 Superspace

Supersymmetry is most compactly and elegantly described in terms of superfields. These are defined in superspace, an "upgrade" of Minkowski space. Fields in Minkowski space are in one-to-one correspondence with its isotropy group. Superfields are in a similar way tight to the symmetry group of superspace.

[7] The designations are not unique in the literature: You find the names "superalgebra" and "super Lie algebra", although mathematically there are subtle differences [99].

The idea of superspace was developed by A. Salam and J. Strathdee in 1974 [446]. Superspace has, besides the coordinates x^μ $(\mu = 0, \ldots, 3)$, which commute, additional anti-commuting coordinates θ^a, $(a = 1, \ldots, 3)$. (Everything can of course be generalized to higher dimensions.)

Remembering that in the case of Minkowski space, the elements of the isometry group are directly linked to the motions in Minkowski space, we expect for the superspace a group element of the form

$$G(x,\theta) = e^{i(x^\mu T_\mu + \bar{\theta}^a Q_a)}. \tag{B.46}$$

Multiplying two group elements we apply the Baker-Campbell-Hausdorff's formula (A.6)

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots}$$

generalized to Grassmann objects. Calculate for instance $G(0,\epsilon)G(x,\theta)$. Here $A = i\bar{\epsilon}^a Q_a$ and $B = i(x^\mu P_\mu + \bar{\theta}^b Q_b)$ for which–due to the Wess-Zumino algebra (specifically (8.58) and (8.57))–one obtains

$$\begin{aligned} [A,B] &= \left[i(\bar{\epsilon}^a Q_a), i(x^\mu P_\mu + \bar{\theta}^b Q_b)\right] = -\left[\bar{\epsilon}^a Q_a, \bar{Q}_b \theta^b\right] \\ &= -\bar{\epsilon}^a \{Q_a, \bar{Q}_b\}\theta^b = -2(\bar{\epsilon}^a (\gamma^\mu)_{ab}\theta^b)P_\mu. \end{aligned}$$

Since the higher nested brackets of A and B vanish in this case, the result is

$$G(0,\epsilon)G(x^\mu,\theta) = e^{i\left[\bar{\epsilon}^a Q_a + x^\mu T_\mu + i\bar{\theta}^b Q_b + i(\bar{\epsilon}^a(\gamma^\mu)_{ab}\theta^b)T_\mu\right]} = G(x + i\bar{\epsilon}\gamma^\mu\theta,\ \theta+\epsilon).$$

This induces a translation in superspace

$$g(\epsilon) : (x^\mu, \theta^a) \to (x^\mu + i\bar{\epsilon}^a(\gamma^\mu)_{ab}\theta^b,\ \theta^a + \epsilon^a). \tag{B.47}$$

As it turns out, in supersymmetry it is much more appropriate and convenient to work with two-component Weyl spinorial coordinates. Superspace is thus "coordinized" by $z^M = (x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$. The equivalent to (B.46) is

$$G(x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}) = e^{i(x^\mu T_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})}$$

with

$$g(\epsilon,\bar{\epsilon}) : (x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}) \to (x^\mu + \xi^\mu,\ \theta^\alpha + \epsilon^\alpha,\ \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}) \tag{B.48}$$

where

$$\xi^\mu = i\epsilon\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\epsilon}. \tag{B.49}$$

The operators that generate these transformations are the supercharges

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \qquad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu \tag{B.50}$$

obeying the anti-commutator relations (8.60).

B.3.2 Superfields

Superfields $\mathsf{F}(z^M)$ are mappings of the superspace to a Grassmann algebra, since a Taylor expansion of $\mathsf{F}(x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$ in the odd coordinates has at most terms quadratic in θ^α and $\bar{\theta}_{\dot{\alpha}}$. Therefore, a general superfield can be written as

$$\mathsf{F}(x, \theta, \bar{\theta}) = f(x) + \theta\, \phi(x) + \bar{\theta}\, \bar{\chi}(x) + \theta^2\, m(x) + \bar{\theta}^2\, \bar{n}(x) + (\theta\sigma^\mu\bar{\theta})\, v_\mu(x) \tag{B.51}$$

$$+\, \theta^2\bar{\theta}\, \bar{\lambda}(x) + \bar{\theta}^2\theta\, \psi(x) + \theta^2\bar{\theta}^2\, d(x). \tag{B.52}$$

Here and in the following, I comply with the prevailing convention that when indices are not written explicitly, undotted indices are meant to be descending and dotted indices are meant to be ascending; see (B.15, B.18).

It is easy to verify that the sum of two superfields is again a superfield. Also, a linear combinations of superfields is again a superfield. But notice that also the product of two superfields is again a superfield. The derivative of a superfield is defined through $\delta\mathsf{F} = \delta z^M \frac{\partial \mathsf{F}}{\partial z^M}$ and

$$\frac{\partial}{\partial z^M}(\mathsf{F}_1\mathsf{F}_2) = \frac{\partial \mathsf{F}_1}{\partial z^M} F_2 + (-1)^{|z^M|}\mathsf{F}_1\frac{\partial \mathsf{F}_2}{\partial z^M}.$$

The superfield (B.51) contains as component fields

$$\begin{aligned} 4 \text{ complex (pseudo) scalar fields:} &\quad f, m, n, d \\ 4 \text{ Weyl spinor fields:} &\quad (\phi, \psi) \in \left(\frac{1}{2}, 0\right), \qquad (\chi, \lambda) \in \left(0, \frac{1}{2}\right) \\ 1 \text{ Lorentz } 4-\text{vector field:} &\quad v_\mu. \end{aligned}$$

Thus, there are 16 bosonic and 16 fermionic field components. Let us reflect the mass dimensions of the different quantities. The superspace coordinate transformations (B.47) imply that ϵ and θ have the same mass dimension, namely $-1/2$. Assuming that the scalar superfield f has mass dimension $[f] = 0$, we find

$$\left[\{f, \phi, \bar{\chi}, m, \bar{n}, v_\mu, \bar{\lambda}, \psi, d\}\right] = \left\{0, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, 2\right\}. \tag{B.53}$$

Next, we are interested in the behavior of a superfield under a supersymmetry transformation with parameters $\epsilon, \bar{\epsilon}$:

$$\mathsf{F}(x, \theta, \bar{\theta}) \to \mathsf{F}(x + \xi, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}), \tag{B.54}$$

where ξ is the expression (B.49). Again, for most purposes it is sufficient to consider infinitesimal transformations

$$\delta_{\epsilon\bar{\epsilon}}\mathsf{F}(x, \theta, \bar{\theta}) := \left\{\xi^\mu\partial_\mu + \epsilon\frac{\partial}{\partial\theta} + \bar{\epsilon}\frac{\partial}{\partial\bar{\theta}}\right\}\mathsf{F}. \tag{B.55}$$

A straightforward (although quite lengthy) calculation yields the explicit transformations of the component fields. In fact, the transformation rules of the fields can be deduced–up to numerical constants–by observing the balance of mass dimensions and saturation of Lorentz and Weyl indices. Take for instance the transformation of f. It must be linear in ϵ and $\bar{\epsilon}$: $\delta f = f_1\epsilon + f_2\bar{\epsilon}$. Now $[\delta f] = 0$, and from $[\epsilon] = -1/2$ we deduce $[f_i] = +1/2$. The only fields at our disposal carrying this mass dimension are ϕ and $\bar{\chi}$. No derivative (which has $[\partial_\mu] = +1$) applied on any of the fields yields terms with mass dimension $+1/2$. Thus $[\delta f]$ must be a linear combination of $\epsilon\phi$ and $\bar{\epsilon}\bar{\chi}$ with numerical coefficients. For another example, take the component field d for which generically $\delta d = d_1\epsilon + d_2\bar{\epsilon}$. Since $[\delta d] = 2$ we read off $[d_i] = 5/2$. Component fields with this mass dimension do not exist in the superfield F. The only way to generate terms with the requested mass dimension is by taking the derivatives of the fields $\bar{\lambda}$ and ψ. Thus

$$\delta d = d_1^\mu \partial_\mu \bar{\lambda}\epsilon + d_2^\mu \partial_\mu \psi\bar{\epsilon} = \tilde{d}_1 \epsilon\sigma^\mu \partial_\mu \bar{\lambda} + \tilde{d}_2 \bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \psi,$$

where the matrices σ and $\bar{\sigma}$ make their appearance because they are the only objects around by which the Lorentz indices and the Weyl indices can be saturated. (The result that the highest component in the superfield expansion transforms as a derivative will be seen to be of utmost importance in the construction of supersymmetric actions.) The only things left open are numerical constants $\tilde{d}_i$. In a quite similar way, one may derive the transformation rules of the other fields. I refrain from doing this here, since the most generic superfield is not of prime importance–mainly because it is not irreducible.

A superfield is reducible if it contains a subset of fields which is closed under supersymmetry transformations. One may identify irreducible multiplets by imposing restrictions on the superfield that are compatible with supersymmetry. It turns out that two types of specific superfields are sufficient to construct the most general actions in superspace, namely vector and chiral superfields. As will be seen later, this conforms with fields occurring in the Standard Model, namely chiral fermions and vector bosons.

Chiral Superfields

For the specification of a chiral superfield, we need the notion of covariant spinor derivatives, defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu. \tag{B.56}$$

These anti-commute with the generators of the supersymmetry transformations (B.50) and all the Lorentz generators. Furthermore,

$$\{D_\alpha, D_\beta\} = 0 = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} \tag{B.57a}$$

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu. \tag{B.57b}$$

These operators have the same mass dimension as the superymmetry generators, namely $[D] = +1/2$.

A chiral superfield (sometimes called a left-chiral superfield) is indirectly defined by the condition

$$\bar{D}_{\dot{\alpha}} \Phi = 0. \tag{B.58}$$

This constraint on a generic superfield can conveniently be solved in terms of the variables θ and

$$y^\mu := x^\mu + i\theta\sigma^\mu\bar{\theta},$$

since

$$\bar{D}_{\dot{\alpha}} y^\mu := \bar{D}_{\dot{\alpha}}(x^\mu + i\theta\sigma^\mu\bar{\theta}) = 0 \qquad \bar{D}_{\dot{\alpha}}\theta = 0.$$

Thus any superfield depending only on y^μ and θ is a chiral superfield. Expanded in the odd coordinates

$$\begin{aligned}\Phi &= A(y) + \sqrt{2}\,\theta\,\psi(y) + \theta^2\,F(y)\\ &= \left[A(x) + i(\theta\sigma^\mu\bar{\theta})\,\partial_\mu A(x) - \frac{1}{2}(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})\,\partial_\mu\partial_\nu A(x)\right]\\ &\qquad + \left[\sqrt{2}\theta\,\psi(x) + i\sqrt{2}(\theta\sigma^\mu\bar{\theta})\theta\,\partial_\mu\psi(x)\right] + \theta^2\,F(x)\end{aligned} \tag{B.59}$$

where the terms in the squared brackets are Taylor expansion around x - the expansion again yielding only few terms due to the Grassmann character of the θ- and $\bar{\theta}$-coordinates. By using the Fierz identities in the form (B.23) the latter expression can be written as

$$\Phi(x) = A + \sqrt{2}\theta\,\psi + \theta^2\,F + i(\theta\sigma^\mu\bar{\theta})\,\partial_\mu A + \frac{i}{\sqrt{2}}\theta^2(\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi) - \frac{1}{4}\theta^2\bar{\theta}^2\,\partial_\mu\partial^\mu A.$$

On the other hand, this is the most general solution to condition (B.58), since in terms of $(y, \theta, \bar{\theta})$ the spinor derivatives (B.56) are

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu} \qquad\qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}.$$

The chiral superfield (B.59) has in its multiplet two complex scalars A and F, and a Weyl spinor ψ_α. It seems that there is a mismatch between the four bosonic and the two fermionic degrees of freedom. However, as seen later, not all bosonic fields represent physical degrees of freedom since the F-fields do not propagate. In any case, with regard to a supersymmetric extension of the standard model, the chiral superfield contains not only the chiral quarks and leptons (in terms of ψ), but also a scalar (Higgs) boson.

Recalling that θ has mass dimension $[\theta] = -1/2$, and assigning the usual mass-dimension $[A] = +1$ we deduce the (usual) mass dimension $+3/2$ for the fermionic field ψ_α, and the (unusual) mass dimension $[F] = +2$.

The superfield $\Phi^\dagger$ obeys $D_\alpha \Phi^\dagger = 0$. It is a function of $y^{*\mu} := x^\mu - i\theta\sigma^\mu\bar{\theta}$ and $\bar{\theta}$ only, also called a right-chiral or an anti-chiral superfield. The explicit form of $\Phi^\dagger$ in terms of components is obtained from (B.59) by conjugation:

$$\Phi^\dagger = A^*(y^*) + \sqrt{2}\bar{\theta}\ \bar{\psi}(y^*) + \bar{\theta}^2\ F^*(y^*).$$

If a supersymmetry transformation (B.55) is applied to a chiral superfield, it turns out to be convenient to phrase this in the variables y and θ as

$$\begin{aligned}\delta_{\epsilon\bar{\epsilon}}\Phi(y,\theta) &= \{\epsilon\frac{\partial}{\partial\theta} + 2i\theta\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\}\Phi \\ &= \sqrt{2}\ \epsilon\ \psi + 2\epsilon\theta\ F + 2i\theta\sigma^\mu\bar{\epsilon}\ \partial_\mu A + 2\sqrt{2}i\theta\sigma^\mu\bar{\epsilon}\theta\ \partial_\mu\psi \\ &\equiv \delta_{\epsilon\bar{\epsilon}}\ A + \sqrt{2}\ \theta\delta_{\epsilon\bar{\epsilon}}\ \psi + \theta^2\delta_{\epsilon\bar{\epsilon}}\ F.\end{aligned} \tag{B.60}$$

The last line displays the fact that the supersymmetry algebra closes, i.e. that the supersymmetry transformation of a chiral superfield is again a chiral superfield. By comparison of the last two expressions in (B.60) we deduce the transformation behavior of the different fields:

$$\delta_{\epsilon\bar{\epsilon}}A = \sqrt{2}\epsilon\psi \tag{B.61a}$$

$$\delta_{\epsilon\bar{\epsilon}}\psi = \sqrt{2}\epsilon F + i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu A \tag{B.61b}$$

$$\delta_{\epsilon\bar{\epsilon}}F = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi. \tag{B.61c}$$

In order to obtain the transformation for the F field one needs to make use of the identity $\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2$ according to (B.22). Notice that the F field transforms into a total derivative. The transformations (B.61) are related to those of the fields in the Wess-Zumino model (8.44a, 8.49, 8.48) if one defines from S, P and from the F_i the complex fields

$$A := \frac{1}{\sqrt{2}}(S + iP) \qquad F := \frac{1}{\sqrt{2}}(F_1 + iF_2). \tag{B.62}$$

Products of chiral superfields are again chiral superfields. The product of two left-chiral superfields for instance is

$$\begin{aligned}\Phi_i\Phi_j &= (A_i + \sqrt{2}\theta\psi_i + \theta\theta F_i)(A_j + \sqrt{2}\theta\psi_j + \theta\theta F_j) \\ &= A_iA_j\ +\ \sqrt{2}\theta(\psi_iA_j + A_i\psi_j)\ +\ \theta^2(A_iF_j + A_jF_i - \psi_i\psi_j).\end{aligned}$$

It has become conventional in the supersymmetry community to define

$$\Phi_i\Phi_j\big|_F := \int d^2\theta\ \ \Phi_i\Phi_j = A_iF_j + A_iF_j - \psi_i\psi_j.$$

By induction, also the product of any number of left-chiral superfields is a left-chiral superfield. Thus for instance

$$\begin{aligned}\Phi_i\Phi_j\Phi_k\big|_F &:= \int d^2\theta \;\; \Phi_i\Phi_j\Phi_k \\ &= F_iA_jA_k + F_jA_kA_i + F_kA_iA_j - \psi_i\psi_jA_k - \psi_j\psi_kA_i - \psi_k\psi_jA_i.\end{aligned}$$

Everything derived above for (left-)chiral superfields applies *mutatis mutandis* to right-chiral superfields.

Vector Superfields

A superfield is called a vector superfield $V(x,\theta,\bar\theta)$ if it is self-conjugate $(V = V^\dagger)$. (The naming is a little confusing here. The term "vector" superfield originates from the fact that it contains a real Minkowski vector field v_μ.) This restriction cuts the number of fields in the general superfield (B.51) in half. Expressed as a power series expansion in $(\theta^\alpha, \bar\theta_{\dot\alpha})$ a vector superfield can be written as

$$\begin{aligned}V(x,\theta,\bar\theta) &= C(x) + i\theta\chi(x) + i\bar\theta\bar\chi(x) \\ &\quad + \frac{i}{2}\theta^2\left[M(x)+iN(x)\right] - \frac{i}{2}\bar\theta^2\left[M(x)-iN(x)\right] - \theta\sigma^\mu\bar\theta\; v_\mu(x) \\ &\quad + i\theta^2\bar\theta\left[\bar\lambda(x) + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right] - i\bar\theta^2\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right] \\ &\quad + \frac{1}{2}\theta^2\bar\theta^2\left[D(x) + \frac{1}{2}\Box C(x)\right].\end{aligned} \tag{B.63}$$

where C, D, M, N and v_μ are real fields[8]. The notation of the component fields as in (B.63) will become comprehensible immediately. Applying a supersymmetric transformation (B.55) to a vector superfield in order to find the transformations for its component fields is a little elaborate and gives lengthy expressions (as compared to the chiral superfield). In any case, as will become clear in the next section, the only important result is

$$\delta_{\epsilon\bar\epsilon}D = -\epsilon\sigma^\mu\;\partial_\mu\bar\lambda + \bar\epsilon\sigma^\mu\;\partial_\mu\lambda. \tag{B.64}$$

This shows that the component with $\theta^2\bar\theta^2$ transforms into a total derivative.

There are two distinguished vector superfields, both derived from chiral and anti-chiral superfields, and both serving as building blocks of supersymmetric actions. These are for one $(\Phi + \Phi^\dagger)$, and secondly $\Phi^\dagger\Phi$.

[8] In the naming of the component fields, I follow the established notation, which of course is historically influenced.

- Take

$$\Phi + \Phi^\dagger = (A + A^*) + \sqrt{2}(\theta\psi + \bar\theta\bar\psi) + (\theta^2 F + \bar\theta^2 F^*) + i\theta\sigma^\mu\bar\theta\partial_\mu(A - A^*) + \frac{i}{\sqrt{2}}\theta^2\bar\theta\bar\sigma^\mu\partial_\mu\psi + \frac{i}{\sqrt{2}}\bar\theta^2\bar\theta\sigma^\mu\partial_\mu\bar\psi + \frac{1}{4}\theta^2\bar\theta^2\Box(A + A^*).$$

The appearance of a gradient of the scalar field $(A - A^*)$ motivates the definition

$$V \longmapsto V + (\Phi + \Phi^\dagger) \tag{B.65}$$

as the superspace version of a $\mathbf{U(1)}$ gauge transformation. Under this transformation, the component fields in (B.63) transform as

$$\begin{aligned} C &\longmapsto C + (A + A^*) \\ \chi &\longmapsto \chi - i\sqrt{2}\psi \\ (M + iN) &\longmapsto (M + iN) - 2iF \\ v_\mu &\longmapsto v_\mu - i\partial_\mu(A - A^*) \\ \lambda &\longmapsto \lambda \\ D &\longmapsto D. \end{aligned}$$

From this we learn two things: First, the fields λ and D are invariant under (B.65). And secondly, since the fields C, χ, M, N simply transform by shifts, there is a gauge choice through which these fields can be brought to vanish. In this *Wess-Zumino gauge* the vector superfield simplifies to

$$V(x, \theta, \bar\theta)\big|_{WZ} = -\theta\sigma^\mu\bar\theta v_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2 D. \tag{B.66}$$

Of course, in choosing this gauge, supersymmetry is broken, but it still allows the “ordinary” gauge transformations $v_\mu \mapsto v_\mu + \partial_\mu f$. Later, in gauging a supersymmetric theory, use will be made of the exponential e^V, which in the WZ-gauge becomes the finite expression

$$e^V\big|_{WZ} = 1 - \theta\sigma^\mu\bar\theta v_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2\left[D - \frac{1}{2}v_\mu v^\mu\right] \tag{B.67}$$

because $V^k\big|_{WZ} = 0$ for $k > 2$.

- Since $\Phi^\dagger\Phi$ is obviously self-conjugate, it is a vector superfield *per se*. For reasons explained below, one is interested only in the $\theta^2\bar\theta^2$-term:

$$\begin{aligned} \Phi^\dagger\Phi\big|_D &:= \int d^2\theta d^2\bar\theta\ \Phi^\dagger\Phi \\ &= F^*F + \frac{1}{4}A^*\Box A + \frac{1}{4}\Box A^* A - \frac{1}{2}\partial_\mu A^*\partial^\mu A \\ &\qquad + \frac{i}{2}(\partial_\mu\bar\psi\bar\sigma^\mu\psi - \bar\psi\bar\sigma^\mu\partial_\mu\psi). \end{aligned} \tag{B.68}$$

B.3.3 Superactions

In Sect. 5.3 action functionals were constructed for fields which are irreducible under Poincaré transformations. The respective Lagrangians became largely fixed by power counting arguments. With the techniques from superspace and superfields, one can bring forth super-Lagrangians and superactions in a similar manner.

By definition of variational symmetry, we want the action to be invariant under SuSy transformations. This is satisfied if $\mathcal{L}$ transforms into a total derivative. As a matter of fact, $\delta\mathcal{L} \neq 0$, since otherwise this would imply that $[\delta_1, \delta_2]\,\mathcal{L} \sim \partial\mathcal{L} = 0$, or that $\mathcal{L}$ must be a constant. In the previous treatment of superfields, we already met objects which transform into a total derivative, namely the highest components of chiral superfields (the "F-term") and of vector superfields (the "D-term"). Thus schematically a viable action is

$$S = \int d^4x \Big(\int d^2\theta\, \mathcal{L}_F + \int d^2\bar{\theta}\, \mathcal{L}_F^\dagger + \int d^2\theta d^2\bar{\theta}\, \mathcal{L}_D \Big) \tag{B.69}$$

where $\mathcal{L}_F$ and $\mathcal{L}_D$ are a chiral and a vector superfield.

Lagrangian Components Constructed from Chiral Superfields

As shown, products of chiral superfields (and products of anti-chiral superfields) are chiral (anti-chiral) superfield themselves. These are thus possible components in $\mathcal{L}_F$. However, multiplying more then three chiral superfields results in terms with mass dimension $d > 4$, which would lead to non-renormalizable interactions. Thus there are three possible contributions to $\mathcal{L}_F$ which may be combined to

$$\mathcal{L}_F = \mathcal{W}(\Phi_i) := c_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k, \tag{B.70}$$

called the *superpotential*[9].

What about kinetic energy terms, that is terms with derivatives? Derivatives are present in the D-component of $\Phi^\dagger\Phi$, see (B.68). If this is taken as $\mathcal{L}_D$ in (B.69), it allows for a straightforward interpretation in terms of interacting fields: First of all it contains the proper kinetic terms for the scalar A and the fermion ψ. Furthermore, there is no kinetic term for F. This field is not propagating; it is an auxiliary field. As a matter of fact, dimensional arguments exclude higher powers of $\Phi^\dagger\Phi$, since $\left[\Phi^\dagger\Phi\right] = 2$.

Thus we arrive at the action that describes the interacting field theory of a chiral superfield:

$$S = \int d^8z\;\; \Phi^\dagger\Phi + \int d^6z\, \mathcal{W}(\Phi_i) + \int d^6\bar{z}\, \mathcal{W}^\dagger(\Phi_i). \tag{B.71}$$

The simplest supersymmetric theory is the one consisting of the kinetic term only: The Lagrangian for the free massless Wess-Zumino model (8.43) is simply $\Phi^\dagger\Phi|_D$

[9] This superpotential is by no means related to the superpotential arising in the context of Klein-Noether identities and for pseudo-tensors in GR.

with the identifications (B.62) except for total derivatives and after using the "field equation" $F = 0$. And we regain the massive and interacting Wess-Zumino model (8.52) by adding a superpotential, i.e.

$$\begin{aligned}\mathcal{L}_{WZ} &= \Phi^\dagger\Phi\big|_D + [(\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3) + h.c.]\big|_F \\ &= F^*F - |\partial_\mu A|^2 + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \\ &\quad + \left[\left(m\{AF - \psi^2\} + g\,\{FA^2 - \psi^2\, A\}\right) + h.c.\right],\end{aligned} \tag{B.72}$$

again up to a total derivative. The field equations with respect to F and F^* are

$$[\mathcal{L}_{WZ}]^F = F^* + mA + gA^2 = 0, \qquad [\mathcal{L}_{WZ}]^{F^*} = F + m^*A^* + g^*A^{*2} = 0$$

which shows that the auxiliary fields can be expressed by the scalar fields. In observing that the terms depending on F, F^* in (B.72) can be rewritten as

$$\begin{aligned}&FF^* + F(mA + gA^2) + F^*(m^*A^* + g^*A^{*2}) \\ &\qquad\qquad = |F + m^*A^* + gA^{*2}|^2 - |(mA + gA^2)|^2,\end{aligned}$$

and that the first terms vanishes on-shell, (B.72) becomes

$$\mathcal{L}_{WZ} = -|\partial_\mu A|^2 + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - m(\psi^2 + \bar{\psi}^2) - g\psi^2\, A - g\bar{\psi}^2\, A^* - V(A, A^*). \tag{B.73}$$

Note that the scalar potential $V(A, \bar{A}) = |(mA + gA^2)|^2 \geqslant 0$, and that one is no longer free to add an arbitrary constant as one could do in a non-supersymmetric theory. Note further that supersymmetry enforces relations among masses and couplings

$$m_A = m = m_\psi \qquad \lambda_{A^4} = \lambda_{\psi\psi A}.$$

The expression (B.68) can be generalized to a term built from different chiral superfields and thus allows for a kinetic term

$$\mathcal{L}^H = H^{ij}\Phi_i^\dagger\Phi_j\big|_D$$

where H must be a positive definite Hermitean matrix. By taking appropriate linear combinations of the superfields, one may assume that $H^{ij} = \delta^{ij}$:

$$\mathcal{L}^{kin} = \sum_i \Phi_i^\dagger\Phi_i\big|_D. \tag{B.74}$$

The most general supersymmetric Lagrangian (built out of chiral and anti-chiral superfields) is made from the sum of this kinetic term and the superpotential (B.70):

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^{kin} + \left[\mathcal{W}(\Phi_i)\big|_F + h.c.\right] \\ &= F_i^*F_i - |\partial_\mu A_i|^2 + i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i + \left[\frac{\partial\mathcal{W}(A_i)}{\partial A_j}F_j - \frac{1}{2}\frac{\partial^2\mathcal{W}(A_i)}{\partial A_j\partial A_k}\psi_j\psi_k + \text{h.c.}\right].\end{aligned} \tag{B.75}$$

Again the F-fields are non-propagating due to their Euler-Lagrange equations

$$F_k^* + \left[\frac{\partial \mathcal{W}(A_i)}{\partial A_k}\right] = 0, \qquad F_k + \left[\frac{\partial \mathcal{W}(A_i)}{\partial A_k}\right]^* = 0. \tag{B.76}$$

Plugging this back into (B.75) one obtains

$$\mathcal{L} = -|\partial_\mu A_i|^2 + i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i - \left|\frac{\partial \mathcal{W}(A_i)}{\partial A_k}\right|^2 - \left[\frac{1}{2}\frac{\partial^2 \mathcal{W}(A_i)}{\partial A_j \partial A_k}\psi_j \psi_k + \text{h.c.}\right]. \tag{B.77}$$

Again we detect in the third term

$$V = \left|\frac{\partial \mathcal{W}}{\partial A_k}\right|^2 = \sum_k |(c_k + m_{kj}A_j + g_{kji}A_j A_i)|^2$$

the scalar masses and their self-interaction, and in the last term

$$\left[\frac{\partial^2 \mathcal{W}}{\partial A_j \partial A_k}\psi_j \psi_k + \text{h.c.}\right] = (m_{jk} + 2g_{jki}A_i)\psi_j \psi_k$$

we see the fermion masses and three-point couplings between two fermions and a scalar. With regard to the Standard Model this "smells" like the Yukawa interaction between a Higgs boson and two fermions.

Vectorfield Lagrangians

In the previous subsection, the most general supersymmetric Lagrangian containing scalar and fermionic fields was constructed. As an intermediate step towards supersymmetric Yang-Mills theories, the next task is the construction of terms containing the typical field-strengths for vector fields. As will be seen these can be derived from the expressions

$$W_\alpha := -\frac{1}{4}\bar{D}^2 D_\alpha V \qquad \overline{W}^{\dot{\alpha}} := +\frac{1}{4}D^2 \bar{D}^{\dot{\alpha}} V, \tag{B.78}$$

where V is a vector superfield (and $D^2 := D^\alpha D_\alpha$, $\bar{D}^2 := \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}$). From their algebra (B.57) the derivatives can be shown to obey $D_\alpha D^2 \equiv 0 \equiv \bar{D}^{\dot{\alpha}}\bar{D}^2$. This in turn entails that the fields W_α and $\bar{W}^{\dot{\alpha}}$ are chiral and antichiral, respectively. Under the SuSy gauge transformations (B.65), W_α transforms into

$$W'_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V - \frac{1}{4}\bar{D}^2 D_\alpha \Phi - \frac{1}{4}\bar{D}^2 D_\alpha \Phi^\dagger.$$

The last term vanishes since $\Phi^\dagger$ is anti-chiral. But the second term also vanishes because of

$$\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} D_\alpha \Phi = \bar{D}_{\dot{\alpha}}\{\bar{D}^{\dot{\alpha}}, D_\alpha\}\Phi = -2i\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\alpha\dot{\beta}}\bar{D}_{\dot{\alpha}}\partial_\mu \Phi = 0.$$

Thus W_α is SuSy gauge invariant ($W'_\alpha = W_\alpha$). The same can be proven for $\overline{W}_{\dot\alpha}$. The derivation of the explicit expressions for these objects in terms of ordinary fields turns out to be rather elaborate (see e.g. [533]). In the Wess-Zumino gauge, the result is

$$W_\alpha = -i\lambda_\alpha(y) + D(y)\theta_\alpha - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta^2(\sigma^\mu\partial_\mu\bar\lambda(y))_\alpha \tag{B.79a}$$

$$\overline{W}^{\dot\alpha} = i\bar\lambda^{\dot\alpha}(y^*) + D(y^*)\bar\theta^{\dot\alpha} + i(\bar\sigma^{\mu\nu}\bar\theta)^{\dot\alpha}F_{\mu\nu}(y^*) - i\bar\theta^2(\bar\sigma^\mu\partial_\mu\bar\lambda(y^*))^{\dot\alpha}. \tag{B.79b}$$

The vector field v_μ only appears in the form $F_{\mu\nu} := \partial_\mu v_\nu - \partial_\nu v_\mu$. This justifies calling the W fields supersymmetric field strengths.

Quite in analogy with "ordinary" gauge fields, it makes sense to build from the W-fields quadratic expressions $W^\alpha W_\alpha$ and $\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}$. These are chiral or anti-chiral superfields, respectively, whose F-terms are suited to define a kinetic Lagrangian

$$\mathcal{L}_k := \frac{1}{4}(W^\alpha W_\alpha|_F + \bar W_{\dot\alpha}\bar W^{\dot\alpha}|_F) = \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar\lambda. \tag{B.80}$$

This Lagrangian component reproduces the kinetic term for an Abelian gauge theory (the photon v_μ) together with a "photino" λ. Since there is no kinetic term for D, this auxiliary superfield can be removed from the theory by its field equation.

Local Gauge Symmetry by Minimal Coupling

Let us next mimic the procedure of introducing a locally gauge-invariant interaction term amongst the photon and the Fermi fields via the introduction of covariant derivatives. A global $\mathbf{U}(\mathbf{1})$-transformation of a chiral superfield is given by

$$\Phi'_k = e^{-iq_k\Lambda}\Phi_k,$$

where q_k is the $\mathbf{U}(\mathbf{1})$-charge of Φ_k. The Fermi fields are components in the Lagrangian (B.75). The kinetic terms $\Phi_k^\dagger\Phi_k|_D$ are globally gauge invariant. However, the part in (B.75) with the superpotential $\mathcal{W}$ is not invariant in general. Its gauge invariance depends on the coefficients in (B.70): For $\mathcal{W}$ to be $\mathbf{U}(\mathbf{1})$-invariant, we must demand

$$c_i = 0 \qquad \text{if} \qquad q_i \neq 0 \tag{B.81a}$$

$$m_{ij} = 0 \qquad \text{if} \qquad q_i + q_j \neq 0 \tag{B.81b}$$

$$g_{ijk} = 0 \qquad \text{if} \qquad q_i + q_j + q_k \neq 0. \tag{B.81c}$$

In turning the global invariance into a local invariance by allowing the gauge parameter Λ to become spacetime dependent, we first observe that the transformed chiral superfields fields Φ'_k are chiral only if Λ is chiral: $\bar D^{\dot\alpha}\Lambda = 0 = D_\alpha\Lambda^\dagger$. The kinetic energy term transforms as

$$\Phi'^\dagger_k\Phi'_k = \Phi^\dagger_k\Phi_k e^{iq_k(\Lambda^\dagger - \Lambda)} \qquad (\text{no sum with respect to k}).$$

The extra factor may be compensated by introducing the vector superfield V with a transformation behavior according to (B.65): $V' = V + i(\Lambda^\dagger - \Lambda)$. Then the Lagrangian

$$\mathcal{L}_G = \frac{1}{4}(W^\alpha W_\alpha|_F + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_F) + \Phi_k^\dagger e^{q_k V}\Phi_k|_D + \left[\mathcal{W}(\Phi_i)\big|_F + h.c.\right] \tag{B.82}$$

is invariant under local $\mathbf{U(1)}$ gauge transformations–provided of course that (B.81) is fulfilled. The first term contains the gauge-invariant kinetic energy expression for the $\mathbf{U(1)}$-gauge fields. The second term contains the gauge-invariant kinetic energy expression for the matter fields. A somewhat tedious calculation yields (in the WZ gauge) for each summand in this expression

$$\begin{aligned}\Phi^\dagger e^{gV}\Phi\big|_D &= |F|^2 + (D_\mu A)^* D^\mu A - i\bar{\psi}\sigma^\mu D_\mu\psi + q|A|^2 D \\ &\quad + iq\sqrt{2}(A^*\lambda\psi - \bar{\lambda}\bar{\psi}A)\end{aligned} \tag{B.83}$$

with the covariant derivative $D_\mu := \partial_\mu + igv_\mu$. (Just for clarification: The F in this expression is the field strength with respect to the vector field v_μ - and not the auxiliary field. The field A is a scalar field as it arises from the Fourier expansion of a chiral superfield according to (B.59).)

This shows that there is a fermion field λ acting as the supersymmetric partner of the gauge boson v_μ. Therefore it is justified to call λ the "photino"–or more general–"gaugino". Both v_μ and λ transform in the same way under the gauge group. And both the photon and the photino fields are massless in this theory. Furthermore, aside of the (minimal) couplings of the gauge field v_μ to the fermionic matter field ψ - as familiar from gauge invariance–there is an additional interaction between ψ, the scalar field A, and the gaugino λ. The strength of this interaction is also given by the gauge coupling g.

Supersymmetric Yang-Mills Lagrangian

The previous considerations can be generalized to non-Abelian groups; see the original articles by S. Ferrara and B. Zumino [172] and by A. Salam and J. Strathdee [447]. This generalization refers to the vector superfield which now becomes Lie-algebra valued

$$V = V^a T^a,$$

and to W-fields generalized to

$$W_\alpha := -\frac{1}{4\,g}\bar{D}^2 e^{-gV} D_\alpha e^{gV} \qquad \overline{W}^{\dot{\alpha}} = \frac{1}{4\,g} D^2 e^{gV}\bar{D}^{\dot{\alpha}} e^{-gV}.$$

Under the transformations $e^V \to e^{i\Lambda^\dagger} e^V e^{-i\Lambda}$, the superfields transform covariantly as

$$W_\alpha \to e^{i\Lambda} W_\alpha^{-i\Lambda} \qquad \bar{W}^{\dot{\alpha}} \to e^{i\Lambda^\dagger}\overline{W}\dot{\alpha}e^{-i\Lambda^\dagger}.$$

In expanding e^V in the Wess-Zumino gauge according to (B.67), one obtains after some moderate calculation the generalization of (B.79)

$$W_\alpha = -i\lambda_\alpha(y) + D(y)\theta_\alpha - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta^2(\sigma^\mu D_\mu\bar{\lambda}(y))_\alpha$$

where now the field strength and the covariant derivative take the well-known form

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig\,[v_\mu, v_\nu] \qquad D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - ig\left[v_\mu, \bar{\lambda}\right].$$

The supersymmetric free Yang-Mills action is

$$S_{\text{S-YM}} = -\frac{1}{4}\int d^4x d^2\theta \ \mathrm{Tr}\ W^\alpha W_\alpha = \int d^4x\ \mathrm{Tr}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}D^2\right].$$

In addition to the familiar kinetic term for the gauge fields, this expression contains a kinetic energy term for the gauginos, as well as the coupling of the gauginos to the gauge fields–its strength dictated by the group structure functions. There is no kinetic term for the D-field, which can be eliminated as an auxiliary field.

A (chiral) matter field Φ can be minimally coupled to the Yang-Mills field by putting it into the adjoint representation of the gauge group, and replacing (B.71) by

$$S_{\text{matter}} = \int d^8z \ \ \mathrm{Tr}\ \ \Phi^\dagger e^{-gV}\Phi + \int d^6z \ \ \mathcal{W}(\Phi_i) + \int d^6\bar{z} \ \ \mathcal{W}^\dagger(\Phi_i).$$

B.4 *Supergroups

The classification of “ordinary” Lie groups can be extended to graded Lie groups (’supergroups’). Similar to “ordinary” Lie groups one finds families of simple supergroups. There are two main series, which are called orthosymplectic $\mathbf{OSp(N/M)}$ and super-unitary $\mathbf{SU(N/M)}$. Additionally there are exceptional simple groups. The following is only a sketch of the representation theory of supergroups; for details see e.g. Section 3.1 in [507] or [191].

B.4.1 OSp(N/M) and the Super-Poincaré Algebra

The graded Lie group $\mathbf{OSp(N/M)}$ can be understood from its building blocks $\mathbf{O(N)}$ and $\mathbf{Sp(N)}$. As explained in App. A, the group $\mathbf{O(N)}$ is represented by the real orthogonal $N \times N$ matrices that leave invariant the expression $\sum^{N} x_i x_i$, and the symplectic group $\mathbf{Sp(M)}$ is represented by those $M \times M$ matrices that leave $\sum^{M} \theta_a C_{ab} \theta_b$ invariant (where C_{ab} is a real antisymmetric matrix and the θ_a are anticommuting). The orthosymplectic group $\mathbf{OSp(N/M)}$ is the set of matrices that leaves invariant the "line element"

$$\sum^{N} x_i x_i \; + \; \sum^{M} \theta_a C_{ab} \theta_b .$$

The elements of the orthosymplectic group can be organized into block form:

$$\begin{pmatrix} O(N) & A \\ B & Sp(M) \end{pmatrix} .$$

Here the entries on the diagonal are bosonic, whereas A and B are fermionic, but of no further interest here. We have

$$\mathbf{O(N)} \otimes \mathbf{Sp(M)} \subset \mathbf{OSp(N/M)}.$$

In supergravity one is interested in the orthosymplectic groups $\mathbf{OSp(N/4)}$. Its building block $\mathbf{Sp(4)}$ is isomorphic to $\mathbf{O(3,2)}$ (see e.g. [507]). By a group contraction, the $\mathbf{OSp(N/4)}$ becomes the Poincaré group. The $\mathbf{O(N)}$-part describes the extensions from $N = 1$ to $N = 8$.

B.4.2 SU(N/M) and the Super-Conformal Algebra

The matrices of $\mathbf{SU(N/M)}$ leave invariant the expression

$$\sum^{N} (z_i)^* z_i \; + \; \sum^{M} (\theta_a)^* \delta_{ab} \theta_b \qquad \text{with} \qquad \delta_{ab} = \pm \delta_{ab}.$$

It holds that

$$\mathbf{SU(N)} \otimes \mathbf{SU(M)} \otimes \mathbf{U(1)} \subset \mathbf{SU(N/M)}.$$

If one takes for the first factor the special unitary group $\mathbf{SU(2,2)}$ the supergroup $\mathbf{SU(2,2/1)}$ has–after group contraction–the algebra of super-conformal symmetry. The expression left invariant is

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 + \theta_1^* \theta^1 + \theta_2^* \theta^2 - \theta_3^* \theta^3 - \theta_4^* \theta^4 .$$

This super-conformal group–also used as a gauge group of supergravity–has in its algebra aside from the 15 generators $T_\mu, M_{\mu\nu}, C_\mu, S$ of the conformal group (see Sect. 3.5.1) and aside from the fermionic generators Q^a, further spinorial elements and–for closure of the algebra–also another bosonic element; the full algebra and two feasible representations of the generators can be found for instance in Sect. 14.5 of [359].

Appendix C
Symmetries and Constrained Dynamics

Symmetries make Lagrangians look compact and aesthetic, but–as will be seen in this section–they render Hamiltonians ugly. This is because the standard procedure–the Legendre transformation–of going from the configuration-velocity space to the phase space is obstructed. As pointed out in Subsect. 3.3.3, the Noether identities imply in the case of a local symmetry that the canonically conjugate momenta cease to be independent: They are constrained; see (3.80). Thus fundamental physics, abundant in local symmetries, is full of constraints. This point becomes virulent if one wants to step from the classical theory to its quantized version in a canonical way.

This appendix has three sections. In the first one, the basic terminology is exposed, more detailed for systems with finitely many degrees of freedom, and a little sketchy for field theories. The second section treats Yang-Mills type theories including good-old electrodynamics, and the last one deals with re-parametrization invariant–or–generally covariant theories, in this case the relativistic point particle and general relativity. The plain existence of two separate sections devoted to those theories describing particle fields on the one hand side and gravity on the other hand may be seen as an indication that the unification of both worlds is still not reached, despite some (or, if you like, many) formal similarities in their Hamiltonian formulation.

Most of the material in this appendix is treated in more detail in [479]. The core techniques for constrained dynamics are dealt with in [551]. A compact overview is given by [126]. More advanced techniques, especially if it comes to the quantization of constrained systems by BRST-BFV-techniques, can be found in [213], [266].

C.1 Constrained Dynamics

C.1.1 Singular Lagrangians

Assume a classical theory with a finite number of degrees of freedom q^k $(k = 1, \ldots, N)$ defined by its Lagrange function $L(q, \dot{q})$ with the equations of motion

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Fundamental Theories of Physics 176, DOI 10.1007/978-94-007-7642-5,

$$\begin{aligned}[L]_k &:= \frac{\partial L}{\partial q^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} = \left(\frac{\partial L}{\partial q^k} - \frac{\partial^2 L}{\partial \dot{q}^k \partial q^j}\dot{q}^j\right) - \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j}\ddot{q}^j \\ &=: V_k - W_{kj}\ddot{q}j = 0. \end{aligned} \tag{C.1}$$

For simplicity, it is assumed that the Lagrange function does not depend on time explicitly; all the following results can readily be extended. As demonstrated in the main text, a crucial role is played by the matrix (sometimes called the "Hessian")

$$W_{kj} \;:=\; \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j}. \tag{C.2}$$

If $\det W = 0$, not only the Lagrangian but the system itself is termed 'singular', and 'regular' otherwise. This appendix deals exclusively with singular systems.

From the definition of momenta by

$$p_k(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^k}, \tag{C.3}$$

we immediately observe that only in the regular case can the $p_k(q, \dot{q})$ be solved for all the velocities in the form $\dot{q}^j(q, p)$ - at least locally.

In the singular case, $\det W = 0$ implies that the $N \times N$ matrix W has a rank R smaller than N - or that there are $P = N\text{-}R$ null eigenvectors ξ^k_ρ:

$$\xi^k_\rho \, W_{kj} \equiv 0 \qquad \text{for} \qquad \rho = 1, \ldots, P \; (= N - R). \tag{C.4}$$

This rank is independent of which generalized coordinates are chosen for the Lagrange function. The null eigenvectors serve to identify those of the equations in (C.1) which are not of second order. By contracting these with $\ddot{q}^j$ we get the P on-shell equations

$$\chi_\rho = \xi^k_\rho \, V_k(q, \dot{q}) = 0.$$

In analogy with the terminology introduced in the next subsection, these may be called 'primary Lagrangian constraints'. Being functions of $(q, \dot{q})$ they are not genuine equations of motion but–if not fulfilled identically–they restrict the dynamics to a subspace within the configuration-velocity space (or in geometrical terms, the tangent bundle $T\mathbb{Q}$). For reasons of consistency, the time derivative of these constraints must not lead outside this subspace. This condition possibly enforces further Lagrangian constraints and by this a smaller subspace of allowed dynamics, etc. Thus eventually one derives a chain of Lagrangian constraints that define the "true" subspace in which a consistent dynamics occurs. I shall not go into further details here (see for instance [344], [478], [479]), but rather leave the Lagrangian description in favor of the Hamiltonian treatment.

The previous considerations are carried over to a field theory with a generic Lagrangian density $\mathcal{L}(Q^\alpha, \partial_\mu Q^\alpha)$. Rewrite the field equations similar to (C.1)

$$[\mathcal{L}]_\alpha := \frac{\partial \mathcal{L}}{\partial Q^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial Q^\alpha_{,\mu}} = \left(\frac{\partial \mathcal{L}}{\partial Q^\alpha} - \frac{\partial^2 \mathcal{L}}{\partial Q^\alpha_{,\mu} \partial Q^\beta} Q^\beta_{,\mu}\right) - \frac{\partial^2 \mathcal{L}}{\partial Q^\alpha_{,\mu} \partial Q^\beta_{,\nu}} \; Q^\beta_{,\mu\nu}$$
$$:= \mathcal{V}_\alpha - \mathcal{W}^{\mu\nu}_{\alpha\beta} \, Q^\beta_{,\mu\nu}. \tag{C.5}$$

Depending now on the choice of a time variable T (e.g. $T = x^0, \; T = x^0 - x^{D-1}, \ldots$) the Hessian is defined by

$$\overset{T}{\mathcal{W}}_{\alpha\beta} := \frac{\partial^2 \mathcal{L}}{\partial(\partial_T Q^\alpha)\partial(\partial_T Q^\beta)}. \tag{C.6}$$

If the rank of this matrix is $R < N$, it has $P = N\text{-}R$ null eigenvectors

$$\overset{T}{\xi}{}^\alpha{}_\rho \overset{T}{\mathcal{W}}_{\alpha\beta} \equiv 0. \tag{C.7}$$

Observe that the Hessian (and by this also the zero eigenvectors) depend on the selection of the time variable. This indeed has the direct consequence that the number of zero eigenvectors and of (primary) constraints depends on this choice.

C.1.2 Constraints as a Consequence of Local Symmetries

In investigating the consequences of Noether's second theorem, we found that the momenta are not independent but obey constraints; see (3.80). Here the argumentation will be slightly modified: The Noether identities for local symmetries of the form (3.67) are according to (3.74)

$$[\mathcal{L}]_\alpha (\mathcal{A}^\alpha_r - Q^\alpha_{,\mu} \mathcal{D}^\mu_r) - \partial_\mu([\mathcal{L}]_\alpha \mathcal{B}^{\alpha\mu}_r) = 0.$$

Let us identify the terms with the highest possible derivatives of the fields Q^α by using the expression (C.5) which already isolates the second derivatives. A further derivative possibly originates from the last term in the Noether identity. It reads $\mathcal{B}^{\alpha\mu}_r \mathcal{W}^{\lambda\nu}_{\alpha\beta} \, Q^\beta_{,\lambda\nu\mu}$. This term must vanish itself for all third derivatives of the fields, and therefore

$$\mathcal{B}^{\alpha(\mu}_r \mathcal{W}^{\lambda\nu)}_{\alpha\beta} = 0,$$

where the symmetrization goes over μ, λ, ν. Among these identities is the Hessian, and we find

$$\overset{T}{\mathcal{B}}{}^\alpha{}_r \, \overset{T}{\mathcal{W}}_{\alpha\beta} = 0. \tag{C.8}$$

Thus the non-vanishing $\overset{T}{\mathcal{B}}_r$ are null-eigenvectors of the Hessian. And comparing this with (C.7) there must be linear relationships

$$\overset{T}{\mathcal{B}}{}^\alpha{}_r \;=\; \lambda^\rho_r \; \overset{T}{\xi}{}^\alpha{}_\rho$$

with coefficients λ^{ρ}_{r}. In case all or some of the $\overset{T}{\mathcal{B}}{}^{\alpha}{}_{r}$ are zero, we can repeat the previous argumentation by singling out the terms with second derivatives, and find again that the Hessian has a vanishing determinant.

We conclude that every action which is invariant under local symmetry transformations necessarily describes a singular system. This, however, should not lead to the impression that any singular system exhibits local symmetries. As a matter of fact, a system can become singular just by the choice of the time variable.

C.1.3 Rosenfeld-Dirac-Bergmann Algorithm

Since the fields and the canonical momenta are not independent, they cannot be taken as coordinates in a phase space as one is accustomed in the unconstrained case. This difficulty was known already by the end of the 1920's, and after unsatisfactory attempts by eminent physicists such as W. Pauli, W. Heisenberg, and E. Fermi this problem was attacked by L. Rosenfeld. He undertook the very ambitious effort of obtaining the Hamiltonian version for the Einstein-Maxwell theory as a preliminary step towards quantization [439]. But only in the late forties and early fifties did the Hamiltonian version of constrained dynamics acquire a substantially mature form due to P.A.M. Dirac on the one hand and to P. Bergmann and collaborators on the other hand; for historical aspects, see [449], [450].

In order to get an easier access on the notions of constrained Hamiltonian dynamics, let us restrict ourselves again to a system with finitely many degrees of freedom.

Primary Constraints

The rank of the Hessian (C.2) being $R = N\text{-}P$ implies that–at least locally–the Eqs. (C.3) can be solved for R of the velocities in terms of the positions, some of the momenta and the remaining velocities. Furthermore, there are P relations

$$\phi_{\rho}(q,p) = 0 \qquad \rho = 1,\ldots,P \tag{C.9}$$

which restrict the dynamics to a subspace $\Gamma_P \subset \Gamma$ of the full phase space Γ. These relations were dubbed *primary constraints* by Anderson and Bergmann [9], a term suggesting that there are possibly secondary and further generations of constraints.

For many of the calculations below, one needs to set regularity conditions, namely (1) the Hessian of the Lagrangian has constant rank, (2) there are no ineffective constraints, that is constraints whose gradients vanish on Γ_P, (3) the rank of the Poisson bracket matrix of constraints remains constant in the stabilization (RDB) algorithm described below.

Weak and Strong Equations

It will turn out that even in the singular case, one can write the dynamical equations in terms of Poisson brackets. But one must be careful in interpreting them in the presence of constraints. In order to support this precaution, Dirac [124] contrived the concepts of "weak" and "strong" equality.

If a function $F(p,q)$ which is defined in the neighborhood of Γ_P becomes identically zero when restricted to Γ_P it is called "weakly zero", denoted by $F \approx 0$:

$$F(q,p)\big|_{\Gamma_P} = 0 \longleftrightarrow F \approx 0.$$

(Since in the course of the algorithm the constraint surface is possibly narrowed down, a better notation would be $F \approx_{|\Gamma_P} 0$.) If the gradient of F is also identically zero on Γ_P, F is called "strongly zero", denoted by $F \simeq 0$:

$$F(q,p)\big|_{\Gamma_P} = 0 \qquad \Big(\frac{\partial F}{\partial q^i}, \frac{\partial F}{\partial p_k}\Big)\big|_{\Gamma_P} = 0 \longleftrightarrow F \simeq 0.$$

It can be shown (see e.g. [479]) that if F vanishes weakly, it strongly is a linear combination of the functions defining the constraint surface–or -

$$F \approx 0 \longleftrightarrow F - f^\rho \phi_\rho \simeq 0.$$

Indeed, the subspace Γ_P can itself be defined by the weak equations

$$\phi_\rho \approx 0, \tag{C.10}$$

since due to the regularity assumptions, the constraints do not vanish strongly.

Canonical and Total Hamiltonian

Next we introduce the "canonical" Hamiltonian

$$H_C = \; p_i \dot{q}^i - L(q,\dot{q}).$$

Its variation yields

$$\delta H_C = \; (\delta p_i)\dot{q}^i + \; p_i \delta\dot{q}^i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i = \dot{q}^i \, \delta p_i - \frac{\partial L}{\partial q^i}\, \delta q^i$$

(after using the definition of momenta), revealing the remarkable fact that the canonical Hamiltonian can be written in terms of q's and p's. No explicit dependence on any velocity variable is left, despite the fact that the Legendre transformation is non-invertible[1]. Observe, however, that the expression for δH_C given in terms of

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[1] In geometrical terms, this is explained by the Legendre projectability of the Lagrangian energy function; see Section C.5.1.

the variations δq^i and δp_i does not allow the derivation of the Hamilton equations of motion, since the variations are not independent due to the primary constraints (C.10). In order that these be respected, the variation of H_C needs to be performed together with Lagrange multipliers. Thus we introduce the “total” Hamiltonian

$$H_T = H_C + u^\rho \phi_\rho \tag{C.11}$$

with arbitrary multiplier functions u^ρ in front of the primary constraint functions. Varying the total Hamiltonian with respect to (u, q, p) we obtain the constraints (C.10) and

$$\frac{\partial H_C}{\partial p_i} + u^\rho \frac{\partial \phi_\rho}{\partial p_i} = \dot{q}^i \tag{C.12a}$$

$$\frac{\partial H_C}{\partial q^i} + u^\rho \frac{\partial \phi_\rho}{\partial q^i} = -\frac{\partial L}{\partial q^i} = -p_i \tag{C.12b}$$

where the last relation follows from the definition of momenta and the Euler-Lagrange equations. This recipe of treating the primary constraints with Lagrange multipliers sounds reasonable and may work, but it leaves one unsure about a mathematical justification. This justification must guarantee that the Hamiltonian description of the dynamics is ultimately equivalent to the Lagrangian formalism; a proof is given in [32]. The Eqs. (C.12) are reminiscent of the Hamilton equations for regular systems. However, there are extra terms depending on the primary constraints and the multipliers. Nevertheless, (C.12) can be written in terms of Poisson brackets, provided one adopts the following convention: Consider

$$\{F, H_T\} = \{F, H_C + u^\rho \phi_\rho\} = \{F, H_C\} + u^\rho \{F, \phi_\rho\} + \{F, u^\rho\}\phi_\rho.$$

Since the multipliers u_ρ are not phase-space functions, the Poisson brackets $\{F, u^\rho\}$ are not defined. However, these appear multiplied with constraints and thus the last term vanishes weakly. Therefore the dynamical equations for any phase-space function $F(q, p)$ can be written as

$$\dot{F}(p, q) \approx \{F, H_T\}. \tag{C.13}$$

Before continuing to scrutinize how many of the functions u^ρ can be determined, and before we discover that the undetermined ones are connected to local symmetries of the theory in question, let us relate the primary constraints to the eigenvectors of the Hessian. The primary constraints can be thought of as functions of the velocities: $\phi_\rho = \phi_\rho(q, \partial L/\partial \dot{q})$. Then

$$0 \approx \frac{\partial \phi_\rho}{\partial \dot{q}^k} = \frac{\partial \phi_\rho}{\partial p_i}\frac{\partial p_i}{\partial \dot{q}^k} = \frac{\partial \phi_\rho}{\partial p_i} W_{ik}.$$

Thus $\partial \phi_\rho / \partial p_i$ is a null eigenvector of the Hessian and can be expressed as a linear combination of the null eigenvectors from (C.4). They even may serve to canonize the null eigenvectors:

$$\xi^i_\rho \equiv \frac{\partial \phi_\rho}{\partial p_i}. \tag{C.14}$$

A more precise definition in terms of Legendre projectability is given in Sect. C.5.

Stability of Constraints

For consistency of a theory, one must require that the primary constraints are conserved during the dynamical evolution of the system:

$$0 \overset{!}{\approx} \dot{\phi}_\rho \approx \{\phi_\rho, H_C\} + u^\sigma \{\phi_\rho, \phi_\sigma\} := h_\rho + C_{\rho\sigma} u^\sigma . \tag{C.15}$$

There are essentially two distinct situations, depending on whether the determinant of $C_{\rho\sigma}$ vanishes (weakly) or not

- $\det C \neq 0$: In this case (C.15) constitutes an inhomogeneous system of linear equations with solutions

  $$u^\rho \approx -\bar{C}^{\rho\sigma} h_\sigma$$

  where $\overline{C}$ is the inverse of the matrix C. Therefore, the Hamilton equations of motion (C.13) become

  $$\dot{F} \approx \{F, H_C\} - \{F, \phi_\rho\} \bar{C}^{\rho\sigma} \{\phi_\sigma, H_C\},$$

  which are free of any arbitrary multipliers.
- $\det C \approx 0$: In this case, the multipliers are not uniquely determined and (C.15) is only solvable if the h_ρ fulfill certain relations, derived as follows: Let the rank of C be M. This implies that there are $(P\text{-}M)$ linearly-independent null eigenvectors, i.e. $w^\rho_\alpha C_{\rho\sigma} \approx 0$ from which by (C.15) we find the conditions

  $$0 \overset{!}{\approx} w^\rho_\alpha h_\rho .$$

  These either are fulfilled or lead to a certain number S' of new constraints

  $$\phi_{\bar{\rho}} \approx 0 \qquad \bar{\rho} = P+1, \ldots, P+S'$$

  called "secondary" constraints according to Anderson and Bergmann [9]. The primary and secondary constraints define a hypersurface $\Gamma_2 \subseteq \Gamma_P$. In a further step–for consistency again–one has to check that the original and the newly generated constraints are conserved on Γ_2. This might imply another generation of constraints, called tertiary constraints, defining a hypersurface $\Gamma_3 \subseteq \Gamma_2$, etc., etc. It seems harder to formulate the algorithm in all its details (these may be found in [479]) than to apply it to the field theories of interest. In most physically relevant cases, the algorithm terminates with the secondary constraints, and there are only few cases where also at most tertiary constraints appear.

The algorithm terminates when the following situation is attained: There is a hypersurface Γ_C defined by the constraints

$$\phi_\rho \approx_{|\Gamma_C} 0 \quad \rho = 1, \ldots, P \quad \text{and} \quad \phi_{\bar{\rho}} \approx_{|\Gamma_C} 0 \quad \bar{\rho} = P+1, \ldots, P+S. \tag{C.16}$$

The first set $\{\phi_\rho\}$ contains all P primary constraints, the other set $\{\phi_{\bar{\rho}}\}$ comprises all secondary, tertiary, etc. constraints, assuming there are S of them. It turns out to be convenient to use a common notation for all constraints as $\phi_{\hat{\rho}}$ with $\hat{\rho} = 1, \ldots, P+S$. Furthermore, for very left null-eigenvector $w_\alpha^{\hat{\rho}}$ of the matrix

$$\hat{C}_{\hat{\rho}\rho} = \{\phi_{\hat{\rho}}, \phi_\rho\} \tag{C.17}$$

the conditions $w_\alpha^{\hat{\rho}}\{\phi_{\hat{\rho}}, H_C\} \approx_{|\Gamma_C} 0$ are fulfilled. For the multiplier functions u^ρ, the equations

$$\{\phi_{\bar{\rho}}, H_C\} + \{\phi_{\bar{\rho}}, \phi_\rho\}u^\rho \approx_{|\Gamma_C} 0, \tag{C.18}$$

hold. In the following, weak equality $\approx$ is always understood with respect to the "final" constraint hypersurface Γ_C. This hypersurface is also termed 'reduced phase-space'. Its description in local coordinates and its symplectic structure are far from obvious.

First- and Second-Class Constraints

We are still curious about the fate of the multiplier functions. Our curiosity leads to the notion of first- and second-class objects.

Some of the Eqs. (C.18) may be fulfilled identically, others represent conditions on the u^ρ. The details depend on the rank of the matrix $\hat{C}$ defined by (C.17). If the rank of $\hat{C}$ is P, all multipliers are fixed. If the rank of $\hat{C}$ is $K < P$ there are P-K solutions of

$$\hat{C}_{\hat{\rho}\rho}V_\alpha^\rho = \{\phi_{\hat{\rho}}, \phi_\rho\}V_\alpha^\rho \approx 0. \tag{C.19}$$

The most general solution of the linear inhomogeneous Eqs. (C.18) is the sum of a particular solution U^ρ and a linear combination of the solutions of the homogeneous part:

$$u^\rho = U^\rho + v^\alpha V_\alpha^\rho \tag{C.20}$$

with arbitrary coefficients v^α. Together with ϕ_ρ, also the linear combinations

$$\phi_\alpha := V_\alpha^\rho \phi_\rho \tag{C.21}$$

constitute constraint functions. According to (C.19), these have the property that their Poisson brackets with all constraints vanish on the constraint surface.

A phase-space function $\mathcal{F}(p, q)$ is said to be *first class* (FC) if it has a weakly vanishing Poisson bracket with all constraints in the theory:

$$\{\mathcal{F}(p, q), \phi_{\hat{\rho}}\} \approx 0.$$

If a phase-space object is not first class, it is called *second class* (SC). Due to the definitions of weak and strong equality a first-class quantity obeys the strong equation

$$\{\mathcal{F}, \phi_{\hat{\rho}}\} \simeq f_{\hat{\rho}}^{\hat{\sigma}}\ \phi_{\hat{\sigma}},$$

from which by virtue of the Jacobi identity we infer that the Poisson bracket of two FC objects is itself an FC object:

$$\begin{aligned}\{\{\mathcal{F},\mathcal{G}\},\phi_{\hat{\rho}}\} &= \{\{\mathcal{F},\phi_{\hat{\rho}}\},\mathcal{G}\} - \{\{\mathcal{G},\phi_{\hat{\rho}}\},\mathcal{F}\} \\ &= f_{\hat{\rho}}^{\hat{\sigma}}\{\phi_{\hat{\sigma}},\mathcal{G}\} + \{f_{\hat{\rho}}^{\hat{\sigma}},\mathcal{G}\}\phi_{\hat{\sigma}} - g_{\hat{\rho}}^{\hat{\sigma}}\{\phi_{\hat{\sigma}},\mathcal{F}\} - \{g_{\hat{\rho}}^{\hat{\sigma}},\mathcal{F}\}\phi_{\hat{\sigma}} \approx 0.\end{aligned}$$

It turns out to be advantageous to reformulate the theory completely in terms of its maximal number of independent FC constraints and the remaining SC constraints. Assume that this maximal number is found after building suitable linear combinations of constraints. Call this set of FC constraints Φ_I $(I = 1, \ldots, L)$ and denote the remaining second class constraints by χ_A. Evidence that one has found the maximal number of FC constraints is the non-vanishing determinant of the matrix built by the Poisson brackets of all second class constraints

$$(\Delta_{AB}) = \{\chi_A, \chi_B\}. \tag{C.22}$$

If this matrix were singular, then there would exist a null vector $0 \approx e^A\Delta_{AB} \approx \{e^A\chi_A, \chi_B\}$, meaning that the constraint $e^A\chi_A$ commutes with all SC constraints. Since by the definition of first class the Poisson brackets with all FC constraints also vanish, $e^A\chi_A$ would be a first-class constraint itself, contradicting the assumption that all independent first-class constraints were found already. As a corollary we find that the number of SC constraints must be even[2], because otherwise $\det\Delta = 0$.

Rewriting the total Hamiltonian (C.11) with the aid of (C.20) as

$$H_T = H' + v^\alpha\phi_\alpha \qquad \text{with} \qquad H' = H_C + U^\rho\phi_\rho, \tag{C.23}$$

we observe that H' is itself first class, and that the total Hamiltonian is a sum of a first class Hamiltonian and a linear combination of primary first class constraints (PFC).

Consider again the system of Eqs. (C.18). They are identically fulfilled for the FC constraints. For a SC constraint, we can write these equations as

$$\{\chi_A, H_C\} + \Delta_{AB}\, u^B \approx 0$$

with the understanding that $u^B = 0$ if χ^B is a secondary constraint (SC). For the other multipliers we obtain

$$u^B = \overline{\Delta}^{BA}\ \{\chi_A, H_c\} \qquad \text{for} \qquad \chi_B \quad \text{primary}. \tag{C.24}$$

[2] This holds true only for finite-dimensional (and bosonic) systems.

where $\overline{\Delta}$ is the inverse of Δ. As a result, we find that all multipliers belonging to the primary second-class constraints in H' of (C.23) are determined, and that only the v^α are left open: There are as many arbitrary functions in the Hamiltonian as there are (independent) primary first-class constraints (PFC).

Inserting the solutions (C.24) into the Hamilton Eqs. (C.13), they become

$$\dot{F}(p,q) \approx \{F, H_T\} \approx \{F, H_C\} + \{F, \phi_\alpha\}v^\alpha - \{F, \chi_A\}\overline{\Delta}^{AB}\{\chi_B, H_C\}. \tag{C.25}$$

C.1.4 First-Class Constraints and Symmetries

"First-Class Constraints are Gauge Generators": Maybe . . . , . . . Some

It was argued that a theory with local variational symmetries necessarily is described by a singular Lagrangian and that it acquires constraints in its Hamiltonian description. In the previous section, we at last saw the essential difference between regular and singular systems in that for the latter, there might remain arbitrary functions as multipliers of primary first-class constraints. An educated guess leads us to suspect that these constraints are related to the local symmetries on the Lagrange level. This guess points in the right direction, but things aren't that simple. Dirac, in his famous lectures [126], introduced an influential invariance argument (in more detail described below) by which he conjectured that also secondary first-class constraints lead to invariances. His argumentation gave rise to the widely-held view that "first-class constraints are gauge generators". Aside from the fact that Dirac did not use the term "gauge" anywhere in his lectures, later work on relating the constraints to variational symmetries revealed that a detailed investigation on the full constraint structure of the theory in question is needed. The following analysis of Dirac's arguments is heavily influenced by [408].

Dirac's Conjecture and Variational Symmetries

Dirac's deliberations in [126] are as follows (in his own words): "For a general dynamical variable g with initial value g_0, its value at time δt is

$$g(\delta t) = g_0 + \delta t\, \dot{g} = g_0 + \delta t\, \{g, H_T\} = g_0 + \delta t\big[\{g, H'\} + v^\alpha\{g, \phi_\alpha\}\big].$$

The coefficients v are completely arbitrary and at our disposal. Suppose we take different values, v', for these coefficients. That would give a different $g(\delta t)$, the difference being $\Delta g(\delta t) = \delta t(v'^\alpha - v^\alpha)\{g, \phi_\alpha\}$. We may write this as

$$\Delta g(\delta t) = \epsilon^\alpha\{g, \phi_\alpha\}, \tag{C.26}$$

where

$$\epsilon^\alpha = \delta t(v'^\alpha - v^\alpha)$$

is a small arbitrary number, small because of the coefficient δt and arbitrary because the v's and v''s are arbitrary. We can change all our Hamiltonian variables in accordance with the rule (C.26) and the new Hamiltonian variables will describe the same state. This change in the Hamiltonian variables consists in applying an infinitesimal contact transformation with a generating function $\epsilon^\alpha \phi_\alpha$. We come to the conclusion that the ϕ_α's, which appeared in the theory in the first place as the primary first-class constraints, have this meaning: *as generating functions of infinitesimal contact transformations, they lead to changes in the q's and p's that do not affect the physical state*." (pp. 20-21 in [126]; italics by Dirac himself)

Dirac next calculates the commutator of two contact transformation with parameter ϵ, ϵ',

$$(\Delta_\epsilon \Delta_{\epsilon'} - \Delta_{\epsilon'} \Delta_\epsilon) g = \epsilon^\alpha {\epsilon'}^\beta \{g, \{\phi_\alpha, \phi_\beta\}\}$$

to show that $\{\phi_\alpha, \phi_\beta\}$ is also a generating function of an infinitesimal contact transformation in the sense outlined above. Since the ϕ_α are FC constraint functions their Poisson bracket is–as shown before–strongly equal to a linear combination of FC constraints. Since in general within this linear combination also secondary FC constraints can appear, one may conjecture that *all* FC constraints give rise to "changes in the q's and p's that do not affect the physical state"–and this is known as "Dirac's conjecture". It also led Dirac to introduce an "extended" Hamiltonian which contains all FC constraints

$$H_E = H_T + v^{\alpha'} \phi_{\alpha'} = H' + v^I \Phi_I; \tag{C.27}$$

and he remarks: "The presence of these further terms in the Hamiltonian will give further changes in g, but these further changes in g do not correspond to any change of state and so they should certainly be included, even though we did not arrive at these further changes of g by direct work from the Lagrangian" ([126], p.25). Although, as shown below, this extended Hamiltonian has some interesting features, it is not logically substantiated by the consistent derivation of phase-space dynamics from configuration-velocity space dynamics.

Dirac connected first-class constraints with "changes in the q's and p's that do not affect the physical state" in an infinitesimal neighborhood of the initial conditions. Somehow, it became common wisdom to call these "gauge transformations"; but this has to be taken with caution. The term "gauge transformation" is otherwise mainly used in the context of Yang-Mill theories where it has the meaning of "local phase transformations". However by some, it is also used for reparametrization-invariant theories; there it has the meaning of "general coordinate transformations".

Lagrangian and Hamiltonian Symmetries

Before trying to specify the relation between symmetries in configuration-velocity space and those in phase space one needs to state which symmetries one is talking about in both spaces. In Subsects. 2.2.2 - 2.2.4 the Noether symmetries are defined with respect to the Lagrangian and restricted to point transformations. The Lie symmetries refer to point transformations of the second-order equations of motion. We saw that even for seemingly simple examples (like the free nonrelativistic particle) the discovery of these symmetries is far from trivial. How does one find all invariances of an action for more complicated cases, if not already obvious by construction of the action? There are examples from supersymmetry where the action was known first, but the symmetries were found only later. If it comes to the Hamiltonian version of a theory it is not at all obvious how its symmetries (which are defined below) are related to the Lagrangian symmetries, especially in the case of singular systems. It is not evident *a priori* that all Lagrangian symmetries are Hamiltonian symmetries and *vice versa*. There even are counterexamples where Lagrangian symmetries in the tangent space cannot be projected to Hamiltonian symmetries in the cotangent space, and where Hamiltonian symmetries do not have a Lagrangian counterpart.

In order to simplify the following consideration consider classical mechanics. In asking for Lagrangian symmetries one seeks all transformations $\delta_\epsilon q^i = f^i(q, \dot{q}, \ddot{q}, \ldots \epsilon^r, \dot{\epsilon}^r, \ddot{\epsilon}^r, \ldots)$ that leave the action $S_L = \int dt\, L(q, \dot{q})$ invariant. (Here, for simplicity, I refer to transformations of the dependent variables only.) In turns out that for a complete and consistent investigation of tangent-space symmetries and their relation to cotangent-space symmetries one must allow also for "trivial" transformations

$$\delta_\rho q^i = \int dt'\, \rho^{ij}(t, t') \frac{\delta S_L}{\delta q^j(t')} = [L]_k\, \rho^{ik}$$

where the coefficient functions ρ are antisymmetric: $\rho^{ij}(t, t') = \rho^{ji}(t', t)$. These transformations are trivial in the sense that they vanish on-shell.

For regular systems we derived in Sect. 2.2.3 that for the Hamiltonian form of the action $S_H = \int dt\, \big(p_i q^i - H(q, p)\big)$ Noether symmetry transformations can on-shell be expressed as canonical transformations. Also in the singular case, we are interested in finding a symmetry generator such that $g(\underline{\epsilon}) = g_I(q, p)\underline{\epsilon}^I$ with $\delta_{\underline{\epsilon}}(-) = \{-, g(\underline{\epsilon})\}$. Again there are trivial transformations

$$\delta_{\xi,\alpha} q^i = EM(q^j)\xi_j{}^i + EM(p_j)\alpha^{ji} \qquad \delta_{\xi,\beta} p_i = EM(p_j)\xi^j{}_i + EM(q^j)\beta_{ji}$$

with antisymmetric matrices α and β and where $EM(F) := \dot{F} - \{F, H\}$ are the Euler-Hamilton expressions, that is $EM(F) = 0$ is the Hamilton equation for the phase-space variable F. As it turns out, sometimes one needs to invoke trivial transformations in order to fully restore an equivalence between Lagangian and Hamiltonian constraints [364].

Under certain regularity restrictions and under conditions of projectability, the symmetries for singular systems can be algorithmically constructed from the chain of Lagrangian constraints required by the quest for time conservation of the primary (Lagrangian) constraints [82].

The authors of [267] base their derivation of symmetry transformations on an observation by C. Teitelboim[3]: Under the assumption that only first-class constraints

[3] in his unpublished 1973 PhD thesis "The Hamiltonian structure of spacetime".

are present, the action with the extended Hamiltonian

$$S_E = \int d^4x(p_i\dot{q}^i - H_E) = \int d^4x(p_i\dot{q}^i - H_C - v^I\Phi_I) \tag{C.28}$$

changes only by a boundary term under the transformations

$$\delta_\epsilon q^i = \{q^i, \Phi_I\}\epsilon^I, \qquad \delta_\epsilon p_i = \{p_i, \Phi_I\}\epsilon^I \tag{C.29}$$

if the multipliers transform as

$$\delta_\epsilon v^I = \dot{\epsilon}^I + v^K\epsilon^J C_{KJ}^{\ \ I} - \epsilon^K H_K^I,$$

and where

$$\{\Phi_I, \Phi_J\} =: C_{IJ}^{\ \ K}\Phi_K \qquad \{H_C, \Phi_I\} =: H_I^K\Phi_K. \tag{C.30}$$

Under further specified regularity conditions, relations between Lagrangian symmetries and phase-space symmetries are derived, and it turns out that the generators for phase-space symmetries are linear combinations of all first class constraints, with the peculiarity that the coefficients depend on the multipliers v^I.

Indeed, in the the literature you will find many articles in which the authors argue with the extended Hamiltonian. Since up to now there is no consensus about the relevance of this heuristically introduced object, it might be worthwhile how the previous consideration changes in case one starts with the total Hamiltonian

$$S_T = \int d^4x(p_i\dot{q}^i - H_T) = \int d^4x(p_i\dot{q}^i - H_C - u^\rho\phi_\rho). \tag{C.31}$$

Assume that it is invariant with respect to the transformations (C.29). First, we have

$$\delta_\epsilon(p_i\dot{q}^i) = \dot{\epsilon}^I\Phi_I + \frac{d}{dt}\left[p_i\delta_\epsilon q^i - \epsilon^I\Phi_I\right]$$

and

$$\delta_\epsilon H_T = \epsilon^I\{H_C, \Phi_I\} + \phi_\rho\delta_\epsilon u^\rho + u^\rho\epsilon^I\{\phi_\rho, \Phi_I\}.$$

Since the Φ_I are FC constraints, we find

$$\{\phi_\rho, \Phi_I\} =: C_{\rho I}^{\ \ \hat{\rho}}\phi_{\hat{\rho}} \qquad \{H_C, \Phi_I\} =: H_I^{\hat{\rho}}\phi_{\hat{\rho}}$$

where the summation (with respect to $\hat{\rho}$) extends over all constraints defining the final constraint surface in phase space. Thus as a condition for the invariance of the action (C.31), we derive

$$0 \stackrel{!}{=} \dot{\epsilon}^I\Phi_I - \epsilon^I G_I^{\hat{\rho}}\phi_{\hat{\rho}} - \phi_\rho\delta_\epsilon u^\rho \qquad \text{with} \qquad G_I^{\hat{\rho}} = H_I^{\hat{\rho}} + u^\rho C_{\rho I}^{\ \ \hat{\rho}}. \tag{C.32}$$

This is, as it stands, a relation involving constraints of different characteristics, the first term involving all first-class constraints, the second containing all constraints, and the third primary constraints only. In this form one cannot say whether it has solutions for the symmetry parameters $\underline{\epsilon}^I$, or how many solutions there are. But there is a specific case in which the previous expression can be analyzed further: If all constraints are first-class primary, the transformations of the multipliers can be expressed as $\delta_{\underline{\epsilon}} u^I = \dot{\underline{\epsilon}}^I - \underline{\epsilon}^J G^I_J$. Notice that in this case the total and the extended Hamiltonians are identical and we are back at the previous consideration.

In [411], the condition for $G_\zeta(t)$ to be a canonical generator of infinitesimal phase-space transformations in the sense that

$$\delta_\zeta g = \{g, G_\zeta(t)\} \tag{C.33}$$

is derived as

$$\frac{\partial G_\zeta}{\partial t} + \{G_\zeta, H_T\} \simeq \text{PFC}$$

where PFC stands for a linear combination of primary first-class constraints. Since $H_T = H' + v^\alpha \phi_\alpha$, and remembering that the functions v^α which multiply the primary first-class constraints are arbitrary, we find the three conditions

$$G_\zeta(t) \qquad \text{is a first-class function} \tag{C.34a}$$

$$\frac{\partial G_\zeta}{\partial t} + \{G_\zeta, H'\} \simeq \text{PC} \tag{C.34b}$$

$$\{G_\zeta, PFC\} \simeq \text{PC}. \tag{C.34c}$$

where PC stands for a linear combinations of primary constraints, and PFC for any of the primary first class constraints.

Relating Lagrangian and Hamiltonian Symmetries

We saw that for all theories and models of fundamental physics the infinitesimal transformations on fields Q^α are of the generic form (3.67), that is

$$\delta_\epsilon Q^\alpha = \mathcal{A}^\alpha_r(Q) \cdot \epsilon^r(x) + \mathcal{B}^{\alpha\mu}_r(Q) \cdot \epsilon^r_{,\mu}(x) + \ldots, \tag{C.35}$$

and these are assumed to leave the action $S = \int d^D x \mathcal{L}(Q, \dot{Q})$ of the theory in question invariant. So why at all should the transformations (C.26)–and tentatively extended to all first-class constraints–investigated by Dirac in an infinitesimal neighborhood of the initial conditions, that is

$$\delta_\epsilon Q^\alpha = \epsilon^I \{Q^\alpha, \Phi_I\} \tag{C.36}$$

be related to the transformations (C.35) of a variational symmetry? Is there a mapping between the parameter functions ϵ^r and $\underline{\epsilon}^I$? Can one specify an algorithm to calculate the generators of Noether symmetries in terms of constraints? Or, in starting from the symmetry generating function G with the properties (C.34), is there a relation to the Dirac transformations (C.36) in terms of first-class constraints?

It seems that the very first people to address these questions were J.L. Anderson and P.G. Bergmann [9]–even before the Hamiltonian procedure for constrained systems was fully developed. N. Mukundan [366] started off from the chain (3.76) of Noether identities and built symmetry generators as linear combinations of first class primary and secondary constraints from them, assuming that no tertiary constraints are present. L. Castellani [79] devised an algorithm for calculating symmetry generators for local symmetries, implicitly neglecting possible second-class constraints. His algorithm was proven to be correct for those Noether transformations that are projectable from tangent to cotangent space.

The set of cotangent-space symmetry transformations characterized by (C.34) includes transformations that do not have a counterpart in tangent space. As shown in [33], the generator for those Noether symmetries that are projectable to phase space are characterized by

$$G_\zeta(t) \qquad \text{is a first-class function} \tag{C.37a}$$

$$\frac{\partial G_\zeta}{\partial t} + \{G_\zeta, H_C\} = \mathrm{PC} \tag{C.37b}$$

$$\{G_\zeta, PC\} = \mathrm{PC}. \tag{C.37c}$$

These conditions are more restrictive then (C.34) in that (1) "ordinary" equalities arise instead of strong ones, (2) instead of H' the canonical Hamiltonian enters the condition, (3) the last Poisson bracket contains all primary constraints (PC) and not only the PFC's.

In observing that for both Yang-Mills theories and for general relativity, the infinitesimal symmetry transformations can be expanded in terms of infinitesimal parameters and their derivatives, Anderson and Bergmann [9] already assumed an expansion

$$G_\zeta = \sum_{k=0}^{N} \zeta^{(k)}(t) G_k(q,p) \qquad \text{with} \qquad \zeta^{(k)}(t) := \frac{d^k \zeta}{dt^k}. \tag{C.38}$$

(In its full glory the ζ carries a further index denoting different symmetries.) If this *ansatz* is plugged into (C.37) then due to the arbitrariness of the ζ, one obtains

$$\{G_k, PC\} \simeq \mathrm{PC} \tag{C.39}$$

and the conditions for Noether projectable transformations is expressed as the chain

$$G_N = \mathrm{PFC} \tag{C.40a}$$

$$G_{k-1} + \{G_k, H_C\} = \mathrm{PC}, \qquad \mathrm{k} = \mathrm{K}, \ldots, 1 \tag{C.40b}$$

$$\{G_0, H_C\} = \mathrm{PC}. \tag{C.40c}$$

Thus one may start with taking G_N to be identified with an arbitrarily chosen primary first-class constraint. Next G_{N-1} is found to be a secondary first-class constraint (up to PC pieces), etc. One expects that any independent PFC can be taken as the head of the chain (C.40a). However, both conceptually and algorithmically, it makes a difference how the chain algorithm is started. It is even not clear from the very outset that solutions of (C.39) and (C.40) exist. It was shown in [227] under which conditions one can–at least locally–construct a complete set of independent symmetry transformations according to (C.38). As a corollary of this proof it is found that the number of independent Lie symmetry transformations is identical to the number of primary first-class constraints.

The symmetry parameter $\underline{\epsilon}^I$ in the generating function $g = \underline{\epsilon}^I \Phi_I$ can be expressed by the symmetry parameter ζ in the generator G_ζ. This was made specific for pure first-class systems in [101] and later in this appendix it is exemplified for the relativistic particle.

Finally, it remains to be settled how the Dirac-type transformations (C.26) are related to the transformations (C.35). For both transformations, it holds that

$$\delta g(t) = \{g, G_\zeta(t)\} = \sum_{k=0}^{N} \zeta^{(k)}(t)\{g, G_k\}.$$

Dirac's transformations are characterized by infinitesimal time intervals δt near $t = 0$. In order to guarantee $\delta g(0) = 0$, one must impose $\zeta^{(k)}(0) = 0 \ \forall k$. Then to first order in δt,

$$\zeta^{(k)}(\delta t) = \zeta^{(k)}(0) + \delta t \cdot \zeta^{(k+1)}(0)$$

implying

$$\begin{aligned} \zeta^{(k)}(\delta t) &= 0 \qquad \text{for} \qquad k = 0, \ldots, N-1 \\ \zeta^{(N)}(\delta t) &= \delta t \cdot \zeta^{(N+1)}(0) \end{aligned}$$

where the value of $\zeta^{(N+1)}(0)$ is arbitrary. Choosing $\zeta^{(N+1)}(0) = (v'^{\alpha} - v^{\alpha})$ yields a transformationfor each α of the form

$$\delta g(t) = \zeta\{g, G_N\}$$

which is a Dirac-type transformation (C.26), since, due to (C.40), G_N is a primary first class function.

Observables

A classical *observable* $\mathcal{O}(t)$ is defined as a phase space function that is invariant under the transformations generated by the $G(t)$, that is for which

$$\{\mathcal{O}(t), G(t)\} \approx 0. \tag{C.41}$$

This is equivalent to the vanishing of the Poisson bracket of an observable with all first-class constraints:

$$\{\mathcal{O}(t), \Phi_I\} \approx 0.$$

This property incidentally makes the discussion of whether to employ the total or the extended Hamiltonian superfluous: Although, in general, the equations of motions are different in both cases for generic fields, they are identical for observables.

C.1.5 Second-Class Constraints and Gauge Conditions

The previous subsection dealt at length with first-class constraints because they are related to variational symmetries of the theory in question. Second-class constraints χ_A enter the Hamiltonian equations of motion (C.25) without arbitrary multipliers. If there are no first-class constraints the dynamics is completely determined by

$$\dot{F}(p,q) \approx \{F, H_T\} \approx \{F, H_C\} - \{F, \chi_A\}\overline{\Delta}^{AB}\{\chi_B, H_C\}$$

without any ambiguity.

Dirac Bracket

Dirac introduced in [124] a "new P.b.":

$$\{F, G\}^* := \{F, G\} - \{F, \chi_A\}\ \overline{\Delta}^{AB}\ \{\chi_B, G\} \tag{C.42}$$

nowadays called the *Dirac bracket* (DB). Sometimes for purposes of clarity it is judicious to indicate in the notation $\{F, G\}^*_\Delta$ that the DB is built with respect to the matrix Δ. The Dirac bracket satisfies the same properties as the Poisson bracket, i.e. it is antisymmetric, bilinear, and it obeys the product rule and the Jacobi identity. Furthermore, the DBs involving SC and FC constraints obey

$$\{F, \chi_A\}^* \equiv 0 \qquad \{F, \Phi_I\}^* \approx \{F, \Phi_I\}.$$

Thus when working with Dirac brackets, second-class constraints can be treated as strong equations. The equations of motion (C.25) written in terms of DBs are

$$\dot{F}(p,q) \approx \{F, H_T\}^*. \tag{C.43}$$

For later purposes let me mention an iterative property of the DB: First define the Dirac bracket for a subset of the SC constraints (denoted by $\chi_{A'}$) with $\det\{\chi_{A'}, \chi_{B'}\} \neq 0$:

$$\{F, G\}^*_{\Delta'} := \{F, G\} - \{F, \chi_{A'}\}\overline{\Delta'}^{A'B'}\{\chi_{B'}, G\}.$$

Next select further second-class constraints $\chi_{A''}$, such that $\{\chi_{A''}\} \cap \{\chi_{A'}\} = \{\}$, $\Delta''_{A''B''} = \{\chi_{A''}, \chi_{B''}\}$ with $\det \Delta''_{A''B''} \neq 0$ and

$$\{F, G\}^*_{\Delta''} := \{F, G\}^*_{\Delta'} - \{F, \chi_{A''}\}^*_{\Delta'} \overline{\Delta''} A''B'' \{\chi_{B''}, G\}^*_{\Delta'}.$$

This iterative process can be pursued until all second-class constraints are exhausted. The result is the same as (C.42), that is as one would have computed the DB with respect to the full set of second-class constraints.

Just as Poisson brackets have a geometric and group-theoretical meaning, also the Dirac brackets are more than just a convenient compact notation; see e.g. [479].

Gauge Fixing

The existence of unphysical symmetry transformations notified by the presence of first-class constraints may make it necessary to impose conditions on the dynamical variables. This is specifically the case if the observables cannot be constructed explicitly–and this is notably true in general for Yang-Mills and for gravitational theories. These extra conditions are further constraints

$$\Omega_a(q, p) \approx 0, \tag{C.44}$$

where now weak equality refers to the hypersurface Γ_R defined by the weak vanishing of all previously found first- and second-class constraints, that is the hypersurface Γ_C together with the constraints (C.44). The idea is that the quest for stability of these constraints, namely

$$0 \stackrel{!}{\approx} \dot{\Omega}_a \approx \{\Omega_a, H_C\} + \{\Omega_a, \phi_\alpha\} v^\alpha - \{\Omega_a, \chi_A\} \overline{\Delta}^{AB} \{\chi_B, H_C\}$$

is meant to uniquely determine the multiplier v^α. At least for finite-dimensional systems, the previous condition can be read as a linear system of equations which has unique solutions if the number of independent gauge constraints is the same as the number of primary FC constraints and if the gauge constraints are chosen so that the determinant of the matrix

$$\Lambda_{\beta\alpha} := \{\Omega_\beta, \phi_\alpha\} \tag{C.45}$$

does not vanish[4]. In this case the multipliers are fixed to:

$$v^\alpha = \overline{\Lambda}^{\alpha\gamma} \big[- \{\Omega_\gamma, H_C\} + \{\Omega_\gamma, \chi_A\} \overline{\Delta}^{AB} \{\chi_B, H_C\} \big].$$

If there are no second-class constraints the Hamilton equations can be shown (see e.g. [479]) to be equivalent to

$$\dot{F}(p, q) \approx \{F, H_C\}^*_{\hat{\Lambda}} \qquad \text{with} \qquad \hat{\Lambda} = \begin{pmatrix} \{\phi_\alpha, \phi_\beta\} & \{\phi_\alpha, \Omega_\beta\} \\ \{\Omega_\alpha, \phi_\beta\} & \{\Omega_\alpha, \Omega_\beta\} \end{pmatrix}. \tag{C.46}$$

[4] This determinant is related to the Faddeev-Popov determinant which plays a central rôle in the path integral formulation of gauge systems; see App. D.1.3.

It is quite obvious how, by making use of the iterative property of the Dirac bracket, one can extend this to situations with SC constraints: Build the Dirac brackets with respect to an enlarged matrix containing both the Δ-matrix from the Poisson brackets of the SC constraints and the matrix Λ made from the gauge fixings and the primary first-class constraints.

Some remarks concerning the choice of gauge constraints Ω^α :

- The condition of a non-vanishing determinant $(\det(\Lambda_{\alpha\beta}) \neq 0)$ is only a sufficient condition for determining the arbitrary multipliers connected with the primary FC constraints.
- The gauge constraints must not only be such that the "gauge" freedom is removed (this is guaranteed by the non-vanishing of $\det \Lambda$), but also the gauge constraints must be accessible, that is for any point in phase space with coordinates (q, p), there must exist a transformation $(q, p) \to (q', p')$ such that $\Omega_\alpha(q', p') \approx 0$. This may be achievable only locally.
- We will see that in case of reparametrization invariance (at least one of) the gauge constraints must depend on the parameters explicitly–and not only on the phase-space variables.
- Especially in field theories it may be the case that no globally admissible (unique and accessible) gauge constraints exist. An example is given by the Gribov ambiguities, as they were first found in in Yang-Mills theories.

C.1.6 Constraints in Field Theories: Some Remarks

The canonical formulation for singular Lagrangians was derived in the previous section for systems with a finite number of degrees of freedom. However, the motivation for the work of Rosenberg, the Syracuse group around Bergmann, and of Dirac were constraints in field theory (or even more specifically: in general relativity). Although many of the previous techniques and results can be directly transferred to field theories, one should be aware about some peculiarities.

First, constraints in a field theory are in general no longer functions of the phase-space fields, but they may also depend on spatial derivatives:

$$\phi_\rho[Q, \Pi, \partial_i Q, \partial_i \Pi] = 0.$$

Here "spatial" (and the index i) refer to $(D\text{-}1)$ dimensions other than the chosen time variables. In the instant form (with $x^0 \doteq T$) it has the meaning of the Euclidean "spatial". As a consequence, constraints in field theories are no longer only pure algebraic relations among phase-space variables, but can very well be differential equations. Furthermore, with $\phi_\rho \approx 0$, also spatial derivatives and integrals of ϕ_ρ are weakly zero. Constraints in a field theory must be understood as having not only a discrete index (ρ) but additionally a continuous one, namely one for each point in spacetime.

In the context of identifying the constraints and the multiplier functions we need to solve linear equations which contain matrices defined in terms of Poisson brackets of constraints:

$$P_{rs}(\bar{x},\bar{y}) = \{\phi_r(x), \phi_s(y)\}_{x^T=y^T}.$$

If P is non-singular, its inverse $\bar{P}^{rs}$ is not uniquely defined by

$$\int dy\ P_{rs}(x,y)\bar{P}^{st}(y,z) = \delta_r t\delta(x-z)$$

and, in case $\det P = 0$, the null eigenvectors of P are also not defined uniquely.

We saw that a main step towards the final Hamiltonian for constrained systems is to identify the maximal number of first-class constraints and to distinguish these from the rest of all constraints. This task is purely algebraic for non-field theories. In field theories, first-class constraints may also be built by taking into account for instance spatial derivatives of other constraints.

C.1.7 Quantization of Constrained Systems

Already from "ordinary" quantum mechanics–and without phase-space constraints–we know that there is no royal road[5] connecting a classical theory with its quantum counterpart.

The "Dirac quantization" for systems with constraints largely builds on the Hilbert space method sketched in Sect. 4.1.4. (switching from classical observables to Hermitean operators in a Hilbert space, replacing Poisson brackets of phase-space objects by commutators of observables):

$$\{F,G\} \longrightarrow \frac{1}{i\hbar}[\hat{F},\hat{G}]. \tag{C.47}$$

However, one has to keep in mind that due to the existence of constraints, the Hilbert space in which the quantum operators act is itself restricted–and this can be described explicitly only for specific cases. In any case, one may try one of the following:

1. Imagine having fixed the multipliers "in front" of the constraints by one way or another. Call the full set of constraints (including the gauge constraints) $\mathcal{C}_\alpha$. Use the substitution rule (C.47) as if there were no constraints. Require that matrix elements built with the constraints vanish. That is, define the Hilbert space $\mathcal{H}_P$ implicitly by those states which obey

$$\langle\psi'|\hat{\mathcal{C}}_\alpha|\psi\rangle\ = 0. \tag{C.48}$$

[5] This phrase is said to stem from Euclid, when he was asked by the ruler Ptolemy I Soter if there was a shorter road to learning geometry than through Euclid's Elements: "There is no royal road to geometry".

Aside from difficulties arising from finding the quantum counterparts $\hat{\mathcal{C}}_\alpha$ of the phase-space constraints, this condition is not very handy because you have to know the full spectrum of the Hamiltonian before you can decide which states do belong to $\mathcal{H}_P$.

2. Instead of (C.48) one may consider the stronger conditions

$$\hat{\mathcal{C}}_\alpha |\psi\rangle = 0.$$

But indeed this is much to strong, since for consistency one has to demand that also $[\hat{\mathcal{C}}_\alpha, \hat{\mathcal{C}}_\beta]|\psi\rangle = 0$, and this does not hold in general. It at most holds weakly for first-class constraints ϕ_α because $\{\phi_\alpha, \phi_\beta\} = g^\gamma_{\alpha\beta}\phi_\gamma$. But even in this case, one must allow for the possibility that the quantum analogue of the Poisson brackets does not neatly read $[\hat{\phi}_\alpha, \hat{\phi}_\beta] = \hat{g}^\gamma_{\alpha\beta}\hat{\phi}_\gamma$, since the structure coefficients g may become operators.

3. In the case of second-class constraints χ_A, there is no way to impose consistently the conditions $\hat{\chi}_A|\psi\rangle = 0$. This would make sense only if the operators $\hat{\chi}_A$ themselves vanish. Nevertheless, this opens another perspective of quantizing a system with phase-space constraints. We know that after complete gauge fixing, all constraints become second class and that, when using Dirac brackets instead of Poisson brackets, the constraints can be set to zero identically. Thus it is suggestive to replace the "rule" (C.47)

$$\{F, G\}^* \longrightarrow \frac{1}{i\hbar}[\hat{F}, \hat{G}]. \tag{C.49}$$

But of course the problems with operator ordering remain. Furthermore, the quantization may work in one gauge, but not in another.

Path Integral

Assume we are dealing with a theory with N degrees of freedom that has first-class constraints ϕ_α $(\alpha = 1, \ldots M)$ only. According to [165], we may formally introduce the path-integral

$$\int \mathcal{D}p\mathcal{D}q\mathcal{D}\lambda \, \exp\left\{i \int_0^T [p_k \dot{q}^k - H(p,q) - \lambda^\alpha \phi_\alpha(p,q)]\right\}$$
$$= \int \mathcal{D}p\mathcal{D}q \, \delta\big(\phi_\alpha(p,q)\big) \exp\left\{i \int_0^T [p_k \dot{q}^k - H(p,q)]\right\}.$$

Introduce M complementary gauge constraints $\Omega_\alpha(p,q) = 0$, such that the original symmetry is completely broken. As mentioned, a necessary condition is $\det\{\Omega_\alpha, \phi_\beta\} \neq 0$. (For simplicity we may assume that there are no Gribov ambiguities.) Then, as shown by Faddeev,

$$\int \mathcal{D}p\mathcal{D}q\ \delta\big(\phi_\alpha(p,q)\big)\delta\big(\Omega_\alpha(p,q)\big)\det\{\Omega_\alpha,\phi_\beta\}\exp\left\{i\int_0^T[p_k\dot{q}^k - H(p,q)]\right\}$$
$$= \int \mathcal{D}p^*\mathcal{D}q^*\ \exp\left\{i\int_0^T[p_j^*\dot{q}^{*j} - H(p^*,q^*)]\right\}$$

where the (q^*, p^*) are the coordinates in the $(N\text{-}M)$-dimensional reduced phase space. Faddeev's idea was extended to theories with quite general gauge constraints and additional second-class constraints by Batalin, Vilkovisky, Fradkin, Fradkina, ... in the seventies (more details may be found [479], Chap. IV).

Field-Antifield Formalism

This was the forerunner of BRST ghost/anti-ghost techniques, and their specific embodiment for constrained dynamics; see [261], [265]. These techniques enable the quantization of any kind of system with variational symmetries in the broadest sense (reducible gauge algebra, soft algebra, open algebra, etc.)

C.2 Yang-Mills Type Theories

In order to get a feeling about the generic Hamiltonian structure of Yang-Mills fields coupled to spinor fields, we first consider vacuum electrodynamics, next the Maxwell-Dirac theory ("classical QED"), and then the pure Yang-Mills theory.

C.2.1 Electrodynamics

Start from the Lagrangian $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, and choose the instant form $(x^0 \mathrel{\hat{=}} T)$. The Hessian

$$\mathcal{W}^{\mu\nu} := \frac{\partial^2\mathcal{L}}{\partial(\partial_0 A_\mu)\partial(\partial_0 A_\nu)} = \eta^{0\mu}\eta^{0\nu} - \eta^{\mu\nu}\eta^{00},$$

has rank $(D\text{-}1)$ in D dimensions. Thus, one primary constraint can be expected. From

$$\Pi^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} = \partial^\mu A^0 - \partial^0 A^\mu, \tag{C.50}$$

the momentum canonically conjugate to A_0, namely

$$\phi_1 := \Pi^0(x) \approx 0 \tag{C.51}$$

is detected as the primary constraint. The canonical Hamiltonian is

$$\mathcal{H}_C = \Pi^\mu \dot{A}_\mu - \mathcal{L} = \frac{1}{2}\Pi_i\Pi^i - A_0\,\partial_i\Pi^i + \frac{1}{4}F_{ik}F^{ik} + \partial_i(\Pi^i A_0). \tag{C.52}$$

The total Hamiltonian $\mathcal{H}_T = \mathcal{H}_C + u_1\Pi^0$ contains the arbitrary function $u_1(x)$ in correlation with the primary constraint. Because of the canonical Poisson brackets $\{A_\mu(x), \Pi^\nu(y)\}_{x^0=y^0} = \delta_\mu^\nu\,\delta^3(\vec{x}-\vec{y})$–all others vanishing–one finds

$$0 \stackrel{!}{\approx} \dot{\phi}_1 = \{\Pi^0, H_T\} = -\{\Pi^0, \int d^3x\; A_0\partial_i\Pi^i\} = \partial_i\Pi^i.$$

and thus discovers the secondary constraint

$$\phi_2 := \partial_i\Pi^i \approx 0. \tag{C.53}$$

Requiring that the secondary constraint vanish weakly does not enforce further constraints. Since $\{\phi_1, \phi_2\} = \{\Pi^0, \partial_i\Pi^i\} = 0$ the two constraints are first class. Observe that because of $\Pi^i = -F^{0i} = -E^i$ the constraint $\partial_i\Pi^i \approx 0$ is simply Gauß' Law, that is the first of the Maxwell Eqs. (3.1a).

Section C.1.5 dealt with the relationship of first-class constraints and symmetry generators. One may try to confine oneself to the only PFC and set $\Pi^0 = G_0$ in (C.40a). However, for this choice, $\{G_0, H_C\} \neq$ PC. The next attempt is $\Pi^0 = G_1$ and indeed this gives the solution

$$G_\zeta = \int d^3x(\dot{\zeta}\Pi^0 - \zeta\partial_i\Pi^i) = \int d^3x\;(\partial_\mu\zeta)\Pi^\mu + \text{b.t.} \tag{C.54}$$

The transformations mediated by this generator on the phase-space variables are

$$\begin{aligned}
\delta_\zeta A_\mu &= \{A_\mu, G_\zeta\} = \{A_\mu(x), \int d^3y\;(\partial_\nu\zeta)\Pi^\nu\}\\
&= \int d^3y\;\partial_\nu\zeta(y)\;\{A_\mu(x), \Pi^\nu(y)\} = \partial_\mu\zeta\\
\delta_\zeta\Pi^\mu &= \{\Pi^\mu, G_\zeta\} = \{\Pi^\mu(x), \int d^3y\;(\partial_\nu\zeta)\Pi^\nu\} = 0.
\end{aligned}$$

The transformation of the A_μ are indeed the Noether symmetries (5.47) identifying the infinitesimals as $\zeta = (1/e)\theta$–e being the charge. The cotangent space transformation of Π^μ is completely compatible with the transformation in tangent space, since by the definition (C.50), one obtains $\delta\Pi^\mu = \partial^\mu\delta A^0 - \partial^0\delta A^\mu = 0$.

So far, the treatment of Hamiltonian electrodynamics fully respects the gauge symmetry of the theory. The gauge symmetry can be broken by choosing two gauge conditions Ω_I (two, because both first-class constraints make their appearance in G_ϵ according to (C.54)), such that the matrix $\{\Omega_I, \Phi_J\}$ is nonsingular. An admissible gauge choice is e.g. the radiation gauge

$$\Omega_1 := A_0 \approx 0 \qquad \Omega_2 := \partial_i A_i \approx 0.$$

The matrix $\hat{\Lambda}$ defined in (C.46) is

$$\hat{\Lambda}(\vec{x}, \vec{y})) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\nabla^2 \\ -1 & 0 & 0 & 0 \\ 0 & \nabla^2 & 0 & 0 \end{pmatrix} \delta(\vec{x} - \vec{y}).$$

Indeed this matrix is nonsingular; it can also be shown that by this choice the original gauge freedom $\delta A_\mu = \partial_\mu \epsilon$ is completely broken [479]: the function $\epsilon(x)$ is fully determined by the radiation gauge. The Dirac brackets is shaped by the inverse of $\hat{\Lambda}$, which is

$$\hat{\Lambda}^{-1} = \begin{pmatrix} 0 & 0 & -\delta(\vec{x} - \vec{y}) & 0 \\ 0 & 0 & 0 & -\frac{1}{4\pi|\vec{x}-\vec{y}|} \\ \delta(\vec{x} - \vec{y}) & 0 & 0 & 0 \\ 0 & \frac{1}{4\pi|\vec{x}-\vec{y}|} & 0 & 0 \end{pmatrix}.$$

From this, the fundamental Dirac brackets are calculated as

$$\{A_\mu(x), \Pi^\nu(y)\}^* \ = \ (\delta_\mu^\nu \ + \delta_\mu^0 \eta^{\nu 0})\delta^3(\vec{x} - \vec{y}) \ - \partial_\mu \partial^\nu \ \frac{1}{4\pi|\vec{x} - \vec{y}|}$$

–all others vanishing. The DB's different from the PB's are

$$\begin{aligned} &\{A_0(x), \Pi^\nu(y)\}^* \ = \ 0 \qquad \{A_\mu(x), \Pi^0(y)\}^* \ = \ 0 \\ &\{A_i(x), \Pi_j(y)\}^* \ = \ \delta_{ij} \ \delta^3(\vec{x} - \vec{y}) \ - \partial_i \partial_j \ \frac{1}{4\pi|\vec{x} - \vec{y}|} =: \delta_{ij}^{\mathrm{tr}}(\vec{x} - \vec{y}). \end{aligned}$$

The first two of these bracket relations are in accordance with $\Pi^0 = 0 = A_0$, and the latter one is in line with $\partial_i \Pi_i = 0 = \partial_i A_i$. Here $\delta_{ij}^{\mathrm{tr}}(\vec{x} - \vec{y})$ denotes the transversal delta function which was introduced *ad hoc* in previous treatments of electrodynamics.

While in the case of vacuum electrodynamics it is possible to identify the true degrees of freedom and thus to specify explicitly the reduced phase space, this is no longer true in non-Abelian gauge theories.

If *in lieu* of the instant form, the front form (that is $T \hat{=} x^+ = (x^0 - x^3)$) had been chosen, then instead of one primary constraint there would have been three primary constraints and one secondary constraint. Two linear combinations can be made first class and two are second class. The latter serve to define the Dirac brackets, and the Hamiltonian formulation in front form eventually turns out to be completely equivalent to the one in instant form, for details see [479].

C.2.2 Maxwell-Dirac Theory

Our starting point is the QED-Lagrangian (5.51)

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + e\bar{\psi} A_\mu \gamma^\mu \psi. \tag{C.55}$$

for which the canonical momenta (in the instant form) are computed as

$$\Pi^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu}$$

$$\pi^\alpha := \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\alpha)} = -i(\bar{\psi}\gamma^0)^\alpha \qquad \bar{\pi}^\alpha := \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi}_\alpha)} = 0.$$

We spot the primary constraints directly:

$$\phi := \Pi^0 \approx 0 \qquad \phi^\alpha := \pi^\alpha + i(\bar{\psi}\gamma^0)^\alpha \approx 0 \qquad \bar{\phi}^\alpha := \bar{\pi}^\alpha \approx 0. \tag{C.56}$$

The canonical Hamiltonian density is found to be

$$\mathcal{H}_C = \mathcal{H}_C^{ED} - i\bar{\psi}\gamma^k\partial_k\psi + m\bar{\psi}\psi + e\bar{\psi}A_\mu\gamma^\mu\psi,$$

where $\mathcal{H}_C^{ED}$ is the canonical Hamiltonian density for the free electromagnetic field (C.52). The total Hamiltonian is thus

$$H_T = H_C + \int d^3x\,(u\phi + \phi_\alpha v_\alpha + w_\alpha\bar{\phi}_\alpha) \tag{C.57}$$

with multiplier functions $\{u,\ v_\alpha,\ w_\alpha\}$. Stabilization of the primary constraints requires

$$0 \overset{!}{\approx} \{\phi, H_T\} = \partial_k\Pi^k + e\bar{\psi}\gamma^0\psi \tag{C.58a}$$

$$0 \overset{!}{\approx} \{\phi_\alpha, H_T\} = i\partial_k(\bar{\psi}\gamma^k)_\alpha + m\bar{\psi}_\alpha - e(\bar{\psi}\gamma_\mu A^\mu)_\alpha - i(w\gamma_0)_\alpha \tag{C.58b}$$

$$0 \overset{!}{\approx} \{\phi_\alpha, H_T\} = i\partial_k(\gamma^k\psi)_\alpha - m\psi_\alpha + e(\gamma_\mu A^\mu\psi)_\alpha - i(\gamma_0 v)_\alpha. \tag{C.58c}$$

To calculate these expressions, one needs the Bose-Fermi Poisson brackets (see App. B.2.4.), that is specifically the non-vanishing equal-time brackets

$$\{A_\mu(x), \Pi^\nu(y)\} = \delta_\mu^\nu\,\delta(\vec{x}-\vec{y}) \tag{C.59}$$

$$\{\psi_\alpha, \pi^\beta\} = -\delta_\alpha^\beta\,\delta(\vec{x}-\vec{y}) \qquad \{\bar{\psi}_\alpha, \bar{\pi}^\beta\} = -\delta_\alpha^\beta\,\delta(\vec{x}-\vec{y}). \tag{C.60}$$

Conditions (C.58b) and (C.58c) can be fulfilled by an appropriate choice of the multiplier functions w_α and v_α. On the other hand, (C.58a) constitutes a secondary constraint

$$\phi_G := \partial_k\Pi^k + e\bar{\psi}\gamma^0\psi \approx 0. \tag{C.61}$$

Observe that this is again Gauß's law, now with the charge density $j_0 = e\bar{\psi}\gamma^0\psi$. Requiring next that the time derivative of ϕ_G vanish on the constraint surface does not lead to further constraints–the straightforward but tedious calculation is suppressed here. Thus at the end of the algorithm we are left with a set of 10 constraints (in four dimensions): $\{\phi, \phi^\alpha, \bar{\phi}^\alpha, \phi_G\}$ (or $10 \times \infty$–to be pedantic)

What about the maximal number of first-class constraints that can be built from these constraints? One immediately finds that the constraint function ϕ has a vanishing Poisson bracket with all other nine constraints. Thus, $\Phi_1 = \phi \approx 0$ is a first-class constraint. A further one is found to be the combination

$$\Phi_2 := \phi_G + ie(\phi^\alpha \psi_\alpha + \bar{\psi}_\alpha \bar{\phi}^\alpha) = \partial_k \Pi^k - ie(\pi^\alpha \psi_\alpha + \bar{\pi}^\alpha \bar{\psi}_\alpha).$$

Dirac brackets are to be built with the second-class constraint functions ϕ^α and $\bar{\phi}^\alpha$. The matrix corresponding to (C.22) is

$$\Delta(\vec{x},\vec{y}) = -i \begin{pmatrix} 0 & (\gamma^0)^T \\ \gamma^0 & 0 \end{pmatrix} \delta(\vec{x}-\vec{y}) \quad \Delta^{-1}(\vec{x},\vec{y}) = i \begin{pmatrix} 0 & \gamma^0 \\ (\gamma^0)^T & 0 \end{pmatrix} \delta(\vec{x}-\vec{y})$$

and the Dirac bracket becomes

$$\begin{aligned} \{F(x), G(y)\}^*_\Delta = \{F(x), G(y)\} - i(\gamma^0)_{\alpha\beta} \int dz \; \{F(x), \phi^\alpha\}\{\bar{\phi}^\beta, G(y)\} \\ - i(\gamma^0)_{\alpha\beta} \int dz \; \{F(x), \bar{\phi}^\alpha\}\{\phi^\beta, G(y)\}. \end{aligned}$$

The Dirac brackets differing from the fundamental brackets (C.59) are

$$\{\bar{\psi}_\alpha, \bar{\pi}^\beta\}^*_\Delta = 0 \tag{C.62a}$$

$$\{\psi_\alpha, \bar{\psi}_\beta\}^*_\Delta = -i(\gamma^0)_{\alpha\beta}\, \delta(\vec{x}-\vec{y}) \tag{C.62b}$$

which are compatible with the constraints $\phi^\alpha \approx 0 \approx \bar{\phi}^\alpha$ and confirm that one is allowed to interpret the second-class constraints as strong equations. Finally the gauge generator is

$$G_\zeta = \int d^3x \left[\dot{\zeta}\Pi^0 - \zeta\big(\partial_k \Pi^k - ie(\pi^\alpha \psi_\alpha + \bar{\pi}^\alpha \bar{\psi}_\alpha)\big)\right].$$

C.2.3 Non-Abelian Gauge Theories

The Yang-Mills Lagrangian

$$L = -\frac{1}{4} \int d^3x \; F^a_{\mu\nu} F^{\mu\nu}_a \qquad \text{with} \qquad F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g A^b_\mu A^c_\nu f^{bca}$$

contains the momenta (canonically conjugate to the A^a_μ)

$$\Pi^\mu_a = -F^{0\mu}_a. \tag{C.63}$$

Here, we assume again the instant form ($x^0 = T$) and thus the fundamental Poisson brackets are

$$\{A^a_\mu(x), \Pi^\nu_b\}_{x^0=y^0} = \delta^\nu_\mu \delta^a_b \delta(\vec{x}-\vec{y}).$$

The antisymmetry of the field strength components gives the primary constraints

$$\phi_a = \Pi^0_a \approx 0. \tag{C.64}$$

The canonical Hamiltonian

$$H_C = \int d^3x\, (\Pi^\mu_a \dot{A}^a_\mu - L) = \int d^3x\, \big(\Pi^k_a \dot{A}^a_k - \frac{1}{2}\Pi^\mu_a \Pi^a_\mu + \frac{1}{4} F^a_{kl} F^{kl}_a\big)$$

becomes independent of the "velocities", due to $\dot{A}^a_k = F^a_{0k} + \partial_k A^a_0 + g A^b_0 A^c_k f^{bca}$:

$$H_C = \int d^3x\, \big(\frac{1}{2}\Pi^\mu_a \Pi^a_\mu + \frac{1}{4} F^a_{kl} F^{kl}_a - A^a_0(\partial_k \Pi^k_a - g f_{ac}{}^b A^c_k \Pi^k_b) + b.t.\big) \tag{C.65}$$

(after another partial integration). The algorithm for finding possible further constraints starts with requiring that

$$0 \overset{!}{\approx} \{\phi_a, H_T\} = \{\phi_a, H_C\} + \{\phi_a, \int d^3x\; u^b\; \phi_b\} =: \varphi_a$$

which leads to the secondary constraints

$$\varphi_a = \partial_k \Pi^k_a - g f_{ac}{}^b A^c_k \Pi^k_b \approx 0. \tag{C.66}$$

With the gauge group covariant derivative (5.63), this constraint can be written as $\varphi_a = D_k \Pi^k_a$, which directly shows the generalization from the secondary constraint in the Abelian (Gauß's Law) to the non-Abelian case. Checking for stability of the constraints φ_a as $0 \overset{!}{\approx} \{\varphi_a, H_T\} \approx \{\varphi_a, H_C\} = g f_{ab}{}^c A^b_0\; \varphi_c$, one observes that this vanishes on the hypersurface defined by $\phi_a \approx 0 \approx \varphi_a$. Thus, there are no further constraints. Both the primary and secondary constraints are first class, since

$$\{\Pi^0_a(x), D_i \Pi^i_a(y)\} = 0,$$
$$\{D_i \Pi^i_a(x), D_j \Pi^j_b(y)\} = -g f_{ab}^{\;\;c}\; (D_k \Pi^k_c(x))\; \delta(\vec{x} - \vec{y}) \approx 0.$$

In order to find the gauge generators one may start the chain (C.40) with $G_{1(a)} = \Pi^0_a$ and

$$G_{0(a)} = -\{\Pi^0_a, H_C\} + \lambda_{ab}\Pi^0_b = -\varphi_a + \lambda_a{}^b \Pi^0_b.$$

The coefficients $\lambda_a{}^b$ in the linear combination of the primary first-class constraints are determined from

$$PC \overset{!}{=} \{G_{0(a)}, H_C\} = \{\varphi_a, H_C\} + \lambda_a{}^b\{\Pi^0_b, H_C\} = g f_{ac}{}^b A^c_0 \varphi_b + \lambda_a{}^b \varphi_b$$

with the result

$$G_\zeta = \int d^3x (\dot{\zeta^a} \Pi^0_a - \zeta^a \big(D_i \Pi^i_a - g f_{ac}{}^b A^c_0 \Pi^0_b\big)) = \int d^3x (D_\mu \zeta^a) \Pi^\mu_a + \text{b.t.} \tag{C.67}$$

The transformations generated by G_ζ

$$\delta_\zeta A^a_\mu = \{A^a_\mu(x), \int d^3y\ (D_\nu \zeta^b)\Pi^\nu_b\} = D_\mu \zeta^a$$

$$\delta_\zeta \Pi^\mu_a = \{\Pi^\mu_a(x), \int d^3y\ (D_\nu \zeta^b)\Pi^\nu_b\}$$

$$= \int d^3y\ \Pi^\nu_b(y)\{\Pi^\mu_a(x), g f_{cd}{}^b A^c_\nu \zeta^d(y)\} = -g f_{ad}{}^b \Pi^\mu_b \zeta^d$$

correspond to the Noether symmetry transformations (5.62) if $\zeta^a = (1/g)\theta^a$, those for the momenta being compatible with the identifications (C.63).

C.3 Reparametrization-Invariant Theories

The terms in this section heading are essentially synonymous to "generally covariant" or "diffeomorphism invariant", the prime example from fundamental physics being general relativity. But even classical physics can be reformulated in a reparametrization-invariant way, and this has various benefits, see Chap. 3 of [441]. Many of the results in Subsect. C.3.1 hold independently of the specific example. Some of these are directly visible in the most simple case ("GR in one dimension"), the relativistic point particle. In stepping up to higher dimensions, one would arrive at relativistic string models, at membranes, and many other extended entities which are more or less fashionable in modern theoretical physics.

The reason why Yang-Mills type theories are treated in a section separate from reparametrization-invariant theories is substantiated by the fact that there is an essential difference in their Hamiltonian description: As will be demonstrated, the total Hamiltonian for reparametrization-invariant theories is a sum of constraints. This odd feature led to a plethora of prejudices, misunderstandings and still unsettled questions in the Hamiltonian treatment of generally covariant systems, and as such for the canonical formulation of GR.

In this section, I start with disclosing some basic features of reparametrization-invariant theories. Then the Hamiltonian analysis of the relativistic particle serves as a preliminary stage to canonical GR. In this book, I only sketch the treatment of vacuum GR (ADM approach for metric gravity, tetrad gravity, Ashtekar variables). But I close the section by presenting some essential peculiarities in the treatment of GR in the presence of Yang-Mills and/or Dirac fields.

C.3.1 Immediate Consequences of Reparametrization Invariance

The immediate consequences of reparametrization invariance are explicitly demonstrated to hold for systems with a finite number of degrees of freedom, but can straightforwardly shown to retain validity also for field theories.

1. Consider a system with an action $S = \int_{t_1}^{t_2} dt\ L(q^k, \dot{q}^k)$. If the Lagrange function fulfills

$$L(q, \lambda \dot{q}) = \lambda L(q, \dot{q}), \tag{C.68}$$

then the action is invariant with respect to the reparametrization $t \to \hat{t}(t)$. In order to interpret $\hat{t}(t)$ again as "time", the transformations need to be restricted to strictly monotonous functions, or $d\hat{t}/dt > 0$. Furthermore, $\hat{t}(t_i) = t_i$, that is, the boundary is mapped to itself.
2. The relation (C.68) expresses the fact that the Lagrange function is homogeneous (of first degree) in the velocities. Differentiate both sides with respect to λ:

$$\frac{\partial L(q, \lambda \dot{q})}{\partial \lambda \dot{q}^j} \dot{q}^j = L(q, \dot{q})$$

and use (C.68) again to derive (after renaming/rescaling the velocities)

$$\frac{\partial L}{\partial \dot{q}^j} \dot{q}^j - L = 0. \tag{C.69}$$

On the other hand if (C.69) holds true, we regain (C.68) by integration.
3. The condition (C.69) is equivalent to a vanishing canonical Hamilton function (by definition of the canonical momenta as $p_j = \partial L / \partial \dot{q}^j$):

$$H_C = p_j \dot{q}^j - L = 0. \tag{C.70}$$

4. Differentiate (C.69) with respect to the velocities:

$$0 = \frac{\partial}{\partial \dot{q}^k} \left[\frac{\partial L}{\partial \dot{q}^j} \dot{q}^j - L \right] = W_{kj} \dot{q}^j + \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial \dot{q}^k}$$

from which it is seen that the Hessian has the zero eigenvector $(\dot{q}^j)$:

$$W_{kj} \dot{q}^j = 0. \tag{C.71}$$

5. The invariance with respect to the reparametrization $t \to \hat{t}$ brings about a Noether identity. Instead of deriving it according to the procedure of Sect. 3.3., proceed in the following way: Contract the Euler-Lagrange derivatives $[L]_i$ with $\dot{q}^i$, i.e. consider

$$[L]_i \dot{q}^i = \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right] \dot{q}^i = \frac{\partial L}{\partial q^i} \dot{q}^i - \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i \right] + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i.$$

Now make use of (C.69) in the second term to replace it by $\frac{d}{dt} L$. Next compute the time derivative of the Lagrangian in order to find that it cancels with the two other terms. Thus

$$[L]_i \dot{q}^i = 0 \tag{C.72}$$

is the Noether identity.

6. With a similar consideration, we derive the variation $\bar{\delta}L$ assuming that the q^i transform as scalars under the reparametrization; infinitesimally:

$$\delta t = \epsilon \qquad \text{and} \qquad \delta q^i = 0, \qquad \bar{\delta} q^i = -\dot{q}_i \epsilon.$$

From this

$$\bar{\delta} L = \frac{\partial L}{\partial q^i} \bar{\delta} q_i + \frac{\partial L}{\partial \dot{q}^i} \bar{\delta} \dot{q}^i = -\frac{\partial L}{\partial q^i} \dot{q}^i \epsilon - \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i \epsilon - \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i \dot{\epsilon}.$$

Now using (C.69), the last term becomes $(-L\dot{\epsilon})$ and the two first terms add up to $(-\dot{L}\epsilon)$. Thus, independently of the specific or explicit form of the Lagrangian, it transforms as a scalar density:

$$\bar{\delta} L = -\frac{d}{dt}(L\epsilon). \tag{C.73}$$

All previous results can be extended to reparametrization-invariant field theories in D dimensions. The homogeneity conditions (C.68) and (C.69) become

$$\mathcal{L}(Q, \lambda_\mu^\nu \partial_\nu Q) = \det(\lambda) \mathcal{L} \quad \longleftrightarrow \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu Q^\alpha)} \partial(\partial_\nu Q^\alpha) = \delta_\mu^\nu \mathcal{L}.$$

From these, one derives the vanishing of the canonical Hamiltonian according to (C.70) and the singularity of the Hessian according to (C.71) (depending on one's choice of the temporal parameter). The D Noether identities (C.72) correspond to

$$[L]_\alpha \partial_\mu Q^\alpha = 0.$$

Finally for $\delta x^\mu = \epsilon^\mu$, (C.73) becomes generalized to

$$\bar{\delta} \mathcal{L} = -\partial_\mu (\mathcal{L} \epsilon^\mu).$$

C.3.2 Free Relativistic Particle

World-Line Description

The trajectory of a massive relativistic particle can be written in parametric form $x^\mu(\tau)$ where τ labels the trajectory. The action from which to derive its equation of motion may be taken to be proportional to the length of the world line as

$$S = -m \int d\tau (\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2}. \tag{C.74}$$

As shown in Subsect, 3.2.3 this reproduces in the nonrelativistic limit the dynamics for the free particle. The Lagrangian to (C.74) is homogeneous in the velocities $L(\lambda \dot{x}^\mu) = \lambda L(\dot{x}^\mu)$. The $(D \times D)$ Hessian associated with this Lagrangian is

$$W_{\mu\nu} := \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = \frac{m}{(\dot{x}^2)^{3/2}}(\dot{x}_\mu \dot{x}_\nu - \eta_{\mu\nu}\, \dot{x}^2),$$

and it has rank $(D\text{-}1)$. The zero eigenvector is easily recognized as being proportional to $\dot{x}^\mu$ in accordance with (C.71). The Euler-Lagrange derivatives are

$$[L]_\mu = -\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}\,\ddot{x}^\nu = -W_{\mu\nu}\ddot{x}^\nu$$

from which the identity $[L]_\mu \dot{x}^\mu \equiv 0$ immediately results, revealing that only $(D\text{-}1)$ equations of motion are independent. This is off course the Noether identity, corresponding to (C.72), and stemming from the invariance of the action under re-parametrization, infinitesimally

$$\delta\tau = \epsilon(\tau) \qquad \text{and} \qquad \delta_\epsilon x^\mu = 0 \qquad \bar{\delta}_\epsilon x^\mu = -\dot{x}^\mu \epsilon, \tag{C.75}$$

where $\bar{\delta} L = -d/d\tau(\epsilon L)$, again in compliance with the generic results of the previous subsection.

Let us proceed to the Hamiltonian formulation for this system: The canonical momenta to the x^μ are

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -m\frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}. \tag{C.76}$$

We expect one primary constraint since the rank of the Hessian is 1. This constraint can either be found by solving the $p_\mu(\dot{x}^\nu)$ for as many "velocities" as possible, or by observing that

$$p_\mu p^\mu = m^2 \frac{\dot{x}_\mu \dot{x}^\mu}{\dot{x}^2} = m^2.$$

This is just the mass-shell condition, in this context to be understood as a constraint among the momenta which we write as

$$H_\perp := \frac{1}{2}(p^2 - m^2) \approx 0. \tag{C.77}$$

In going through the Rosenfeld-Dirac-Bergmann procedure, we first find that the canonical Hamiltonian $H_C = p_\mu \dot{x}^\mu - L$ vanishes identically–in agreement with the findings in the previous paragraph. Thus the total Hamiltonian is proportional to the constraint: $H_T = vH_\perp$ with the multiplier v. No further constraints exist.

In what sense is the constraint $H_\perp$ a symmetry generator? In order to answer this question we need to relate the infinitesimal transformation in phase space

$$\delta_\xi x^\mu := \{x^\mu, \xi H_\perp\} = \xi p^\mu \qquad \delta_\xi p_\mu := \{p_\mu, \xi H_\perp\} = 0$$

to the reparametrization (C.75). Thus due to (C.76) we seek the correspondence

$$-\xi m(\dot{x}^2)^{-1/2}\dot{x}^\mu \longleftrightarrow -\dot{x}^\mu \epsilon.$$

This demonstrates that the infinitesimal parameter ξ mediating the symmetry transformation in phase space must depend on $\dot{x}^\mu$, that is the "time" derivative of the "fields" x^μ. This is a typical feature of reparametrization-invariant theories (and as such also for general relativity) which seems to be little known. As pointed out by P. Bergmann and A. Komar [42], the action (C.74) is not only invariant with respect to transformations $\delta\tau = \epsilon(\tau)$, but also to those where $\delta\tau = \epsilon\,[\tau, x^\mu]$, that is where ϵ is a functional of τ and x^μ. In phase space, however, these "field"-dependent transformations are not arbitrary. In this case of a relativistic particle we have $\delta\tau = (\dot{x}^2)^{-1/2}\epsilon(\tau)$. The deeper reason of this dependency is the demand for projectability of the Lagrangian transformation $\delta_\epsilon x^\mu$ under the Legendre transformation (see C.5.1.): With the zero eigenvector $(\dot{x}^\mu)$, the operator for checking projectability is $\Gamma = \dot{x}^\nu\ \partial/\partial\dot{x}^\nu$ according to (C.119), so that

$$0 \stackrel{!}{=} \Gamma(\delta_\epsilon x^\mu) = \dot{x}^\nu \frac{\partial}{\partial\dot{x}^\nu}(-\dot{x}^\mu\epsilon) = -\dot{x}^\mu\epsilon - \dot{x}^\nu\dot{x}^\mu\frac{\partial\epsilon}{\partial\dot{x}^\nu}.$$

This has indeed the solution $\epsilon \propto (\dot{x}^2)^{-1/2}\xi(\tau)$. The variation of the momenta (expressed in configuration-velocity space variables) becomes

$$\bar{\delta}_\epsilon p_\mu = \frac{\partial p_\mu}{\partial\dot{x}^\nu}\bar{\delta}_\epsilon\dot{x}^\nu = \frac{\partial p_\mu}{\partial\dot{x}^\nu}(-\dot{x}^\nu\epsilon)^{\cdot} = -\frac{\partial p_\mu}{\partial\dot{x}^\nu}\ddot{x}^\nu\epsilon - \frac{\partial p_\mu}{\partial\dot{x}^\nu}\dot{x}^\nu\dot{\epsilon} = [L]_\mu\epsilon - (W_{\mu\nu}\dot{x}^\nu)\dot{\epsilon}.$$

The second term vanishes identically. But the first term does not: The transformation mediated by the first-class constraint is compatible with the Noether symmetry transformation only on-shell. This is typically the case for reparametrization-invariant theories, in contrast to Yang-Mills type theories, the deeper reason being that the Hamiltonian is quadratic in the momenta.

A point of confusion is the seemingly double role of the constraint as a Hamiltonian and as a generator of symmetry transformations. The vanishing of the total Hamiltonian is discussed with terms like "frozen time" or "nothing happens". This attitude comes about from the formal similarity of the two entities by choosing $\xi = (v/2)$. Although both have the same mathematical form, the gauge generator and the Hamiltonian act in different spaces: The gauge generator takes a complete solution and maps a point in the space of solutions to another point in this space, i.e. another solution, while the Hamiltonian takes initial data in this point and maps these to later data in the same point. This issue of time is not specific to the relativistic particle but arises in all generally covariant theories, and has been clarified in [412].

As for gauge fixing, one first verifies that any gauge constraint function Ω necessarily must depend on the parameter τ since it must be conserved (weakly), that is

$$0 \stackrel{!}{\approx} \frac{d\Omega}{d\tau} = \frac{\partial\Omega}{\partial\tau} + \{\Omega, H_T\} = \frac{\partial\Omega}{\partial\tau} + vp^\mu\frac{\partial\Omega}{\partial x^\mu}.$$

A non-vanishing solution for the free parameter v exists only if both $\frac{\partial\Omega}{\partial\tau} \neq 0$ and if $\frac{\partial\Omega}{\partial x^\mu} \neq 0$, at least for one index μ. Of course one can imagine many gauge conditions fulfilling these requests. A choice which eventually leads to simple equations of motions is $\Omega = x^0$-τ, from which $v = 1/p^0$. The ensuing Hamilton function

$$H = \frac{1}{2\,p^0}(p^2 - m^2)$$

in turn entails the simple Hamilton equations of motion $\dot{x}^\mu = p^\mu/p^0$ and $\dot{p}_\mu = 0$.

In order to complete the story, we might construct the Dirac brackets with respect to the two constraints Ω and $H_\perp$. I leave it as a simple exercise to show that the fundamental Dirac brackets become

$$\{x^\mu, x^\nu\}^*_\Delta = 0 \qquad \{p_\mu, p_\nu\}^*_\Delta = 0 \qquad \{x^\mu, p_\nu\}^*_\Delta = \delta^\mu_\nu - \frac{p^\mu}{p^0}\,\delta^0_\nu$$

which indeed allows one to take the constraint as a strong equation: $\{x^\mu, p^2 - m^2\}^*_\Delta = 0$.

Einbein Description

The dynamics of the relativistic particle can also be derived from the Lagrange function

$$L(\dot{x}^\mu, e) = \frac{1}{2e}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{m^2}{2}e. \tag{C.78}$$

This Lagrangian contains a further function e, which one may call the einbein[6], because of a resemblance with the notion of a vierbein field, for instance in 4D general relativity. The Euler-Lagrange derivatives are

$$[L]_\mu = -\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{d}{d\tau}\left(\frac{\dot{x}_\mu}{e}\right) \qquad\qquad [L]_e = \frac{\partial L}{\partial e} = -\frac{1}{2e^2}\dot{x}^2 + \frac{1}{2}m^2.$$

Observe that the Euler-Lagrange equation for the einbein is the Lagrangian primary constraint. On-shell with $[L]_e = 0$ holds

$$e \doteq \pm\frac{\sqrt{\dot{x}^2}}{m},$$

and replacing this in the original Lagrangian (C.78), one indeed returns to the description of the relativistic particle with the action (C.74)[7]. Notice that this holds only for $m \neq 0$. And also, be aware that both Lagrangians for the massive relativistic particles are equivalent only with respect to the classical solutions, but not necessarily equivalent in their quantum version. The Lagrangian (C.78) has the advantage of being free of the nasty square root and of having a zero mass limit. Being quadratic in the velocities, it can be brought into to a path-integral formulation one is accustomed to from scalar and vector field theories.

[6] If the relativistic particle is treated in the ADM manner (see C.3.3.) as one-dimensional gravity, the e is more appropriately interpreted as the lapse function N.

[7] This substitution is allowed in the sense that it does not change the dynamic content of the model, since the field equation for e itself is used to express e by the other fields. The field e is indeed an auxiliary field; see Sect. 3.3.4.

The Euler derivatives are not independent but obey the identity

$$[L]_\mu \dot{x}^\mu - e\partial_\tau [L]_e \equiv 0.$$

This "smells" like a Noether identity, and indeed it can be rewritten in a form that allows one to read off the infinitesimal transformations that stand behind the symmetry:

$$\big([L]_\mu \dot{x}^\mu + [L]_e \dot{e}\big) - \partial_\tau \big([L]_e e\big) \equiv 0.$$

Compare this with the generic expression (3.84) of the Noether identity to find that for $\delta\tau = \epsilon$ the "fields" transform as $\delta x^\mu = 0$ and $\delta e = -e\dot{\epsilon}$, or

$$\bar{\delta} x^\mu = -\dot{x}^\mu \epsilon \qquad \bar{\delta} e = -e\dot{\epsilon} - \dot{e}\epsilon = -\partial_\tau(e\epsilon). \tag{C.79}$$

The Lagrangian (C.78) is quasi-invariant with respect to these transformations

$$\begin{aligned}\delta L = \frac{\partial L}{\partial \dot{x}^\mu}\bar{\delta}\dot{x}^\mu + \frac{\partial L}{\partial e}\bar{\delta} e &= -\Big(\frac{\partial L}{\partial \dot{x}^\mu}\ddot{x}^\mu + \frac{\partial L}{\partial e}\dot{e}\Big)\epsilon - \Big(\frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\mu + \frac{\partial L}{\partial e}e\Big)\dot{\epsilon}\\ &= -\dot{L}\epsilon - \Big(\frac{\dot{x}^2}{e} - \frac{\dot{x}^2}{2e^2}e + \frac{m^2}{2}e\Big)\dot{\epsilon} = -\partial_\tau(L\epsilon).\end{aligned}$$

Thus this model exhibits features of reparametrization invariance considered in the previous subsection. But observe that the Lagrangian (C.78) is not homogeneous of first degree, and thus we cannot infer immediately that the canonical Hamiltonian vanishes. But we will see shortly that the total Hamiltonian vanishes weakly, since it becomes a sum of first-class constraints.
The momenta canonically conjugate to x^μ and e are

$$p_\mu := \frac{\partial L}{\partial \dot{x}^\mu} = \eta_{\mu\nu}\frac{\dot{x}^\nu}{e} \qquad P := \frac{\partial L}{\partial \dot{e}} = 0. \tag{C.80}$$

The latter is the primary constraint we are inclined to expect from the invariance related to (C.79). The canonical Hamiltonian is

$$H_C = \dot{x}^\mu p_\mu + \dot{e}P - L = ep^2 - \frac{1}{2}\frac{e^2\,p^2}{e} - \frac{1}{2}m^2 e = \frac{1}{2}e(p^2 - m^2),$$

and thus

$$H_T = \frac{1}{2}e(p^2 - m^2) + uP. \tag{C.81}$$

The requirement of time conservation of the primary constraint leads to

$$0 \overset{!}{\approx} \{P, H_T\} = -\frac{1}{2}(p^2 - m^2) := -H_\perp.$$

This is a secondary constraint–and in accordance with the general findings–the phase-space version of the primary Lagrangian constraint $[L]_e = 0$. Actually it is identical to the primary Hamiltonian constraint derived from the action (C.74). Both constraints are first class, so that in the end the total Hamiltonian (C.81) is a linear combination of first-class constraints. The equations of motion induced by this Hamiltonian are

$$\dot{x}^\mu = ep^\mu \qquad \dot{p}_\mu = 0 \qquad \dot{e} = u \qquad \dot{P} = -\frac{1}{2}(p^2 - m^2) \approx 0.$$

The equation of motion for x^μ reproduces the momentum in (C.80), the equation of motion for the einbein e determines the multiplier u.
Both constraints enter the symmetry generator

$$G_\zeta = H_\perp \zeta + P\dot{\zeta}.$$

As in the situation of the worldline description of the relativistic particle, the phase-space symmetry generator can be used to generate Noether symmetries only for a specific choice of descriptors ζ, namely from

$$\delta_\zeta x^\mu = \{x^\mu, G_\zeta\} = \eta^{\mu\nu} p_\nu \zeta \qquad \delta_\zeta e = \{e, G_\zeta\} = \dot{\zeta}$$

we get only the correspondence with (C.79) if $\zeta = -e\epsilon$. Again, the descriptor in the symmetry generator depends on fields, here on the einbein e. And again this directly follows from the demand for projectability, which in this case amounts to

$$0 \stackrel{!}{=} \frac{\partial}{\partial \dot{e}} \bar{\delta} e = -e \frac{\partial}{\partial \dot{e}} \dot{\epsilon} - \epsilon - \dot{e} \frac{\partial}{\partial \dot{e}} \epsilon,$$

and which indeed has the solution $\epsilon = \xi(\tau)/e$. The variation of the momenta vanish on-shell, i.e. either on solutions or on the constraint surface.

In order to complete the story, let us investigate the invariance of the action

$$S_T = \int d\tau \Big(p_\mu \dot{x}^\mu + P\dot{e} - \frac{e}{2}(p^2 - m^2) - u\Phi_1 \Big)$$

under the transformations canonically resulting from the generator $\delta_{\underline{\epsilon}}(-) = \{-, \Phi_I\}\underline{\epsilon}^I$ with the two first class constraints $\Phi_1 = P$ and $\Phi_2 = 1/2(p^2 - m^2)$. Following the argumentation after (C.31), the condition (C.32) reads in this case

$$0 \stackrel{!}{=} \dot{\epsilon}^1 \Phi_1 + \dot{\epsilon}^2 \Phi_2 - \epsilon^1 \Phi_2 - \delta_\epsilon u \, \Phi_1 \ ,$$

and thus $\delta_{\underline{\epsilon}} u = \dot{\underline{\epsilon}}^1$ and $\dot{\underline{\epsilon}}^2 = \underline{\epsilon}^1$, this being consistent with previous results.

Supergravity in One Dimension

Off course the Hamiltonian approach can (and has) also been applied to supersymmetric theories. Although this is not the place to display for example the canonical treatment of generic (or even specific) supergravity theories, we will be able to identify some typical features of Hamiltonian supersymmetry with one-dimensional models. “Supergravity in one dimension” has also been titled “The spinning classical particle", although the spin is commonly understood as an outgrowth of relativistic quantum physics. This section will be a bit sketchy, since otherwise it could become a book of its own.

"Just for fun", consider the following Lagrange function

$$L = \frac{1}{2}\Big\{\frac{\dot{x}^2}{e} - i\psi_\mu\dot{\psi}^\mu - \frac{2i}{e}\chi(\psi_\mu\dot{x}^\mu)\Big\} \tag{C.82}$$

in terms of commuting "fields" x^μ, e and anti-commuting fields ψ^μ, χ. Instead of dealing immediately with the (quasi-)invariances of the Lagrangian (which you most probably would not be able to guess) we turn directly to the phase space and discover the invariances from the constraints. The momenta are[8]

$$p_\mu := \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{e}(\dot{x}^\mu - i\chi\psi_\mu) \qquad p_e := \frac{\partial L}{\partial \dot{e}} = 0$$
$$\pi_\mu := \frac{\partial L}{\partial \dot{\psi}^\mu} = \frac{i}{2}\psi_\mu \qquad \pi_\chi := \frac{\partial L}{\partial \dot{\chi}} = 0.$$

From this, one reads off the primary constraints

$$\phi_e := p_e \approx 0 \qquad \phi_\mu := \pi_\mu - \frac{i}{2}\psi_\mu \approx 0 \qquad \phi_\chi := \pi_\chi \approx 0.$$

The canonical Hamiltonian and the total Hamiltonian are found to be

$$H_C = \frac{e}{2}p^2 + i\chi(p_\mu\psi^\mu) \qquad H_T = H_C + u\phi_e + v^\mu\phi_\mu + v\phi_\chi$$

with an even multiplier u and odd multipliers (v^μ, v). According to the Rosenfeld-Dirac-Bergmann procedure the time (τ) derivatives of the primary constraints are required to vanish weakly. In order to implement these requirements, calculate the Poisson brackets among the constraints, using the canonical Bose-Fermi Poisson brackets

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu \qquad \{e, p_e\} = 1 \qquad \{\psi^\mu, \pi_\nu\} = -\delta^\mu_\nu \qquad \{\chi, \pi\} = -1$$

–all others vanishing. From these, we find the only non-vanishing brackets among the primary constraints: $\{\phi_\mu, \phi_\nu\} = i\eta_{\mu\nu}$. Next, require

$$\begin{aligned}
0 \overset{!}{\approx} \{\phi_e, H_T\} &= \{p_e, H_C\} = -\frac{1}{2}p^2 =: \hat{\phi} \\
0 \overset{!}{\approx} \{\phi_\mu, H_T\} &= \{\phi_\mu, i\chi(p\psi)\} - \{\phi_\mu, \phi_\nu\}v^\nu = ip^\mu\chi - iv^\mu \\
0 \overset{!}{\approx} \{\phi_\chi, H_T\} &= \{\pi_\chi, H_C\} = i(p\psi) =: \bar{\phi}.
\end{aligned}$$

The first and the last requirement result in two secondary constraint functions $\hat{\phi}$ and $\bar{\phi}$. The further requirements lead to a determination of the multipliers v^μ as

[8] Observe that all over this book, derivatives are taken "from the right" in the sense of App. B.2.3.

$v^\mu = p^\mu \chi$. Tertiary constraints do not arise since $\{\hat{\phi}, H_T\} = 0$ and $\{\bar{\phi}, H_T\} = i\{\, p\psi, v^\mu \pi_\nu\} = ip_\mu v^\mu = ip^2\chi \approx 0$.

The constraints $\phi_e, \phi_\chi, \hat{\phi}$ are immediately recognized as first-class constraints. The constraint $\bar{\phi}$ is not first class, since $\{\bar{\phi}, \phi_\mu\} = ip_\mu$. However, the linear combination of second-class constraints

$$\Omega := \bar{\phi} + p^\mu \phi_\mu$$

is first class: Its Poisson brackets with $\phi_e, \phi_\chi, \hat{\phi}$ vanish and

$$\begin{aligned} \{\Omega, \phi_\mu\} &= ip^\nu\{\psi_\nu, \phi_\mu\} + p^\nu\{\phi_\nu, \phi_\mu\} = ip^\nu\{\psi_\nu, \pi_\mu\} + ip_\mu = 0 \\ \{\Omega, \Omega\} &= \{ip\psi, p\phi\} - \{p\phi, ip\psi\} + \{p\phi, p\phi\} \propto p^2 \approx 0. \end{aligned}$$

With this, the total Hamiltonian can be written as a sum of first-class constraints:

$$H_T = -e\hat{\phi} + \chi\Omega + u\phi_e + v\phi_\chi.$$

Dirac brackets are built with respect to the second-class constraints. The matrix inverse to $(\Delta) = \{\phi_\mu, \phi_\nu\}$ is $(\Delta^{-1}) = -i\eta_{\mu\nu}$ and therefore the Dirac brackets are

$$\{A, B\}^*_\Delta = \{A, B\} + i\{A, \phi_\lambda\}\{\phi^\lambda, B\}$$

specifically

$$\{A, H_T\}^*_\Delta = \{A, H_T\} + i\{A, \phi_\lambda\}\{\phi^\lambda, H_T\} = \{A, H_T\}.$$

Since the only fields which have non-vanishing Poisson brackets with the second-class constraints ϕ_μ are ψ^μ and π_ν, the sole changes in the canonical brackets are

$$\{\psi_\mu, \psi_\nu\}^*_\Delta = i\eta_{\mu\nu} \qquad \{\psi_\mu, \pi_\nu\}^*_\Delta = -\frac{1}{2}\eta_{\mu\nu} \qquad \{\pi_\mu, \pi_\nu\}^*_\Delta = -\frac{i}{4}\eta_{\mu\nu}.$$

The Hamilton equations of motion derived from $\dot{F} = \{F, H_T\}^*$ read

$$\dot{x}^\mu = ep^\mu + i\chi\psi^\mu \qquad \dot{p}_\mu = 0 \tag{C.83a}$$

$$\dot{e} = u \qquad \dot{p_e} \approx 0 \tag{C.83b}$$

$$\dot{\psi}^\mu = v^\mu = \chi p^\mu \qquad \dot{\pi}_\mu = (i/2)v^\mu = (i/2)\chi p^\mu \tag{C.83c}$$

$$\dot{\chi} = v \qquad \dot{\pi}_\chi \approx 0. \tag{C.83d}$$

The appearance of two primary first class constraints indicates the presence of symmetries. Since the PFC ϕ_e is even and ϕ_χ is odd, we expect that each of them takes part in generating even and odd symmetry transformations, respectively. Thus we will go through the gauge generating algorithm (C.40) for the two primary first-class constraints separately.

In starting with the PFC $\phi_e = p_e$, we observe from $\{\phi_e, H_T\} = \hat{\phi}$ that this does not yield a PC. Therefore ϕ_e is unsuited for being the generator component G_0. Thus we start the algorithm with $G_1 = \phi_e$ and in the next step we determine $G_0 = -\{\phi_e, H_T\} + PC = -\hat{\phi} + PC$. Here; PC could be any linear combination of primary constraints. However, since $\{\hat{\phi}, H_T\} = 0$ the simplest solution[9] is $G_0 = -\hat{\phi}$. We thus have

$$G_\zeta = \dot{\zeta}\, G_1 + \zeta\, G_0 = \dot{\zeta}\, p_e + \zeta\, \frac{1}{2} p^2.$$

These generate transformations $\delta_\zeta F = \{F, G_\zeta\}$:

$$\begin{aligned} &\delta_\zeta x^\mu = \zeta p^\mu \qquad \delta_\zeta e = \dot{\zeta} \qquad \delta_\zeta \psi^\mu = 0 = \delta_\zeta \chi \\ &\delta_\zeta p_\mu = \delta_\zeta p_e = \delta_\zeta \pi_\mu = \delta_\zeta \pi_\chi = 0. \end{aligned} \tag{C.84}$$

As for the fermionic symmetry, we start the algorithm by choosing the PFFC $G_1 = \pi_\chi$. Then $G_0 = -\{G_1, H_C\} + PC = \bar{\phi} + PC$. Since $\bar{\phi}$ itself does not obey (C.40c) we are forced to add a PC part, tentatively of the form $G_0 = \bar{\phi} + w^\mu \phi_\mu + \alpha p_e + \beta p_\chi$ with bosonic parameter/fields w^μ, β and fermionic parameter α. In order that this G_0 satifies (C.40c) the parameter are determined and $G_0 = \Omega + 2i\chi p_e$. Therefore

$$G_\theta = \dot{\theta}\, G_1 + \theta\, G_0 = \dot{\theta}\, \pi_\chi + \theta\, (\Omega + 2i\chi p_e),$$

generating

$$\begin{aligned} &\delta_\theta x^\mu = i\theta\psi^\mu \qquad \delta_\theta e = 2i\theta\chi \qquad \delta_\theta \psi^\mu = \theta p^\mu \qquad \delta_\theta \chi = \dot{\theta} \\ &\delta_\theta p_\mu = \delta_\theta p_e = \delta_\theta \pi_\chi = 0 \qquad \delta_\theta \pi_\mu = \frac{i}{2}\theta p_\mu. \end{aligned} \tag{C.85}$$

The Lagrange function (C.82) is indeed quasi-invariant under the transformations (C.85):

$$\delta_\zeta L = \frac{d}{d\tau}\Big(\frac{\zeta}{e}(L + \frac{i}{2}\psi_\mu \dot{\psi}^\mu)\Big) \tag{C.86}$$

and under the transformations (C.85) as

$$\delta_\theta L = \frac{d}{d\tau}\Big(\frac{\theta}{e}(\frac{i}{2}\psi_\mu \dot{x}^\mu)\Big).$$

Noticing that in the absence of fermionic fields, the action (C.82) reduces to the action of a massless relativistic particle, and that this action is reparametrization-invariant under the transformations $\bar{\delta}_\epsilon x^\mu = -\epsilon \dot{x}^\mu$ and $\bar{\delta}_\epsilon e = -(\dot{\epsilon e})$, we are tempted to recover these again. We observe that

$$\delta_\zeta x^\mu = \zeta p^\mu = \frac{\zeta}{e}\dot{x}^\mu - i\zeta\chi\psi^\mu.$$

[9] Indeed, one could add the primary constraint $(\psi\phi + \chi\pi_\chi)$ which at the very end modifies the ensuing infinitesimal transformations by on-shell terms.

In identifying $\epsilon = -\zeta/e$, the first term is recognized as a reparametrization. The second term is a supersymmetry transformation with $\theta = \zeta\chi$. This suggests that we interpret the δ_ζ-transformations as a mixture of a genuine reparametrization $\bar{\delta}_\epsilon$ with a specific supersymmetry transformation $\delta_{\theta(\epsilon)}$:

$$\hat{\delta}_\epsilon := \delta_{(\zeta=-e\epsilon)} + \delta_{\theta=-e\chi\epsilon}. \tag{C.87}$$

This results in

$$\delta_\epsilon x^\mu = -\epsilon\dot{x}^\mu \quad \delta_\epsilon e = -\epsilon\dot{e} - e\dot{\epsilon} \quad \delta_\epsilon\psi^\mu = -\chi\epsilon p^\mu \simeq -\epsilon\dot{\psi}^\mu \quad \delta_\epsilon\chi = -\epsilon\dot{\chi} - \chi\dot{\epsilon}.$$

Here the $\simeq$ symbol has the meaning "being the same on the solutions of the Hamilton equations of motion" (C.83).

In conclusion, I want to stress, that we found the symmetry transformations that leave the Lagrangian (C.82) quasi-invariant by analyzing the Hamiltonian constraints. A proper mixing of the symmetry transformations provided by the canonical generators results in symmetry transformations of the Lagrangian. The mixing of different transformations is typical of any generally covariant theory with additional local symmetries. The mathematical reason behind this is the substitution of Lie derivatives by group-covariant Lie derivatives as explained in Appendix F. This mixing of symmetries will be seen again in the case of coupling a Yang-Mills theory to the gravitational field.

The reparametrizations and local supersymmetry transformations form a group: Denoting $\tilde{\delta}_j\mathrm{F} = \hat{\delta}_{\epsilon_j}\mathrm{F} + \delta_{\theta_j}\mathrm{F}$, one finds for all fields $\mathrm{F} = \{x^\mu, e, \psi^\mu, \chi\}$ that for two transformations $\tilde{\delta}_1$ and $\tilde{\delta}_2$ the commutator $(\tilde{\delta}_1\tilde{\delta}_2 - \tilde{\delta}_2\tilde{\delta}_1)\mathrm{F} = \tilde{\delta}_3\mathrm{F}$ obey

$$\epsilon_3 = (\dot{\epsilon}_1\epsilon_2 - \dot{\epsilon}_2\epsilon_1) + \frac{2i}{e}\theta_1\theta_2 \qquad \chi_3 = (\dot{\theta}_2\epsilon_1 - \dot{\theta}_1\epsilon_2) + \frac{2i\chi}{e}\theta_1\theta_2. \tag{C.88}$$

The Lagrangian (C.82) is more than just a toy example and exhibits many features of supersymmetric theories. For instance, the commutator (C.88) reveals that the product of two supersymmetry transformations makes up for a reparametrization (i.e. a coordinate transformation). It also exhibits the fact that for reparametrization-invariant theories, we cannot avoid having field-dependent structure functions (here depending on e and χ). The example also illustrates that in order to describe "classical spin", it does not suffice to include just one fermionic field ψ^μ into the Lagrangian (C.82) for the relativistic particle. An additional field (χ) is needed in order to obtain a group of symmetry transformations. In the late 1970's, other descriptions for "supergravity in one dimension " were discussed, also for the massive case. The underlying Lagrangians although being related to (C.82), suffered from being quasi-invariant only "on-shell" (i.e. on the solutions of the equations of motion), or from requiring *ad hoc* assumptions in order to assure congruity of the Lagrangian and Hamiltonian description; see e.g. [480].

If aside from the coordinate τ, odd coordinates θ are introduced, one can write down a compact action for "superfields" $X^\mu(z)$ and $E(z)$ which is invariant under reparametrizations in the superspace in terms of coordinates $z = (\tau, \theta)$; see for instance Sect. 3.7 in [226].

The relativistic (super)particle was here only treated classically. A thorough description of its quantization is given in [263]. The most advanced quantization procedure by BRST techniques can be found in [505].

C.3.3 Metric Gravity

In this subsection, only vacuum metric gravity is treated. Some remarks on both tetrad gravity and on Yang-Mills and Dirac fields coupled to the gravitational field are made in the following subsections.

P.A.M. Dirac was the first to find the Hamiltonian for vacuum metric gravity [125]. His tedious derivation obtained a more streamlined form with the (3+1)-decomposition, today known by the acronym ADM for its three authors R. Arnowitt, S. Deser, and C.W. Misner [15]. Also its coordinate-free variant was investigated [283]. For years, this was the established version, until A. Ashtekar discovered a set of fields by which the Hamiltonian structure of general relativity acquires a form similar to that of Yang-Mills gauge theories [16]. These "new" variables are the ones preferred today, also for loop quantum gravity.

In the following I will, contrary to the conventions at other places of this book, use the "mostly plus" metric $\mathrm{diag}\, g = (-1, +1, +1, +1)$ because it is the one made use of in the influential ADM paper.

The $3+1$ Decomposition

Arnowitt, Deser and Misner started with a 3+1 split of the metric:

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

i.e.

$$(g_{\mu\nu}) = \begin{pmatrix} -N^2 + N^i N^j g_{ij} & g_{ij} N^j \\ g_{ij} N^j & g_{ij} \end{pmatrix} \tag{C.89}$$

where the indices $i, j, \ldots$ run from 1 to 3. The objects N and N^i are called the *lapse function* and the *shift functions*, respectively. The geometric interpretation is as follows: Fix a hypersurface $x^0 = \mathrm{const}$ and call it Σ. In the coordinates x^i, the three-dimensional metric h_{ij} induced on Σ coincides with the three-dimensional part of the four-dimensional metric, namely $h_{ij} = g_{ij}$. Let h^{ij} be the inverse to the metric on Σ:

$$h_{ij} h^{jk} = \delta_i^j$$

from which

$$h^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}, \qquad \text{or explicitly} \qquad (g^{\mu\nu}) = \begin{pmatrix} -1/N^2 & N^j/N^2 \\ N^i/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix}.$$

The lapse and the shift functions are thus

$$N := \frac{1}{\sqrt{-g^{00}}} \qquad N^i := -\frac{g^{0i}}{g^{00}}. \tag{C.90}$$

We notice further $\sqrt{-g} = N\sqrt{h},$ where $g := \det(g_{\mu\nu}),\ h := \det(h_{ij})$. The idea is to use the three metric, and the lapse and the shift as "configuration" variables. As we will see, the canonically conjugate momenta to the three-metric are related to the extrinsic curvature of the 3-dimensional hypersurface Σ.

In choosing coordinates ξ^i for the hyper-surface Σ, the metric h_{ij} induced from the spacetime metric $g^{\mu\nu}$ is

$$h_{ij} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} := g_{\mu\nu}\ x^\mu_i\ x^\nu_j.$$

To regain the full geometry we need to add a normal vector n^μ which fulfills

$$g_{\mu\nu}n^\mu\ n^\nu = -1, \qquad g_{\mu\nu}n^\mu x^\nu = 0.$$

to the three tangent vectors x^μ_i. Now it can be shown that

$$(n^\mu) = -\frac{(g^{0\mu})}{\sqrt{(-g^{00})}} = \frac{1}{N}(1, -N^i). \tag{C.91}$$

The embedding of Σ in the original 4-dimensional space is characterized by the extrinsic curvature

$$K_{ij} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - \partial_0 h_{ij}\right), \tag{C.92}$$

where the slash stands for the covariant derivative in the 3-metric. According to the Gauss-Codazzi equations[10] the Riemann curvature tensors of the full and the embedded space are

$$\begin{aligned} {}^{(4)}R_{ijkl} &= {}^{(3)}R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} \\ {}^{(4)}R_{0jkl} &= K_{jl|k} - K_{jk|l}. \end{aligned}$$

This already suffices to express the Hilbert Lagrangian within the 3+1-split. The result is:

$$\mathcal{L} = \frac{1}{2\kappa}\sqrt{-g}\ {}^{(4)}R = \frac{1}{2\kappa}\ Nh^{1/2}({}^{(3)}R\ + K_{ij}K^{ij}\ - K^2) + \text{b.t.}. \tag{C.93}$$

This is the starting point for the Hamiltonian approach by the Rosenfeld-Dirac-Bergmann algorithm[11].

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[10] For details see e.g. [230] or App.E of [479].

[11] The 3+1 decomposition is of genuine importance also in other cases and specifically for numerical relativity [230].

The Hamiltonian for the Hilbert-Einstein Action

The Lagrange density (C.93) is a functional of the six metric components h_{ij} and the lapse and the shift functions (N, N^i) which are contained in the extrinsic curvature (C.92). Let us use as a short-hand notation $(N^\alpha) = (N, N^i)$ (while remaining aware that this is not a 4-vector). Their canonical momenta are

$$P_\alpha := \frac{\partial \mathcal{L}}{\partial(\partial_0 N^\alpha)} = 0. \tag{C.94}$$

The canonical momenta with respect to h_{ij} are

$$p^{ij} := \frac{\partial \mathcal{L}}{\partial(\partial_0 h_{ij})} = -\frac{1}{2\kappa} h^{1/2}\big(K^{ij} - K h^{ij}\big). \tag{C.95}$$

This expression can be solved for the K^{ij}, giving

$$K^{ij} = -2\kappa\, h^{-1/2}\left(p^{ij} - \frac{1}{2} p h^{ij}\right),$$

where $p := p^{ij} h_{ij}$. And, recalling (C.92), the "velocities" $\partial_0 h_{ij}$ can therefore be expressed by the momenta

$$h_{ij,0} = -4\kappa\, N h^{-1/2}\left(p^{ij} - \frac{1}{2} p h^{ij}\right) - (N_{i|j} + N_{j|i}).$$

Thus there are the four primary constraints

$$\phi_\alpha := P_\alpha \approx 0. \tag{C.96}$$

Calculating the canonical Hamiltonian density (and after a further partial integration and disregarding a surface term) one eventually obtains

$$\begin{aligned} \mathcal{H}_C &= N \mathcal{H}_\perp + N^i \mathcal{H}_i \\ &= N\left[(2\kappa)\, h^{-1/2}(p^{ij} p_{ij} - \frac{1}{2} p^2) - \frac{h^{1/2}}{2\kappa}\, {}^{(3)}R\right] + N^i\left[-2 p_i{}^j{}_{|j}\right]. \end{aligned} \tag{C.97}$$

The total Hamiltonian is

$$H_T = H_C + \int d^3x\; v^\alpha P_\alpha$$

with multipliers v^α. The consistency (or stabilization) condition $\{P_\alpha,\ H_T\} \overset{!}{\approx} 0$ based on the canonical Poisson brackets

$$\begin{aligned} \{N^\alpha(x),\ P_\beta(y)\}_{x^o = y^o} &= \delta^\alpha_\beta \delta(\vec{x}, \vec{y}) \\ \{h_{ij}(x), p^{kl}(y)\ \}_{x^o = y^o} &= \frac{1}{2}(\delta^k_i \delta^l_j + \delta^l_i \delta^k_j)\delta(\vec{x}, \vec{y}) \end{aligned}$$

yields the secondary constraints

$$\mathcal{H}_\perp \approx 0 \qquad\qquad \mathcal{H}_i \approx 0, \tag{C.98}$$

known as the *Hamiltonian constraint* $\mathcal{H}_\perp$ and the *momentum constraints* $\mathcal{H}_i$. There are no further constraints since the Poisson brackets of these constraints vanish on the constraint surface due to the Poisson brackets

$$\begin{aligned}
\{\mathcal{H}_\perp(x),\ \mathcal{H}_\perp(y)\} &= -\sigma[h^{ij}\mathcal{H}_j(x) + h^{ij}\mathcal{H}_j(y)]\partial_i\delta(x,y) \\
\{\mathcal{H}_i(x),\ \mathcal{H}_\perp(y)\} &= \mathcal{H}_\perp\partial_i\delta(x,y) \\
\{\mathcal{H}_i(x),\ \mathcal{H}_j(y)\} &= [\mathcal{H}_j(x)\partial_i + \mathcal{H}_i(y)\partial_j]\delta(x,y).
\end{aligned} \tag{C.99}$$

The factor σ in the first Poisson bracket is introduced to show the dependence of the constraint algebra on the metric signature $\mathrm{diag}(\sigma,+1,+1,+1)$. Certainly, in GR we are dealing with the Lorentzian case $\sigma = -1$, but we will see later in the Ashtekar version of canonical GR how one can "play" with this signature together with another parameter[12].

The Hamiltonian is–up to boundary terms–a linear combination of constraints, and this can be shown to be typical for all reparametrization-invariant theories. We saw it already for the relativistic particle, other examples are the relativistic string and–amazingly–any theory derivable from an action if the coordinates are treated as configuration space variables. In all these cases the Hamiltonian can be written in the form

$$H = \int d^{D-1}x\ (N^\alpha\mathcal{H}_\alpha + v^\alpha P_\alpha) + b.t.,$$

where $\mathcal{H}_\alpha = (\mathcal{H}_\perp, \mathcal{H}_i)\ \ i = 1,\ldots,D$-1. It is remarkable, that although the explicit form of the $\mathcal{H}_\alpha$ is different in the diverse examples, they always obey the algebra (C.99). This universal structure of the Hamiltonian and the constraint algebra reflect the criteria to embed a spatial surface within the spacetime manifold; that is, they are valid if the theory is consistent with the foliation of spacetime into arbitrary spacelike hypersurfaces as shown by S. A. Hojman, K. Kuchař, and C. Teitelboim [271], see also [485]. Be aware that the structure of the Hamiltonian and the constraint algebra holds true for any diffeomorphism-invariant theory of gravity formulated in a Riemann space, that is specifically for many modifications and extensions of GR described in Section 7.6. The structural changes that arise in Riemann-Cartan spaces are handled below in Subsect. C.3.4. on tetrad gravity.

It is definitely crucial that (C.99) is not the algebra of the diffeomorphism group (see App. A.2.6). Only the spatial constraints form the algebra (A.20). The Poisson brackets among the constraints $\mathcal{H}_\perp$ in (C.99) reveal that the structure coefficient depends on the $(D$-1$)$-metric. This is one of the reasons why the canonical quantization

[12] The algebra (C.99) is defined for other values of σ as well. It considerably simplifies for $\sigma = 0$; this corresponds to the limit in which the speed of light is zero, characteristic for the Carroll kinematical group; see Subsect. 3.4.5.

program of gravity ran aground. Only for specific cases in which gravity is coupled to (macroscopic) matter it is known that by forming new constraint functions from the $\mathcal{H}_\alpha$ it is possible to get rid of the metric-dependence in the algebra. For instance in the original dust model of Brown and Kuchař [59] the weight-two scalar combination $\mathcal{G} = \mathcal{H}_\perp^2 - h^{ij}\mathcal{H}_i\mathcal{H}_j$ has strongly vanishing Poisson brackets with itself. The full algebra corresponds to the semidirect product of the Abelian algebra generated by $\mathcal{G}$ and the algebra of spatial diffeomorphisms generated by $\mathcal{H}_i$. The reason behind introducing matter/dust is that it serves as a material reference system which breaks the diffeomorphism invariance and de-parametrizes the theory.

For later purposes the constraint algebra (C.99) is compactly denoted as

$$\{\mathcal{H}_\alpha(x), \mathcal{H}_\beta(y)\}_{x^0=y^0} = \int d^{D-1}z\, C^\gamma_{\alpha\beta}(x,y,z)\mathcal{H}_\gamma$$

or, for short

$$\{\mathcal{H}_\alpha, \mathcal{H}_{\beta'}\} = C^{\gamma''}_{\alpha\beta'}\mathcal{H}_{\gamma''}. \tag{C.100}$$

Gauge Generators and Observables

Already at the example of the relativistic particle, it was mentioned that the symmetry transformations in the cotangent space are a restricted subset of the symmetry transformations in the tangent space. As pointed out by P. Bergmann and A. Komar [42] the Hilbert-Einstein Lagrangian is quasi-invariant not only with respect to diffeomorphism (locally: $\hat{x}^\mu = \hat{x}^\mu(x)$) but to the even larger group **Q** with transformations of the form $\hat{x}^\mu = \hat{x}^\mu[x, g_{\rho\sigma}]$. In addition, they showed that in phase space, these generically field-dependent functionals must be restricted. In its infinitesimal form $\hat{x}^\mu = x^\mu + \epsilon[x, g_{\rho\sigma}]$, this restriction can be expressed as a condition of the functional dependence of ϵ on the fields as

$$\epsilon^\mu[x, g_{\rho\sigma}] = n^\mu\xi^0 + \delta^\mu_i\xi^i \qquad \text{where} \qquad \xi^\alpha = \xi^\alpha[g_{ij}, K^{ij}]. \tag{C.101}$$

And it has been shown in [451] that only these restricted symmetry transformations can be realized as canonical transformations in phase space. Namely, for this realizability the commutator $[\delta_1, \delta_2]x^\mu$ of two symmetry transformations $\delta_i x^\mu = -\epsilon^\mu_i(x, g(x))$ must not depend on time derivatives of the ϵ_i. As shown in [413] the deeper reason is again the projectability issue[13]. The null vectors of the Hessian for the Hilbert-Einstein action (expressed in the 3+1 decomposition) are spanned by

$$\Gamma_\lambda = \frac{\partial}{\partial \dot{N}^\lambda}.$$

The condition $\Gamma_\lambda(\delta_\epsilon g_{\mu\nu}) = 0$ leads to a set of differential equations for $\epsilon[x, g_{\rho\sigma}]$ having (C.101) as solution. Thus the transformations must depend explicitly on the

[13] In another guise, this was disclosed also in [329].

lapse and shift function via the normal vector (C.91). Further the descriptors ξ^α are allowed to depend only on the three-geometry or the $(D\text{-}1)$ geometry, respectively. We saw this kind of transformation already with the relativistic particle: There the "normal" vector has only one component, namely $n^0 = 1/N$ and (C.101) simplifies to $\epsilon(x, N) = (1/N)\xi(x)$, where N corresponds to the "einbein" e. The transformations (C.101) form the Bergmann-Komar group $\mathbf{BK}$. This is a subgroup of $\mathbf{Q}$ as is $\mathbf{Diff}(\mathbf{M})$. However, $\mathbf{Diff}(\mathbf{M})$ is not a subgroup of $\mathbf{BK}$! I point out here that the "gauge choice" $(N = 1, N^i = 0)$, very often seen in the literature, blurs the distinction between these symmetry groups.

The symmetry generator for the Bergmann-Komar transformations can be derived by the algorithm (C.40) and is, according to [79] and [413], given by

$$G_\xi = -\big(\mathcal{H}_\alpha + N^{\gamma''} C^{\beta'}_{\alpha\gamma''} P_{\beta'}\big)\xi^\alpha - P_\alpha \dot{\xi^\alpha}, \tag{C.102}$$

where

$$\dot{\xi^\alpha} = \frac{d}{dt}\xi^\alpha = \frac{\partial}{\partial t}\xi^\alpha + N^{\beta'}\{\xi^\alpha, \mathcal{H}_{\beta'}\}.$$

Observe that the lapse and the shift functions are integral building blocks in this generator–appearing in a specific combination of the first-class constraints. Observables $\mathcal{O}$ are those objects which have weakly vanishing Poisson brackets with the symmetry generators: $\{\mathcal{O}, G_\xi\} \approx 0$. The problem of determining observables in general relativity is as old as GR itself. Only for some spacetimes with special asymptotic behavior or additional Killing symmetries have observables been constructed explicitly. Nevertheless, one can establish a formal expression to derive and prove their properties, like their equations of motion, the relation of their Poisson brackets to Dirac brackets, etc. This was done in [416] with the aid of the symmetry generator (C.102) together with "gauge" conditions[14]. Solutions of the field equations which are related by a symmetry operation can only be discriminated by coordinate conditions in the tangent space or by "gauge" conditions χ^α in the cotangent space. These are (locally) unique and complete if they explicitly depend on coordinates, e.g.

$$\chi^\alpha = x^\alpha - X^\alpha, \tag{C.103}$$

where each of the X^α is a spacetime scalar and one must require $\det(\{X^\alpha, \mathcal{H}_\beta\}) \neq 0$.

A formal expression for the observable adjoined to a phase space quantity Φ is obtained through a transformation to coordinates intrinsically defined by (C.103):

$$\mathcal{O}_\Phi = \exp(\{-, G_{\bar{\xi}}\})\Phi|_{\bar{\xi}=\chi}. \tag{C.104}$$

Next, introduce the nonsingular matrix $\mathcal{A}^\alpha{}_\beta := \{X^\alpha, \mathcal{H}_\beta\}$ with its inverse $\mathcal{B}^\alpha{}_\beta := (\mathcal{A}^{-1})^\alpha{}_\beta$. (It might technically be hard or even impossible to find the inverse, but

[14] I use quotation marks here because the Bergmann-Komar group is not the gauge group of GR in the sense of a Yang-Mills gauge theory.

remember that we are seeking a formal definition.) Notice that the requirement $\dot{\chi} \approx 0 = \delta_0^\alpha - \mathcal{A}^\alpha{}_\beta N^\beta$ determines the lapse and shift: $N^\alpha = \mathcal{B}^\alpha{}_0$. Now define

$$\bar{\mathcal{H}}_\alpha := \mathcal{B}^\beta{}_\alpha \left(\mathcal{H}_\beta - \mathcal{B}^\gamma{}_\delta N^\epsilon \{\mathcal{A}^\delta{}_\epsilon, \mathcal{H}_\beta\} P_\gamma \right),$$

$$\bar{P}_\alpha := \mathcal{B}^\beta{}_\alpha P_\beta, \qquad G_{\bar{\xi}} = \bar{\mathcal{H}}_\alpha \bar{\xi}^\alpha + \bar{P}_\alpha \dot{\bar{\xi}}^\alpha.$$

Then for any phase-space function Φ, the observable in the gauge χ is

$$\mathcal{O}_\Phi[\chi] \approx \exp(\{-, G_{\bar{\xi}}\})\Phi_{|\bar{\xi}\to\chi} = \Phi + \{\Phi, G_{\bar{\xi}}\}_{|\bar{\xi}\to\chi} + \frac{1}{2!}\{\{\Phi, G_{\bar{\xi}}\}, G_{\bar{\xi}}\}_{|\bar{\xi}\to\chi} + \ldots.$$

For phase-space functions not depending on the lapse and the shift, this expression can be written as

$$\mathcal{O}_\Phi[\chi] = \Phi + \chi^\alpha\{\Phi, \overline{\mathcal{H}}_\alpha\} + \chi^\alpha\chi^\beta \frac{1}{2!}\{\{\Phi, \overline{\mathcal{H}}_\alpha\}, \overline{\mathcal{H}}_\beta\} + \ldots.$$

or in a compact notation

$$\mathcal{O}_\Phi[\chi] =: \sum_{k=0}^{\infty} \frac{1}{k!}\chi^{(k)}\{\Phi, \bar{\mathcal{H}}\}_{(k)}.$$

It was shown in [416] that the time rate of change of an observable is

$$\frac{d}{dt}\mathcal{O}_\Phi \approx \mathcal{O}_{\{\Phi, \bar{\mathcal{H}}_0\}}.$$

Thus, contrary to claims often made in the community, observables can be time dependent. Furthermore, there is a Taylor expansion

$$\mathcal{O}_\Phi \approx \sum_{k=0}^{\infty} \frac{t^k}{k!} C_k \qquad \text{with} \qquad C_k[\Phi, \chi] = \{\Phi, \bar{\mathcal{H}}_0\}_{(k)}. \tag{C.105}$$

The constants C_k depend, as indicated, on the gauge choice, which in turn enters in the definition of $\bar{\mathcal{H}}$. These constants may be either real constant numbers or constants of motion. Quite interestingly, the observables contain generators of rigid Noether symmetries: if C_k is a nontrivial constant of motion, the Noether charge is

$$Q_k := C_k + U^\alpha_{k\beta} N^\beta P_\alpha,$$

where the U-coefficients are indirectly defined from $\{C_k, \mathcal{H}_\beta\} =: U^\alpha_{k\beta}\mathcal{H}_\alpha$. For every Φ the infinitesimal symmetry transformations are thus given by $\delta_k\Phi = \epsilon\{\Phi, Q_k\}$. In general, the explicit form of the Noether charges (and their related symmetries) depends on the choice of the gauge function χ.
The mapping $\Phi \to \mathcal{O}_\Phi$ is not a canonical transformation $(\{\mathcal{O}_\Phi, \mathcal{O}_{\Phi'}\} \neq \mathcal{O}_{\{\Phi,\Phi'\}})$. However,

$$\{\mathcal{O}_\Phi, \mathcal{O}_{\Phi'}\} \approx \mathcal{O}_{\{\Phi,\Phi'\}^*_\Delta}$$

where the Dirac bracket is defined with respect to the matrix $\Delta_{AB} = \{\varphi_A, \varphi_B\}$, built from the Poisson brackets of the set of constraints $\varphi_A = \{\mathcal{H}_\alpha, \mathcal{P}_\alpha, \chi^\alpha, \dot{\chi}^\alpha\}$.

Example: Observables for Finite-Dimensional Systems

Although the previous considerations referred to gravity as a field theory, they can be made explicit for finite-dimensional systems. As specific cases we will consider the free relativistic particle in Minkowski space and an example from cosmology. These can be derived from an action

$$S[q^\alpha, N] = \int dt\ N\left[\frac{1}{2N^2}G_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - U(q)\right]. \tag{C.106}$$

with a non-singular matrix $G_{\alpha\beta}$. The canonical momenta are $p_\alpha = N^{-1}G_{\alpha\beta}\dot{q}^\beta$, $P \approx 0$. By this, the total Hamiltonian is $H_T = N\mathcal{H}(q,p) + \lambda P$, with

$$\mathcal{H} = \left[\frac{1}{2}\bar{G}^{\alpha\beta}p_\alpha p_\beta + U(q)\right],$$

where $\bar{G}^{\alpha\beta}$ is the inverse to $G_{\alpha\beta}$.

Several of the previous expressions simplify in one dimension. The gauge condition (C.103) will be written as $\chi = t - T(q, N, p, P)$ and the $\mathcal{A}$ matrix simplifies to $\mathcal{A}(q,p) := \{T, \mathcal{H}\}$. Furthermore,

$$\begin{aligned}
&G_\xi = P\dot{\xi} + \mathcal{H}\xi \qquad \overline{\mathcal{H}} = \mathcal{A}^{-1}\big(\mathcal{H} - \mathcal{A}^{-1}N\{\mathcal{A},\mathcal{H}\}P\big)\\
&\mathcal{O}_\Phi[\chi] \approx \sum_{k=0}^{\infty}\frac{1}{k!}t^k C_k[\Phi,\chi] \qquad \text{with} \qquad C_k[\Phi,\chi] = \{\Phi, \bar{\mathcal{H}}\}_{(k)}\\
&Q_k = C_k + U_k NP \qquad U_k\mathcal{H} = \{C_k, \mathcal{H}\}.
\end{aligned}$$

(A) Relativistic particle in Minkowski space

Comparison with (C.78) yields $G_{\alpha\beta} \Leftrightarrow \eta_{\mu\nu}$, $U(q) \Leftrightarrow \frac{1}{2}m^2$, leading to $\mathcal{H} = \frac{1}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2)$. Since the Poisson brackets of p_μ and P with the G_ξ vanish, these momenta are recognized as observables. The rescaling function $\mathcal{A}$ is

$$\mathcal{A} = \{T, \mathcal{H}\} = \frac{\partial T}{\partial q^\mu}\{q^\mu, \mathcal{H}\} = \frac{\partial T}{\partial q^\mu}p_\mu.$$

For a gauge choice, let $T(q)$ be a linear combination of the fields q^μ: $T(q) = \alpha_\mu q^\mu = (\alpha q)$, where the α_μ do not depend on the q^μ. For this choice $\mathcal{A} = (\alpha p)$. Since $\{\mathcal{A}, \mathcal{H}\} = 0$, furthermore $\overline{\mathcal{H}} = (\alpha p)^{-1}\mathcal{H}$ and $G_{\bar{\xi}} = (\alpha p)^{-1}\dot{\bar{\xi}}P + \bar{\xi}\overline{\mathcal{H}}$. For the nested Poisson brackets $\{q^\mu, \overline{\mathcal{H}}\}_{(k)}$, we calculate

$$\begin{aligned}
&\{q^\mu, \overline{\mathcal{H}}\}_{(1)} = \{q^\mu, \overline{\mathcal{H}}\} \approx (\alpha p)^{-1}\{q^\mu, \mathcal{H}\} = (\alpha p)^{-1}p^\mu\\
&\{q^\mu, \overline{\mathcal{H}}\}_{(j)} = 0 \qquad \text{for} \qquad j \geq 2.
\end{aligned}$$

Therefore the observables associated to q^μ are

$$\mathcal{O}_{q^\mu}[\chi] \approx q^\mu + \chi\frac{p^\mu}{(\alpha p)} = q_\mu - \frac{p^\mu}{(\alpha p)}(\alpha q) + \frac{p^\mu}{(\alpha p)}\,t.$$

Notice especially that $\mathcal{O}_{\alpha q} = t$, which is consistent with the gauge choice $t = (\alpha q)$. From the expression $\mathcal{O}_{q^\mu}$ we read off the conserved quantities:

$$C_0[q^\mu] = q^\mu - (\alpha p)^{-1} p^\mu (\alpha q) \qquad\qquad C_1[q^\mu] = (\alpha p)^{-1} p^\mu .$$

Since we already know that p^μ is conserved, the object C_1 offers no new information. However, multiplying C_0 with the constant (αp) we get the constants of motion

$$C^\mu := (\alpha p) C_0[q^\mu] = (\alpha_\nu p^\nu) q^\mu - (\alpha_\nu q^\nu) p^\mu = \alpha_\nu (p^\nu q^\mu - q^\nu p^\mu) = \alpha_\nu M^{\nu\mu},$$

thus the C^μ are just linear combinations of the angular momenta $M^{\mu\nu}$. Indeed the $C_i[q^\mu]$ can directly be identified with Noether charges since $\{C_i[q^\mu], \mathcal{H}\} = 0$. If now for instance we choose $\alpha = (1, 0, 0, 0)$, we have aside from the space-time translations $T_\mu := p_\mu$ also the boosts $K^j := q^0\, p^j - q^j p^0$, and their algebra delivers the generator of space rotations J^k.
Finally, for completeness, let us determine the observable associated to N. These are derived with the aid of $\{N, G_{\bar\xi}\} = (\alpha p)^{-1} \dot{\bar\xi}$ and $\{\{N, G_{\bar\xi}\}, G_{\bar\xi}\} = 0$ as

$$\mathcal{O}_N = N + \{N, G_{\bar\xi}\}_{\bar\xi = \chi} = N + (\alpha p)^{-1} \dot\chi = N + (\alpha p)^{-1} (1 - (\alpha \dot q)) = (\alpha p)^{-1}$$

which again is consistent with the gauge choice: $N = \mathcal{A}^{-1}$.

Instead of the Minkowski metric any other background metric may be used. The example of a relativistic particle in an anti-de Sitter background is treated in [416]. Let me remark that in any case, those constants of motion that are linear in the momenta are directly related to the Killing symmetries of the background: Let $K^\mu(q) p_\mu$ be a constant of motion. Then

$$\{K^\mu p_\mu, \mathcal{H}\} = \{K^\mu p_\mu, \frac{1}{2}(g^{\rho\sigma} p_\rho p_\sigma + m^2)\} = -(\pounds_K g^{\rho\sigma}) p_\rho p_\sigma$$

where $\pounds_K$ is the Lie derivative with respect to the vector field $K = K^\mu \partial_\mu$. Therefore $\{K^\mu p_\mu, \mathcal{H}\} = 0$ is equivalent to $\pounds_K g^{\rho\sigma} = 0$.

(B) Minisuperspace cosmology
One speaks of minisuperspace[15] gravity if the symmetries imposed upon spacetime makes the system finite dimensional. In cosmology for instance, assuming spatial homogeneity ("looking the same at each location"), the most general line element can be written[16] as $ds^2 = -N^2(t) dt^2 + h_{ij}(t) dx^i dx^j$. An important class are the Bianchi-type cosmologies. These are classified according to their isotropy groups. If the structure constants of the associated Lie algebras fulfill $f_{ab}^{\ \ b} = 0$, one is allowed to perform a "symmetry reduction", that is to insert the metric into the action. And it is known that for a large subset of Bianchi-type cosmologies the action can always be brought into the generic form (C.106).

[15] The name was coined by Ch. Misner; despite the "super" it has nothing at all to do with supersymmetry.

[16] In a mathematically correct way, spatial homogeneity–as well as isotropy–is to be defined via Killing symmetries; see [514], [444].

Astrophysical observations are in good agreement with a cosmological model which is not only spatially homogeneous but also spatially isotropic ("looking the same in each direction"). This is based on the Robertson-Walker metric for which

$$ds^2 = -N^2dt^2 + a^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2))$$

with the scale factor $a(t)$ describing the expansion of our universe. Inserting this for example into the action describing gravity coupled to a scalar field φ, one obtains

$$S = S_g + S_m = \frac{3}{\kappa}\int dt\ N\big(-\frac{a\dot{a}^2}{N^2} + ka - \frac{\Lambda a^3}{3}\big) + \frac{1}{2}\int dt\ Na^3\big(\frac{\dot{\varphi}^2}{N^2} - m^2\varphi^2\big).$$

Here $k = (0, \pm 1)$ characterizes the topology of the three-space, Λ is the cosmological constant, and m the mass of the scalar field; for a derivation see [309]. By denoting $q^0 = a$, $q^1 = \varphi$ you may realize that this has the form of (C.106). In particular, for flat universe without a cosmological constant and a massless scalar, the Lagrangian for this model becomes

$$L = \frac{1}{2N}\Big(-\frac{12a}{\kappa}\,\dot{a}^2 + a^3\dot{\varphi}^2\Big) = \frac{1}{2N}e^{-3\Omega}\Big(-\frac{12a}{\kappa}\,\dot{\Omega}^2 + \dot{\varphi}^2\Big) \quad \text{with} \quad a = e^{-\Omega}. \tag{C.107}$$

(We could even get rid of the numerical factor before $\dot{\Omega}^2$ by rescaling Ω.) To see the generality of this approach, let us consider the so-called anisotropic Bianchi type-I cosmology, defined by the metric

$$ds^2 = -N^2dt^2 + e^{-2\Omega}(e^{2\beta_x}dx^2 + e^{2\beta_y}dy^2 + e^{2\beta_z}dz^2)$$

with the restriction $\beta_x + \beta_y + \beta_z = 0$. By steps similar to those taken in the previous case, one derives the Lagrangian

$$L = \frac{1}{2N}\omega^{-1}(\Omega)\left(-\dot{\Omega}^2 + \dot{\beta}_+^2 + \dot{\beta}_-^2 + \dot{\varphi}^2\right),$$

with $\omega = e^{3\Omega}$ and $\beta_+ = \frac{1}{2}(\beta_x + \beta_y)$, $\beta_- = \frac{1}{2\sqrt{3}}(\beta_x - \beta_y)$. The Hamiltonian is determined to be $H = N\mathcal{H} + \lambda P$ with

$$\mathcal{H} = \frac{1}{2}\omega(\Omega)(-p_\Omega^2 + p_+^2 + p_-^2 + p_\varphi^2).$$

Similar to the relativistic particle, all momenta are observables, independent of a gauge choice. Next, let us find the observables associated to $\Omega, \beta_\pm, \varphi$. Choosing for instance the gauge $\chi = t - \Omega = 0$, gives $\mathcal{A} = -\omega p_\Omega$ and $\overline{\mathcal{H}} = \mathcal{A}^{-1}\mathcal{H} - 3NP$. From this, a straightforward calculation yields observables

$$\mathcal{O}_\Omega = t \qquad \mathcal{O}_\Phi = \Big(\Phi + \Omega\,\frac{p_\Phi}{p_\Omega}\Big) - \frac{p_\Phi}{p_\Omega}\,t \qquad \mathcal{O}_N = -\frac{1}{p_\Omega}e^{-3t}$$

where Φ stands for the set $\Phi = \{\beta_+, \beta_-, \varphi\}$. Notice that the expression for $\mathcal{O}_\Omega$ is compatible with the gauge condition, and the one for $\mathcal{O}_N$ is consistent with

$N = \mathcal{A}^{-1}$ (in the chosen gauge). The two coefficients in $\mathcal{O}_\Phi$ are used to calculate the Noether charges. Introducing a notation by which the three functions in Φ are called Φ_K and their momenta p_K the Noether charges are

$$P_\Omega := p_\Omega - 3NP \qquad P_K := p_K \qquad Q_K := (p_\Omega \Phi_K + p_K \Omega) - 3\Phi_K NP.$$

The algebra closes with further charges $R_{KL} = \{Q_K, Q_L\} = p_K \Phi_L - p_L \Phi_K$ and the full algebra becomes

$$\begin{aligned}
\{P_i, P_j\} &= 0 & P_i &\in \{P_\Omega, P_K\} \\
\{P_\Omega, Q_K\} &= -P_K & \{P_\Omega, R_{KL}\} &= 0 \\
\{P_K, Q_L\} &= -\delta_{KL} P_\Omega & \{P_J, R_{KL}\} &= \delta_{JK} P_L - \delta_{JL} P_K \\
\{Q_K, Q_L\} &= R_{KL} & \{Q_J, R_{KL}\} &= \delta_{JK} Q_L - \delta_{JL} Q_K \\
\{R_{IJ}, R_{KL}\} &= \delta_{IL} R_{JK} - \delta_{JL} R_{IK} + \delta_{JK} R_{IL} - \delta_{IK} R_{JL}.
\end{aligned}$$

It can be verified that this is isomorphic to the conformal algebra $\mathfrak{c}(2,1)$. And also one may verify that the Lagrangian (C.107) is quasi-invariant under the transformations generated by the Noether charges $C_k \in \{P_\Omega, P_K, Q_K, R_{KL}\}$.

Ashtekar Variables

The previously-described ADM approach to canonical GR prevailed up to the 1980's. In 1986, A. Ashtekar introduced what is now sometimes called "new variables", which make it possible to cast canonical GR in a form in which it more closely resembles canonical Yang-Mills theories. In my presentation I follow [309].

Basic entities for constructing the Ashtekar variables are "triads" h^i_a spanning the three-geometry similar to the tetrads describing the four-geometry. Thus the triads obey

$$h_{ij} h^i_a h^j_b = \delta_{ab}.$$

Furthermore, one can introduce related objects h^a_j "living" in the cotangent space. These fulfill $h^a_j h^i_a = \delta^j_i$ and $h^a_j h^j_b = \delta^a_b$. How to go from the tetrads e^I_μ and the normal n^μ to the x^0=const hypersurface to the triads is elucidated in [262]. Instead of the triads themselves consider their "densitized" variants

$$\tilde{h}^i_a := \sqrt{h} h^i_a \qquad \text{with} \qquad \sqrt{h} = \det(h^b_j).$$

It can be shown that the extrinsic curvature (C.92) expressed as

$$K^a_i := K_{ij} h^{ja}$$

is canonically conjugate to $\tilde{h}^i_a$. Further there is the identity

$$\mathcal{G}_a := \epsilon_{abc} K^a_i \tilde{h}^{ci} \equiv 0, \tag{C.108}$$

which in the Rosenfeld-Dirac-Bergmann algorithm become constraints (called the "Gauß-constraints"). These constraints generate $\mathbf{SO(3)}$-rotations, symmetries originating from the use of triads.

The canonical variables considered since the work of Ashtekar [16] are the

$$H_a^i = \frac{1}{8\pi\beta}\tilde{h}_a^i(x)$$

together with a mixture of K_i^a and $\tilde{h}_a^i(x)$ into a connection A_i^a defined by

$$GA_i^a(x) := \Gamma_i^a(x) + \beta K_i^a. \tag{C.109}$$

Newton's constant G needs to enter so that the objects GA_i^a acquire the dimension of an inverse length–like a Yang-Mills connection. Furthermore, β is an arbitrary non-vanishing complex number (the *Barbero-Immirzi parameter*[17]). In (C.109) Γ_i^a is related to the $3D$ spin connection ω_{iab} as

$$\Gamma_i^a = -\frac{1}{2}\omega_{ibc}\epsilon^{abc}.$$

The configuration variables H_a^i and the momenta A_i^a are canonically conjugate:

$$\{H_b^j(x), A_i^a(y)\} = \delta_i^j\delta_a^b\delta(x,y) \qquad \{H_a^i, H_b^j\} = 0 = \{A_i^a, A_j^b\}.$$

How do the constraints look in the new variables? The Gauß constraints (C.108) are equivalent to

$$\hat{\mathcal{G}}_a = \partial_i H_a^i + G\epsilon_{abc}A_i^b H^{ci} := \mathcal{D}_i H_a^i \approx 0. \tag{C.110}$$

This is formally similar to the Gauß constraint (C.66) of Yang-Mills theory. From the commutator of two covariant derivatives, one calculates the curvature/field-strength associated to the connection/potential A_j^a as

$$F_{ij}^a = 2G\partial_{[i}A_{j]}^a + G^2\epsilon_{abc}A_b^j A_b^c.$$

The metric gravity constraints from (C.97) become proportional to

$$\hat{\mathcal{H}}_\perp = -(\det H_a^i)^{-1/2}\Bigg[\frac{\sigma}{2}\epsilon_{abc}H^{ai}H^{bj}F_{ij}^c \\ + \frac{1-\beta^2\sigma}{\beta^2}H_{[a}^i H_{b]}^j(GA_i^a - \Gamma_i^a)(GA_j^b - \Gamma_j^b)\Bigg] + (GC) \tag{C.111a}$$

$$\hat{\mathcal{H}}_i = F_{ij}^a H_a^j + (GC) \tag{C.111b}$$

where (GC) stands for a linear combination of the Gauß constraints and where $\sigma = -1/+1$ for the Lorentzian/Euclidean case. Thus the second term in (C.111a) drops

[17] In the current literature, it is $\gamma = -1/\beta$ which is called the Barbero-Immirzi parameter.

out for the choice $\beta = \pm i$ in the Lorentzian case and $\beta = \pm 1$ in the Euclidean case. Ashtekar originally used $\beta = i$, with the consequence that the connection (C.109) becomes complex. Thus one needed to introduce reality conditions; technicalities going beyond the scope of this book. Later J.F. Barbero introduced with $\beta = 1$ canonical variables similar to the ones by Ashtekar, but with the advantage of being real, and G. Immirzi pointed out, that any choice for the complex β is allowed. The price to be paid is a loss of simplicity, as can be seen at the Hamiltonian constraint (C.111a). Anyhow, being born into the world, the Barbero-Immirzi parameter β has given rise to many research activities. Among others came the observation that different Hamiltonian forms of GR can be derived from one and the same action by appropriate choices of a parameter [273]. This *Holst action* is an extension of the Hilbert-Palatini action derived from (7.66)

$$S_{\text{Holst}} = \int d^4x\, e\, e_I^\mu e_J^\nu \Big(R^{IJ}{}_{\mu\nu}(\omega) + \frac{\gamma}{2} \epsilon^{IJ}{}_{KL} R^{KL}_{\ \mu\nu}(\omega) \Big), \tag{C.112}$$

(here $2\kappa = 1$) in which, as indicated the tetrads and the spin connections are varied independently. For a constant parameter γ the second term does not influence the field equations since it is a boundary term. In point of fact, the γ term seems to play a similar role as the θ term in non-Abelian gauge theories. The choices $\gamma = \{0, i, 1\}$ lead to the ADM, the Ashtekar, and the Barbero Hamiltonian for GR, respectively. For arbitrary choices of γ, the canonical formulation of the Holst action leads to a family of $SU(2)$ connection formulations of GR Hamiltonians.

Boundary Conditions and Surface Terms

In the sequel, I will deal solely with canonical general relativity in the ADM approach, although the procedures and findings described here apply *mutatis mutandis* to the Ashtekar formulation (see [487]) and to canonical tetrad gravity (see [47] and references therein).

The central finding of ADM is the Hamiltonian (C.97)

$$H_{ADM}\left[N, N^i\right] = \int_\Sigma d^3x \, \left(N\mathcal{H}_\perp + N^i \mathcal{H}_i\right)$$

as a sum of constraints. Bryce DeWitt was the first to realize that this is only true if the three-manifold Σ is compact without a boundary [118]. For the asymptotically flat case, in particular, DeWitt added a surface term at infinity

$$E_{ADM}\left[h_{ij}\right] = \oint d^2 s_k (h_{ik,i} - h_{ii,k}) \tag{C.113}$$

to the ADM-Hamiltonian so that

$$H_{DW} = H_{ADM} + E_{ADM}\left[h_{ij}\right]$$

coincides asymptotically with the known total energy expression from linearized GR. The role of surface terms was further clarified in pioneering work by T. Regge and C. Teitelboim [427]. They showed that the GR vacuum field equations do in fact not inevitably follow from the Hamiltonian H_{ADM}. The variation of this Hamiltonian is generically given by

$$\delta H_{ADM} = \int d^3x \ \left[A^{ij}(x)\delta h_{ij}(x) + B_{ij}(x)\delta p^{ij}(x)\right] + \oint d^2s_k \ C^k .$$

The explicit form of the C^k does not matter here; they are given as Eq. (2.6) in [427]. In any case, only if the surface term vanishes, are the functional derivatives of H_{ADM} defined, and only in this case are we allowed to conclude from Hamilton's variational principle that

$$A^{ij} =: \frac{\partial H_{ADM}}{\partial h_{ij}} = -\dot{p}^{ij} \qquad B_{ij} =: \frac{\partial H_{ADM}}{\partial p^{ij}} = \dot{h}_{ij} .$$

Notice that otherwise, we not only would find the wrong field equations, but we would get no consistent field equations at all! In the case of closed spaces, the surface integral vanishes, and thus this problem does not arise. For open spaces, however, this is not true in general. Regge and Teitelboim cured this insufficiency for asymptotically flat spaces by amending the ADM Hamiltonian through a surface term, that is by starting from a modified Hamiltonian

$$H\left[N, N^i\right] = H_{ADM}\left[N, N^i\right] + \oint_{\mathcal{S}} B(N, N^i) d\mathcal{S}$$

where $B(N, N^i)$ is linear in its arguments and the integral is understood as the limit $r \to \infty$ on spheres of radius r in the asymptotically flat region of Σ. They were not only able to show that there is a choice of B such that the modified Hamiltonian is functionally differentiable, but that by appropriate assumptions about the lapse function the Poincaré charges at spatial infinity can be defined–reproducing among other the ADM energy (C.113). Here are some more details: DeWitt already claimed that for asymptotically flat geometries, the falloff behavior of the three-metric for $r \to \infty$ must be $h_{ij} \sim \delta_{ij} + \mathcal{O}(r^{-1})$, and that the momenta p^{ij} behave as r^{-2}. Regge and Teitelboim argue that lapse and shift must behave as

$$N^\mu \overset{r \to \infty}{\sim} \alpha^\mu + \beta^\mu_i x^i$$

where α^μ describes space-time translations, $\beta_{ij} = -\beta_{ji}$ spatial rotations, and $\beta^\perp_i$ boosts. Given this falloff behavior the variation of the ADM Hamiltonian results as

$$\delta H_{ADM} = \int d^3x \ \left[A^{ij}(x)\delta h_{ij}(x) + B_{ij}(x)\delta p^{ij}(x)\right] + \alpha^\mu \delta \check{P}_\mu - \frac{1}{2}\beta^\mu_i \delta \check{M}^i_\mu$$

with

$$\check{P}^{\perp} = \oint_{r\to\infty} d^2 s_k (h_{ik,i} - h_{ii,k}), \qquad \check{P}^i = -2 \oint_{r\to\infty} d^2 s_k \; p^{ki}$$
$$\check{M}^{ij} = -2 \oint_{r\to\infty} d^2 s_k (x^i p^{kj} - x^j p^{ki}),$$
$$\check{M}_{\perp j} = -2 \oint_{r\to\infty} d^2 s_k \left[x_j (h_{ik,i} - h_{ii,k}) - h_{jk} + h_{ii}\delta_{jk} \right].$$

These can be identified with the total energy, the total momentum, the total angular momentum and the generator of boosts respectively. Specifically $\check{P}^{\perp} \equiv E_{ADM}\,[h_{ij}]$. Now it is obvious how to arrive at a Hamiltonian that may serve to derive the GR Hamilton equations:

$$H_{RT} = H_{ADM} - \alpha^{\mu} \check{P}_{\mu} + \frac{1}{2} \beta^{\mu}_{i} \check{M}^{i}_{\mu}. \tag{C.114}$$

It is really striking that the quest for mathematical consistency–namely the existence of functional derivatives for the action–yields physically desirable results about conserved quantities at infinity!

The analysis of Regge and Teitelboim was improved by R. Beig and N. Ó Murchadha [39]: Amongst other things they derived falloff conditions for the lapse function and the shift vector from internal consistency (compatibility of the boundary conditions and the evolution equations) instead of the previous more or less *ad hoc* assumed behavior, and they corrected the expression for the boost generator $\check{M}_{\perp j}$. Furthermore, “Precise meaning is given to the statement that, as a result of these boundary conditions, the Poincaré group acts as a symmetry group on the phase space of general relativity” in a fairly abstract mathematical manner.

Arnowitt, Deser, and Misner had by laxity (common in those “ancient times”) discarded any total derivative terms in deriving “their” Hamiltonian. Only during the last two decades has the role of boundary terms become recognized as pivotal. ADM even started with the wrong action: The Hilbert action does not lead to the GR vacuum field equations in the case of open geometries. Instead, as explained in Sect. 7.5.2, one has to consider the “Trace-K” action

$$S_K = \frac{1}{2\kappa} \int d^4x \,\sqrt{-g} R + \frac{1}{\kappa} \int_{\partial\Omega} d^3x \;\, \sqrt{\epsilon h} K$$

which is the sum of the Hilbert action and the Gibbons-Hawking-York boundary term (7.72). Curiosity provokes the question as to which Hamiltonian results if one starts with this action and keeps a record of all boundary terms arising in between (essentially by performing partial derivatives). The gratifying answer was given in [252]: The resulting Hamiltonian is the same as the one established within the logic of Regge and Teitelboim in case of asymptotically flat spacetimes. The answer is even more satisfying in that S. Hawking and G. Horowitz could write down the gravitational Hamiltonian for more general boundary conditions.

The extension of the results of Regge and Teitelboim from asymptotic Minkowski to (A)dS spacetimes has been carried out by M. Henneaux and C. Teitelboim [264], and to Einstein-Cartan and teleparallel gravitational theories by M. Blagojević and I. Nicolić; see Chap. 6 in [47].

C.3.4 Tetrad Gravity

Constraints in Riemann-Cartan Gravity Theories

Instead of immediately treating the Hamiltonian formulation of GR coupled to Dirac fields, let me address the generic structure of the Hamiltonian and the algebra of constraints in theories with both curvature and torsion. Specifically for Poincaré gauge theories, the Hamiltonian version was tackled in a series of publications by M. Blagojević, I.A. Nicolić and M. Vasilić; their findings are summarized in [46], [47] and in the chapter "Hamiltonian Structure" of [48].

The gravitational part of the theory is given in terms of the vierbeins e^K_μ and the spin connections $\omega^{IJ}_{\ \ \mu}$. Their derivatives enter the Lagrangian $e\mathcal{L}_G(R^{IJ}_{\ \ \mu\nu}, T^I_{\ \mu\nu})$ only through the curvature[18] $R^{IJ}_{\ \ \mu\nu}$ and the torsion $T^I_{\ \mu\nu}$, and these derivatives appear antisymmetrized. Therefore the momenta canonically conjugate to e^K_0 and $\omega^{IJ}_{\ \ 0}$ vanish. These are primary constraints

$$\phi^0_K = \pi^0_K \approx 0 \qquad \phi_{IJ}^{\ \ 0} = \pi_{IJ}^{\ \ 0} \approx 0$$

which are always present independent of the specific form of the Lagrangian. They are called "sure" constraints by Blagojević *et.al.* to contrast them from "if" constraints which occur for certain parameter combinations in PG theories. The total Hamiltonian is thus given by $\mathcal{H}_T = \mathcal{H}_C + u^K_0\phi^0_K + (1/2)u^{IJ}_{\ \ 0}\pi_{IJ}^{\ \ 0} + (u\cdot\phi)$ where the last term stands symbolically for the if-primaries. Now as it turns out, the canonical Hamiltonian can always be written as

$$\mathcal{H}_C = e^K_0\mathcal{H}_K - \frac{1}{2}\omega^{IJ}_{\ \ 0}\mathcal{H}_{IJ} + \partial_i D^i = N\mathcal{H}_\perp + N^i\mathcal{H}_i - \frac{1}{2}\omega^{IJ}_{\ \ 0}\mathcal{H}_{IJ} + \partial_i D^i, \tag{C.115}$$

with the lapse function and the shift functions now expressed by the vierbeins and the normal n^K, so that $\mathcal{H}_\perp = n^K\mathcal{H}_K$ and $\mathcal{H}_i = e^K_i\mathcal{H}_K$. Any tangent space object, say V_K, is expressible by its parallel and its orthogonal components:

$$V_K = n_K V_\perp + V_{\bar{K}} \longleftrightarrow V_\perp = n_K V^K, \qquad V_{\bar{K}} = (\delta^I_K - n^I n_K)V_K.$$

The Hamiltonian (C.115) has a striking similarity with the ADM expression for metric gravity–aside from the $\mathcal{H}_{IJ}$ -term. One gets immediately the sure secondary constraints

$$\mathcal{H}_\perp \approx 0 \qquad \mathcal{H}_i \approx 0 \qquad \mathcal{H}_{IJ} \approx 0. \tag{C.116}$$

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[18] Here, I refrain from indicating by a tilde that in the present context we are dealing with the Riemann-Cartan curvature.

From the Lagrangian perspective, we know that the theory is invariant with respect to diffeomorphism and local Lorentz rotations. And being aware that a variational symmetry makes itself felt in the Hamiltonian by the existence of first-class constraints, we expect that the secondary constraints are first-class and will be part of a phase-space symmetry generator. This indeed turns out to be true.

It is possible to derive the explicit (phase-space) expressions of the constraints (C.116) for any gravitational Lagrangian. I renounce to write it down here, you find this in the literature cited above. A peculiarity was found for certain PGT's, in that the number and type of constraints may be different in different regions of phase space. In [88] the origin of this degenerate behavior was analyzed, with the prospect to get a viable PGT with fewer parameters.

Matter Constraints

In the literature cited before it is shown that the Hamiltonian for minimally coupled matter has the same structure as the gravitational part (C.115). With the conventions to rewrite the matter part of the Lagrangian

$$e\mathcal{L}_M(Q, \nabla_\mu Q) = e\overline{\mathcal{L}}_M(Q, \nabla_{\bar{K}} Q, \nabla_\perp Q; n^K)$$

and using the factorization of the determinant as $e = NJ$, where J does not depend on e_0^K, the matter field momenta become

$$P = \frac{\partial(e\mathcal{L}_M)}{\partial(\partial_0 Q)} = J\frac{\partial\overline{\mathcal{L}}_M}{\partial(\nabla_\perp Q)}.$$

Now the matter part of the canonical Hamiltonian is found to be similar to (C.115) with

$${}^M\mathcal{H}_\perp = P\nabla_\perp Q - J\overline{\mathcal{L}}_M, \qquad {}^M\mathcal{H}_i = P\nabla_i Q, \qquad {}^M\mathcal{H}_{IJ} = P\Sigma_{IJ}Q$$

where Σ_{IJ} is the spin matrix which enters because $\nabla_0 Q = N\nabla_\perp Q + N^i\nabla_i Q = \partial_0 Q + \frac{1}{2}\omega^{IJ}{}_0\Sigma_{IJ}Q$. Notice that only the constraint ${}^M\mathcal{H}_\perp$ depends on the Lagrangian, while the others do not. And be aware that the matter Lagrangian also may give rise to further primary constraints.

Einstein-Cartan-Dirac Interaction

Now we could turn to the example of coupling fermions to GR. However, in the context of the previous results, GR is not a natural theory, because its action corresponds to the Einstein-Cartan action plus a multiplier term taking account of the vanishing torsion. Therefore, "for the sake of simplicity" let us investigate the interaction of a Dirac spinor with Einstein-Cartan gravity.

The gravitational part is

$$S_{EC} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R = -\frac{1}{8\kappa}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^K_\rho e^L_\sigma R^{IJ}_{\mu\nu}$$
$$= -\frac{1}{4\kappa}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}_{IJKL}\, e^K_\rho e^L_\sigma(\partial_\mu\omega^{IJ}{}_\nu + \omega^I{}_{K\mu}\omega^{KJ}{}_\mu).$$

with the abbreviation $\epsilon^{\mu\nu\rho\sigma}_{IJKL} := \epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}$. The momenta are

$$\pi^\mu_K = \frac{\partial L_{EC}}{\partial(\partial_0 e^K_\mu)} = 0 \qquad\qquad \pi_{IJ}{}^\mu = \frac{\partial L_{EC}}{\partial(\partial_0\omega^{IJ}_\mu)} = \frac{1}{2\kappa}\epsilon^{0\mu\rho\sigma}_{IJKL}\, e^K_\rho e^L_\sigma.$$

Thus not only the π^0_K and $\pi_{IJ}{}^0$ do vanish (as derived before independently of any Lagrangian), but there are further primary constraints

$$\phi^i_K = \pi^i_K \approx 0 \qquad\qquad \phi_{IJ}{}^k = \pi_{IJ}{}^k - \frac{1}{2\kappa}\epsilon^{0k\rho\sigma}_{IJKL}\, e^K_\rho e^L_\sigma.$$

For the free EC theory, all together there are 40 primary constraints. This is already severe but is not yet the end, since we need to guarantee that the constraints are stable in time. This gives rise to 16 secondary constraints. It turns out that 10 of the primary constraints and 10 of the secondary constraints are first class. With the number $N_{FC} = 20$ and $N_{SC} = 36$ and bearing in mind that one gauge condition corresponds to each first-class constraint, one finds for the $N = 40$ field components (in configuration space) the number of degrees of freedom $2N-(2N_{FC}+N_{SC}) = 4$ in phase space. (For details see [46], [47].)

As for the matter part we have $eL_M = e(i\bar\psi\gamma^\mu\nabla_\mu\psi - m\bar\psi\psi)$. The momenta π and $\bar\pi$, canonically conjugate to ψ and $\bar\psi$, respectively are part of the primary constraints $\phi = \pi + (i/2)J\bar\psi\gamma^\perp \approx 0$ and $\bar\phi = \bar\pi \approx 0$. We met these constraints in (C.56) where (in the coupled Maxwell-Dirac case) they turned out to be second-class.

In using now the previous results for the gravitational part (with the specific primary constraint structure of the EC theory) and the results for the matter part (with the primary constraints resulting from the definition of the momenta canonically conjugate to the Dirac fields) you are able to write down the explicit form of the canonical Hamiltonian. Next you need to start the RDB algorithm. In a first step, you need to investigate the stability of the 48 primary constraints. You will find that–incorporating the 8 extra constraints–the previous count remains valid, but that due to the presence of fermions the explicit form of the first- and second-class constraints is modified.

You see that the Hamiltonian analysis of the Einstein-Cartan-Dirac theory requires some ingenuity. This was already realized when people formulated this program for the tetrad version of GR. Here you might think of a first-order approach (related to the previous procedure) or a second-order approach where you express the spin connection by the tetrads and their derivatives. Both approaches, surprisingly, did not immediately lead to the same results. The comparison became even more problematic, if in the course of the calculations one took over findings from metric gravity. A résumé is given in [85]. One now understands that different procedures are interrelated by canonical transformations, and these are mediated by boundary terms in the action. But I am still not sure that all the diverse approaches for tetrad GR in their mutual relationships are fully understood.

C.3.5 Einstein-Dirac-Yang-Mills-Higgs Theory

According to present-day knowledge fundamental physics is characterized by a Lagrangian with gravitational, gauge boson, fermionic and Higgs fields. Thus this appendix may be expected to close with the Hamiltonian for today's "World Theory". Because of the presence of fermionic fields, the Hamiltonian has to be formulated with tetrads, and you have already realized in the previous subsection that things will become quite intricate. Therefore in this subsection I we will be far less ambitious than the title announces and only show a glimpse at what happens in the case of scalar and vector fields coupled to gravity.

Constraint Structure

As already mentioned, for any generally covariant theory, the algebra of constraints (C.99) is the same, although the explicit form of the constraints themselves will differ. This led C. Teitelboim to reflect about the "Hamiltonian Structure of Space-Time" [486].

Let us consider the case of a theory of integer-spin fields coupled to gravity. Denote by $\{q_\alpha, p^\alpha\}$ the set of all canonical phase-space variables. It is quite obvious that because the Lagrangian is a sum of a gravitational and a matter part, the ADM-type action has the generic form

$$S = \int d^4x \left[\dot{q}_\alpha p^\alpha - N\mathcal{H}_\perp - N^i\mathcal{H}_i - \lambda^I\mathcal{C}_I\right]$$

where the last term is a linear combination of constraints possibly arising from the local symmetry of the matter theory. If split into the gravitational and the matter part, this becomes

$$S = \int d^4x \left[\dot{h}_{ij}p^{ij} + \dot{Q}_a P^a - N({}^G\mathcal{H}_\perp +^M \mathcal{H}_\perp) - N^i({}^G\mathcal{H}_i +^M \mathcal{H}_i) - \lambda^I\mathcal{C}_I\right].$$

But it is less obvious that the constraints $\mathcal{H}_\perp$ and $\mathcal{H}_i$ acquire the generic form

$$\begin{aligned} \mathcal{H}_\perp(x) &= G^{\alpha\beta}(q)p_\alpha p_\beta + \sqrt{-h}\,U(q,\partial_i q) \\ \mathcal{H}_i(x) &= \mathcal{D}_{i\alpha}\,p^\alpha \end{aligned} \tag{C.117}$$

for both the gravitational and the matter part. Here, G is a nonsingular metric, and $\mathcal{D}$ a differential operator. For pure gravity, where the phase space variables are

$\{h_{ij}, p^{ij}\}$, we identify

$$G^{\alpha\beta} \leftrightarrow G^{ijkl} := \frac{1}{2\sqrt{-h}}(h^{ik}h^{jl} + h^{il}h^{jk} - h^{ij}h^{kl})$$

$$U(q) \leftrightarrow -\frac{1}{2\kappa}\ {}^{(3)}R$$

$$\mathcal{H}_i \leftrightarrow -2p_i{}^j{}_{|j}.$$

For a massive scalar field $Q \simeq \varphi,\ P \simeq \pi$:

$${}^M\mathcal{H}_\perp = \frac{1}{2\sqrt{-h}}\pi^2 + \frac{1}{2}\sqrt{-h}(h^{ij}\ \varphi_{,i}\varphi_{,j} + m^2\varphi^2)$$

$${}^M\mathcal{H}_i = \varphi_{,i}\pi.$$

And for a Yang-Mills field with $Q_\alpha \simeq A^a_i,\ P^a \simeq \Pi^a_i$:

$${}^M\mathcal{H}_\perp = \frac{1}{2\sqrt{-h}}\ h^{ij}\ \gamma_{ab}\ \Pi^a_i\Pi^b_j + \frac{1}{2}\sqrt{-h}\ h^{ij}h^{kl}\ \gamma_{ab}\ F^a_{ik}F^b_{jl}$$

$${}^M\mathcal{H}_i = F^a_{ij}\ \Pi^j_a$$

$$\lambda^\alpha\mathcal{C}_\alpha \leftrightarrow A^a_0\ (\Pi^i_{a\ |i} - f^c{}_{ab}\ A^b_i\ \Pi^i_c)$$

where γ_{ab} is the Cartan-Killing metric of the internal gauge group.

Projectability and Mixed Symmetry Transformations

To safeguard the equivalence of the tangent space and the cotangent space description of singular systems not only on the level of dynamical equations but also with respect to symmetries, one must investigate the projectability of the tangent space symmetry transformations under the Legendre map. We saw in the case of pure gravity that the transformation parameter has to depend on the lapse and shift functions in the specific form (C.101). A peculiarity arises if a matter theory which itself exhibits local symmetries is coupled to the gravitational field.

Take for example the Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4}\sqrt{|g|}F^a_{\mu\nu}F^b_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\gamma_{ab}.$$

which exhibits primary phase-space constraints

$$\phi_a := \Pi^0_a = \frac{\partial\mathcal{L}_{YM}}{\partial\dot{A}^a_0} = \sqrt{|g|}F^b_{\mu\nu}g^{0\mu}g^{0\sigma}\gamma_{ab} \approx 0.$$

With the zero eigenvectors of the Hessian $W^{\mu\nu}_{ab}$ found from

$$\xi^b_{a\mu} = \frac{\partial\phi_a}{\partial\Pi^\mu_b} = \delta^b_a g_{0\mu}$$

according to (C.14), the condition for Legendre projectability is controlled by the operator $\Gamma_a = \partial/\partial\dot{A}_0^a$. The Lagrangian $\mathcal{L}_{YM}$ is invariant with respect to the gauge transformations

$$\delta_G(\theta)A_\mu^a = -\partial_\mu\theta^a - \theta^b A_\mu^c f^{bca}$$

and this is clearly projectable: $\Gamma_a(\delta_G(\theta)A_\mu^b) = 0$.

Under diffeomorphism the Yang-Mills fields transform as

$$\delta_D(\epsilon)A_\mu^a = -A_{\mu,\nu}^a\epsilon^\nu - A_\nu^a\epsilon^\nu_{,\mu}.$$

For projectability we now demand that this symmetry variation does not depend on $\dot{N}^\alpha$ and $\dot{A}_0^a$. This is true for the transformation of the spatial components, but not for the components A_0^a:

$$\delta_D(\epsilon)A_0^a = -\dot{A}_0^a\epsilon^0 - A_{0,k}^a\epsilon^k - A_0^a\dot{\epsilon}^0 - A_k^a\dot{\epsilon}^k.$$

This cannot be cured by a functional dependency of ϵ as in (C.101), nor by contemplating a dependency of ϵ on $\dot{A}_0^a$. But instead, it was found [451] that together with gauge transformations a modified diffeomorphism can be realized in the phase space. This modified diffeomorphism is a mixture of a diffeomorphism and a specific gauge transformation[19]:

$$\hat{\delta}(\epsilon) := \delta_D(\epsilon) - \delta_G(A_\nu^a\epsilon^\nu).$$

The modified transformation $\hat{\delta}(\epsilon)$ is projectable if the ϵ^ν are expressed by the "descriptors" ξ^ν according to (C.101); for further details see [415], also for the derivation of the symmetry generators. Let me remark here that it was already pointed out in [288] that the infinitesimal coordinate transformation of the gauge fields can be expressed as

$$\delta_D(\epsilon)A_\mu^a = F_{\mu,\nu}^a\epsilon^\nu - D_\mu(A_\nu^a\epsilon^\nu).$$

The second term is the same ϵ-dependent gauge transformation that is subtracted in the definition of $\hat{\delta}(\epsilon)$.

C.4 Alternative Approaches

- L. Faddeev and R. Jackiw proposed an alternative to the Dirac procedure in dealing with constraints. Indeed, the first sentence in [166] is unusually harsh: "It appears that some of our contemporary colleagues are unaware of modern, mathematically

[19] A mixture of diffeomorphisms with gauge transformations seems to be necessary or at least advantageous in other instances too, see e. g. the introduction of group covariant Lie derivatives in App. F.

based, approaches to quantization, especially of constrained systems. They keep the prejudice that Dirac's method with its Dirac bracketing and categorization of constraints as first or second class, primary or secondary, 'is mandatory'". Faddeev and Jackiw start[20] from the canonical Hamiltonian H_C and the set of primary constraints $\phi^\rho = 0$ (remember that the latter are directly derived from the definition of the canonical momenta), and write the Lagrangian as

$$L = p_i \dot{q}^i - H_C(q, p) - u^\rho \phi_\rho \tag{C.118}$$

with a set of multipliers u^ρ. Next, one eliminates as many variables as there are primary constraints by plugging these into the Lagrangian, resulting in a new Lagrangian $L' = a_s(x)\dot{x}^s - H(x)$, where x^s are the coordinates remaining after the reduction by the constraint solutions. Now according to a theorem by Darboux, it is always possible to change variables $x^s \to Q^r, P_r, Z^a$ such that the Lagrangian becomes

$$L' = P_r \dot{Q}^r - H'(Q^r, P_r, Z^a).$$

If H' does not depend on any of the Z^a (for which their canonical counterpart is missing), one is ready: The Q^r and P_r are the coordinates in the reduced phase space (which even has a genuine symplectic structure). Otherwise, it might be possible to eliminate some of the variables Z^a:

$$\frac{\partial H'}{\partial Z^a} = 0 \Longleftrightarrow Z^{a_1} = f^{a_1}(Q^r, P_r, Z^{a_2}), \qquad f_{a_2}(Q^r, P_r) = 0$$

and one can argue that this gives rise to a Lagrangian

$$L'' = P_r \dot{Q}^r - H''(Q^r, P_r) - Z^{a_2} f_{a_2}(Q^r, P_r).$$

This has a structure similar to (C.118) and the algorithm can be restarted. The procedure terminates with the identification of the symplectic coordinates in the reduced phase. Notice that the Faddeev-Jackiw procedure in each of its steps aims at getting rid of unphysical variables. In contrast, the Rosenberg-Dirac-Bergmann procedure keeps all original variables and aims at a consistent picture of constraints that are conserved in time. And it preserves in its first-class constraints all variational symmetries of the original action. In contrast, the Faddeev-Jackiw method seems to lose the traces of local symmetries. However, nothing really gets lost. The two procedures were shown to be equivalent [204]–with a slight exception which is related to the Dirac conjecture. Nevertheless it is disputable which of the two, FJ or RDB, is to be preferred. It is true that the FJ method leads to the true degrees of freedom, but in order to disentangle them you need to solve constraints and to find the Darboux coordinates. Thus it is no surprise that the FJ procedure did not supersede the RDB method.

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[20] I take the description of the FJ algorithm from [204].

- The Faddeev-Jackiv method aims at reducing the original phase space in each of the algorithmic steps. In quite a different spirit a method inspired also by Faddeev was developed in a series of articles by I. A. Batalin, E.S. Fradkin, T.E. Fradkina and I.V. Tyutin (thereafter termed the BFFT formalism), designed for turning second-class constraints in a theory into first-class ones; the motivation being that first-class constraints have a more handy behavior both in canonical and in path-integral quantization. The central point in the BFFT procedure is to enlarge the phase space by as many canonical variables as there are second-class constraints (remember that their number is even in case of finite-dimensional bosonic systems) and to introduce new constraints as a power series in the second-class constraints. Requiring these new constraints to be first class leads to consistency conditions for the expansion. At a first sight, this philosophy sounds a little weird. But the notion "original phase space" has no meaning at all. Might not the second-class constrained system be a gauge-fixed version of a larger gauge theory? For a review of the BFFT formalism in which you also find the reference to the original articles, see [23].
- Let me also mention an approach by J. Klauder [311], which more directly addresses quantization than the other procedures where a consistent classical canonical theory is established prior to quantization. Klauder formally constructs from the constraints a projection operator to the reduced phase space in the context of phase-space coherent states. With this, he can treat path integrals without introducing gauge constraints, the Faddeev-Popov determinant, or Dirac brackets. Thus far, only few physically relevant models could be treated explicitly.

C.5 Constraints and Presymplectic Geometry

In this section those changes will be addressed which arise in a geometry-based description for singular systems, as opposed to the description for regular systems in Subsect. 2.1.5.

C.5.1 Legendre Projectability

For singular systems, the Legendre map $\mathcal{F}L$ is no longer an isomorphism of the tangent and the cotangent bundles. There exists a nontrivial kernel for the pullback of $\mathcal{F}L^*$, spanned by the vector fields

$$\Gamma_\rho = \xi_\rho^k \frac{\partial}{\partial \dot{q}^k} \tag{C.119}$$

where the ξ_ρ are zero eigenvectors of the Hessian W; compare with (C.4). The pullback of a function g through $\mathcal{F}L$ is denoted as

$$\mathcal{F}L^* g(q,p) = g\big(q, \hat{p} = \frac{\partial L}{\partial \dot{q}}\big).$$

Since the image $\mathcal{F}L(TQ)$ is the primary constraint surface $\Gamma_P \in T^*Q$, we have $\mathcal{F}L^*(\phi_\rho) = \phi_\rho(q,\hat{p}) \equiv 0$. In differentiating the constraint $\phi_\rho(q,\hat{p}(q,\dot{q}))$ with respect to the velocities, one finds that the vector field components ξ^ρ_k in (C.119) are related to the cotangent bundle primary constraints ϕ_ρ by

$$\xi^i_\rho = \mathcal{F}L^*\big(\frac{\partial \phi_\rho}{\partial p_i}\big), \tag{C.120}$$

a little sloppily stated as (C.14).

The vector fields Γ_ρ are instrumental in finding out which structures are *Legendre projectable*. For a function $f_L : T\mathbb{Q} \to \mathbb{R}$ the projectability condition reads

$$\Gamma_\rho \cdot\ f_L = 0 \qquad \text{for} \qquad \forall \rho \Leftrightarrow \exists f_H \qquad \text{such that} \qquad \mathcal{F}L^*(f_H) = f_L.$$

In taking for instance the Lagrangian energy E one finds immediately that this is Legendre projectable:

$$\Gamma_\rho \cdot E = \xi^k_\rho\, \frac{\partial}{\partial \dot{q}^k}\Big(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\Big) = \xi^k_\rho\, \Big(\frac{\partial^2\, L}{\partial \dot{q}^i \partial \dot{q}^k}\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\delta^k_i - \frac{\partial L}{\partial \dot{q}^k}\Big) = 0.$$

Therefore a Hamiltonian exists such that $\mathcal{F}L^*(H_C) = E$. This is the canonical Hamiltonian in the Rosenfeld-Dirac-Bergmann algorithm. But also the so-called total Hamiltonian $H_T = H_C + u^\rho \phi_\rho$ obeys $\mathcal{F}L^*(H_C) = E$ because of $\mathcal{F}L^*(\phi_\rho) = 0$.

A useful and convenient vector field is the evolution operator K which transforms a cotangent-space function $f(q,p)$ into its time derivative in the tangent space:

$$K \cdot f = \dot{q}^i\ \mathcal{F}L^*\Big(\frac{\partial f}{\partial q^i}\Big) + \frac{\partial L}{\partial q^i}\mathcal{F}L^*\Big(\frac{\partial f}{\partial p_i}\Big) + \mathcal{F}L^*\Big(\frac{\partial f}{\partial t}\Big). \tag{C.121}$$

Since the two-form ω_L (see 2.30) is only presymplectic (closed, but degenerate) the solution of $i_X\omega_L = dE$ is no longer unique. It is true that solutions have the form[21]

$$X_E = \dot{q}^i \frac{\partial}{\partial q^i} + a^i(q,\dot{q})\frac{\partial}{\partial \dot{q}^i}$$

but, contrary to the regular case with (2.28), the functions a^i are not completely determined. Namely if X_E is a solution, so is $X_E + Y$, where the vector field Y belongs to the kernel of ω_L, that is $i_Y\omega_L = 0$. In [414] one finds a derivation that this kernel is provided by Γ_ρ and by

$$\Delta_{\bar{\mu}} = \xi^k_{\bar{\mu}} \frac{\partial}{\partial q^k} + \Big(K\big(\frac{\partial \phi_{\bar{\mu}}}{\partial p_k}\big) - \mathcal{F}L^*\big(\frac{\partial \phi'_{\bar{\mu}}}{\partial p_k}\big)\Big)\frac{\partial}{\partial \dot{q}^k}, \tag{C.122}$$

where this expression is defined for all indices $\bar{\mu}$ for which the constraints $\phi_{\bar{\mu}}$ are first class. The $\phi'_{\bar{\mu}}$ are secondary constraints: $\phi'_{\bar{\mu}} := \{\phi_{\bar{\mu}}, H\}$.

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[21] Again assuming the absence of any explicit time dependence.

As demonstrated in [32], the Lagrangian and Hamiltonian dynamics relate to each other by

$$\dot{q}^k = \mathcal{F}L^*\{q^k, H_C\} + v^\mu \mathcal{F}L^*\{q^k, \phi_\mu\} \tag{C.123a}$$

$$\frac{\partial L}{\partial q^k} = \mathcal{F}L^*\{p_k, H_C\} + v^\mu \mathcal{F}L^*\{p_k, \phi_\mu\}. \tag{C.123b}$$

On applying the operators Γ_ρ to (C.123a), we find that because of (C.120), and due to the fact that $\Gamma_\rho \circ \mathcal{F}L^* = 0$, the functions v^μ are not projectable, but rather $\Gamma_\rho v^\mu = \delta^\mu_\rho$. With the aid of (C.123), the defining equation for the operator K can be written

$$K \cdot f = \mathcal{F}L^*\{f, H_C\} + v^\mu \mathcal{F}L^*\{f, \phi_\mu\} + \mathcal{F}L^*\Big(\frac{\partial f}{\partial t}\Big). \tag{C.124}$$

C.5.2 Symmetry Transformations in the Tangent and Cotangent Bundle

We saw that for regular systems, a constant of motion associated with a Noether symmetry can be understood as a generator of the symmetry in phase space. This ceases to be true in general, since a Noether symmetry transformation may not be projectable onto phase space.

The following exposition essentially follows [33], [411] and [413]. The Noether invariance identity in (2.50) was split as

$$V_i \delta q^i + \dot{q}^i \frac{\partial J_L}{\partial q^i} + \frac{\partial J_L}{\partial t} \equiv 0 \tag{C.125a}$$

$$\frac{\partial J_L}{\partial \dot{q}^i} - W_{ik}\delta q^k \equiv 0. \tag{C.125b}$$

Here, I forego indicating the symmetry transformation by the letter S on δq^k since the following refers to symmetry transformations only. (Also, I assume that the variation is the active one.) Instead, the current is written with a subscript L to indicate that it is a function in $\mathcal{F}(T\mathbb{Q})$. For regular systems, the identity (C.125b) can directly be solved to yield explicit expressions of the infinitesimal symmetry transformations. In the singular case, after multiplying (C.125b) with the zero eigenvectors ξ^ρ one obtains the result that

$$\Gamma_\rho \cdot J_L = 0 \ ,$$

and therefore a G exists such that $J_L = \mathcal{F}L^*(G)$. This makes it possible to rewrite (C.125b) as

$$W_{ik}\Big(\mathcal{F}L^*\Big(\frac{\partial G}{\partial p_k}\Big) - \delta q^k\Big) = 0 \ ,$$

revealing that the factor multiplying the Hessian must be a linear combination of zero eigenvectors, or

$$\delta q^k = \mathcal{F}L^*\Big(\frac{\partial G}{\partial p_k}\Big) + \xi^k_\sigma\, r^\sigma(q,\dot q,t) = \mathcal{F}L^*\{q^k, G\} + r^\sigma\ \mathcal{F}L^*\{q^k,\phi_\sigma\} \quad \text{(C.126)}$$

with functions $r^\rho(q,\dot q,t)$. We observe that the transformations δq^k are projectable only if $\Gamma_\rho r^\sigma = 0$. If the functions r^σ are not projectable, there is a variational symmetry in the tangent bundle which is not the pull-back of a Hamiltonian symmetry transformation. In the sequel, to keep things simple, only Noether projectable symmetry transformations will be treated; for the general case see [411].

Plugging the expression for δq^k into (C.125a) results in

$$\begin{aligned}
&\Big(\frac{\partial L}{\partial q^i} - \dot q^k \frac{\partial \hat p_i}{\partial q^k}\Big)\mathcal{F}L^*\Big(\frac{\partial G}{\partial p_i}\Big) + V_i\xi^i_\sigma r^\sigma + \dot q^i \frac{\partial J_L}{\partial q^i} + \frac{\partial J_L}{\partial t}\\
&\quad = \frac{\partial L}{\partial q^i}\mathcal{F}L^*\Big(\frac{\partial G}{\partial p_i}\Big) + \dot q^i \mathcal{F}L^*\Big(\frac{\partial G}{\partial q^i}\Big) + \mathcal{F}L^*\Big(\frac{\partial G}{\partial t}\Big) + r^\sigma\chi_\sigma\\
&\quad = K\cdot G + r^\sigma \chi_\sigma = 0.
\end{aligned}$$

In this expression, the

$$\chi_\sigma := \xi^k_\sigma V_k = K\cdot\phi_\sigma$$

are the primary Lagrange constraints. This can be seen from (C.120) and from

$$0 = \frac{\partial \phi_\mu}{\partial q^i} + \frac{\partial \phi_\mu}{\partial \hat p^k}\frac{\partial \hat p^k}{\partial q^i}, \qquad \text{which is} \qquad \mathcal{F}L^*(\frac{\partial \phi_\mu}{\partial q^i}) = -\xi^k_\mu\ \mathcal{F}L^*(\frac{\partial^2 L}{\partial q^i \partial \dot q^k});$$

i.e.

$$\begin{aligned}
K\cdot\phi_\mu &= \dot q^i\ \mathcal{F}L^*\Big(\frac{\partial \phi_\mu}{\partial q^i}\Big) + \frac{\partial L}{\partial q^i}\mathcal{F}L^*\Big(\frac{\partial \phi_\mu}{\partial p_i}\Big)\\
&= \xi^k_\mu\Big(-(\frac{\partial^2 L}{\partial q^i\partial \dot q^k}) + \frac{\partial L}{\partial q^i}\Big) = \xi^k_\mu V_k = \chi_\mu.
\end{aligned}$$

Now, in the case of projectability, the condition derived for G to be the generator of a symmetry transformation in phase space can be written as

$$K\cdot(G + r^\sigma\phi_\sigma) = 0. \quad \text{(C.127)}$$

The transformations induced in the tangent bundle can be expressed as

$$\delta q^k = \mathcal{F}L^*\{q^k, \tilde G\} \qquad \text{with} \qquad \tilde G := G + r^\sigma_H \phi_\sigma.$$

Furthermore, it can be shown, that

$$\delta \hat p_k = V_{kj}\delta q^j + W_{kj}\delta \dot q^j = \mathcal{F}L^*\{q_k, \tilde G\} + [L]_j \frac{\partial \delta q^j}{\partial \dot q^k};$$

and thus the phase-space induced transformations for the momenta are (in general) "correct" on solutions only. We saw this previously in the example of the relativistic particle. A similar term, depending on the functional dependence of the symmetry transformation was shown to arise in the regular case; see Sect. 2.2.3.
Expressing the action of K by (C.124), the condition (C.127) becomes

$$\mathcal{F}L^*\{\tilde{G}, H_C\} + v^\mu \mathcal{F}L^*\{\tilde{G}, \phi_\mu\} + \mathcal{F}L^*\Big(\frac{\partial \tilde{G}}{\partial t}\Big) = 0.$$

Acting on this with Γ_ρ, yields for projectable transformations

$$\mathcal{F}L^*\{\tilde{G}, H_C\} + \mathcal{F}L^*\Big(\frac{\partial \tilde{G}}{\partial t}\Big) = 0 \qquad \mathcal{F}L^*\{\tilde{G}, \phi_\mu\} = 0.$$

This is equivalent to

$$\{G, H_C\} + \frac{\partial G}{\partial t} = PC \qquad \{G, \phi_\mu\} = PC.$$

These conditions for relating Lagrangian to Hamiltonian symmetries hold both for global and local transformations. For local transformations they lead to detailed conditions for the coefficients G_k in the expansion $G(\epsilon) = \sum \epsilon^{(k)} G_k$; compare (C.38) and C.4.1.

I should mention that a related geometry-based relationship of Noether symmetry transformations and infinitesimal canonical transformations by first-class constraints is derived in [329].

The Algebra of Symmetry Transformations

Next, let us compare the commutator algebra of transformations in configuration space with the Poisson bracket algebra of generators in phase space. Based on previous findings the transformations are related to each other by

$$\delta^L q^k = \mathcal{F}L^*(\delta^H q^k) \qquad\qquad \delta^L \hat{p}_k = \mathcal{F}L^*(\delta^H p_k) + [L]_j \frac{\partial \delta q^j}{\partial \dot{q}^k}.$$

For the commutator of two Hamiltonian transformations the Jacobi identity implies

$$[\delta_1^H, \delta_2^H] q^k = \{q^k\{G_2, G_1\}\}.$$

And the commutator of two Lagrangian transformations is calculated to be

$$[\delta_1^L, \delta_2^L] q^k = \mathcal{F}L^*(\{q^k\{G_2, G_1\}\}) - [L]_i W_{jl}\left(\frac{\partial^2 G_1}{\partial p_i \partial p_l}\frac{\partial^2 G_2}{\partial p_k \partial p_j} - \frac{\partial^2 G_2}{\partial p_i \partial p_l}\frac{\partial^2 G_1}{\partial p_k \partial p_j}\right).$$

The extra term is an antisymmetric combination of the equations of motion, called "trivial" transformation in Sect. C.1.4. This term vanishes off-shell only if (one of) the generators is linear in the momenta[22]. Typically this is the case for Yang-Mills type theories, but in reparametrization-invariant theories the generators always contain the Hamiltonian constraint $\mathcal{H}_\perp$, which is quadratic in the momenta; see (C.117) and the expressions immediately following.

C.5.3 Constraint Stabilization

The Rosenfeld-Dirac-Bergmann algorithm was formulated in purely geometric terms by M.J. Gotay, J.M. Nester and G. Hinds [228], for its essence see also [72], [479]. They start from the observation that the Legendre map $\mathcal{F}L$ is no longer a diffeomorphism, but a map into $T^*\mathbb{Q}$. The image $\mathcal{F}L(T\mathbb{Q})$ of $T\mathbb{Q}$ under $\mathcal{F}L$ may be assumed to be a submanifold $\Gamma_P \in T^*\mathbb{Q}$. However, in general it is not a symplectic manifold but merely presymplectic (closed, but degenerate). The two-form ω_P inherited from the inclusion map $j : \mathcal{F}L(T\mathbb{Q}) \to T^*\mathbb{Q}$ as $j^*\omega = \omega_P$ gives rise to the equation

$$(i_X\omega_P - dH_C)\big|_{\Gamma_P} = 0.$$

This looks like a Hamilton equation, but does not have a unique solution for the vector field X. It serves as the starting point of an algorithm in which successively the constraint surfaces are refined.

Is there an inherent meaning of primary/secondary, of first-/second-class constraint classification and of the Dirac bracket? As demonstrated in [411], the first- and second-class classification distinguishes projectable and non-projectable primary Lagrange constraints. As for the Dirac bracket: Assume that the second-class constraints define locally a submanifold $\bar{\Gamma}$ of the full phase space $T^*\mathbb{Q}$. The inclusion map $\bar{j} : \bar{\Gamma} \to T^*\mathbb{Q}$ induces on $\bar{\Gamma}$ a two-form $\bar{\omega} = \bar{j}^*\omega$. It can be shown that $\bar{\omega}$ is a symplectic form on $\bar{\Gamma}$ and that the Dirac bracket is the Poisson bracket with respect to this symplectic form.

[22] This result was also noted in [364].

Appendix D
*Symmetries in Path-Integral and BRST Quantization

The path-integral formulation of quantum theories goes back to ideas of P.A.M. Dirac in the 1930's, and it was elaborated by R. Feynman in the 1940's within his doctoral thesis, see [175]. At first, path-integral techniques looked like just another approach to quantum physics, but today, specifically when it comes to quantum field theory, they are judged to be the most appropriate means to tackle conceptual issues. Path-integral methods allow one to directly determine Lorentz covariant Feynman rules of a theory from its Lagrangian. This became feasible also for non-AbeIian gauge theories and general relativity after the work of L. Faddeev and V. Popov in the 1960's. And G. 't Hooft's derivation of the Feynman rules for spontaneously broken gauge theories was likewise carried out in the path-integral formulation[1]. Today, with the better understanding of the BRST symmetries, any theory based on a classical or a pseudo-classical action (involving also fermionic variables in the sense of App. B.2.4) can be treated by path integrals.

The intuitive ideas on path integrals do come from Feynman's thought experiments about interference patterns in double-slit configurations. These led to postulates as follows

- The probability of a particle moving from point a to point b is given by the absolute square of the complex transition function $K(b,a)$: $P(b,a) = |K(b,a)|^2$.
- The transition function obeys $K(c,a) = \sum K(c,b)K(b,a)$ where the sum is taken over all path connecting a and c, as well all intermediate points b.
- The transition function is given by a sum of phase factors as

$$K(b,a) = \sum_{paths(ab)} \gamma e^{iS(path)/\hbar} \tag{D.1}$$

  where S is the classical action, evaluated along a path, and the sum extends over all paths connecting a and b. The normalization constant γ can be determined from the second postulate.

We expect a transition from quantum mechanics to classical mechanics by taking the limit $\hbar \to 0$. All those paths for which the action is large are suppressed by large

[1] see [569] for a topical point of view.

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fluctuations of the phase. The paths that dominate the sum are those for which the action is small. In the limit $\hbar \to 0$ the path with the smallest value of S becomes relevant. But this is just the classical trajectory. In this sense the "sum-over-histories" approach offers an explanation why nature (classically) obeys extremum principles.

D.1 Basics

D.1.1 Path-integral Formulation of Quantum Mechanics

We begin by calculating the probability amplitude for a process in which a particle starts from the position q' at time t' and is found at q'' at some later time t''. In the Schrödinger picture, the time evolution of states is described by $|q,t\rangle = e^{iHt}|q\rangle$. Thus

$$\langle q'',t''|q',t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle.$$

Let us now see in which sense this probability amplitude is evaluated by summing over all possible paths that connect (q',t') and (q'',t''). First, divide the time interval $(t''-t')$ into equidistant segments:

$$\delta t = t_{n+1} - t_n = \epsilon \qquad t_0 = t' \qquad t_N = t''.$$

Sandwich the completeness relation $\int dq \ |q,t\rangle\langle q,t| = 1$ $(N$-1) times into the amplitude:

$$\begin{aligned}\langle q'',t''|q',t'\rangle &= \int dq_1 \ldots dq_{N-1}\, \langle q'',t''|q_{N-1},t_{N-1}\rangle \ldots \langle q_2,t_2|q_1,t_1\rangle\langle q_1,t_1|q',t'\rangle \\ &= \prod_{n=1}^{N-1}\int dq_n\, \langle q''|e^{-iH\epsilon}|q_(N-1)\rangle \ldots \langle q_2|e^{-iH\epsilon}|q_1\rangle\langle q_1|e^{-iH\epsilon}|q'\rangle.\end{aligned} \tag{D.2}$$

Now take the most simple Hamiltonian imaginable, namely the one for a free relativistic particle: $H = \hat{p}^2/2\ m$. Consider any one of the inner products $\langle q_{n+1}|\exp\{-i\epsilon\hat{p}^2/2\ m\}|q_n\rangle$; insert another complete set of states according to $\int \frac{dp}{2\pi}\ |p\rangle\langle p| = 1$ to arrive at

$$\begin{aligned}\langle q_{n+1}|\exp\{-i\epsilon\frac{\hat{p}^2}{2\ m}\}|q_n\rangle &= \int \frac{dp}{2\pi}\langle q_{n+1}|\exp\{-i\epsilon\frac{\hat{p}^2}{2\ m}\}|p\rangle\langle p|q_n\rangle \\ &= \int \frac{dp}{2\pi}\exp\{-i\epsilon\frac{p^2}{2\ m}\}\langle q_{n+1}|p\rangle\langle p|q_n\rangle \\ &= \int \frac{dp}{2\pi}\exp\{-i\epsilon\frac{p^2}{2\ m}\}\exp\{ip(q_{n+1}-q_n)\}.\end{aligned} \tag{D.3}$$

The last step in this manipulation made use of $\langle q|p\rangle = e^{ipq}$. The resulting expression for each of the inner products is a Gaussian integral with respect to p.

Let me next give the Gaussian integral formula in a quite general form: For a real, symmetric $N \times N$ matrix A_{ij} with positive eigenvalues, and for a vector x_i

$$\prod_{k=1}^{N} \int_{-\infty}^{+\infty} dx_k \; e^{-\frac{1}{2}x\cdot A\cdot x + J\cdot x} = \left(\frac{(2\pi)^N}{\det A}\right)^{1/2} e^{\frac{1}{2}J\cdot A^{-1}\cdot J}. \tag{D.4}$$

Assume that this Gauss integration holds also for imaginary A and J, and formally replace $A \to iA$, $J \to iJ$. Then

$$\prod_{k=1}^{N} \int_{-\infty}^{+\infty} dx_k \; e^{-\frac{i}{2}x\cdot A\cdot x + iJ\cdot x} = \left(\frac{(-2\pi i)^N}{\det A}\right)^{1/2} e^{-\frac{1}{2}J\cdot A^{-1}\cdot J}. \tag{D.5}$$

This assumption has to be taken with caution. One may be more careful by either taking formally $(i\epsilon)$ to be real, or by putting in by hand a factor $e^{-\zeta p^2}$ into (D.3), with the option $\zeta \to 0$ at the end of the calculation. In any case, identifying in (D.3) $A = \epsilon/m$ and $J = (q_{n+1} - q_n)$ the p integration yields with (D.5)

$$\langle q_{n+1}| \exp\{-i\epsilon \frac{\hat{p}^2}{2\,m}\}|q_n\rangle = \frac{1}{2\pi}\left(\frac{2\pi m}{i\epsilon}\right)^{1/2} \exp\{\frac{im}{2\epsilon}(q_{n+1} - q_n)^2\}.$$

Inserting these results into (D.2), we finally arrive at

$$\langle q''|e^{-iH(t''-t')}|q'\rangle = \left(\frac{-im}{2\pi\epsilon}\right)^{N/2} \prod_{n=1}^{N-1} \int dq_n \; \exp\{i\epsilon(m/2)\sum_{k=1}^{N}[(q_{k+1} - q_k)/\epsilon]^2\}.$$

Letting $\epsilon = \delta t \to 0$ we can replace

$$[(q_{k+1} - q_k)^2/\delta t]^2 \;\to\; \dot{q}^2 \qquad \delta t \sum_{k=1}^{N-1} \;\to\; \int_{t'}^{t''} dt.$$

Furthermore, define the integral over paths as

$$\int \mathcal{D}q = \lim_{N\to\infty} \left(\frac{-imN}{2\pi(t'' - t')}\right)^{N/2} \left(\prod_{n=1}^{N-1} \int dq_n\right), \tag{D.6}$$

so that the previous result can be written

$$\langle q''|e^{-iH(t''-t')}|q'\rangle = \int \mathcal{D}q \exp\left\{i \int_{t'}^{t''} dt \; \frac{1}{2}m\dot{q}^2\right\}.$$

The expression (D.6) for the measure $\mathcal{D}q$ needs some getting used to. Observe that it is an integral over functions $q_n(t)$, therefore instead of the physically motivated "path integral", one is mathematically dealing with a "functional integral".

If instead of the Hamiltonian for the free particle, one takes the more general $H = \hat{p}^2/2\,m + V(\hat{q})$, the transition amplitude would become

$$\langle q''|e^{-iH(t''-t')}|q'\rangle = \int \mathcal{D}q \exp\left\{i\int_{t'}^{t''} dt\,[\frac{1}{2}m\dot{q}^2 - V(q)]\right\} = \int \mathcal{D}q\; e^{iS(q,\dot{q})}. \tag{D.7}$$

This then demonstrates explicitly what is meant by the sum over all paths in (D.1). Compared to (D.1), there is a factor $1/\hbar$ missing in the exponential. But not only must it be there for dimensional reasons, we also can recover it from the calculations by noticing that the correct normalization condition is $\langle q|p\rangle = e^{ipq/\hbar}$ entailing the completeness relation $\int \frac{dp}{2\pi\hbar}\,|p\rangle\langle p| = 1$; compare with (4.15).

The expression (D.7) contains the action $S(q,\dot{q})$ in its Lagrangian form. But we see more or less immediately that the transition to the Hamiltonian form $S(q,p)$ is simply mediated by a path integral in momentum space, namely

$$\begin{aligned}\int \mathcal{D}q e^{iS} &= \int \mathcal{D}q \exp\left\{i\int_{t'}^{t''} dt\,[\frac{1}{2}m\dot{q}^2 - V(q)]\right\}\\ &= \int \mathcal{D}q\mathcal{D}p \exp\left\{i\int_{t'}^{t''} dt\,[p\dot{q} - \frac{p^2}{2\,m} - V(q)]\right\}\\ &= \int \mathcal{D}q\mathcal{D}p \exp\left\{i\int_{t'}^{t''} dt\,[p\dot{q}(q,p) - H(p,q)]\right\}.\end{aligned}$$

The exact procedure is to insert into the left-hand side a sequence of N Gaussian integrations over p_k by using (D.4) and to reabsorb the constants in the limit $N\to\infty$ in the definition of $\mathcal{D}p$, similar to (D.6). Is it not intriguing to recover the Legendre transformation as a p-integration in the path integral? (This, however, only holds for Hamiltonians quadratic in the momenta.)

Now consider instead of the previous transition amplitude the more general $\langle q'',t''|\hat{q}(t_1)|q',t'\rangle$ (where $t' < t_1 < t''$):

$$\langle q'',t''|\hat{q}(t_1)|q',t'\rangle = \langle q'',t''|e^{-iH(t'-t_1)}\hat{q}e^{-iH(t_1-t'')}|q',t'\rangle = \int \mathcal{D}q\mathcal{D}p\, q(t_1)e^{iS[q,p]}.$$

Moreover, one finds that

$$\int \mathcal{D}q\mathcal{D}p\; q(t_1)q(t_2)e^{iS[q,p]} = \langle q'',t''|P\hat{q}(t_1)\hat{q}(t_2)|q',t'\rangle,$$

where P denotes time-ordering:

$$P(\hat{q}(t_k)\hat{q}(t_l)) = \begin{cases} \hat{q}(t_k)\hat{q}(t_l) & \text{for} \quad t_k < t_l,\\ \hat{q}(t_l)\hat{q}(t_k) & \text{for} \quad t_k > t_l.\end{cases}$$

A further notion is the *generating functional*

$$\langle q'',t''|q',t'\rangle_J = \int \mathcal{D}q\;\exp\left\{i\int dt\,[L(q,\dot{q}) + Jq]\right\}$$

with its functional derivatives

$$\frac{\delta}{\delta J(t_1)}\langle q'', t''|q', t'\rangle_J = i\int \mathcal{D}q\mathcal{D}p\; q(t_1)\exp\left\{i\int dt\;[L(q,\dot q)+Jq]\right\}$$

$$\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}\langle q'', t''|q', t'\rangle_J = (i)^2\int \mathcal{D}q\mathcal{D}p\; q(t_1)q(t_2)\exp\left\{i\int dt\;[L(q,\dot q)+Jq]\right\}.$$

From this, one observes that the generating functional yields

$$\langle q'', t''|P(\hat q(t_1)\ldots\hat q(t_n))|q', t'\rangle = (-i)^n\frac{\delta^n}{\delta J(t_1)\ldots\delta J(t_n)}\langle q'', t''|q', t'\rangle_J\Bigg|_{J=0}. \tag{D.8}$$

In general, one might be interested in the transition of an initial state $|I\rangle$ (not necessarily a position) to a final state $|F\rangle$. This is

$$\langle F, t''|I, t'\rangle = \int dq''\; dq'\;\langle F|q''\rangle\langle q''|e^{-iH(t''-t')}|q'\rangle\langle q'|I\rangle.$$

A special name is given to the transition amplitude in case $|I\rangle$ and $|F\rangle$ are the ground state and t' and t'' refer to the infinite past and infinite future, respectively:

$$\langle 0|0\rangle = \lim_{t'\to-\infty}\;\lim_{t''\to+\infty}\langle 0, t''|0, t'\rangle.$$

This cumbersome expression can be avoided by replacing H by $(1-i\epsilon)H$ (see e.g. [470], Sect. 6; [421], Sect. 2.3) Define

$$Z[J] := \int \mathcal{D}q\mathcal{D}p\exp\left\{i\int_{-\infty}^{+\infty} dt\;[p\dot q-(1-i\epsilon)H(p,q)+Jq]\right\};$$

then $\langle 0|0\rangle = Z[0]$. In subsequent expressions the $(i\epsilon)$ will often be omitted unless it is needed for explicit calculations.

Example: Harmonic Oscillator

Let me state some results for the harmonic oscillator[2] which exhibits very many features of functional integrals for relativistic field theories. The Hamiltonian is

$$H(p,q) = \frac{1}{2\,m}p^2 + \frac{1}{2}m\omega^2q^2.$$

Here the $(1\text{-}i\epsilon)$ prescription amounts to the replacement $m\to(1+i\epsilon)m$ and $m\omega^2\to(1-i\epsilon)m\omega^2$, and thus, after switching to the Lagrangian:

$$Z[J] = \int \mathcal{D}q\exp\left\{i\int_{-\infty}^{+\infty} dt\;[\frac{1}{2}(1+i\epsilon)m\dot q^2 - \frac{1}{2}(1-i\epsilon)m\omega^2q^2 + Jq]\right\}.$$

[2] details in e.g. [470], Sect. 7

Here, we directly see the physical meaning of the Jq term as an external force on the oscillator. In going over to the Fourier components, e.g.

$$\tilde{J}(E) := \frac{1}{\sqrt{2\pi}} \int dt\; J(t) e^{-iEt},$$

the generating functional can be expressed as (from now on $m = 1$)

$$\begin{aligned} Z[J] &= Z(0) \exp\Big\{ -\frac{i}{2} \int_{-\infty}^{+\infty} dE\; \frac{\tilde{J}(E)\tilde{J}(-E)}{E^2 - \omega^2 + i\epsilon} \Big\} \\ &= Z(0) \exp\Big\{ -\frac{i}{2} \int_{-\infty}^{+\infty} dt dt'\; J(t) G(t-t') J(t') \Big\}. \end{aligned}$$

Here,

$$G(t-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, \frac{e^{-i(t-t')E}}{E^2 - \omega^2 + i\epsilon} = \frac{1}{2i\omega} [\theta(t) e^{-i\omega t} + \theta(-t) e^{i\omega t}]$$

is the Green's function for the operator $\big(\frac{d^2}{dt^2} + \omega^2\big)$:

$$\big(\frac{d^2}{dt^2} + \omega^2\big) G(t) = -\delta(t).$$

We may also verify the relations between the Green's function and the time ordered products (D.8). Consider for instance the two-point function

$$\begin{aligned} \langle 0|P(\hat{q}(t_1)\hat{q}(t_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(t_1)}\frac{1}{i}\frac{\delta}{\delta J(t_2)} \langle 0|0\rangle_J \Big|_{J=0} \\ &= \frac{1}{i}\frac{\delta}{\delta J(t_1)} \Big[\int_{-\infty}^{+\infty} dt dt'\; G(t_2 - t') J(t') \Big] Z[J] \Big|_{J=0} \\ &= \Big[\frac{1}{i} G(t_2 - t_1) + (\text{terms in } J) \Big] Z[J] \Big|_{J=0} = -iG(t_2 - t_1). \end{aligned}$$

This can be generalized to express the n-point function in terms of the Green's function.

D.1.2 Functional Integrals in Field Theory

The path integral can be immediately generalized to more than just one degree of freedom and it can be written in a characteristic way as

$$\int \mathcal{D}q e^{iS(q,\dot{q})} = \int \Big(\prod_{k=1}^{N} \mathcal{D}q_k \Big) e^{iS(q_k, \dot{q_k})},$$

where again $S(q_k, \dot{q_k})$ is the classical action and the path integral extends over all degrees of freedom.

The next step is to generalize the path-integral description to a field theory. The fields in fundamental physics are either bosonic (with spin 0, 1, 2) or fermionic (with spin $\frac{1}{2}$, $\frac{3}{2}$).

Bosonic Fields

Let us begin with the free scalar field theory described by the Lagrangian

$$\mathcal{L}_f = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2).$$

The generating functional is defined as

$$Z_f[J] = \int \mathcal{D}\varphi \ \exp i \int d^4x [\mathcal{L}_f(\varphi, \partial\varphi) + J\varphi], \tag{D.9}$$

where the $J(x)$ denotes an arbitrary classical source field. The symbol $\mathcal{D}\varphi$ indicates that at each point of spacetime, one integrates over all possible values of the fields $\varphi(x)$. By defining the Fourier components as

$$\tilde{J}(k) := \frac{1}{(2\pi)^2} \int d^4x \ J(x) e^{-ikx}$$

one can essentially mimic the calculations of the one-dimensional harmonic oscillator to arrive at

$$Z_f[J] = Z_f(0) \exp\Big\{ -\frac{i}{2} \int_{-\infty}^{+\infty} dxdy \ J(x) \triangle_F(x-y) J(y) \Big\}. \tag{D.10}$$

Here,

$$\triangle_F(x-y) = \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \ \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}$$

is the Green's function for the Klein-Gordon operator

$$-(\partial_\mu \partial^\mu + m^2) \triangle_F(x) = \delta^4(x). \tag{D.11}$$

This result can be obtained formally also in the following way: Write the derivative term in the Lagrangian as $(\partial\varphi)^2 = \partial(\varphi\partial\varphi) - \varphi \ \partial^2\varphi$. Then (D.9) becomes–up to a boundary term–

$$Z_f[J] = \int \mathcal{D}\varphi \ \exp i \int d^4x [-\frac{1}{2}\varphi(\partial_\mu\partial^\mu + m^2)\varphi + J\varphi].$$

This is a Gaussian integral (D.5) with the identification $A = -(\partial_\mu \partial^\mu + m^2)$. If we baldly extend the Gaussian integral from the case of finite discrete integration to the continuous case, we need to identify the inverse of the Klein-Gordon operator A. But, according to (D.11) this is none other than $\triangle_F(x)$, and we arrive directly back at (D.10). Observe that the normalization of Z is formally identical to

$$Z[0] = [\det(\partial^2 + m^2)]^{-1/2},$$

a quantity which is not well-defined, but also not needed in explicit calculations of physically-measurable quantities such as cross sections and decay rates. This is because we always calculate vacuum expectation values

$$\langle \hat{O} \rangle := \frac{\langle 0|\hat{O}|0\rangle}{\langle 0|0\rangle},$$

and here, the dependence on $\langle 0|0\rangle = Z[0]$ cancels out because it is in both the numerator and the denominator.
Similarly to the case of the harmonic oscillator

$$\langle 0|P(\hat{\varphi}(x_1)(\hat{\varphi}(x_2))|0\rangle = -i\triangle(x_2 - x_1),$$

and (D.8) generalizes to

$$\langle 0|P(\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n))|0\rangle = (-i)^n \frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)} Z[J]\bigg|_{J=0}. \tag{D.12}$$

This reproduces the techniques developed in the early 1950's by J. Schwinger of "probing" a quantum field theory with a varying external source.

Extending the free scalar theory by an interaction Lagrangian term to the action $S = \int d^4x\, \mathcal{L}_f + \int d^4x\, \mathcal{L}_{int}$, we may write

$$\exp i(S_f + S_{int}) = e^{iS_f} \sum_{n=0}^{\infty} \frac{i^n}{n!} (\mathcal{L}_{int})^n.$$

After expanding $\mathcal{L}_{int}$ in terms of φ, the path integral for the full theory is a sum of terms proportional to

$$\int \mathcal{D}\varphi\, e^{iS_f} \varphi(x_1)\ldots.\varphi(x_k).$$

These are related to the k-point functions which can be symbolized by Feynman diagrams and associated rules. For details, see again any book on QFT.

The previous considerations for a scalar field can in principle be extended to massive spin-1 and spin-2 fields (let them be called generically Φ) by rewriting the free Lagrangian in the form $\Phi\mathcal{O}\Phi$ with a differential operator $\mathcal{O}$, then performing a Gaussian integration with respect to Φ, identifying the inverse of $\mathcal{O}$ with its Green's function. For instance for the Proca theory with

$$\begin{aligned}\mathcal{L}_P &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \\ &= \frac{1}{2}A_\mu\left[(\partial_\lambda\partial^\lambda + m^2)\eta^{\mu\nu} - \partial^\mu\partial^\nu\right]A_\nu + b.t. \\ &= A_\mu \mathcal{O}^{\mu\nu} A_\nu + b.t.\end{aligned}$$

the inverse of $\mathcal{O}^{\mu\nu}(x)$ is given by

$$D_{\nu\lambda}(x) = \int \frac{d^4k}{(2\pi)^2} D_{\nu\lambda}(k)e^{ikx} \quad \text{and} \quad D_{\nu\lambda}(k) = \frac{-\eta_{\nu\lambda} + k_\nu k_\lambda/m^2}{k^2 - m^2}. \tag{D.13}$$

Observe that for the Maxwell theory with $m^2 = 0$, the inverse of $[(\partial\partial)\ \eta^{\mu\nu} - \partial^\mu\partial^\nu]$ does not exist. Since the spin-1 and spin-2 fields of fundamental physics are gauge bosons–and as such massless–they need a special treatment, for instance by the Faddeev-Popov method. Before giving an example of this, a few remarks about fermionic fields and functional integrals are in order.

Fermionic Fields

For fermionic fields $\psi(x)$, the generating functional can be defined by using Grassmann variables and be manipulated by integration over Grassmann variables (see App. B.2.):

$$Z_F[\eta, \overline{\eta}] = N \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\{\mathcal{L}_F(x) + \overline{\eta}\psi + \bar{\psi}\eta\}}$$

with the external currents η and $\overline{\eta}$, where $\mathcal{L}_F$ might for instance be the Dirac Lagrangian $\mathcal{L}_D = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$. This can be integrated formally by using (B.41), resulting in

$$Z_D[\eta, \overline{\eta}] = \exp\left[i\int d^4x d^4y\ \overline{\eta}(x)S(x-y)\eta(y)\right],$$

where $S(x)$ is the inverse of the Dirac operator, that is $(i\gamma^\mu\partial_\mu - m)S(x) = \delta^4(x)$, which yields the Feynman propagator for the Dirac field:

$$S_{\alpha\beta}(x-y) = i\langle 0|P\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle.$$

D.1.3 Faddeev-Popov Ghost Fields in Theories with Local Symmetries

Yang-Mills Theories

Consider a Yang-Mills theory gauge theory of fermions, starting from

$$Z = \int \mathcal{D}A\ \mathcal{D}\bar{\psi}\ \mathcal{D}\psi e^{i\{S_{YM}[A] + S_F[A,\psi]\}} \tag{D.14}$$

with the Yang-Mills action and the minimally-coupled fermion action

$$S_{YM}[A] = -\frac{1}{4\,g^2}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} \quad S_F[A,\psi] = i\bar{\psi}\gamma^\mu(\partial_\mu + iA^a_\mu T_a)\psi - m\bar{\psi}\psi.$$

Here, $\mathcal{D}[A]$ symbolizes the rather cumbersome measure $\prod_{\mu,a,x} dA^a_\mu(x)$. As it stands, the expression (D.14) for Z is infinite, because in integrating over all fields one overcounts by picking contributions from all those fields which are related to each other by a gauge transformation

$$A^\Omega_\mu = \Omega A_\mu \Omega^{-1} + i(\partial_\mu \Omega)\Omega^{-1};$$

compare (5.61). Hence, in a first step one must "divide out" this infinite factor stemming from the gauge-orbit volume. A formal method developed by L. Faddeev and V. Popov [167] is based on path-integral adapted gauge fixing. Write the gauge fixing conditions generically as $G[A] = 0$ (here I completely dispense with indices, both for the gauge fields and for enumerating the gauge fixing conditions). Off course it is possible to insert a delta-function $\delta(G[A])$ into the path integral, but this would change the measure of integration (in the sense of (D.16)). Instead, define the *Faddeev-Popov determinant* $\triangle^G_{FP}(A)$ implicitly by

$$1 = \triangle^G_{FP}(A)\int \mathcal{D}\Omega\;\delta(G[A^\Omega]). \tag{D.15}$$

Here $\mathcal{D}\Omega = \prod_x d\Omega(x)$ is the invariant group measure obeying $\mathcal{D}\Omega = \mathcal{D}(\Omega'\Omega)$. This so-called Haar measure exists for compact groups, see e.g. [248]. The Faddeev-Popov determinant depends apparently on the gauge condition, but actually, it is gauge invariant: Calculate $\triangle^G_{FP}$ for a gauge-shifted field as

$$(\triangle^G_{FP})^{-1}(A^{\Omega'}) = \int \mathcal{D}\Omega'\;\delta(G[A^{\Omega'\Omega}]) = \int \mathcal{D}(\Omega'\Omega)\;\delta(G[A^{\Omega'\Omega}]) = (\triangle^G_{FP})^{-1}(A).$$

Insert the "1" defined in terms of (D.15) into (D.14):

$$Z = \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\Big(\triangle^G_{FP}\int \mathcal{D}\Omega\;\delta(G[A^\Omega])\Big)e^{i\{S_{YM}[A]+S_F[A,\psi]\}}.$$

Now perform a gauge transformation $A \to A^{\Omega^{-1}}$ on this path integral. At first one might ask whether the measure is invariant. Under an infinitesimal gauge transformation $(A^a_\mu)_\epsilon = A^a_\mu + \partial_\mu\epsilon^a - f^{abc}A^c_\mu\epsilon^b$, the measure with respect to the vector fields changes as

$$(\mathcal{D}A)_\epsilon = \prod_{\mu,a,x}(dA^a_\mu)_\epsilon = \mathcal{J}_\epsilon \prod_{\mu,a,x} dA^a_\mu$$

where $\mathcal{J}_\epsilon$ is the determinant of the Jacobian

$$\frac{\delta(A^a_\mu)_\epsilon}{\delta A^b_\nu} = \delta^4(x-y)\delta^\nu_\mu\big[\delta^a_b - f^a{}_{bc}\epsilon^c\big].$$

To first order in ϵ, we have $\det \mathcal{J}_\epsilon = 1 - \mathrm{tr}(gf^a{}_{bc}\epsilon^c)$, and this is unity because of the antisymmetry of the structure constants. Let us assume that also the fermionic part $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ is invariant, although some intricacies are encountered in the verification. These will be dealt with in Sect. D.2.4. The part involving the action is of course invariant, and so is the Faddeev-Popov determinant. The factor $G[A^\Omega]$ simply changes into $G[A]$. Therefore, in

$$Z = \Big(\int \mathcal{D}\Omega\Big) \int \mathcal{D}A\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \triangle_{FP}^G\, \delta(G[A])e^{i\{S_{YM}[A]+S_F[A,\psi]\}}$$

the (infinite) group volume $\int \mathcal{D}\Omega$ can be factored out. Since we already saw that a factor which does not depend on the fields cancels out from vacuum expectation values, we may simply omit it and there is no longer the problem of overcounting. Observe that the gauge condition is embedded in the path integral in form of a delta function, but that the Faddeev-Popov procedure provides the correct measure. Thus far, however, the Faddeev-Popov determinant is only defined implicitly by (D.15). In this expression, $\delta(G[A^\Omega])$ appears. Remember that the delta function relation for a function $f(x\ )$ in one variable with $f(a) = 0$ (assume that there is only one such a) obeys

$$\delta(f(x)) = \frac{\delta(x-a)}{f'(x)}.$$

This can be generalized to the functionals $G[A^\Omega]$ that are zero for $\Omega = \Omega_0$, as:

$$\delta(G[A^\Omega]) = \delta(\Omega - \Omega_0) \det \left| \frac{\delta G[A^\Omega](x)}{\delta\Omega(x')} \right|^{-1}. \tag{D.16}$$

Write the gauge-field transformations in their infinitesimal form $(\Omega = 1 + i\epsilon^a T^a)$ and expand $G[A^\Omega]$ to give:

$$G[A^\Omega(x)] = G[A(x)] + \int d^4y\, [M^G(x,y)]\epsilon(y) + \mathcal{O}(\epsilon^2). \tag{D.17}$$

The derivative of this functional with respect to the $\epsilon' s$ is simply the determinant of M^G. Therefore

$$\triangle_{\mathrm{FP}}^G = \det M^G = \int \mathcal{D}\eta\, \mathcal{D}\tilde{\eta}\, \exp i\Big(\int d^4x\, d^4y\, \tilde{\eta}(x) M^G(x,y)\eta(y)\Big), \tag{D.18}$$

where in the last step the *Faddeev-Popov ghost fields*[3] η and $\tilde{\eta}$ were introduced in order to convert the Faddeev-Popov determinant into a Gaussian integral. Finally, also the gauge fixing term $\delta(G[A])$ can be promoted to an exponential, with the result

$$Z = \int \mathcal{D}A\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}\gamma\, \mathcal{D}\eta\, \mathcal{D}\tilde{\eta} e^{i\{S_{YM}[A]+S_F[A,\psi]+S_{\mathrm{gf}}+S_{\mathrm{ghost}}\}}, \tag{D.19}$$

---

[3] These are called ghosts since they are Grassmann even fields with the wrong, namely Fermi-Dirac, statistics. They were introduced into theoretical physics already by R. Feynman [177] and B. DeWitt [118].

and with the gauge fixing and the ghost action terms

$$S_{\text{gf}} = \int d^4x \;\; \gamma^a G^a[A] \tag{D.20a}$$

$$S_{\text{ghost}} = \int d^4x \;\; d^4y \;\; \tilde{\eta}^a(x) M^G_{ab}(x,y) \eta^b(y). \tag{D.20b}$$

To get the final message after all of these formal manipulations: The generating functional (D.19) defines a quantum field theory with fermions ψ, gauge bosons A, ghosts $\eta, \tilde{\eta}$ and the fields γ. The "rules of the game" allow us to derive the Feynman graphs in which all of these entities participate, order by order. The diagrams and the calculational recipes involved with them depend on the chosen gauge. For instance, in the Lorenz gauge $G^a \cong L^a = \partial^\mu A^a_\mu$ the Faddeev-Popov matrix is calculated starting from (D.17)

$$\begin{aligned} L^a[A(\epsilon)] &= \partial^\mu A^a_\mu(\epsilon) = \partial^\mu (A^a_\mu + \partial_\mu \epsilon^a - f^{abc} A^b_\mu \epsilon^b) \\ &= \partial^\mu A^a_\mu + (\delta^a_b \partial^\mu \partial_\mu - f^{abc} A^c_\mu \partial^\mu) \epsilon^b \\ &= L^a[A] + (M^L)^a{}_b \epsilon^b = L^a[A] + \int d^4y \; (M^L(y))^a{}_b \delta^4(x-y) \epsilon^b, \end{aligned}$$

from which

$$(M^L(x,y))^a{}_b = \partial^\mu (\partial_\mu \delta^{ab} - f^{abc} A^c_\mu) \delta^4(x-y),$$

such that in this gauge, the ghost part of the action becomes

$$\begin{aligned} S^{(L)}_{\text{ghost}} &= -\int d^4x \;\; \partial^\mu \tilde{\eta}^a(x) \{ \partial_\mu \eta^a(x) - g f^{abc} A^c_\mu(x) \eta^b(x) \} \\ &= -\int d^4x \;\; \partial^\mu \tilde{\eta}^a(x) D_\mu \eta^a. \end{aligned} \tag{D.21}$$

In the Abelian case, there is no coupling between the ghost and the gauge fields, and the contribution of the ghosts to the path integral is just an (irrelevant) constant. This is the reason why the Feynman rules for QED could be derived before Faddeev and Popov entered the scene. But it should be mentioned that it is not the non-Abelian character of the theory which necessitates non-trivial ghosts. For instance, in the axial gauge the ghosts can be integrated away both in the Abelian and non-Abelian gauge theory; see e.g. [421], (see Sect. 8.1).

In most of the literature, you will find the gauge condition implemented in another way then with the fields γ^a in the action part S_{gf} above. This approach is based on the extension of the Lagrangian by terms with auxiliary fields that may represent the gauge fixing. In Sect. 3.3.4, this was explained for the Lorenz gauge in case of electrodynamics, with the Lagrangian

$$\mathcal{L}_\xi = -\frac{1}{2} \partial^\mu A^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + B(\partial^\mu A_\mu) + \frac{\xi}{2} B^2 + A_\mu j^\mu.$$

Extending (D.14) by auxiliary B^a terms, we also introduce a further integration over these fields:

$$Z = \int \mathcal{D}A\, \mathcal{D}B\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi e^{i\{S_{YM}[A]+\int d^4x\, \left[\frac{\xi}{2}B^2+BG\right]+S_F[A,\psi]\}}.$$

But the B integrations can immediately be executed with (D.4) to yield a term proportional to

$$S'_{\text{gf}} = -\frac{1}{2\xi}\int d^4x\, G^a G^a.$$

Since the most common gauge fixings are linear in the gauge fields (or derivatives thereof), this trick provides a further quadratic term in A^a_μ to the effective action. We furthermore find the link $\gamma^a = -1/(2\xi)G^a$.

The above derivation of the Faddeev-Popov integral can be rendered more geometric by introducing a metric and an affine connection in the space of gauge fields. This has the merit that one directly arrives at the gauge invariant and gauge fixing independent effective action (this term being explained in D.2.2.), first advocated by G.A. Vilkovisky; see e.g. [322].

Gravity: Summing Over Geometries

For gravity the first detailed investigation with functional integrals was that of Misner (1957). For the metric field, the path integral formally becomes

$$Z[g_{\mu\nu}] = \int \mathcal{D}g\; e^{iS[g_{\mu\nu}]}.$$

As in any other theory, the measure $\mathcal{D}g = \prod_{\mu,\nu,x}\; dg_{\mu\nu}(x)$ needs to be defined. This amounts to a summation over all geometries (and maybe also all topologies). But again, as in the Yang-Mills case, one must be aware of the problem of overcounting. In this case, the naive sum over geometries embraces all metrics which are related by diffeomorphism. The Faddeev-Popov method may be employed, but since the diffeomorphism group is not compact, the notion of an invariant group measure is dubious, to put it mildly [36]. In any case, there are various techniques by which functional-integral methods can lead to quantum gravity results, even despite the fact that gravitation is not renormalizable according to dimensional arguments, see e.g. [309].

D.2 Noether and Functional Integrals

The title of this section is obviously nonsense, since E. Noether lived and worked at least one generation before R. Feynman. But what I would like to stress here is how the Noether theorems make their appearance in functional integrals.

D.2.1 Noether Currents and Ward-Takahashi-Slavnov-Taylor Identities

The functional integral is formulated in terms of the classical action. Since Noether symmetries are defined with reference to an action, we expect that these symmetries can be traced more or less directly into the quantum level. Let us assume that we are dealing with a theory with fields Φ_α, defined by its action $S[\Phi]$. Let us first investigate the transformation of the path integral under infinitesimal transformations $\Phi'(x) = \Phi(x) + \delta\Phi$ which we later assume to be symmetry transformations of the classical theory. We begin with the path integral

$$Z[J] = \int \mathcal{D}\Phi \; e^{i(S+J\Phi)} \tag{D.22}$$

where $J\Phi = \int d^4x \; J_\alpha \Phi_\alpha(x)$, and the sum extends over all indices that are necessary to fully specify Φ_α with its Lorentz or internal group transformation behavior. The value of $Z[J]$ does not change when changing the integration variable from Φ to Φ', provided that the measure is invariant ($\mathcal{D}\Phi' = \mathcal{D}\Phi$), which is true in most cases. Thus

$$0 = \delta Z[J] = i \int \mathcal{D}\Phi \; e^{i(S+J\Phi)} \int d^4x \; \big(\frac{\delta S}{\delta \Phi_\alpha} + J_\alpha\big)\delta\Phi_\alpha(x). \tag{D.23}$$

If we take the functional derivative of this expression with respect to $J_\beta(x')$ and then set $J = 0$, we find

$$0 = \int \mathcal{D}\Phi \; e^{iS} \int d^4x \; \left[i\frac{\delta S}{\delta\Phi_\alpha(x)} \Phi_\beta(x') + \Phi_\beta(x')\delta_{\alpha\beta}\delta^4(x,x') \right] \delta\Phi_\alpha(x).$$

In the case that $\delta\Phi$ is a symmetry transformation $\delta_S\Phi$, it has an associated current J^μ_S, and we have from (3.45) $\frac{\delta S}{\delta\Phi_\alpha}\delta_S\Phi_\alpha = -\partial_\mu J^\mu_S$, which can be inserted in the first term of the previous expression

$$0 = \int \mathcal{D}\Phi \; e^{iS} \int d^4x \; \left[-i\partial_\mu J^\mu_S(x) \; \Phi_\beta(x') + \Phi_\alpha(x')\delta^4(x,x')\delta_S\Phi_\alpha(x) \right],$$

or, written in terms of vacuum expectation values

$$i\partial_\mu \langle 0|PJ^\mu_S(x)\Phi_\beta(x')|0\rangle = \langle 0|P\Phi_\alpha(x')\delta^4(x,x')\delta_S\Phi_\alpha(x)|0\rangle.$$

In taking not only one but n functional derivatives of (D.23) with respect to $J_{\alpha_k}(x_k)$, one accordingly arrives at

$$\begin{aligned} &i\partial_\mu \langle 0|PJ^\mu_S(x)\Phi_{\alpha_1}(x_1)\ldots\Phi_{\alpha_n}(x_n)|0\rangle \\ &\qquad = \sum_{k=1}^{n} \delta^4(x - x_k)\langle 0|P\Phi_{\alpha_1}(x_1)\ldots\delta_S\Phi_{\alpha_k}(x)\ldots\Phi_{\alpha_n}(x_n)|0\rangle. \end{aligned}$$

In QED these identities between correlation functions are called Ward-Takahashi identities, because their consequences were already known from work of J.C. Ward and Y. Takahashi of the early 1950's concerning renormalizability issues. Their non-Abelian generalization goes under the name Slavnov-Taylor identities. Generically, they show that in the quantum field theory the Noether currents, located inside of correlation functions, are conserved up to terms depending on the detailed form of the symmetry transformations $\delta_S \Phi$.

D.2.2 Quantum Action and its Symmetries

Definition and Properties of the Quantum Action

A pivotal role in this book about symmetries is played by the classical action. Only if a theory can be derived from minimizing an action can one make use of all the powerful consequences of variational symmetries. The corresponding object on the quantum level is the *quantum action*, also called the "(quantum) effective action" (and sometimes even "effective" action, although the latter could be confused with the action of an effective field theory).

Again, we start of with the generic path integral (D.22). Define the vacuum expectation value of the $\hat{\Phi}^\alpha$ in the presence of the current J as ϕ_J^α:

$$\phi_J^\alpha(x) = \frac{\langle 0|\hat{\Phi}^\alpha(x)0\rangle_J}{\langle 0|0\rangle_J} = -\frac{i}{Z[J]}\frac{\delta}{\delta J_\alpha(x)}Z[J] = \frac{\delta}{\delta J_\alpha(x)}W[J]. \tag{D.24}$$

Here the W-functional is related to $Z[J]$ by

$$Z[J] = e^{iW[J]} \qquad W[J] = -i\ln Z[J]. \tag{D.25}$$

In every textbook on quantum field theory it is shown that while $Z[J]$ enfolds all Feynman graphs, the functional $W[J]$ generates all connected graphs. The relation (D.24) can be inverted to read

$$\phi_J^\alpha(x) = \phi^\alpha \qquad \text{if} \qquad J_\alpha(x) = J_{\phi^\alpha}(x).$$

The quantum action is defined as

$$\Gamma[\phi] = W[J_\phi] - \int d^4x \; J_{\phi^\alpha}(x)\phi^\alpha(x). \tag{D.26}$$

This is a functional of the mean field $\phi^\alpha(x)$ through the implicit dependence of J on the ϕ's. The variational derivative of the quantum action is

$$\frac{\delta\Gamma[\phi]}{\delta\phi^\alpha(x)} = -J_{\phi^\alpha}(x). \tag{D.27}$$

This justifies calling $\Gamma[\phi]$ an "effective action": The values of the $\phi^\alpha(x)$ in the absence of a current J are given by the stationary points of Γ

$$\frac{\delta\Gamma[\phi]}{\delta\phi^\alpha(x)} = 0.$$

This is reminiscent to the derivation of the classical field equations from the action as $\delta S/\delta\phi_c^\alpha = 0$. And indeed, the classical action is the leading term in an $^1/_\hbar$ expansion of the quantum action:

$$\Gamma[\phi] = \frac{1}{\hbar}S[\phi] + \sum_{k=1}^{\infty}(\hbar)^{(k-1)}\Gamma_k[\phi].$$

Symmetries of the Quantum Action

As it turns out, for establishing the renormalizability of a theory, one needs to understand how symmetries of the classical action relate to the symmetries of the quantum action. Hereafter, I will follow the argumentation of [524], Sect. 16.4: Denote again all fields in the theory by Φ and consider infinitesimal transformations $\Phi^\alpha \to \phi^\alpha + \delta\Phi^\alpha[x;\Phi]$. The generating function (D.22) transforms as

$$Z[J] \to \int \mathcal{D}(\Phi + \delta\Phi)\ \exp i\{S[\Phi + \delta\Phi] + J(\Phi + \delta\Phi)\}.$$

If we assume that both the action and the measure are invariant, the previous expression becomes

$$\begin{aligned} Z[J] \to & \int \mathcal{D}\Phi\ e^{\{iS[\Phi] + iJ(\Phi+\delta\Phi)\}} \\ &= Z[J] + i\epsilon \int \mathcal{D}\Phi \int d^4y\ \delta\Phi^\alpha[y;\Phi] J_\alpha(y)\ e^{\{iS[\Phi]+iJ\Phi\}}. \end{aligned}$$

From the invariance of $Z[J]$, we obtain the condition

$$\int d^4y\ \langle\delta\Phi^\alpha(y)\rangle_J J_\alpha(y) = 0$$

where

$$\langle\delta\Phi^\alpha(y)\rangle_J := \frac{1}{Z[J]}\int \mathcal{D}\Phi \int d^4y\ \delta\Phi^\alpha[y;\Phi]\ e^{\{iS[\Phi]+iJ\Phi\}}.$$

Due to (D.27), this condition can be written as

$$\int d^4y\ \langle\delta\Phi^\alpha(y)\rangle_{J_\Phi} \frac{\delta\Gamma[\Phi]}{\delta\Phi^\alpha(y)} = 0, \tag{D.28}$$

revealing that the quantum action is invariant under the infinitesimal transformation $\Phi^\alpha \to \Phi^\alpha + \langle \delta\Phi^\alpha(y)\rangle_{J_\Phi}$. In general, this is not the same invariance transformation as for the classical action. An important exception are linear transformations

$$\delta\Phi^\alpha[x;\phi] = s^\alpha(x) + \int d^4y\, t^\alpha{}_\beta(x,y)\Phi^\beta(y)$$

for which, because of $\langle \Phi^\beta(y)\rangle_J = \Phi^\beta(y)$, we find $\langle \delta\Phi^\alpha(x)\rangle_{J_\Phi} = \delta\Phi^\alpha[x;\phi]$. For these transformations, the quantum action is invariant (provided that both the classical action and the measure are invariant). And indeed the linear transformations are those we encounter in Yang-Mills theories and gravitational theories where

$$\delta\Phi^\alpha = \int d^4y \big[\mathcal{A}^\alpha_r \delta(x-y) + \mathcal{B}^{\alpha\mu}_r \partial_\mu \delta(x-y) + \ldots\big] \cdot \epsilon^r$$

and the coefficients in front of the delta functions are at most linear in the fields.

D.2.3 BRST Symmetries

Symmetries Out of Nowhere?

A major technical advance in dealing with Yang-Mills theories and gravitational theories came with the observation that after gauge fixing, the path integral exhibits a global symmetry. This was discovered by C. Becchi, A. Rouet, R. Stora, and independently by I.V. Tyutin, shortly after the Faddeev-Popov recipe was published. These authors observed that the path integral (D.19) is invariant under the BRST transformations mediated by a constant odd Grassmann number χ

$$\begin{aligned}
\delta_\chi A^a_\mu &= \chi D_\mu \eta^a \\
\delta_\chi \psi &= \chi (T^a \eta^a)\psi \\
\delta_\chi \eta^a &= -\frac{\chi}{2} f^{abc}\eta^b\eta^c \\
\delta_\chi \tilde{\eta}^a &= -\chi\gamma^a \\
\delta_\chi \gamma^a &= 0.
\end{aligned}$$

At first sight these transformations seem a bit patchy, but they do not really come out of nowhere: Acting on the vector fields and the spinors, the transformations δ_χ are recognized as "ordinary" gauge transformation $A^a_\mu \to A^a_\mu + D_\mu\theta^a$ and $\psi \to \psi + (T^a\theta^a)\psi$ with $\theta^a = \chi\eta^a$. The form of the transformations on the other fields is related to the nilpotency of the BRST transformations.

Nilpotency of BRST Transformations

Remarkably, the δ_χ-transformation is nilpotent: $\delta_\chi \delta_{\chi'} X = 0$, where X is any product of the fields from the set $\Psi = \{A^a_\mu, \psi, \eta^a, \tilde{\eta}, \gamma\}$. This can be proved in two steps. Start by investigating the action of a second BRST transformation on the fields themselves. In writing $\delta_\chi \Psi =: \chi s \Psi$ we show that $\delta_\chi s\Psi = 0$. This is obvious for γ and $\tilde{\eta}^a$. For η^a with $s\eta^a = -\frac{\theta}{2} f^{abc}\eta^b\eta^c$:

$$\delta_\chi s\eta^a \propto f^{abc}\chi\{f^{bde}\eta^d\eta^e\eta^c + f^{cde}\eta^b\eta^d\eta^e\} \equiv 0,$$

because of the Grassmann odd nature of the $\eta' s$ and the Jacobi identities of the structure coefficients. For $s\psi$:

$$\begin{aligned}
\delta_\chi s\psi &= \delta_\chi((T^a\eta^a)\psi) = (T^a\delta_\chi\eta^a)\psi - (T^a\eta^a)\delta_\chi\psi \\
&= -(T^a\frac{\chi}{2}f^{abc}\eta^b\eta^c)\psi - (T^a\eta^a)\chi(T^b\eta^b)\psi \\
&= \chi\big(T^aT^b\eta^a\eta^b - \frac{1}{2}T^a f^{abc}\eta^b\eta^c\big)\psi.
\end{aligned}$$

Since the first term in the bracket is anti-symmetric under the exchange of η^a and η^b it may be replaced by $(1/2)\ [T^a, T^b]\ \eta^a\eta^b = (1/2)f^{abc}\ T^c\ \eta^a\eta^b$, and then it cancels against the second term. Finally, for sA^a_μ:

$$\begin{aligned}
\delta_\chi sA^a_\mu &= \delta_\chi\{\partial_\mu\eta^a - f^{abc}A^c_\mu\eta^b\} \\
&= -\frac{\chi}{2}\partial_\mu(f^{abc}\eta^b\eta^c) - \chi f^{abc}(D_\mu\eta^c)\eta^b + \frac{\chi}{2}f^{abc}A^c_\mu f^{bde}\eta^d\eta^e \\
&= -\frac{\chi}{2}f^{abc}\Big\{\partial_\mu(\eta^b\eta^c) + 2(\partial_\mu\eta^c)\eta^b\Big\} \\
&\qquad\qquad + \frac{\chi}{2}f^{abc}\Big\{2\ f^{cde}A^e_\mu\eta^d\eta^b + f^{bde}A^c_\mu\eta^d\eta^e\Big\} \equiv 0.
\end{aligned}$$

The first term vanishes because of the anti-symmetry of the structure constants in the last two indices, the second vanishes due to the Jacobi identity. This concludes the demonstration that the BRST transformation is nilpotent on any of the fields from the set Ψ. Next, take a product of two fields. We find:

$$\begin{aligned}
\delta_\chi(\Psi\Psi') &= (\delta_\chi\Psi)\Psi' + \Psi(\delta_\chi\Psi') \\
&= \chi\{(s\Psi)\Psi' + (-1)^{|\Psi|}\Psi(s\Psi')\} = \chi s(\Psi\Psi').
\end{aligned}$$

Making use of $\delta_\chi(s\Psi) = 0 = \delta_\chi(s\Psi')$, we derive

$$\begin{aligned}
\delta_\chi s(\Psi\Psi') &= (s\Psi)(\delta_\chi\Psi') + (-1)^{|\Psi|}(\delta_\chi\Psi)(s\Psi') \\
&= (s\Psi)(\chi s\Psi') + (-1)^{|\Psi|}(\chi s\Psi)(s\Psi') \\
&= \chi\{(-1)^{|s\Psi|}(s\Psi)(s\Psi') + (-1)^{|\Psi|}(s\Psi)(s\Psi')\} = 0,
\end{aligned}$$

since $|s\Psi| = |\Psi| + 1$. This extends to products of more than two of the Ψ fields and finalizes the proof of the nilpotency of the BRST transformations. The argumentation can be reversed, in the sense that demanding the nilpotency of BRST transformations on the Ψ fields leads to the BRST transformation of the ghost fields (see [470], Sect. 74).

Now let us verify the invariance of (D.19) under the BRST transformations. It was mentioned that the transformation δ_χ, acting on the gauge boson fields and the fermion fields, can be understood as a gauge transformation. Therefore the Yang-Mills action S_{YM} and the fermion part S_F is obviously invariant. What about the (quasi)-invariance of the gauge fixing and the ghost Lagrangians? For the sake of simplicity, let us investigate this for the case of the Lorenz gauge: We have

$$\begin{aligned}
\delta_\chi \mathcal{L}_{\text{gf}} &= \delta_\chi(\gamma^a \partial^\mu A^a_\mu) = \gamma^a \partial^\mu \delta_\chi A^a_\mu = \gamma^a \partial^\mu (\chi D_\mu \eta^a) \\
\delta_\chi \mathcal{L}_{\text{ghost}} &= -\delta_\chi(\partial^\mu \tilde{\eta}^a D_\mu \eta^a) = -\delta_\chi(\partial^\mu \tilde{\eta}^a) D_\mu \eta^a - \partial^\mu \tilde{\eta}^a \delta_\chi (D_\mu \eta^a) \\
&= \chi(\partial^\mu \gamma^a) D_\mu \eta^a,
\end{aligned}$$

where the term $\delta_\chi(D_\mu \eta^a)$ drops out because of the nilpotency of the BRST transformations. Therefore, at the end of the day,

$$\delta_\chi(\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}) = \partial^\mu(\chi \gamma^a D_\mu \eta^a)$$

which is a total derivative. From this we realize that in the Lorenz gauge the gauge fixing condition and the ghost contribution are interwoven in such a way that $(\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}})$ is quasi-invariant. This holds true for any gauge choice; see again [524], Sect. 15.7).

Last but not least, the BRST-invariance proof is complete only if the measure in (D.19) is invariant. This may be verified by computing the Jacobian of the BRST transformations on the set of fields Ψ near the identity:

$$\mathcal{J}_\chi = \operatorname{sdet}\big[\delta^\beta_\alpha + \frac{\partial(\delta_\chi \Psi_\alpha)}{\partial \Psi_\beta}\big] = 1 + \operatorname{str}\big[\frac{\partial(\delta_\chi \Psi_\alpha)}{\partial \Psi_\beta}\big];$$

(for the definition and properties of the supertrace and the superdeterminant see App. B.2.2.) The transformations on the $\tilde{\eta}^a$ and the γ^a fields do not contribute to the trace, which means that the measures $\mathcal{D}\tilde{\eta}^a$ and $\mathcal{D}\gamma^a$ are separately BRST invariant. The remaining terms read

$$\mathcal{J}_\chi = 1 + \operatorname{str}\big[\frac{\partial(\delta_\chi A^a_\mu)}{\partial A^d_\nu} + \frac{\partial(\delta_\chi \eta^a)}{\partial \eta^d}\big] = 1 + \chi \ \operatorname{tr}\big[f^{adc}\delta_{\mu\nu}\eta^c - f^{adc}\eta^c\big] = 1.$$

The BRST transformations are global transformations. They give rise to Noether currents

$$J^\mu_\chi = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\alpha)} \delta_\chi \Psi^\alpha =: \chi j^\mu_B,$$

and thus to the BRST charge $Q = \int j_B^0 d^3x$ which because of $\delta\chi\delta\chi = 0$ satisfies the nilpotency condition

$$Q^2 = 0.$$

The BRST charge is a sublime device to identify the physical states among the Fock-space states $|\Omega\rangle$ generated in the usual way. As will be shown below, the condition $Q|\Omega\rangle = 0$ properly selects the physical states of the system. Therefore, BRST quantization has established itself as a mathematically rigorous approach to quantizing field theories with a local symmetry.

BRST Symmetries as the Quantum Equivalent to Classical Symmetries

Although the BRST symmetry was discovered in the Faddeev-Popov path integral formulation of non-Abelian gauge theories, it developed an independent existence as it became possible to introduce the symmetries and their associated fields already on the level of the classical theory.

Let me outline in which sense "two quantum symmetries are associated with a classical symmetry" [5]; also [34]. In the condensed DeWitt notation, the gauge transformations on the fields $\Phi = \{A, \psi\}$ were written in Sect. 3.3.5 as

$$\delta\Phi^\alpha = \mathcal{R}_r^\alpha \cdot \epsilon^r. \tag{D.29}$$

To simplify matters, assume that there are only bosonic symmetries, such that the symmetry parameters ϵ are even Grassmann variables. Further, assume that the algebra closes on-shell; that is, we are dealing with a closed algebra (not necessarily a Lie algebra). Then the conditions (3.89) read

$$\mathcal{R}_{r,\beta}^\alpha \mathcal{R}_s^\beta - \mathcal{R}_{s,\beta}^\alpha \mathcal{R}_r^\beta = -\mathcal{R}_t^\alpha f^t{}_{rs}.$$

Now introduce for each of the ϵ^r a partner c^r with Grassmann parity opposite to ϵ^r. Thus in this assumed case, the c^r are Grassmann odd. Then the previous closure relations and the Jacobi identity become

$$\mathcal{R}_{s,\beta}^\alpha \mathcal{R}_r^\beta c^r c^s = \frac{1}{2}\mathcal{R}_t^\alpha f^t{}_{rs} c^r c^s \qquad (f^r{}_{st,\alpha}\mathcal{R}_p^\alpha - f^r{}_{qt} f^q{}_{ps}) c^p c^s c^t = 0. \tag{D.30}$$

These relations even hold if in the set of infinitesimal parameters there were Grassmann odd symmetry parameters ϵ^r (together with their Grassmann even partner c^r)–this is one of the strengths of introducing the c^r. Furthermore, in the case of a (super)-Lie algebra with $f^r{}_{st,\alpha} = 0$ the second relation implies the (graded) Jacobi-identity for the structure constants.

But the trick of using the c^r as a bookkeeping device is only the beginning of a fascinating story of defining the quantum symmetries. For this purpose, consider pairs of independent ghost fields c^r and $\tilde{c}^r$. (Very often in the literature the $\tilde{c}^r$ are called anti-ghosts. This could be confusing since they are in no sense the anti-particles of the ghosts c^r.) Define the BRST and anti-BRST transformations (denoted by s and $\bar{s}$)

$$s\Phi^\alpha = \mathcal{R}^\alpha_r c^r \qquad \bar{s}\Phi^\alpha = \mathcal{R}^\alpha_r \tilde{c}^r \tag{D.31a}$$

$$sc^r = -\frac{1}{2}f^r{}_{st}c^s c^t \qquad \bar{s}\tilde{c}^r = -\frac{1}{2}f^r{}_{st}\tilde{c}^s\tilde{c}^t \tag{D.31b}$$

$$s\tilde{c}^r + \bar{s}c^r = -\frac{1}{2}f^r{}_{st}c^s\tilde{c}^t. \tag{D.31c}$$

If one introduces an additional field b^r (corresponding to the γ-fields in the gauge fixing action part (D.20a)) having the same Grassmann parity as the ϵ^r, the previous set of transformation is completed to

$$s\tilde{c}^r = b^r \qquad \bar{s}c^r = -b^r - f^r{}_{st}\tilde{c}^s c^t$$

$$sb^r = 0 \qquad \bar{s}b^r = -f^r{}_{st}\tilde{c}^s b^t - \frac{1}{2}\mathcal{R}^\alpha_q f^r{}_{st,\alpha}c^q\tilde{c}^s c^t.$$

One finds that the transformations s and $\bar{s}$ fulfill the nilpotency relations

$$s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$$

as a consequence of the closure property and the Jacobi condition in the form (D.30). The nilpotency of the generators is indeed equivalent to the closure relation when they act on the fields Φ, while it is equivalent to the Jacobi identity when they act on ghosts and anti-ghosts. Thus, one can define the gauge symmetry either by the classical gauge Eqs. (D.29) completed by the closure and Jacobi relations, or by the set of BRST and anti-BRST transformations together with the nilpotency relations. Observe the course of events: One started from a theory which is invariant under local symmetry transformations and thus has a redundant set of fields. Instead of resolving this redundancy one introduces more fields. In this way, the local symmetry disappears, there is no redundancy any longer; instead there is a global symmetry with a Noether charge that can be used directly to define the Hilbert space of the quantized theory. This equivalence between the local gauge invariance in the original theory and the global nilpotent symmetries remarkably holds in case of closed algebras and therefore not only for Yang-Mills type gauge theories but also for general relativity. Another remarkable result concerns the path integral measure

$$\mathcal{D}\Phi := \mathcal{D}\Phi^\alpha\mathcal{D}c^r\mathcal{D}\tilde{c}^r\mathcal{D}b^r.$$

It turns out that $\mathcal{D}\Phi$ is invariant under s- and $\bar{s}$-transformations iff $f^r{}_{rt} = 0$, and this is true if the symmetry group is compact. This reflects factoring out of the Haar measure in the Faddeev-Popov procedure.

A BRST charge can be established for any gauge symmetry with a Lie algebra $[\Gamma_r, \Gamma_s] = f^t{}_{rs}\Gamma_t$ by a simple recipe: Introduce two sets of anticommuting operators c^r and $\tilde{c}_r$ which obey the commutation relation $\{c^r, \tilde{c}_s\} = \delta^r_s$. (If interpreted as a Poisson bracket, the c and $\bar{c}$ are canonically conjugate variables.) Then

$$Q = c^r(\Gamma_r - \frac{1}{2}f^t{}_{rs}c^s\tilde{c}_t)$$

is nilpotent because of the Jacobi identity of the structure constants and the oddness of the ghosts and anti-ghosts. As charges, the Γ_r generate gauge transformations: $\delta\Psi^\alpha = [\Gamma_r, \Psi^\alpha]\,\epsilon^r = \mathcal{R}^\alpha_r \epsilon^r$. The transformations mediated by Q

$$\delta_Q \Psi^\alpha := \{Q, \Psi^\alpha\} = [\Gamma_r, \Psi^\alpha]\, c^r = s\Psi^\alpha$$
$$\delta_Q c^r := \{Q, c^r\} = -\frac{1}{2} f^r{}_{st} c^s c^t = s c^r$$
$$\delta_Q \tilde{c}_r := \{Q, \tilde{c}_r\} = \Gamma_r - f^t{}_{rs} c^s \tilde{c}_t$$

reproduce previous results as in (D.31). Define the ghost number operator

$$N_g = \sum_{r=1}^{\dim G} c^r \tilde{c}_r .$$

Its commutator with the BRST charge is $[N_g, Q] = Q$. All those states $|\Phi\rangle$ of a Hilbert space are called BRST invariant for which $Q|\Phi\rangle = 0$. Because of the nilpotency of Q, one traces two kinds of BRST-invariant states: (1) any state of the form $|\Phi\rangle = Q|\Omega\rangle$. The states $|\Phi\rangle$ and $|\Omega\rangle$ differ by one unit in ghost number. The state $|\Phi\rangle$ has zero norm because $\langle\Omega|Q^\dagger Q|\Omega\rangle = \langle\Omega|Q^2|\Omega\rangle = 0$. (2) states $|\Phi\rangle$ with

$$Q|\Phi\rangle = 0 \qquad |\Phi\rangle \neq Q|\Omega\rangle \qquad \text{for all } \Omega \text{ of the Hilbert space;}$$

these are the physically relevant ones. It also makes sense to term all those states $|\Phi\rangle$ and $|\Phi'\rangle$ equivalent for which $|\Phi'\rangle = |\Phi\rangle + Q|\Omega\rangle$. The equivalence classes are called BRST cohomology classes[4]. Now consider specifically all those states $|\Phi\rangle^0$ which do have zero ghost charge. These states are annihilated by the $\tilde{c}_r$ and we have $Q|\Phi\rangle^0 = c^r\Gamma_r|\Phi\rangle^0 = 0$. Therefore these BRST-invariant states are also invariant under the action of the Lie algebra.

Antibracket, Antifields and Gauge Theory Quantization

The title of this (admittedly small) subsection is identical with that of a review article [226] on what is also known as the Batalin-Vilkovisky quantization–treated likewise in the textbook [524]. It can be thought of as a further generalization of the previously sketched BRST formalism for gauge theories with closed algebras to those with an open algebra or with so-called reducible gauge symmetries. Thus, it seems to be the most general quantization procedure for field theories invariant under those symmetry transformations (D.29) that close as a group. It has its merits specifically in gravity and supergravity.

[4] Indeed there is a large similarity of the BRST operator Q and the de Rham operator d. Therefore, the previous considerations mimic the distinction of closed and of exact forms; see App. E.2.4.

D.2.4 Fujikawa: Fermionic Path Integrals and Anomalies

In the previous sections, we came several times to the question of whether the path-integral measure with respect to the fermions is invariant under certain transformations. The answer depends on the type of transformations. This sounds obvious, but was not really deeply investigated before the work of K. Fujikawa in 1979 [199]. In his textbook, A. Zee tells us that some time ago, you could find people that were not fond of the path-integral approach, and that they felt vindicated in that the path integrals seemingly could not exhibit the breaking of chiral invariance; [567], p. 278. Fujikawa however realized that a chiral transformation gives rise to a change in the measure of the path integral over the fermion fields, and that the Jacobian yields exactly the term historically found from triangle diagrams. In deriving this, Fujikawa worked with the Euclidean path integral[5], because in four Euclidean dimensions the Dirac operator is Hermitean, and this allows for a rigorous derivation of the anomaly. For an elaborate account of anomalies in quantum field theory see [37].

Consider a unitary transformation on fermionic fields

$$\psi(x) \to U(x)\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)\ (\gamma^0 U^\dagger(x)\gamma^0).$$

Since for fermionic variables, the measure transforms with the inverse of the transformation matrix (see App. B.2.3.), we get

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to (\text{Det}\,\mathcal{U}\ \text{Det}\,\bar{\mathcal{U}})^{-1}\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

where

$$\mathcal{U}_{xn,ym} := U(x)_{nm}\delta^4(x-y) \qquad \overline{\mathcal{U}}_{xn,ym} := [\gamma^0 U^\dagger(x)\gamma^0]_{nm}\delta^4(x-y).$$

Here the indices (n,m) run over Dirac spin indices and possible species ("flavor") indices. In case that $U(x)$ is a unitary non-chiral transformation $\mathcal{U}(x) = \exp\{i\lambda(x)T\}$ with T a Hermitean matrix not involving γ^5, we have $\overline{\mathcal{U}}\mathcal{U} = 1$. This entails $\text{Det}\ \mathcal{U}\ \text{Det}\ \overline{\mathcal{U}} = 1$, and therefore for non-chiral transformations, the measure is invariant. This is true in particular for gauge transformations where T is any one of the gauge group generators T^a. For chiral transformations, we write

$$\mathcal{U}(x) = \exp\{i\gamma^5\lambda(x)T\}.$$

Since γ^5 anti-commutes with γ^0, the matrix $\mathcal{U}$ is (pseudo)-Hermitean:

$$\overline{\mathcal{U}} = \mathcal{U}$$

and the measure picks up a factor $(\text{Det}\ \mathcal{U})^{-2}$: $\mathcal{D}\psi\mathcal{D}\bar{\psi} \to (\text{Det}\ \mathcal{U})^{-2}\mathcal{D}\psi\mathcal{D}\bar{\psi}$. We may calculate

$$\text{Det}\,\mathcal{U} = \exp\text{Tr}\ \{\ln\mathcal{U}\} = \exp\text{Tr}\ \{\ln(1+(\mathcal{U}-1))\} \sim \exp\text{Tr}(\mathcal{U}-1),$$

---

[5] I did not mention the technical trick of a "Wick rotation" $t \leftrightarrow i\tau$ which serves to provide a path integral with a better behavior, but is not completely well defined because of the question of whether an analytical continuation always exists.

which holds for infinitesimal chiral transformations

$$[\mathcal{U}-1]_{xn,ym} \propto i\lambda(x)[\gamma^5\, T]_{nm}\delta^4(x-y).$$

Therefore,

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to e^{i\int d^4x\;\lambda(x)\mathcal{A}(x)}\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

with the anomaly function

$$\mathcal{A}(x) = -2\mathrm{Tr}_{(n,m)}\,[\gamma^5\, T]\delta^4(x-y)$$

and with the trace to be taken both over Dirac and flavor indices.

The variation of the generating function is

$$\begin{aligned}&\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{i\int d^4x\;\mathcal{L}(x)}\\ &\qquad \to \mathcal{D}\psi\mathcal{D}\bar{\psi} \times e^{i\int d^4x\;\lambda(x)\mathcal{A}(x)} \times \exp\{i\int d^4x\;[\mathcal{L}(x)-\lambda(x)\partial_\mu J^\mu_A]\},\end{aligned}$$

where $J^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$ is the classical axial current (6.35). In order that the generating function is invariant one must choose $\partial_\mu J^\mu_A = \mathcal{A}(x)$.

Now of course the tricky point is to calculate the anomaly function $\mathcal{A}(x)$. As a matter of fact, it is not well defined by the previous expression, since the trace vanishes and the delta function is infinite. Therefore it needs to be regularized. Since this is not intended to become a book on quantum field theory, I will not replicate the regularization procedure here (see the original article by Fujikawa or Sect. 22.2 in Weinberg's book) but only state the result

$$\mathcal{A}(x) = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}_a F^{\rho\sigma}_b\;\mathrm{Tr}(T^aT^bT)$$

where 'Tr' is the trace over the flavor indices.

Appendix E
*Differential Geometry

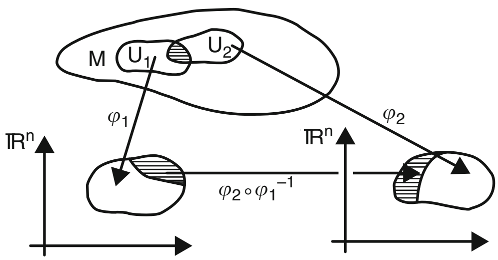
This appendix introduces more or less informally and rather sketchy the differential geometric notions that can help to understand gauge and gravity theories on an abstract level. This demonstrates anew the "unreasonable effectiveness of mathematics in the natural sciences" (E. Wigner). For further details, and especially for proofs, I refer to [317] (by many regarded as the most comprehensive description of the field on an advanced level), [143] (describing in a compact way the full set of tools needed in mathematical physics), [187] (maybe more mathematics- than physics-oriented), [285] (high-level with detailed proofs, but with many examples, also from physics), [459] (a strongly physics-oriented overview), [499] (an excellent, but rather dense presentation of fibre bundles in fundamental physics). A rich arsenal of material, especially if it comes to the use of differential forms in contemporary theoretical physics is found in [259]. A good introductory chapter on differential geometry, topology and fibre bundles can be found in [37], and the relations between notions in physics and those of topology and geometry are exemplified in [369]. In most of this literature you find topics that are not treated in this appendix, especially the astounding relations which exist between topological invariants and the local geometry of manifolds.

But before plunging into manifolds and fibre bundles, let me motivate shortly why in physics these mathematical concepts are required. The arena for talking about physics is (still) space and time. It seems natural to represent spacetime by a set of 'spacetime points'[1]. But in physics we need more structure, since we want to talk about space-time regions, and about points that are in a certain neighborhood, or even ordered. This necessitates to give spacetime a topology. Since the days of Newton, physics became successful in describing dynamical evolution in terms of differential equations in spacetime. Therefore an additional structure (differentiability) needs to be imposed on the mathematical spaces involved. In order to identify events in spacetime we use coordinates, but of course these are only a technical means. The dynamics of a physical system is independent of coordinates, and the description in terms of coordinates may even hide certain structural features of the system or pretend

[1] Quotation marks are used here, because general relativity reveals that this reading is to naive, and that importance has to be given to relations between events; see e.g. [441].

K. Sundermeyer, *Symmetries in Fundamental Physics,*
Fundamental Theories of Physics 176, DOI 10.1007/978-94-007-7642-5,

Fig. E.1 Manifold



features which are pure coordinate effects. Adding coordinate independence to the previous structures immediately leads to the mathematical notion of differentiable manifolds. In present-day fundamental physics we meet further structures like gauge groups, connections and curvatures. And as will be pointed out in this appendix, the appropriate arena to deal with these structures are fibre bundles.

E.1 Differentiable Manifolds

E.1.1 From Topological Spaces to Differentable Manifolds

It is assumed that the reader is acquainted with the notion of a topological space as a structure on which one can define a neighborhood and continuous functions. A *homeomorphism* between two topological spaces is a 1-1 map $\varphi : X \to Y$ for which both φ and its inverse φ^{-1} are continuous. If φ and φ^{-1} are continuously differentiable then φ is called a *diffeomorphism.*

A D-dimensional manifold M^D is a topological space[2] that locally has the properties of a D-dimensional Euclidean space $\mathbb{R}^D$: A neighborhood of a point in M^D can continuously be mapped in a one-to-one way to the neighborhood of a point in $\mathbb{R}^D$. To be more precise, introduce a *chart* $(U_\alpha, \varphi_\alpha)$ as a homeomorphism φ_α from an open set $U_\alpha \subset M^D$ into an open set $R_\alpha \subset \mathbb{R}^D$. Two charts are compatible if the overlap maps are diffeomorphisms $(\varphi_1 \cdot \varphi_2^{-1} \in C^\infty,\ \varphi_2 \cdot \varphi_1^{-1} \in C^\infty)$ unless $U_1 \cap U_2 = \emptyset$; see Fig. E.1. A set of compatible charts covering M^D is called an atlas. In every chart the manifold can be equipped with a coordinate system: for $x \in M^D$ the coordinates are $x^\mu = \varphi(x) \in \mathbb{R}^D$.

The naming makes it clear what one is aiming at. For instance the surface of a sphere, although not being homeomorphic to a plane, locally has enough smoothness to be mapped into an atlas. One chart is not sufficient since there will always be a point on the sphere that cannot be projected to the plane.

In this appendix I will only treat finite-dimensional manifolds. One possibility of extending the notion of manifolds to infinite dimensions is to consider Banach

[2] More rigourosly, it is assumed that it is a connected, Hausdorff topological space. But since the following presentation is without proofs, these technicalities are mentioned here only for purists.

manifolds modeled on Banach spaces. It is also assumed that we are throughout this appendix dealing with C^∞ manifolds (as in the previous definition). In certain contexts it might suffice that the charts are C^k-related. (In the sequel all maps in C^∞ manifolds will be called smooth.) Also complex manifolds are investigated in mathematics and applied to modern theoretical physics (catchword: Kähler manifolds). In these the transition functions are required to be analytic.

E.1.2 Tensor Bundles

On a manifold one can erect tensor bundles as "superstructures" by starting with defining the tangent and cotangent spaces of a manifold.

Tangent Bundle and Vector Fields

We are interested in the notion of vectors on a manifold M (henceforth I will mostly drop the index for the dimension of the manifold and for the Euclidean space). The idea is to introduce these as tangent vectors of curves 'through' $x \in M$: A curve through[3] a point x is a smooth mapping of an interval $I = [0,1] \subset \mathbb{R}$ to the manifold:

$$C : \mathbb{I} \to M \qquad t \mapsto C(t) \qquad \text{with} \qquad C(0) = x.$$

The coordinates of this curve are $x^\mu(C(t))$, and the tangent vector to the curve is

$$\frac{d}{dt} x^\mu(C(t)).$$

Since one can have more then one curve with $C(0) = x$, the proper definition is: *A tangent vector* at $x \in M$ is an equivalence class of curves in M, where the equivalence relation between two curves is that they are tangent at the point x.
Another–equivalent–definition is to understand a tangent vector as a directional derivative: Consider functions $f \in \mathcal{F}M$ that is $f : M \to \mathbb{R}$. The change of f along a curve is given by

$$\frac{d}{dt} f(C(t)), \qquad \text{locally} \qquad \frac{\partial}{\partial x^\mu} f \, \frac{dx^\mu(C(t))}{dt}.$$

In defining

$$X = (X^\mu \partial_\mu) \qquad \text{with} \qquad X^\mu = \frac{dx^\mu(C(t))}{dt}$$

we can write $\frac{d}{dt} f(C(t)) = Xf$. For every point along the curve we take this expression to define the differential operator X_x as the tangent vector to the manifold

[3] Observe that a curve is a map, thus what is meant, is the image of the curve through x.

in $x \in M$. All tangent vectors at a point in the manifold can be shown to build a vector space $\mathfrak{X}_x M$ isomorphic to $\mathbb{R}^D$. The natural basis in $\mathfrak{X}_x M$ is the coordinate or *holonomic* basis $\{\partial_\mu\}$. But of course any other (*anholonomic*) basis $\{e_I\}$ can be chosen where $\partial_\mu = e^I_\mu \, e_I$ with an invertible $D \times D$ matrix e^I_μ–in physical contexts called a D-bein. Every element $X_x \in T_x M$ can be written as

$$X_x = (X^\mu \partial_\mu)_x = (X^I e_I)_x \qquad \text{with} \qquad X^\mu = X^I \, e^\mu_I .$$

The X^μ correspond to what in the component language is called a covariant vector. This can directly be verified by calculating, how the vector changes from one coordinate chart to another overlapping one.

The space $\mathfrak{X}_x M$ can be given a Lie algebra structure:

$$[,] : \mathfrak{X}_x M \otimes \mathfrak{X}_x M \to \mathfrak{X}_x M \qquad \text{with}$$
$$[X_x, Y_x] = Z_x = (Z^\nu \partial_\nu)_x = \left[(X^\mu Y^\nu_{,\mu} - Y^\mu X^\nu_{,\mu}) \partial_\nu \right]_x . \tag{E.1}$$

This algebra is infinite-dimensional and closely related to the group of diffeomorphisms of the manifold. The algebra (E.1), evaluated on the basis vectors, results in $[\partial_\mu, \partial_\nu] = 0$ for the holonomic basis, but in any anholonomic basis we have $[e_I, e_J] = C^K{}_{IJ} \, e_K$ instead.

Finally define the *tangent bundle* $TM = \cup_x T_x M$ in which at each point in the (base) manifold the tangent space at this point is attached as a fibre. (More on fibre bundles in the last part of this appendix.) Then the space of all vector fields $\mathfrak{X}M$ is interpreted as a section $s : M \to TM$ in this bundle, where s is a C^∞ map.

Cotangent Bundle and One-Forms

The vector space $T^*_x M$, called the cotangent space to the manifold at one of its points x, is defined as the vector space dual to $T_x M$. That is, if as a basis in $T_x M$ one chooses the holonomic $(\partial_\mu)_x$ and calls $(dx^\mu)_x$ the dual basis in $T^*_x M$ we have by definition $dx^\mu(\partial_\nu) = \langle dx^\mu, \partial_\nu \rangle = \delta^\mu_\nu$. For generic elements $X \in T_x M$ and $\omega \in T^*_x M$ holds

$$\alpha(X) = \langle \alpha, X \rangle = \alpha_\mu X^\nu \langle dx^\mu, \partial_\nu \rangle = \alpha_\mu X^\mu .$$

This result is independent of the basis chosen. For anholonomic bases ϑ^I in $T^*_x M$ holds $\langle \vartheta^I, e_J \rangle = \delta^I_J$. In this case the anholonomic and holonomic bases are related by $\vartheta^I = e^I_\mu dx^\mu$.

In analogy to previous definitions one obviously can build the vector space $\mathfrak{X}^* M$. The bundle formed from a manifold and its pointwise attached cotangent spaces is called the *cotangent bundle*. Sections of the cotangent bundle are called *one-forms*. The components of a one-form are identified with those of a contravariant vector[4]. Any cotangent bundle is a symplectic manifold; see App. E.1.4.

[4] … that is, a contravariant vector is not a vector in the language and domain of smooth manifolds.

Tensor Fields

From the tangent and cotangent space of a manifold M at a point x (for short written as $T = T_xM$ and $T^* = T^*_xM$) one can *via* multilinear mappings

$$\underbrace{T^* \times T^* \times \ldots T^*}_{\text{r}} \times \underbrace{T \times T \times \ldots T}_{\text{s}} \to \mathbb{R}$$

define r-times covariant and s-times contravariant tensor spaces of the manifold M at a point x ; the pair (r, s) defining the order of the tensor space. In taking together tensor spaces of a given order from all points of the manifold one obtains the tensor bundle T^r_sM. Sections in this bundle are *tensor fields*.

As common in most of the mathematical literature, a tensor field is understood to take vector fields and one-forms as arguments to deliver (locally) its components: For $t^r_s \in T^r_sM$

$$t^r_s = t^{I_1 \cdot\cdot I_r}_{J_1 \cdot\cdot J_s}\, \vartheta^{J_1} \otimes \cdot \cdot \otimes \vartheta^{J_s} \otimes e_{I_1} \otimes \cdot \cdot \otimes e_{I_r} \qquad t^r_s(X_1, \cdot\cdot, X_r, \omega_1, \cdot\cdot, \omega_s) = t^{I_1 \cdot\cdot I_r}_{J_1 \cdot J_s}.$$

In the majority of considerations in this book, the notions of a tangent and a cotangent bundle suffice. Furthermore, a prominent role is played by bundles $\Omega_pM \subset T^0_pM$. The bundles Ω_pM contain all totally antisymmetric p-maps $T^* \times T^* \times \ldots T^* \to \mathbb{R}$. Sections in Ω_pM are called (differential) p-forms. More about differential forms below in E.2.

Pull-Back and Push-Forward

Frequently we are asking for the transformation of vector fields or forms (or tensor fields in general) of a manifold M with respect to diffeomorphisms $\varphi : M \to N$. The mappings

$$\varphi_* : T^r_sM \to T^r_sN \qquad \text{and} \qquad \varphi^* : T^r_sN \to T^r_sM$$

are called *push-forward* and *pull-back*, respectively. Instead of defining them generically here, let me state what they are in case of vector fields and one-forms: For $\varphi_* : T_xM \to T_{\varphi(x)}N$ define for $X \in T_xM$ the push-forward φ_*X by

$$\varphi_*X\,[f] = X\,[f \circ \varphi] \qquad \text{in coordinates} \qquad \varphi_*X^\nu(y) = X^\mu \frac{\partial y^\nu}{\partial x^\nu}(x).$$

For $\varphi^* : T^*_{\varphi(x)}N \to T^*_xM$ define the pull-back in such a way that for an arbitrary vector field $X = X^\nu\partial_\nu \in T_xM$ and a one-form $\alpha = \alpha_\mu dx^\mu \in T^*_xM$ holds $\langle\varphi^*\alpha, X\rangle = \langle\alpha, \varphi_*X\rangle$. By comparing $\langle\varphi^*\alpha, X\rangle = \varphi^*\alpha_\mu(x)X^\mu(x)$ with $\langle\alpha, \varphi_*X\rangle = \alpha_\nu(y)$ the pull-back for the components is

$$\varphi^*\alpha_\mu(x) = \alpha_\nu(y(x))\frac{\partial y^\nu}{\partial x^\mu}.$$

This can readily be generalized to p-forms. The pull-back on forms has the properties

$$d(\varphi^*\alpha) = \varphi^*d\alpha\ , \quad \varphi^*(\alpha_p \wedge \beta_q) = \varphi^*\alpha_p \wedge \varphi^*\beta_q, \quad (\varphi \cdot \varphi')^* = \varphi^{'*} \cdot \varphi^*. \tag{E.2}$$

E.1.3 Flows and the Lie Derivative

Vector fields on a manifold generate flows: Let C be a curve on the manifold parametrized as $C(t) \in M$ and let the vector $X(C(t))$ be tangent to the curve, that is $\dot{C} = X(C(t))$. Then C is called the integral curve of the vector field X. This obviously characterizes dynamical systems in which an integral curve is called a trajectory. Given a vector field it is always possible to find locally a unique integral curve.

Flows are also called one-parameter groups of diffeomorphisms: Let Φ_t be (for every t) a diffeomorphism $\Phi_t : M \to M$ with $\Phi_0 = \mathrm{id},\ \Phi_t \cdot \Phi_{t'} = \Phi_{t+t'}$. The vector field associated to this diffeomorphism is

$$\frac{d}{dt}\Phi_t(x)\Big|_{t=0} = X_x^{\Phi}. \tag{E.3}$$

A vector field X on a manifold induces a differential operator $\pounds_X$–called the *Lie derivative*, and having the meaning of the derivative in the direction of the vector field. It is defined to operate on all tensor fields on the manifold M and transforms a geometric object of a certain kind (in terms of its covariant and contravariant indices) into an object of the same kind.

In the simplest case it acts on functions $f \in \mathcal{F}M$:

$$(\pounds_X f)_x = \frac{d}{dt}\Big(\big(f(\Phi_t(x))\big)\Big|_{t=0}$$

or in coordinates/charts for which $X = X^\mu \partial_\mu$

$$\pounds_X f = X^\mu \frac{\partial f}{\partial x^\mu}.$$

The differential operator $\pounds_X$ has all properties of a derivative, that is (i) $\pounds_X(f_1 + f_2) = \pounds_X f_1 + \pounds_X f_2$ (linearity) (ii) $\pounds_X(f_1 \cdot f_2) = f_1 \pounds_X \cdot f_2 + f_2 \cdot \pounds_X f_1$ (Leibniz rule).

The Lie derivative of a vector field Y with respect to a vector field X is defined to be the vector field Z by

$$Z = \pounds_X Y = \pounds_X \pounds_Y - \pounds_Y \pounds_X := [X, Y]\,.$$

In a chart with $X = X^\mu \partial_\mu,\ Y = Y^\nu \partial_\nu$

$$[X, Y]\ f = \left[X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right] \frac{\partial}{\partial x^\nu}\ f.$$

This $[\cdot, \cdot]$ composition is identical with the one in (E.1).

The Lie derivative on a p-form is defined in the next section.

The invariance of a tensor field $T \in T_s^r M$ under a flow mediated by a vector field X is synonymous with $\pounds_X T = 0$.

E.1.4 Symplectic Manifolds

Definition: A symplectic manifold (M, ω) is a smooth manifold M with a symplectic form ω, that is a non-degenerate closed two-form.

Although differential forms are presented in more details in the next section, the terms 'two-form', 'closed' and 'non-degenerate' can already be explained here: A two-form is related to an antisymmetric tensor field in $T^*M \otimes T^*M$. In a coordinate basis it has the generic form $\omega = \frac{1}{2}\omega_{\mu\nu}(dx^\mu dx^\nu - dx^\nu dx^\mu) =: \frac{1}{2}\omega_{\mu\nu}\ dx^\mu \wedge dx^\nu$. The two form is *closed* if its exterior derivative (explained below) vanishes. In components the exterior derivative of ω is: $d\omega = \frac{1}{3!}\omega_{\mu\nu,\lambda}\ dx^\lambda \wedge dx^\mu \wedge dx^\nu$. Therefore closedness is synonomous with $\omega_{[\mu\nu,\lambda]} = 0$ (in all coordinate systems). In a coordinate-free language, a two-form takes two vectors as input and outputs a real number: $\omega(X, Y) \mapsto \mathbb{R}$. The two-form is defined as *non-degenerate* if $\omega(X, Y) = 0$ for $\forall\ Y$ implies $X = 0$. In the local coordinates used before, non-degeneracy is synonymous to $\det(\omega_{\mu\nu}) \neq 0$.

Prominent examples of symplectic manifolds are the cotangent bundles T^*M of a manifold. In order to keep contact to applications in physics, we select especially T^*Q, the phase space to a configuration space Q in classical mechanics. In coordinates (q^k, p_k) on T^*Q, the canonical symplectic form is $\omega = dp_k \wedge dq^k$.

Local symmetries in fundamental physics typically imply that the dynamics happens on a subset of the phase space. This subset is in general not a symplectic manifold, but only a pre-symplectic one, characterized by a two-form which is closed but degenerate.

E.2 Cartan Calculus

E.2.1 Differential Forms

Differential p-forms ω on a D-dimensional manifold are elements of $\Omega_p M \subset T_p^* M$: If ϑ^I is chosen as the basis in the cotangent space T^*M, a basis in $\Omega_p M$ is provided by

$$\vartheta^{I_1 I_2 \ldots I_p} := \vartheta^{[I_1} \otimes \vartheta^{I_2} \otimes \ldots \otimes \vartheta^{I_p]}$$

where the [] denotes complete anti-symmetrization including a factorial, e.g.

$$\vartheta^{IJ} = \frac{1}{2!}(\vartheta^I \otimes \vartheta^J - \vartheta^J \otimes \vartheta^I).$$

A p-form thus has the generic appearance

$$\alpha_p = \frac{1}{p!}\hat{\alpha}_{I_1 I_2 \ldots I_p}\vartheta^{I_1 I_2 \ldots I_p}.$$

with $\hat{\alpha}_{I_1 I_2 \ldots I_p} \in \mathcal{F}M$. (If it is clear from the context neither the subscript indicating the form degree nor the hat symbol on the components will be mentioned.) In the space

$$\Omega M := \bigoplus_{p=0}^{D} \Omega_p M$$

one defines several operations

- Exterior product $\wedge : \Omega_p M \times \Omega_q M \to \Omega_{p+q} M$ defined on the basis as

$$\vartheta^{I_1 I_2 \ldots I_p} \wedge \vartheta^{J_1 J_2 \ldots J_q} := \vartheta^{I_1 I_2 \ldots I_p J_1 J_2 \ldots J_q}$$

  from which

$$\alpha_p \wedge \beta_q = (-1)^{pq} \alpha_q \wedge \beta_p .$$

- Exterior derivative $d : \Omega_p M \to \Omega_{p+1} M$ with

$$df = f_{,\mu} dx^\mu$$

  and the properties

$$\begin{aligned} d(\alpha_p \wedge \beta_q) &= d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q \\ dd\alpha &= 0 \\ d\alpha_D &= 0. \end{aligned}$$

- Contraction with a vector field $i_Y : \mathfrak{X} \times \Omega_p M \to \Omega_{p-1} M$
  Let the vector field $Y \in \mathfrak{X}$ locally be given by $V = Y^I e_I$. Then the actions on functions and basic one-forms defined by

$$i_Y f = 0 \qquad i_Y \vartheta^I = Y^I$$

  allow to calculate the action of i_Y on any differential form. One easily proves that

$$\begin{aligned} i_Y(\alpha_p \wedge \beta_q) &= i_Y \alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge i_Y \beta_q \\ i_X i_Y + i_Y i_X &= 0 \\ i_{fY} &= f i_Y . \end{aligned}$$

  In denoting $i_I := i_{e_I}$ one gets the useful formulae

$$\begin{aligned} i_I \vartheta^J &= \delta_I^J \\ \vartheta^I \wedge i_I \alpha_p &= p\, \alpha_p \\ i_I(\vartheta^I \wedge \alpha_p) &= (D-p)\alpha_p \\ i_{I_p} i_{I_{p-1}} \ldots i_{I_1}\, \alpha_p &= \hat{\alpha}_{I_1 I_2 \ldots I_p} . \end{aligned}$$

- Lie derivative with respect to a vector field $\mathcal{L}_Y : \mathfrak{X} \times \Omega_p M \to \Omega_p M$

$$\mathcal{L}_Y \alpha := (i_Y d + d i_Y)\alpha$$
$$\mathcal{L}_Y(\alpha_p \wedge \beta_q) = \mathcal{L}_Y \alpha_p \wedge \beta_q + \alpha_p \wedge \mathcal{L}_Y \beta_q$$

from which

$$\mathcal{L}_Y d = d\mathcal{L}_Y \qquad \mathcal{L}_Y i_Y = i_Y \mathcal{L}_Y \qquad i_{[X,Y]} = [i_X, \mathcal{L}_Y].$$

E.2.2 Differentiation with Respect to a Form

Let F be a function of a form α, then we take the convention

$$\delta F = \delta\alpha \wedge \frac{\partial F}{\partial \alpha},$$

that is the variation is always to the left. This gives for instance to

$$F = \alpha_p \wedge \beta_q : \qquad \frac{\partial F}{\partial \alpha_p} = \beta_q \qquad \frac{\partial F}{\partial \beta_q} = (-1)^{pq}\alpha_p.$$

This convention has also to be obeyed for the chain rule:

$$F(\alpha) = f(g(\alpha)) : \qquad \frac{\partial F}{\partial \alpha} = \frac{\partial g}{\partial \alpha} \wedge \frac{\partial f}{\partial g}.$$

The maps $\mathrm{Der} := \{d, i_Y, \mathcal{L}_Y\}$ act on a function of forms as

$$\mathrm{Der}\ \ F(\alpha) = \mathrm{Der}\ (\alpha) \wedge \frac{\partial F}{\partial \alpha}.$$

E.2.3 Hodge Duality*

The Hodge duality operation can be defined for orientable (semi)-Riemannian manifolds. A manifold M^D is *orientable*, if there exists a global non-vanishing D-form

$$\alpha_D = \frac{1}{D!}\hat{\alpha}_{I_1 I_2 \ldots I_D}\vartheta^{I_1 I_2 \ldots I_D}.$$

A metric $g = g_{IJ}\vartheta^I \otimes \vartheta^J$ is an element of $T_2^* M$. On an orientable semi-Riemannian manifold the canonical D-form (volume form) is

$$\eta := \sqrt{|\, g\,|}\vartheta^1 \wedge \vartheta^2 \ldots \wedge \vartheta^D. \tag{E.4}$$

The Hodge $*$ operation is a mapping $*: \Omega_p M \to \Omega_{D-p} M$ defined on the chain of forms (occasionally called *Trautmann forms*)

$$\eta = *1, \quad \eta^I = *\vartheta^I, \quad \ldots, \quad \eta^{IJK\ldots} = *\vartheta^{IJK\ldots}, \quad \ldots, \quad \eta^{I_1 I_2 \ldots I_D} = *\vartheta^{I_1 I_2 \ldots I_D}.$$

Sometimes it is more useful to work with the forms $\eta_{IJK\ldots}$. They are successively derived from the D-form η as $\eta_I = i_I \eta, \ \ \eta_{IK} = i_K i_I \eta, \ldots,$

$$i_I \eta_{I_1 I_2 \ldots I_p} = \eta_{I_1 I_2 \ldots I_p I}$$

such that $\eta_{I_1 I_2 \ldots I_D}$ is the Levi-Civita tensor in D dimensions. The set of η's may serve as bases of D, D-1, .. forms. They satisfy the identities

$$\vartheta^I \wedge \eta_{I_1 I_2 \ldots I_p} = p\, \delta^I_{[I_p} \eta_{I_1 I_2 \ldots I_{p-1}]};$$

specifically (and especially in four dimensions) one often needs

$$\begin{aligned}
\vartheta^I \wedge \eta_J &= \delta^I_J \eta \\
\vartheta^I \wedge \eta_{JK} &= \delta^I_K\, \eta_J - \delta^I_J\, \eta_K \\
\vartheta^I \wedge \eta_{JKL} &= \delta^I_J\, \eta_{KL} + \delta^I_K\, \eta_{LJ} + \delta^I_L\, \eta_{JK} \\
\vartheta^I \wedge \eta_{JKLM} &= \delta^I_M\, \eta_{JKL} - \delta^I_L\, \eta_{JKM} + \delta^I_K\, \eta_{JLM} - \delta^I_J\, \eta_{KLM}.
\end{aligned}$$

Without proof I state the following relations (σ being the signature of the metric):

$$*(*\alpha_p) = (-1)^{p(D-p)+\sigma}\ \alpha_p \tag{E.5a}$$

$$\alpha_p \wedge *\beta_p = \beta_p \wedge *\alpha_p \tag{E.5b}$$

$$i_I * \alpha = (-1)^p * (\vartheta_I \wedge \alpha) \tag{E.5c}$$

$$\vartheta_I \wedge *\alpha = (-1)^{p+1} * i_I \alpha. \tag{E.5d}$$

Some Useful Variations

In dealing with Lagrangians and symmetries one typically performs variations. We need

$$\delta\eta_{I_1 I_2 \ldots I_p} = \delta\vartheta^I \wedge \eta_{I_1 I_2 \ldots I_p I} = \delta\vartheta^I \wedge i_I \eta_{I_1 I_2 \ldots I_p} \tag{E.6a}$$

$$\delta(\alpha \wedge *\beta) = \delta\alpha \wedge *\beta + \alpha \wedge *\delta\beta + \alpha \wedge (\delta\vartheta^I \wedge i_I * \beta) - \alpha \wedge *(\delta\vartheta^I \wedge i_I \beta) \tag{E.6b}$$

$$\delta(i_I \alpha) = i_I \delta\alpha - (i_I \delta e^J) \wedge i_J \alpha. \tag{E.6c}$$

One often needs (E.6b) for expressions $\delta(\alpha \wedge *\alpha)$. By the use of (E.5b) this becomes

$$\delta(\alpha \wedge *\alpha) = 2\delta\alpha \wedge *\alpha + \delta\vartheta^I \wedge [i_I(\alpha \wedge *\alpha) - 2 i_I \alpha \wedge *\alpha]. \tag{E.7}$$

E.2.4 Integration of Differential Forms and Stokes's Theorem

Integration requires an integration measure: This must transform in an appropriate way under a change of integration variables: For example a volume element $dx^1 dx^2$ in two dimensions transforms under a change $x^i \to y^k(x)$ as $dx^1 dx^2 = \det\left(\frac{\partial x}{\partial y}\right) dy^1 dy^2$, that is with the Jacobi determinant. In the exterior calculus, the canonical D-form (E.4) transforms correctly as a volume element as can be seen by writing it in local coordinates as $\eta = dx^1 \wedge dx^2 \cdots \wedge dx^D = \frac{1}{D}\epsilon_{\mu_1 \cdots \mu_D} dx^{\mu_1} \cdots dx^{\mu_D}$. This allows one to define for any D-form $\omega = \hat{\omega}\eta$ the integral in the domain of a chart $(U_\alpha, \varphi_\alpha)$ with coordinates $(x^1, \cdots, x^D)$ by the ordinary integral:

$$\int_{U_\alpha} \hat{\omega}\eta = \int_{\varphi_\alpha(U_\alpha)} \hat{\omega}(\varphi_\alpha^{-1}(x))\; dx^1 \cdots dx^D = \int_{R_\alpha} \hat{\omega} dV.$$

The integral of ω over the whole manifold is finally given by a finite sum of integrations from different charts:

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \hat{\omega}_\alpha \eta.$$

Stokes' Theorem

Let M be a D-dimensional orientable compact manifold with a non-empty boundary ∂M and let α be a $(D-1)$ form. Then

$$\int_M d\alpha = \int_{\partial M} \alpha. \tag{E.8}$$

This generalizes integration theorems we are acquainted with from one, two, and three dimensions:

In the 1-dimensional case take $M = [a,b]$, $\partial M = \{a,b\}$. Now $\alpha(x)$ is a scalar function with $d\alpha(x) = \frac{d\alpha}{dx}dx$ and Stokes' theorem is nothing but ordinary integration:

$$\int_M d\alpha = \int_a^b \frac{d\alpha}{dx} dx = \alpha(b) - \alpha(a) = \int_{\partial M} \alpha.$$

In the 2-dimensional case take for M the disk $M = \{(x_1, x_2) | x_1^2 + x_2^2 \leqslant 1\}$, such that $\partial M = \{(x_1, x_2) | x_1^2 + x_2^2 = 1\}$ is a circle. For the one-form $\alpha = \alpha_1(x) dx^1 + \alpha_2(x) dx^2 := \vec{A}(x) d\vec{x}$ we have generically

$$d\alpha = \frac{\partial \alpha_j}{\partial x^i} dx^i dx^j := \frac{1}{2}\epsilon_{ijk} B_k dx^i dx^j = \vec{B} d\vec{f}$$

with $B_k = \epsilon_{kij}\partial_i A_j, \quad df_k = \frac{1}{2}\epsilon_{kij} dx^i dx^j$. From this, we regain what one learns as Stokes' theorem in electrodynamics

$$\int_M d\alpha = \int_{\text{disk}} (\vec{\nabla} \times \vec{A}) d\vec{f} = \int_{\text{circle}} \vec{A}(x) d\vec{x} = \int_{\partial M} \alpha$$

in terms of the vector potential $\vec{A}$ and the magnetic field $\vec{B}$.
Quite in analogy one recovers in three dimensions

$$\int_M d\alpha = \int_{\text{ball}} (\vec{\nabla}\vec{E})dV = \int_{\text{sphere}} \vec{E}(x)d\vec{f} = \int_{\partial M} \alpha$$

with a two form $\alpha = \frac{1}{2}\alpha_{ik}(x)dx^i dx^j := \vec{E}(x)d\vec{f}$ and its derivative $d\alpha = (\partial_k E_k)dV$. If $\vec{E}$ is interpreted as the electric field, this becomes what in electrodynamics is called Gauss' law.

Coderivative or Adjoint Exterior Derivative

In the space of p-forms one can define an inner product by

$$(\alpha_p, \beta_p) := \int_M \alpha_p \wedge *\beta_p.$$

Because of (E.5b), the inner product is symmetric, $(\alpha_p, \beta_p) = (\beta_p, \alpha_p)$, and positive definite: $(\alpha_p, \alpha_p) \geqslant 0$, with equality only for $\alpha_p = 0$. Now consider

$$(d\alpha_{p-1}, \beta_p) = \int_M d\alpha_{p-1} \wedge *\beta_p = \int_M d(\alpha_{p-1} \wedge *\beta_p) + (-1)^p \int_M \alpha_{p-1} \wedge d * \beta_p.$$

The first term vanishes due to Stokes' theorem (E.8) in the case of a manifold without boundary ($\partial M = 0$). The coderivative of a form is defined as a map $d^\dagger : \Omega_p M \to \Omega_{p-1} M$, where

$$d^\dagger \alpha_p := (-1)^{D(p+1)+\sigma+1} * d * \alpha_p \tag{E.9}$$

such that for a manifold without boundary by (E.5a), $(d\alpha_{p-1}, \beta_p) = (\alpha_{p-1}, d^\dagger \beta_p)$. From the external derivative and the coderivative one defines the Laplace-deRham operator as a map $d : \Omega_p M \to \Omega_p M$ by

$$\triangle := dd^\dagger + d^\dagger d.$$

One easily shows that it commutes with the Hodge star, the external derivative, and the coderivative. Furthermore–and this is important for many applications–it is a positive operator:

$$(\alpha_p, \triangle\alpha_p) = (\alpha_p, dd^\dagger\alpha_p) + (\alpha_p, d^\dagger d\alpha_p) = (d^\dagger\alpha_p, d^\dagger\alpha_p) + (d\alpha_p, d\alpha_p) \geqslant 0.$$

From this follows that $\triangle\alpha_p = 0$ is equivalent to $d\alpha_p = 0$ together with $d^\dagger\alpha_p = 0$.

E.2.5 Poincaré Lemma and de Rham Cohomology

Harmonic, Closed, Coclosed and Exact Forms

Definitions: A p-form α is called *harmonic* if $\triangle\alpha_p = 0$, *closed* if $d\alpha = 0$, and *coclosed* if $d^\dagger\alpha = 0$. Furthermore α_p is called *exact*, if there exists globally a form β_{p-1} such that $\alpha_p = d\beta_{p-1}$. Because of $d^2 = 0$ any exact form is a closed form. In analogy, α_p is called *coexact*, if there exists globally a form β_{p+1} such that $\alpha_p = d^\dagger\beta_{p+1}$.

Due to a theorem by W.V.A. Hopf, any p-form on a compact orientable manifold without boundary can be uniquely decomposed as

$$\alpha_p = d\beta_{p-1} + d^\dagger\gamma_{p+1} + \delta_p,$$

where δ_p is harmonic.

The question of whether a closed form is exact depends on the topology of the domain in question: There is a lemma by Poincaré stating that any closed form is locally exact, and that on $\mathbb{R}^D$ (or on contractible manifolds) every closed form with $p > 0$ is globally exact; for a proof see e.g. [187].

Take as an example the non-contractible manifold $M = \mathbb{R}^2\backslash\{0\}$ and a one-form $\alpha = \alpha_i dx^i$ with

$$\alpha_1 = -\frac{x^2}{r^2}, \qquad \alpha_2 = \frac{x^1}{r^2} \qquad r^2 = (x^1)^2 + (x^2)^2.$$

The form α is closed since

$$d\alpha = \Big(\frac{\partial}{\partial x^k}\frac{x^k}{r^2}\Big)dx^1 dx^2 = 0.$$

We can write $\alpha = d\beta = d(\arctan\frac{x^2}{x^1})$, but $\beta(x^1, x^2)$ is not differentiable on the whole manifold M. Thus α is not exact on M. Exactness can however be achieved if one cuts the plane–for instance along the negative x^1 axis. Therefore on $M' = \mathbb{R}^2\backslash\{(x^1, x^2)|x^1 \leqslant 0, x^2 = 0\}$ the one-form α is exact.

De Ram Cohomology

The question under which circumstances a closed form is exact, arises in various physical contexts. So for instance gauge equivalent gauge potentials differ by an exact form. Another example is a boundary term in an action which, although it does not influence the dynamical equations, nevertheless relates to the boundary behavior of the fields and to the canonical structure of the theory. Therefore an understanding, whether and "how much" closed-ness differs from exact-ness is relevant to questions of invariance and symmetries. The appropriate mathematics is the de Rham cohomology. At first some definitions:

Define as the p-cocycles of a manifold M all its closed forms, and as p-coboundaries its exact forms:

$$Z^p(M) = \{\alpha \in \Omega_p M | \, d\alpha = 0\} \qquad B^p(M) = \{\alpha \in \Omega_p M | \alpha = d\beta\}.$$

Since the exterior derivative is a mapping $d : \Omega_p \to \Omega_{p+1}$ we observe that $Z^p(M)$ is the kernel and $B^p(M)$ is the image of d with:

$$\mathrm{Im}\ (d : \Omega_{p-1} \to \Omega_p) \ \subset \ \mathrm{Ker}\ (d : \Omega_p \to \Omega_{p+1}).$$

Obviously $B^p(M) \subseteq Z^p(M)$ because every exact p-form is closed. But when are the sets of closed and exact p-forms on a manifold equal? A measure for the deviation of exact-ness from closed-ness is the *de Rham cohomology group*, defined as the quotient

$$H^p(M) := Z^p(M)/B^p(M).$$

What does this amount to in specific examples? (1) At first consider $H^0(M)$. Since for functions $f \in \Omega_0$, only sense can be given to $B^0(M) = \{0\}$. On the other hand, a function is closed iff $df = \frac{\partial f}{\partial x^k} dx^k = 0$, that iff its partial derivatives vanish in any coordinate system. If M is connected, a closed form can only be a constant, i.e. a real number. Therefore with $Z^0(M) \cong \mathbb{R}$ we get $H^0(M) \cong \mathbb{R}$ for all connected manifolds. If M has k connected components, results $H^0(M) \cong \mathbb{R}^k$. (2) Due to the Poincaré lemma, $Z^p(\mathbb{R}^D) \cong B^p(\mathbb{R}^D)$ for $p > 0$, it follows that $H^p(\mathbb{R}^D) \cong \{0\}$ (except for $H^0(\mathbb{R}^D) \cong \mathbb{R}$). (3) Taking instead of $\mathbb{R}$ the manifold S one can prove that $H^1(S) \cong \mathbb{R}$, exhibiting the other topology of the circle compared to the real line. Remarkably, although defined in terms of the differentiable structure on the manifold, the de Rham cohomology groups solely depend on the topological structure. The dimension of $H^p(M)$ is called the p-th Betti number of M. The Euler characteristic of a manifold, a pure topological quantity depending on the number of vertices, edges and faces of a polyhedron homeomorphic to M can be expressed as the alternating sum of Betti numbers; for more details see e.g. [187].

E.3 Manifolds with Connection

Connections, covariant derivatives, curvature, and torsion for Riemann-Cartan geometries are treated in Subsect. 7.3. But these notions can be broadened to have a meaning for more general manifolds and for fibre bundles. The most general definitions are stated here in an axiomatic style. But I also indicate that these lead to the notions as commonly used in Riemann-Cartan geometries.

Definition: Linear Connection

A linear connection ∇ on a differentiable manifold M is a mapping

$$\nabla : T^q_p M \to T^q_{p+1} M$$

which (1) is linear: $\nabla(t_p^q + s_p^q) = \nabla\ t_p^q + \nabla\ s_p^q$; (2) behaves as a derivation: $\nabla(t_p^q \otimes s_{p'}^{q'}) = \nabla t_p^q \otimes s_{p'}^{q'} + t_p^q \otimes \nabla s_{p'}^{q'}$; (3) acts on functions f as $\nabla f = df$; (4) commutes with contractions: $(\nabla\langle X, \omega\rangle)(Y) = (\nabla X)(\omega, Y) + (\nabla\omega)(X, Y)$.

E.3.1 Linear Connection on Tensor Fields

... on Vector Fields

The definition of a linear connection if applied to a vector field X results at first in

$$\nabla X = \nabla(X^I e_I) = dX^I \otimes e_I + X^I \nabla e_I.$$

Now ∇e_I must be a T_1^1 tensor:

$$\nabla e_I = \Gamma^K{}_{JI}\ \vartheta^J \otimes e_K = \omega^K{}_I \otimes e_K$$

with the connection one-form ω for which $\omega^K{}_I(e_J) = \Gamma^K{}_{JI}$. The connection on a vector field can then be written as

$$\nabla X = (dX^I + X^K \Gamma^I{}_{JK}\ \vartheta^J) \otimes e_I =: (\nabla_I X^K)\vartheta^I \otimes e_K,$$

with $\nabla_I X^K = e_I \cdot X^K + X^J \Gamma^K{}_{IJ}$.

... on Covector Fields aka One-Forms

Because of condition (4) in the definition of a linear connection, one gets on the basic one-form fields: $(\nabla\vartheta^I)(e_J, e_K) = -(\nabla e_J)(\vartheta^I, e_K) = -\Gamma^I{}_{KJ}$, or

$$\nabla\vartheta^I = -\Gamma^I{}_{KJ}\ \vartheta^K \otimes \vartheta^J = -\vartheta^K \otimes \omega^I{}_K.$$

Then, for any one-form $\omega = \omega_I \vartheta^I$ one obtains from $\nabla\omega = d\omega_I \otimes \vartheta^I + \omega_I \nabla\vartheta^I$

$$\nabla\omega = (\nabla_J \omega_I)\vartheta^J \otimes \vartheta^I = (e_J \cdot \omega_I - \omega_K \Gamma^K{}_{IJ})\ \vartheta^J \otimes \vartheta^I.$$

... on Higher Tensor Fields

On arbitrary higher tensor fields, the linear connection can be derived from ∇e_I and $\nabla\vartheta^I$ by applying the product rule (2) in the definition of the linear connection.

E.3.2 Covariant Derivative

The covariant derivative ∇_Y along a vector field Y is a mapping $\nabla_Y : \mathfrak{X}M \times T^q_p M \to T^q_p M$ which (1) is linear; (2) behaves as a derivation; (3) acts on functions as $\nabla_Y f = Y \cdot f = (df)(Y)$; (4) obeys $\nabla_{fY} = f\nabla_Y$; (5) commutes with contractions: $\nabla_Y(\langle X, \omega\rangle) = \langle \nabla_Y X, \omega\rangle + \langle X, \nabla_Y \omega\rangle$.
Defining

$$\nabla_{e_I} e_J = \Gamma^K{}_{IJ}\, e_K = \omega^K{}_J(e_I)\, e_K$$

one can–due to the defining properties of the covariant derivative–derive its action of any tensor fields

$$\nabla_Y X = Y^I(\nabla_{e_I} X^J)\, e_J \qquad\qquad \nabla_Y \omega = -Y^I(\nabla_{e_I}\omega_J)\, \vartheta_J$$

$$(\nabla_Y t)(X_1, \cdot\cdot, X_q, \omega_1, \cdot\cdot, \omega_p) = Y \cdot t(X_1, \cdot\cdot, X_q, \omega_1, \cdot\cdot, \omega_p)$$
$$-t(\nabla_Y X_1, \cdot\cdot, X_q, \omega_1, \cdot\cdot, \omega_p) - \cdots - t(X_1, \cdot\cdot, X_q, \nabla_Y \omega_1, \cdot\cdot, \omega_p) - \cdots.$$

E.3.3 Torsion and Curvature

Torsion

The torsion is defined by

$$T(X, Y) := (\nabla_X Y - \nabla_Y X) - [X, Y]\,. \tag{E.10}$$

This a mapping $T : \mathfrak{X}M \times \mathfrak{X}M \to \mathfrak{X}M$. From its definition, the torsion (1) is linear; (2) antisymmetric; (3) obeys $T(fX, gY) = fgT(X, Y)$. Both $(\nabla_X Y - \nabla_Y X)$ and $[X, Y]$ are vector fields, but only the specific combination (E.10) has the property (3)

For a basis $\{e_K\}$ in $\mathfrak{X}M$, the torsion becomes due to previous results

$$T(e_I, e_J) := \nabla_{e_I} e_J - \nabla_{e_J} e_I - [e_I, e_J] = \Gamma^K_{IJ} e_K - \Gamma^K_{JI}\, e_K - C^K_{IJ}\, e_K =: T^K_{IJ}\, e_K,$$

and for instance in a holonomic basis $\{\partial_\lambda\}$, one verifies $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$ as the expression for the torsion in a Riemann-Cartan space.

The torsion can also be interpreted as a vector-valued 2-form $T \in \Omega_2(M, \mathfrak{X}M)$, that is

$$T(X, Y) = \Theta^K(X, Y)\, e_K$$
$$\Theta^K = \Theta^K(e_I, e_J)\vartheta^I \otimes \vartheta^J = \Theta^K(e_I, e_J)\, \vartheta^I \wedge \vartheta^J = \frac{1}{2} T^K{}_{IJ}\, \vartheta^I \wedge \vartheta^J.$$

In reformulating the defining equation for the torsion by the one-forms ϑ and ω one obtains its differential form expression (E.11) below. The essential steps for this derivation are as follows:

$$\begin{aligned}T(X,Y) &= T(X^I e_I, Y^J e_J)\\ &= \left[\nabla_X Y^K - \nabla_Y X^K - [X,Y]^K\right] e_K + \left[Y^K \nabla_X - X^K \nabla_Y\right] e_K.\end{aligned}$$

Now, with $\nabla_X Y^K = X\cdot\vartheta^K(Y)$ and $\nabla_X e_K = \omega^L{}_K(X) e_L$ the previous expression becomes

$$\begin{aligned}T(X,Y) = &\left[X\cdot\vartheta^K(Y) - Y\cdot\vartheta^K(X) - \vartheta^K([X,Y])\right] e_K\\ &+ \left[\vartheta^K(Y)\omega^L{}_K(X) - \vartheta^K(X)\omega^L{}_K(Y)\right] e_L.\end{aligned}$$

The first term can be identified with $(d\vartheta^K)(X,Y)e_K$ and the second one with $(-(\vartheta^K \wedge \omega^L{}_K)(X,Y))e_L$, and therefore indeed

$$T(X,Y) = \left[d\vartheta^K - \vartheta^L \wedge \omega^K{}_L\right](X,Y)\, e_K.$$

Curvature

The curvature is defined by

$$R(X,Y,Z) := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

Thus it is a mapping $R : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$. From its definition, the curvature (1) is linear; (2) antisymmetric in its first two arguments; (3) obeys $R(fX, gY, hZ) = fgh\ R(X,Y,Z)$. The name curvature may be justified in that $R(\partial_\mu,\partial_\nu,\partial_\rho) = R^\sigma{}_{\mu\nu\rho}\partial_\sigma$. Like for the torsion also the curvature can be interpreted as a vector-valued two-form:

$$R(X,Y,e_I) = \Omega_I^K(X,Y)e_K \qquad \Omega_I^K = \Omega_I^K(e_J,e_L)\vartheta^J\wedge\vartheta^L = \frac{1}{2}R^K_{IJL}\vartheta^J\wedge\vartheta^L.$$

and again, a half-page calculation then gives the expression of the curvature in terms of differential forms, namely (E.12).

Bianchi Identities

Torsion and curvature are defined in terms of the D-beins ϑ^I and spin connections $\omega^I{}_J$ through the Cartan structural equations

$$\Theta^I = d\vartheta^I + \omega^I{}_J\wedge\vartheta^J = \frac{1}{2}T^I{}_{KJ}\vartheta^{KJ} \tag{E.11}$$

$$\Omega^I{}_J = d\omega^I{}_J + \omega^I{}_K\wedge\omega^K{}_J = \frac{1}{2}R^I{}_{JKL}\vartheta^{KL}. \tag{E.12}$$

Taking the exterior derivative of these equations one arrives at the *Bianchi identities*

$$\begin{aligned}
d\Theta^I &= d\omega^I{}_J \wedge \vartheta^J - \omega^I{}_J \wedge d\vartheta^J \\
&= d\omega^I{}_J \wedge \vartheta^J - \omega^I{}_J \wedge (\Theta^J - \omega^J{}_K \wedge \vartheta^K) \\
&= \Omega^J{}_J \wedge \vartheta^J - \omega^I{}_J \wedge \Theta^J \\
d\Omega^I{}_J &= d\omega^I{}_K \wedge \omega^K{}_J - \omega^I{}_K \wedge d\omega^K{}_J \\
&= (\Omega^I{}_K - \omega^I{}_L \wedge \omega^L{}_K) \wedge \omega^K{}_J - \omega^I{}_K \wedge (\Omega^K{}_J - \omega^K{}_L \wedge \omega^L{}_J) \\
&= \omega^K{}_J \wedge \Omega^I{}_K - \omega^I{}_K \wedge \Omega^K{}_J.
\end{aligned}$$

The covariant derivative D with respect to the spin-connection is

$$\begin{aligned}
Df^{I_1..I_k}_{J_1,..,J_l} = df^{I_1,..,I_k}_{J_1,..,J_l} + \omega^{I_1}_L \wedge f^{L,..,I_k}_{J_1,..,J_l} \quad + .. + \quad \omega^{I_k}_L \wedge f^{I_1,..,L}_{J_1,..,J_l} \\
- \omega^L_{J_1} \wedge f^{I_1,..,I_k}_{L,..,J_l} \quad - .. - \quad \omega^L_{J_k} \wedge f^{I_1,..,I_k}_{J_1,..,L}.
\end{aligned}$$

Applying the covariant derivative twice we obtain for a form with an upper Lorentz index

$$\begin{aligned}
DDf^I &= D(df^I + \omega^I{}_J \wedge f^J) \\
&= \omega^I{}_J \wedge df^J + d(\omega^I{}_J \wedge f^J) + \omega^I{}_K \wedge (\omega^K{}_J \wedge f^J) \\
&= d\omega^I{}_J \wedge f^J + \omega^I{}_K \wedge \omega^K{}_J \wedge f^J + \omega^I{}_J \wedge df^J - \omega^I{}_J \wedge df^J = \Omega^I{}_J \wedge f^J.
\end{aligned}$$

Similarly one derives:

$$\begin{aligned}
DDf^I &= \Omega^I{}_J \wedge f^J, \\
DDg_K &= -\Omega^J{}_K \wedge g_J, \\
DDh^I{}_K &= \Omega^I{}_J \wedge h^J{}_K - \Omega^L{}_K \wedge h^I{}_L.
\end{aligned} \tag{E.13}$$

In terms of the covariant derivative the Bianchi identities become

$$D\Theta^I = \Omega^I{}_J \wedge \vartheta^J \qquad D\Omega^I{}_J = 0. \tag{E.14}$$

The torsion itself can be written as $\Theta^I = D\vartheta^I$, but observe that $\Omega^I{}_J \neq D\omega^I{}_J$.

E.4 Lie Groups

A D-dimensional Lie group **G** has both properties of a D-dimensional manifold and a D-parameter group with a compatibility of the differentiability in the manifold with the group composition: The group operation induces a C^∞ map of the manifold into itself. All the previously introduced definitions for manifolds can be taken over, but the surplus group structure allows to sharpen and refine the terminology. Points in the manifold G are now the group elements $g \in G$, and as we will see, the Lie algebra generators X_a are the basis of a vector space isomorphic to T_eG, that is the tangent space of the Lie group at the group identity e.

E.4.1 Lie Algebra

In App. A.2, the Lie algebra associated to a Lie group $\mathbf{G}$ is introduced by expanding group elements near the group identity e in a Taylor series in local coordinates, and by relating the group structure to generators. Here it will be shown that the relation of a Lie group to its associated Lie algebra and the representation issues can be understood in generic geometrical terms.

Any element g of G maps $h \in G$ to either $h \mapsto gh$ (*left translation by g*) or to $h \mapsto hg$ (*right translation by g*). For left translations[5]:

$$L_g : G \to G \qquad L_g h = gh.$$

These maps induce push-forward maps in tangent spaces: $L_{g^*} : T_h G \to T_{gh} G$. A vector field $\bar{X}$ on G is called left-invariant if L_{g^*} maps $\bar{X}$ at h to $\bar{X}$ at gh:

$$L_{g^*} : \bar{X}_h = \bar{X}_{L_g h} = \bar{X}_{gh} \qquad \forall \ g, h \in G.$$

A vector $A \in T_e G$ generates a unique left-invariant vector field on G by $(X_A)_g = L_{g^*} A$.

The left-invariant vector fields on G form a D-dimensional real vector space. Furthermore, the Lie bracket of two left-invariant vector fields is left-invariant, because of

$$L_{g^*}[\bar{X}, \bar{Y}]_h = L_{g^*}(\bar{X}_h \bar{Y}_h - \bar{Y}_h \bar{X}_h) = [L_{g^*}\bar{X}_h, L_{g^*}\bar{Y}_h] = [\bar{X}_{gh}, \bar{Y}_{gh}] = [\bar{X}, \bar{Y}]_{gh}.$$

The Lie algebra $\mathfrak{g}$ associated to the Lie group G is defined by the Lie algebra of the left-invariant vector fields. Since any left-invariant vector field is uniquely defined by its value at the group unit element $(\bar{X}_g = L_{g^*}\bar{X}_e)$, the Lie algebra $\mathfrak{g}$ is as a vector space isomorphic to the tangent space $T_e G$ to the group manifold G at the unit element.

Let $X_a \quad (a = 1, \ldots, D)$ be a basis in $\mathfrak{g}$. Then holds $[X_a, X_b] = f^c{}_{ab} X_c$ with structure constants $f^c{}_{ab}$ which completely characterize the Lie algebra (and also the group in the neighborhood of the identity). The algebra $\mathfrak{g}$ inherits the properties of the group G near the identity element.

It is also possible to define left-invariant forms of G: $\bar{\omega} \in \Omega_p G$ is called left-invariant if

$$L_g^* \, \bar{\omega}_{gh} = \bar{\omega}_h \qquad L_g^* : T_{gh}^* G \to T_h^* G.$$

The vector space $\mathfrak{g}^*$ of left-invariant one-forms is dual to $\mathfrak{g}$: If $\bar{X} \in \mathfrak{g}$ and $\bar{\omega} \in \mathfrak{g}^*$, the internal product $\bar{\omega}(\bar{X}) = \langle \bar{\omega}, \bar{X} \rangle$ is constant on G. Let now let ω^a be a basis in $\mathfrak{g}^*$. Then by the duality property $\langle \omega^a, X_b \rangle = \delta^a_b$. Now consider the exterior derivative of the two-form $d\omega$ evaluated in the X^a basis:

$$d\omega^a(X_b, X_c) = X_b(\langle \omega^a, X_c \rangle) - X_c(\langle \omega^a, X_\rangle) - \langle \omega^a, [X_b, X_c] \rangle.$$

---

[5] In the following only those structures defined by left translations will be treated. The ones for right translations are defined in analogy.

The first two terms vanish because the arguments are constants, the last term can be expressed via the structure constants, with the result $d\omega^a(X_b, X_c) = -f^a{}_{bc}X_c$. On the other hand, $(\omega^d \wedge \omega^e)(X_b, X_c) = \delta^d_b\delta^e_c - \delta^d_c\delta^e_b$. Combining this with the previous expression results in the Maurer-Cartan (structure) equation

$$d\omega^a + \frac{1}{2}f^a{}_{bc}\,\omega^b \wedge \omega^c = 0 \tag{E.15}$$

holding for any left-invariant one-form. This equation is to be interpreted as the counterpart in $\mathfrak{g}^*$ of $[X_a, X_b] = f^c{}_{ab}X_c$ which are valid in $\mathfrak{g}$.
The canonical one-form or *Maurer-Cartan form* $\hat{\Omega}$ on $\mathbf{G}$ is a left-invariant Lie algebra valued one-form, defined by

$$\hat{\Omega} = X_a \otimes \theta^a$$

where $\{\theta^a\}$ is a basis in T^*_gG. The Maurer-Cartan form moves a tangent vector $X_g \in T_gG$ back to the unit: $\hat{\Omega}_g(X_g) = L_{g^{-1}*}X_g = X_e$. For this the Maurer-Cartan equation simply reads $d\hat{\Omega}+\frac{1}{2}\hat{\Omega}\wedge\hat{\Omega} = 0$. The Maurer-Cartan form plays an important role in advanced calculations of Faddeev-Popov ghosts and the BRST quantization procedure of theories with local symmetries.

E.4.2 Group Covariant Derivative

Some of the next definitions and terms can potentially only be fully appreciated in terms of the fibre bundle understanding of gauge theories.

In physics we deal with "fields" Ψ^α which transform according to (irreducible) representations of G, that is $\Psi^\alpha \in \Omega(M, V_d)$, where V_d is the representation space (and $\alpha = 1, \ldots, d$). Under infinitesimal group transformations

$$g = 1 + \gamma = 1 + \gamma^a \otimes X_a$$

the Ψ transform as

$$\delta_\gamma\,\Psi = \hat{\gamma}\Psi, \qquad \text{i.e.} \qquad \delta_\gamma\,\Psi^\alpha = (\hat{\gamma})^\alpha{}_\beta\,\Psi^\beta = \gamma^a(\hat{X}_a)^\alpha{}_\beta\,\Psi^\beta, \tag{E.16}$$

where $(\hat{X}_a)^\alpha{}_\beta$ are d-dimensional representation matrices for the generators X_a.

Since for spacetime-dependent parameter $\gamma^a(x)$ the derivative $d\Psi^\alpha$ does not transform in the same way as Ψ^α itself, a covariant derivative is introduced by help of a connection $A = A^a \otimes X_a \in \Omega_1(M, \mathfrak{g})$, i.e. the components A^a of the Lie-algebra valued vectorfield A are one-forms. The definition is $D\Psi = d\Psi + A \wedge \Psi$, or explicitly

$$D\Psi^\alpha := d\Psi^\alpha + (\hat{A})^\alpha{}_\beta \wedge \Psi^\beta = d\Psi^\alpha + A^a \wedge (\hat{X}_a)^\alpha{}_\beta\,\Psi^\beta. \tag{E.17}$$

Requiring $\delta_\gamma D\Psi = \hat{\gamma} D\Psi$ the connection must transform according to $\delta_\gamma A = -d\hat{\gamma} - [A, \hat{\gamma}]$, which, if split as $\delta_\gamma A = \delta_\gamma A^a \otimes X_a + A^a \otimes \delta_\gamma X_a$ is

$$\delta_\gamma A^a = -d\gamma^a \qquad \text{and} \qquad \delta_\gamma X^a = [\gamma, X^a].$$

Keeping the basis fixed, we also define for the components the variation

$$\hat{\delta}_\gamma A^a = -d\gamma^a - f^a{}_{bc} A^b \gamma^c.$$

Originally defined on elements from $\Omega(M, V_d)$, it is possible to extend the covariant derivative to forms $\alpha \in \Omega(M, \mathfrak{g})$, by defining

$$D\alpha := d\alpha + [A, \alpha] \qquad \text{or} \qquad (D\alpha)^a := d\alpha^a + f^a{}_{bc} A^b \wedge \alpha^c; \tag{E.18}$$

symbolically written as $D = d + [A, -]$. By this we have

$$\delta_\gamma A = -D\gamma. \tag{E.19}$$

E.4.3 Group Curvature

Applying the covariant derivative twice we have (at first in a matrix notation)

$$\begin{aligned} DD\Psi &= D(d\Psi + A \wedge \Psi) = d(d\Psi + A \wedge \Psi) + A \wedge (d\Psi + A \wedge \Psi) \\ &= d(A \wedge \Psi) + A \wedge d\Psi + A \wedge (A \wedge \Psi). \end{aligned}$$

The first two terms are nothing but $dA \wedge \Psi$, the last one can be reformulated as

$$\begin{aligned} A \wedge (A \wedge \Psi) &= A \wedge (A^a \wedge \hat{X}_a \Psi) = A^b \wedge A^a \wedge \hat{X}_b \hat{X}_a \Psi \\ &= \frac{1}{2} A^b \wedge A^a \wedge \left[\hat{X}_b, \hat{X}_a\right] \Psi = \frac{1}{2}(A \wedge A) \wedge \Psi. \end{aligned}$$

This gives rise to the definition of a curvature

$$F = dA + \frac{1}{2} A \wedge A \qquad \text{or} \qquad F^a := dA^a + \frac{1}{2} f^a{}_{bc} A^b \wedge A^c, \tag{E.20}$$

such that

$$DD\Psi^\alpha := (\hat{F})^\alpha{}_\beta \wedge \Psi^\beta.$$

The covariant derivative of the curvature is calculated as

$$\begin{aligned} DF &= D\big(dA + \frac{1}{2} A \wedge A\big) = d\big(dA + \frac{1}{2} A \wedge A\big) + A \wedge \big(dA + \frac{1}{2} A \wedge A\big) \\ &= \frac{1}{2} d(A \wedge A) + A \wedge dA + \frac{1}{2} A \wedge (A \wedge A). \end{aligned}$$

The first two terms cancel and the last term vanishes because of the Jacobi identity:

$$A \wedge (A \wedge A) = \frac{1}{2} A \wedge A^a \wedge A^b \otimes [X_a, X_b] = \frac{1}{4} A^c \wedge A^a \wedge A^b \otimes [X_c, [X_a, X_b]] = 0.$$

Thus there is the Bianchi-identity

$$DF \equiv 0.$$

The transformation of the curvature is calculated as

$$\delta_\gamma F = d\delta_\gamma A + A \wedge \delta_\gamma A = D\delta_\gamma A = -DD\gamma = -[F, \gamma],$$

and thus F transforms homogeneously under the group.

E.4.4 Isometries and Coset Manifolds

Killing Vector Fields and Isometries

For a space with metric $g_{\mu\nu}(x)$, a coordinate transformation $x \to x'$ is called an isometry iff $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$, which can also be written $\bar{\delta} g_{\mu\nu} = 0$. Since a metric transforms like a tensor, i.e.

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x),$$

the isometry condition can equivalently be expressed as

$$g_{\mu\nu}(x) \stackrel{!}{=} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x').$$

In general, this is a complicated system of differential equations for the isometric coordinate transformations $x(x')$. Since we are looking for an isometry group, it is possible and allowed to investigate infinitesimal coordinate transformations

$$x'^\mu = x^\mu + \xi^\mu \qquad \text{with} \qquad |\xi^\mu| \ll 1,$$

for which the isometry condition becomes

$$\begin{aligned} g_{\mu\nu}(x) &\stackrel{!}{\simeq} (\delta^\rho_\mu + \xi^\rho_{,\mu})(\delta^\sigma_\nu + \xi^\sigma_{,\nu}) g_{\rho\sigma}(x + \xi) \\ &\simeq (\delta^\rho_\mu \delta^\sigma_\nu + \delta^\rho_\mu \xi^\sigma_{,\nu} + \delta^\sigma_\nu \xi^\rho_{,\mu}) \big(g_{\rho\sigma}(x) + \xi^\lambda g_{\rho\sigma,\lambda}\big). \end{aligned}$$

In order that this equation is fulfilled one arrives at the infinitesimal form of the isometry condition:

$$g_{\mu\nu,\lambda} \xi^\lambda + g_{\mu\lambda} \xi^\lambda_{,\nu} + g_{\lambda\nu} \xi^\lambda_{,\mu} = 0. \tag{E.21}$$

Any solution ξ of (E.21) is called a Killing vector (field) of the metric space. Let $\{k_A\}$ be the complete set of linear independent Killing vectors of the manifold $(A = 1, \ldots N)$. The coordinate change can be expressed as $\xi^\mu = \epsilon^A k_A^\mu$.

Some further remarks on Killing vectors:

- In a Riemann space V the condition (E.21) can be written in terms of covariant derivatives as

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0.$$

- Since isometries form a continuous group they have an associated Lie algebra with generators X_A fulfilling $[X_A, X_B] = f_{AB}{}^{C} X_C$.
- The generators can be expressed by the Killing vectors as $X_A := k_A^\mu \partial_\mu$. Therefore

$$k_A^\mu\, \partial_\mu k_B^\nu - k_B^\mu\, \partial_\mu k_A^\nu = f_{AB}^{\ C} k_C^\nu. \tag{E.22}$$

- The Killing condition can also be formulated by the Lie derivative as $\pounds_\xi g_{\mu\nu} = 0$. The Lie bracket of two Killing vector fields is again a Killing vector. The Killing vector fields on a Riemann manifold V thus form a Lie subalgebra of vector fields on V. This is the Lie algebra of the isometry group of V.
- The maximal number of linear independent Killing vectors in D dimensions is $\frac{D(D+1)}{2}$.

These notions shall be illustrated at the example of S^2, the surface of a sphere with radius r. In polar coordinates $(x_1 = \theta, x_2 = \varphi)$ the metric is given by

$$g_{ij} = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

There are three Killing Eqs. (E.21) corresponding to

$$\xi^\theta_{,\theta} = 0 \qquad \sin^2\theta\, \xi^\varphi_{,\theta} + \xi^\theta_{,\varphi} = 0 \qquad \cos\theta\, \xi^\theta + \sin\theta\, \xi^\varphi_{,\varphi} = 0.$$

The solutions of these differential equations

$$\xi^\theta = A\sin(\varphi + \varphi_0) \qquad\qquad \xi^\varphi = a + A\cos(\varphi + \varphi_0)\cot\theta$$

depend on three free parameter (A, φ_0, a) giving rise to three independent Killing vectors, which may be written as

$$k_1 = \begin{pmatrix} \sin\varphi \\ \cos\varphi\cot\theta \end{pmatrix} \qquad k_2 = \begin{pmatrix} \cos\varphi \\ -\sin\varphi\cot\theta \end{pmatrix} \qquad k_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{E.23}$$

The generators $X_A = k_A^\theta\, \partial_\theta + k_A^\varphi\, \partial_\varphi$ obey the algebra

$$[X_A, X_B] = \epsilon_{ABC} X_C$$

which is isomorphic to $\mathfrak{so}(3)$. This result can be extended to the D-dimensional sphere for which the isometry group is $\mathbf{SO(D+1)}$.

Manifolds which host their maximal number of independent Killing vectors are called *maximally symmetric*. For these the Riemann curvature tensors are uniquely specified by their Riemann curvature as

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}\{g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\}. \tag{E.24}$$

For D=1 this statement is also true because any one-dimensional Riemann manifold is flat.

Group Action

In the context of symmetries we understand symmetry operations as elements of (symmetry) groups. These operate on entities, which generically are sets. A *left action* of a group $\mathbf{G}$ on a set S is a map

$$\Phi : \mathbf{G} \times S \to S$$

such that for $s \in S$ and the unit element $e \in G$ the identity axiom $\Phi(e, s) = s$ and the associativity axiom $\Phi(g \circ h, s) = \Phi(g, \Phi(h, s))$ are obeyed. For a *right action* holds instead $\Phi(g \circ h, s) = \Phi(h, \Phi(g, s))$. The set S is also called a left (respectively right) G-set. In most applications the sets have also manifold properties and we talk of *G-spaces*. A group action is termed

- *transitive* if for every pair $(s, s') \in S \times S$ there is a group element g, such that $\Phi(g, s) = s'$
- *effective* if $\Phi(g, s) = s \quad \forall s$ implies $g = e$
- *free* if $\Phi(g, s) = s$ forsome s implies $g = e$.

For every $s \in S$ define the *stabilizer* subgroup[6] as

$$\mathbf{Stab_S}(s) := \{g \in \mathbf{G} |\ \Phi(g, s) = s\}.$$

Thus the stabilizer of a point s contains all symmetry transformations in $\mathbf{G}$ which leave s invariant. It is indeed a group since the product of any two leaving the point fixed also leaves it fixed, and since the identity of $\mathbf{G}$ is in this set. This amounts to identifying the set S with the coset space

$$S = \mathbf{G}/\mathbf{Stab_S}.$$

The stabilizer is a subgroup of $\mathbf{G}$ which, however, in general is not invariant. The action of $\mathbf{G}$ on S is free iff all stabilizers are trivial.

A closely related notion is the *orbit*: The orbit of a point $s \in S$ is the subset of S to which s can be moved by the elements of $\mathbf{G}$:

$$\mathrm{Orb}_G\ (s) := \{\Phi(g, s) |\ g \in \mathbf{G}\} \subseteq S.$$

[6] in other contexts also called the isotropy group or the little group.

Two points s and s' may be defined as being equivalent $s \sim s'$ iff there exists a $g \in \mathbf{G}$ such that $\Phi(g, s) = s'$. The orbits are therefore recognized as equivalence classes under this relation: s and s' are equivalent iff $\mathrm{Orb}_G\,(s) = \mathrm{Orb}_G\,(s')$.

Homogeneous and Isotropic Spaces

Let $\mathbf{G}$ be the Lie group of isometries of a Riemann manifold V with metric g. As seen before, the Lie algebra associated to $\mathbf{G}$ is that of the Killing vector fields of g. The manifold is said to be *homogeneous* if its isometry group acts transitively on it. (The term homogeneous reflects that the geometry is the same everywhere in V). The Riemann manifold V is said to be *isotropic* around a point p if there is a nontrivial stabilizer $\mathbf{H_p}$. In textbooks on differential geometry one finds proofs of: A homogeneous space that is isotropic about one point is maximally symmetric. A space that is isotropic around every point, is homogeneous.

In the example, the isometry group of the sphere S^2 is the rotation group $\mathbf{SO(3)}$. It acts transitively on S^2. For every point p on the sphere there exists a subgroup $\mathbf{H_p} \cong \mathbf{SO(2)}$ that leaves p fixed. Forming left and right cosets of $\mathbf{SO(2)} \subset \mathbf{SO(3)}$ one finds that they are identical. Therefore the factor group $\mathbf{SO(3)}/\mathbf{SO(2)}$ is a group. Since we are dealing with Lie groups the factor group itself can be assigned a topology, giving it the properties of a *coset space*. Indeed one can show that

$$S^D = \mathbf{SO(D+1)}/\mathbf{SO(D)}.$$

Further Definitions

Let M be a left G-space: $(g, x) \mapsto gx$. Then a left-invariant vector field X_V generated by $V \in T_eG$ gives rise to an *induced vector field*

$$X_V(x) = \frac{d}{dt}\exp(tV)x\big|_{t=0}.$$

The homomorphism

$$\mathrm{ad}_h : G \to G \qquad g \mapsto hgh^{-1}$$

is called an *adjoint representation* of G. It induces the tangent space maps $ad_{h^*} : T_gG \to T_{hgh^{-1}}G$, and specifically the adjoint map in the Lie algebra

$$\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g} \qquad V \mapsto hVh^{-1}.$$

For a matrix group, this becomes explicitely $\mathrm{Ad}_h(V) = hVh^{-1}$.

E.5 Fibre Bundles

The notion of a fibre bundle generalizes that of a Cartesian product. For the motivation and illustration of fibre bundles, all textbooks use a cylinder and the Möbius strip as examples: Start with a circle S^1 (called the base space) and attach to each point of it a closed interval $D \subset \mathbb{R}$, the fibre. By simply taking the Cartesian product $S^1 \times D$ the geometrical interpretation is that of a cylinder. But this glueing together of the two manifolds can be done in another way: By one twist the resulting object receives the topology of a Möbius strip. Both the cylinder and the Möbius strip are *locally trivial*, that is, locally homoeomorphic to $S^1 \times D$.

We met already another example, namely the tangent bundle in which the tangent spaces $T_x(M)$, at points x of a manifold M represent the fibres. And there are further examples abound: Aristotelean space-time E is a *trivial bundle*, namely the Cartesian product $E = T \times S$ of a base space ("time" $T = E^1$) and a three-dimensional space $S = E^3$. Galileian space-time, in which one allows for Galilei-boosts mixing space and time, can be described by a projection $\pi : E \to T$ that associates to any event $p \in E$ the corresponding instant of time $t = \pi(p) \in T$. In the language of fibre bundles, E is called the *total space*, and T the *base space*. The set $\pi^{-1}(t)$ (that is all events that happen simultaneously with p) is called the *fibre* over t. Each fibre is isomorphic to $\mathbb{R}^3$, called the *typical fibre* or *standard fibre*. The Galileian bundle can be trivialized, i.e. there is a smooth map $h : E \to T \times \mathbb{R}^3$

In the context of gauge symmetries the bundle language allows for a proper understanding of gauge potentials, connections, and curvatures. In the main text various structural similarities in the description of Yang-Mills and of gravitational theories became visible, but the dissimilarities are most properly recognized in the language of fibre bundles.

E.5.1 Definition, Various Types, and Examples of Fibre Bundles

Definition of a Fibre Bundle

A fibre bundle is a structure[7] (E, π, M, F, G), where the elements are (1) a topological space E, the total space; (2) a topological space M, the base space; (3) a surjection called the projection $\pi : E \to B$; (4) a topological space F, the standard fibre. This is homeomorphic to the inverse images $\pi^{-1}(x) := F_x$ with $x \in M$; the F_x are called fibres; (5) a *structure group* G of homeomorphism of the fibre F, the fibres being left G-spaces; (6) A set of open neighborhoods U_α covering B, for which for every U_α there is a homeomorphism

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \quad \text{where} \quad \pi \cdot \varphi_\alpha^{-1}(x, f) = x \quad x \in U_\alpha \subset B,\ f \in F.$$

[7] Strictly speaking, the whole complex $\mathbb{E} = (E, \pi, M, F, G)$ constitutes a fibre bundle, but one lazily also calls E the fibre bundle.

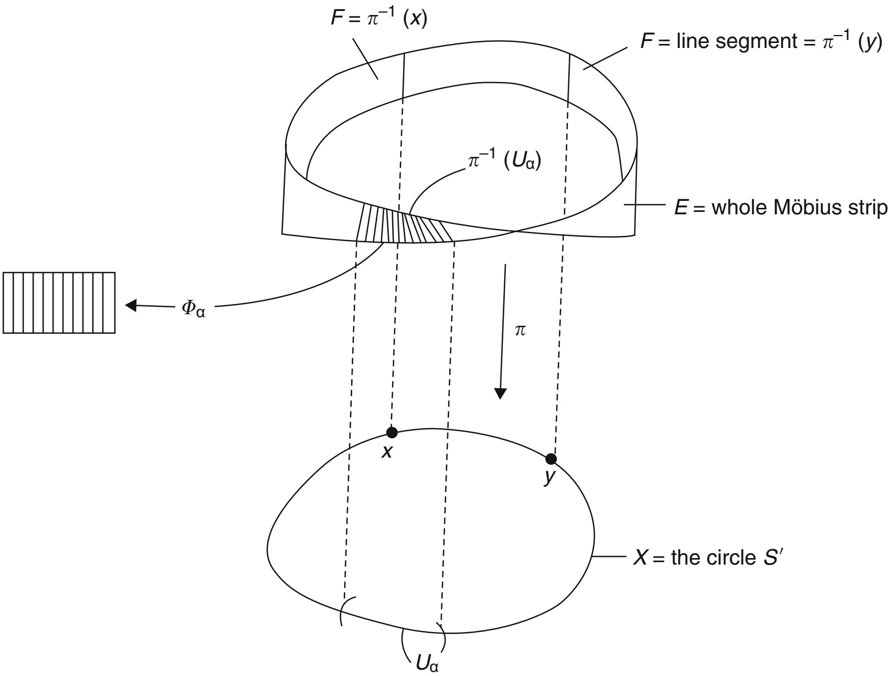


Fig. E.2 The Möbius strip as a fibre bundle

The latter feature expresses the intuitive notion of *local triviality*.

These terms are illustrated for the Möbius strip[8] in Fig. E.2.

The structure group G originates from the transitions from one set of local bundle coordinates $(U_\alpha, \varphi_\alpha)$ to another set (U_β, φ_β). If $U_\alpha \cap U_\beta \neq \emptyset$ it makes sense to define $g_{\alpha\beta} = \varphi_\alpha \cdot \varphi_\beta^{-1}$, which are homeomorphic mappings $F \to F$. These *transition functions* constitute the structure group G. They obey what is called a *cocycle condition*: $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Thus specifically $g_{\alpha\alpha} = e$, and taking $\alpha = \gamma$: $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$. By this it is guaranteed that the local pieces of a bundle are glued together in a consistent manner. Of course the topological and geometrical properties of a fibre bundle should not depend on the choice of $(U_\alpha, \varphi_\alpha)$. A change from φ to φ' induces a change in the group element from $g_{\alpha\beta}$ to $g'_{\alpha\beta} = \varphi'_\alpha \cdot \varphi'^{-1}_\beta$. In order that the $g'_{\alpha\beta}$ constitute the structure group, there must exist $\lambda \in G$ such that $g'_{\alpha\beta} = \lambda_\alpha^{-1}\, g_{\alpha\beta}\lambda_\beta$.

The structure group carries information on the global properties of the fibre bundle. For example, the cylinder and the Möbius strip differ from each other in that for the cylinder the structure group is built by the group unit alone, whereas the structure group for the Möbius strip is isomorphic to Z_2, containing also the twist mapping.

[8] taken from [369]

When is a fibre bundle globally trivial? It is easily shown that global triviality can be obtained if and only if all transition functions can be chosen to be the identity map. Furthermore, if the base space is a contractible manifold (that is, if it can continuously be shrunk to a point inside the manifold itself) any fibre bundle with this base space is globally trivial. The more general answer is as follows: If and only if the principal bundle associated to a fibre bundle admits a global section, both bundles are globally trivial. This needs some further terminology and definitions.

A *global section*–or section[9] for short–in the bundle (E, π, M, F, G) is defined as a continuous map s from the base space to the total space, $s : M \to E$ such that $\pi \cdot s(x) = x$ for $\forall\ x \in M$. Denote the set of sections by $\Gamma(M, E)$. If a section can only be defined for an open set $U \subset M$, this is called a local section, denoted by $\Gamma(U, E)$.

Vector, Tangent, Frame, Principal, Associated . . . . Bundles

By imposing additional structures or conditions on a fibre bundle one arrives at specific bundles:

A *vector bundle* (E, π, M, V, G) of rank n is a fibre bundle where the fibre is an n-dimensional vector space V and where the structure group acts linearly on the fibres; thus $G \subseteq GL(n)$.

The *tangent bundle* to a manifold M^D is an example of a vector bundle. It is obtained by choosing the fibre F at a point x of a base manifold M^D to be the tangent space $T_x M^D$. Locally the tangent bundle has the direct product structure $T(U_\alpha) \cong \mathbb{R}^D \times \mathbb{R}^D$, and the structure group is $G = GL(D, \mathbb{R})$. Quite in analogy, one can build the *cotangent bundle*.

Related (below the term 'associated bundle' will be defined) to the tangent bundle TM is the *linear frame bundle* LM. A 'linear frame' at $x \in M$ is an ordered basis $\{X_1, \ldots, X_D\}$ of $T_x M$. Denote the collection of all linear frames in x by $L_x M$; then the set union over all points of the manifold is called the linear frame bundle LM^D. Any frame can be reached by acting with an element of $GL(D, \mathbb{R})$ on a fixed frame. Let this group act on LM to the right by

$$R_a : LM \to LM \quad (x, \{X_1, \ldots, X_D\}) \mapsto (x, \{a^i_1 X_i, \ldots, a^i_D X_i\}) \quad a^k_i \in GL(D, \mathbb{R}).$$

This group action is free and thus the quotient $LM/GL(D, \mathbb{R})$ is a manifold. Define a map from this manifold to M, such that the orbit through $u \in L_x M$ is mapped to $x \in M$. By this the orbit space can be identified with M and there is a natural projection $\pi : LM \to M$. By construction, this defines a bundle in which the fibre is $GL(D, \mathbb{R})$, that is the structure group itself. A linear frame bundle is an example of a principal bundle, defined below.

For any fibre bundle $\mathbb{E} := (E, \pi, M, F, G)$ one can construct a correlated *principal bundle* $\mathbb{P}(\mathbb{E}) = (P, \pi, M, G)$, characterized by replacing the fibre with the

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[9] sometimes also called 'cross-section'

structure group ($F \equiv G$). The principal bundle contains the information of whether a fibre bundle can be trivialized globally.

A more formal definition: A *principal G-bundle* (P, π, M, G) is a fibre bundle which is a right G-space such that the group action is free and $\pi(pg) = \pi(p)$ for $p \in P$ and $g \in G$. When restricted to a fibre $\pi^{-1}(x)$, the action is also transitive. This implies that the fibre (i.e. the orbit of any point x) is isomorphic to the structure group itself, and the base space is isomorphic to the orbit space: $M \cong P/G$.

Indeed the homogeneous space construction directly leads to a principal bundle: Due to a theorem by Cartan, any topologically closed subgroup H of a Lie group G is a Lie group, and therefore the quotient $M = G/H$, called a homogeneous space, is a manifold. The map $\pi : G \to G/H$ is a projection. And since $\pi^{-1}(x)$ of any point $x \in M$ is diffeomorphic to H, this is the fibre. The right action $(h, g) \mapsto gh$ satisfies $\pi(gh) = \pi(g)$ by definition.

Any principal G-bundle (P, π, M, G) has fibre bundles $\mathbb{E}_P$ associated to it. For every choice of a left G-space fibre F, the *associated bundle* $\mathbb{E}_P$ with fibre F has the same base space and structure group as the principal bundle. It can be constructed by considering $P \times F$ and the equivalence relation $(pg, f) \sim (p, gf)$ for $p \in P,\ g \in G,\ f \in F$. The total space of $\mathbb{E}_P$ is built by the equivalence classes of $(P \times F)/G$, and the projection in E_P is $\pi_E\left[(p, f)\right] = \pi_P(p)$. By this, for example the linear frame bundle is recognized as the principal bundle associated to the tangent bundle. A specific vector bundle $\mathbb{V}$ associated to a principal bundle P can be constructed from a linear representation $\rho : G \to V$ of the structure group G on a finite-dimensional vector space V. The representation map ρ defines in a canonical way the associated vector bundle $\mathbb{V} = (P \times V)/G$ by the equivalence relation $(p, \rho(g) \cdot v) \simeq (p \cdot g, v)$ for all $p \in P,\ v \in V, g \in G$.

E.5.2 Connections in Fibre Bundles

Connections are needed if we want to describe how geometric objects (tensor fields, say) are transported from one point in the geometric space to another. In Riemann-Cartan geometry, which is the arena of gravitational theories, a connection Γ is introduced in terms of parallel transport, and it entails the notion of a Γ-covariant derivative, of curvature R and torsion T. In Yang-Mills type theories a connection A is understood as a one-form field. It is introduced in order to take care of consistent local transformations of derivatives of matter fields, with an ensuing definition of an A-covariant derivative and of field strengths F associated to the gauge fields A. Indeed, there is an astounding likeness of these notions, and the language of fibre bundles reveals the similarities (and the dissimilarities as well.)

There are various equivalent ways to define connections in fibre bundles. The authors of [143] distinguish the ‘parallel transport’, ‘tangent space’, ‘cotangent space’, ‘axiomatic’, and ‘change of frame’ approaches. Here I sketch the parallel transport approach in which parallel transport is defined ‘vertically’ for the base manifold and extended ‘horizontally’ to a principal bundle, and the cotangent space approach in which a one-form connection introduced that is “fed” by vertical vector fields only. The axiomatic approach is based on the definition of connections as in App. E.3.

Parallel Transport in a Principal Bundle

Consider a principal bundle (P, π, M, G) and assume that the base space is a manifold. Since the Lie group is a manifold too, the principal bundle is a manifold itself, and it makes sense to talk about tangent and cotangent spaces at points $p \in P$, and of tangent and cotangent bundles of the principal bundle.

To prepare the notion of parallel transport, we choose a curve C in M and extend this to a curve $\tilde{C}$ in the total space P. A natural way to do so is to define vectors tangent to $\tilde{C}$: Given a vector $X_x \in T_x M$ tangent to C at a point x along the curve, the curve $\tilde{C}$ through the fibre attached to this point is a ‘lift’ of X_x, namely an element of $T_p P$ for some $p \in P$ with $\pi(p) = x$. In order to lift a vector in $T_C M$ to TP one proceeds as follows: Decompose $T_p P$ at every point p into a subspace of vectors tangent to the fibre, called the vertical subspace $ver_p P$, and a complement $hor_p P$, called the horizontal subspace. A choice of a connection is now equivalent to a choice of the horizontal subspace, since the vertical subspace acts along the fibres: $ver_p P = \{X \in T_p P | \pi_* X = 0\}$. Since $\dim\, P = \dim\, M + \dim\, G$ and $\dim\, ver_p P = \dim\, G$, the horizontal space has $\dim\, hor_p P = \dim\, M$.

In short, the connection on a principal bundle is a $\mathfrak{g}$-valued one form which projects $T_p P$ to $ver_p P \cong \mathfrak{g}$. The full definition is: A *connection* on P is defined as a smooth and unique split of the tangent space $T_p P$ at each point $p \in P$

$$T_p P = ver_p P \oplus hor_p P \qquad X = verX + horX$$

where $X \in T_p P, \quad verX \in ver_p P, \quad horX \in hor_p P$, such that the choice of a horizontal subspace at a point p determines all horizontal subspaces at points pg:

$$hor_{pg} P = R_{g^*} hor_p P \qquad\qquad \text{for every} \qquad g \in G,$$

where R_{g^*} is push-forward of the right translation in G. This is called the *equivariance condition*.

Parallel transport is then defined as a horizontal lift: If $C : [0, 1] \to M$ is a curve in the base manifold, its horizontal lift $\tilde{C} : [0, 1] \to P$ obeys (1) $\pi(\tilde{C}) = C$, (2) all vectors X_P tangent to $\tilde{C}$ are horizontal: $X_P \in hor_{\tilde{C}} P$.

Given the notion of parallel transport in $\mathbb{P}$ one is able to define parallel transport on any fibre bundle associated to $\mathbb{P}$. Take for example the canonical vector bundle $\mathbb{V}$ with the representation space of the structure group: If $c_P(t)$ is a horizontal lift of a curve $c(t)$ in the base space, the curve $c_V(t)$ is a horizontal lift in $\mathbb{V}$ if $c_V(t) = [(c_P(t), v)]$, where v is a constant element of V.

Connection as a Lie-Algebra Valued One-Form

A connection in a principal bundle can also be characterized by a one-form ω with values in $\mathfrak{g} \cong T_e G$. Explicitly: Define the connection one-form $\omega \in \Gamma(T^* P) \otimes \mathfrak{g}$

on P by the projection of the tangent space T_pP onto the vertical subspace ver_pP satisfying

$$\omega(\hat{X}_A) = A \qquad \text{for every} \qquad A \in \mathfrak{g}$$
$$R_g^*\omega = Ad_{(g^{-1})}\omega \qquad \text{for every} \qquad g \in G.$$

Here $\hat{X}_A$ is the vector field on P induced by the the exponential map $R_g(\exp tA)$, R_g^* is the pull-back of the right-action $R_g : p \mapsto pg$ and $\mathrm{Ad} : A \mapsto hAh^{-1}$ is the automorphism of $\mathfrak{g}$ associated to $h \in G$ by the adjoint representaion of G in its algebra. The second relation serves to show that the definition of the connection is equivalent to its definition *via* parallel transport. Namely, for an arbitrary vector $X_p \in T_pP$ it reads $R_g^*\omega_p(X) = \omega_{pg}(R_{g^*}X) = g^{-1}\omega(X)g$, and it can be shown that the horizontal subspace defined by the kernel of the map $\omega : T_pP \to \mathfrak{g}$,

$$hor_pP = \{X \in T_pP|\omega(X) = 0\}$$

satisfies the equivariance condition $hor_{pg}P = R_{g^*}hor_pP$.

The explicit form of the connection and its dependency on A can be verified to be:

$$\omega = g^{-1}dg + g^{-1}Ag. \tag{E.25}$$

Covariant Derivative and Curvature

In order to define the curvature on a (principal) bundle we first need a definition of the covariant derivative: Consider a Lie-algebra valued q-form $\alpha \in \Omega_qP \otimes \mathfrak{g} = \Omega(P, \mathfrak{g})$ and decompose it as $\alpha = \alpha^a \otimes X_a$ where α^a is an "ordinary" q-form and the set $\{X_a\}$ constitutes a basis of $\mathfrak{g}$. Furthermore let $X_1, \cdots, X_{q+1} \in T_pP$ be tangent vectors on P. The *exterior covariant derivative* is defined by

$$D\alpha(X_1, \cdots, X_{q+1}) = d_P\alpha\ (hor\ X_1, \cdots, hor\ X_{q+1})$$

where $d_P\alpha = (d_P\alpha^a) \otimes X_a$ is the exterior derivative on P.

The curvature 2-form Ω of a connection ω on the bundle P is defined by

$$\Omega = D\omega \in \Omega_2P \otimes \mathfrak{g}. \tag{E.26}$$

The curvature 2-form has the property $R_g^*\Omega = g^{-1}\Omega g$ and can be shown to fulfill the Cartan structure equation $\Omega = d_P\omega + \omega \wedge \omega$.

E.5.3 Yang-Mills Gauge Field Theory in Fibre Bundle Language

How do the mathematical bundle notions relate to the physics language used to describe Yang-Mills gauge theories or gravitational theories. In this subsection this will be answered for a Yang-Mills theory.

Indeed the "stage" for gauge theories are principal bundles and their associated vector bundles. The idea is to enlarge the space-time arena M, taken to be for instance 4-dimensional Minkowski space, by attaching to each spacetime point a fibre, each fibre being homoemorphic to the gauge group.

Matter fields are represented by sections in real or complex vector bundles $\mathbb{V}$ associated with the principal bundle $\mathbb{P}$. They are defined by local sections $s : U \to P$ (where $U \subset M$). The section provides a local trivialization of V by fixing D independent basis vectors $e_\alpha : U \to V$. By this the fibre V_x obtains a linear basis $\{e_\alpha\}_x$. The fields $\Psi : U \to V$ are then uniquely represented by $\Psi(x) = \Psi^\alpha(x)\{e_\alpha\}_x$.

What in physics is called "local gauge transformation" corresponds to an automorphism in the principal bundle: Consider mappings $G(p) : P \to P$ which are equivariant, i.e. $G(gp) = gG(p)$ for $g \in G$, and which preserve each fibre, meaning that they act trivially on the base space: $(\pi \circ G(p) = \pi)$. These maps constitute the group $\mathfrak{G}$ of local gauge transformations. It has group elements depending on spacetime. It is therefore an infinite-dimensional group. From its definition, $\mathfrak{G}$ is a subgroup of the automorphism group $Aut(P)$. It has a subgroup $\mathfrak{G}_\circ$ (vertical automorphism) with group elements mediating "pure gauge transformations". In general, the group $\mathfrak{G}$ can be taken (locally) to be isomorphic to the semidirect product of the covariance group $Cov(M)$ of the base manifold and the structure group:

$$\mathfrak{G} \cong Cov(M) \ltimes G. \tag{E.27}$$

For an $SU(N)$ Yang-Mills theory in 4-dimensional Minkowski space this reads $\mathfrak{G} \cong ISO(1,3) \ltimes SU(N)$. It is important not to confuse the two groups G and $\mathfrak{G}$. The structure group G constitutes the fibre and is of relevance when it comes to define the connection. The group of local gauge transformations $\mathfrak{G}$ (with the exception of $\mathfrak{G}_0$) mediates via the gauge principle the interaction of matter fields with the gauge fields.

In the language of infinitesimal transformations the Ψ transform as (E.16): $\delta_\gamma \Psi^\alpha = (\hat{\gamma})^\alpha{}_\beta \ \Psi^\beta = \gamma^a (\hat{X}_a)^\alpha{}_\beta \ \Psi^\beta$, where $(\hat{X}_a)^\alpha{}_\beta$ are d-dimensional representation matrices for the generators X_a of the structure group. This is often termed "gauge transformations of the first kind".

The gauge potentials and gauge field strengths are related to the connection one-form and the curvature two-form, respectively, in the principal bundle. The relation becomes explicit as follows: The connection form (E.25) and the curvature form (E.26) are defined globally on the principal bundle. The gauge potential $\mathcal{A}$ and the field strength $\mathcal{F}$ are differential forms in the base manifold. If the principal bundle is non-trivial, a relation between the forms in the principal bundle and its base manifold can only be achieved locally. To understand its meaning in geometric terms let us consider at first the relation between the connection one-form A and the gauge potential $\mathcal{A}$: Take an open subset U of the base space and choose a local section s on U. The *gauge potential* $\mathcal{A}$ is defined as $\mathcal{A} := s^*\omega \in \Gamma(U, T^*M) \otimes \mathfrak{g}$, that is as the pull-back map from P to M.

This shall be made more explicit (according to Sect. 7.10 in [369]): Let the local coordinates in P be given by $p = (x, g),\ x \in M,\ g \in G$. Choose

$$\partial^{\alpha\beta} := \frac{\partial}{\partial g_{\alpha\beta}} \ \in ver_{(x,g)}P \qquad \tilde{\partial}_\mu := \frac{\partial}{\partial x^\mu} + C_{\mu\alpha\beta}\frac{\partial}{\partial g_{\alpha\beta}} \ \in hor_{(x,g)}P$$

as a basis in ver_pP and hor_pP, respectively. Then $X = X_{\alpha\beta}\partial^{\alpha\beta} + X^\mu\tilde{\partial}_\mu$ is a generic vector in the tangent space of P. The connection form (E.25) is locally

$$\omega_p = g^{-1}_{\alpha\beta}\ dg_{\alpha\beta} + g^{-1}_{\alpha\beta}\ A^a_\mu\ (T_a)^{\alpha\gamma}\ dx^\mu\ g_{\gamma\beta}$$

where (T_a) are representation matrices for the generators of G. Now

$$\langle\omega_p, X_p\rangle = g^{-1}_{\alpha\beta}\ \left(X_{\alpha\beta} + X^\mu C_{\mu\alpha\beta}\right) + g^{-1}_{\alpha\beta}\ A^a_\mu\ (T_a)^{\alpha\gamma}\ g_{\gamma\beta}\ X^\mu.$$

For $X_p \in hor_pP$, for which all coefficients $X_{\alpha\beta}$ vanish, we require $\langle\omega_p, horX_p\rangle = 0$. And since this must hold for all vectors in hor_pP, we get $g^{-1}_{\alpha\beta}\left[C_{\mu\alpha\beta} +\ A^a_\mu\ (T_a)^{\alpha\gamma}\right]\ g_{\gamma\beta}$. After multiplication with g we get a relation that fixes the coefficients $C_{\mu\alpha\beta}$ in the basis vectors for hor_pP, that is the coefficients in the derivative $\tilde{\partial}_\mu$:

$$C_{\mu\alpha\beta} = -A^a_\mu\ (T_a)^{\alpha\gamma}\ g_{\gamma\beta}.$$

From this we sense that the derivative $\tilde{\partial}_\mu$ corresponds to what physicists call a covariant derivative in a gauge theory.

Quite in analogy to the gauge potential, the *Yang-Mills field strength* is defined as $\mathcal{F} := s^*\Omega \in \Gamma(U, \Omega_2 B) \otimes \mathfrak{g}$. And indeed from the Cartan structure equation we derive

$$\mathcal{F} = s^*\Omega = s^*d_P\omega + s^*\omega \wedge \omega = d(s^*\omega) + s^*\omega \wedge s^*\omega = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

(Here use was made of (E.2).) Notice that both the gauge potential and the field strength are defined with respect to a chart U in the base manifold and with respect to a section in this chart. Consider now two intersecting charts $U \cap U' \neq \emptyset$ for which $s' = sg$. One shows that the corresponding gauge potentials are related by $\mathcal{A}' = g^{-1}\mathcal{A}g + g^{-1}dg$, also called "gauge transformations of the second kind". This reveals that gauge freedom corresponds to the freedom to choose local coordinates on a principal bundle. Similarly one finds for the field strengths defined on two charts U_α and U_β with $\mathcal{F}_\alpha = s^*_\alpha\Omega$ on U_α, etc., the compatibility relation $\mathcal{F}_\beta = g^{-1}_{\alpha\beta}\mathcal{F}_\alpha g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, where $g_{\alpha\beta}$ are the transition functions.

A *pure gauge* is defined by

$$\mathcal{A} = g^{-1}dg \quad \Longleftrightarrow \mathcal{F} \quad = 0.$$

which are inner automorphism in the gauge algebra.

Table E.1 Fibre bundle and gauge field entities

Fibre bundle	Gauge field theory
principal bundle $\mathbb{P}$	"stage" of gauge field theory
base space M	spacetime
structure group G	gauge group
connection on $\mathbb{P}$	gauge potential
curvature on $\mathbb{P}$	field strength
vector bundle $\mathbb{V}$ associated to $\mathbb{P}$	"stage" of matter fields
section in $\mathbb{V}$	matter field
automorphisms of $\mathbb{P} \in \mathfrak{G}$	local gauge transformations
vertical automorphisms of $\mathbb{P} \in \mathfrak{G}_0$	pure gauge transformations
covariant derivative	dynamical coupling

Table (E.1), which is adapted from other similar comparisons found in the literature (originating from [556]) summarizes the previous statements[10].

The $U(1)$ principal bundle for electrodynamics and the $SU(N)$ bundles for non-Abelian gauge theories can be trivialized if the base manifold is the full Minkowski space. Thus the bundle approach seems to be nothing but an overload in formalism. But this is no longer true if the base space is non-contractible. An example is the Aharonov-Bohm effect, where in its most simple configuration the base space is a 2-dimensional plane from which a disc is cut out. Another example is electromagnetism with monopoles in which the base space is $\mathbb{R}^2 \times S^2$. Bundle techniques became important also for other topological configurations like instantons in non-Abelian gauge theories. And the non-existence of a global gauge choice in Yang-Mills theories, known as Gribov ambiguity by physicists, relates to the fact that the principal bundle on which gauge choices correspond to sections, is non-trivial; see [285].

Even for globally trivial bundles the fibre bundle approach to Yang-Mills gauge field theories and gravitational theories (if formulated in terms of metric and tetrads) is not simply an overload in formalism, but is a means to isolate the 'physically significant' parts of these theories and distinguish these from their 'merely mathematical' aspects [244].

E.5.4 Metric and Tetrad Gravity on a Bundle

Metric Manifolds

Gravitational theories as compared to Yang-Mills theories have an additional structure, namely a metric. The existence of a metric on a manifold allows to establish additional mappings between different bundles erected over a base space.

A *Riemannian metric* g_x is a tensor on $T_xM \otimes T_xM$ which is (i) symmetric: $g_x(X,Y) = g_x(Y,X)$; (ii) positive definite: $g_x(X,X) \geqslant 0$, and $g_x(X,X) = 0$

[10] Notice, that according to this table, we have a quite generic definition of symmetry transformations as "automorphisms in bundles".

only for $X = 0$. A metric is *pseudo-Riemannian* if instead of (ii) holds (iii) if $g_x(X,Y) = 0$ for $\forall X \in T_xM \Rightarrow Y = 0$.

We recall that on arbitrary manifolds, an inner product can be defined as a map

$$\langle -,-\rangle : T_x^*M \otimes T_xM \to \mathbb{R} \qquad \langle \omega, X\rangle = \langle \omega_\mu dx^\mu, X^\nu \partial_\nu\rangle = \omega_\mu X^\mu .$$

The metric defines an inner product between two vectors $X, Y \in T_x(M)$ by $g_x(X,Y) : T_xM \otimes T_xM \to \mathbb{R}$. This gives rise to a map

$$g_x(X,-) : T_xM \to \mathbb{R} \qquad Y \mapsto g_x(X,Y).$$

This map can be identified with a one-form: $g_x(X,-) \leftrightarrow \omega$. Since on the other hand, the one-form ω induces a unique vector X by $\langle \omega, Y\rangle = g_x(X,Y)$, the metric makes available an isomorphism

$$T_xM \cong T_x^*M.$$

Locally the metric is represented as $g_x = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$, and the isomorphism between vectors and forms becomes

$$\omega_\mu = g_{\mu\nu}X^\nu, \;\; X^\mu = g^{\mu\nu}\omega_\nu \qquad \text{and} \quad g(X,Y) = \langle \omega, Y\rangle = \omega_\mu X^\mu = g_{\mu\nu}X^\nu Y^\mu .$$

Raising and lowering indices with a metric is of course familiar from Riemannian geometry. But the previous expression makes it clear that a Riemann contravariant vector is the component of a vector in a tangent space, and that the covariant vector is the component of a one-form in a cotangent space. It is only by the metric that one can be expressed by the other.
The metric components $g_{\mu\nu}$ are defined with respect to base vectors in the tangent spaces, e.g. in the holonomic basis by $g_{\mu\nu}(x) = g_x(\partial_\mu, \partial_\nu)$, or in an anholonomic basis by $g_{IJ} = g(e_I, e_J)$. (From now on I suppress the reference to the point x, but keep in mind that the subsequent procedure[11] is bound to work for each point separately.) If one switches to another basis, in matrix notation $e' = \Upsilon e$, the metric components change as $g' = \Upsilon^T g \Upsilon$. Now by choosing $\Upsilon = O \cdot D$, where O is an orthogonal matrix ($O^T = O^{-1}$) and D is a diagonal matrix, one can achieve that g' becomes diagonal with either +1 or -1 on the diagonal. By an appropriate choice of the orthonormal matrix one can arrange that all the negative values appear first. This shows that any metric vector space locally has a basis on which the metric tensor has the diagonal form $(\eta) = \mathrm{diag}(-1, \cdot\cdot, -1, 1, \cdot\cdot, 1)$. The basis in which this is true is said to be *orthonormal*.

In the main text (see Subsect. 7.3.) the essentials of Riemann-Cartan geometry are explained, including its description in terms of tetrads. In the language of fibre bundles the

$$e_I = e_I^\mu \partial_\mu \qquad \text{with} \qquad \{e_I^\mu\} \in GL(D, \mathbb{R})$$

[11] commonly known as Gram-Schmidt algorithm

constitute a frame of basis vectors. According to the Gram-Schmidt algorithm these can be made orthogonal: $g_x(e_I, e_J) = e^\mu_I e^\nu_J g_{\mu\nu}(x) = \eta_{IJ}$. By this, the D-beins e^μ_I are only defined up to $SO(D-1,1)$ transformations in the case of a Lorentzian metric and $SO(D)$ in the case of a Riemannian metric. If at every point of a base space is attached an orthonormal frame as a fibre, this results in the orthonormal frame bundle $OM \subset LM$. This is a principal bundle with structure group $O(D)$. The tetrads are interpreted as sections in the associated tangent bundle.

Soldering in Gravitational Theories

The geometry of a linear frame bundle LM associated to the tangent bundle of a manifold M is closely tied to the geometry of the base manifold. This can be expressed by means of a one-form θ on LM called the solder form. Let $u \in L_x M$ be a frame at $x \in M$. It was mentioned previously that the map $u : \mathbb{R}^D \to T_x M$ is a linear isomorphism. The solder form of the linear frame bundle is the R^D-valued one-form

$$\theta_u(Y) = u^{-1}\pi_*(Y).$$

Here $Y \in T_{(x,u)}LM$ is a tangent vector to LM at the point (x, u), and π is the projection $LM \to M$. The solder form vanishes on vectors tangent to the fiber and it is equivariant

$$\theta_u(Y^V) = 0 \qquad R^*_g\theta = g^{-1}\theta$$

where R_g is right translation by $g \in GL(D, \mathbb{R})$. The form θ gives rise to a soldering of the base space and the fibres.

The soldering also becomes visible in the possibility to choose locally an orthonormal basis in the tangent and the cotangent spaces. The tetrads act in the same way as the soldering form: They transform the coordinate basis to an orthonormal basis. And there is still another view on soldering: The metric tensor which lives in the base space, can be expressed through the tetrads, which as explained before are sections in a tensor bundle. In Riemann geometry this soldering is even stronger, since the connections (living in the principal bundle) can uniquely be expressed by derivatives of the metric.

GR as a Gauge Theory–a Fibre Bundle Perspective

The two previous subsections revealed that despite there are similarities of Yang-Mills theories and gravitational theories, the fibre bundle language clearly uncovers dissimilarities. I couldn’t state this better than A. Trautman [499]:

“For me, a gauge theory is any physical theory of a dynamical variable which, at the classical level, may be identified with a connection on a principal bundle. The structure group G of the bundle P is the group of gauge transformations of the first kind; the group $\mathfrak{G}$ of gauge transformations of the second kind may be identified with a subgroup of the group $\mathrm{Aut}\ P$ of all automorphisms of P. In this sense, gravitation is a gauge theory; the basic gauge field is a linear connection ω (or a connection closely related to a linear connection). $\cdots$ The most important difference between gravitation and other gauge theories is due to the soldering of the bundle of frames LM to the base manifold M. The bundle LM is constructed in a natural and unique way from M, whereas a noncontractible M may be the base of inequivalent bundles in the same structure group. $\cdots$ The soldering form θ leads to torsion which has no analog in nongravitational theories. Moreover, it affects the group $\mathfrak{G}$, which now consists of the automorphisms of LM preserving θ. This group contains no vertical automorphisms other than the identity; it is isomorphic to the group $\mathrm{Diff}\ M$ of all diffeomorphisms of M. In a gauge theory of the Yang-Mills type over Minkowski space-time, the group $\mathfrak{G}$ is isomorphic to the semidirect product of the Poincaré group by the group $\mathfrak{G}_0$ of vertical automorphisms of P. In other words, in the theory of gravitation, the group $\mathfrak{G}_0$ of 'pure gauge' transformations reduces to the identity; all elements of $\mathfrak{G}$ correspond to diffeomorphisms of M."

I leave this here with only few commentaries, since most terms in this quotation were defined before. The relevant principal bundle in gravitational theories is the orthonormal frame bundle. As the structure group one may for instance take the Lorentz group, but as discussed in Subsect. 7.6.3, other gauge groups might be taken as well. The role of "matter" fields in Yang-Mills theories is in gravitational theories taken on by the tetrad fields. The group $\mathfrak{G}$ of local gauge transformations is due to (E.27) $\mathfrak{G} \cong \mathrm{Diff}(M) \ltimes SO(1,3) \cong \mathrm{Diff}(M)$. These structures demonstrate the benefits of the fibre bundle language if one clearly discriminates the objects involved: There are three different groups: (i) the covariance group of the spacetime manifold $\mathrm{Diff}(M)$, (ii) the structure group G = SO(3,1), (iii) the group of local gauge transformations $\mathfrak{G}$. The diffeomorphism group carries no physical meaning. Its appearance reflects the option of choosing coordinates in the base manifold. The peculiarity of gravitational theories is that the group $\mathfrak{G}$ is isomorphic to the covariance group $\mathrm{Diff}(M)$. This becomes still more intricate, if for the structure group one would take the Poincaré group instead of the Lorentz group. Then even $G \cong \mathrm{Diff}(M)$ (as can be understood from the fact that local translations correspond to local diffeomorphisms).

These subtleties reflect again the still persistent debate on a "principle" of general covariance and on the interrelation of covariance and invariance (see Subsect. 7.5.3). The insights which can be gained by the fibre bundle view of gravitational theories are discussed in [244].

Appendix F
*Symmetries in Terms of Differential Forms

Today most of the theoretical research articles on symmetries in fundamental physics are written in terms of differential forms and the exterior calculus. Especially in the discussion of symmetries in gravitational actions (and their extension to supergravity), the argumentation in terms of the exterior calculus is preferable, because otherwise one easily gets lost in the tensor notation. In this appendix I describe the structural relations encoded in the "world action" in a very condensed form. More (or rather, all) of the details–especially with respect to Poincaré gauge theories and metric affine gauge theories–may be found in [259].

F.1 Actions and Field Equations

F.1.1 The "World" Action

The classical action, in D-Dimensions, written as

$$S = \int d^D x \, \mathcal{L}_c(Q^A, Q^A_{,\mu}) \tag{F.1}$$

is a real quantity and a scalar with respect to Lorentz transformations. As described, the Lagrange density $\mathcal{L}_c$ is a function depending on a collection of fields–generically called Q^A–and their first derivatives only. In this appendix, where everything is cast in terms of differential forms, this restriction comes quite naturally. Higher-derivative theories can in principle be expressed in the language of differential forms by the introduction of Lagrange multipliers. As elucidated in the main text, the functional form of $\mathcal{L}_c$ may also be restricted by (dimensional) renormalization requirements, implying restrictions on the dependence of the Lagrangian on the fields and their derivatives.

In fundamental physics we are dealing with specific variants of fields, where each variant is characterized by its behavior under Lorentz transformations, gauge group transformations and diffeomorphisms.

K. Sundermeyer, *Symmetries in Fundamental Physics,*
Fundamental Theories of Physics 176, DOI 10.1007/978-94-007-7642-5,

In terms of differential forms the previous coordinate-referred action is

$$S = \int_{M^D} \mathcal{L}(Q, dQ). \tag{F.2}$$

Here $\mathcal{L} \in \Omega_D(M, R)$ is a D-form, which locally can be written with the volume form η as:

$$\mathcal{L} = \mathcal{L}_c\, \eta \equiv \mathcal{L}_c\, \vartheta^0 \wedge \vartheta^1 \wedge \ldots \wedge \vartheta^{D-1},$$

where the ϑ^I constitute a basis in T^*M. This shows the (local) relation between the coordinate action (F.1) and the form action (F.2). Observe that in (F.2) there is no explicit reference to coordinates, thus this formulation of the action is by shape and construction invariant with respect to general coordinate transformations–so long as $\mathcal{L}$ transforms as a Riemannian scalar.

In the following, we deal with the fields $Q = \{\vartheta, \omega, A, \psi, \phi\}$, that is, tetrads ϑ (1-forms), spin-connections ω (1-forms), Yang-Mills gauge fields A (1-forms), fermion fields ψ (0-forms), and Higgs scalars ϕ (0-forms). According to present knowledge, the "world action" splits into a (G)ravitational and a (M)atter part[1]:

$$S[\vartheta, \omega, A, \psi, \phi] = S_G[\vartheta, \omega] + S_M[\vartheta, \omega, A, \psi, \phi]. \tag{F.3}$$

We may assume that because of diffeomorphism invariance the gravitational part of the Lagrangian contains the dependence on the derivatives of the tetrads and on the spin-connection only via the torsion Θ and the curvature Ω (both being 2-forms)[2]

$$\Theta^I = d\vartheta^I + \omega^I{}_J \wedge \vartheta^J = \frac{1}{2} T^I_{KJ} \vartheta^{KJ} \tag{F.4a}$$

$$\Omega^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K_J = \frac{1}{2} R^I_{JKL} \vartheta^{KL}. \tag{F.4b}$$

Therefore,

$$\mathcal{L}_G(\vartheta, d\vartheta, \omega, d\omega) = \tilde{\mathcal{L}}_G\big(\vartheta, \Theta(\vartheta, d\vartheta, \omega), \Omega(\omega, d\omega), \lambda\big). \tag{F.5}$$

Here, the extra variable(s) λ are introduced as non-dynamical multiplier fields in the case of any constraints that might be imposed on the theory. Take for instance the goal to preclude torsion right from the beginning. Then $\tilde{\mathcal{L}}_G = \tilde{\tilde{\mathcal{L}}}_G(\vartheta, \Theta, \Omega) + \lambda_I \Theta^I$, so that the Euler-Lagrange equations for λ result in $\Theta^I = 0$.

---

[1] Although perhaps anchored in our minds by a scientific education, this might not be true–see for instance the equivalence of Brans-Dicke theory with GR: Brans-Dicke theory contains a scalar field. Another example is the CDJ form of GR with its scalar field η; see Sect. 7.5.2 "Optional forms of actions for GR".

[2] Similar to the case of local phase invariance (see Sect. 5.3.4) we might prove this assumption by Noether's second theorem, or the Klein-Noether identities.

We also assume minimal coupling of matter to gravity, that is a matter Lagrangian in which the dependence on the spin connection stems solely from replacing the derivative of the "matter" fields $\Phi = \{A, \psi, \phi\}$ by the covariant derivatives:

$$\mathcal{L}_M(\vartheta, \omega, \Phi, d\Phi) = \tilde{\mathcal{L}}_M(\vartheta, \Phi, D\Phi) \tag{F.6}$$

where the covariant derivative is defined as

$$D\Phi^\alpha = d\Phi^\alpha + \omega^I{}_J \wedge (\hat{X}_I{}^J)^\alpha{}_\beta \Phi^\beta,$$

with the matrices $\hat{X}$ originating from the representation of the Lorentz group to which the fields belong. In case of non-minimal coupling the matter Lagrangian can also depend on $d\vartheta$ and $d\omega$, but in order to be diffeomorphism-invariant, this dependence comes only via the torsion and the curvature[3].

General Relativity

Most of the following is universal in the sense that it makes no difference which gravitational Lagrangian is selected. The Lagrangian (F.5) covers for instance GR, the Einstein-Cartan theory, and Poincaré gauge theories in their most general form.

Taking as an example the gravitational part of general relativity with its Hilbert-Einstein action (including a cosmological term Λ) we find

$$S_{HE} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(R-2\Lambda) = \frac{1}{2\kappa}\int_M (R-2\Lambda)\,\eta = \frac{1}{2\kappa}\int *(R-2\Lambda), \tag{F.7}$$

where R is the Riemann curvature scalar with $*R = \Omega^{IJ} \wedge \eta_{IJ}$ because of

$$\begin{aligned}\Omega^{IJ} \wedge \eta_{IJ} &= \frac{1}{2}R^{IJ}{}_{KL}\vartheta^K \wedge \vartheta^L \wedge \eta_{IJ} \\ &= \frac{1}{2}R^{IJ}{}_{KL}\vartheta^K \wedge (\delta^L_J\eta_I - \delta^L_J\eta_I) = -R^{IJ}{}_{KI}\vartheta^K \wedge \eta_J = -R^{IJ}{}_{JI}\eta = R\eta.\end{aligned}$$

Maxwell and Yang-Mills

In order to see the relation between the component and the form notations, we consider Maxwell's theory in Minkowski space. The vector potential may be expressed as a one-form

$$A = A_\mu dx^\mu = A_I\vartheta^I$$

[3] A dependence of $\mathcal{L}_M$ on ω can be shown to be ruled out by requiring Lorentz invariance [31]

and the (two-form) field strength is

$$F := dA = A_{\mu,\nu}\, dx^\nu \wedge dx^\mu = \frac{1}{2}(A_{\mu,\nu} - A_{\nu,\mu})dx^\nu \wedge dx^\mu = \frac{1}{2}F_{\nu\mu}\, dx^\nu \wedge dx^\mu.$$

We immediately recover from $dF = d(dA) \equiv 0$ the homogeneous Maxwell Eqs. (3.1c, 3.1d). Furthermore, the gauge invariance of the field strength is more or less trivial: with the choice of a different vector potential $A^\Lambda = A + d\Lambda$, the field strength is unchanged: $F = F^\Lambda$. A gauge choice amounts to impose a condition on Λ. Take for instance the Lorenz gauge condition

$$\triangle\Lambda = -d^\dagger A \tag{F.8}$$

where $\triangle = dd^\dagger + d^\dagger d$ is the Laplace-Beltrami operator and $d^\dagger$ the coderivative (E.9). Then $d^\dagger A^\Lambda = d^\dagger A + d^\dagger d\Lambda = d^\dagger A + \triangle\Lambda = 0$, revealing that in this gauge the gauge potential has to fulfill $d^\dagger A = 0$. You may convince yourself that in 4-dimensional Minkowski space $d^\dagger A = -\partial_\mu A^\mu$.

The coordinate version of the ED Lagrangian is

$$\mathcal{L}_c = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu.$$

In terms of a D-form $\mathcal{L} = \mathcal{L}_c\eta$, we expect a kinetic term proportional to the "square" of the field strength. Since the field strength is itself a 2-form, there is only the $(D\text{-}2)$-form $*F$ available to build the "square":

$$\begin{aligned} F \wedge *F &= (\frac{1}{2}F_{IJ}\vartheta^I \wedge \vartheta^J) \wedge *(\frac{1}{2}F_{KL}\vartheta^K \wedge \vartheta^L) = \frac{1}{4}F_{IJ}F^{KL}\,\vartheta^{IJ} \wedge \eta_{KL} \\ &= \frac{1}{4}F_{IJ}F^{KL}\,\vartheta^I \wedge 2\delta^J_{[L}\eta_{K]} = \frac{1}{4}F_{IJ}F^{KL}\,\vartheta^I \wedge (\delta^J_L\eta_K - \delta^J_K\eta_L) \\ &= \frac{1}{2}F_{IJ}F^{IJ}\,\eta. \end{aligned}$$

Thus the action for electrodynamics in terms of forms is

$$S_{ED}[\vartheta, A] = \int_M (-\frac{1}{2}F \wedge *F + A \wedge *J). \tag{F.9}$$

This is immediately extended to free Yang-Mills theory, since extra group indices on the gauge fields are simply add-ons and do not interfere with the form type: If $\mathbf{G}$ is the gauge group, the gauge fields A are $\mathfrak{g}$-valued one-forms: $A \in \mathfrak{g} \otimes \Omega_1(M)$. According to (E.20) the field strength in the non-Abelian theory is the 2-form $F = dA + \frac{1}{2}A \wedge A$. The Yang-Mills action is

$$S_{YM}[\vartheta, A] = \int_M (-\frac{1}{2}\mathrm{tr}(F \wedge *F) + A \wedge *J) \tag{F.10}$$

where tr is the trace on the representation of $\mathfrak{g}$ with the conventional normalization condition $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. Furthermore, it is to be understood that also J is a Lie algebra valued one form, e.g. $J = J^a_\mu T_a dx^\mu$.

Scalars

Let us next examine the field theory for a neutral scalar field: The field ϕ is a zero-form. A generic Lagrangian is

$$\mathcal{L} = \frac{1}{2}(d\phi \wedge *d\phi) - V(\phi).$$

The potential can for instance be of the $\lambda\phi^4$ type:

$$V(\phi) = \frac{1}{2}m^2(\phi \wedge *\phi) + \frac{\lambda}{4!}(\phi \wedge *\phi) \wedge *(\phi \wedge *\phi).$$

The interaction of the scalar field with a Yang-Mills field is described by minimal coupling, that is by the replacement

$$d\phi \ \rightarrow \overset{A}{D}\phi = d\phi + A \wedge \phi,$$

(here a coupling constant g is absorbed into the A-field) where $\overset{A}{D}$ is the covariant derivative with respect to the gauge group connection A. The precise meaning of this modified exterior derivative will be defined below in F.4. Therefore

$$S_H[\vartheta, A, \phi] = \int_M \frac{1}{2}(\overset{A}{D}\phi \wedge * \overset{A}{D}\phi) - V(\phi). \tag{F.11}$$

Fermions

As exhibited in the main text, fermion fields ψ transform with the spinor representation of the Lorentz-group. They additionally may have values in the representation of the gauge group. Thus a covariant derivative is

$$D_\mu\psi = \partial_\mu\psi + \omega^I{}_{J\mu}L^J{}_I\psi + A^a_\mu T^a\psi \qquad \leftrightarrow \qquad D\psi = d\psi + \omega \wedge \psi + A \wedge \psi,$$

where $L^J{}_I$ are the generators of the Lorentz group and T^a the generators of the gauge group, both taken in the representations to which ψ belongs. Defining $\not{D}\Psi = \gamma^I e^\mu_I D_\mu\psi$ the fermion sector of the theory has the action

$$S_F[\vartheta, \omega, A, \phi, \psi] = \int_{M^D} d^4x \ |e|(\bar{\psi}\not{D}\psi + Y(\phi, \bar{\psi}, \psi)). \tag{F.12}$$

Here spinors are considered to be "ordinary" zero-forms. In the mathematical literature you can find approaches to understanding spinors as Clifford-valued differential forms; see e.g. [434].

F.1.2 Field Equations

Field Equations in Terms of Forms

The field equations are found by the principle of stationary action: The generic variation of the Lagrangian in (F.2) is

$$\delta\mathcal{L} = \delta Q \wedge \frac{\partial\mathcal{L}}{\partial Q} + \delta(dQ) \wedge \frac{\partial\mathcal{L}}{\partial(dQ)}$$

(here all contingent indices on the fields Q are suppressed.) The variation δ is meant to be the "active" variation; thus it commutes with the exterior derivative: $(\delta d = d\delta)$. Reshuffling derivatives, $\delta\mathcal{L}$ can be written as

$$\delta\mathcal{L} = \delta Q \wedge \mathcal{E} + d\Theta(Q, \delta Q) \tag{F.13}$$

with the Euler derivatives

$$\mathcal{E} := \frac{\delta\mathcal{L}}{\delta Q} = \frac{\partial\mathcal{L}}{\partial Q} - (-1)^q d\frac{\partial\mathcal{L}}{\partial(dQ)} \tag{F.14}$$

(where q is the form degree of Q) and the exact form

$$\Theta = \delta Q \wedge \frac{\partial\mathcal{L}}{\partial(dQ)},$$

in other contexts also called the *symplectic potential*; see App. F.3.1. If we introduce the field momenta

$$P := \frac{\partial\mathcal{L}}{\partial(dQ)} \tag{F.15}$$

which are $(D\text{-}q\text{-}1)$-forms, the Euler derivatives and the symplectic potentials become

$$\mathcal{E} = \frac{\partial\mathcal{L}}{\partial Q} - (-1)^q dP \qquad \Theta = \delta Q \wedge P.$$

The symplectic potential depends (locally) on the dynamical fields and linearly on their variations. "Normally", this term is thrown away as being "nothing but" a boundary term. But we are aware that this can be somewhat premature, since even for a compact spacetime M with boundary ∂M the action $S = \int_M \mathcal{L}$ with

$$\delta S = \int_M \delta\mathcal{L} = \int_M \delta Q \wedge \mathcal{E} + \int_{\partial M} \Theta(Q, \delta Q) \tag{F.16}$$

will, in general, not be extremized by solutions of $\mathcal{E} = 0$. Only if Θ vanishes on ∂M does the requirement $\delta S \stackrel{!}{=} 0$ imply the vanishing of the Euler derivatives:

$$\mathcal{E}_{\{\vartheta,\omega,A,\phi,\psi\}} = 0,$$

and these are the field equations[4].

We know that an action of a theory is not unique in that we can add surface terms. And it is stressed in the main part of this book that these terms are not at all void of a physical interpretation. Therefore, these will be traced in this appendix in terms of differential forms: Consider instead of the Lagrangian D-form $\mathcal{L}$ the Lagrangian $\mathcal{L}' = \mathcal{L} + d\mathcal{K}$. Then

$$\delta\mathcal{L}' = \delta\mathcal{L} + \delta d\mathcal{K} = [\delta Q \wedge \mathcal{E} + d\Theta] + d\delta\mathcal{K} = \delta Q \wedge \mathcal{E} + d[\Theta + \delta\mathcal{K})].$$

Thus, if the Lagrangian is altered by addition of an exact form, this leaves the Euler derivatives untouched, but causes a shift in the symplectic potential: $\Theta \to \Theta + \delta\mathcal{K}$. Strictly speaking, in general $\mathcal{L}'$ describes a physical system different from the one described by $\mathcal{L}$, because the boundary conditions are changed. In the case that $\mathcal{K}$ depends only on the fields Q and not on its derivatives we have $\delta\mathcal{K} = \delta Q \wedge \frac{\partial\mathcal{K}}{\partial Q}$. Then both Lagrangians lead to the same field equations $\mathcal{E} = 0$ if δQ vanishes on ∂M. In that case, we may call the Lagrangians $\mathcal{L}'$ and $\mathcal{L}$ solution and boundary equivalent. Adding a term $d\mathcal{K}$ to the Lagrangian with $\mathcal{K}$ depending on the derivative of the fields is for instance possible for $\mathcal{K} = -(Q \wedge P)$. Then

$$\delta\mathcal{L}' = \delta Q \wedge \mathcal{E} - d[Q \wedge \delta P].$$

The field equations are unchanged (the Lagrangians are solution-equivalent). However, in this case the field equations only derive from the variation of the Lagrangian if the boundary condition is $\delta P = 0$; and therefore boundary equivalence of $\mathcal{L}$ and $\mathcal{L}'$ is no longer valid, in general. This is the field-theoretic generalization of the case of finite-dimensional systems, as derived in Sect. 2.1.4; see (2.24). Aside from the possible presence of a $d\mathcal{K}$ term, the symplectic potential Θ is not unique for another reason: One can add to it an exact D-1 form $\Theta \to \Theta + dY(Q, \delta Q)$ without altering (F.13).

On the other hand, these considerations point out how boundary conditions on fields and surface terms in an action functional can be balanced [287]: Suppose that (F.16) does not allow one to infer the field equations $\mathcal{E} = 0$ because the surface term with Θ does not vanish for the boundary conditions on ∂M (called X). We seek a modified action S_X such that the field equations result whenever δQ satisfies the conditions X on ∂M. A sufficient condition for S_X to exist is the existence of a (D-1) form $B(Q)$ and a (D-2) form $\mu(Q, \delta Q)$ such that

$$\Theta(Q, \delta Q)_{|\partial M} = \delta B(Q) + d\mu(Q, \delta Q)_{|\partial M}$$

for all variations δQ that satisfy X. Then

$$S_X = \int_M \mathcal{L} - \int_{\partial M} B.$$

[4] Some authors prefer to call the Hodge dual of these expressions field equations ($*\mathcal{E} = 0$). This can be advantageous since the components of these forms directly compare to the field equations in local frames.

Gravitational and Matter Field Equations

With the split (F.3), the variation splits into

$$\delta\mathcal{L} = \left[\delta\vartheta \wedge \frac{\delta\mathcal{L}_G}{\delta\vartheta} + \delta\omega \wedge \frac{\delta\mathcal{L}_G}{\delta\omega}\right] + \left[\delta\vartheta \wedge \frac{\delta\mathcal{L}_M}{\delta\vartheta} + \delta\omega \wedge \frac{\delta\mathcal{L}_M}{\delta\omega} + \delta\Phi \wedge \frac{\delta\mathcal{L}_M}{\delta\Phi}\right] + b.t.$$
$$= \left[\delta\vartheta^I \wedge E_I + \delta\omega^I{}_J \wedge C_I{}^J\right] + \left[-\delta\vartheta^I \wedge T_I - \delta\omega^I{}_J \wedge S_I{}^J + \delta\Phi \wedge \frac{\delta\mathcal{L}_M}{\delta\Phi}\right] + b.t.$$

with the $(D\text{-}1)$ forms

$$E_I := \frac{\delta\mathcal{L}_G}{\delta\vartheta^I} = \frac{\partial\mathcal{L}_G}{\partial\vartheta^I} + d\frac{\partial\mathcal{L}_G}{\partial(d\vartheta^I)} \tag{F.17a}$$

$$C_I{}^J := \frac{\delta\mathcal{L}_G}{\delta\omega^I{}_J} = \frac{\partial\mathcal{L}_G}{\partial\omega^I{}_J} + d\frac{\partial\mathcal{L}_G}{\partial(d\omega^I{}_J)} \tag{F.17b}$$

also called the *Einstein forms* and the *Cartan forms*, respectively, and the matter energy-momentum and the spin form[5]

$$T_I := -\frac{\delta\mathcal{L}_M}{\delta\vartheta^I} = -\frac{\partial\mathcal{L}_M}{\partial\vartheta^I} \tag{F.18a}$$

$$S_I{}^J := -\frac{\delta\mathcal{L}_M}{\delta\omega^I{}_J} = -\frac{\partial D\Phi}{\partial\omega^I{}_J} \wedge \frac{\partial\mathcal{L}_M}{\partial D\Phi} = -\hat{X}_I{}^J\Phi \wedge P_\Phi. \tag{F.18b}$$

Therefore the Euler derivatives become

$$\mathcal{E}_\alpha := \frac{\delta\mathcal{L}_M}{\delta\Phi^\alpha} \qquad \mathcal{E}_I := E_I - T_I \qquad \mathcal{E}_I{}^J := C_I{}^J - S_I{}^J. \tag{F.19}$$

The matter field equations $\mathcal{E}_\alpha = 0$ are $(D\text{-}q_\alpha)$-forms; the field equations $\mathcal{E}_I = 0$ and $\mathcal{E}_I{}^J = 0$ are $(D\text{-}1)$-forms.

Gravitational Vacuum Field Equations

The gravitational vacuum field equations, that is the equations $E_I = 0$ and $C_I{}^J = 0$, are not manifestly covariant, because of the explicit appearance of the spin connections $\omega^I{}_J$. Therefore, these expressions shall be rewritten in terms of torsion and curvature according to (F.5). Instead of varying the Lagrangian with respect to the tetrads and the spin-connections, one has alternatively

$$\delta\tilde{\mathcal{L}}_G = \delta\vartheta^I \wedge \frac{\partial\tilde{\mathcal{L}}_G}{\partial\vartheta^I} + \delta\Theta^I \wedge \frac{\partial\tilde{\mathcal{L}}_G}{\partial\Theta^I} + \delta\Omega^I{}_J \wedge \frac{\partial\tilde{\mathcal{L}}_G}{\partial\Omega^I{}_J} + \delta\lambda \wedge \frac{\partial\tilde{\mathcal{L}}_G}{\partial\lambda}. \tag{F.20}$$

[5] I use essentially the notation of [318].

First note that

$$\frac{\partial\tilde{\mathcal{L}}_G}{\partial\Theta^I} = \frac{\partial\mathcal{L}_G}{\partial d\vartheta^I} =: p_I \qquad\qquad \frac{\partial\tilde{\mathcal{L}}_G}{\partial\Omega^I{}_J} = \frac{\partial\mathcal{L}_G}{\partial d\omega^I{}_J} =: \pi_I{}^J, \tag{F.21}$$

where p_I and $\pi_I{}^J$ are the momenta conjugate to ϑ_I and $\omega^I{}_J$. Furthermore,

$$\delta\Theta^I = \delta d\vartheta^I + \delta\omega^I{}_J \wedge \vartheta^J + \omega^I{}_J \wedge \delta\vartheta^J = D\delta\vartheta^I + \delta\omega^I{}_J \wedge \vartheta^J \tag{F.22a}$$

$$\delta\Omega^I{}_J = \delta d\omega^I{}_J + \delta\omega^I{}_K \wedge \omega^K{}_J + \omega^I{}_K \wedge \delta\omega^K{}_J = D\delta\omega^I{}_J. \tag{F.22b}$$

(Incidentally notice that although the derivative d commutes with the variation δ, the covariant derivative D does not.) From this,

$$\delta\Theta^I \wedge p_I = \delta\vartheta^I \wedge Dp_I + \delta\omega^I{}_J \wedge \vartheta^J \wedge p_I + d(\delta\vartheta^I \wedge p_I)$$
$$\delta\Omega^I{}_J \wedge \pi_I{}^J = \delta\omega^I{}_K \wedge D\pi_I{}^J + d\big(\delta\omega^I{}_J \wedge \pi_I{}^J\big).$$

Collecting all the pieces, (F.20) can be written as

$$\begin{aligned}\delta\widetilde{\mathcal{L}}_G = \delta\vartheta^I \wedge \big(\frac{\partial\tilde{L}_G}{\partial\vartheta^I} + Dp_I\big) + \delta\omega^I{}_J \wedge \big(D\pi_I{}^J + \vartheta^{[J} \wedge p_{I]}\big) \\ + \delta\lambda \wedge \frac{\partial\tilde{\mathcal{L}}_G}{\partial\lambda} + d\big(\delta\vartheta^I \wedge p_I + \delta\omega^I{}_J \wedge \pi_I{}^J\big),\end{aligned} \tag{F.23}$$

and therefore the vacuum gravitational-field equations can also be written in the manifestly covariant form

$$E_I = \frac{\partial\tilde{\mathcal{L}}_G}{\partial\vartheta^I} + Dp_I = 0 \qquad C_I{}^J = D\pi_I{}^J + \vartheta^{[J} \wedge p_{I]} = 0 \tag{F.24}$$

together with constraint equations $\frac{\partial\tilde{\mathcal{L}}_G}{\partial\lambda} = 0$. For later purposes, I introduce the abbreviations

$$e_I = \frac{\partial\tilde{\mathcal{L}}_G}{\partial\vartheta^I} \qquad c_I{}^J = \vartheta^{[J} \wedge p_{I]}. \tag{F.25}$$

As an example, for the Hilbert-Einstein Lagrangian $\tilde{\mathcal{L}}_G \to \mathcal{L}_{HE} = \frac{1}{2\kappa}\Omega^{IJ} \wedge \eta_{IJ} + \lambda_I\Theta^I$ we have $p_I = 0$, $\pi_{IJ} = \frac{1}{2\kappa}\eta_{IJ}$. The field equations become

$$E_I = \Omega^{KJ} \wedge \frac{\partial}{\partial\vartheta^I}\eta_{KJ} = \Omega^{KJ} \wedge \eta_{KJI} = 0 \qquad C_{IJ} = D\eta_{IJ} = 0$$

together with the equation of vanishing torsion. As a matter of fact, the field equations $D\eta_{IJ} = 0$ again imply, due to $D\eta_{IJ} = \eta_{IJK} \wedge \Theta^K$, that the torsion must vanish. (This is true only in the vacuum theory without spin currents acting as sources.). Regarding the vanishing torsion in the first set of field equations, the components of (the dual of) the $(D\text{-}1)$-form equation $E_I = 0$ are indeed identical to Einstein's vacuum field equations.

Maxwell and Yang-Mills Field Equations

Electrodynamics is defined by (F.9). Varying this action for a fixed source current, we get

$$\begin{aligned}-\delta F \wedge *F + \delta A \wedge *J &= -\delta(dA) \wedge *F + \delta A \wedge *J = -d(\delta A) \wedge *F + \delta A \wedge *J \\ &= -d(\delta A \wedge *F) - \delta A \wedge \big(d * F - *J\big).\end{aligned}$$

Dropping the first term, we get the field equations $(d * F = *J)$ which in 4-dimensional Minkowski space (in which $** = 1$ and $d^\dagger = *d*$) can be written

$$d^\dagger F = J.$$

Because of $d^\dagger d^\dagger = 0$, we immediately obtain current conservation: $d^\dagger J = 0$.

The equations $d^\dagger F = J$ together with $dF = 0$ comprise the full set of Maxwell's equations. The latter are fulfilled for $F = dA$ (at least locally). In the Lorenz gauge (F.8), for which $d^\dagger A = 0$, we get from $d^\dagger F = d^\dagger dA = \triangle A = J$ the other relation which the gauge potential must fulfill.

Let me also outline how to arrive from the Yang-Mills action (F.10) to the field equations. We now have

$$\begin{aligned}\delta F \wedge *F &= \delta\Big(dA + \frac{1}{2}[A, A]\Big) \wedge *F = \Big(\delta dA + [\delta A, A]\Big) \wedge *F \\ &= d(\delta A \wedge *F) + \delta A \wedge \Big(d * F + [A, *F]\Big)\end{aligned}$$

from which it is readily understood that the field equations are $D*F = *J$. Just as for the exterior derivative, a coderivative to $D^\dagger$ can be introduced. In 4dim-Minkowski space this amounts to $D^\dagger = *D*$, and the Yang-Mills field equations read $D^\dagger F = J$. One can show that $D^\dagger D^\dagger F = 0$, and therefore the current is covariantly conserved.

Einstein-Maxwell Field Equations

The Einstein-Maxwell theory is defined by the action

$$S[\vartheta, \omega, A] = S_{HE}[\vartheta, \omega] + S_{MW}[\vartheta, A] = \frac{1}{2\kappa}\int_M \Omega^{IJ} \wedge \eta_{IJ} - \frac{1}{2}\int_M F \wedge *F.$$

In the previous passage, we calculated

$$\delta\mathcal{L}_{HE} = \frac{1}{2\kappa}\delta\vartheta^L \wedge \Omega^{IJ} \wedge \eta_{IJL}.$$

Here the term proportional to $\delta\omega^{IJ}$ is dropped because it is proportional to the torsion. This vanishes in the vacuum theory and in this case also because the Maxwell-Lagrangian does not depend on the spin connection. Now use (E.7) to derive

$$\delta\mathcal{L}_{MW} = -\frac{1}{2}\delta(F \wedge *F) = -\delta F \wedge *F - \frac{1}{2}\delta\vartheta^L \wedge [F \wedge i_L * F - i_L F \wedge *F].$$

The matter-field equations are enclosed in the first term and these were derived previously as $d * F = 0$, which are the Maxwell equations[6] as the components of the 1-form

$$d^\dagger F = 0. \tag{F.26}$$

The terms in $\delta\mathcal{L}_{HE} + \delta\mathcal{L}_{MW}$ which are proportional to $\delta\vartheta^L$, constitute the gravitational field equations

$$\frac{1}{2\kappa}\Omega^{IJ} \wedge \eta_{IJL} - \frac{1}{2}[F \wedge i_L * F - i_L F \wedge *F] = 0, \tag{F.27}$$

from which one reads off the energy-momentum current for the electromagnetic field:

$$T_L^{(ED)} = -\frac{1}{2}[F \wedge i_L * F - i_L F \wedge *F].$$

Field Equations Derived by Palatini Variation

In some situations, it turns out to be advantageous to derive the field equations by the Palatini variations[7]. Take for instance the action for flat-space Maxwell vacuum theory

$$S[A, F] = \int_M (\frac{1}{2}F \wedge *F - dA \wedge *F) = \int_M \overline{\mathcal{L}}$$

where A and F are now understood to be independent variables. The Euler-Lagrange equations derive from

$$\begin{aligned}\delta\overline{\mathcal{L}} &= F \wedge \delta * F - \delta dA \wedge *F - dA \wedge \delta * F \\ &= (F - dA)\delta * F - \delta A \wedge d * F - d(\delta A \wedge *F)\end{aligned}$$

as $F = dA$ and $d{*}F = 0$. The latter is the flat-space variant of (F.26), and the former relates the field strength to the connection. Since $\Pi = -\frac{1}{2} * F$ is the momentum canonically conjugate to A, the Lagrangian can also be written as $\overline{\mathcal{L}}[F(A), \Pi] = F \wedge \Pi$.

In the case of gravitational theories, one can verify that the variation of the Lagrangian

$$\overline{\mathcal{L}}_G(\vartheta^K, \omega^{KL}, p_K, \pi_{KL}) = \Theta^I \wedge p_I + \Omega^{IJ} \wedge \pi_{IJ} - \Lambda(p_K, \pi_{KL}), \tag{F.28}$$

---

[6] This highly compact notation reveals again that physics became simpler with the notion of symmetries; compare this with the "original" Maxwell equations written in terms of derivatives of the components of the E- and B-fields with respect to time and to the (x, y, z)- coordinates. And with (F.26), you even have electrodynamics coupled to a gravitational field.

[7] This is sometimes also called "first-order" variation. In the context of differential forms, this terminology might lead to confusion since we already have a first-order formalism in treating the tetrads and the spin connections as independent objects.

explicitly (but suppressing indices)

$$\delta\overline{\mathcal{L}}_G(\vartheta,\omega,p,\pi) = \delta\Theta\wedge p \;+\; \delta\Omega\wedge\pi \;+\; \Theta\wedge\delta p \;+\; \Omega\wedge\delta\pi \\ -\;\delta p\wedge\frac{\partial\Lambda}{\partial p} \;-\; \delta\pi\wedge\frac{\partial\Lambda}{\partial\pi}$$

can be rewritten as

$$\delta\overline{\mathcal{L}}_G = \delta\vartheta^I\wedge E_I + \delta\omega^I{}_J\wedge C^I{}_J + (\Theta - \frac{\partial\Lambda}{\partial p})\wedge\delta p + (\Omega - \frac{\partial\Lambda}{\partial\pi})\wedge\delta\pi \tag{F.29}$$

and thus leads to the field Eqs. (F.24) together with

$$\frac{\delta\overline{\mathcal{L}}_G}{\delta p_I} = \Theta^I - \frac{\partial\Lambda}{\partial p_I} = 0 \qquad \frac{\delta\overline{\mathcal{L}}_G}{\delta\pi_{IJ}} = \Omega^{IJ} - \frac{\partial\Lambda}{\partial\pi_{IJ}} = 0.$$

Here we see one of the advantages of the Palatini formalism, in that restrictions on the geometry (like for instance vanishing torsion) can be imposed on the Lagrangian by selecting the potential Λ to be independent of the associated momenta. Specifically GR is defined by

$$\overline{\mathcal{L}}_{GR} = \Theta^I\wedge p_I + \Omega^{IJ}\wedge\pi_{IJ} - \Lambda^{IJ}\wedge\big(\pi_{KL} - \frac{1}{2\kappa}\eta_{KL}\big).$$

The p variation leads to vanishing torsion, the variation of the multipliers Λ enforce that $\pi_{KL} = \frac{1}{2\kappa}\eta_{KL}$ in accordance with previous findings. Furthermore, the variation of π results in $\Omega = V$, and the variation of the spin connection ω leads to vanishing p, again in agreement with previous findings.

The Palatini description allows a more or less straightforward transition to the Hamiltonian; more about this is given in Sect. F.3.1.

F.2 Symmetries

F.2.1 Generic Variational Symmetries

Noether Currents and Superpotentials

In the previous section, we considered arbitrary variations of the action and derived the field equations–provided boundary terms drop out. In a shorthand notation due to (F.13), the generic variation of the Lagrangian is $\delta\mathcal{L} = \delta Q\wedge\mathcal{E} + d\Theta(Q,\delta Q)$. A variational symmetry is present if the Lagrangian is quasi-invariant, that is, if under an infinitesimal transformation $\delta_\epsilon Q$

$$\delta_\epsilon\mathcal{L} = dF_\epsilon. \tag{F.30}$$

This is by virtue of (F.13) equivalent to

$$dF_\epsilon = \delta_\epsilon Q \wedge \mathcal{E} + d\Theta(Q, \delta_\epsilon Q) \qquad \text{or} \qquad \delta_\epsilon Q \wedge \mathcal{E} + dJ_\epsilon \equiv 0, \tag{F.31}$$

with the Noether current (which is a $(D\text{-}1)$-form)

$$J_\epsilon := \Theta(Q, \delta_\epsilon Q) - F_\epsilon = \delta_\epsilon Q \wedge P - F_\epsilon. \tag{F.32}$$

If a Lagrangian $\mathcal{L}$ obeys a symmetry condition (F.30), the solution-equivalent Lagrangian $\mathcal{L}' = \mathcal{L} + d\mathcal{K}$ obeys $\delta_\epsilon \mathcal{L}' = d(F_\epsilon + \delta_\epsilon \mathcal{K})$, showing that the theory described by $\mathcal{L}'$ is also quasi-invariant. The ambiguity in Θ gives rise to an ambiguity in the Noether current as

$$J_\epsilon \to J_\epsilon + \delta_\epsilon \mathcal{K} + dY(Q, \delta_\epsilon Q).$$

On solutions (denoted as $\mathcal{E} \doteq 0$), for every symmetry transformation characterized by the parameter ϵ, the Noether current is conserved:

$$dJ_\epsilon \doteq 0.$$

Therefore, on-shell, every Noether current can be derived from a *superpotential* $(D\text{-}2)$-form U_ϵ, so that

$$J_\epsilon \doteq dU_\epsilon.$$

The superpotential is not unique, since it can be replaced by $U_\epsilon + dV_\epsilon$ (and it also inherits the ambiguities from the currents).

Klein-Noether identities

Assuming that the symmetry variation can be written in the form[8]

$$\delta_\epsilon Q = \epsilon \mathcal{A} + d\epsilon \wedge \mathcal{B}, \qquad F_\epsilon = \epsilon\sigma + d\epsilon \wedge \rho, \tag{F.33}$$

we also expand the Noether current as

$$J_\epsilon = \epsilon j + d\epsilon \wedge k = \epsilon(j - dk) + d(\epsilon k).$$

From (F.32) and (F.33), one finds the current "components"

$$j = \mathcal{A} \wedge P - \sigma \qquad k = \mathcal{B} \wedge P - \rho. \tag{F.34}$$

[8] This assumption is justified in the case of all fundamental interactions.

Now inserting (F.33) into the identity (F.31), one obtains

$$\begin{aligned} 0 &\equiv (\epsilon\mathcal{A} + d\epsilon \wedge \mathcal{B}) \wedge \mathcal{E} + d(\epsilon(j - dk)) \\ &= \epsilon(\mathcal{A} \wedge \mathcal{E} + dj) + d\epsilon \wedge [\mathcal{B} \wedge \mathcal{E} + j - dk] \end{aligned}$$

leading to the Klein-Noether identities

$$\mathcal{A} \wedge \mathcal{E} + dj \equiv 0 \tag{F.35a}$$

$$\mathcal{B} \wedge \mathcal{E} + j - dk \equiv 0. \tag{F.35b}$$

In the language of differential forms, we obtain solely these two sets of identities. A third one, corresponding to (3.71a) in the component notation, is "automatically" present, since dk is a closed form. Observe that the second identity is present only for local symmetries $(d\epsilon \neq 0)$. In any case, the current j is on-shell conserved:

$$dj = -\mathcal{A} \wedge \mathcal{E} \doteq 0.$$

Since aside from this expression for dj we also have, according to (F.34),

$$dj = d(\mathcal{A} \wedge P) - d\sigma$$

we can equate these expressions to arrive at

$$\begin{aligned} &\mathcal{A} \wedge \frac{\partial \mathcal{L}}{\partial Q} - (-1)^q \mathcal{A} \wedge dP + d\mathcal{A} \wedge P + (-1)^q \mathcal{A} \wedge dP - d\sigma \\ &\qquad\qquad = \mathcal{A} \wedge \frac{\partial \mathcal{L}}{\partial Q} + d\mathcal{A} \wedge P - d\sigma \equiv 0. \end{aligned}$$

In the case of local symmetries we also realize that on-shell

$$j \doteq dk.$$

From (F.35), the Noether identities result immediately:

$$\mathcal{A} \wedge \mathcal{E} - d(\mathcal{B} \wedge \mathcal{E}) \equiv 0. \tag{F.36}$$

Inserting the second Klein-Noether identity into (F.34), we find from

$$J_\epsilon \equiv -\epsilon(\mathcal{B} \wedge \mathcal{E}) + d(\epsilon k), \tag{F.37}$$

that the Noether current itself can be written as a linear combination of field equations and the total derivative of a superpotential $U_\epsilon = \epsilon k$. This was first discovered for diffeomorphism symmetries in GR by P. Bergmann and coworkers around 1950; but as we have just derived, it holds for any local symmetry. Observe that according to (F.37), for the on-shell conservation of the Noether current, it suffices that the field equations are fulfilled for those fields which transform with derivatives of the symmetry parameter ϵ, that is for which $\mathcal{B} \neq 0$ in (F.33). Also keep in mind that the $(D\text{-}1)$ form F_ϵ is not unique since it changes with surface terms in the action. Therefore, also the current components j and k are not unique. It is true that for a given action one can calculate the F_ϵ. For Yang-Mills theory $\delta_\theta \mathcal{L} = 0$ (and thus $F_\theta = 0$), and for general relativity (and all diffeomorphism covariant theories) $\delta_\xi \mathcal{L} = \pounds_\xi \mathcal{L}$, and thus $F_\xi = i_\xi \mathcal{L}$.

Noether Charges

We identified two currents that are conserved on-shell, namely the "full" Noether currents J_ϵ and its "component" j. In taking the latter, define quantities

$$q = \int_S j = \int_{\partial S} k$$

by integrating the $(D\text{-}1)$-form j over a $(D\text{-}1)$-dimensional spatial hypersurface S. On shell, this current component can be derived from a superpotential $(D\text{-}2)$-form as $j = dk$. Therefore the quantity q can be expressed by an integral over the spatial boundary ∂S. But one will find that in general these q do not transform as scalars w.r.t. the symmetries unless one imposes restrictions on the allowed symmetry transformations on the boundary ∂S. In YM-theories the quantities q are gauge dependent and in GR they are coordinate dependent. Thus they do not qualify as conserved charges. Instead take the full Noether current and define for each parameter set ϵ an associated conserved charge by

$$C[\epsilon] = \int_S J_\epsilon = \int_{\partial S} U_\epsilon. \tag{F.38}$$

This seems to open a treasure chest of charges. Which ones can be given a meaning depends on the theory in question. Specifically for generally covariant theories one is interested to identify those charges which correspond to conserved energy and momentum.

In fundamental physics we have two kinds of invariances, either "geometric" ones, associated to coordinate symmetries, or "internal" symmetries, associated to gauge symmetries. The coordinate symmetries we are interested in are the local Lorentz transformations (related to the use of tetrads) and general coordinate transformations. These will be dealt with in the next two subsections.

F.2.2 Lorentz Transformations

Since the action is a Lorentz scalar, the theory is quasi-invariant with respect to Lorentz transformations

$$\delta_L(\lambda)\mathcal{L} = dF_\lambda.$$

In the following, the case of an invariant Lagrangian, that is $dF_\lambda = 0$, is assumed. A generalization to the case of quasi-invariance is possible without difficulty. It is needed for instance in teleparallel theories, see [387]. An infinitesimal Lorentz transformation is parameterized as $\Lambda^I{}_J = \delta^I_J + \lambda^I{}_J$ with $\lambda^{IJ} + \lambda^{JI} = 0$. The λ^{IJ} are coefficients of vector fields λ in the Lie-algebra associated to the Lorentz group: $\lambda = \lambda^{IJ} \otimes X^{IJ}$, where the generators X^{IJ} constitute a basis in the algebra;

for more details about these mathematical structures see App. F.4. below. Under the Lorentz transformations, the fields transform as

$$\begin{aligned}\delta_L(\lambda)\Phi^\alpha &= \lambda^\alpha{}_\beta \\ \Phi^\beta, \delta_L(\lambda)\vartheta^I &= \lambda^I{}_J\, \vartheta^J, \\ \delta_L(\lambda)\omega^I{}_J &= -D\lambda^I{}_J = -d\lambda^I{}_J - \omega^I{}_K\lambda^K{}_J + \lambda^I{}_K\omega^K{}_J. \end{aligned} \tag{F.39}$$

Here $\lambda^\alpha{}_\beta = (\lambda^I{}_J \hat{X}_I{}^J)^\alpha{}_\beta$ is the Lorentz generator for the matter fields (i.e. matrices of the representation to which Φ^α belongs). The group composition (near the identity) is characterized by the commutator of two infinitesimal Lorentz transformations:

$$[\delta_L(\lambda), \delta_L(\lambda')] = \delta_L([\lambda, \lambda']), \tag{F.40}$$

where the commutator of the two parameters is taken in the algebra $\mathfrak{so}(3,1)$.

Currents for Local Lorentz Symmetry

The current corresponding to (F.32) is

$$\begin{aligned}J_\lambda = \delta_\lambda Q \wedge P &= \lambda^\alpha{}_\beta\, \Phi^\beta \wedge P_\alpha + \lambda^I{}_J\, \vartheta^J \wedge p_I - D\lambda^I{}_J \wedge \pi_I{}^J \\ &= -\lambda^I{}_J S_I{}^J + \lambda^I{}_J c_I{}^J - d(\lambda^I{}_J \wedge \pi_I{}^J) + \lambda^I{}_J \wedge D\pi_I{}^J\end{aligned}$$

with the definition of the spin momentum S from (F.18b) and the abbreviation for c from (F.25). Then, because of (F.24) and (F.19), this can finally be cast in the form

$$J_\lambda = \lambda^I{}_J \mathcal{E}_I{}^J - d(\lambda^I{}_J \wedge \pi_I{}^J). \tag{F.41}$$

This is of course compatible with the general finding (F.37) and with the variation of the fields corresponding to (F.33). The superpotential is $U_\lambda = -\lambda^I{}_J \wedge \pi_I{}^J$. The Noether currents are conserved on the solutions of the field equations for the spin connections.

Klein-Noether Identities for Local Lorentz Symmetry

From (F.13), we find directly $\delta_L(\lambda)\mathcal{L} = \delta_L(\lambda)Q \wedge \mathcal{E} + d\big(\delta_L(\lambda)Q \wedge P\big) = 0$, and thus the identity

$$\begin{aligned}&\lambda^\alpha{}_\beta\, \Phi^\beta \wedge \mathcal{E}_\alpha + \lambda^I{}_J \vartheta^J \wedge \mathcal{E}_I - D\lambda^I{}_J \wedge \mathcal{E}_I{}^J \\ &\qquad + d\big(\lambda^\alpha{}_\beta\, \Phi^\beta \wedge P_\alpha + \lambda^I{}_J \vartheta^J \wedge p_I - D\lambda^I{}_J \wedge \pi_I{}^J\big) \equiv 0.\end{aligned} \tag{F.42}$$

In order to go from this expression to the Klein-Noether identities, we shift the covariant derivative away from the parameter λ by making use of

$$D\lambda^I{}_J \wedge Z_I{}^J = d(\lambda^I{}_J \wedge Z_I{}^J) - \lambda^I{}_J \wedge DZ_I{}^J.$$

This is valid because $\lambda^I{}_J \wedge Z_I{}^J$ is a Lorentz scalar for arbitrary $Z_I{}^J$. Then (F.42) can be written as

$$\lambda^I{}_J\, A_I{}^J + d(\lambda^I{}_J\, B_I{}^J) \equiv 0, \tag{F.43}$$

with

$$A_I{}^J = (\hat{X}_I{}^J)^\alpha{}_\beta\, \Phi^\beta \wedge \mathcal{E}_\alpha + \vartheta^{[J} \wedge \mathcal{E}_{I]} + D\mathcal{E}_I{}^J \equiv 0, \tag{F.44a}$$

$$B_I{}^J = (\hat{X}_I{}^J)^\alpha{}_\beta\, \Phi^\beta \wedge P_\alpha + \vartheta^{[J} \wedge p_{I]} + D\pi_I{}^J - \mathcal{E}_I{}^J \equiv 0. \tag{F.44b}$$

The first line is the expected result: These are the Noether identities originating from local Lorentz invariance. And since by the definition of the spin form (F.18b) and the Cartan form from (F.24), $B_I{}^J = -S_I{}^J + C_I{}^J - \mathcal{E}_I{}^J$, the second set of identities (F.44b) yields the field Eqs. (F.19) with respect to the spin connections. There is additional information hidden in the consistency condition following from taking the covariant derivative of (F.44b) and inserting this into the Noether identities (F.44a):

$$D(S_I{}^J - C_I{}^J) \equiv -\vartheta^{[J} \wedge (T-E)_{I]} - (\hat{X}_I{}^J)^\alpha{}_\beta\, \Phi^\beta \wedge \mathcal{E}_\alpha.$$

In case of vacuum gravity, this amounts to a relation between the Cartan and the Einstein form,

$$DC_I{}^J + \vartheta^{[J} \wedge E_{I]} \equiv 0 \tag{F.45}$$

and for a matter-field in a gravitational background field to

$$DS_I{}^J + \vartheta^{[J} \wedge T_{I]} \equiv -(\hat{X}_I{}^J)^\alpha{}_\beta\, \Phi^\beta \wedge \mathcal{E}_\alpha. \tag{F.46}$$

The superpotential becomes $U_\lambda = -\lambda^I{}_J\, \pi_I{}^J$, so that charges

$$C\left[\lambda^I{}_J\right] = -\int_{\partial S} \lambda^I{}_J \pi_I{}^J = -\frac{1}{2\kappa}\int_{\partial S} \lambda^I{}_J\, \eta_I{}^J \tag{F.47}$$

result, where the last expression on the right-hand side are the charges from the Einstein-Cartan theory.

F.2.3 Diffeomorphisms

General coordinate transformations are described infinitesimally

$$\delta_D(\xi) = \pounds_\xi, \qquad \text{where} \qquad \xi = \xi^\mu \partial_\mu = \xi^I e_I$$

is a vector field[9]. They form a group whose algebra near the identity transformation is encoded in the commutator

$$\begin{aligned}[][\delta_D(\xi), \delta_D(\xi')] &= [\pounds_\xi, \pounds_{\xi'}] = [di_\xi, \pounds_{\xi'}] + [i_\xi d, \pounds_{\xi'}] \\ &= d[i_\xi, \pounds_{\xi'}] + [i_\xi, \pounds_{\xi'}]d = di_{[\xi,\xi']} + i_{[\xi,\xi']}d = \pounds_{[\xi,\xi']},\end{aligned}$$

that is

$$[\delta_D(\xi), \delta_D(\xi')] = \delta_D([\xi, \xi']). \tag{F.48}$$

For the commutator of a general coordinate transformation and a Lorentz transformation, one finds

$$[\delta_D(\xi), \delta_L(\lambda)] = \delta_L(\pounds_\xi \, d\lambda) = \delta_L(i_\xi \, d\lambda). \tag{F.49}$$

If the Lagrangian is a weight-one scalar, it transforms as

$$\delta_\xi \mathcal{L} = \pounds_\xi \mathcal{L} = i_\xi d\mathcal{L} + di_\xi \mathcal{L} = di_\xi \mathcal{L},$$

since $d\mathcal{L} = 0$, $\mathcal{L}$ being a D-form. Thus we identify $F_\xi = i_\xi \mathcal{L}$ in (F.30).

Diffeomorphism Currents

The current corresponding to (F.32) can be rewritten as

$$\begin{aligned}J_\xi &= \delta_\xi Q \wedge P - i_\xi \mathcal{L} = \pounds_\xi Q \wedge P - i_\xi Q \wedge \frac{\partial \mathcal{L}}{\partial Q} - i_\xi dQ \wedge \frac{\partial \mathcal{L}}{\partial dQ} \\ &= di_\xi Q \wedge P - i_\xi Q \wedge \frac{\partial \mathcal{L}}{\partial Q} = -i_\xi Q \wedge \mathcal{E} + d(i_\xi Q \wedge P),\end{aligned}$$

or explicitly as

$$J_\xi = -\xi^I \mathcal{E}_I - i_\xi \omega^I{}_J \wedge \mathcal{E}_I{}^J - i_\xi \Phi^\alpha \wedge \mathcal{E}_\alpha + dU_\xi. \tag{F.50}$$

Again, the Noether current can be represented as a sum of a terms that vanish on-shell and a superpotential $U_\xi = i_\xi Q \wedge P$, such that $dJ_\xi \doteq 0$.

From (F.31), we immediately see that in this case of diffeomorphisms

$$dJ_\xi = -\delta_\xi Q \wedge \mathcal{E} = -\pounds_\xi Q \wedge \mathcal{E}.$$

As usual, this vanishes on-shell. But it also vanishes for vacuum gravity if $\pounds_\xi\{\vartheta, \omega\} = 0$, that is in the case of Killing symmetries.

[9] This corresponds to a local transformation $x^\mu \to x^\mu - \xi^\mu$. That is, a minus sign is chosen in this appendix, in order to avoid an abundance of minus signs in subsequent expressions. This differs from the convention used in the main text.

The expression for the current (F.50) in the case of vacuum gravity and in the case of matter in a fixed gravitational background becomes:

$$J^v_\xi = -\xi^I E_I - i_\xi \omega^I{}_J \wedge C_I{}^J + dU^v_\xi$$
$$J^M_\xi = \xi^I T_I + i_\xi \omega^I{}_J \wedge S_I{}^J - i_\xi \Phi^\alpha \wedge \mathcal{E}_\alpha + dU^M_\xi .$$

If the spin current vanishes, we have in particular

$$J^M_I = T_I - i_I \Phi^\alpha \wedge \mathcal{E}_\alpha + d(i_I \Phi \wedge P_\Phi). \tag{F.51}$$

Thus, on-shell ($\mathcal{E}_\alpha \doteq 0$), the Noether current J^M_I is identical to the Hilbert matter energy-momentum current T_I plus an exact form.

Gravitational Klein-Noether Identities for Diffeomorphism Symmetry

Covariant Lie derivatives
Let us first consider a pure gravitational theory in order to motivate the introduction of a covariant Lie derivative. Starting from (F.20) and using the definitions of the momenta according to (F.21) and the abbreviations from (F.25), the condition for diffeomorphism invariance specializes to

$$\pounds_\xi \vartheta^I \wedge e_I + \pounds_\xi \Theta^I \wedge p_I + \pounds_\xi \Omega^I{}_J \wedge \pi_I{}^J - \pounds_\xi \tilde{\mathcal{L}}_G \equiv 0. \tag{F.52}$$

This does not look very revealing; it even is not manifestly Lorentz invariant. In view of an improvement, introduce a covariant derivative with respect to an optional Lorentz connection $\Gamma = \Gamma^I{}_J \Lambda^I{}_J$

$$\overset{\Gamma}{D} = d + \Gamma$$

together with a covariant Lie derivative referring to this connection:

$$\overset{\Gamma}{\pounds}_X \mathcal{F} = (i_X \overset{\Gamma}{D} + \overset{\Gamma}{D} i_X)\mathcal{F} = (i_X(d+\Gamma) + (d+\Gamma)i_X)\mathcal{F} = \pounds_X \mathcal{F} + i_X(\Gamma \mathcal{F}).$$

General properties of these covariant derivatives for arbitrary Γ are investigated in [386]. A quite natural choice is $\Gamma = \omega (= \omega^I{}_J \Lambda_I{}^J)$. Then $\overset{\omega}{D} = D$ and

$$\overset{\omega}{\pounds}_X = \pounds_X + i_X \omega^I{}_J \Lambda^I{}_J . \tag{F.53}$$

The Lorentz-covariant Lie derivative obeys (in this compact notation $\Omega = \Omega^I{}_J \Lambda_I{}^J$)

$$[i_X, \overset{\omega}{\pounds}_Y]\mathcal{F} = i_{[X,Y]}\mathcal{F}$$
$$\left[\overset{\omega}{\pounds}_Y, D\right]\mathcal{F} = i_X \Omega \wedge \mathcal{F}$$
$$\left[\overset{\omega}{\pounds}_X, \overset{\omega}{\pounds}_Y\right]\mathcal{F} = \overset{\omega}{\pounds}_{[X,Y]} \mathcal{F} + i_Y i_X \Omega \wedge \mathcal{F}.$$

Now it is obvious to define a modified diffeomorphism:

$$\overset{\omega}{\delta}_D(\xi) := \overset{\omega}{\mathcal{L}}_\xi = \delta_D(\xi) + \delta_L(i_\xi\omega) \tag{F.54}$$

which is a mixture of a general coordinate transformation and a specific Lorentz transformation for which

$$[\overset{\omega}{\delta}_D(\xi), \delta_L(\lambda)] = \delta_L(\overset{\omega}{\mathcal{L}}_\xi\,\lambda) \qquad \left[\overset{\omega}{\delta}_D(\xi), \overset{\omega}{\delta}_D(\xi')\right] = \overset{\omega}{\delta}_D([\xi,\xi']) + \delta_L(i_{\xi'} i_\xi \Omega).$$

The Lorentz-covariant Lie derivative acts on the fields as

$$\begin{aligned}
\overset{\omega}{\mathcal{L}}_\xi\,\vartheta^I &= i_\xi\Theta^I + D i_\xi\vartheta^I \\
\overset{\omega}{\mathcal{L}}_\xi\,\Theta^I &= i_\xi\Omega^I{}_J \wedge \vartheta^J + \xi^J\Omega^I{}_J + D i_\xi\Theta^I \\
\overset{\omega}{\mathcal{L}}_\xi\,\Omega^I{}_J &= D i_\xi\Omega^I{}_J \wedge P_I{}^J.
\end{aligned}$$

Substituting the Lorentz-covariant Lie derivative into (F.52), one finds after some algebra

$$\overset{\omega}{\mathcal{L}}_\xi\,\vartheta^I \wedge e_I + \overset{\omega}{\mathcal{L}}_\xi\,\Theta^I \wedge p_I + \overset{\omega}{\mathcal{L}}_\xi\,\Omega^I{}_J \wedge \pi_I{}^J - \overset{\omega}{\mathcal{L}}_\xi\,\tilde{\mathcal{L}}_G \equiv i_\xi\omega^I{}_J A_I{}^J \equiv 0, \tag{F.55}$$

where $A_I{}^J \equiv 0$ is the (Lorentz) Noether identity (F.44a). Indeed, it is shown in [386] that (F.52) has the same appearance for any covariant Lie derivative.

Noether Identities

As for the Lorentz symmetry case, we aim at writing the previous identity in the form $A^G + dB^G = \xi^I A_I^G + d(\xi^I B_I^G) \equiv 0$, and then inferring from the arbitrariness of the ξ^I that the expressions A_I and B_I must themselves vanish. (The upper index G, standing for "gravitation", is superfluous here, but it is introduced because later the corresponding expressions for the matter fields are investigated.) The different terms in (F.55) are exploited as

$$\begin{aligned}
\overset{\omega}{\mathcal{L}}_\xi\,\vartheta^I \wedge e_I &= i_\xi\Theta^I \wedge e_I + d(i_\xi\vartheta^I \wedge e_I) - \xi^I \wedge De_I \\
\overset{\omega}{\mathcal{L}}_\xi\,\Theta^I \wedge p_I &= i_\xi\Omega^I{}_J \wedge c_I{}^J + \xi^J\Omega^I{}_J \wedge p_I + d(i_\xi\Theta^I \wedge p_I) + i_\epsilon\Theta^I \wedge Dp_I \\
\overset{\omega}{\mathcal{L}}_\xi\,\Omega^I{}_J \wedge \pi_I{}^J &= d(i_\xi\Omega^I{}_J \wedge \pi_I{}^J) + i_\xi\Omega^I{}_J \wedge D\pi_I{}^J \\
\overset{\omega}{\mathcal{L}}_\xi\,\tilde{\mathcal{L}}_G &= (i_\xi D + D i_\xi)\tilde{\mathcal{L}}_G = D i_\xi\tilde{\mathcal{L}}_G = d i_\xi\tilde{\mathcal{L}}_G.
\end{aligned}$$

Collecting all the terms, one eventually obtains

$$A^G = i_\xi\Theta^I \wedge (e_I + Dp_I) + i_\xi\Omega^I{}_J \wedge (c^J{}_I + D\pi_I{}^J) - \xi^I \wedge De_I + \xi^J\Omega^I{}_J \wedge p_I.$$

This can be expressed as per (F.24, F.25) in terms of (the vacuum gravity) Euler derivatives as

$$A^G = i_\xi\Theta^I \wedge E_I + i_\xi\Omega^I{}_J \wedge C_I^J - \xi^I \wedge DE_I$$

since $\Omega^J{}_I \wedge p_J = -DDp_I$. Furthermore,

$$B^G = \xi^I \wedge (E_I - Dp_I) + i_\xi \Theta^I \wedge p_I + i_\xi \Omega^I{}_J \wedge \pi^J{}_I - i_\xi \tilde{\mathcal{L}}_G.$$

By writing

$$i_\xi F = i_{(\xi^I e_I)} F = \xi^I i_{(e_I)} F =: \xi^I i_I F,$$

both A^G and B^G can be expanded as

$$A_I^G = i_I \Theta^J \wedge E_J + i_I \Omega^K{}_J \wedge C_K^J - DE_I \equiv 0 \tag{F.56a}$$

$$B_I^G = (E_I - Dp_I) + i_I \Theta^J \wedge p_J + i_I \Omega^K{}_J \wedge \pi^J{}_K - i_I \tilde{\mathcal{L}}_G \equiv 0. \tag{F.56b}$$

The first set of these relations represents the Noether identities of any generally covariant vacuum gravity theory with respect to the diffeomorphism symmetry. This can also be read as another relation between the Einstein and the Cartan forms:

$$DE_I = i_I \Theta^J \wedge E_J + i_I \Omega^K{}_J \wedge C_K^J. \tag{F.57}$$

The second set of identities in (F.56) yields an alternative definition of the Einstein form:

$$E_I = D\Big(\frac{\partial \tilde{\mathcal{L}}_G}{\partial \Theta^I}\Big) - i_I \Theta^J \wedge \Big(\frac{\partial \tilde{\mathcal{L}}_G}{\partial \Theta^J}\Big) - i_I \Omega^K{}_J \wedge \Big(\frac{\partial \tilde{\mathcal{L}}_G}{\partial \Omega^K{}_J}\Big) + i_I \tilde{\mathcal{L}}_G.$$

Noether Identities for Diffeomorphism Symmetries in Full

If matter is coupled to gravity[10], we need to explore the full theory defined by a Lagrangian $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M$. Then (F.55) is augmented by "matter terms":

$$\begin{aligned}
&\overset{\omega}{\pounds}_\xi\, \vartheta^I \wedge e_I + \overset{\omega}{\pounds}_\xi\, \Theta^I \wedge p_I + \overset{\omega}{\pounds}_\xi\, \Omega^I{}_J \wedge \pi_I{}^J - \overset{\omega}{\pounds}_\xi\, \tilde{\mathcal{L}}_G \\
&\qquad - \overset{\omega}{\pounds}_\xi\, \vartheta^I \wedge T_I + \overset{\omega}{\pounds}_\xi\, \Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial \Phi^\alpha} + \overset{\omega}{\pounds}_\xi\, D\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{D\Phi^\alpha} - \overset{\omega}{\pounds}_\xi\, \tilde{\mathcal{L}}_M \\
&\qquad\qquad = A + dB = (A^G + dB^G) + (A^M + dB^M) \equiv 0.
\end{aligned}$$

Here the first line is identical to $(A^G + dB^G)$, that is the expression derived in the previous paragraph. The second line is evaluated as

$$\begin{aligned}
A^M + dB^M = &- i_\xi \Theta^I \wedge T_I - D i_\xi \vartheta^I \wedge T_I + i_\xi D\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial \Phi^\alpha} + D i_\xi \Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial \Phi^\alpha} \\
&+ i_\xi DD\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{D\Phi^\alpha} + D i_\xi D\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{D\Phi^\alpha} - d i_\xi \tilde{\mathcal{L}}_M.
\end{aligned}$$

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[10] Observe how this formulation is self-evident.

After some algebra (shifting the D derivatives and relating the curvature term to the covariant derivative of the matter field equation), one arrives at

$$A^M + dB^M = -\, i_\xi \Theta^I \wedge T_I + \xi^I DT_I - i_\xi \Omega^I{}_J \wedge S_I{}^J + (-1)^{|\Phi^\alpha|} i_\xi \Phi^\alpha \wedge D\mathcal{E}_\alpha$$
$$+\, i_\xi D\Phi^\alpha \wedge \mathcal{E}_\alpha + d\Big(i_\xi \Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial \Phi^\alpha} + i_\xi D\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial D\Phi^\alpha} - \xi^I T_I - i_\xi \tilde{\mathcal{L}}_M\Big).$$

Taking together the contributions for the gravitational part (according to (F.56)) and the “matter part” results in:

$$A_I = i_I \Theta^J \wedge \mathcal{E}_J + i_I \Omega^K{}_J \wedge \mathcal{E}_K{}^J - D\mathcal{E}_I + (-1)^{|\Phi^\alpha|} i_I \Phi^\alpha \wedge D\mathcal{E}_\alpha$$
$$+\, i_I D\Phi^\alpha \wedge \mathcal{E}_\alpha \equiv 0 \qquad \text{(F.58a)}$$

$$B_I = (\mathcal{E}_I - Dp_I) + i_I \Theta^J \wedge p_J + i_I \Omega^K{}_J \wedge \pi^J{}_K + i_I \Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial \Phi^\alpha}$$
$$+\, i_I D\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial D\Phi^\alpha} - i_I \tilde{\mathcal{L}} \equiv 0. \qquad \text{(F.58b)}$$

The first expression $A_I \equiv 0$ is the complete diffeomorphism Noether identity. Observe that this holds for any gravitational theory built solely from curvature and torsion and which couples minimally to arbitrary matter fields, as long as they transform according to a representation of the Lorentz group.

Field Theory in a Fixed Gravitational Background

Various conclusions can be drawn from the identities (F.58) if the gravitational field is not dynamical; they become

$$A_I^M = -i_I \Theta^J \wedge T_J - i_I \Omega^K{}_J \wedge S_K{}^J - DT_I + (-1)^{|\Phi^\alpha|} i_I \Phi^\alpha \wedge D\mathcal{E}_\alpha$$
$$+\, i_I D\Phi^\alpha \wedge \mathcal{E}_\alpha \equiv 0 \qquad \text{(F.59a)}$$

$$B_I^M = -T_I + i_I \Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial \Phi^\alpha} + i_I D\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial D\Phi^\alpha} - i_I \tilde{\mathcal{L}}_M \equiv 0. \qquad \text{(F.59b)}$$

Therefore, on the solutions of the matter field equations, we find from (F.59a) the energy-momentum “conservation law”

$$DT_I \doteq i_I \Theta^J \wedge T_J + i_I \Omega^K{}_J \wedge S_K{}^J. \qquad \text{(F.60)}$$

This can be considered together with the angular-momentum “conservation law” (F.46), namely $DS_I{}^J \doteq -\vartheta^{[J} \wedge T_{I]}$. Both of these relations can also be derived from the identities (F.45, F.57) and the field equations.

In Minkowski space, (F.60) becomes $dT_\mu = 0$, and therefore (F.46) can be written as

$$dS^{\mu\nu} + \frac{1}{2} dx^\mu \wedge T^\nu - \frac{1}{2} dx^\nu \wedge T^\mu = d(S^{\mu\nu} + \frac{1}{2} x^\mu \wedge T^\nu - \frac{1}{2} x^\nu \wedge T^\mu)$$
$$=: dJ^{\mu\nu} = 0.$$

From this, we recover the angular momentum part and the spin part in the total angular momentum density $J^{\mu\nu}$.

The identity (F.58b) on a fixed gravitational background becomes

$$T_I = i_I \Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial \Phi^\alpha} + i_I D\Phi^\alpha \wedge \frac{\partial \tilde{\mathcal{L}}_M}{\partial D\Phi^\alpha} - i_I \tilde{\mathcal{L}}, \tag{F.61}$$

an expression for the energy-momentum current defined in terms of the matter fields. Notice that originally it is defined via the derivative of the matter Lagrangian with respect to the tetrads; compare (F.18a)[11]

Example: Klein-Gordon Field Coupled to Einstein-Cartan Gravity

The previous expressions are quite generic and hold for all theories with Lagrangians of the form $\mathcal{L} = \mathcal{L}_G(\vartheta, \Theta, \Omega) + \mathcal{L}_M(\vartheta, \Phi, D\Phi)$. In order to comprehend their essence, let us consider the simplest case:

$$\mathcal{L}_G = \frac{1}{2\kappa}\Omega^{IJ} \wedge \eta_{IJ} \qquad \mathcal{L}_M = \frac{1}{2}(d\phi \wedge * d\phi) - V \qquad V = \frac{1}{2}m^2(\phi \wedge *\phi).$$

This describes a massive scalar field in a Einstein-Cartan theory (without a cosmological constant). The variation of the gravitational part was already treated previously with the result

$$\delta\mathcal{L}_G = \frac{1}{2\kappa}\delta\vartheta^I \wedge \Omega^{KJ} \wedge \eta_{KJI} + \delta\omega^I_J \wedge D\eta_I{}^J + dB_G,$$

with an exact form B_G which has no influence on the field equations. From this we read off the Einstein and the Cartan forms according to (F.17):

$$E_I = \frac{1}{2\kappa}\Omega^{JK} \wedge \eta_{JKI} \qquad C_K{}^J = \frac{1}{2\kappa}D\eta_K{}^J.$$

The variation of the matter Lagrangian is

$$\begin{aligned}\delta\mathcal{L}_M &= \frac{1}{2}\delta(d\phi) \wedge * d\phi \\ &= \delta d\phi \wedge * d\phi + \frac{1}{2}\delta\vartheta^I \wedge [(-1) d\phi \wedge i_I * d\phi - i_I d\phi \wedge * d\phi] - \delta V.\end{aligned}$$

Furthermore,

$$\delta V = \frac{1}{2}m^2\ \delta(\phi \wedge *\phi) = m^2\ \delta\phi \wedge *\phi + \frac{1}{2}m^2\ \delta\vartheta^I \wedge \phi \wedge i_I * \phi$$

so that

$$\delta\mathcal{L}_M = -\delta\varphi \wedge (d * d\phi + m^2 * \phi) - \delta\vartheta^I \wedge T_I + dB_M$$

[11] This reflects the astounding finding in Sect. 7.5.2 that the Belinfante energy-momentum tensor for a matter theory is derived by coupling the matter fields to gravity.

with the matter energy-momentum current according to (F.18a)

$$T_I = \frac{1}{2}\left[d\phi \wedge i_I * d\phi + i_I d\phi \wedge *d\phi\right] + \frac{1}{2}m^2\, \phi \wedge i_I * \phi. \tag{F.62}$$

The spin current $S_I{}^J$ vanishes identically. Collecting all terms, the Euler-Lagrange equations are found:

$$\begin{aligned}
\mathcal{E}_\phi &= -d * d\phi - m^2 * \phi = 0 \\
\mathcal{E}_I &= \frac{1}{2\kappa}\Omega^{JK} \wedge \eta_{JKI} - T_I = 0 \\
\mathcal{E}_K{}^J &= \frac{1}{2\kappa} D\eta_K{}^J = 0.
\end{aligned}$$

We verify that the equation $\mathcal{E}_\phi = 0$ is equivalent to the Klein-Gordon equation: Write

$$d * d\phi = d * \phi_{,\mu} dx^\mu = d(\phi_{,\mu} * dx^\mu) = \phi_{,\mu\nu} dx^\nu \wedge *dx^\mu = \phi_{,\mu\nu} g^{\nu\mu}\eta.$$

Then $\mathcal{E}_\phi = -(\partial_\mu \partial^\mu + m^2)\phi\, \eta$, and $*\mathcal{E}_\phi = 0$ is indeed the Klein-Gordon equation. Furthermore, the energy-momentum tensor (F.62) can be rewritten as

$$T_I = \frac{1}{2}[2 i_I d\phi \wedge *d\phi - i_I(d\phi \wedge *d\phi)] + i_I V = i_I d\phi \wedge *d\phi - i_I \mathcal{L}_M,$$

and is then identical to the canonical energy-momentum tensor for the scalar field. This complies with (F.51) in that $i_I \phi \equiv 0$.

F.3 Gravitational Theories

F.3.1 Energy-Momentum Conservation

Since in Minkowski space energy-momentum conservation has its roots in the invariance of the field theory actions with respect to spacetime transformations, it makes sense to try to identify energy-momentum as conserved charges adjoined to diffeomorphism currents. And since in all other theories energy is always the numerical value of the Hamiltonian, it makes sense to look for an energy-momentum derived from a Hamiltonian.

From (F.50) (and the preceding equation) we identify the diffeomorphism current

$$J_\xi = \pounds_\xi Q \wedge P - i_\xi \mathcal{L} = -i_\xi Q \wedge \mathcal{E} + d(i_\xi Q \wedge P). \tag{F.63}$$

leading according to (F.38) to diffeomorphism charges

$$C[\xi] = \int_{\partial S} U_\xi = \int_{\partial S} (i_\xi Q \wedge P).$$

Symplectic Approach à la Nester

The following is largely adopted from [373]; J.M. Nester and his research group opened up another approach towards a comprehensive understanding of the long-term problem of energy-momentum conservation in general relativity with its relations to the notions of quasilocal energy, boundary conditions in the GR action, and pseudo energy-momentum tensors; see also [87], [372]. They specifically place the Hamiltonian of a theory at the center of their considerations.

In the main text, we explained that a Hamiltonian in a field theory is defined only with respect to the choice of a time variable; see Subsect. 3.4.4. In geometrical terms this means that spacetime is foliated into spacelike surfaces Σ_t (parametrized by t) together with a transversal timelike vector field N, such that $i_N\, dt = 1$. Then any form α can be decomposed into its "time" component $\hat{\alpha} := i_N\alpha$ and its "spatial" projection $\underline{\alpha} = \alpha - dt \wedge \hat{\alpha}$. Specifically,

$$\mathcal{L} = dt \wedge i_N\mathcal{L}.$$

Take for instance the first-order (or Palatini) version of the Maxwell Lagrangian: $\overline{\mathcal{L}}_{ED} = F \wedge \Pi = dA \wedge \Pi$, where Π is the momentum canonically conjugate to A. From this Lagrangian one can construct the Hamiltonian three form

$$\mathcal{H}_{ED}(N) = \pounds_N A \wedge \Pi - i_N\overline{\mathcal{L}}_{ED}. \tag{F.64}$$

Explicitly,

$$\begin{aligned}\mathcal{H}_{ED}(N) &= \pounds_N A \wedge \Pi - i_N dA \wedge \Pi - F \wedge i_N\Pi \\ &= -d i_N A \wedge \Pi - F \wedge i_N\Pi \\ &= -i_N A \wedge d\Pi - F \wedge i_N\Pi + d(i_N A \wedge \Pi).\end{aligned} \tag{F.65}$$

Now, also in the case of gravitational fields it is possible to rewrite the Lagrangian (F.28) as

$$i_N\overline{\mathcal{L}}_G = \pounds_N\vartheta \wedge p + \pounds_N\omega \wedge \pi - \mathcal{H}_G(N) = \pounds_N Q \wedge P - \mathcal{H}_G(N), \tag{F.66}$$

where Q stands for the fields in the set $\{\vartheta^K, \omega^K{}_L\}$ and P for their conjugate momenta $\{p_K, \pi_K{}^L\}$. The explicit expression for the Hamiltonian $\mathcal{H}_G(N)$ is calculated to be

$$\begin{aligned}\mathcal{H}_G(N) = &- \Theta \wedge i_N p - \Omega \wedge i_N\pi - i_N\vartheta \wedge Dp - i_N\omega \wedge \big(D\pi + [\vartheta \wedge p]\big) \\ &+ i_N\Lambda\ + d\mathcal{B}(N),\end{aligned}$$

with

$$\mathcal{B}(N) = i_N\vartheta \wedge p + i_N\omega \wedge \pi = i_N Q \wedge P. \tag{F.67}$$

The gravitational Hamiltonian can also be written as

$$\mathcal{H}_G(N) = -i_N Q \wedge \frac{\delta\overline{\mathcal{L}}_G}{\delta P} - \frac{\delta\overline{\mathcal{L}}_G}{\delta Q} \wedge i_N P + d\mathcal{B}(N).$$

The expression (F.66) together with previous findings allows us to derive conditions which must be met in order that the Hamiltonian field equations can be derived. First, vary (F.66)

$$\delta(i_N\overline{\mathcal{L}}_G) = (\delta\pounds_N Q)\wedge P + \pounds_N Q\wedge\delta P - \delta\mathcal{H}_G(N).$$

and compare this with $i_N(\delta\overline{\mathcal{L}}_G)$ using (F.29):

$$\begin{aligned} i_N\delta\overline{\mathcal{L}}_G &= i_N d(\delta Q\wedge P) + [\text{on} - \text{shell terms}] \\ &\doteq \pounds_N(\delta Q\wedge P) - d i_N(\delta Q\wedge P) \\ &= (\pounds_N\delta Q)\wedge P + \delta Q\wedge\pounds_N P - d i_N(\delta Q\wedge P). \end{aligned}$$

Since N is not varied, $\delta i_N\overline{\mathcal{L}}_G = i_N\delta\overline{\mathcal{L}}_G$, and thus

$$\delta\mathcal{H}_G(N) = \pounds_N Q\wedge\delta P - \delta Q\wedge\pounds_N P + d\mathcal{C}(N) \qquad \mathcal{C}(N) = i_N(\delta Q\wedge P). \tag{F.68}$$

Therefore, if the surface term $d\mathcal{C}(N)$ vanishes, we are allowed to derive from the previous expression the Hamiltonian field equations

$$\pounds_N Q^\alpha = \frac{\delta\mathcal{H}_G(N)}{\delta P_\alpha} \qquad \pounds_N P_\alpha = -\frac{\delta\mathcal{H}_G(N)}{\delta Q^\alpha}.$$

This is not only reminiscent of the findings by T. Regge and C. Teitelboim [427], but it also offers the possibility of introducing appropriate closed forms in the gravitational Lagrangians in order to cope with different asymptotic spacetime geometries.

Observe that the Hamiltonian $\mathcal{H}_G(N)$ defined by (F.66) is identical with the diffeomorphism Noether current J_N (F.63) with the superpotential $\mathcal{B}(N) = U_{(\xi=N)}$. Therefore,

$$d\mathcal{H}_G(N) = [\text{linear combination of field equations}]\,, \tag{F.69}$$

and by Stokes' theorem the integrated Hamiltonian is

$$H(N) = \int_\Sigma [\text{linear combination of field equations}] + \int_{\partial\Sigma}\mathcal{B}(N).$$

Therefore $H(N)$, which can be identified with the energy in the finite region Σ (a three-dimensional spatial hypersurface) is numerically determined solely by the boundary term, and is thus a quasilocal energy-momentum.

However, the Hamiltonian is not unique, since one can add to it a total derivative without affecting the conservation law (F.69). This amounts to a change of $\mathcal{B}(N)$ and $\mathcal{C}(N)$. Instead of interpreting this as a weakness (indeed this indeterminacy is common to any Noether current), it opens an arena for defining different gravitational energies in a controlled way, where the control is due to boundary terms. As a matter of fact, it is found that with the further requirement of covariant boundaries there are only two possibilities for the $\mathcal{B}(N)$. I refrain from giving these expressions here (they are defined in terms of a background metric) and refer to [87], [372], were you may also find details for the case of general relativity. With the different boundary terms, they reproduce the various pseudotensors, superpotentials, and the Komar and the ADM mass. This “covariant Hamiltonian formulation”, as it is called by Nester *et al.*, also allows us to understand anew the role of the Gibbons-Hawking-York boundary term in the gravitational Lagrangian and the Brown-York approach towards quasi-local energies.

On this level of consideration, the phase-space constraints which result from local invariances, in the sense discussed in App. C, are not yet visible. The constraints are found in the differential form notation by varying

$$S = \int dt \int_{\Sigma_t} \pounds_N Q^\alpha \wedge P_\alpha - \mathcal{H}_G(N).$$

with respect to the temporal components $\hat{Q}^\alpha, \hat{P}_\alpha$, and the time-evolution equation results from the variations with respect to the spatial components $\underline{Q}^\alpha, \underline{P}_\alpha$.

Symplectic Approach à la Wald

The proper technical basis for the following Noether charge approach was developed by J. Lee and R. Wald [329], originally not in the language of differential forms, in an attempt to understand the relationship between local symmetries occurring in a Lagrangian formulation of a field theory, and the corresponding constraints present in a phase-space formulation. Later, these techniques gave an impetus for the derivation of the black-hole thermodynamics laws directly from diffeomorphism invariance of a theory [515]. The techniques were refined in [516] in order to become applicable to defining counterparts to conserved quantities at null infinity.

In the approach of Wald *et al.*, considerable weight is given to the symplectic potential Θ as it appears in the variation of a Lagrangian

$$\delta\mathcal{L} = \delta\Phi \wedge \mathcal{E} + d\Theta(Q, \delta Q). \tag{F.70}$$

It serves to define the ’symplectic current’ ω :

$$\omega(Q; \delta_1 Q, \delta_2 Q) := \delta_1\Theta(Q, \delta_2 Q) - \delta_2\Theta(Q, \delta_1 Q). \tag{F.71}$$

For its exterior derivative one finds, by using (F.70)

$$d\omega = -\delta_2 Q \wedge \delta_1\mathcal{E} + \delta_1 Q \wedge \delta_2\mathcal{E}.$$

This vanishes on-shell and/or for variations $\delta_1 Q,\ \delta_2 Q$ that solve the linearized field equations. Furthermore, define the symplectic form relative to a Cauchy surface Σ as

$$\Omega(Q; \delta_1 Q, \delta_2 Q) := \int_\Sigma \omega(Q; \delta_1 Q, \delta_2 Q).$$

This form is independent of Σ if $Q,\ \delta_1 Q,\ \delta_2 Q$ are solutions–provided that either Σ is compact or the fields satisfy appropriate asymptotic conditions. The form Ω is degenerate (presymplectic) on the space $\mathcal{F}$ of all field configurations, but after factoring $\mathcal{F}$ by the degeneracy subspace of Ω, one can define a phase space Γ; for details see [329]. A Hamiltonian H_ξ conjugate to a vector field ξ^μ is a function on Γ whose pull-back to $\mathcal{F}$ satisfies

$$\delta H_\xi = \Omega(\Phi; \delta Q, \pounds_\xi Q). \tag{F.72}$$

The previous considerations apply to any field theory with local symmetries. Next, consider specifically diffeomorphism invariant theories: Diffeomorphisms are generated by smooth vector fields ξ^μ. Every vector field generates a local symmetry and gives rise to a Noether current $(D\text{-}1)$-form

$$J_\xi := \Theta(Q, \pounds_\xi Q) - i_\xi \mathcal{L}.$$

We verify that from $dJ_\xi = d\Theta(Q, \pounds_\xi Q) - di_\xi\mathcal{L} = \pounds_\xi \mathcal{L} - \pounds_\xi Q \wedge \mathcal{E} - di_\xi \mathcal{L} = -\pounds_\xi Q \wedge \mathcal{E}$, the Noether currents are closed forms for all ξ whenever the field equations are fulfilled (i.e. $\mathcal{E} \doteq 0$) or whenever ξ is a symmetry (i.e $\pounds_\xi Q = 0$). Therefore, on solutions, the Noether current is exact: $J_\xi = dU_\xi$ with $(D\text{-}2)$-forms U_ξ. It is shown in [287]) that on-shell

$$J_\xi = dU_\xi + \xi^\mu C_\mu.$$

where $C_\mu = 0$ are the constraint equations of the theory. The variation of the Noether current is (on-shell)

$$\begin{aligned}
\delta J_\xi &= \delta\Theta(Q, \pounds_\xi Q) - i_\xi \delta\mathcal{L} = \delta\Theta(Q, \pounds_\xi Q) - i_\xi d\Theta(Q, \delta Q) \\
&= \delta\Theta(Q, \pounds_\xi Q) - \pounds_\xi \Theta(Q, \delta Q) + d(i_\xi \Theta(Q, \delta Q)) \\
&= \omega(Q; \delta Q, \pounds_\xi Q) + d(i_\xi \Theta(Q, \delta Q))
\end{aligned}$$

where in the last step the symplectic current according to (F.71) is introduced. Therefore

$$\omega(Q; \delta Q, \pounds_\xi Q) = \delta J_\xi - d(i_\xi \Theta) = \xi^\mu \delta C_\mu + \delta dQ_\xi - d(i_\xi \Theta)$$

which integrates to

$$\Omega(Q; \delta Q, \pounds_\xi Q) = \int_\Sigma \xi^\mu \delta C_\mu + \int_{\partial\Sigma} (\delta Q_\xi - i_\xi \Theta).$$

If a D-1 form B with

$$\delta \int_{\partial\Sigma} i_\xi B = \int_{\partial\Sigma} i_\xi \Theta$$

can be found, then a Hamiltonian exists according to (F.72), and is given by

$$H_\xi = \int_\Sigma \xi^\mu C_\mu + \int_{\partial\Sigma} (Q_\xi - i_\xi \mathrm{B}).$$

Examples:

- The energy of a system can be defined if a Hamiltonian conjugate to "time translations" $\xi^\mu = t^\mu$ exists
- If a Hamiltonian exists that is conjugate to a rotation the angular momentum can be defined as

$$J = -H_\varphi = -\int_{\partial\Sigma} Q_\varphi$$

where φ is assumed to be tangent to Σ, so that the B-contribution vanishes.

The previous definitions and the ensuing relations hold for all diffeomorphism invariant theories. Take now as an example general relativity with the Hilbert Lagrangian four-form components $\mathcal{L} = \frac{1}{16\pi}\epsilon_{\mu\nu\rho\sigma}R$. The components of the symplectic 3-form are

$$\Theta_{\mu\nu\rho} = \frac{1}{16\pi}\epsilon_{\lambda\mu\nu\rho}g^{\lambda\sigma}g^{\kappa\tau}(D_\kappa\delta g_{\sigma\tau} - D_\sigma\delta g_{\kappa\tau}).$$

From this, one derives the Noether current and the Noether charge (components)

$$(J_\xi)_{\mu\nu\rho} = \frac{1}{8\pi}\epsilon_{\lambda\mu\nu\rho}D_\sigma(D^{[\sigma}\xi^{\lambda]}) \qquad (Q_\xi)_{\mu\nu} = -\frac{1}{16\pi}\epsilon_{\mu\nu\rho\sigma}D^\rho\xi^\sigma.$$

If spacetime is asymptotically flat, the angular momentum defined by this charge becomes identical to the ADM expression, and when (ξ) is an asymptotic Killing vector, it agrees with Komar's expression. Furthermore, for asymptotic time translations, expressions for the B can be found, such that a Hamiltonian exists and that the energy becomes the ADM energy.

F.3.2 Gravitational Theories Beyond Einstein

As stated before, the previous considerations are generic and apply not only to general relativity. In principle you can "build" your own action functionals by building wedge products out of the forms $\vartheta, \Theta, \Omega$, such that the product is a D-form. But by far not all of these mathematically admissible gravitational theories describe our world. They must be at least as good as GR in the sense of reproducing the experimental data. From the point of geometry, a "natural" candidate is the Einstein-Cartan theory, which as explained in Subsect. 7.3.4 is a theory in a Riemann-Cartan space. This geometry specializes to a Riemann space for vanishing torsion, and it reduces to a Weizenböck's teleparallel space in the case of vanishing torsion. From the motivation to understand gravitational theories as a gauge theory of the Poincaré group a Lagrangian emerged that has terms linear and quadratic in the curvature, and terms quadratic in the torsion; in a shorthand notation $R + T^2 + R^2$. Explicitly this is a linear combination of ten terms with, in principle, ten independent coefficients. This 10-parameter Lagrangian can of course also be expressed in terms of differential forms, and–to stress this point again–all the previous results on the Noether currents with respect to local Lorentz and diffeomorphism invariance hold true. The theories with vanishing curvature and those with vanishing torsion were–in terms of the exterior calculus–further investigated in [318]. The Noether identities for the teleparallel theory, which is a specific linear combination of the three torsion T^2 terms (more about this below), are treated in [387].

The component form of the full PGT theory was formulated explicitly by K. Hayashi and T. Shirafuji [253]. In order to better understand its structure, they split the curvature and the torsion into its irreducible components with regard to the Lorentz group. This was reformulated by R. Wallner in the language of the exterior calculus [517]: In 4D, the torsion can be written as

$$\Theta^I = \Gamma^I + \frac{1}{3}\,\vartheta^I \wedge \Lambda + \frac{1}{3}\; * (\Xi \wedge \vartheta^I) =^{(1)} \Theta^I +^{(2)} \Theta^I +^{(3)} \Theta^I .$$

Here Λ is the "trace" part (a 1-form)

$$\Lambda := i_I \Theta^I i_I \qquad \left(\Theta^I - \frac{1}{3}\vartheta^I \wedge \Lambda\right) = 0$$

and Ξ is the 1-form "axial" part

$$\Xi := *(\Theta^I \wedge \vartheta_I) \qquad \left(\Theta^I - \frac{1}{3} * (\Xi \wedge \vartheta^I)\right) \wedge \vartheta_I = 0.$$

Then Γ^I can be proven to be traceless with a vanishing axial part:

$$i_I \Gamma^I = 0 \qquad\qquad \Gamma^I \wedge \vartheta_I = 0.$$

Thus, torsion Θ^I with its 24 components is decomposed in terms of the Λ (4 components), the Ξ (4 components) and the Γ^I (16 components).

Also, the curvature with its 36 components can be decomposed into $1+6+9+9+10+1$ irreducible components.

The U_4 objects can be split into V_4 objects plus further contributions by splitting the Riemann-Cartan connection $\tilde{\omega}^I{}_J$ into the Riemann connection $\omega^I{}_J$ and the contortion $K^I{}_J$ as

$$\tilde{\omega}^I{}_J = \omega^I{}_J + K^I{}_J \qquad \text{where} \qquad d\vartheta^I + \omega^I{}_J \wedge \vartheta^J = 0.$$

With this, the U_4 curvature and torsion become

$$\Theta^I = K^{IJ} \wedge \vartheta_J \qquad\qquad \tilde{\Omega}^I{}_J = \Omega^I{}_J + DK^I{}_J + K^I{}_L \wedge K^L{}_J \tag{F.73}$$

where D is the Riemann covariant derivative.

If we specifically consider the $R + T^2$ Lagrangian (without a cosmological constant), it has four terms, namely

$$\mathcal{L} = \sum_i^4 a_i\,\mathcal{L}_i = a_1(\Gamma^I \wedge *\Gamma_I) + a_2(\Lambda \wedge *\Lambda) + a_3(\Xi \wedge *\Xi) + a_4\,\tilde{\Omega}_{IJ} \wedge \eta^{IJ}.$$

The tilde notation in $\mathcal{L}_4$ indicates that the curvature form lives in Riemann-Cartan space; indeed this Lagrangian alone describes the Einstein-Cartan theory. The (Einstein-Cartan) R term in the Lagrangian can be split into an (Einstein) R term plus further contributions by the previous splitting of the Riemann-Cartan connection: The Lagrangian term $\mathcal{L}_4$ becomes

$$\tilde{\Omega}_{IJ} \wedge \eta^{IJ} = \Omega_{IJ} \wedge \eta^{IJ} + K_{IL} \wedge K^L_{\ J} \wedge \eta^{IJ} + d(K_{IJ} \wedge \eta^{IJ}).$$

The second term in this expression can be written in terms of torsion. This is a three-page exercise in the exterior calculus with the resulting identity

$$\tilde{\Omega}_{IJ} \wedge \eta^{IJ} = \Omega_{IJ} \wedge \eta^{IJ} + (\Gamma^I \wedge *\Gamma_I) - \frac{2}{3}(\Lambda \wedge *\Lambda) + \frac{1}{6}(\Xi \wedge *\Xi) + \text{ exact form.}$$

Teleparallel GR

This leads to the remarkable observation that in a spacetime with vanishing curvature $\tilde{\Omega}$, the Hilbert-Einstein action is–up to a boundary term–equivalent to a specific T^2 theory, called the teleparallel equivalent of general relativity ($GR_{\|}$). In any teleparallel theory, one can find (under the condition that the manifold is parallelizable) an orthonormal frame in which the connection one-forms vanish; see e.g. Chapter 3 in [47]. This frame is unique only up to constant rotations. The advantage of this frame is that the torsion looks like a gauge field strength: $F^I := \Theta^I = d\vartheta^I$. It can indeed be interpreted as a field strength of a gauge theory of the translation group giving rise to a covariant derivative $d + \vartheta$. (Strictly speaking, this description of teleparallel theories differs in some aspects from the "proper" approach in which in the PGT Lagrangian a term with Lagrangian multipliers forces the curvature to become zero; see [387].)

Write the Lagrangian of the teleparallel equivalent of GR as

$$\mathcal{L}_{GR_{\|}} = -\frac{1}{2}F^I \wedge \Pi_I + \mathcal{L}_M(\vartheta, \Phi, D\Phi), \quad \Pi_I = \frac{1}{\kappa} * \left({}^{(1)}F_I - 2\,{}^{(2)}F_I - \frac{1}{2}\,{}^{(3)}F_I\right)$$

with ${}^{(2)}F^I = \frac{1}{3}\vartheta^I \wedge i_I F^I$, ${}^{(3)}F^I = \frac{1}{3}i^I(\vartheta^J \wedge F_J)$, ${}^{(1)}F^I = F_I - {}^{(2)}F_I - {}^{(3)}F_I$. The variations of the Lagrangian with respect to the coframe ϑ and the field strength F results in the field equations

$$\Sigma_I + T_I - d\Pi_I = 0$$

with the matter energy-momentum $T_I := \frac{\delta \mathcal{L}_M}{\delta \vartheta}$, and the gravitational energy-momentum

$$\Sigma_I := \frac{\partial \mathcal{L}_G}{\partial \vartheta} = i_I \mathcal{L}_G + i_I F^J \wedge \Pi_J = \frac{1}{2}\left[i_I F^J \wedge \Pi_J - F^J \wedge i_I \Pi_J\right].$$

Notice that this term has a structure similar to the energy-momentum expressions for the matter fields. One thus could be tempted to use the exact form $J_I := \Sigma_I + T_I$ to define the total four-momentum

$$P_I = \int_S (\Sigma_I + T_I) \doteq \int_{\partial S} \Pi_I.$$

However, to qualify as representing a genuine conserved quantity, this must be covariant under both general coordinate and local Lorentz transformations. Covariance under general coordinate transformations is guaranteed, since P_I is expressed in terms of differential forms. But as shown in [387], under a Lorentz transformation

$$P_I' = \int_{\partial S} (\Lambda^{-1})^J{}_I \Pi_J - \frac{1}{2\kappa} \int_{\partial S} (\Lambda^{-1})^J{}_I (\Lambda^{-1})^K{}_L \, d\Lambda^L{}_N \wedge \eta_J{}^N{}_K .$$

Thus the "energy-momentum" transforms as a vector only under those Lorentz transformations for which $d\Lambda$ vanishes on the boundary ∂S.

And of course we meet again the arbitrariness of allowing a closed form to be added to the current J_I: The changes $\Sigma_I \to \Sigma_I + d\Psi_I$ and $\Pi_I \to \Pi_I + \Psi_I$ leave the field equations untouched, but give rise to a change in the object P_I. This change can be compensated by a surface term $d\Psi$ in the action with $\frac{\partial \Psi}{\partial \vartheta^I} = \Psi_I$. Again, we see the interplay of boundary terms and conserved quantities.

F.3.3 Topological Terms

A "topological term" or "topological invariant" or "topological density" is the name for geometry-based surface terms in an action functional. The differential-form language facilitates a proper bookkeeping of topological terms.

Lanczos-Lovelock(-Cartan) Gravity

Differential forms permit an elegant formulation of Lanczos-Lovelock gravity in D dimensions as

$$\mathcal{L}_L = \sum_p \gamma_p \, \epsilon_{I_1 \dots I_D} \, \Omega^{I_1 I_2} \wedge \dots \wedge \Omega^{I_{2\,p-1} I_{2\,p}} \wedge \vartheta^{I_{2\,p+1} \dots I_D} , \tag{F.74}$$

with (coupling) constants γ_p. Assuming the absence of torsion, the field equations are

$$\sum_p \gamma_p (D - 2\,p) \epsilon_{I_1 \dots I_D} \Omega^{I_1 I_2} \wedge \dots \wedge \Omega^{I_{2\,p-1} I_{2\,p}} \wedge \vartheta^{I_{2\,p+1} \dots I_{D-1}} = 0.$$

Following [570], denote by $\mathcal{L}_{(C,V)}$ a summand in the Lanczos-Lovelock Lagrangian with the number of C curvature forms and $V = D$-2 C vielbein forms. Thus for $D = 2$ there are two terms

$$\mathcal{L}_{(0,2)} = \epsilon_{IJ} \vartheta^{IJ} = \eta \qquad \mathcal{L}_{(1,0)} = \epsilon_{IJ} \Omega^{IJ} .$$

The first one relates to a cosmological constant, while the second is proportional to the Euler density (to be explained below) in two dimensions. The corresponding

Lagrangian term is a topological invariant. For $D = 3$, we have the terms $\mathcal{L}_{(0,3)} = \eta$, $\mathcal{L}_{(1,1)} = \epsilon_{IJK}\Omega^{IJ} \wedge \vartheta^K$, which are the cosmological constant term and the Hilbert Lagrangian. For $D = 4$ we find

$$\mathcal{L}_{(0,4)} = \eta, \qquad \mathcal{L}_{(1,2)} = \epsilon_{IJKL}\Omega^{IJ} \wedge \vartheta^{KL}, \qquad \mathcal{L}_{(2,0)} = \epsilon_{IJKL}\Omega^{IJ} \wedge \Omega^{KL}$$

corresponding to the cosmological term, the Hilbert term and the Euler density in four dimensions. This pattern is repeated for higher dimensions: $\mathcal{L}_{(0,D)}$ is always the cosmological term in a Lagrangian, $\mathcal{L}_{(1,D-2)}$ the Hilbert Lagrangian, and for D even $\mathcal{L}_{(\frac{D}{2},0)}$ is the D-dimensional Euler density.

In Lanczos-Lovelock gravity, it was originally assumed that the underlying geometry is Riemannian. A Lanczos-Lovelock Lagrangian within Riemann-Cartan geometry was derived in [351]. This means that in a sum extending (F.74) additional terms are allowed, namely all those which are Lorentz invariant in this extended context. These are not only terms involving torsion explicitly, but also terms which vanish for vanishing torsion due to Bianchi identities. What does this give in four dimensions? The authors of [351] find three additional terms in the Lanczos-Lovelock Lagrangian:

$$\Omega^{IJ} \wedge \vartheta_{IJ}, \qquad \Theta^I \wedge \Theta_I, \qquad \Omega^{IJ} \wedge \Omega_{IJ}.$$

The latter is the 4D-Pontryagin density which is locally exact:

$$P_4 := \Omega^{IJ} \wedge \Omega_{IJ} = d\Big(\omega^I{}_J \wedge d\omega^J{}_I + \frac{2}{3}\omega^I{}_J \wedge \omega^J{}_K \wedge \omega^K{}_I\Big).$$

A linear combination of the other two terms is again a topological term

$$NY_4 := \Theta^I \wedge \Theta_I + \Omega_{IJ} \wedge \vartheta^{IJ} = d\Big(\vartheta^I \wedge \Theta_I\Big)$$

known as the Nieh-Yan density. Together with the previously derived Euler density

$$E_4 = \epsilon_{IJKL}\, \Omega^{IJ} \wedge \Omega^{KL} = d\Big(\omega^{IJ} \wedge (\Omega_{KL} - \frac{1}{3}\omega^K{}_M \wedge \omega^{LM})\epsilon_{IJKL}\Big)$$

we have three topological terms in 4D. The three-forms from which they are derived by applying an external derivative are called Chern-Simons forms. Written in components,

$$E_4 = (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\eta$$

which shows that the Euler 4-form density has the Gauß-Bonnet tensor as components.

Playing with Topological Terms

The three topological terms in 4D are unsuited as action forms, because they would constitute surface terms (they are locally exact). However, as we are aware of the discussion of boundary terms in the main text, they may be suitable to serve different (partly interdependent) purposes. A boundary term

1. may be necessary in order to allow the definition of a functional integration (Regge/Teitelboim);
2. must be included into the action in order to cancel terms due to non-trivial boundary conditions (Gibbons, Hawking, York);
3. can serve as the generator of a canonical transformation;
4. signals a non-trivial topology–this is the case if the corresponding densities are not globally exact.

As for item 4.) above, there is the mathematical notion of cohomology classes which relates integrals of the topological terms to topologies of the manifold. And with regard to item 2.), a simple line of reasoning reveals how in terms of differential forms boundary terms are to be handled: Start from the Hilbert-Einstein Lagrangian (assuming the curvature being defined with respect to the Levi-Civita connection) and rewrite it as

$$\begin{aligned}\Omega^{IJ} \wedge \eta_{IJ} &= (d\omega^{IJ} + \omega^{I}{}_{K} \wedge \omega^{KJ}) \wedge \eta_{IJ} \\ &= d(\omega^{IJ} \wedge \eta_{IJ}) + \omega^{IJ} \wedge d\eta_{IJ} + \omega^{I}{}_{K} \wedge \omega^{KJ} \wedge \eta_{IJ}.\end{aligned}$$

Since the combination

$$\Omega^{IJ} \wedge \eta_{IJ} - d(\omega^{IJ} \wedge \eta_{IJ})$$

involves only first derivatives of the tetrads, it is reasonable to add to the GR action a term

$$-\frac{1}{2}\int_M d(\omega^{IJ} \wedge \eta_{IJ}) = -\frac{1}{2}\int_{\partial M} \omega^{IJ} \wedge \eta_{IJ}.$$

It was demonstrated in [128] that this term is identical to the surface term considered in [211], [562]. This surface term was then used by the same author for the purpose 3.), namely to reproduce the canonical transformation from the tetrad variables and their momenta to Ashtekar's "new" canonical variables (see C 3.3.). The previous findings, which are valid only for the case of vanishing torsion were extended to the case with torsion. Here, it turned out that one needs to add (1/2 of) the Nieh-Yan term in order to recover the canonical transformations leading to Ashtekar's variables.

The most general action describing vacuum GR in first-order formalism and being compatible with diffeomorphism invariance and Lorentz invariance, can be written (following [431]):

$$S\,[\vartheta, \omega] = \int \alpha_1\ \eta_{IJ} \wedge \Omega^{IJ} + \alpha_2\ \vartheta^{IJ} \wedge \Omega_{IJ} + \alpha_3\ P_4 + \alpha_3\ E_4 + \alpha_5\ NY_4 + \alpha_6\ \eta.$$

The first term is the Hilbert-Palatini action, where α_1 is related to the gravitational constant. The first two terms together are proportional to the Holst action (C.112), and α_2 is proportional to the Barbero-Immirzi parameter. The terms with $\alpha_3, \alpha_4, \alpha_5$ are the Pontryagin, the Euler, and the Nieh-Yan topological densities in 4D, respectively. Finally, the α_6 term represents the cosmological constant.

Even and Odd Spacetime Dimensions

In the section on Poincaré gauge theories, I point out the fact that the Einstein-Cartan theory in 4D is not invariant with respect to translations. This can easily be exhibited in differential forms: For translations with an infinitesimal parameter ϵ^I the vierbeins and the curvature transform as

$$\delta\vartheta^I = D\epsilon^I = d\epsilon^I + \omega^I{}_K\epsilon^K \qquad \delta\Omega^{IJ} = 0 \quad \text{since} \quad \delta\omega^I{}_K = 0.$$

For the 4D Einstein-Cartan theory $\mathcal{L}_{(1,2)} = \epsilon_{IJKL}\, \vartheta^I \wedge \vartheta^J \wedge \Omega^{KL}$, we find

$$\begin{aligned}\delta\mathcal{L}_{(1,2)} &= 2\epsilon_{IJKL}(D\epsilon^I) \wedge \vartheta^J \wedge \Omega^{KL} \\ &= d(2\epsilon_{IJKL}\, \epsilon^I\vartheta^J \wedge \Omega^{KL}) + 2\epsilon_{IJKL}\, \epsilon^I D(\vartheta^J \wedge \Omega^{KL}) \\ &= d\,\left(2\epsilon_{IJKL}\epsilon^I\vartheta^J \wedge \Omega^{KL}\right) + 2\epsilon_{IJKL}\, \epsilon^I\Theta^J \wedge \Omega^{KL}\end{aligned}$$

where the Bianchi identity $D\Omega^{KL} = 0$ was used. This is an exact form only if the torsion vanishes. On the other hand, for the 3D Einstein-Cartan theory:

$$\delta\mathcal{L}_{(1,1)} = 2\,\epsilon_{IJK}(D\epsilon^I) \wedge \Omega^{KL} = d(2\epsilon_{IJK}\epsilon^I\Omega^{KL}).$$

These results can be applied to any dimension with the insight that the Einstein Lagrangian terms $\mathcal{L}_{(2n+1,1)}$ in Lanczos-Lovelock-Cartan gravity are translation invariant. For gravity and supergravity in odd dimensions, see the lecture notes [565].

F.4 Gauge Theories

In this section[12] we will restrict ourselves to the matter Lagrangian $\mathcal{L}_M$ in (F.3), that is we assume a fixed background gravitational field. The collection of matter fields, previously called Φ^A is now split in two sets, namely "particle fields" $\{\Psi^\alpha\}$ and "force fields" $\{A^a\}$. There might be different fields Ψ, each with their own index range, dictated by the gauge symmetry group $\mathbf{G}$ as explained below. In the following, we will deal only with one variant of Ψ fields, because the others may simply be added.

[12] The mathematics needed is outlined in App. E.5.

F.4.1 Global Symmetry

Assume that

$$S_P = \int \mathcal{L}_P(\Psi^\alpha, d\Psi^\alpha, \vartheta, \omega) \tag{F.75}$$

is invariant under infinitesimal Lie-group transformations. This means the following: Call the infinitesimal group parameter $\gamma = \gamma^a \otimes X_a = \gamma^a X_a$ where the X_a are generators of $\mathfrak{g}$, the Lie algebra associated to the symmetry group **G**. The index a runs from 1 to the dimension of the group manifold. Thus γ is a Lie-algebra valued vector field. The p-form Ψ^α is considered as an element of $\Omega_p(M, V)$, where V is a representation space of **G** (and $\alpha = 1, \ldots, \dim\, V$). Thus $\Psi^\alpha \in \Omega(M, V) = \oplus\Omega_p(M, V)$. Observe that this has consequences for the Lagrangian form $\mathcal{L}$, which must be an element of $\Omega_D(M, R)$. Let for instance

$$\mathcal{L} = \Psi_p \wedge \tilde{\Psi}_{D-p} = \Psi_p^a V_a \wedge \tilde{\Psi}_{D-p}^b V_b := \Psi_p^a \wedge \tilde{\Psi}_{D-p}^b (V_a, V_b).$$

To obtain an element of $\Omega_D(M, R)$, it is necessary to have an inner product for the basis vectors in V:

$$(\ ,\ ) : V \times V \to R, \qquad \text{for example} \qquad (V_a, V_b) = \delta_{ab}.$$

A **G** gauge transformation does not affect the tetrad and the spin connection in (F.75). On the Ψ-fields they are acting as

$$\delta_G(\gamma)\Psi^\alpha = \hat{\gamma}^\alpha{}_\beta \Psi^\beta = (\gamma^a \hat{X}_a)^\alpha{}_\beta \Psi^\beta, \tag{F.76}$$

where $\hat{X}_a$ are the generators of **G** in the Ψ representation. Let us subsequently write $\delta_G(\gamma) = \delta_\gamma$. (Quasi)-invariance of (F.75) means that $\delta_\gamma \mathcal{L}_P = dF_\gamma$, which by virtue of (F.32) is equivalent to the existence of an on-shell exact form J_γ

$$J_\gamma := \delta_\gamma \Psi^\alpha \wedge P_\alpha - F_\gamma \qquad \text{with} \qquad dJ_\gamma \doteq 0. \tag{F.77}$$

In the following, just for simplification at this point, we assume invariance of the action (F.75), that is $F_\gamma \equiv 0$. We might distinguish global and local gauge invariance, the former characterized by constant parameters γ^a. In this case:

$$J_\gamma = \gamma^a \wedge j_a \qquad \text{with} \qquad j_a := (\hat{X}_a)^\alpha{}_\beta \Psi^\beta \wedge P_\alpha.$$

F.4.2 Local Gauge Transformations

The globally-invariant action (F.75) is turned into a locally-invariant action by replacing the "ordinary" derivative by the covariant derivative through the introduction of a gauge field A^a and by introducing a kinetic term for the gauge field. This new action is then

$$S_M = \int \mathcal{L}_M = \int \mathcal{L}_P(\Psi^\alpha, D\Psi^\alpha, \vartheta, \omega) + \mathcal{L}_F(A, dA, \vartheta). \tag{F.78}$$

The local symmetry with a spacetime dependent γ^a, together with

$$\delta_\gamma \Psi^\alpha = (\hat{\gamma})^\alpha{}_\beta \Psi^\beta \qquad \delta_\gamma A^a = -D\gamma^a$$

(according to (E.16, E.19)), converts (F.31) into

$$\begin{aligned} 0 \equiv \delta_\gamma \mathcal{L}_M &= \hat{\gamma}^\alpha{}_\beta \Psi^\beta \wedge \mathcal{E}_\alpha - D\gamma^a \wedge \mathcal{E}_a + dJ_\gamma, \\ J_\gamma &= \hat{\gamma}^\alpha{}_\beta \Psi^\beta \wedge P_\alpha - D\gamma^a \wedge P_a \end{aligned} \tag{F.79}$$

where

$$\mathcal{E}_\alpha = \frac{\delta \mathcal{L}_M}{\delta \Psi^\alpha} = \frac{\delta \mathcal{L}_P}{\delta \Psi^\alpha} \qquad \mathcal{E}_a = \frac{\delta \mathcal{L}_M}{\delta A^a} = \frac{\partial \mathcal{L}_P}{\partial A^a} + \frac{\delta \mathcal{L}_F}{\delta A^a} \tag{F.80}$$

$$P_\alpha = \frac{\partial \mathcal{L}_M}{\partial (d\Psi^\alpha)} = \frac{\partial \mathcal{L}_P}{\partial (D\Psi^\alpha)} \qquad P_a = \frac{\partial \mathcal{L}_M}{\partial (dA^a)} = \frac{\partial \mathcal{L}_F}{\partial (dA^a)}. \tag{F.81}$$

By reshuffling the D derivatives, (F.79) becomes

$$\hat{\gamma}^\alpha{}_\beta \Psi^\beta \wedge \mathcal{E}_\alpha + \gamma^a D\mathcal{E}_a + d(\hat{\gamma}^\alpha{}_\beta \Psi^\beta \wedge P_\alpha - \gamma^a \mathcal{E}_a + \gamma^a DP_a) = 0.$$

As exerted previously, writing this in the form $\gamma^a A_a + d(\gamma^a B_a) = 0$, gives rise to the identities

$$A_a = (\hat{X}_a)^\alpha{}_\beta \Psi^\beta \wedge \mathcal{E}_\alpha + D\mathcal{E}_a = 0 \tag{F.82a}$$

$$B_a = (\hat{X}_a)^\alpha{}_\beta \Psi^\beta \wedge P_\alpha - \mathcal{E}_a + DP_a = 0. \tag{F.82b}$$

We guess from the known result of the Lorentz Noether current that the Noether gauge current can be expressed in terms of the gauge field equations and a superpotential as

$$J_\gamma = \gamma^a \mathcal{E}_a - d(\gamma^a P_a)$$

and confirm this by the use of (F.82b).

Clearly (F.82a) is the Noether identity, constituting a relation among the Euler derivatives of $\mathcal{L}_M$. The other identity (F.82b) will be further analyzed, since it restricts $\mathcal{L}_F$: From the second expression in (F.80) one derives

$$\mathcal{E}_a = \frac{D\Psi^\alpha}{\partial A^a} \wedge \frac{\partial \mathcal{L}_P}{D\Psi^\alpha} + \frac{\delta \mathcal{L}_F}{\delta A^a} = (\hat{X}_a)^\alpha{}_\beta \Psi^\beta \wedge P_\alpha + \frac{\delta \mathcal{L}_F}{\delta A^a}.$$

Therefore (F.82b) boils down to a condition on $\mathcal{L}_F$

$$-\frac{\partial \mathcal{L}_F}{\partial A^a} - d\frac{\partial \mathcal{L}_F}{\partial (dA^a)} + D\frac{\partial \mathcal{L}_F}{\partial (dA^a)} \stackrel{!}{=} 0,$$

or, making use of (E.18),

$$-\frac{\partial \mathcal{L}_F}{\partial A^a} + f_{bc}{}^a A^b \wedge \frac{\partial \mathcal{L}_F}{\partial A^c} \stackrel{!}{=} 0.$$

The solution of this differential equation reveals that the dependence of $\mathcal{L}_F$ on the gauge fields A^a and their derivatives is only through the field strength $F^a := dA^a + \frac{1}{2} f_{bc}{}^a A^b \wedge A^c$.

A further restriction on $\mathcal{L}_F$ stems from the requirement that it should be a scalar with respect to the gauge group **G**. A natural choice is

$$\mathcal{L}_F = g_{ab} F^a \wedge {*F^b}$$

where g_{ab} is an invariant metric in **G**.

Group Covariant Diffeomorphisms

The group properties of the gauge transformations are characterized by the commutators of infinitesimal group transformations. We have

$$[\delta_G(\gamma), \delta_G(\gamma')] = \delta_G([\gamma, \gamma']), \tag{F.83}$$

where the commutator is taken in the algebra of **G**. In analogy to the case of local Lorentz transformations (F.49) the commutator of group transformations and diffeomorphisms is

$$[\delta_D(\xi), \delta_G(\gamma)] = \delta_G(i_\xi d\gamma). \tag{F.84}$$

As with the "mixture" of diffeomorphisms with the Lorentz transformations we may define a modified diffeomorphism. Its form can be derived by observing that

$$\pounds_\xi A = i_\xi dA + d i_\xi A = i_\xi \big(F - \frac{1}{2}(A \wedge A)\big) + \big(D i_\xi A - A \wedge i_\xi A\big) = i_\xi F + D(i_\xi A),$$

where in this case the covariant derivative is defined with respect to the group connection A. Notice that the last term is nothing but a gauge transformation. The modified diffeomorphism is thus reasonably defined as

$$\overset{G}{\pounds}_\xi := \delta_D(\xi) + \delta_G(i_\xi A) \tag{F.85}$$

in terms of a modified Lie derivative

$$\overset{G}{\pounds}_\xi := i_\xi \overset{G}{D} + \overset{G}{D} i_\xi .$$

And again, as in the mixture of diffeomorphisms with Lorentz transformations, one derives

$$[i_X, \overset{G}{\pounds}_Y]\Phi = i_{[X,Y]}\Phi$$

$$\big[\overset{G}{\pounds}_Y, \overset{G}{D}\big]\Phi = i_X F \wedge \Phi$$

$$\big[\overset{G}{\pounds}_X, \overset{G}{\pounds}_Y\big]\Phi = \overset{G}{\pounds}_{[X,Y]}\Phi + i_Y i_X F \wedge \Phi.$$

For the modified diffeomorphism $\overset{G}{\delta}_D(\xi) = \overset{G}{\pounds}_\xi$, it holds that

$$[\overset{G}{\pounds}_\xi, \delta_G(\lambda)] = \delta_G(\overset{G}{\pounds}_\xi \lambda) \qquad \big[\overset{G}{\pounds}_\xi, \overset{G}{\pounds}_\xi\big] = \overset{G}{\pounds}_{[\xi,\xi']} + \delta_G(i_{\xi'} i_\xi F).$$

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