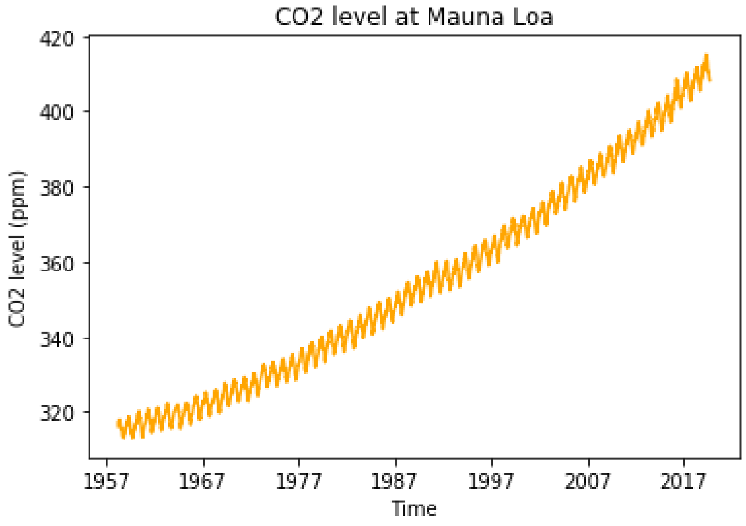
I. Setting Up

In [1]:
```
# import relevant packages
import numpy as np
import scipy. stats as sts
import matplotlib.pyplot as plt
import pandas as pd
import pystan
import datetime as dt
import seaborn
%matplotlib inline
```

In [2]:
```
# Load the data
dataset = pd.read_csv('weekly_in_situ_co2_mlo.csv')
# Convert time to date format
dataset['Time'] = pd.to_datetime(dataset['Time'])
# Plot the data
plt.plot(dataset['Time'], dataset['CO2'], color='orange')
plt.title('CO2 level at Mauna Loa')
plt.xlabel('Time')
plt.ylabel('CO2 level (ppm)')
plt.show()
size = len(dataset.iloc[:, 1])
print('Size of dataset: ', size)
```




```
Size of dataset:  3139
```

In [3]:
```
# Create a new column to store time differences
dataset['t'] = [0 for i in range(len(dataset.iloc[:, 1]))]
for i in range(len(dataset.iloc[:, 0])):
    dataset.iloc[i, 2] = (dataset.iloc[i, 0] - dataset.iloc[0, 0]).days
# Split the dataset to train and test
train = dataset.iloc[:2100, :]
test = dataset.iloc[2100: , :]
```

In [4]:
```
# Take a look at the dataset
dataset.tail()
```

Out[4]:

	Time	CO2	t
3134	2019-08-31	409.32	22435
3135	2019-09-07	408.80	22442
3136	2019-09-14	408.61	22449
3137	2019-09-21	408.50	22456
3138	2019-09-28	408.28	22463

In [51]:
```
# Create a dataframe for time in the future
future = pd.DataFrame()
future['Time'] = pd.date_range(start='2019-10-05', end='2058-01-07', fre
q = 'W')
future['t'] = [0 for i in range(len(future['Time']))]
for i in range(len(future['t'])):
    future.iloc[i, 1] = (future.iloc[i, 0] - dataset.iloc[0, 0]).days
```

In [6]:
```
total_time = dataset['t'].append(future['t'])
total_time_date = dataset['Time'].append(future['Time'])
# Take a look at future time
future.tail()
```

Out[6]:

	Time	t
1992	2057-12-09	36415
1993	2057-12-16	36422
1994	2057-12-23	36429
1995	2057-12-30	36436
1996	2058-01-06	36443

II. Stan Models

1. Construct the models

In [7]:
```
# Stan data
stan_data = {
    # Train, test, future time and data
    'cd_data': train.iloc[:, 1],
    'time': train.iloc[:, 2],
    'N': len(train.iloc[:, 1]),
    'M': len(test.iloc[:, 1]),
    'test_time': test.iloc[:, 2],
    'future_time': future['t'],
    'L': len(future['t']),
    'T': len(dataset['t']) + len(future['t']),
    'total_time': total_time
}
```

In [8]:
```
stan_code1 = """

data {
    int<lower=0> N;
    real<lower=0> cd_data[N];
    int<lower=0> time[N];
    int<lower=0> M;
    int<lower=0> test_time[M];
    int<lower=0> L;
    int<lower=0> future_time[L];
    int<lower=0> T;
    int<lower=0> total_time[T];
}

parameters {
    real<lower=0> c;
    real<lower=0> c0;
    real<lower=0> c1;
    real<lower=0> A;
    real<lower=0, upper=pi()> phi;
    real<lower=0> sigma;

}

model {
    c ~ cauchy(0, 1);
    c0 ~ cauchy(0, 1);
    A ~ cauchy(0, 1);
    phi ~ cauchy(0, 1);
    sigma ~ normal(0, 1);
    for (i in 1:N) {
        cd_data[i] ~ normal(c + c0*time[i] + A*cos(2*pi()*time[i] / 365.
25 + phi), sigma);
    }
}

generated quantities {
    real cd_test[M];
    real cd_future[L];
    real trend[T];

    for(j in 1:M) {
        cd_test[j] = normal_rng(c + c0*test_time[j] + A*cos(2*pi()*test_
time[j] / 365.25 + phi), sigma);
    }


    for (k in 1:L) {
        cd_future[k] = normal_rng(c + c0*future_time[k] + A*cos(2*pi()*f
uture_time[k] / 365.25 + phi), sigma);
    }


    for (q in 1:T) {
        trend[q] = normal_rng(c + c0*total_time[q], sigma);
    }

}

"""
```

In [9]:
```
stan_code2 = """

data {
    int<lower=0> N;
    real<lower=0> cd_data[N];
    int<lower=0> time[N];
    int<lower=0> M;
    int<lower=0> test_time[M];
    int<lower=0> L;
    int<lower=0> future_time[L];
    int<lower=0> T;
    int<lower=0> total_time[T];
}

parameters {
    real<lower=0> c;
    real<lower=0> c0;
    real<lower=0> c1;
    real<lower=0> A;
    real<lower=0, upper=2*pi()> phi;
    real<lower=0> sigma;

}

model {
    c ~ cauchy(0, 1);
    c0 ~ cauchy(0, 1);
    c1 ~ cauchy(0, 1);
    A ~ normal(2, 1);
    phi ~ uniform(0, 2*pi());
    sigma ~ normal(0, 0.25);
    for (i in 1:N) {
        cd_data[i] ~ normal(c + c0*time[i] + c1*time[i]^2 + A*cos(2*pi()
*time[i] / 365.25 + phi), sigma);
    }
}

generated quantities {
    real cd_test[M];
    real cd_future[L];
    real trend[T];

    for(j in 1:M) {
        cd_test[j] = normal_rng(c + c0*test_time[j] + c1*test_time[j]^2
 + A*cos(2*pi()*test_time[j] / 365.25 + phi), sigma);
    }


    for (k in 1:L) {
        cd_future[k] = normal_rng(c + c0*future_time[k] + c1*future_time
[k]^2 + A*cos(2*pi()*future_time[k] / 365.25 + phi), sigma);
    }


    for (q in 1:T) {
        trend[q] = normal_rng(c + c0*total_time[q] + c1*total_time[q]^2,
sigma);
    }

}

"""
```

In [10]:
```
stan_code3 = """

data {
    int<lower=0> N;
    real<lower=0> cd_data[N];
    int<lower=0> time[N];
    int<lower=0> M;
    int<lower=0> test_time[M];
    int<lower=0> L;
    int<lower=0> future_time[L];
    int<lower=0> T;
    int<lower=0> total_time[T];
}

parameters {
    real c;
    real c0;
    real<lower=0> c1;
    real<lower=0> A;
    real<lower=0, upper=pi()> phi;
    real<lower=0> sigma;

}

model {
    c ~ normal(0, 0.5);
    c0 ~ normal(0, 0.25);
    c1 ~ normal(0, 1);
    A ~ normal(0, 1);
    phi ~ normal(0, 1);
    sigma ~ normal(0, 0.25);
    for (i in 1:N) {
        cd_data[i] ~ normal(c + c0*time[i] + c1*time[i]^2 + A*cos(2*pi()
*time[i] / 365.25 + phi), sigma);
    }
}

generated quantities {
    real cd_test[M];
    real cd_future[L];
    real trend[T];

    for(j in 1:M) {
        cd_test[j] = normal_rng(c + c0*test_time[j] + c1*test_time[j]^2
 + A*cos(2*pi()*test_time[j] / 365.25 + phi), sigma);
    }


    for (k in 1:L) {
        cd_future[k] = normal_rng(c + c0*future_time[k] + c1*future_time
[k]^2 + A*cos(2*pi()*future_time[k] / 365.25 + phi), sigma);
    }


    for (q in 1:T) {
        trend[q] = normal_rng(c + c0*total_time[q] + c1*total_time[q]^2,
sigma);
    }

}

"""
```

In [11]:
```
stan_code4 = """

data {
    // train, test, future time and data
    int<lower=0> N;
    real<lower=0> cd_data[N];
    int<lower=0> time[N];
    int<lower=0> M;
    int<lower=0> test_time[M];
    int<lower=0> L;
    int<lower=0> future_time[L];
    int<lower=0> T;
    int<lower=0> total_time[T];
}

parameters {
    // unknown quantities
    // coefficients for quadratic trend
    real<lower=0> c;
    real<lower=0> c0;
    real<lower=0> c1;
    // amplitude and phase for seasonal changes
    real<lower=0> A;
    real<lower=0, upper=pi()> phi;
    // standard deviation for noise
    real<lower=0> sigma;

}

model {
    // priors for the parameters
    c ~ cauchy(0, 1);
    c0 ~ cauchy(0, 1);
    c1 ~ cauchy(0, 1);
    A ~ normal(2, 1);
    phi ~ uniform(0, pi());
    sigma ~ normal(0, 0.25);
    // likelihood function
    for (i in 1:N) {
        cd_data[i] ~ normal(c + c0*time[i] + c1*time[i]^2 + A*cos(2*pi()
*time[i] / 365.25 + phi), sigma);
    }
}

generated quantities {
    // generated data
    real cd_test[M];
    real cd_future[L];
    real trend[T];

    // replicate test data
    for(j in 1:M) {
        cd_test[j] = normal_rng(c + c0*test_time[j] + c1*test_time[j]^2
 + A*cos(2*pi()*test_time[j] / 365.25 + phi), sigma);
    }

    // generate future data
    for (k in 1:L) {
        cd_future[k] = normal_rng(c + c0*future_time[k] + c1*future_time
[k]^2 + A*cos(2*pi()*future_time[k] / 365.25 + phi), sigma);
    }
```

```
    // generate the quadratic trend
    for (q in 1:T) {
        trend[q] = normal_rng(c + c0*total_time[q] + c1*total_time[q]^2, sigma);
    }

}

"""
```

2. Run The Models

In [12]:

```
# Compile the model
stan_model1 = pystan.StanModel(model_code=stan_code1)
stan_model2 = pystan.StanModel(model_code=stan_code2)
stan_model3 = pystan.StanModel(model_code=stan_code3)
stan_model4 = pystan.StanModel(model_code=stan_code4)
```

```
INFO:pystan:COMPILING THE C++ CODE FOR MODEL anon_model_b9a5ac600848c71
88fe63710dec54dac NOW.
INFO:pystan:COMPILING THE C++ CODE FOR MODEL anon_model_d5528f4956cd6f7
8c79322f12e3a3750 NOW.
INFO:pystan:COMPILING THE C++ CODE FOR MODEL anon_model_0ea129f1af6feca
a097637a2248e0a30 NOW.
INFO:pystan:COMPILING THE C++ CODE FOR MODEL anon_model_1d5e238ee890d6e
492bbe5c7675ac229 NOW.
```

In [13]:
```
# Get the posteriors for the parameters
parameters_mod1 = ['c', 'c0', 'A', 'phi', 'sigma']
parameters = ['c', 'c0', 'c1', 'A', 'phi', 'sigma']
results1 = stan_model1.sampling(data=stan_data)
results2 = stan_model2.sampling(data=stan_data)
results3 = stan_model3.sampling(data=stan_data)
results4 = stan_model4.sampling(data=stan_data)
print('Model 1', results1.stansummary(pars=parameters_mod1))
print('Model 2', results2.stansummary(pars=parameters))
print('Model 3', results3.stansummary(pars=parameters))
print('Model 4', results4.stansummary(pars=parameters))
```

```
WARNING:pystan:Maximum (flat) parameter count (1000) exceeded: skipping
diagnostic tests for n_eff and Rhat.
To run all diagnostics call pystan.check_hmc_diagnostics(fit)
WARNING:pystan:3869 of 4000 iterations ended with a divergence (96.7
%).
WARNING:pystan:Try running with adapt_delta larger than 0.8 to remove t
he divergences.
WARNING:pystan:Maximum (flat) parameter count (1000) exceeded: skipping
diagnostic tests for n_eff and Rhat.
To run all diagnostics call pystan.check_hmc_diagnostics(fit)
WARNING:pystan:27 of 4000 iterations saturated the maximum tree depth o
f 10 (0.675 %)
WARNING:pystan:Run again with max_treedepth larger than 10 to avoid sat
uration
WARNING:pystan:Maximum (flat) parameter count (1000) exceeded: skipping
diagnostic tests for n_eff and Rhat.
To run all diagnostics call pystan.check_hmc_diagnostics(fit)
WARNING:pystan:Maximum (flat) parameter count (1000) exceeded: skipping
diagnostic tests for n_eff and Rhat.
To run all diagnostics call pystan.check_hmc_diagnostics(fit)
```

```
Model 1 Inference for Stan model: anon_model_b9a5ac600848c7188fe63710de
c54dac.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

        mean se_mean     sd   2.5%    25%    50%    75%  97.5%  n_eff
Rhat
c     310.56    0.01   0.09  310.4  310.5 310.55 310.61 310.73     58
1.05
c0    3.6e-3  8.4e-7 9.1e-6 3.6e-3 3.6e-3 3.6e-3 3.6e-3 3.6e-3    118
1.04
A       2.59    0.01   0.05   2.47   2.55   2.59   2.61    2.7     27
1.1
phi   1.6e-3  2.5e-4 5.6e-4 6.6e-4 1.1e-3 1.5e-3 2.2e-3 2.4e-3      5
1.8
sigma   2.05    0.01   0.03   1.99   2.03   2.05   2.08   2.09      7
1.28

Samples were drawn using NUTS at Fri Dec 20 15:05:02 2019.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
Model 2 Inference for Stan model: anon_model_d5528f4956cd6f78c79322f12e
3a3750.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

        mean se_mean     sd   2.5%    25%    50%    75%  97.5%  n_eff
Rhat
c     314.34  2.6e-3   0.08 314.19 314.29 314.34 314.39 314.49    873
1.02
c0    2.2e-3  1.6e-6 2.3e-5 2.1e-3 2.2e-3 2.2e-3 2.2e-3 2.2e-3    223
1.04
c1    9.5e-8 1.6e-10 1.5e-9 9.2e-8 9.4e-8 9.5e-8 9.6e-8 9.8e-8     87
1.05
A       2.67    0.09   0.14   2.47   2.54   2.68    2.8   2.85      2
4.12
phi     2.92    2.06   2.92 2.1e-5 3.3e-4    2.9   5.84   5.86      2 4
17.87
sigma   1.11    0.11   0.16   0.93   0.95   1.11   1.27    1.3      2
10.17

Samples were drawn using NUTS at Fri Dec 20 15:09:36 2019.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
Model 3 Inference for Stan model: anon_model_0ea129f1af6fecaa097637a224
8e0a30.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

        mean se_mean     sd   2.5%    25%    50%    75%  97.5%  n_eff
Rhat
c      23.31    0.01   0.51  22.32  22.95  23.31  23.66  24.31   2471
1.0
c0      0.03  2.0e-6 1.1e-4   0.03   0.03   0.03   0.03   0.03   3133
1.0
c1   2.7e-10 4.6e-122.6e-107.9e-128.2e-111.9e-103.7e-109.9e-10   3263
1.0
A       0.63    0.01   0.48   0.02   0.24   0.52    0.9   1.76   2318
1.0
phi     0.67    0.01   0.56   0.02   0.23   0.54   0.99   2.07   2950
1.0
```

```
sigma   39.29  2.3e-3   0.13  39.04   39.2  39.29  39.38  39.54   3336
1.0

Samples were drawn using NUTS at Fri Dec 20 15:13:41 2019.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
Model 4 Inference for Stan model: anon_model_1d5e238ee890d6e492bbe5c767
5ac229.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

        mean se_mean     sd   2.5%    25%    50%    75%  97.5%  n_eff
Rhat
c     314.35  1.7e-3   0.09 314.18 314.29 314.35 314.41 314.52   2747
1.0
c0    2.2e-3  5.5e-7 2.6e-5 2.1e-3 2.1e-3 2.2e-3 2.2e-3 2.2e-3   2225
1.0
c1    9.5e-8 3.5e-11 1.6e-9 9.2e-8 9.4e-8 9.5e-8 9.6e-8 9.8e-8   2192
1.0
A       2.54  7.2e-4   0.04   2.46   2.51   2.54   2.56   2.61   2899
1.0
phi   4.9e-4  1.5e-5 5.0e-4 7.4e-6 1.3e-4 3.3e-4 6.8e-4 1.8e-3   1097
1.01
sigma   1.27  3.2e-4   0.02   1.23   1.26   1.27   1.28   1.31   3649
1.0

Samples were drawn using NUTS at Fri Dec 20 15:18:54 2019.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

In [14]:

```
# Extract samples
samples1 = results1.extract()
samples2 = results2.extract()
samples3 = results3.extract()
samples4 = results4.extract()
```

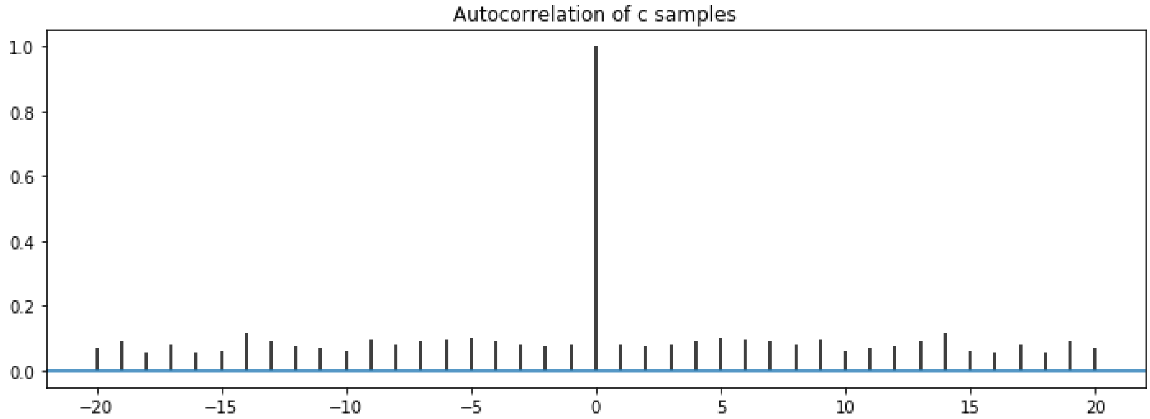
III. Model Comparisons

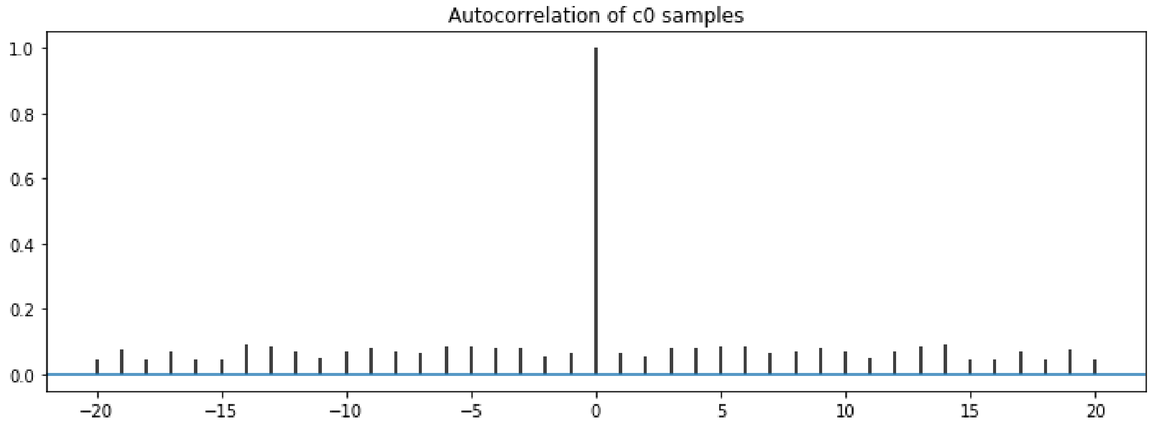
1. Sampling

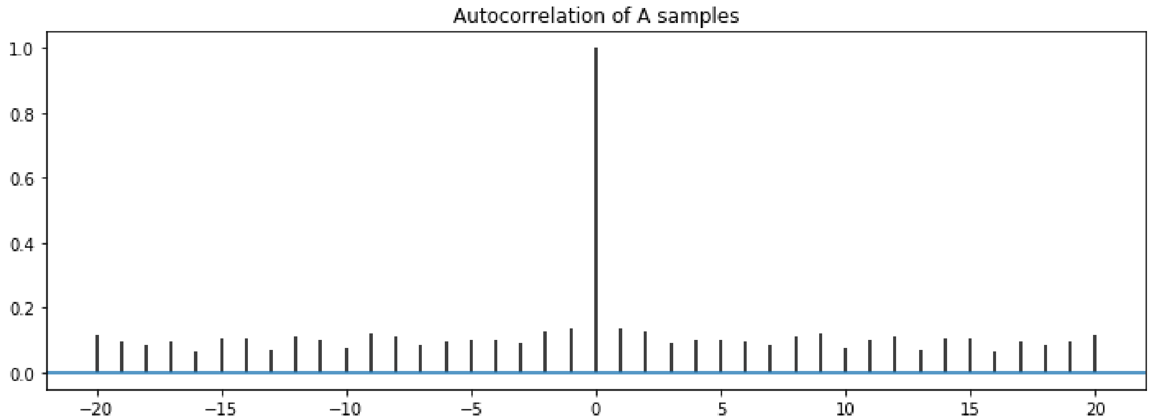
In [15]:
```
# Plot autocorrelation among samples for each parameter
def plot_acf(parameters, samples):
    '''
    Plot the autocorrelation function for a series x. This corresponds t
o the
    acf() function in R. The series x is detrended by subtracting the me
an of
    the series before computing the autocorrelation.
    '''
    from scipy import signal

    for param in parameters:
        plt.figure(figsize=(12, 4))
        plt.acorr(
        samples[param], maxlags=20, detrend=lambda x: signal.detrend(x,
type='constant'))
        plt.title(f'Autocorrelation of {param} samples')
    plt.show()

# Make pair plot of the posteriors over all parameters of the model.
def pairplot(parameters, samples):
    df = pd.DataFrame(
        data=np.transpose([samples[param] for param in parameters]),
        columns=parameters)
    seaborn.pairplot(df, size=2.5, plot_kws={'marker': '.', 'alpha': 0.2
5})
    plt.show()
```

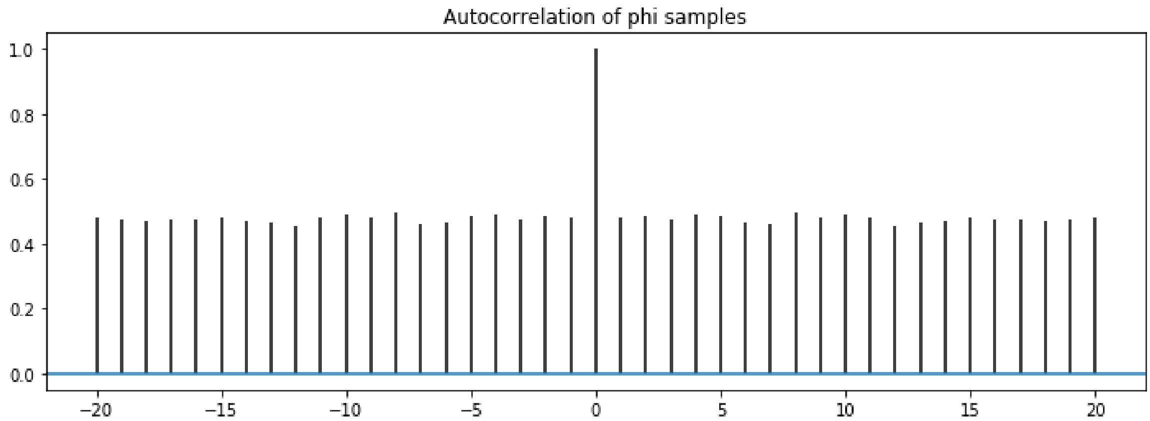
In [45]:
```
print('Model 1')
plot_acf(parameters_mod1, samples1)
pairplot(parameters_mod1, samples1)
```

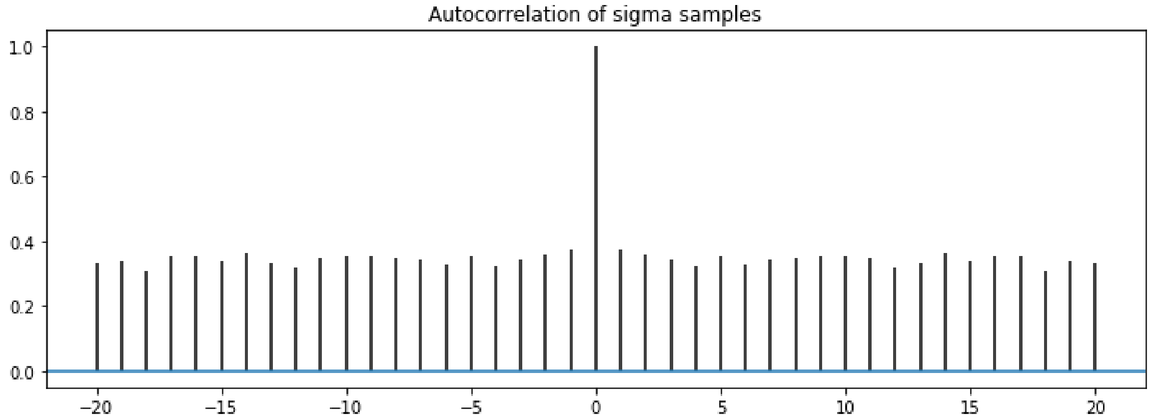
Model 1

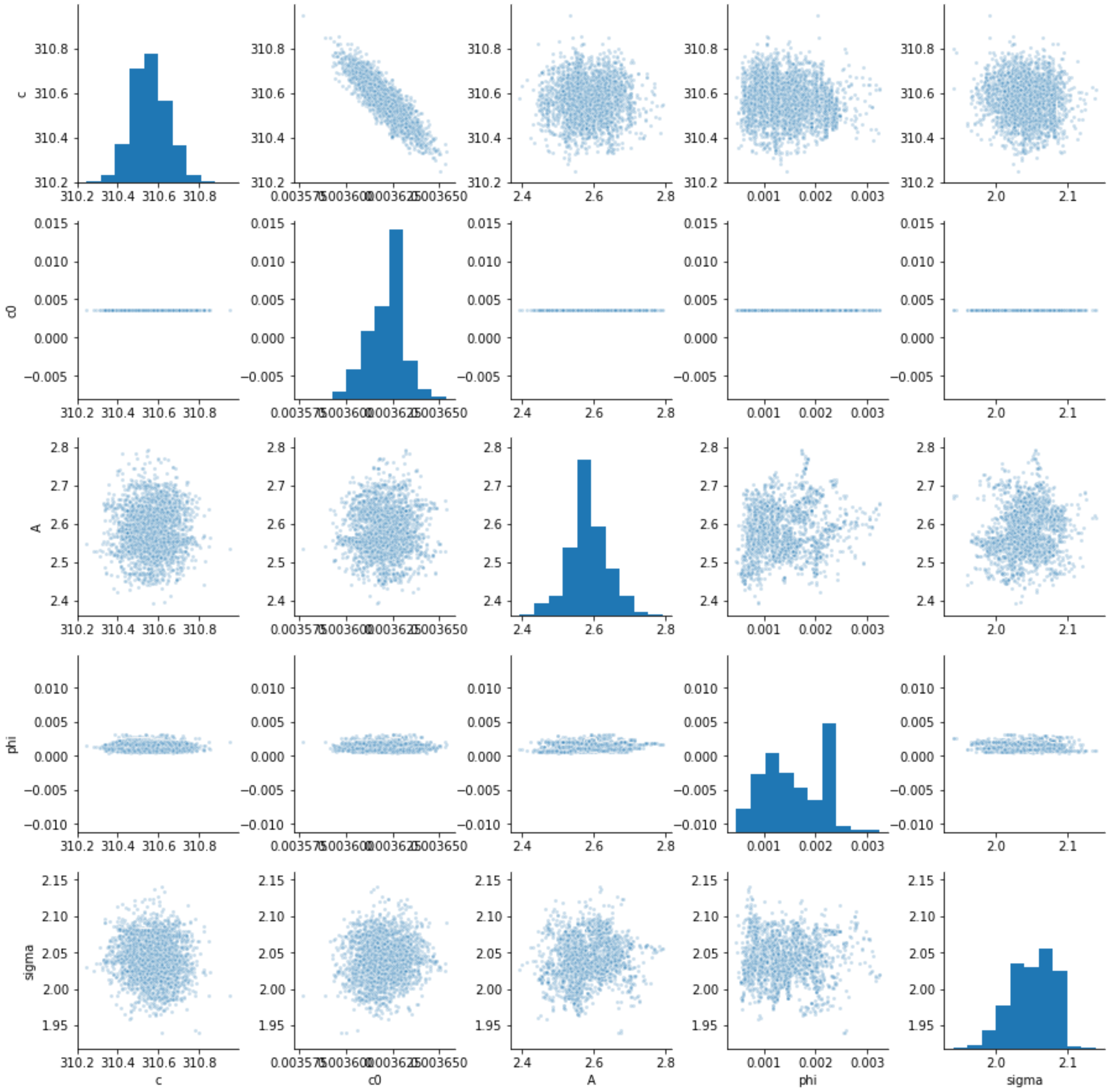

Autocorrelation of c samples


Autocorrelation of c0 samples


Autocorrelation of A samples

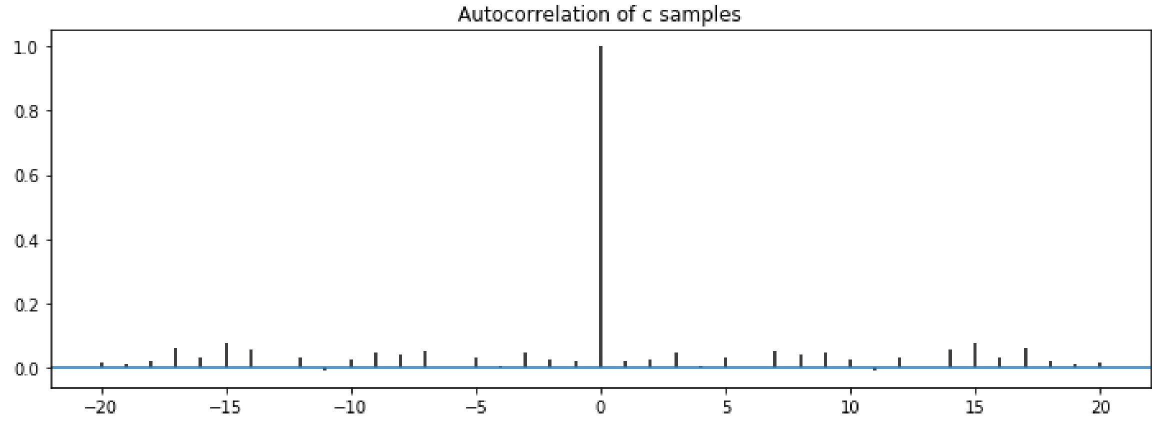

Autocorrelation of phi samples

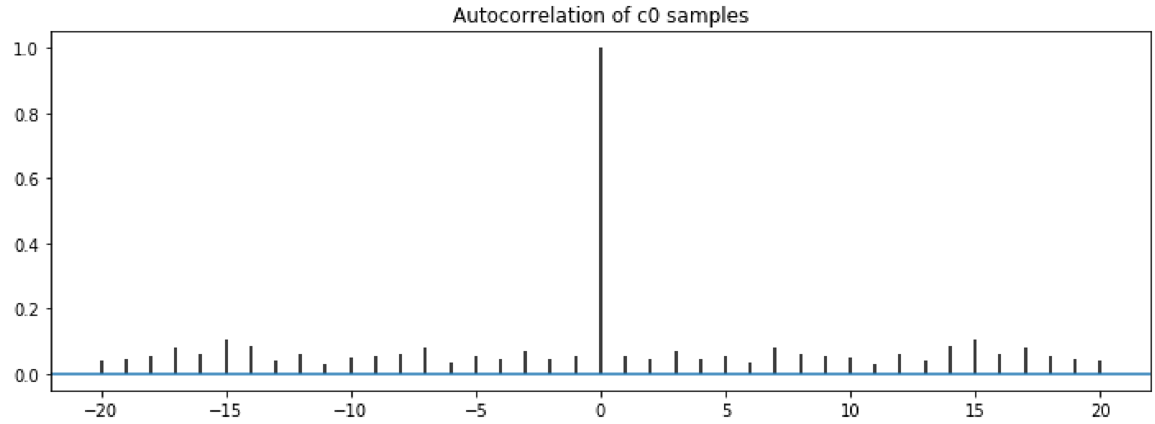

Autocorrelation of sigma samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20

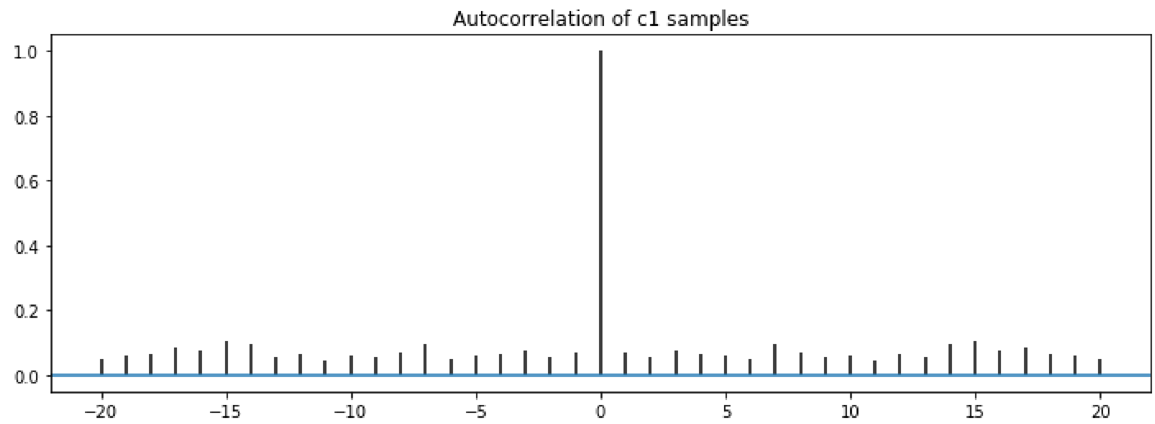

c
c0
A
phi
sigma

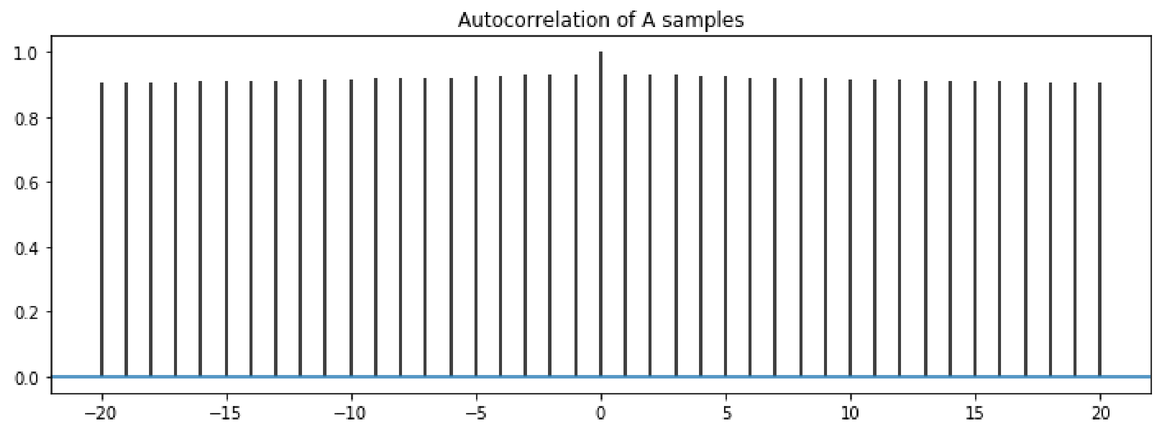
In [44]:
```
print('Model 2')
plot_acf(parameters, samples2)
pairplot(parameters, samples2)
```

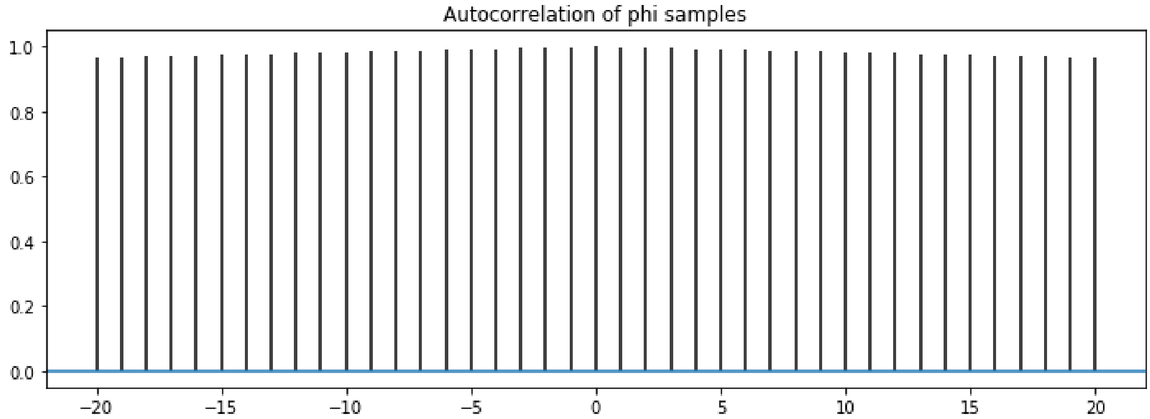
Model 2

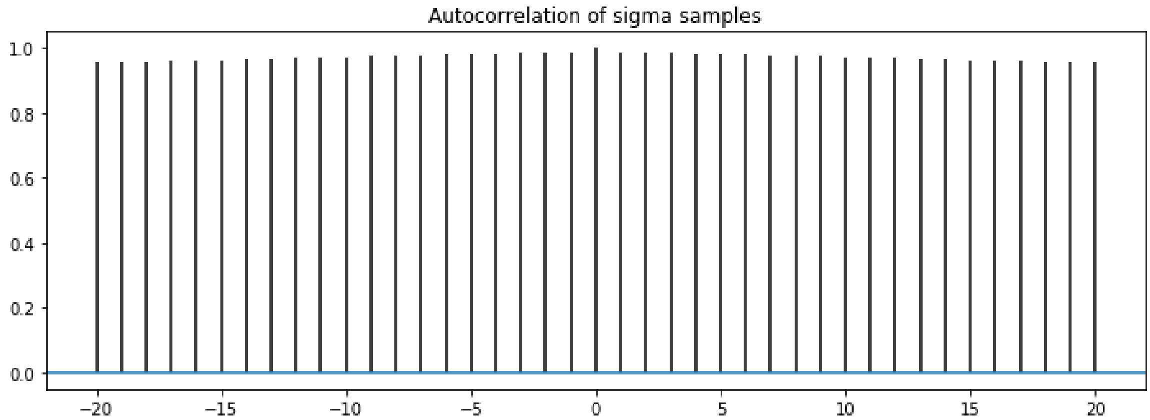

Autocorrelation of c samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20

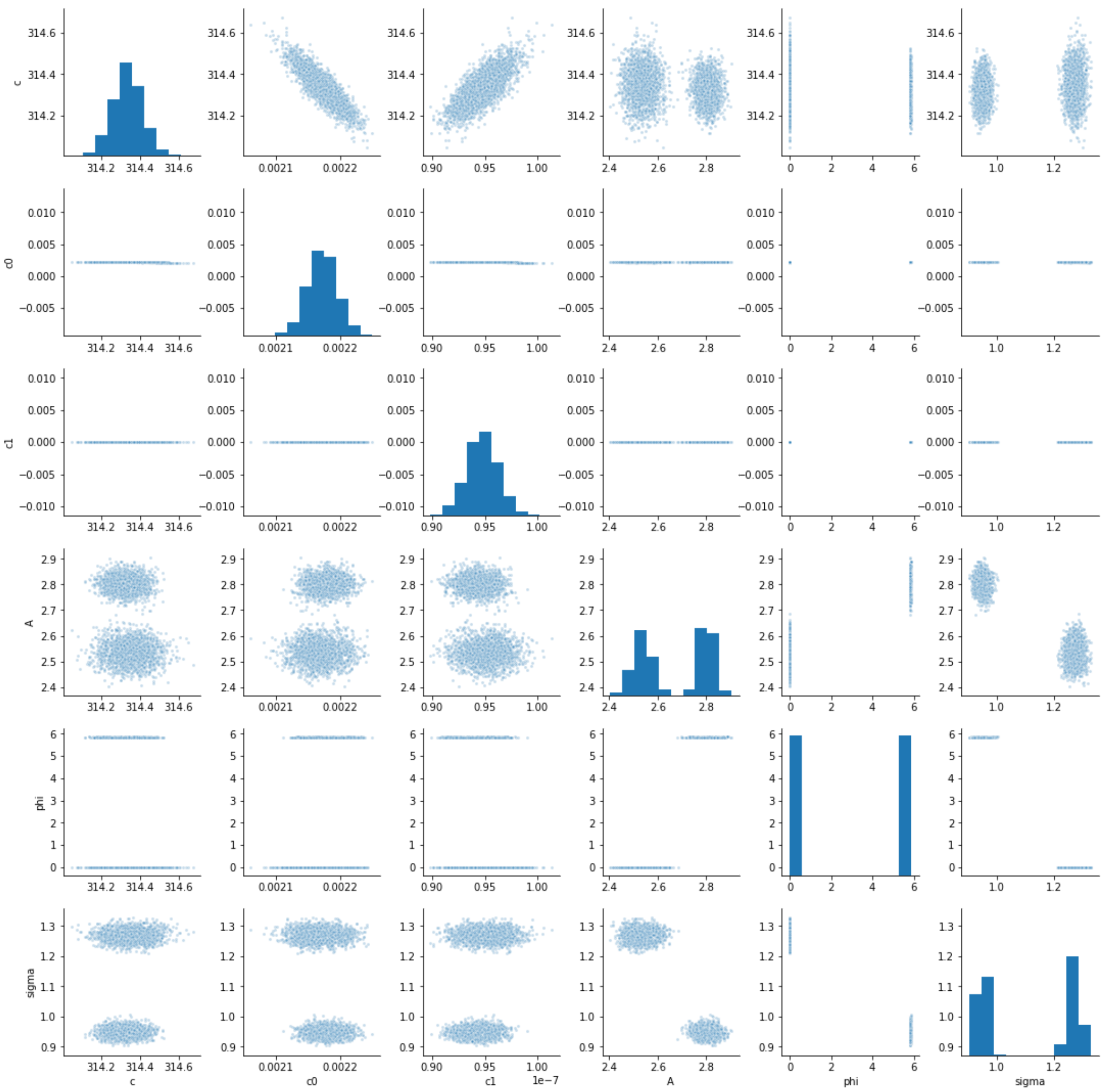

Autocorrelation of c0 samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20


Autocorrelation of c1 samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20


Autocorrelation of A samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20

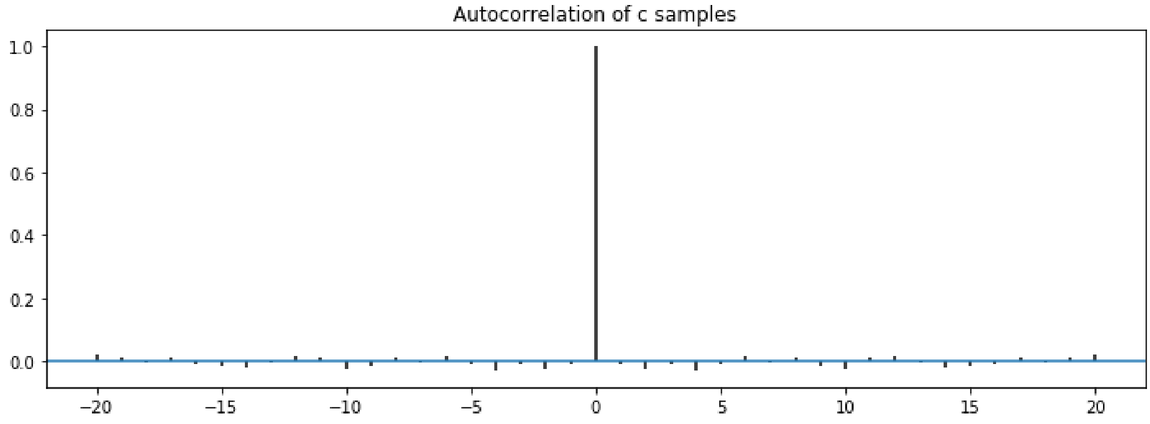

Autocorrelation of phi samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20

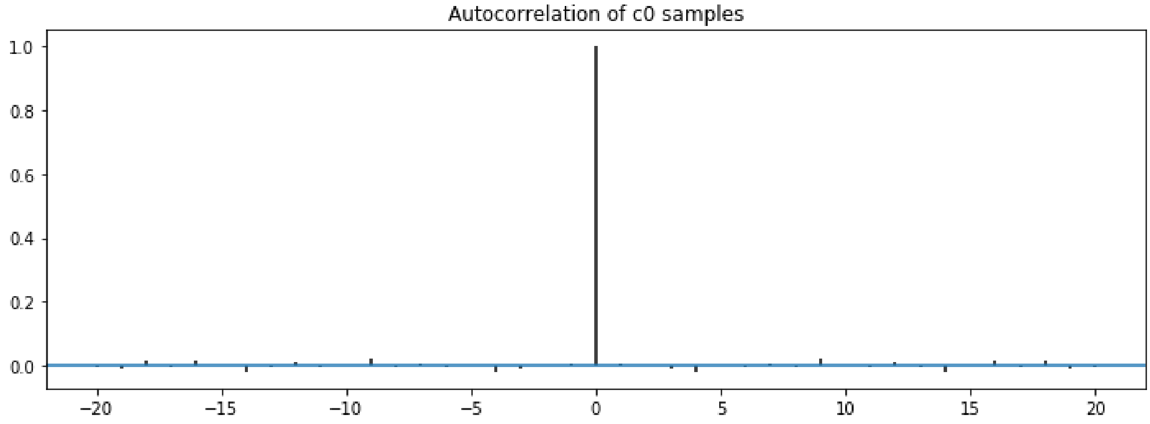

Autocorrelation of sigma samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20

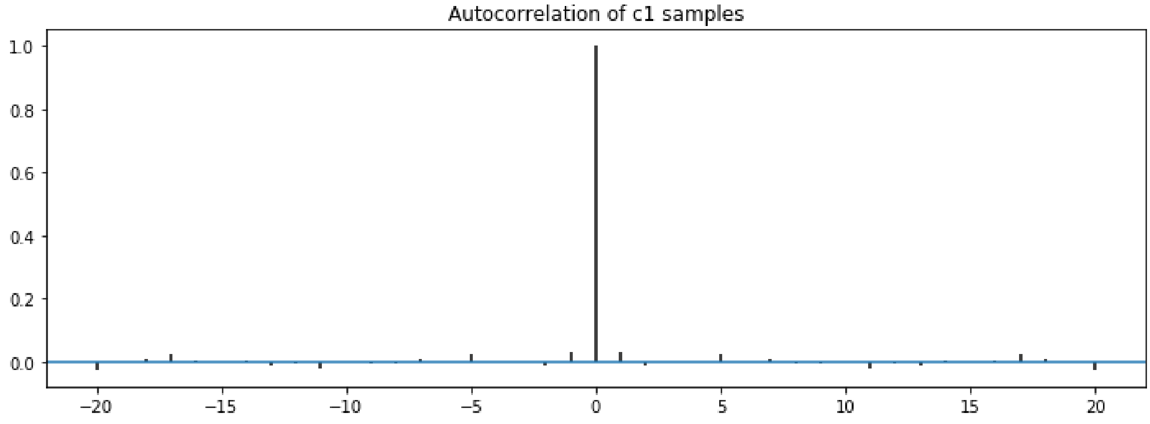

c
c0
c1
A
phi
sigma
1e−7

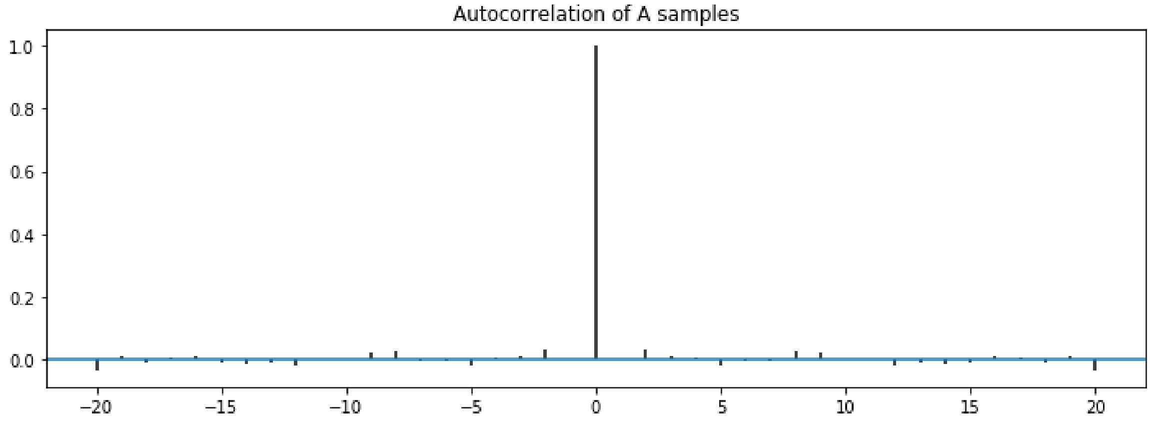
In [46]:
```
print('Model 3')
plot_acf(parameters, samples3)
pairplot(parameters, samples3)
```

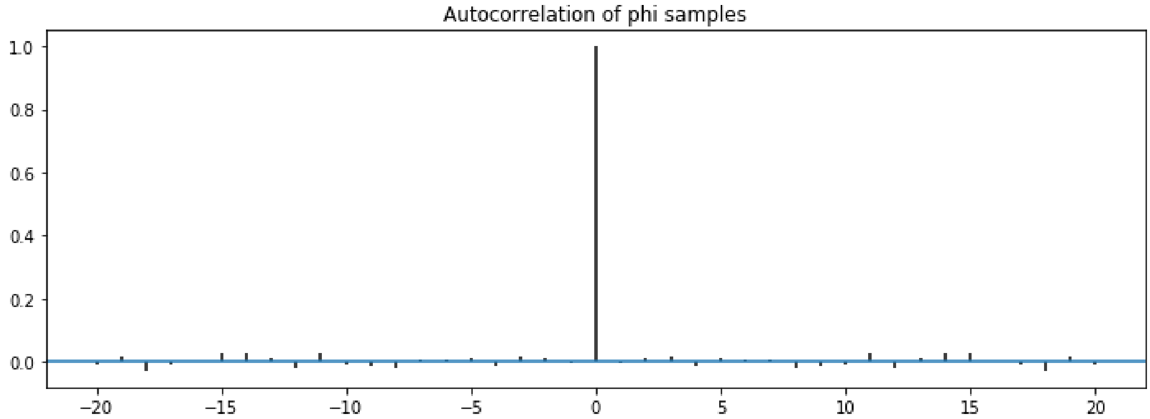
Model 3

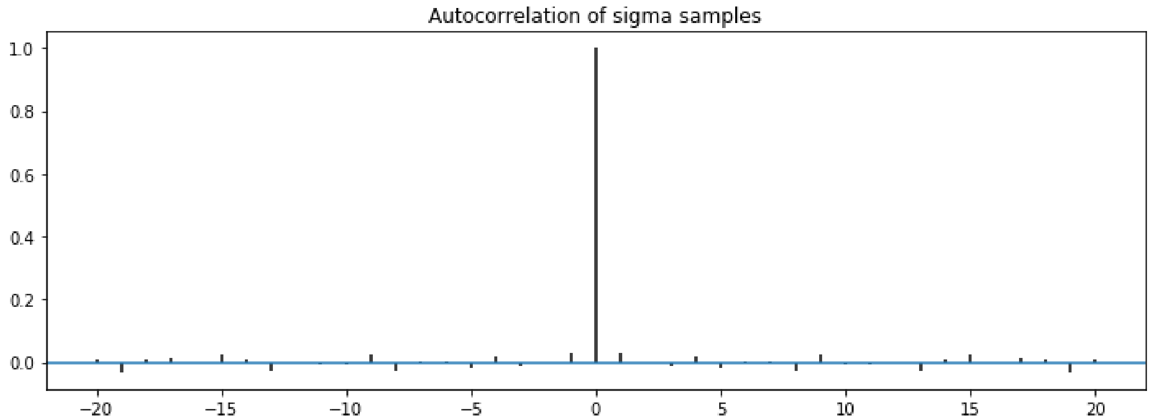

Autocorrelation of c samples

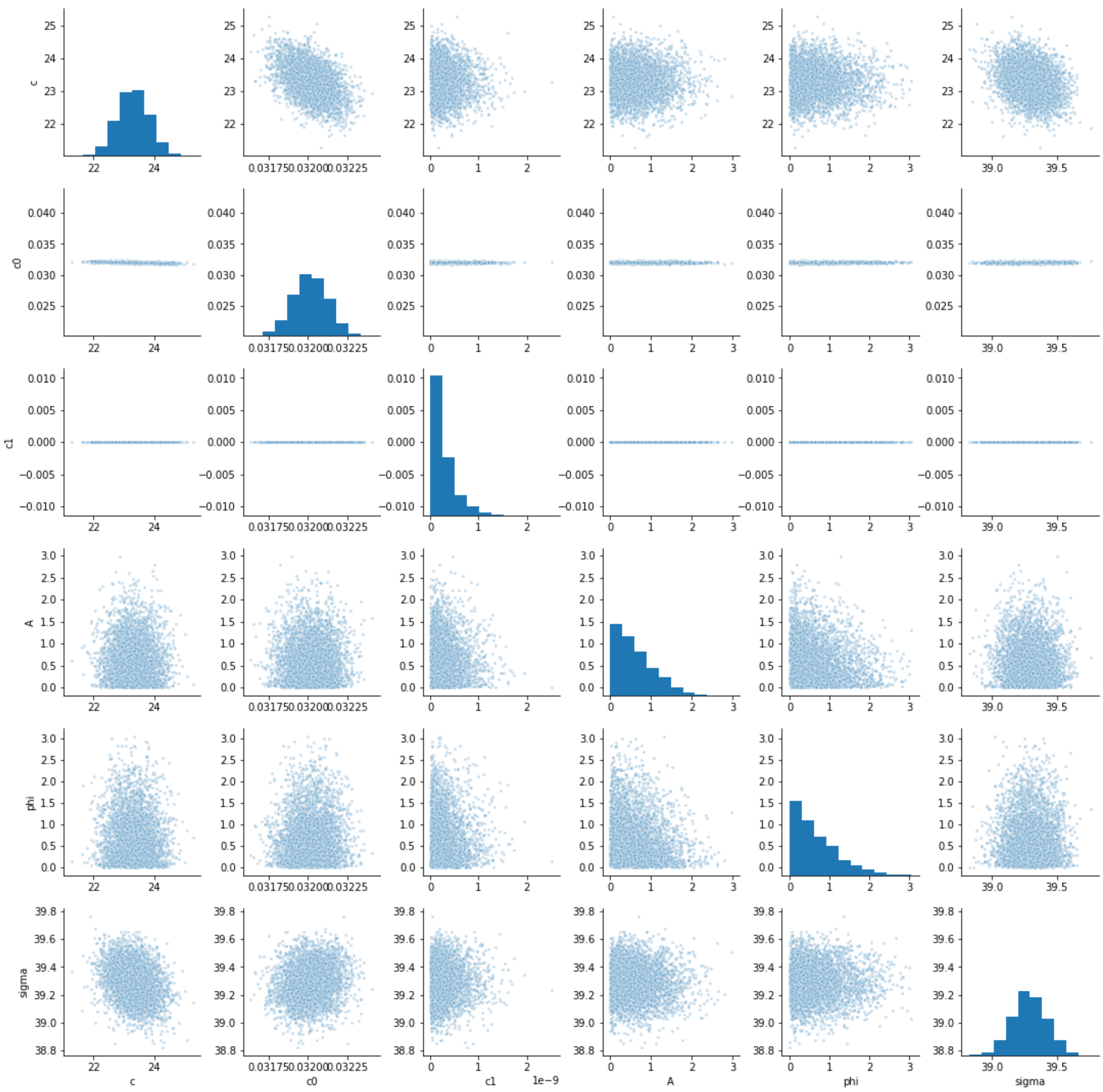

Autocorrelation of c0 samples


Autocorrelation of c1 samples


Autocorrelation of A samples

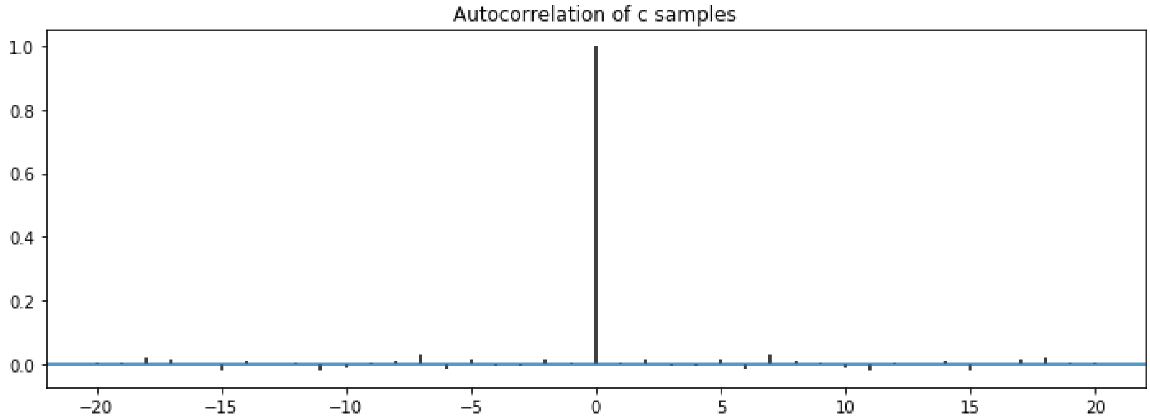

Autocorrelation of phi samples

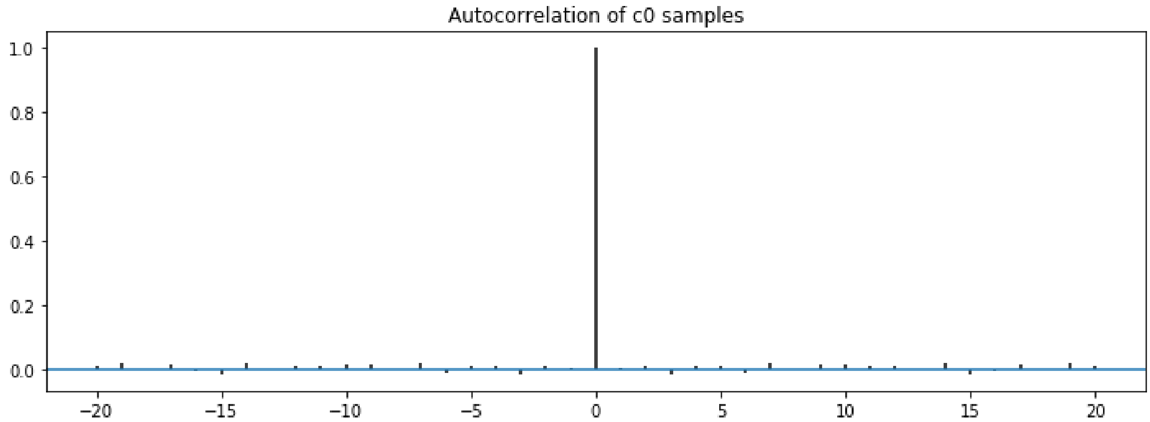

Autocorrelation of sigma samples

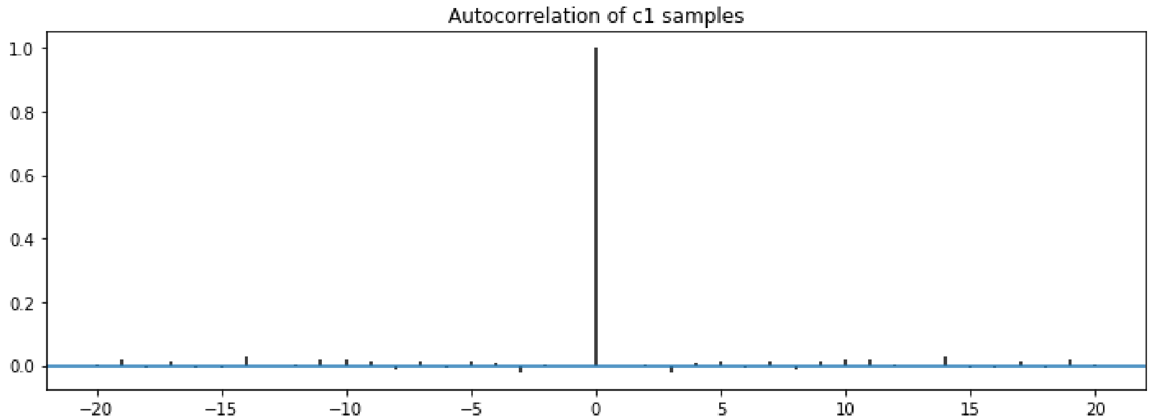

c
c0
c1
A
phi
sigma
1e−9

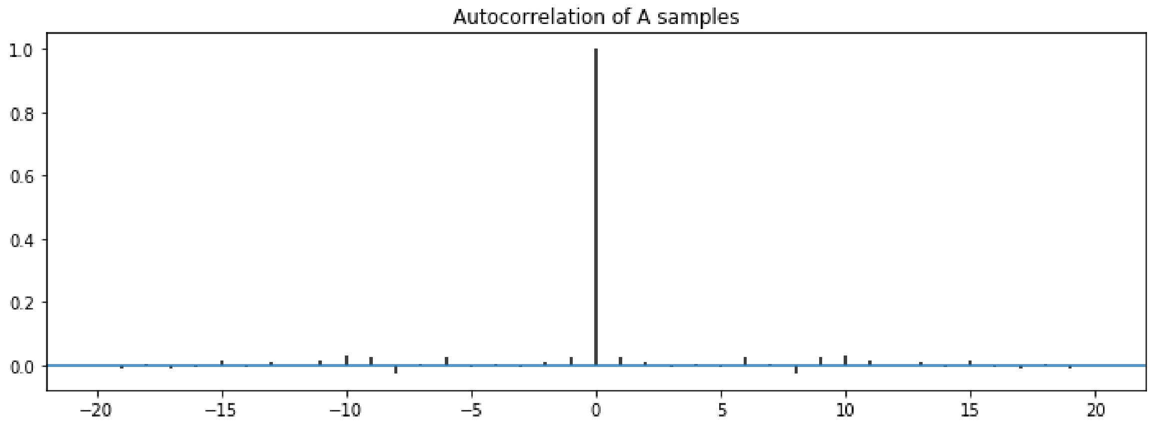
In [47]:
```
print('Model 4')
plot_acf(parameters, samples4)
pairplot(parameters, samples4)
```

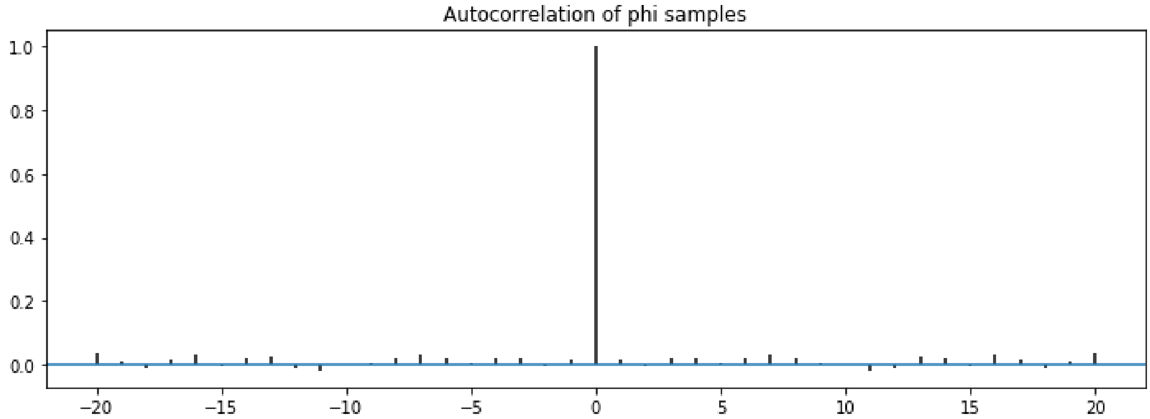
Model 4

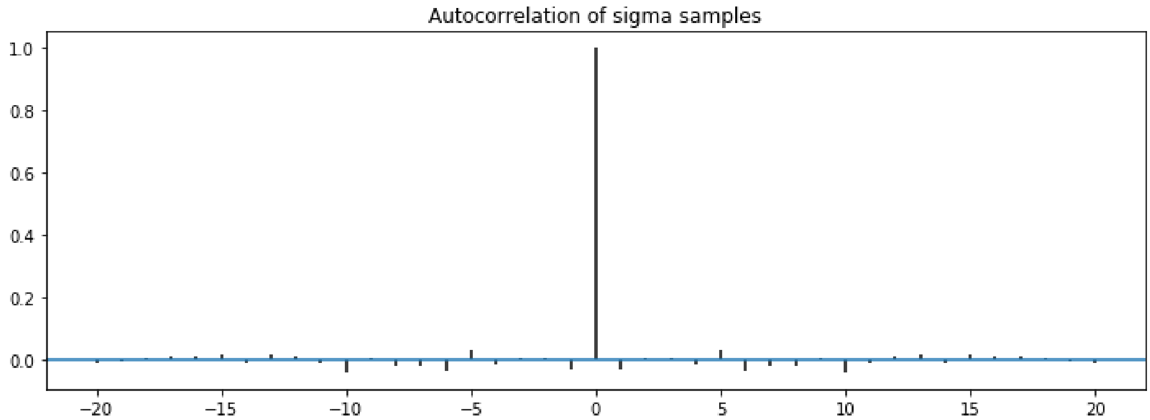

Autocorrelation of c samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20

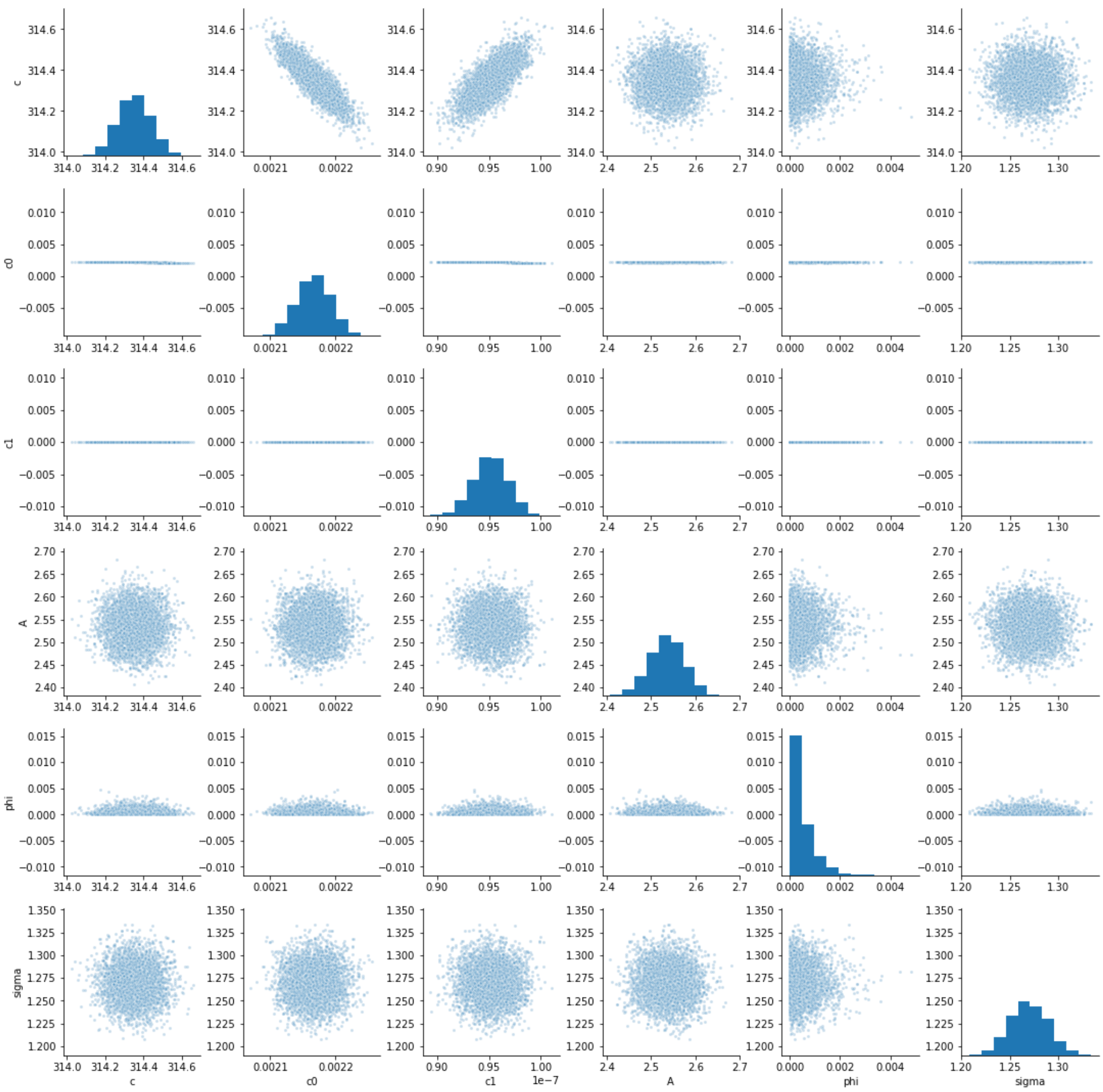

Autocorrelation of c0 samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20


Autocorrelation of c1 samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20


Autocorrelation of A samples
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20


Autocorrelation of phi samples


Autocorrelation of sigma samples

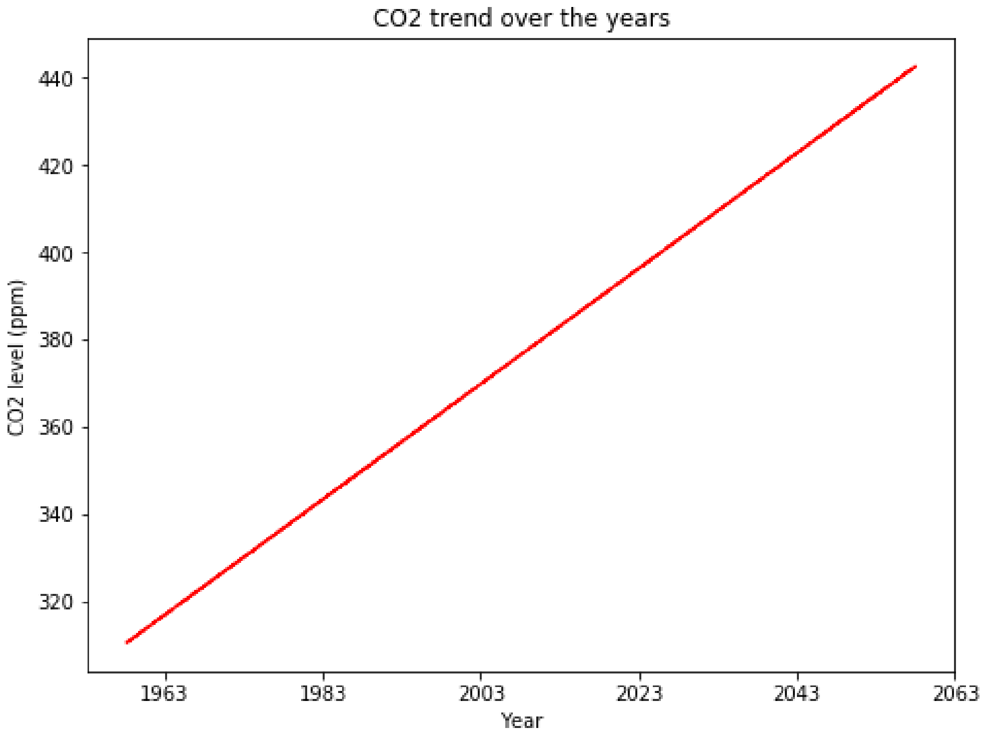

c
c0
c1
A
phi
sigma
1e−7

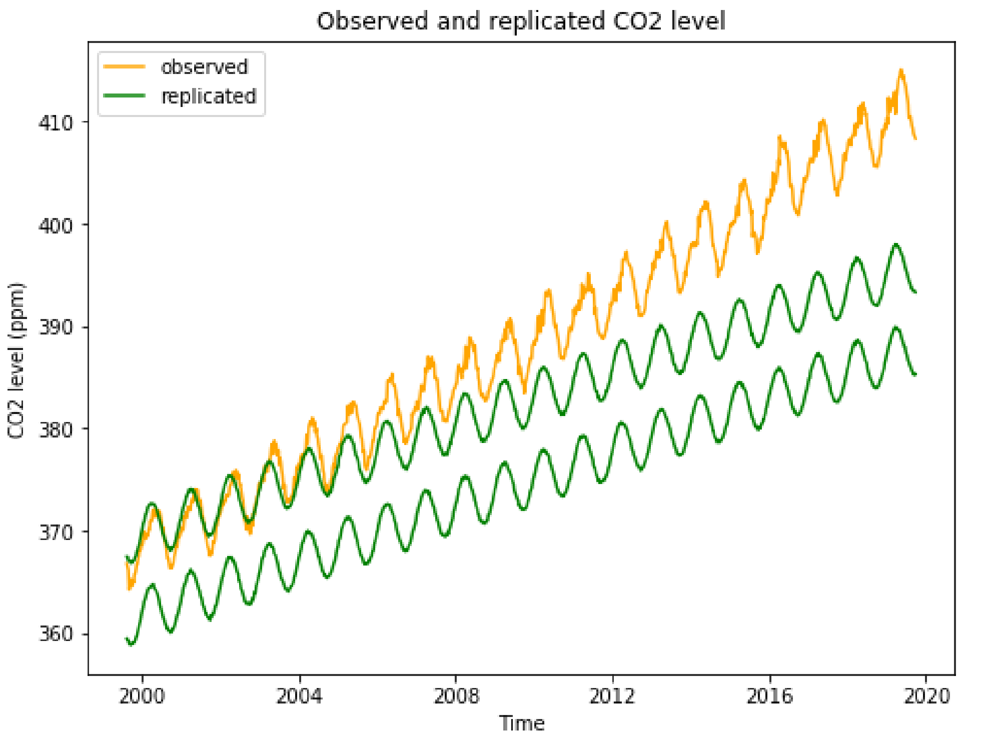
2. Model Accuracy

In [19]:
```
def predict_accuracy(samples):
    # Plot the quadratic trend
    plt.figure(figsize=(8, 6))
    plt.plot(total_time_date, np.percentile(samples['trend'], 50, axis=0
), color='red')
    plt.title('CO2 trend over the years')
    plt.xlabel('Year')
    plt.ylabel('CO2 level (ppm)')

    # Plot 95% confidence interval for replicated data against observed
 data
    plt.figure(figsize=(8, 6))
    plt.title('Observed and replicated CO2 level')
    plt.xlabel('Time')
    plt.ylabel('CO2 level (ppm)')
    # Observed data
    plt.plot(test['Time'], test['CO2'], color='orange', label='observed'
)
    # Replicated data
    plt.plot(test.iloc[:, 0], np.percentile(samples['cd_test'], 2.5, axi
s=0), color='green', label='replicated')
    plt.plot(test.iloc[:, 0], np.percentile(samples['cd_test'], 97.5, ax
is=0), color='green')
    plt.legend()
    plt.show()
```

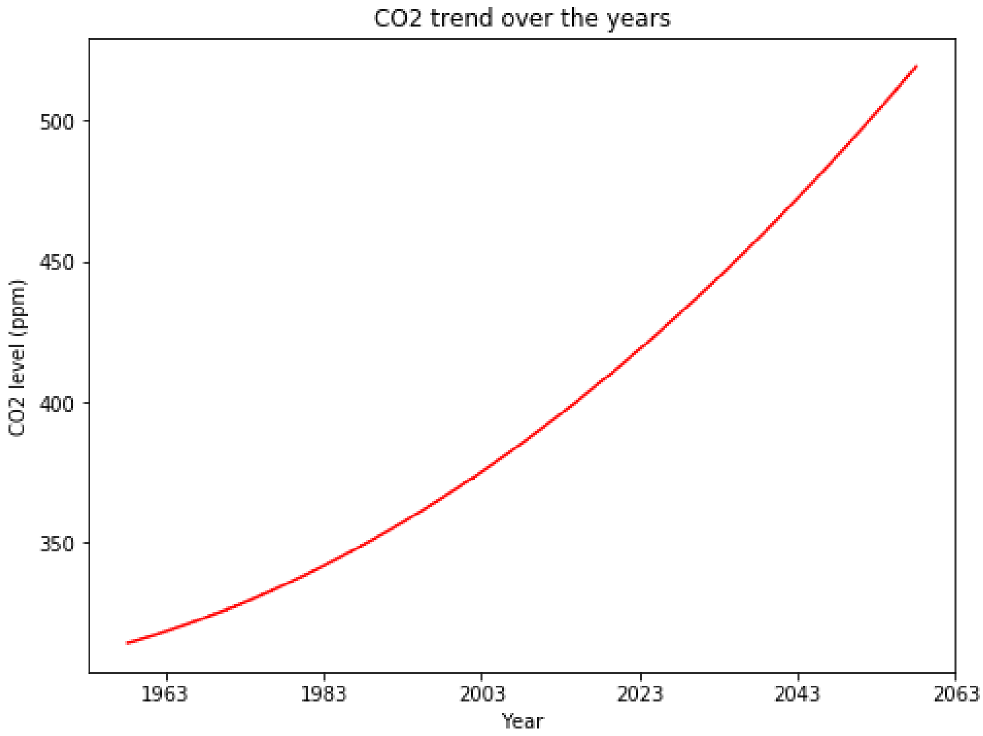
In [43]:
```
print('Model 1')
predict_accuracy(samples1)
print('Model 2')
predict_accuracy(samples2)
print('Model 3')
predict_accuracy(samples3)
print('Model 4')
predict_accuracy(samples4)
```

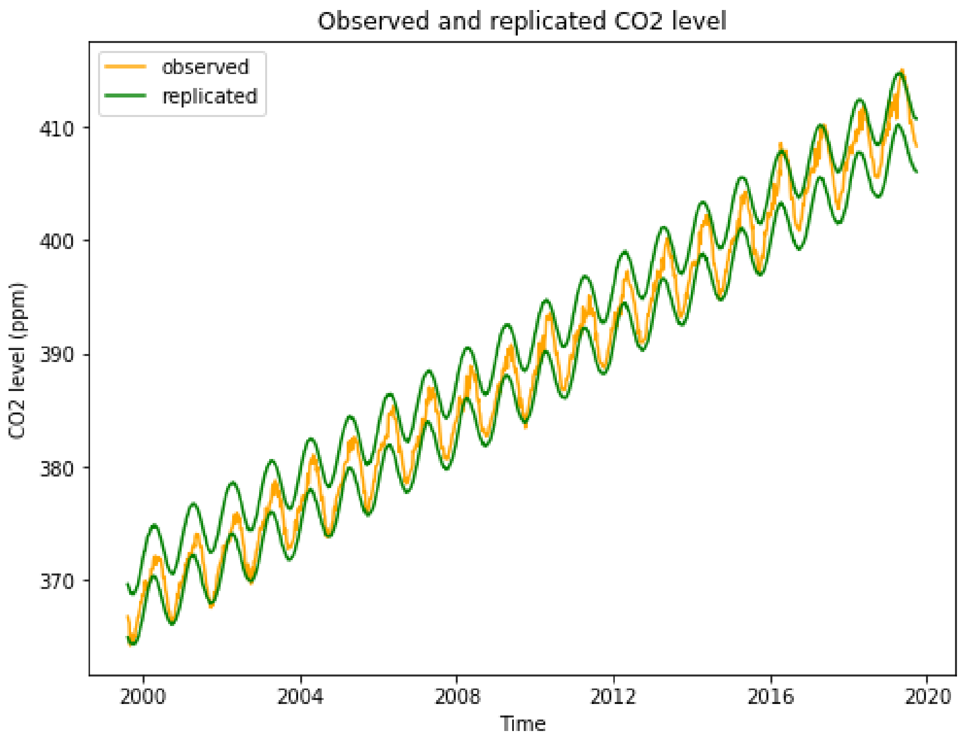
Model 1



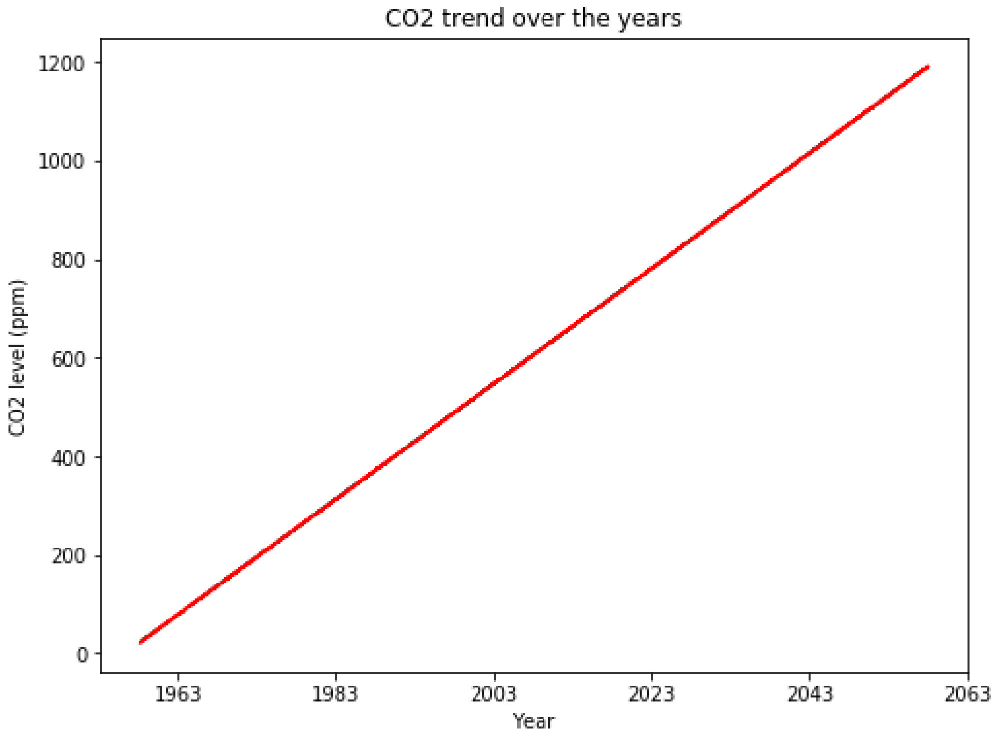


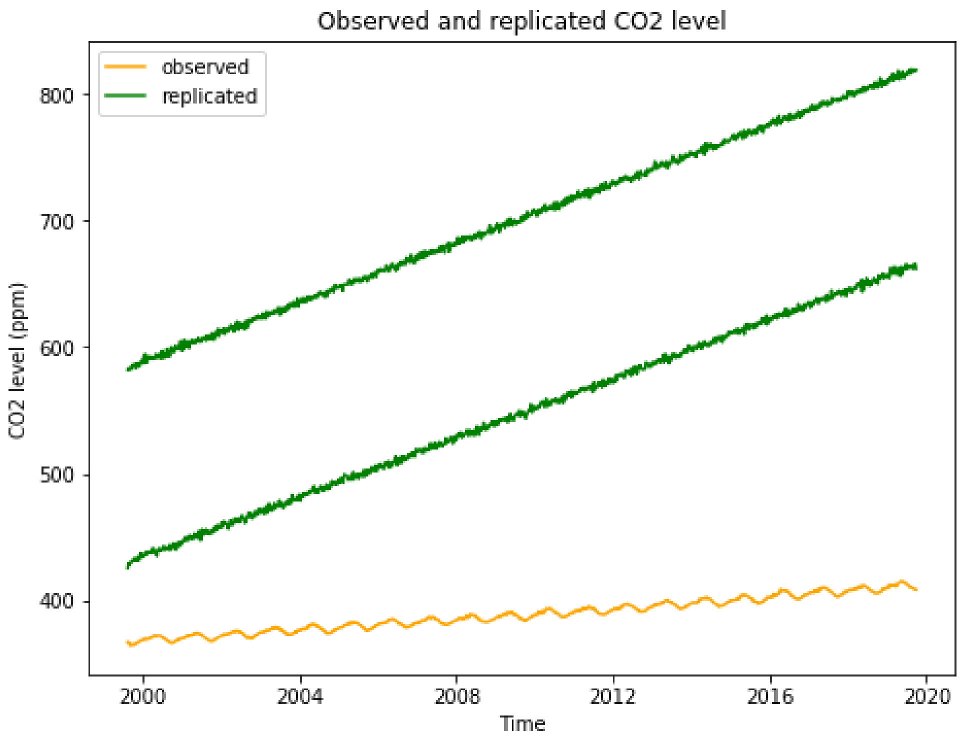
Model 2


CO2 trend over the years
CO2 level (ppm)
500
450
400
350
1963
1983
2003
2023
2043
2063
Year

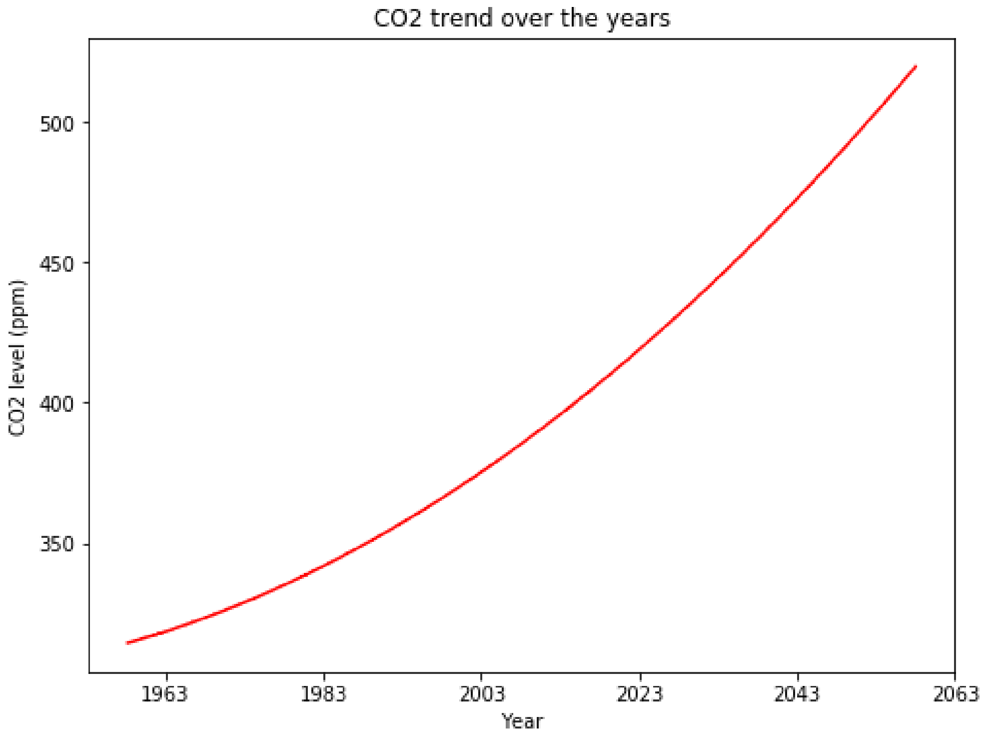

Observed and replicated CO2 level
observed
replicated
CO2 level (ppm)
410
400
390
380
370
2000
2004
2008
2012
2016
2020
Time

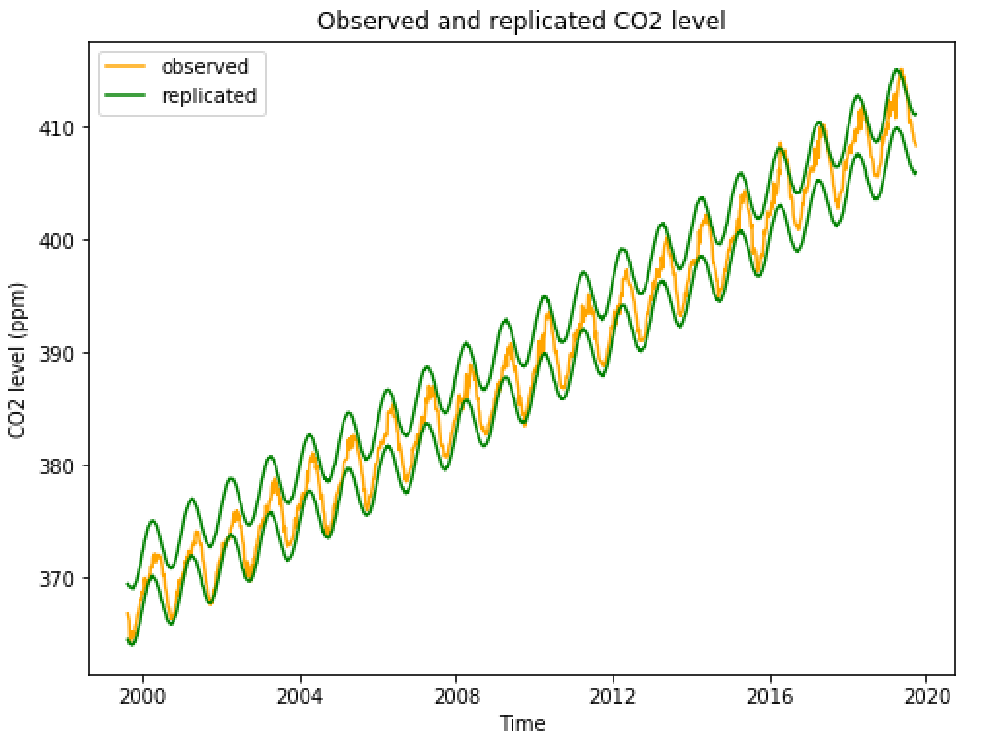
```
Model 3
```


CO2 trend over the years
CO2 level (ppm)
1200
1000
800
600
400
200
0
1963
1983
2003
2023
2043
2063
Year


Observed and replicated CO2 level
observed
replicated
CO2 level (ppm)
800
700
600
500
400
2000
2004
2008
2012
2016
2020
Time

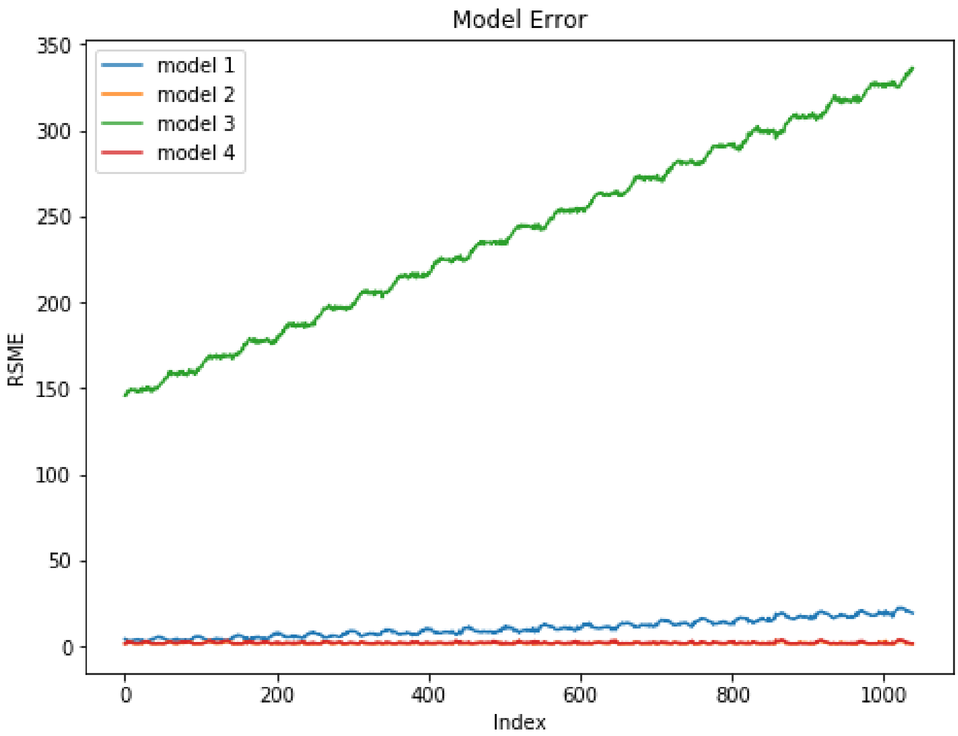
```
Model 4
```


CO2 trend over the years
500
450
400
350
CO2 level (ppm)
1963
1983
2003
2023
2043
2063
Year


Observed and replicated CO2 level
observed
replicated
410
400
390
380
370
CO2 level (ppm)
2000
2004
2008
2012
2016
2020
Time

In [56]:
```
# Error in replication
def RSME(rep, real):
    err = []
    for i in range(len(rep)):
        err.append(np.sqrt(np.mean((rep[i]-real.iloc[i, 1])**2)))
    return err

plt.figure(figsize=(8, 6))
plt.plot(RSME(np.transpose(samples1['cd_test']), test), label='model 1')
plt.plot(RSME(np.transpose(samples2['cd_test']), test), label='model 2')
plt.plot(RSME(np.transpose(samples3['cd_test']), test), label='model 3')
plt.plot(RSME(np.transpose(samples4['cd_test']), test), label='model 4')
plt.title('Model Error')
plt.xlabel('Index')
plt.ylabel('RSME')
plt.legend()
```

Out[56]: <matplotlib.legend.Legend at 0x1a1b5d2160>



III. Prediction

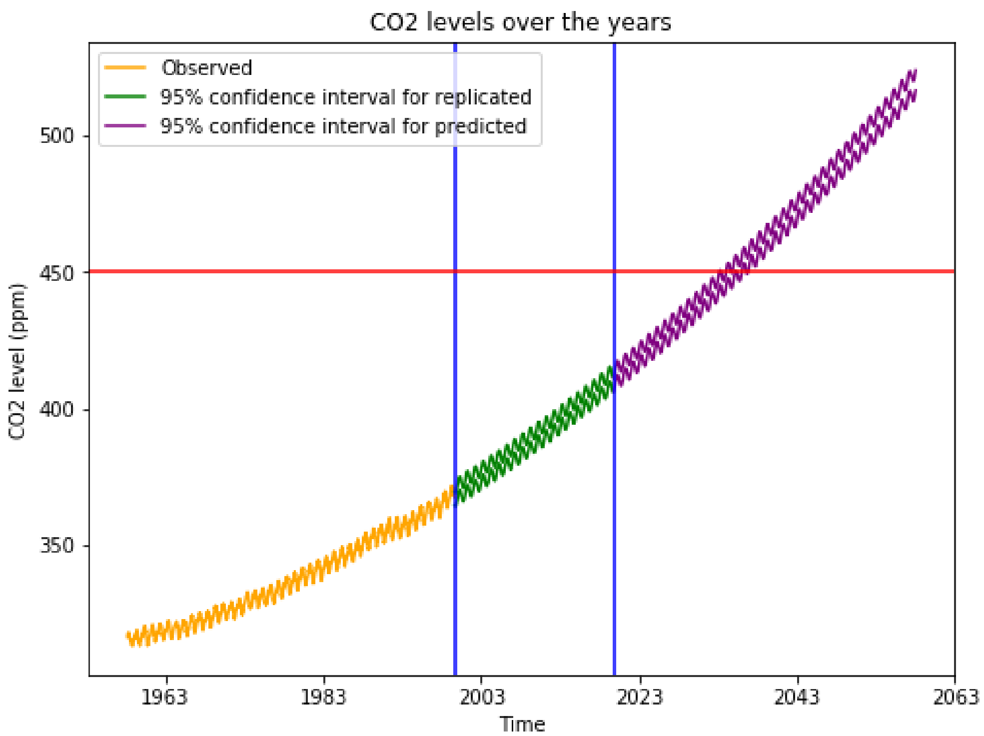
Model 4 was chosen to generate predictions.

In [52]:
```
# Predicted data for the next 40 years
future = pd.concat([future, pd.Series(np.percentile(samples4['cd_future'
], 2.5, axis=0)), pd.Series(np.percentile(samples4['cd_future'], 50, axi
s=0)), pd.Series(np.percentile(samples4['cd_future'], 97.5, axis=0))], a
xis=1)
future = future.rename(columns={0: '2.5%', 1: '50%', 2: '97.5%'})
print(future.head())
print(future.tail())
```

```
         Time      t         2.5%         50%       97.5%
0 2019-10-06  22471  406.050618  408.596559  411.148661
1 2019-10-13  22478  406.072312  408.690201  411.227928
2 2019-10-20  22485  406.261699  408.825368  411.230880
3 2019-10-27  22492  406.475821  408.973243  411.658277
4 2019-11-03  22499  406.702131  409.211055  411.868138
           Time      t         2.5%         50%       97.5%
1992 2057-12-09  36415  515.161587  518.645129  522.159974
1993 2057-12-16  36422  515.355586  518.960300  522.505106
1994 2057-12-23  36429  515.624004  519.337323  522.990192
1995 2057-12-30  36436  516.007535  519.718957  523.371680
1996 2058-01-06  36443  516.301202  520.110811  523.618031
```

In [40]:
```
# Plot observed, replicated and predicted data
plt.figure(figsize=(8, 6))
# Observed
plt.plot(train['Time'], train['CO2'], label='Observed', color='orange')
# Replicated
plt.plot(test.iloc[:, 0], np.percentile(samples4['cd_test'], 2.5, axis=0
), label='95% confidence interval for replicated', color='green')
plt.plot(test.iloc[:, 0], np.percentile(samples4['cd_test'], 97.5, axis=
0), color='green')
# Predicted
plt.plot(future['Time'], np.percentile(samples4['cd_future'], 2.5, axis=
0), label='95% confidence interval for predicted', color='purple')
plt.plot(future['Time'], np.percentile(samples4['cd_future'], 97.5, axis
=0), color='purple')
plt.axvline(test.iloc[0, 0], color='blue')
plt.axvline(future.iloc[0, 0], color='blue')
plt.axhline(450, color='red')
plt.title('CO2 levels over the years')
plt.xlabel('Time')
plt.ylabel('CO2 level (ppm)')
plt.legend()
```

Out[40]: <matplotlib.legend.Legend at 0x104bdac50>



In [42]:
```
# When will the apocalypse be?
ind = np.where((np.abs(future.loc[:, '50%'] - 450) <= 0.05))[0][0]
future.iloc[ind-3:ind+7, :]
```

Out[42]:

	Time	t	2.5%	50%	97.5%
749	2034-02-12	27714	446.431932	449.304473	452.152199
750	2034-02-19	27721	446.572689	449.544792	452.273668
751	2034-02-26	27728	446.821899	449.720788	452.544951
752	2034-03-05	27735	447.203147	449.960553	452.757756
753	2034-03-12	27742	447.250089	450.081540	452.870199
754	2034-03-19	27749	447.397352	450.203000	452.958130
755	2034-03-26	27756	447.520084	450.325974	453.150093
756	2034-04-02	27763	447.582369	450.316130	453.078404
757	2034-04-09	27770	447.506649	450.353947	453.184948
758	2034-04-16	27777	447.604363	450.335958	453.143422

In [55]:
```
ind2 = np.where((np.abs(future.loc[:, '2.5%'] - 450) <= 0.1))[0][0]
future.iloc[ind2-3:ind2+7, :]
```

Out[55]:

	Time	t	2.5%	50%	97.5%
802	2035-02-18	28085	449.362167	452.221061	454.999059
803	2035-02-25	28092	449.530847	452.449082	455.282692
804	2035-03-04	28099	449.802648	452.637642	455.511654
805	2035-03-11	28106	449.926910	452.787015	455.623615
806	2035-03-18	28113	450.142758	452.949979	455.853080
807	2035-03-25	28120	450.228778	452.964924	455.882407
808	2035-04-01	28127	450.187944	453.092061	455.941712
809	2035-04-08	28134	450.275390	453.123132	455.878199
810	2035-04-15	28141	450.210750	453.098330	455.802632
811	2035-04-22	28148	450.191453	453.075421	455.771362

In [ ]: