Section 4.4

Part #1

Derivative of $y = e^u$, $y = a^u$

Using limits prove the following.

$$\frac{d}{dx}e^x = e^x$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$\lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

$$\lim_{h \to 0} e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$e^x \cdot (1)$$

$$e^x$$

h	$\frac{e^h - 1}{h}$
-0.01	0.995
-0.001	0.9995
0.01	1.005
0.001	1.0005

Find the derivaitve.

$$y = x^3e^x - x^2e^x$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^3e^x - x^2e^x)$$

$$\frac{dy}{dx} = \left(x^3\frac{d}{dx}(e^x) + e^x\cdot\frac{d}{dx}(x^3)\right) - \left[x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2)\right]$$

$$\frac{dy}{dx} = (x^3\cdot e^x + e^x\cdot(3x^2)) - [x^2\cdot e^x + e^x\cdot 2x]$$

$$\frac{dy}{dx} = (x^3\cdot e^x + 3x^2e^x) - [x^2\cdot e^x + 2xe^x]$$

$$\frac{dy}{dx} = (x^3\cdot e^x + 3x^2e^x) - x^2\cdot e^x - 2xe^x$$

$$\frac{dy}{dx} = x^3\cdot e^x + 2x^2e^x - 2xe^x$$

$$\frac{dy}{dx} = (x^2 + 2x - 2)xe^x$$

What if we want to find the derivative of something like e^{x^2}, e^{x^3+4x} ?

We will substitute u=f(x) for the exponent $(y = e^{f(x)} = e^u)$ then use the chain rule.

$$y = e^{f(x)} = e^u \qquad \frac{dy}{du} = e^u$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(e^u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ Remember Chain Rule

$$\frac{dy}{dx} = \left((e^u) \cdot \frac{du}{dx}\right)$$

$$\frac{dy}{dx} = \left((e^{f(x)}) \cdot \frac{du}{dx}\right)$$

Find the derivative of the following.

a. $y = e^{x^2}$

$u = x^2, \quad \frac{du}{dx} = 2x$

$$y = e^{x^2} = e^u$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(e^u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \cdot (2x)$$

$$\frac{dy}{dx} = e^{x^2} \cdot (2x)$$

$$\frac{dy}{dx} = 2xe^{x^2}$$

b. $y = e^{4x^2 - 9x}$

$u = 4x^2 - 9x, \quad \frac{du}{dx} = 8x - 9$

$$y = e^{4x^2 - 9x} = e^u$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(e^u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \cdot (8x - 9)$$

$$\frac{dy}{dx} = e^{4x^2 - 9x} \cdot (8x - 9)$$

$$\frac{dy}{dx} = (8x - 9)e^{4x^2 - 9x}$$

Review of logs.

Exponential Form

$$2^3 = 8$$

Logarithmic Form

$$\log_2(8) = 3$$

$$\log_4(5) = \frac{\log 5}{\log 4}$$

$$\ln(x) = \log_e(x)$$

$$\log 5^3 = 3\log 5$$

Since we know how to deal with the derivative of $y = e^x$, we will use it in order to learn to deal with bases that are not e, $(y = a^x)$.

$$\underline{y = a^x}, a > 0, a \neq 1$$

$$\ln(y) = \ln(a^x)$$

$$\log_e(y) = \ln(a^x)$$

$$\log_e(y) = \ln(a^x)$$

$$\log_e(y) = x\ln(a)$$

$$\underline{e^{(x\ln(a))} = y}$$

Since $e^{(x\ln(a))} = y$ and $y = a^x$

then $e^{(x\ln(a))} = a^x$

So why did we do all that? Remember we are trying to find the derivative of $y = a^x$ but we only know how to deal with the derivative of $y = e^u$.

If $y = a^x = e^{(x\ln(a))}$

$$\frac{d}{dx}(y) = \frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(x\ln(a))}\right)$$

Let $u = x\ln(a)$ $\quad \frac{du}{dx} = \ln(a)$

$$\frac{d}{dx}(y) = \frac{d}{dx}(a^x) = \frac{d}{dx}(e^u)$$

$$\frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = e^u\cdot\ln(a)$$

$$\frac{dy}{dx} = \frac{d}{dx}(a^x) = e^{x\ln(a)}\cdot\ln(a)$$

$$\frac{dy}{dx} = \frac{d}{dx}(a^x) = a^x\ln(a), \; a > 0, a \neq 1$$

Find the derivative of $y = 3^x$.

$$y = 3^x$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(3^x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(3^x) \qquad \frac{dy}{dx} = \frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{dy}{dx} = 3^x \ln(3)$$

What about if you want to find the derivative of a base that has an exponent of something other than x such as $4^{x^2}, 3^{5x^4+2x}, 9^{\cos(x)}$?

Make a substitution and then use the Chain Rule, $\left(\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\right)$.

Let u=f(x)

So $y=a^{f(x)}=a^u$

$$\frac{d}{dx}(y)=\frac{d}{dx}(a^u)$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{du}=\frac{d}{du}(a^u)=a^u\ln(a)$$

$$\frac{dy}{dx}=(a^u\ln(a))\cdot\left(\frac{du}{dx}\right)=a^{f(x)}\ln(a)\cdot f'(x)$$

Find the derivative of the following.

a. $y = 7^{-2x}$

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(7^{-2x}\right)$$

Let $u = -2x$ $\frac{du}{dx} = -2$

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(7^{-2x}\right) = \frac{d}{dx}7^{u}$$

$$\frac{d}{dx}(y) = \frac{d}{dx}7^{u}$$ Remember $\frac{d}{dx}(a^{u}) = (a^{u}\ln(a))\cdot\left(\frac{du}{dx}\right)$

$$\frac{dy}{dx} = 7^{u}\ln(7)\frac{du}{dx}$$

$$\frac{dy}{dx} = 7^{u}\ln(7)(-2) = -2\ln(7)\left(7^{-2x}\right)$$

b. $y = 5^{\cos(x)}$

Let u = cos(x) $\frac{du}{dx} = -\sin(x)$

$$\frac{dy}{dx} = \frac{d}{dx}5^u$$

$$\frac{dy}{dx} = 5^u \ln(5)\frac{du}{dx}$$

$$\frac{dy}{dx} = 5^{\cos(x)}\ln(5)(-\sin(x))$$

$$\frac{dy}{dx} = -\ln(5)5^{\cos(x)}\sin(x)$$

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2. $2e^{2x}$

4. $-5e^{-5x}$

6. $-\frac{1}{4}e^{-x/4}$

8. $x^2e^x + xe^x - e^x$

10. $2xe^{x^2}$

12. $-9^{-x}\ln 9$

14. $-3^{\cot x}\ln 3 \csc^2 x$