Section 3.2

Differentiability

Remember:

-For a derivative to occur a limit must exist.

-Therefore, the limit from both sides must be the same. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} (\quad) = \lim_{h\to 0^-} (\quad)$

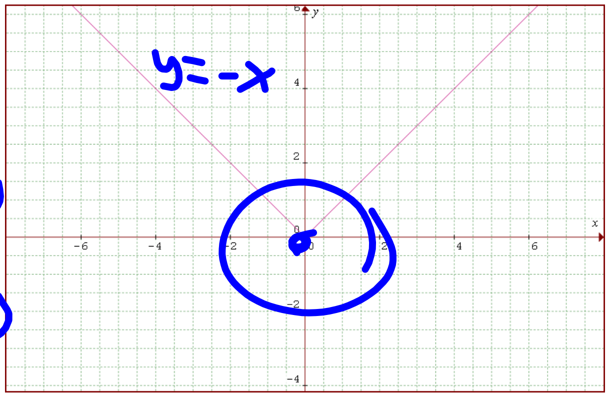
-This is why endpoints do not have a derivative.

Today:

There are four new situations in which a function ***will not*** have a derivative at a point.

Case 1: A Corner

$$y=|x| = \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$$



$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{-(0+h)-0}{h}$$

$$= \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{(0+h)-(0)}{h}$$

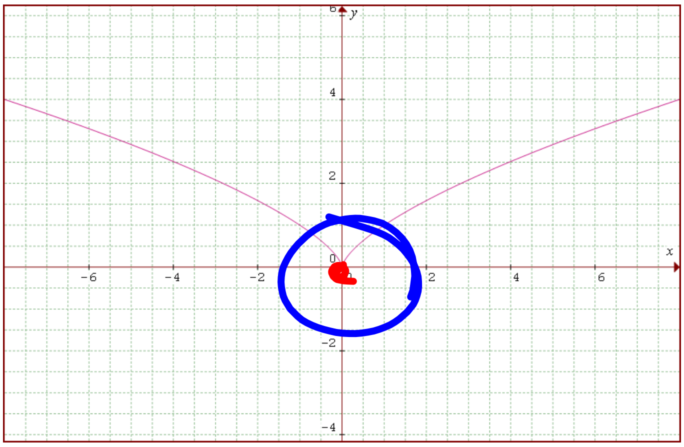
$$= \lim_{h\to 0^+} \frac{h}{h} = 1$$

Since $-1 \neq 1$

The limit DNE, thus the derivative does not exist.

Case 2: A Cusp

$y = x^{2/3}$



$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{(0+h)^{2/3} - 0^{2/3}}{h}$$

$$= \lim_{h \to 0^-} \frac{h^{2/3}}{h} = \lim_{h \to 0^-} \frac{1}{h^{1/3}} = \lim_{h \to 0^-} h^{-1/3}$$

$$= -\infty$$

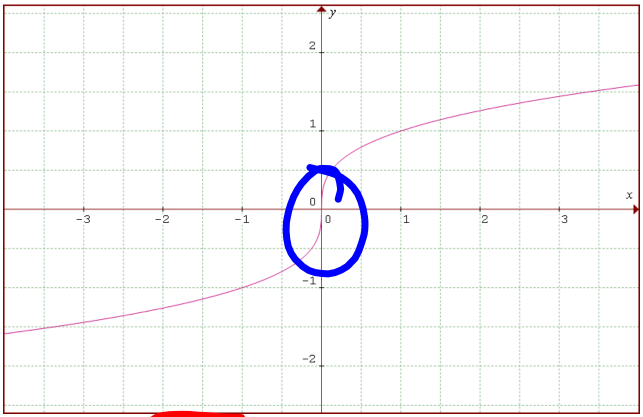
$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{(0+h)^{2/3} - 0^{2/3}}{h}$$

$$= \lim_{h \to 0^+} \frac{h^{\frac{2}{3}}}{h} = \lim_{h \to 0^+} \frac{1}{h^{1/3}} = +\infty$$

$f'(0)$ DNE

Case 3: A Vertical Tangent

$y = \sqrt[3]{x}$



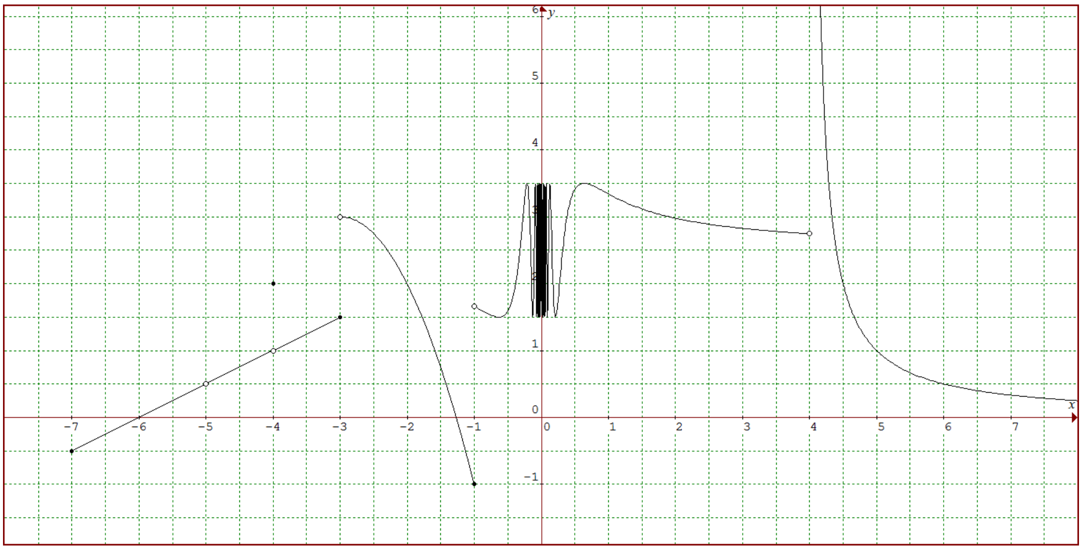
$$\lim_{h \to 0^-} \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h} = \lim_{h \to 0^-} \frac{\sqrt[3]{h}}{h}$$

$$= \lim_{h \to 0^-} \frac{h^{1/3}}{h} = \lim_{h \to 0^-} \frac{1}{h^{2/3}} = +\infty$$

$$\lim_{h \to 0^+} \frac{1}{h^{2/3}} = \infty$$

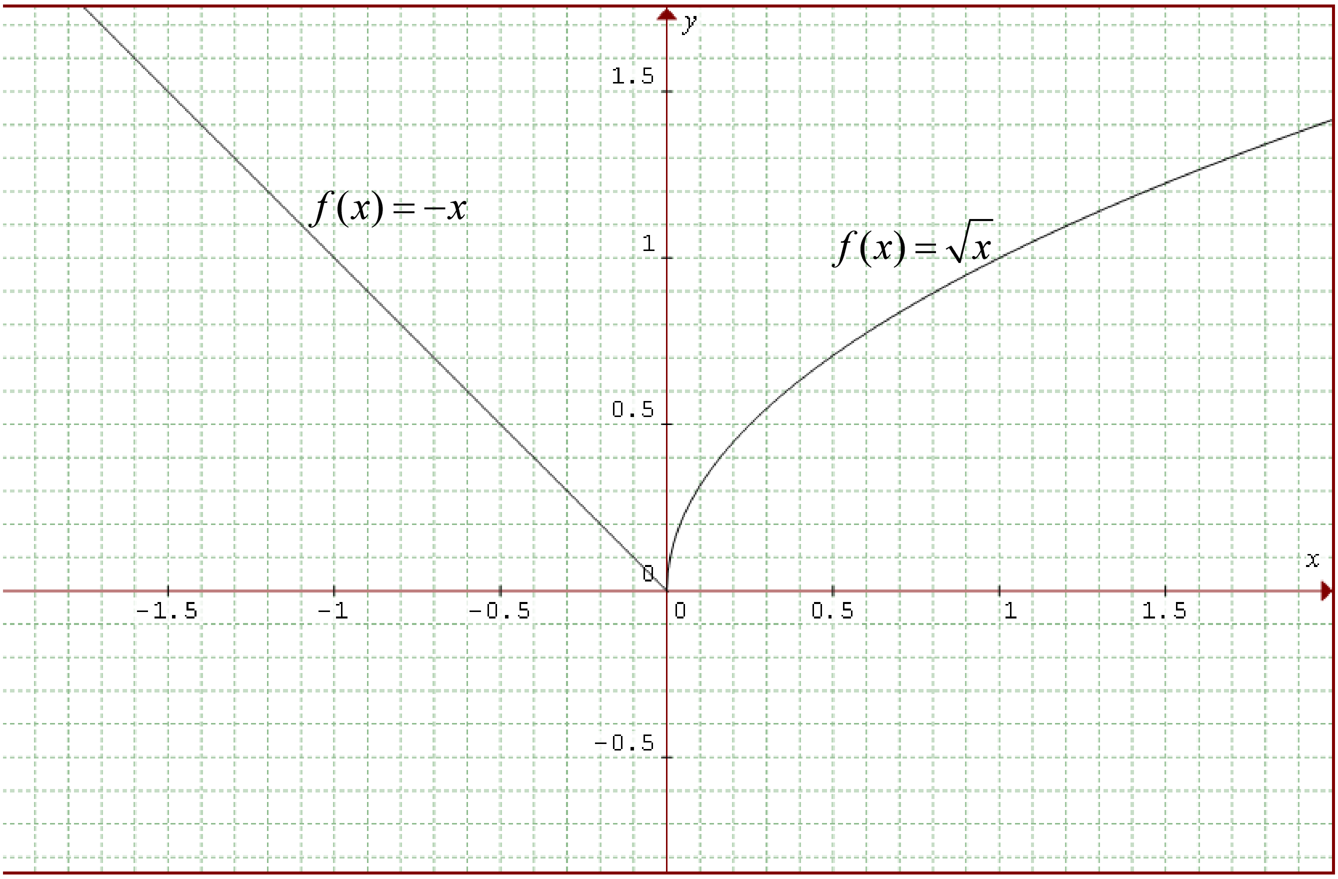
The derivative does not exist at $x = 0$.

Case 4: Discontinuities



No Derivative $x = -5, -4, -3, -1, 0, 4$

Compare right-hand and left-hand derivatives to prove the following functions are not differentiable at the indicated points:

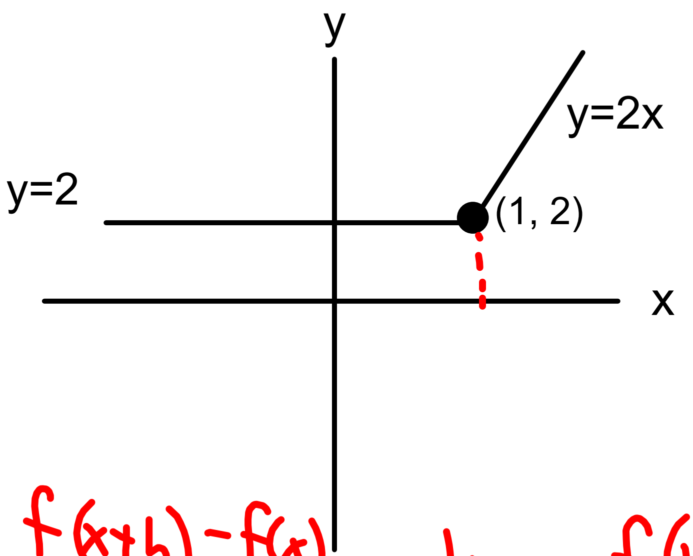


$$\lim_{h \to 0^-} \frac{-(0+h) - (-0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$$\lim_{h \to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}}$$

$$= \infty$$

$f'(0)$ DNE



$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{2-2}{h} = 0$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h)-2(1)}{h}$$

$$\lim_{h\to 0^+} \frac{2+2h-2}{h} = 2$$

Find all points where f(x) is not differentiable:

a) $f(x)=\dfrac{x^4-7x^3+2x^2-18}{x^3-7x^2+12x}$

$x(x^2-7x+12)$

$x(x-3)(x-4) \Leftrightarrow x=0,3,4$

b) $f(x)=\sqrt{|x|}$

$$\lim_{h\to0^-}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0^-}\frac{\sqrt{|0+h|}-\sqrt{|0|}}{h}$$

$$=\lim_{h\to0^-}\frac{\sqrt{|h|}}{h}=-\infty$$

$$\lim_{h\to0^+}\frac{\sqrt{|h|}}{h}=\infty$$

$f'(0)$ D.N.E.

Homework: p.114

#1, 3

#5, 7, 8 → There are some mistakes in the textbook regarding these questions. Part a answers should have the following brackets (#, #) rather than [#, #] and the instructions for b should read continuous but not differentiable excluding endpoints. The proper answers are below.

#11, 15

#31,35

Answers:

5. a. (-3, 2)	7. a. (-3, 3) except x=0	8. a. (-2, 3) except x=-1, 0, 2
b. None	b. None	b. None
c. None	c. x=0	c. x= 0, 2