Lecture 6, Density Matrix, Bloch Sphere, and Quantum Entanglement, Wednesday, Sept. 14

All discussion so far is for an isolated quantum system. However, often we need to study a subsystem of a large quantum system, for which much of what we said need modification. For example, the state is not determined by a vector.

Consider a quantum system consisting of two subsystems A and B. The state space is a *tensor product* of the two individual state spaces for A and B. For example, for a two-spin system, the state space is has $4 = 2 \times 2$ dimensions.

Let us suppose that the quantum state of the system is

$$|\psi\rangle_{AB} = a|+\rangle_A \otimes |+\rangle_B + b|-\rangle_A \otimes |-\rangle_B \tag{38}$$

where $\otimes$ indicates tensor product and will often be omitted without warning. If we measure a general observable of system A, O_A, irrespective of the outcomes of any measurements of B, then we take the expectation value of $O_A \otimes \mathbf{1}_B$,

$$\langle\psi|O_A \otimes \mathbf{1}_B|\psi\rangle = |a|^2\langle+|O_A|+\rangle + |b|^2\langle-|O_A|-\rangle \tag{39}$$

The above result can also be obtained by introducing the density matrix

$$\rho_A = |a|^2|+\rangle_A\ {}_A\langle+| + |b|^2|-\rangle_A\ {}_A\langle-| \tag{40}$$

and then

$$\langle O_A\rangle = \mathrm{Tr}(O_A\rho_A) \tag{41}$$

The density matrix is hermitian, positive definite in the sense that all of its eigenvalue must be real, and has unit trace $\mathrm{Tr}\rho = 1$.

The system A described by ρ_A is not a pure quantum mechanical state although $|\psi\rangle_{AB}$ is. It can be considered as an ensemble, prepared with probability of $|a|^2$ in quantum state $|+\rangle$ and probability of $|b|^2$ in quantum state $|-\rangle$. These probabilities are classical in the sense that there is no interference. In particular, the relative phase between $|+\rangle$ and $|-\rangle$ is completely arbitrary!

Suppose the density matrix has the form,

$$\rho_A = \frac{1}{2}\Big(|+\rangle_A\ {}_A\langle+| + |-\rangle_A\ {}_A\langle-|\Big) \tag{42}$$

If one measures S_z, one gets 1/2 and 1/2 probabilities for $S_z = \pm\hbar/2$. If one measures S_x, one gets the same result. On the other hand, if one has a pure state $|+\rangle_x$, although the S_z measurement also gives the same result, but the S_x measurement always gives $+\hbar/2$, without uncertainty.

Therefore, we can talk about a pure quantum state $|\psi\rangle$ as a vector in the state space. We must also consider the mixed states described by a density matrix, which can be written in general as $\sum_{ij} \rho_{ij}|i\rangle\langle j|$.

Show that if $|\psi\rangle_{AB} = \sum_{i\mu} a_{i\mu}|i\rangle_A \otimes |\mu\rangle_B$, $\rho_{ij} = \sum_\mu a_{i\mu}a^*_{j\mu}$. Show its hermiticity, positive-definiteness, and unity-trace.

If the density matrix has 1 nonzero eigenvalue, it is a pure state. Then it must satisfy $\rho^2 = \rho$. If the density matrix has more than 1 nonzero eigenvalues, it is a mixed state. Density matrix can always be written as a diagonal form in some basis

$$\rho_A = \sum_a p_a|\psi_a\rangle\langle\psi_a| \tag{43}$$

The ensemble is said to be an incoherent superposition of the states $|\psi_a\rangle$ without any fixed-phase relation.

For a spin-1/2 particle, the most general spin density matrix can be written as

$$\rho = \frac{1}{2}(1 + \vec{P}\cdot\vec{\sigma}) \ , \tag{44}$$

where we have used the condition that $\mathrm{Tr}\rho = 1$ and ρ is hermitian. Finally, the determinant $\det\rho = (1/4)(1 - \vec{P}^2)$ must be positive, therefore $\vec{P}^2 \leq 1$. All possible $\vec{P}$ form a solid sphere which is called the Bloch sphere. All states with $P^2 = 1$ are pure. Otherwise, they are mixed states. Any points inside the sphere can be written as a sum of two pure states.

Schmidt decomposition: Any state in A+B system can be written as

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}|i\rangle_A|i'\rangle_B| \tag{45}$$

where $\{|i\rangle\}$ and $\{|i'\rangle\}$ are orthonormal bases in A and B, respectively. The number of non-zero p_i is called Schmidt number. If the Schmidt number is greater than 1, we say $|\psi\rangle_{AB}$ is entangled; and for an entangled state, the density matrix for ρ_A is not pure.

If we start with a large system A+B which is pure, but if A is B is entangled, and if one measures system A only, one finds that the coherence in A is destroyed. We call this situation a *collapse* of the state in system A. State collapse allows us to understand the measurement of quantum system and decoherence.