Stability of graph-dependent switched systems with a simple loop dwell time approach

LINMA2361- Nonlinear dynamical systems

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Contents

1 Introduction

In this project, we are interested in the stability of switched systems in which the switching signal is governed by a graph. Graph-dependent switched systems have many applications in the real world such as in electrical or power grid systems.

But first, let us define a continuous time graph-dependent switched system. It is a piecewise continuous dynamical system with finitely many subsystems and a switched signal which is a piecewise constant function. This signal tells us about the admissible switching from a subsystem to another, the times at which it occurs and in our case here, it corresponds to an infinite path in a directed graph.

The stability of such systems depend not only on the subsystems but also on the switched signal. Indeed, it is possible to find examples of systems with all stable subsystems that are unstable for some signals [**2**]. And on the contrary, it is possible to find signals for which the overall system is stable even though it has unstable subsystems.

To stabilize such systems, there exists different ways to tackle the problem. If the system permits it, we could try to find an input signal based on the subsystems characteristics to stabilize the system and that would work for any switched signals. But here, what we are interested in is to find conditions on switched signals itself such that the switched system is stable (uniformly exponentially).

And for that, we analyze the article "A simple loop in dwell time approach for stability of switched systems" by Nikita Agarwal [**1**].This article is focused on finding conditions on the switching signals to stabilize the system. We will present and simulate here the results that N. Agarwal obtained in her article and we will also provide some proofs and intuitions that the author didn't include in the article. This project will thus follow the structure of the article by providing first some background notions on graphs, graphs-dependent switched systems and switched systems stability. Then, we will consider switched systems with only stable subsystems where we will learn about the dwell time and simple loop dwell time. After that, we will consider more general switched systems with not only stable subsystems but also unstable ones. Then, the concept of flee time will be introduced as well as its equivalent in simple loops. We will complete this report with some commentaries on the article.

2 Background

In this section, we will present the notions that N. Agarwal defined in her article.

Notation (Matrix norm) Through this report, the matrix norm $\|M\|$ is the spectral norm.

2.1 Graphs

Definition 2.1 (Directed graph $\mathcal{G}$)**.** *A directed graph or digraph $\mathcal{G}$ is a set of k vertices and directed edges from one vertex to another ($k \in \mathbb{N}, \boldsymbol{k} = \{1, \dots, k\}$). In the article, the author assumes that there is atmost one edge from one vertex to another.*

Definition 2.2 (Vertices of $\mathcal{G}$)**.** $v(\mathcal{G}) = \{v_1, \dots, v_k\}$

Definition 2.3 (Edges of $\mathcal{G}$)**.** $\mathcal{E}(\mathcal{G}) = \{(i,j) | \exists$ *an edge from vertex* v_i *to* v_j *for* $i, j \in \mathbf{k}\}$

Definition 2.4 (Adjacency matrix of $\mathcal{G}$)**.** $A_{\mathcal{G}} = [a_{ij}]_{i,j=1}^{k}$ *with* $a_{ij} = 1$ *if there is an edge from* v_i *to* v_j *and* $a_{ij} = 0$ *if not.*

Definition 2.5 (Indegree of vertex v_j, $j \in \mathbf{k}$)**.** *The indegree of* $v_j = \sum_{j=1}^{k} a_{ij}$ *is the total number of incoming edges to* v_j*.*

Definition 2.6 (Outdegree of vertex v_i, $i \in \mathbf{k}$)**.** *The outdegree of* $v_i = \sum_{i=1}^{k} a_{ij}$ *is the total number of outgoing edges from* v_i*.*

Definition 2.7 (Path in $\mathcal{G}$)**.** *A path p in $\mathcal{G}$ is a sequence of vertices and directed edges s.t. from each vertex there is an edge to the next vertex in the sequence. It can be denoted either by the sequence of vertices or the sequence of edges. The length of the path $l(p)$ is the number of edges in p.*

Definition 2.8 (Loop)**.** *A loop is a closed path, i.e. with the first and last vertices being the same.*

Definition 2.9 (Simple loop)**.** *A simple loop is a loop with all distinct vertices (except for the first and last).*

Definition 2.10 (Acyclic graph)**.** *An acyclic graph is a graph without loops.*

Definition 2.11 (Strongly connected)**.** *A graph is strongly connected if there is a path from each vertex to every other vertex.*

2.2 Graph-dependent switched systems

Definition 2.12 ($\mathcal{G}$-admissible signal)**.** *A $\mathcal{G}$-admissible signal $\sigma : [0, \infty) \to \{1, \ldots, k\}$ is a right-continuous piecewise constant function with discontinuities at $0 = t_0 < t_1 < \ldots$ s.t. $(\sigma(t_i), \sigma(t_{i+1})) \in \mathcal{E}(\mathcal{G}), \forall i \in \mathbb{N}$ and $t_i \in \mathbb{R}^+$. Let σ_i be the value of σ in the time interval $[t_{i-1}, t_i)$ for $i \in \mathbb{N}_0$. So $\sigma_1, \sigma_2, \ldots, \sigma_n$ is a path in $\mathcal{G}$ of length n. So, a $\mathcal{G}$-admissible signal σ is fully described by the increasing switching times $(t_i)_{i\geq 1}$ and the infinite path in $\mathcal{G}$ $(\sigma_i)_{i\geq 1}$. A path corresponding to the signal $\sigma^{(n)} = \sigma_1\sigma_2 \ldots \sigma_n$ can be described by the edges $e_i = (v_{\sigma_i}, v_{\sigma_{i+1}}), i \geq 1$. And $\delta_i = t_{i+1} - t_i$, the time that the signal spends in the σ_i-th subsystem. $\mathcal{S}_\mathcal{G}$ denotes the collection of all $\mathcal{G}$-admissible signals.*

We will consider k real subsystems with k subsystem matrices $A_1, \ldots A_k \in \mathbb{R}^{D\times D}$ corresponding to the k vertices in the graph $\mathcal{G}$.

The matrices $A_i, i \in \mathbf{k}$ are all diagonalizable. The real Jordan form will be used $A_j = P_j\Lambda_j P_j^{-1}$ where columns of P_j are eigenvectors of A_j with unit norm. Note that if $A = P\Lambda P^{-1}$, then $\|e^{\Lambda s}\| \leq e^{\lambda^* s}$ with $\lambda^* = \max\limits_{d=1,\ldots D} \mathrm{Re}\{\lambda_d\}$ with λ_d the eigenvalues of A.

Definition 2.13 (Stable)**.** *A matrix is called stable if all its eigenvalues $\mathbb{R}e\{\lambda_d\} < 0$ for $d = 1, \ldots D$.*

Definition 2.14 (Unstable)**.** *A matrix is called unstable if for some $d = 1, \ldots D$, $\mathbb{R}\{\lambda_d\} > 0$ and $\mathbb{R}e\{\lambda_d\} \neq 0$ for $d = 1, \ldots D$.*

The switched system that we are interested in is :

$$x'(t) = A_{\sigma(t)}x(t), \quad t \geq 0. \tag{2.1}$$

For $\sigma \in \mathcal{S}_\mathcal{G}$.

Definition 2.15 (Subsystem)**.** *For each $i \geq 1$, with $t \in [t_{i-1}, t_i)$, the system $x'(t) = A_{\sigma_i}x(t)$ is a subsystem of the system (2.1). If A_{σ_i} is stable, so is the subsystem corresponding to it.*

It is assumed that the zeno behaviour (i.e. when the switching times have an accumulation point) doesn't occur so $\lim_{n\to\infty} t_n = \infty$.

2.3 Stability of switched systems

In the article by N. Agarwal, the objective is to find the switching signals $\sigma \in \mathcal{S}_\mathcal{G}$ such that the switched system 2.1 is uniformly exponentially stable, i.e. if

$$\exists \alpha, \beta > 0 \text{ s.t. } \forall x(0) \in \mathbb{R}^D, \|x(t)\| \leq \alpha e^{-\beta t}\|x(0)\| \tag{2.2}$$

Note that since the subsystems are linear systems with real part of eigenvalues of the subsystem matrices being non zero, the only stability that we can have is the uniform exponential stability.

3 Switched system with stable subsystems

In this section, we consider switched systems with only stable susbystems, i.e. $A_1, \ldots A_k$ are Hurwitz.

Definition 3.1 (Dwell time)**.** *The dwell time τ is a minimum time that the signal spends in a stable subsystem.*

It has been proven that if the dwell time is large enough, then the switched system is stable. **[3]**

Notation $\forall i, -\lambda_i^* := \max_{d=1,\dots D} \mathbb{R}\text{e}\{\lambda_{A_i,d}\}$ with $\lambda_{A_i,d}$, the eigenvalues of A_i.

It is known that for $t \in [t_{n-1}, t_n)$,

$$x(t) = e^{A_{\sigma_n}(t-t_{n-1}}e^{A_{\sigma_{n-1}}(t_{n-1}-t_{n-2}} \dots e^{A_{\sigma_1}(t_1}x(0) \tag{3.1}$$

The objective is to obtain an expression like in (2.2). So let us develop the norm of $x(t)$.

$$\|x(t)\| = \|P_{\sigma_n} e^{D_{\sigma_n}(t-t_{n-1})} P_{\sigma_n}^{-1} (\prod_{j=1}^{n-1} P_{\sigma_{n-j}} e^{D_{\sigma_{n-j}}(t_{n-j}-t_{n-j-1})} P_{\sigma_{n-j}}^{-1}) x(0)\| \tag{3.2}$$

$$\leq \|P_{\sigma_n}\| \|P_{\sigma_1}^{-1}\| e^{-\lambda_{\sigma_n}^*(t-t_{n-1})} (\prod_{j=1}^{n-1} \|P_{\sigma_{j+1}}^{-1} P_{\sigma_j}\| e^{-\lambda_{\sigma_j}^*(t_j-t_{j-1})}) \|x(0)\| \tag{3.3}$$

$$\leq \rho e^{-\lambda_{\sigma_n}^*(t-t_{n-1})} a_n^\sigma \|x(0)\| \tag{3.4}$$

$$\leq \rho a_n^\sigma \|x(0)\| \tag{3.5}$$

with $a_n^\sigma = \prod_{j=1}^{n-1} \|P_{\sigma_{j+1}}^{-1} P_{\sigma_j}\| e^{-\lambda_{\sigma_j}^*(t_j-t_{j-1})}$ and $\rho = \max_{i,j\in\mathbf{k}}\{\|P_i^{-1}\|\|P_j\| \,|\exists \text{ path in } \mathcal{G} \text{ from } v_i \text{ to } v_j\}$.[1] Note that ρ is independent of the signal σ, it only depends on the graph $\mathcal{G}$.

The author assumes the following hypothesis.

Hypothesis H1 The graph $\mathcal{G}$ has p simple loops $s_1, \dots s_p, p \geq 1$. And for given $T > 0$, σ is discontinous at some $t \geq T$.

Indeed, if $\mathcal{G}$ has a loop, $\rho \geq 1$ since for any invertible matrices A, B, $\|A\|\|B\|\|B^{-1}\|\|A^{-1}\| \geq \|ABB^{-1}A^{-1}\| = 1$. When $\rho < 1$, $\|x(t)\| \leq \rho^n \|x(0\|$, so the system is always stable.
Intuitively, it is clear that if $\mathcal{G}$ has no loop, every path in $\mathcal{G}$ has length $l(p) \leq k = |\mathcal{G}|$. It means that every signal would be constant eventually as it would reach a final vertex with 0 outdegree. After which, the system stabilizes like a regular linear system. Note that this can also happen when there are loops in the graph. So, to avoid this trivial case, it will be considered that all vertices have an outdegree > 0 and that there is more than just a self-loop.

3.1 Standard decomposition algorithm

Let us consider a graph $\mathcal{G}$, with k vertices $\{v_1, \dots v_k\}$ and a switching signal $\sigma \in \mathcal{S}_\mathcal{G}$ with switching times $(t_i)_{i\geq 1}$ and an infinite path $\sigma^{(n)} = \sigma_1 \dots \sigma_n$. The edges composing this path are $e_i = (v_{\sigma_i}, v_{\sigma_{i+1}}), i \geq 1$ and their corresponding times $\delta_i = t_{i+1} - t_i$.

We will now introduce an algorithm presented by N. Agarwal to decompose the path described by $\sigma^{(n)} = \sigma_1 \dots \sigma_n$ into simple loops and an indecomposable path.

Step 1 Define $P_0 := \sigma_1 \dots \sigma_n$. The corresponding edges are $e_1, \dots e_n$. Let us also define the set of indices of the edges appearing in P_0, $i(P_0) := \{j | e_j \in P_0\}$.
Find k_1 and k_2 such that

$$k_2 \in i(P_0) \text{ is the minimum index at which } \sigma_{k_2} = \sigma_{j+1},\ j < k_2,\ j \in i(P_0)$$

and

$$k_1 \in i(P_0),\ k_1 < k_2,\ \sigma_{k_1} = \sigma_{k_2+1}$$

If there is no (k_1, k_2) corresponding to these criteria, the path P_0 is indecomposable and **STOP** the algorithm.

If there exists such a pair (k_1, k_2), then continue with **Step 2**. In that case, it is clear that $P^0 \cup \sigma_{k_2+1} := \sigma_{k_1}\sigma_{k_1+1} \dots \sigma_{k_2}\sigma_{k_2+1}$ is a simple loop in $\mathcal{G}$.

[1] In the article published by SIAM, the indices i and j were inverted and it would not change the final results. This was confirmed after some mail exchanges with the author.

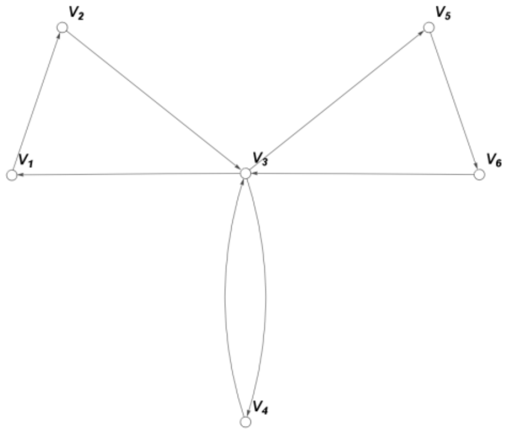

Figure 1: Graph $\mathcal{G}$
[**1**]

Step 2 Redefine $P_0 := P_0 \setminus P^0$. Go back to **Step 1**.

3.2 Classes of switching signals

2 classes of switching signals are considered here depending on whether we want to work with the dwell time τ or the simple loop dwell time τ_c ($c \in \mathbf{p} := \{1, \dots p\}$) corresponding to a lower bound for the total time spent in the loop s_c.

$$\mathcal{S}_{\mathcal{G}}(\tau) = \{\sigma \in \mathcal{S}_{\mathcal{G}} | t_{i+1} - t_i \geq \tau, i \geq 0\} \tag{3.6}$$

and

$$\begin{aligned} \mathcal{S}_{\mathcal{G}}(\tau_1, \dots \tau_p) = \{\sigma \in \mathcal{S}_{\mathcal{G}} | \text{ total time spent by the signal } \sigma \text{ on each simple loop } s_c & \quad (3.7) \\ \text{given by the standard decomposition is } \geq \tau_c, c \in \mathbf{p}\} & \quad (3.8) \end{aligned}$$

3.3 Lower bounds on dwell time and simple loop dwell time

There are several lower bounds on the dwell time that have been derived in other articles[**3**][**4**] such as :

- $\mu_{\mathcal{G}} = \max\limits_{(i,j) \in \mathcal{E}(\mathcal{G})} \frac{\ln \|P_j^{-1} P_i\|}{\lambda_i^*}$
- $\rho^* = \max\limits_{c \in \mathbf{p}} \frac{\sum_{(i,j) \in \mathcal{E}(s_c)} \ln \|P_j^{-1} P_i\|}{\sum_{(i,j) \in \mathcal{E}(s_c)} \lambda_i^*}$
- $\rho_2^* = \max\limits_{(i,j) \in \mathcal{E}(\mathcal{G})} \frac{\ln \|P_j^{-1} P_i\| + \ln \|P_i^{-1} P_j\|}{\lambda_i^* + \lambda_j^*}$

Example 3.2. *We used the same numerical example provided in the article to simulate a trajectory of the following system. Let the graph $\mathcal{G}$ be as in the Figure 1. Note that, we will renumber the nodes starting from 0 to 5 instead of 1 to 6. The subsystems are :*

$$A_0 = \begin{bmatrix} -1.5 & 0 \\ 0 & -1.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -11 & 3 \\ -18 & 4 \end{bmatrix} \tag{3.9}$$

$$A_3 = \begin{bmatrix} 3 & -45 \\ 1 & -11 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 3 & -46 \\ 1 & -11 \end{bmatrix}, \quad A_5 = \begin{bmatrix} -2.1 & 0 \\ 0 & -2.2 \end{bmatrix} \tag{3.10}$$

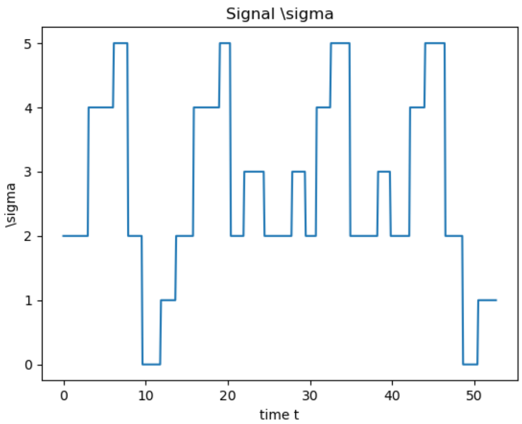

Figure 2: Signal σ

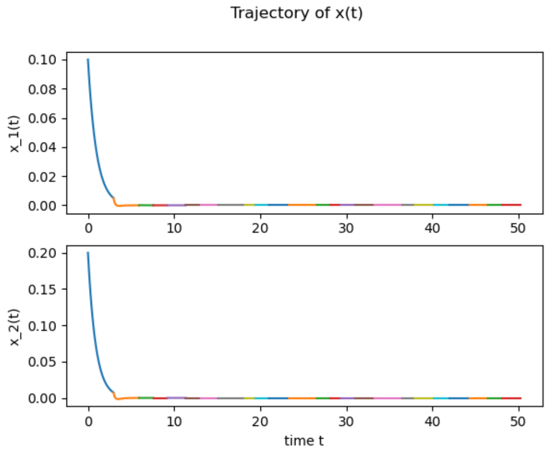

Figure 3: Stable trajectory of x(t)

Here, we used $\rho^* = 1.43454$[2] *to compute a signal* σ *(Figure 2). We then obtained the trajectories described by the Figure 3.*

And if we don't respect the lower bound ρ^**, we obtain the Figures 4 and 5.*

N. Agarwal then, presented in the article the new concept of simple loop dwell time and a lower bound for it.

Let $-\lambda^*_{s_c} = \max_{(i,j)\in\mathcal{E}(s_c)} -\lambda^*_i < 0.$ [3]

$$\nu_{s_c} = \frac{\sum_{(i,j)\in\mathcal{E}(s_c)} \ln \|P_j^{-1}P_i\|}{\lambda^*_{s_c}} \tag{3.11}$$

To simplify the notations, $\nu_c := \nu_{s_c}$.

Note that these lower bounds are expressed with the terms $\|P_j^{-1}P_i\|$. These terms can be thought of as some distance between two subsystems. Indeed, as analyzed in other articles **[3]** **[4]**, the more similar the trajectories produced from the subsystems are, the more rapidly it is possible to switch between them without having instability. And let us remind that it is the eigenvectors that mainly determine the shape of the trajectories in (stable) linear systems.

[2] Note that this is the value computed in the article in SIAM. When trying to recompute ρ^*, we obtained a different value $\rho^* = 1.36724$ which is the same value obtained in another version of the paper (2017). But after discussion with the author, N. Agarwal confirms that it is $\rho^* = 1.43454$ the correct value.

[3] There was a typo in the article published by SIAM, the inequality was inversed.

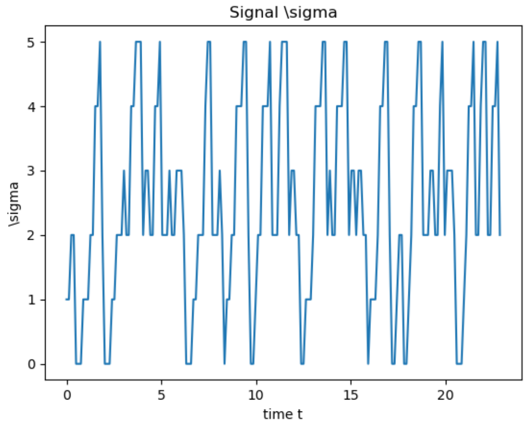

Figure 4: Signal σ

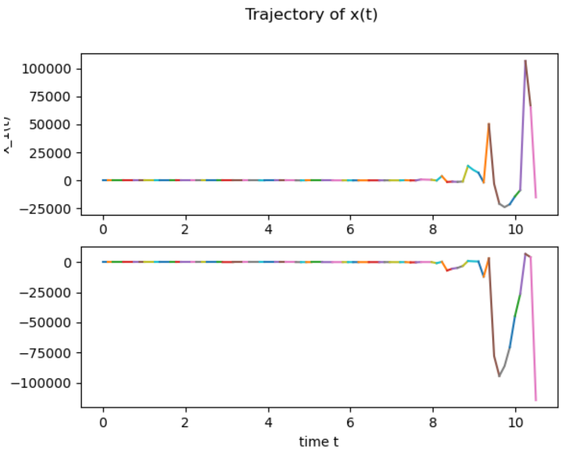

Figure 5: Unstable trajectory of x(t)

With this new quantity, N. Agarwal demonstrated this theorem.

Theorem 3.3. *The switched system (2.1) with switching signal* $\sigma \in \mathcal{S}_{\mathcal{G}}(\tau_1, \dots \tau_p)$ *is stable if for each* $c \in \mathbf{p}$*,* $\tau_c > \nu_c$.

Proof. The standard decomposition of $\sigma^{(n)} = \sigma_1 \dots \sigma_n$ gives a disjoint union of simple loops s_c, n_c^σ times each, $c \in \mathbf{p}$ and an indecomposable path p_n^σ of length $l(p_n^\sigma) \leq k-1$ (with $k = |\mathcal{G}|$).

With this decomposition, the terms in $\ln(a_n^\sigma)$ are separated into simple loops and indecomposable path components.

$$\ln(a_n^\sigma) = \sum_{j=1}^{n-1} (\ln(\|P_{\sigma_{j+1}}^{-1} P_{\sigma_j}\|) - \lambda_{\sigma_j}^*(t_j - t_{j-1})) \tag{3.12}$$

$$= b_n^\sigma + \sum_{c=1}^{p} n_c^\sigma (\sum_{(i,j)\in s_c} \ln(\|P_j^{-1} P_i\|) - \lambda_{s_c}^* \tau_c) \tag{3.13}$$

With b_n^σ corresponding to the indecomposable path component.

If $\tau_c > \frac{\sum_{(i,j)\in\mathcal{E}(s_c)} \ln \|P_j^{-1} P_i\|}{\lambda_{s_c}^*} = \nu_{s_c}$ for each $c \in \mathbf{p}$, then $(\sum_{(i,j)\in s_c} \ln(\|P_j^{-1} P_i\|) - \lambda_{s_c}^* \tau_c) < 0$.

And as σ describes an infinite path, when $n \to \infty$, the number of simple loops $n_c^\sigma \to \infty$ for some loops $c \in \mathbf{p}$ (as the number of nodes is finite and there's no zeno behaviour).

The term b_n^σ remains finite.

This gives $\lim_{n\to\infty} \ln(a_n^\sigma) = -\infty \Leftrightarrow \lim_{n\to\infty} a_n^\sigma = 0$.

Thus the inequality (3.5) becomes :

$$\lim_{n\to\infty} \|x(t)\| = 0 \tag{3.14}$$

Asymptotical stability is obtained.

Moreover it is also true that $a_n^\sigma < e^{b_n^\sigma}$. And so we could say that $a_n^\sigma < e^{-\frac{\ln(b_n^\sigma)t}{t_{n-1}}}$. So we have now :

$$\|x(t)\| \leq \alpha e^{-\beta t} \|x(0)\| \tag{3.15}$$

With $\alpha = \rho > 0$, $\beta = \frac{\ln(b_n^\sigma)t}{t_n} > 0$. So we obtain the uniform exponential stability [4]. □

With this result, it is now possible to switch slowly on some edges of the simple loops and faster on others as only the total time spent in the simple loop is considered. The same reasoning can be used for loops that are composed of several simple loops.

4 Switched system with stable and unstable subsystems

Let us now consider both stable and unstable subsystems.

The hypothesis **H1** is still assumed. The author adds further assumptions.

Hypothesis H2 $A_1, \dots A_r$ are stable diagonalizable matrices and $A_{r+1}, \dots A_k$ are unstable diagonalizable matrices. For the notations, $\lambda_{A_i,d}$ are the eigenvalues of A_i and :

$$-\lambda_i^* := \max_{d=1,\dots D} \mathbb{R}\mathrm{e}\{\lambda_{A_i,d}\} \qquad \text{for } i = 1, \dots r$$

$$\mu_i^* := \max_{d=1,\dots D} \mathbb{R}\mathrm{e}\{\lambda_{A_i,d}\} \qquad \text{for } i = r+1, \dots k$$

[4] In the article, the uniform exponential stability was not demontrated, but only the asymptotical stability

Hypothesis H3 For every $T > 0$, $\exists t, s > T$ s.t. $\sigma(t) \in \{1, \ldots, r\}$ and $\sigma(s) \in \{r+1, \ldots, k\}$.

Indeed, if $\exists T > 0$ s.t. $\sigma(t) \in \{1, \ldots, r\} \forall t \geq T$, then we can just use the results from the previous section with only stable subsystems.
Or if there is one vertex v_i with $i \in \{r+1, \ldots k\}$ that doesn't have an outgoing edge to any vertices in $\{1, \ldots, r\}$. Then it is impossible to have asymptotic stability for any signal $\sigma \in \mathcal{S}_\mathcal{G}$ that takes the value $\sigma = i$ at some point. For that, it is assumed that for each $i \in \{1, \ldots r\} \exists j \in \{r+1, \ldots k\}$ s.t. $\exists$ a path from v_i to v_j. And also for each $i \in \{r+1, \ldots k\} \exists j \in \{1, \ldots r\}$ s.t. $\exists$ a path from v_i to v_j.

4.1 Classes of switching signals

Definition 4.1 (Flee time). *The flee time η is the maximum time that the signal spends in an unstable subsystem.*

There are also 2 classes of switching signals in this section.

$$\mathcal{S}_\mathcal{G}(\tau, \eta) = \{\sigma \in \mathcal{S}_\mathcal{G} | t_{i+1} - t_i \geq \tau \text{ if } \sigma(t_i) \in \{1, \ldots r, \} \tag{4.1}$$
$$t_{i+1} - t_i \leq \eta \text{ if } \sigma(t_i) \in \{r+1, \ldots k\}\} \tag{4.2}$$

With the standard decomposition that gives us the simple loops and an indecomposable path, the following class can be defined.

$$\mathcal{S}_\mathcal{G}(\tau_1, \eta_1, \ldots \tau_p, \eta_p) = \{\sigma \in \mathcal{S}_\mathcal{G} | t_{i+1} - t_i \geq \tau_c \text{ if } \sigma(t_i) \in \{1, \ldots r\} \text{ and } v_{\sigma(t_i)} \in s_c, c \in \mathbf{p}, \tag{4.3}$$
$$t_{i+1} - t_i \leq \eta_c \text{ if } \sigma(t_i) \in \{r+1, \ldots k\} \text{ and } v_{\sigma(t_i)} \in s_c, c \in \mathbf{p}\} \tag{4.4}$$

4.2 Unidirectional ring

We study now the case of the unidirectional ring with k vertices.

First, N. Agarwal presented a proposition concerning the dwell time and flee time.

Proposition 4.2. *The switched system (2.1) with signal $\sigma \in \mathcal{S}_\mathcal{G}(\tau, \eta)$ is stable if $\eta, \tau > 0$ satisfy*

$$\ln(\rho) + \beta < 0 \tag{4.5}$$

Where $\rho = \prod_{(i,j) \in \mathcal{E}(\mathcal{G})} \|P_j^{-1} P_i\|$, $\beta = -(\sum_{i=1}^r \lambda_i^*)\tau + (\sum_{j=r+1}^k \mu_j^*)\eta$.

Proof. Without loss of generality, let us reorder the vertices according to the order in which they appear in the signal σ for the first k switching times. Let $n = vk + w, 0 \leq w < k,$ with $v, w \in \mathbb{N}^+$.

$$\|x(t)\| = \|e^{A_{\sigma_n}(t - t_{n-1})} \ldots e^{A_k(t_k - t_{k-1})} \ldots e^{A_1 t_1} x(0)\| \tag{4.6}$$
$$\leq C(\prod_{i=1}^{w} \|e^{A_{\sigma_{n-i}}(t_{n-i} - t_{n-i-1})}\|)(\prod_{i=1}^{k-1} \|P_{i+1}^{-1} P_i\|^v)\|P_k^{-1} P_1\|^v e^{v(\sum_{i=1}^r -\lambda_i^*)\tau + v(\sum_{i=r+1}^k \mu_i^*)\eta} \tag{4.7}$$
$$\leq C(\prod_{i=1}^{w} \|e^{A_{\sigma_{n-i}}(t_{n-i} - t_{n-i-1})}\|)(\prod_{(i,j) \in \mathcal{E}(\mathcal{G})} \|P_j^{-1} P_i\|^v) e^{v(\sum_{i=1}^r -\lambda_i^*)\tau + v(\sum_{i=r+1}^k \mu_i^*)\eta} \tag{4.8}$$

As $(\prod_{i=1}^w \|e^{A_{\sigma_{n-i}}(t_{n-i} - t_{n-i-1})}\|)$ is finite, it is clear that if we have $\ln(\rho) + \beta < 0$ then

$$\lim_{n \to \infty} (\prod_{(i,j) \in \mathcal{E}(\mathcal{G})} \|P_j^{-1} P_i\|^v) e^{v(\sum_{i=1}^r -\lambda_i^*)\tau + v(\sum_{i=r+1}^k \mu_i^*)\eta} = 0$$

$\square$

Example 4.3. *We will use here an example provided in the article to simulate a trajectory of $x(t)$.*

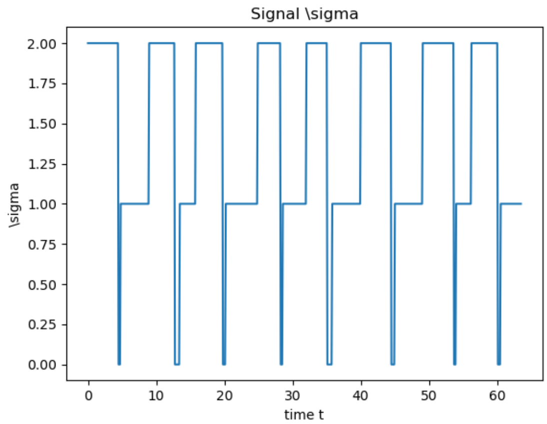

Figure 6: Signal σ in a unidirectional ring

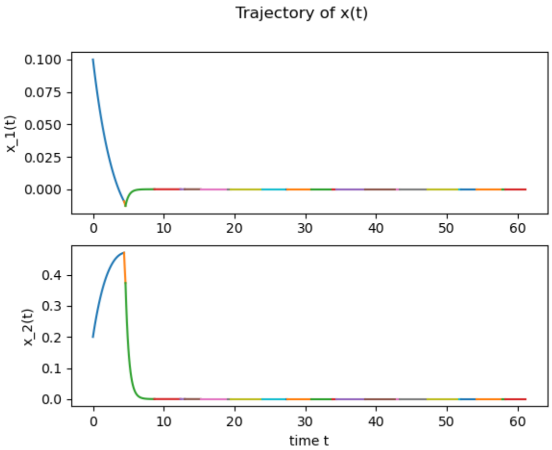

Figure 7: Trajectory of x(t)

So, we consider $\mathcal{G}$ to be a unidirectional ring with $v_0 \to v_1 \to v_2 \to v_0$. With the submatrices

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -0.4 & -0.03 \\ 1.4 & 0.04 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.9 & 0 \\ 2 & -0.6 \end{bmatrix} \tag{4.9}$$

So A_0, A_1 are stable and A_2 unstable. And the conditions given by the proposition is

$$3.38306 - 2.1\tau + 0.9\eta < 0 \tag{4.10}$$

[5]

So, we obtain the signal described by Figure 6 which gives a trajectory as in Figure 7.

Then, another proposition is presented in terms of the simple loop dwell and flee times. But as we are studying the unidirectional ring here, there is only one simple loop s_1.

Proposition 4.4. *The switched system (2.1) with signal $\sigma \in \mathcal{S}_{\mathcal{G}}(\tau_1, \eta_1)$ is stable if $\eta_1, \tau_1 > 0$ satisfy*

$$\ln(\rho) - \lambda^* \tau_1 + \mu^* \eta_1 < 0 \tag{4.11}$$

Where $\rho = \prod_{(i,j)\in\mathcal{E}(\mathcal{G})} \|P_j^{-1} P_i\|$, $-\lambda^* = \max\{-\lambda_1^*, \cdots - \lambda_r^*\}$ and $\mu^* = \max\{\mu_{r+1}^*, \dots \mu_k^*\}$.

[5] This time, when computing the values in the conditions given by the propositions on unidirectional rings, we obtained the same results as in the article. However, there was a typo in the expression of the submatrices as in the article, the $A_1 = \begin{bmatrix} -0.4 & -0.03 \\ 1.43 & 0.4 \end{bmatrix}$ and it is not stable as it was claimed.

4.3 Bipartite graph

Proposition 4.5. *Let $\mathcal{G}$ be a bipartite graph with disjoint classes $\{v_1, \dots v_r\}$ and $\{v_{r+1}, \dots v_k\}$. The switched system (2.1) with signal $\sigma \in \mathcal{S}_\mathcal{G}(\tau, \eta)$ is stable if $\eta, \tau > 0$ satisfy*

$$\ln(\rho_1) + \ln(\rho_2) + \beta < 0 \tag{4.12}$$

Where $\rho_1 = \max\limits_{(i,j)\in\mathcal{E}(\mathcal{G}), i=1,\dots r} \|P_j^{-1}P_i\|$, $\rho_2 = \max\limits_{(i,j)\in\mathcal{E}(\mathcal{G}), i=r+1,\dots k} \|P_j^{-1}P_i\|$ *and* $\beta = -\lambda^*\tau + \mu^*\eta$.

4.4 Arbitrary graph

Now, let us consider an arbitrary graph $\mathcal{G}$ that is a superimposition of the subgraphs $\mathcal{G}_s$ and $\mathcal{G}_u$ with :

$$\mathcal{E}_{\mathcal{G}_s} = \{(i,j) \in \mathcal{E}(\mathcal{G}) | i = 1, \dots r\} \tag{4.13}$$

$$\mathcal{E}_{\mathcal{G}_u} = \{(i,j) \in \mathcal{E}(\mathcal{G}) | i = r+1, \dots k\} \tag{4.14}$$

So the subgraph $\mathcal{G}_s$ corresponds to the stable vertices with all the vertices that can be reached directly from those stable vertices (same idea for the subgraph $\mathcal{G}_u$).

Proposition 4.6. *The switched system (2.1) with signal $\sigma \in \mathcal{S}_\mathcal{G}(\tau, \eta)$ is stable if $\forall c \in \mathbf{p}, \eta, \tau > 0$ satisfy*

$$\ln(\rho_c) - \lambda^*_{s_c}\tau + \mu^*_{s_c}\eta < 0 \tag{4.15}$$

Where $\rho_c = \prod_{(i,j)\in\mathcal{E}(s_c)} \|P_j^{-1}P_i\|$, $-\lambda^*_{s_c} = \sum_{(i,j)\in\mathcal{E}(s_c)\cap\mathcal{E}(\mathcal{G}_s)} -\lambda^*_i$ *and* $\mu^*_{s_c} = \sum_{(i,j)\in\mathcal{E}(s_c)\cap\mathcal{E}(\mathcal{G}_u)} \mu^*_i$.

Proposition 4.7. *The switched system (2.1) with signal $\sigma \in \mathcal{S}_\mathcal{G}(\tau_1, \eta_1, \dots \tau_p, \eta_p)$ is stable if $\forall c \in \mathbf{p}, \eta_c, \tau_c > 0$ satisfy*

$$\ln(\rho_c) - \lambda^*_{s_c}\tau_c + \mu^*_{s_c}\eta_c < 0 \tag{4.16}$$

Where $\rho_c = \prod_{(i,j)\in\mathcal{E}(s_c)} \|P_j^{-1}P_i\|$, $-\lambda^*_{s_c} = \max_{(i,j)\in\mathcal{E}(s_c)\cap\mathcal{E}(\mathcal{G}_s)} -\lambda^*_i$ *and* $\mu^*_{s_c} = \max_{(i,j)\in\mathcal{E}(s_c)\cap\mathcal{E}(\mathcal{G}_u)} \mu^*_i$.

Then, the article provides us 2 approaches to study the stability of switched systems (2.1) with a signal $\sigma \in \mathcal{S}_\mathcal{G}(\tau, \eta)$.

4.4.1 Approach 1 : Number of switches

New quantities are introduced here :

$$N_s(0,t) = \#\{i | 1 \le i \le n, \sigma_i \in \{1, \dots r\}\} \tag{4.17}$$

$$N_u(0,t) = \#\{i | 1 \le i \le n, \sigma_i \in \{r+1, \dots k\}\} \tag{4.18}$$

They correspond to the number of times the signal is in a stable (or unstable) vertex.

The norm of $x(t)$ is redevelopped into :

$$\|x(t)\| \le \|P_{\sigma_n}\|\|P_{\sigma_1}^{-1}\|e^{-\lambda^*_{\sigma_n}(t-t_{n-1})}(\prod_{j=1}^{n-1} \|P_{\sigma_{j+1}}^{-1}P_{\sigma_j}\|e^{-\lambda^*_{\sigma_j}(t_j - t_{j-1})})\|x(0)\| \tag{4.19}$$

$$\le C(e^{-\lambda^*_{\sigma_n}(t-t_{n-1})}\prod_{j=1}^{n-1} \|P_{\sigma_{j+1}}^{-1}P_{\sigma_j}\|e^{-\lambda^*_{\sigma_j}(t_j - t_{j-1})})\|x(0)\| \tag{4.20}$$

$$\le C\rho_1^{N_s(0,t)}\rho_2^{N_u(0,t)}e^{-\lambda^*\tau N_s(0,t) + \mu^*\eta N_u(0,t)}\|x(0)\| \tag{4.21}$$

With $C > 0$ a constant, $\rho_1 = \max\limits_{(i,j)\in\mathcal{E}(\mathcal{G}_s)} \|P_j^{-1}P_i\|$ and $\rho_2 = \max\limits_{(i,j)\in\mathcal{E}(\mathcal{G}_u)} \|P_j^{-1}P_i\|$.

This leads to a new proposition.

Proposition 4.8. *The switched system (2.1) with signal $\sigma \in \mathcal{S}_\mathcal{G}(\tau, \eta)$ is stable if*

$$\lim_{t\to\infty} \sup(N_s(0,t)(\ln\rho_1 - \lambda\tau) + N_u(0,t)(\ln\rho_2 + \mu\eta)) < 0 \tag{4.22}$$

It is easy to observe the similarities between this proposition and the proposition 4.5 concerning bipartite graphs.

With this, the choice of the switching signal is more flexible as we can vary the number of times we switch to a stable system in function of a dwell and flee time that we choose.

4.4.2 Approach 2 : Assume an acyclic $\mathcal{G}_u$

In this second approach, it is assumed that $\mathcal{G}_u$ is acyclic and the matrices $P_1 \dots P_k$ don't have unit norm columns anymore. They can be any eigenvector matrices. For $\sigma \in \mathcal{S}_\mathcal{G}(\tau, \eta), t \in [t_n, t_{n+1})$,

The norm of $x(t)$ is redevelopped into :

$$\|x(t)\| \leq \|P_{\sigma_n}\| \|P_{\sigma_1}^{-1}\| e^{\Lambda_{\sigma_n}(t-t_{n-1})} (\prod_{j=1}^{n-1} \|P_{\sigma_{j+1}}^{-1} P_{\sigma_j}\| e^{\Lambda_{\sigma_j}(t_j - t_{j-1})}) \|x(0)\| \tag{4.23}$$

$$\leq C_1 C_2 a_n^\sigma \|x(0)\| \tag{4.24}$$

With $C_1 = \max_{i,j \in \mathbf{k}} \{ \| |P_j| \| \|P_i^{-1}\| | \exists \text{ a path from} v_j \text{ to } v_i \}$, $C_2 = \max\{ e^{-\tau \lambda, e^{\eta\mu}} \}$ and $a_n^\sigma = \prod_{j=1}^n \|P_{\sigma_{j+1}}^{-1} P_{\sigma_j}\| \|e^{D_{\sigma_j}(t_j - t_{j-1})}\|$.

With this, the terms of a_n^σ are decomposed into terms corresponding to the edges of $\mathcal{G}_s$ and terms corresponding to the edges of $\mathcal{G}_u$.

$$\ln a_n^\sigma \leq N_n \sum_{(i,j) \in \mathcal{G}_s} (\ln \|P_j^{-1} P_i\| - \lambda_i^* \tau) + (n - N_n) \sum_{(i,j) \in \mathcal{G}_u} (\ln \|P_j^{-1} P_i\| + \mu^* \eta) \tag{4.25}$$

Where $N_n = \#\{(\sigma_j, \sigma_{j+1}) \in \mathcal{E}(\mathcal{G}_s) | j \in \mathbf{n}\}$, the number of times the signal is in the subgraph $\mathcal{G}_s$.

Lemma 4.9. *If $\mathcal{G}_u$ is acyclic, $\exists$ eigenvector matrices $Q_1, \dots Q_k$ of $A_1, \dots A_k$ s.t.*

$$\max_{(i,j) \in \mathcal{E}(\mathcal{G}_u)} \|Q_j^{-1} Q_i\| < 1 \tag{4.26}$$

Proof. Suppose $\rho = \max_{(i,j) \in \mathcal{E}(\mathcal{G}_u)} \|P_j^{-1} P_i\| > 1$, the objective is to find a scaling of $P_1, \dots P_k$ to obtain the matrices $Q_1, \dots Q_k$ that respect the inequality $\max_{(i,j) \in \mathcal{E}(\mathcal{G}_u)} \|Q_j^{-1} Q_i\| < 1$.

Choose a $0 < \zeta < 1$ and pose $\kappa = \rho / \zeta > 1$.

Reminder : a topological sorting of a digraph is an ordering of the vertices s.t. if there is an edge from a vertex v_i to v_j, then v_i comes first in the ordering.

Let the topological sorting of $\mathcal{G}_u$ be $v_{a_1}, \dots v_{a_m}$ such that :

- $v_{a_1}, \dots v_{a_{m_1}} \in \{r+1, \dots k\}$ and have indegree $= 0$.
- $v_{a_{m_1+1}}, \dots v_{a_{m_2}} \in \{r+1, \dots k\}$
- $v_{a_{m_2}}, \dots v_{a_m} \in \{1, \dots r\}$

Then, let

- $Q_{a_j} = P_{a_j}, j = 1, \dots m_1$
- $Q_{a_j} = \kappa^{j-m_1} P_{a_j}, j = m_1 + 1, \dots m_2$
- $Q_{a_j} = \kappa^{m_2 - m_1 + 1} P_{a_j}, j = m_2 + 1, \dots m$
- $Q_{a_j} = P_{a_j}, j \in \mathbf{k} \setminus \{a_1, \dots a_m\}$

It is clear that $\|Q_j^{-1} Q_i\| = \tau^{-g} \|P_j^{-1} P_i\|$, for some $g \geq 1$. So $\|Q_j^{-1} Q_i\| \leq \kappa^{-1} \|P_j^{-1} P_i\| \leq \zeta \leq 1$. ☐

This leads to the next proposition.

Proposition 4.10. *The switched system (2.1) with an acyclic subgraph $\mathcal{G}_u$ and signal $\sigma \in \mathcal{S}_\mathcal{G}(\tau, \eta)$ is stable if*[6]

$$\tau > \max_{(i,j) \in \mathcal{E}(\mathcal{G}_s)} \frac{\ln \|Q_j^{-1} Q_i\|}{\lambda_i^*} \quad \textit{and} \quad \eta < \max_{(i,j) \in \mathcal{E}(\mathcal{G}_u)} \frac{\ln \|Q_j^{-1} Q_i\|}{\mu_i^*} \tag{4.27}$$

[6]There were some typo in this expression in the article published by SIAM (inversion of $\mathcal{G}_u$ and $\mathcal{G}_s$).

Proof. In equation (4.25), as $t \to \infty$, $n \to \infty$, then either $N_n \to \infty$ or $n - N_n \to \infty$ or both. So if $\ln \|P_j^{-1} P_i\| - \lambda_i^* \tau < 0, (i,j) \in \mathcal{G}_s$ and $\ln \|P_j^{-1} P_i\| - \mu^* \eta < 0, (i,j) \in \mathcal{G}_u$ then $\lim_{n\to\infty} a_n^\sigma = 0$. And the asymptotic stability is obtained.

For that we need $\ln \|P_j^{-1} P_i\| > 0, (i,j) \in \mathcal{G}_s$ which is guaranteed by the hypothesis **H1**. And we also need $\ln \|P_j^{-1} P_i\| < 0, (i,j) \in \mathcal{G}_u$ which is guaranteed by the lemma 4.9 if we use a scaling of the eigenvector matrices. □

5 Conclusion and comments on the article

After analyzing this article, we learned about the concepts of dwell time, flee time and their simple loops alternatives. N. Agarwal succeeded in providing new bounds on these quantities that are tighter than the previous results in previous articles. There were however some difficulties that we came accross. Indeed, we found some typos in some parts and even though most of them do not change significantly the final results, it was difficult for us to verify the numerical results for example as some of them were not correct (we verified it by exchanging mails with the author). Another remark would be about the hypothesis used. It was not always clear if only some of them were assumed or all of them. Indeed, for us, once introduced, they were necessary for the rest of the article in view of the coherence of the equations. But at some parts, the author only specified to used some of them but not all of them... Finally, we want to comment on the stability focused on here. Indeed, in the SIAM version of the paper, N. Agarwal set the objective to be uniform exponential stability because the susbsystems don't have any eigenvalues with real part equal to 0. That makes us wonder what results can be obtained when we consider subsystems with eigenvalues with real part equal to 0.

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