Operational Research

LINMA2491

Project 2
L-Shaped method for a Farmer's problem

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1 Problem description

The master problem (M1) is given by :

$$
\begin{aligned}
(\mathrm{M1}) : \min_{x,\theta} \quad & 150x_1 + 230x_2 + 260x_3 + \theta \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \leq 500, \\
& \theta \geq \sum_{\omega=1}^{3} (\hat{\pi}_\omega)^T (h - T_\omega x) \quad \text{for each vertex } \hat{\pi} = [\hat{\pi}_\omega]_{\omega=1,2,3} \text{ of } P \quad \text{(Optimality cuts)}, \\
& 0 \geq \sum_{\omega=1}^{3} (\sigma_\omega)^T (h - T_\omega x) \quad \text{for each extreme ray } \sigma = [\sigma_\omega]_{\omega=1,2,3} \text{ of } P \quad \text{(Feasibility cuts)}, \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Or equivalently, in matrix form, we have the following problem (M2)

$$
(\mathrm{M2}) : \min_{x,\theta} \quad \underbrace{\begin{bmatrix} 150 & 230 & 260 \end{bmatrix}}_{c^T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} + \theta \tag{1a}
$$

$$
\text{s.t.} \quad \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq 500, \tag{1b}
$$

$$
\theta \geq \sum_{\omega=1}^{3} (\hat{\pi}_\omega)^T (h - T_\omega x) \quad \text{for each vertex } \hat{\pi} = [\hat{\pi}_\omega]_{\omega=1,2,3} \text{ of } P \quad \text{(Optimality cuts)} \quad , \tag{1c}
$$

$$
0 \geq \sum_{\omega=1}^{3} (\sigma_\omega)^T (h - T_\omega x) \quad \text{for each extreme ray } \sigma = [\sigma_\omega]_{\omega=1,2,3} \text{ of } P \quad \text{(Feasibility cuts)}, \tag{1d}
$$

$$
x_1, x_2, x_3 \geq 0 \tag{1e}
$$

With $P := \{\pi | \pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \omega = 1, 2, 3\}$. The optimality cuts, and feasibility cuts are given by the sub-problems described in the next subsection. The vector h and matrices T_ω will also be described in the next subsection.

1.a Cut generating sub-problem

The sub-problem we are interested in can actually be solved more easily by dividing it into 3 independent sub-problems with $\omega = 1, 2, 3$ which are given by (S_ω):

$$
(S_\omega) : \min_{u_\omega} \quad \underbrace{\begin{bmatrix} 238 & 210 & -170 & -150 & -36 & -10 \end{bmatrix}}_{q^T} \underbrace{\begin{bmatrix} y_{1,\omega} \\ y_{2,\omega} \\ w_{1,\omega} \\ w_{2,\omega} \\ w_{3,\omega} \\ w_{4,\omega} \end{bmatrix}}_{u_\omega}
$$

$$
\text{s.t.} \quad \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}}_{W} \begin{bmatrix} y_{1,\omega} \\ y_{2,\omega} \\ w_{1,\omega} \\ w_{2,\omega} \\ w_{3,\omega} \\ w_{4,\omega} \end{bmatrix} \geq \underbrace{\begin{bmatrix} 200 \\ 240 \\ 0 \\ -6000 \end{bmatrix}}_{h} - T_\omega \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},
$$

$$
u_\omega \geq 0
$$

where

$$T_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3.6 & 0 \\ 0 & 0 & 24 \\ 0 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 20 \\ 0 & 0 & 0 \end{bmatrix} T_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2.4 & 0 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking into account the independence between these 3 sub-problems, makes the calculations easier naturally.

But the problem that we actually want to solve is the general sub-problem taking the 3 scenarios altogether, (S):

$$(S) : \min_{u_G} \quad \frac{1}{3} \underbrace{\begin{bmatrix} q^T & q^T & q^T \end{bmatrix}}_{q_G^T} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{u_G}$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W \end{bmatrix}}_{W_G} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \geq \underbrace{\begin{bmatrix} h \\ h \\ h \end{bmatrix}}_{h_G} - \underbrace{\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}}_{T_G} x,$$

$$u_G \geq 0$$

The duals of the independent small subproblems (S_ω) per scenario $\omega = 1, 2, 3$ are (D_ω):

$$(D_\omega) : \max_{\pi_\omega} \quad \underbrace{\begin{bmatrix} \pi_{1,\omega} & \pi_{2,\omega} & \pi_{3,\omega} & \pi_{4,\omega} \end{bmatrix}}_{\pi_\omega^T} \left(\underbrace{\begin{bmatrix} 200 \\ 240 \\ 0 \\ -6000 \end{bmatrix}}_{h} - T_\omega \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} \right) \tag{2a}$$

$$\text{s.t.} \quad \begin{bmatrix} \pi_{1,\omega} & \pi_{2,\omega} & \pi_{3,\omega} & \pi_{4,\omega} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}}_{W} \leq \underbrace{\begin{bmatrix} 238 & 210 & -170 & -150 & -36 & -10 \end{bmatrix}}_{q^T}, \tag{2b}$$

$$\pi_\omega^T \geq 0 \tag{2c}$$

The dual of (S) is given by (D):

$$(D) : \max_{\pi_G} \quad \underbrace{\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}}_{\pi_G^T} \left(\underbrace{\begin{bmatrix} h \\ h \\ h \end{bmatrix}}_{h_G} - \underbrace{\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}}_{T_G} x \right)$$

$$\text{s.t.} \quad \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \underbrace{\begin{bmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W \end{bmatrix}}_{W_G} \leq \frac{1}{3} \underbrace{\begin{bmatrix} q^T & q^T & q^T \end{bmatrix}}_{q_G^T},$$

$$\pi_G^T \geq 0$$

We know from the course that the vertices of $P = \{\pi | \pi_\omega^T W_\omega \geq p_\omega q_\omega^T, \omega = 1, 2, 3\}$, feasible region of (D), are points of the form $(\frac{1}{3}\hat{\pi}_1, \frac{1}{3}\hat{\pi}_2, \frac{1}{3}\hat{\pi}_3)$, with $\hat{\pi}_\omega$ being a vertex of the feasible region of (D_ω).

And the extreme rays of P can be written as $(0, \ldots, \sigma_\omega, \ldots, 0)$ with σ_ω being an extreme ray of the feasible region of (D_ω).

With these tools, let us remind our objective. We want to solve our Farmer's problem. For that we have a first problem to solve : the master problem (M) which gives a candidate x.

After that, we want to solve the sub-problem (S) which can give us an optimality or feasibility cut or tell us that the solution is already optimal. For that, we will use the dual of (S) : (D). And to facilitate even more the calculations, we will divide (S) into small independent problems each one corresponding to a different scenario : (S_ω), and solve their duals (D_ω). These small duals give solutions, that when combined together give a solution to (D). And so, we can use it to know if our solution is already optimal or not for the Farmer's problem. If not, thanks to the extreme rays or vertices of feasible regions of (D_ω), a feasibility cut or optimality cut is given to the master (M). And another iteration begins until optimality is reached.

2 Feasibility cuts

It is not possible for this Farmer's problem to generate feasibility cuts.

Indeed, in the L-shaped algorithm, a feasibility cut is generated when the primal sub-problem (S) is unfeasible with the candidate x provided by the master problem (M2). This can happen when its dual (D) is either unbounded or unfeasible.

To prove that there will be no feasibility cuts, let us study the small dual problems (D_ω) and show that the problem (D_ω) is always feasible and bounded. As we have explained earlier the results from these problems can be used to infer a conclusion on (D).

First, we observe that the feasible region of (D_ω) is independent of x and randomness. This means that regardless of the scenario ω, if we have a feasible region, the region will remain feasible for every iterations and scenarios.

Now, let us take a closer look at the equation 2b, which describes the feasible region of (D_ω). If we expand the equation, we obtain:

$$\pi_{1,\omega} \leq 238 \tag{3}$$
$$\pi_{2,\omega} \leq 210 \tag{4}$$
$$-\pi_{1,\omega} \leq -170 \qquad \Leftrightarrow \pi_{1,\omega} \geq 170 \tag{5}$$
$$-\pi_{2,\omega} \leq -150 \qquad \Leftrightarrow \pi_{2,\omega} \geq 150 \tag{6}$$
$$-\pi_{3,\omega} - \pi_{4,\omega} \leq -36 \qquad \Leftrightarrow \pi_{3,\omega} + \pi_{4,\omega} \geq 36 \tag{7}$$
$$-\pi_{3,\omega} \leq -10 \qquad \Leftrightarrow \pi_{3,\omega} \geq 10 \tag{8}$$
$$\pi_\omega \geq 0 \tag{9}$$

From this, we observe:

- (D_ω) can't be unfeasible as the feasible region is never empty.
- (D_ω) can't be unbounded. Both $\pi_{1,\omega} \in [170, 280]$ and $\pi_{2,\omega} \in [150, 210]$ are bounded from above. Only $\pi_{3,\omega}$ and $\pi_{4,\omega}$ are unbounded from above. However, if we analyze the cost function of (D_ω) given by equation 2:

  $$\pi_{1,\omega}(200 - T_{[1,1],\omega}x_1) + \pi_{2,\omega}(240 - T_{[2,2],\omega}x_2) + \pi_{3,\omega}(0 - T_{[3,3],\omega}x_3) + \pi_{4,\omega}(-6000 - 0) \tag{10}$$

  Both coefficients of $\pi_{3,\omega}$ and $\pi_{4,\omega}$ are negative (as $T_\omega > 0$ and $x \geq 0$). So if we increase $\pi_{3,\omega}$ and $\pi_{4,\omega}$ to $+\infty$, the cost function will decrease to $-\infty$ and naturally, we can't achieve the maximum that way.

This result is valid for all 3 scenarios and the same can be said for (D).

In conclusion, (D) will never be unbounded, or unfeasible. This means that (S) will never be unfeasible regardless of the candidate x given by the master problem (M). And so, no feasibility cuts will be generated.

3 Optimality cuts

3.a Iteration 1

For the first iteration, the master problem (M2) does not contain any optimality (and feasability) cuts given by equations 1c and 1d yet. The solution of the master problem is:

$$x^1 = [x_1, x_2, x_3] = [0.0, 0.0, 0.0] \tag{11}$$
$$\tag{12}$$

Note that θ is actually an unbounded variable as it corresponds to the expected value function (cost of best possible reaction to x before knowing the realization ω). And this expected value function can be negative.

So for this first iteration, we didn't include the variable θ in the cost function. Indeed, if we did, the problem would become unbounded.

The candidate x^1 will be used in the different sub-problem (D_ω) ($\omega = 1, 2, 3$) in order to obtain π^1_ω:

$$\pi^1_1 = [238.0, 210.0, 36.0, 0.0]$$
$$\pi^1_2 = [238.0, 210.0, 36.0, 0.0]$$
$$\pi^1_3 = [238.0, 210.0, 36.0, 0.0]$$

The first optimality cut is:

$$\theta \geq \frac{1}{3}\pi^1_1(h - T_1x) + \frac{1}{3}\pi^1_2(h - T_2x) + \frac{1}{3}\pi^1_3(h - T_3x) \tag{13}$$
$$\theta \geq -595x_1 - 630x_2 - 720x_3 + 98000.0 \tag{14}$$

3.b Iteration 2

For the second iteration, the master problem (M2) contains the optimality cut given by equation 14. The solution of the master problem is:

$$x^2 = [x_1, x_2, x_3] = [0.0, 0.0, 500.0] \tag{15}$$
$$\theta^2 = 0.00 \tag{16}$$

The candidate x^2 will be used in the different sub-problems (D_ω) ($\omega = 1, 2, 3$) in order to obtain π^2_ω:

$$\pi^2_1 = [238.0, 210.0, 10.0, 26.0]$$
$$\pi^2_2 = [238.0, 210.0, 10.0, 26.0]$$
$$\pi^2_3 = [238.0, 210.0, 10.0, 26.0]$$

The second optimality cut is:

$$\theta \geq \frac{1}{3}\pi^2_1(h - T_1x) + \frac{1}{3}\pi^2_2(h - T_2x) + \frac{1}{3}\pi^2_3(h - T_3x) \tag{17}$$
$$\theta \geq -595x_1 - 630x_2 - 200x_3 - 58000 \tag{18}$$

4 Solution to the Farmer's problem

Our implementation of the L-shaped algorithm gives an optimal solution in 10 iterations.

The optimal cost is -108390, which corresponds to a benefit of 108390.

The optimal solution is

$$x^* = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 170 & 80 & 250 \end{bmatrix}^T$$

.

For scenario 1,

$$u_1^* = \begin{bmatrix} y_{1,1} & y_{2,1} & w_{1,1} & w_{2,1} & w_{3,1} & w_{4,1} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 310 & 48 & 6000 & 0 \end{bmatrix}^T$$

For scenario 2,

$$u_2^* = \begin{bmatrix} 0 & 0 & 225 & 0 & 5000 & 0 \end{bmatrix}^T$$

For scenario 3,

$$u_3^* = \begin{bmatrix} 0 & 48 & 140 & 0 & 4000 & 0 \end{bmatrix}^T$$