LINMA2361- Nonlinear dynamical systems

Homework 2

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To solve this homework assignment, I read and was inspired by the section 3.7 on insect outbreak in the Nonlinear Dynamics and Chaos book by Strogatz [1].

1 Question 1

We are interested in the dynamical system described by :

$$\dot{N} = RN(1 - \frac{N}{K}) - \frac{BN^2}{A^2 + N^2} \tag{1}$$

We want to nondimensionalize the system as it will be easier to analyze. Indeed, here we have 4 parameters : A, B, R, K, we hope that after the nondimensionalization, we will have less parameters.

First, let's divide the equation (1) by B, pose $k = \frac{K}{A}$ and $x = \frac{N}{A}$ (note that $\frac{dx}{dt} = \frac{1}{A}\frac{dN}{dt}$).

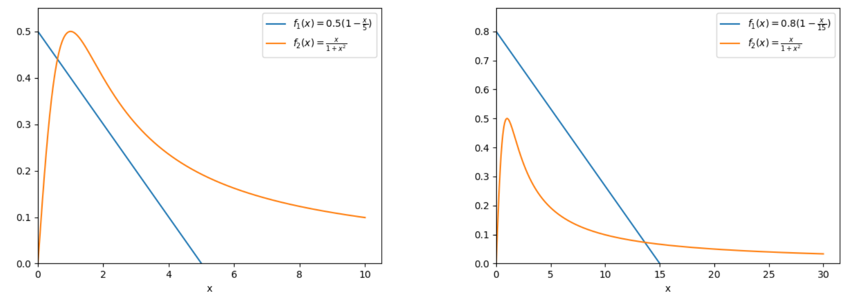
$$\begin{aligned}
\frac{dN}{dt}\frac{1}{B} &= RN\frac{1}{B}(1 - \frac{N}{K}) - \frac{N^2}{A^2 + N^2} \\
\frac{dx}{dt}\frac{A}{B} &= \frac{A}{B}Rx(1 - \frac{Ax}{Ak}) - \frac{A^2x^2}{A^2 + A^2x^2} \\
\frac{dx}{dt}\frac{A}{B} &= \frac{A}{B}Rx(1 - \frac{x}{k}) - \frac{x^2}{1 + x^2}
\end{aligned}$$

Then we pose $\tau = \frac{B}{A}t$. And we want x to be a function of τ. It means that $\dot{x} := \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{dx}{dt}\frac{A}{B}$. We arrive at :

$$\dot{x} = f(x) := rx(1 - \frac{x}{k}) - \frac{x^2}{1 + x^2} \tag{2}$$

With $r = \frac{A}{B}R$.

We only have 2 dimensionless parameters : r and k.



(a) r = 0.5, k = 5 (b) r = 0.8, k = 15

Figure 1: 1 intersection

2 Question 2

Now we want to find a value for r such that there are 4 equilibrium points. Let's first observe that the origin $x = 0$ is an unstable equilibrium point. Indeed, we have $\dot{x}(0) = 0$ and $\ddot{x}(0) = r > 0$.

Then, the 3 remaining equilibrium points must be solutions of :

$$r(1 - \frac{x}{k}) = \frac{x}{1 + x^2} \tag{3}$$

We observe that the right-hand side of the equation is fixed and does not depend on the parameters. The left-hand side is a straight line that moves depending on r and k. The intersections of these 2 functions will give us the equilibrium points.

However, for different values of r and k, we can have 1, 2 or 3 intersections (Figures 1, 2 and 3).

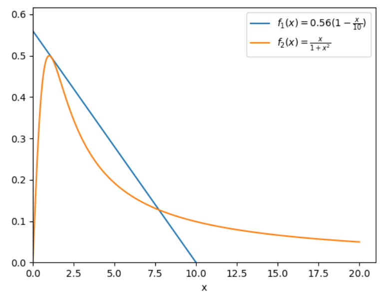
Let's observe the Figure 3 in more details. We will call the 3 intersections A, B and C from left to right. Let's keep the parameter k as it is and move r. When r gets higher, A and B move towards each other until they coincide and we have a saddle-node bifurcation. The same scenario arises when r gets low enough for B and C to get closer. So for a given k, there is a range for r such that there are 3 intersections.

It is clear that if we find when the saddle-node bifurcations occur, we can determine which values of r and k provide 3 intersections.

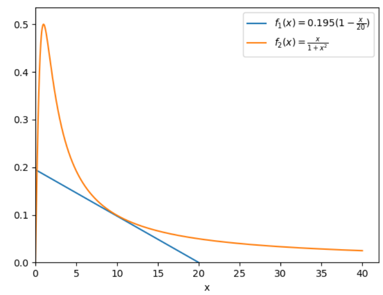
Let us remind that a saddle-node bifurcation occurs when there is a tangent intersection. This requirement is expressed by the following system of equations.

$$\begin{cases} r(1 - \frac{x}{k}) & = \frac{x}{1+x^2} \\ \frac{d}{dx}[r(1 - \frac{x}{k})] & = \frac{d}{dx}[\frac{x}{1+x^2}] \end{cases} \Rightarrow \begin{cases} r(1 - \frac{x}{k}) & = \frac{x}{1+x^2} \\ -\frac{r}{k} & = \frac{1-x^2}{(1+x^2)^4} \end{cases} \tag{4}$$

After substituting the 2nd equation in the 1st for $\frac{r}{k}$, and substituting the re-

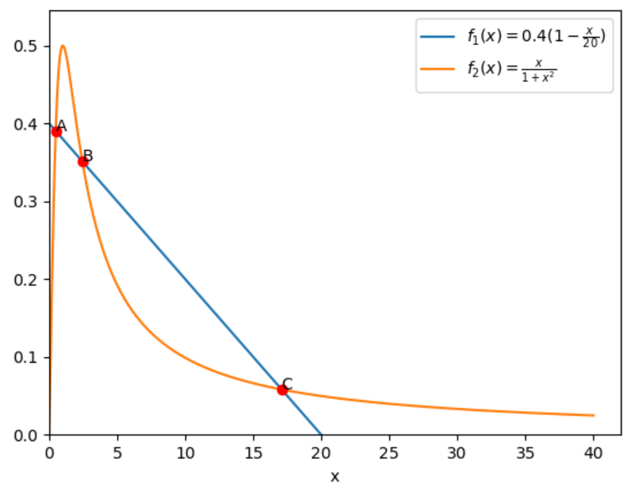


(a) r = 0.56, k = 10



(b) r = 0.195, k = 20

Figure 2: 2 intersections



(a) r = 0.4, k = 20

Figure 3: 3 intersections

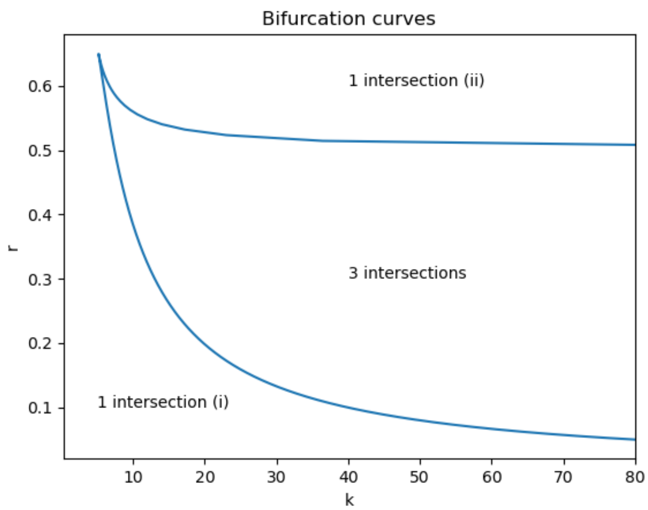


Figure 4: $r(x)$ in function of $k(x)$

sulting r in function of x back into the 2nd equation, we get :

$$\begin{cases} r &= \frac{2x^3}{(1+x^2)^2} \\ k &= \frac{2x^3}{x^2-1} \end{cases} \tag{5}$$

Let's not forget that r and k must be > 0. So we only consider $x > 1$.

For each $x > 1$, we computed the corresponding r and k. The Figure 4 is obtained by plotting $r(x)$ in function of $k(x)$. We can observe that the blue line corresponding to the values at which we have a saddle-node bifurcation (i.e. where we have exactly 2 intersections or equivalently 3 equilibrium points in total for our system with equation (2)) separates the plane into different areas. The area with 3 intersections corresponds to Figure 3, the area with 1 intersection (i) corresponds to Figure 1a and the area with 1 intersection (ii) corresponds to Figure 1b.

So with this Figure 4, we can choose any values for r and k in the area "3 intersections" to have 4 equilibrium points. For example we chose $r = 0.5$ and $k = 15$ on Figure 5. We found 4 equilibrium points : 2 stable ones (at $x \approx 0.74$ and $x \approx 12.65$) and 2 unstable ones (at $x = 0$ and $x \approx 1.6$).

3 Question 3

To highlight the hysteresis that the system can go through, we varied k in a sinusoidal way (Figure 6). We chose $r = 0.58$, $x_0 = 1.1$ and

$$k(t) = 10\sin(t) + 5 \tag{6}$$

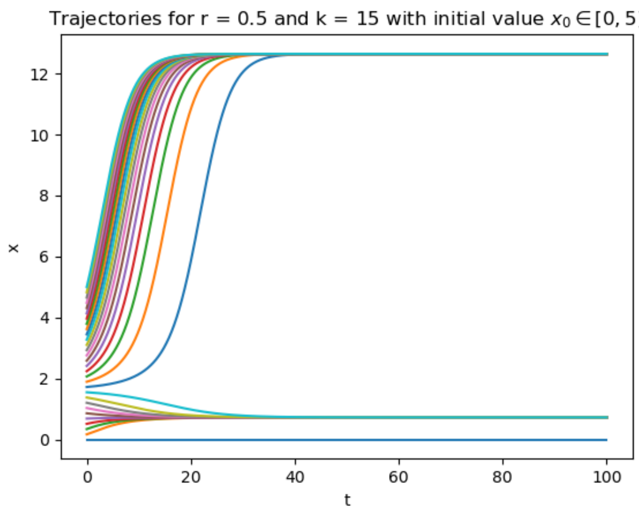


Figure 5: 4 equilibrium points for $(r, k) = (0.5, 15)$

as we wanted our system to go through a saddle-node bifurcation which would cause an outbreak.

Our experiment consisted in letting the system stabilize to an equilibrium. Then, we changed the value of k according to the sinusoidal function. After that, we let the system restabilize again before continuing to change the value of k and so on.

Basically, the trajectory of $x(t)$ is as in Figure 7.

To observe more clearly the evolution of the equilibrium point at which the system stabilizes, we plotted its value in function of k in Figure 8 with the arrows describing the path it takes when k follows Equation (6).

Here, it is even easier to observe that the system undergoes a hysteresis.

4 Question 4

To obtain the stable part of the bifurcation diagram in the (k, x) plane, it suffices to remove the arrows in the Figure 8 and keep only the points. So we obtain the Figure 9.

5 Question 5

One way to obtain the unstable part of the bifurcation diagram is to "invert" the stability of equilibrium points. To do that, we know that for a 1st order 1-D differential equation, when $\dot{x} > 0$ at an equilibrium point, it is unstable. But

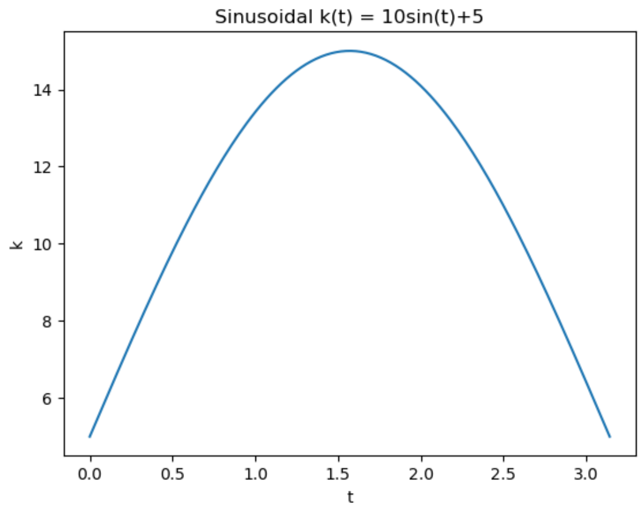


Figure 6: Sinusoidal variation of k

when $\dot{x} < 0$, it is stable. So we will choose

$$g(x) = -f(x) = -rx(1 - \frac{x}{k}) + \frac{x^2}{1 + x^2} \tag{7}$$

The equilibrium points remain the same but their stability is "inverted". So now, doing the same experiment as in Question 3 and with a wise choice for x_0, we obtain the Figure 10.

Putting the stable and unstable part together, we see that they fit in Figure 11.

References

[1] Steven H. Strogatz (1994), Nonlinear Dynamics and Chaos, Perseus Books, Massachussetts, pages 73-79.

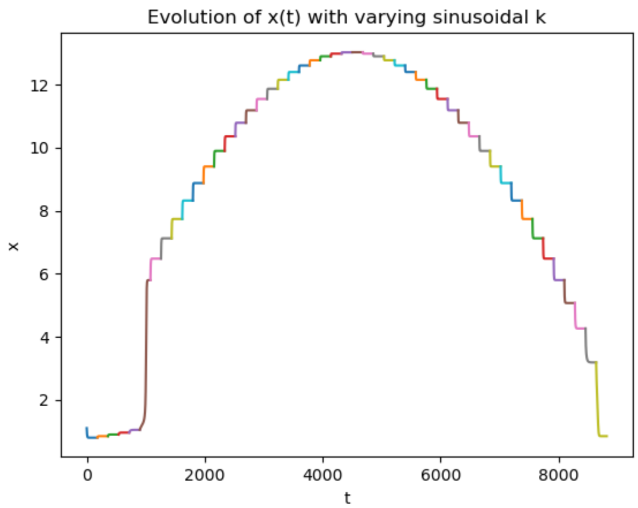


Figure 7: Trajectory of $x(t)$ with $r = 0.58$ and k varying as a sinusoidal. We can see that from an initial point $x_0 = 1.1$, the system stabilizes at $x \approx 0.75$. Then, the color change corresponds to a change in the value of k which gets a bit higher. The system then stabilizes to a slightly higher equilibrium point. This equilibrium point that is slowly getting higher corresponds to the A equilibrium point (on Figure 3). After a while, we observe a big jump (in Figure 7). This is caused by the saddle-node bifurcation. Indeed, it corresponds to the moment when equilibrium points A and B (on Figure 3) get closer until they coincide and then disappear. After that, the only remaining stable equilibrium point is the point C to which the system will stabilize. Then the point C moves as k goes higher. k then gets smaller and the system still stabilizes at C which gets smaller as well. What is interesting to see is that even when k gets back to the value at which there was a big jump earlier, now there is no big jump back. The plot is not "symmetric". The system continues to follow C that gets smaller and smaller until there is a big jump back down. This corresponds to when equilibrium points B and C get closer until they disappear and so the only remaining equilibrium point is A. So the system stabilizes back to A.

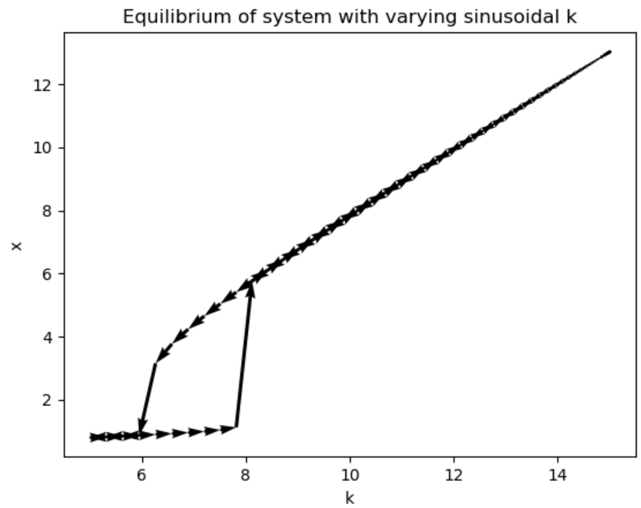


Figure 8: Motion of the equilibrium point in function of k

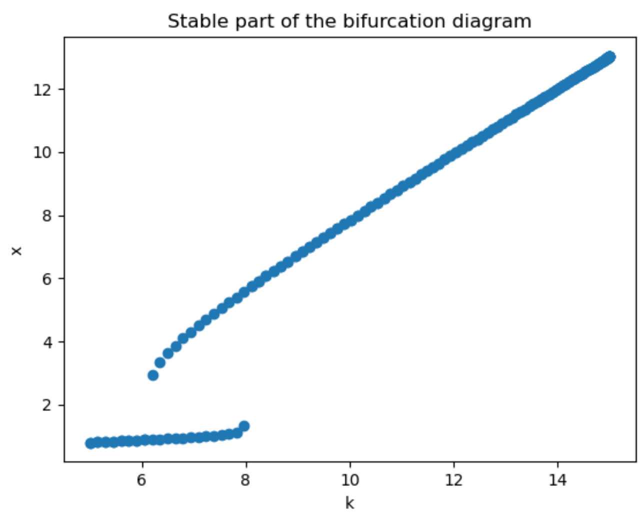


Figure 9: Stable part of the bifurcation diagram of $\dot{x} = f(x)$ in the (k, x) plane

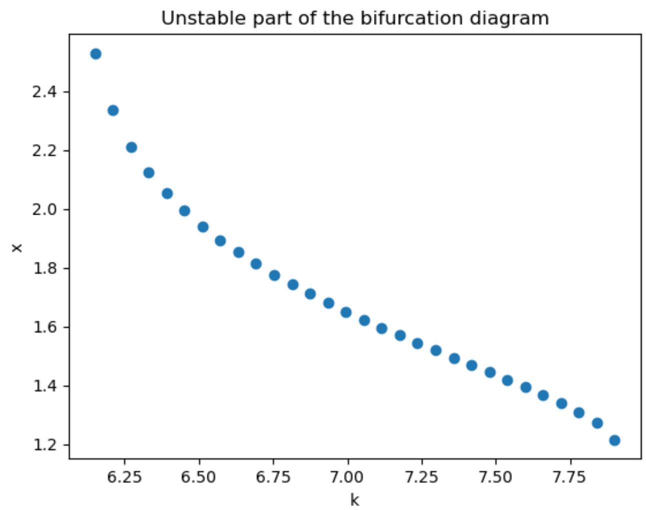


Figure 10: Unstable part of the bifurcation diagram of $\dot{x} = f(x)$ or stable part of the bifurcation diagram of $\dot{x} = g(x)$ in the (k, x) plane

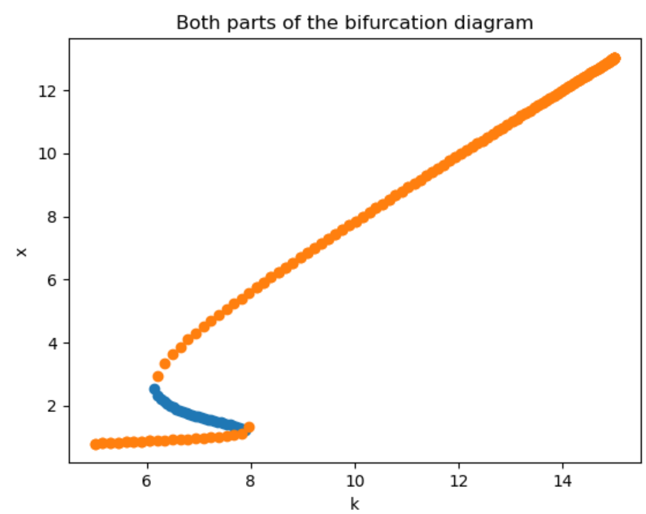


Figure 11: Whole bifurcation diagram of $\dot{x} = f(x)$ in the (k, x) plane