On the Pompeiu problem

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Abstract

It is proved that if $D \subset \mathbb{R}^2$ is diffeomorphic to a disc, and the Fourier transform $\tilde{\chi}(k\alpha) = 0$ for all unit vectors $\alpha \in \mathbb{R}^2$, then D is a disc. Here χ is the characteristic function of D. Various equivalent formulations of the Pompeiu problem are considered.

Key words: The Pompeiu problem; Fourier transforms of characteristic sets; overdetermined boundary value problems
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1 Introduction

The modern formulation of the Pompeiu problem can be given in several equivalent ways. In this paper the two-dimensional problems are considered for simplicity. The history of the Pompeiu problem goes back to 1929, see [2]. Basic known results and references about this problem can be found, for example, in [5], Chapter 11. The papers cited in the bibliography, which is incomplete, deal with various aspects of the Pompeiu problem. Our paper is essentially self-contained.

We assume throughout that $D \subset \mathbb{R}^2$ is a domain diffeomorphic to a disc, and its boundary S is a real analytic convex curve (see [7], where the analyticity assumption is justified). By χ the characteristic function of D and by $\tilde{\chi}(\xi) := \int_D e^{i\xi\cdot x}dx$ its Fourier transform are denoted, respectively. The vector $\xi = k\alpha$, where $k > 0$ is the length of ξ, $\alpha \in S^1$ is a unit vector in $\mathbb{R}^2$, S^1 is a unit circle in $\mathbb{R}^2$, and $\xi \cdot x = (\xi, x)$ is the dot product in $\mathbb{R}^2$.

One of the modern formulations of the Pompeiu problem is the following ([5], [6]):

Formulation 1. *Prove that if*

$$\tilde{\chi}(k\alpha) = 0 \tag{1}$$

for all $\alpha \in S^1$ and a fixed $k > 0$, then D is a disc.

We prove Theorem 1.

Theorem 1. *If (1) holds for all $\alpha \in S^1$ and a fixed $k > 0$, then D is a disc.*

Let us give two other equivalent formulations of the Pompeiu problem. Denote by N the unit normal to S pointing out of D.

Formulation 2. *Suppose that $k > 0$ is fixed and the following problem*

$$(\nabla^2 + k^2)u = 1 \quad in \quad D, \quad u|_S = u_N|_S = 0, \tag{2}$$

has a solution. Prove that then D is a disc.

Formulation 3. *Assume that $f \in L^1_{loc}$, $f \not\equiv 0$, and*

$$\int_D f(y + gx)dx = 0, \quad \forall y \in \mathbb{R}^2, \quad \forall g, \tag{3}$$

where g is an arbitrary rotation. Prove that then D is a disc.

The equivalence of Formulations 1, 2 and 3 is briefly proved below, see also [5], Chapter 11. In this proof 1 stands for Formulation 1, etc.

$1 \Rightarrow 2$. If an entire function of exponential type $\tilde{\chi}(\xi)$ vanishes on the irreducible algebraic variety $\xi^2 = k^2$, then the function $\tilde{u} := \tilde{\chi}(\xi)(\xi^2 - k^2)^{-1}$ is also entire and of the same exponential type. Its Fourier transform $u(x)$ solves problem (2). $\Box$

$2 \Rightarrow 1$. If (2) holds, then multiply (2) by $e^{ik\alpha\cdot x}$, $\alpha \in S^1$, integrate over D, and then by parts, using the boundary conditions (2) and the equation $(\nabla^2 + k^2)e^{ik\alpha\cdot x} = 0$. This yields (1). $\Box$

$3 \Rightarrow 1$. Take the Fourier transform (in the distributional sense) of (3) and get $\overline{\tilde{\chi}(g^{-1}\xi)}\tilde{f}(\xi) = 0$, where the overline stands for the complex conjugate. Therefore, $supp\tilde{f} = \cup_k C_k$, where $C_k = \{\xi : \xi^2 = k^2, \tilde{\chi}|_{\xi^2=k^2} = 0\}$, and the set $\{k\}$ is a discrete finite set of positive numbers. Thus, there is a $k > 0$ such that (1) holds. $\Box$

$1 \Rightarrow 3$. If (1) holds, then there exists an $\tilde{f} \neq 0$ supported on C_k. Then $\overline{\tilde{\chi}(g^{-1}\xi)}\tilde{f}(\xi) = 0$. Taking the inverse Fourier transform of this relation yields (3). $\Box$

In Section 2 we prove Theorem 1. Because Formulation 1 is equivalent to Formulations 2 and 3, the following two theorems will be established if Theorem 1 is proved:

Theorem 2. *If (2) holds, then D is a disc.*

Theorem 3. *If (3) holds, then D is a disc.*

2 Proof of Theorem 1.

Let us describe the steps of our proof.

Step 1: If equation (1) holds for $\alpha \in S^1$, then (1) holds for all complex z_1, z_2 satisfying the equation $z_1^2+z_2^2=1$, where z_1 replaces α_1 and z_2 replaces α_2 in equation (1). Define an algebraic variety

$$M := \{z : z \in \mathbb{C}^2, z_1^2 + z_2^2 = 1\}.$$

Clearly, $S^1 \subset M$.

Step 2: Let $M_1 := \{\eta : \eta \in \mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)\}$. Set $z_1 = -i\eta$. Then (1) yields

$$\int_D e^{\eta x + i(1+\eta^2)^{1/2} y} dxdy = 0, \qquad \forall \eta \in M_1. \tag{4}$$

Step 3: If (4) holds, then

$$\int_D e^{iy} x^n dxdy = 0, \qquad \int_D e^{ix} y^n dxdy = 0, \qquad n = 0, 1, 2, ... \tag{5}$$

Step 4: Let ℓ be an arbitrary unit vector, L_1 be the support line to D at the point $s \in S$ parallel to ℓ, where S is the boundary of D, and L_2 be the support line to D, parallel to L_1, at the point $q \in S$, $q = q(s)$. Let L be the distance between L_1 and L_2, that is, the width of D in the direction orthogonal to ℓ. Since D is strictly convex, one can introduce the equations $y = f(x)$ and $y = g(x)$ of the boundary S between the support points s and q. The graph of f is located above the graph of g. The function f has a unique point of maximum x_1, $x_1 \in (a, b)$, $f(x_1) > f(x)$, where a and b are $x-$coordinates of the points q and s. The function g has a unique point of minimum x_2, $x_2 \in (a, b)$, $g(x) > g(x_2)$. Let us assume that these maximum and minimum are non-degenerate, that is, $f^{''}(x_1) \neq 0$, and $g^{''}(x_2) \neq 0$. Let us write the second formula (5) as

$$\int_D e^{ix} y^n dxdy = \int_a^b e^{ix} \frac{f^{n+1}(x) - g^{n+1}(x)}{n+1} dx = 0, \qquad n = 0, 1, 2, ... \tag{6}$$

The factor $n + 1$ in the denominator can be canceled. The Laplace formula (see, for example, [1]) for the asymptotic of the integral

$$F(\lambda) := \int_a^b \phi(x) e^{\lambda S(x)} dx \sim \Big(- \frac{2\pi}{\lambda S''(\xi)} \Big)^{1/2} \phi(\xi) e^{\lambda S(\xi)}, \qquad \lambda \to \infty,$$

where $\xi \in (a, b)$ is a unique point of non-degenerate maximum of $S(x)$, $S^{''}(\xi) \neq 0$. Let us applying this formula with $S(x) = \ln |f|$, $\lambda = 2m := n +$

$1 \to \infty$, $\phi = e^{ix}$, and take $n = 2m-1$ to ensure that $n+1$ is an even number, so that $\ln f^{n+1}$ and $\ln g^{n+1}$ are well defined. Note that $(\ln |f|)'' = \frac{f''(x_1)}{f(x_1)}$ at the maximum point x_1 since $f'(x_1) = 0$ and $f''(x_1) < 0$. The point of minimum of the function g becomes the point of maximum of g^{2m}. Taking the above into consideration, one obtains from (6) the following formula:

$$\int_D e^{ix} y^n dxdy = \Big[e^{ix_1 + 2m \ln |f(x_1)|} \Big(\frac{2\pi |f''(x_1)|}{(2m-1)|f(x_1)|}\Big)^{1/2} -$$
$$e^{ix_2 + 2m \ln |g(x_2)|} \Big(\frac{2\pi |g''(x_2)|}{(2m-1)|g(x_2)|}\Big)^{1/2}\Big](1 + o(1)) = 0, \quad n \to \infty, \tag{7}$$

where $n = 2m - 1$, $x_1 \in (a, b)$ and $x_2 \in (a, b)$. It follows from the above formula that the expression in brackets, that is, the main term of the asymptotic, must vanish for all sufficiently large m. Therefore,

$$x_1 = x_2 := x_0; \quad |f(x_0)| = |g(x_0)|; \quad |f''(x_0)| = |g''(x_0)|. \tag{8}$$

Also,

$$L = f(x_0) - g(x_0) > 0. \tag{9}$$

Let $R = R(s)$ denote the radius of curvature of the curve S at the point s and let $\kappa = \kappa(s)$ denote the curvature of S at the same point. One has the formula

$$R^{-1} = \kappa = |f''(x_0)|, \tag{10}$$

which follows from the known formula

$$\kappa = \frac{|f''(x_0)|}{[1 + |f'(x_0)|^2]^{3/2}},$$

because $f'(x_0) = 0$ since x_0 is a point of maximum of f.

Step 5: From (8) we derive that

$$L(s) = 2R(s), \qquad \forall s \in S, \tag{11}$$

and

$$L(s) = const. \tag{12}$$

This and (11) imply that

$$R(s) = const. \tag{13}$$

Therefore S is a circle.

This leads to the conclusion of Theorem 1.

In what follows the above steps are discussed in detail. Let us start with **Step 1.**

Lemma 1 *If (1) holds for all $\alpha \in S^1$ and a fixed $k > 0$, then it holds for all $\alpha \in M$ and the same $k > 0$.*

Proof. The function $\tilde{\chi}(\xi)$ is an entire function of $\xi \in \mathbb{C}^2$. Let $\xi = (z_1, z_2)$, $z_j \in \mathbb{C}$, $j = 1, 2$, $z_j = x_j + iy_j$. Assume that $\tilde{\chi}(x_1, x_2) = 0$ if $x_1^2 + x_2^2 = 1$. The function $g(z_1) := \tilde{\chi}(z_1, \sqrt{1 - z_1^2})$ is an analytic function of z_1 for $|z_1| < \delta$, where $\delta < 1$. By our assumption, $g(x_1) = 0$. By the uniqueness theorem for analytic functions of one complex variable, it follows that $g(z_1) = 0$ for $|z_1| < \delta$, and, consequently, everywhere in the set $(z_1, \sqrt{1 - z_1^2})$, where z_1 runs through a set in $\mathbb{C}$ for which $\sqrt{1 - z_1^2}$ is analytic. The union of this set and the set $(z_1, -\sqrt{1 - z_1^2})$ is the variety M. So, if $\tilde{\chi}(x_1, x_2) = 0$ in the set $x_1^2 + x_2^2 = 1$, where x_1 and x_2 are real numbers, then $\tilde{\chi}(z_1, z_2) = 0$ in M. Lemma 1 is proved. $\Box$

Step 1 is completed. $\Box$

Step 2 is obvious and does not require a proof. $\Box$

Step 3. From formula (4) it follows that

$$\sum_{n=0}^{\infty} \frac{\eta^n}{n!} \int_D dxdyx^n e^{i(1+\eta^2)^{1/2}y} = 0 \qquad \forall \eta \in M_1. \tag{14}$$

Let $\eta = 0$. Then $\int_D e^{iy} dxdy = 0$, and the term with $n = 0$ in (14) is gone. Divide (14) with the sum started at $n = 1$ by η and then let $\eta = 0$. Then $\int_D e^{iy} xdxdy = 0$, and the term with $n = 1$ is gone. Continue in this way and get the first formula (5) for all $n = 0, 1, 2, ...$ The second formula in (5) is proved similarly since η and $(1 + \eta^2)^{1/2}$ enter symmetrically in formula (4).

Step 3 is done. $\Box$

Step 4. This step was done above, in the outline of the steps. $\Box$

Step 5. Let $r = r(s)$ be the equation of S, where r is the radius vector of the point $s \in S$. It is known that $r'(s) = t$, where $t = t(s)$ is a unit vector tangential to S. We have chosen s so that t is parallel to ℓ. Since ℓ is arbitrary, the point $s \in S$ is arbitrary. The point $q \in S$, $q = q(s)$, is uniquely determined by the requirement that $t(q) = -t(s)$. Thus, one has

$$(r(q) - r(s), r'(s)) = 0, \qquad (r(q) - r(s), r'(q)) = 0, \quad \forall s \in S, \tag{15}$$

where (u, v) denotes the inner product in $\mathbb{R}^2$. Differentiate the first equation (15) with respect to s and get

$$(r'(q)\frac{dq}{ds} - r'(s), r'(s)) + (r(q) - r(s), r''(s)) = 0, \quad \forall s \in S. \tag{16}$$

Since $r''(s) = \kappa\nu$ where $\nu = \nu(s)$ is the unit normal to S at the point s directed inside D, and $|r(q) - r(s)| = L(s)$, while $r'(q) = -t(s)$, it follows from (16) that

$$-\frac{dq}{ds} - 1 + \kappa(s)L(s) = 0, \quad \forall s \in S. \tag{17}$$

Note that $L(s) = L(q)$ and $\kappa(s) = \kappa(q)$, as follows from (8). Differentiate the second equation (15) with respect to q and get

$$-\frac{ds}{dq} - 1 + \kappa(s)L(s) = 0, \quad \forall s \in S. \tag{18}$$

Compare (17) and (18) and get

$$\frac{ds}{dq} = \frac{dq}{ds} = 1, \qquad \forall s \in S. \tag{19}$$

Therefore equation (17) (or (18)) implies

$$\kappa(s)L(s) = 2. \tag{20}$$

Let us check that $L(s)$=const. Differentiate the relation

$$L^2(s) = |r(q) - r(s)|^2$$

with respect to s. This yields

$$2L(s)L'(s) = 2Re(r'(q) - r'(s), r(q) - r(s)) = -4(t(s), r(q) - r(s)) = 0,$$

because $t(s)$ is orthogonal to $r(q) - r(s)$. Consequently, $L' = 0$ and

$$L(s) = const \qquad \forall s \in S. \tag{21}$$

Equations (20) and (21) imply that $R(s) = const$. This means that D is a disc. Theorem 1 is proved. $\square$

Consequently, Theorems 2 and 3 are also proved. $\square$

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