The Pompeiu problem

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Abstract

Let $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'$, where $\mathcal{S}'$ is the Schwartz class of distributions, and

$$\int_{\sigma(D)} f(x)dx = 0 \quad \forall \sigma \in G, \qquad (*)$$

where $D \subset \mathbb{R}^n$ is a bounded domain, the closure $\bar{D}$ of which is diffeomorphic to a closed ball. Then the complement of $\bar{D}$ is connected and path connected. By G the group of all rigid motions of $\mathbb{R}^n$ is denoted. This group consists of all translations and rotations.

It is conjectured that if $f \neq 0$ and (*) holds, then D is a ball. This conjecture is discussed in detail.

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1 Introduction

In this paper the problem known as the Pompeiu problem is formulated and discussed. This problem originated in a paper [7], of 1929. The problem in a modern formulation is still open.

Dimitrie Pompeiu (1873-1954) was born in Romania, got his Ph.D in 1905 at the Sorbonne, in Paris, under the direction of H.Poincare. He is known mainly for the Pompeiu problem and for the Cauchy-Pompeiu formula in complex analysis.

Let us formulate the Pompeiu problem as it is understood today.

Let $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'$, where $\mathcal{S}'$ is the Schwartz class of distributions, and

$$\int_{\sigma(D)} f(x)dx = 0 \quad \forall \sigma \in G, \tag{1}$$

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where G is the group of all rigid motions of $\mathbb{R}^n$, G consists of all translations and rotations, and $D \subset \mathbb{R}^n$ is a bounded domain, the closure $\bar{D}$ of which is diffeomorphic to a closed ball. Under these assumptions the complement of $\bar{D}$ in $\mathbb{R}^n$ is connected and path connected, see [4], the isotopy extension theorem.

In [7] the following question was raised by D.Pompeiu:

Does (1) imply that $f = 0$?

If yes, then we say that D has P-property (Pompeiu's property), and write $D \in P$. Otherwise, we say that D fails to have P-property, and write $D \in \overline{P}$. Pompeiu claimed in 1929 that every plane bounded domain has $P-$property. This claim turned to be false: a counterexample was given 15 years later in [2]. The counterexample is a domain D which is a disc, or a ball in $\mathbb{R}^n$ for $n > 2$. If D is a ball, then there are $f \not\equiv 0$ for which equation (1) holds. The set of all $f \not\equiv 0$, for which equation (1) holds, was constructed in [8]. There are infinitely many (a continuum) such f. A bibliography on the Pompeiu problem ($P-$problem) can be found in [15] and in [8].

The current formulation of the $P-$problem is the following:

Prove that if $D \in \overline{P}$ then D is a ball.

We use the word ball also in the case $n = 2$, when this word means disc, and discuss the $P-$problem in detail. This problem leads to some problems of general mathematical interest: a symmetry problem for partial differential equations and a problem in harmonic analysis.

Let us make the following standing assumptions:

Assumptions A:

$A_1)$: *D is a bounded domain, the closure of which is diffeomorphic to a closed ball, the boundary S of D is a closed connected C^1-smooth surface,*

$A_2)$: *D fails to have $P-$property.*

Our main conjecture is:

Conjecture 1. *If Assumptions A hold, then D is a ball.*

In Section 2 this Conjecture is discussed. We prove that this Conjecture is equivalent to a symmetry problem for a partial differential equation. Several symmetry problems were studied by the method based on Theorem 2, see [11]-[13].

Conjecture 2. *If problem* (2) *(see below) has a solution, then D is a ball.*

It is also equivalent to the following

Conjecture 3. *If Assumption A_1 holds and the Fourier transform of the characteristic function of D has a spherical surface of zeros, then D is a ball.*

2 Discussion of the Conjectures

It is proved in [14] that if Assumptions A hold, then the boundary S of D is real-analytic. It is proved in Theorem 3, below, that if Assumptions A hold, then the problem

$$(\nabla^2 + k^2)u = 1 \quad \text{in} \quad D, \qquad u\big|_S = 0, \quad u_N\big|_S = 0, \quad k^2 = const > 0, \tag{2}$$

has a solution. In (2) $N = N_s$ is the outer unit normal to S pointing out of D, $s \in S$ is a point on S.

The basic results, on which our discussion of Conjectures 1, 2 and 3 is based, are Theorems 1, 2 and 3. Theorems 1 and 2 were proved originally in [9], [8], and in [10], Chapter 11. A result, equivalent to Theorem 3, has been proved originally in [1] by a considerably longer and more complicated argument. Our proof is borrowed essentially from [10], Chapter 11. In the paper [6] the null-varieties of the Fourier transform of the characteristic function of a bounded domain D are studied. The properties of these varieties and the geometrical properies of D are related, of course, but it is not clear in what way. Conjecture 3, if proved, is an interesting example of such a relation.

To make the presentation essentially self-contained, proofs of all of the basic results are included in this paper.

Theorem 1. *If Assumptions A hold, then*

$$[s, N] = u_N, \quad \forall s \in S, \tag{3}$$

where $[s, N]$ *is the cross product in* $\mathbb{R}^3$*, and* u *is a vector-function that solves the problem*

$$(\nabla^2 + k^2)u = 0 \quad in \quad D, \quad u|_S = 0. \tag{4}$$

If $n = 2$, then D is a plane domain, S is a curve, diffeomorphic to a circle, u is a scalar solution to equation (4), and equation (3) yields $s_1N_2 - s_2N_1 = u_N, \quad \forall s \in S$, where N_j, $j = 1, 2$, are Cartesian coordinates of the unit normal N to S. Indeed, if $n = 2$ then the cross product of two vectors $[s_1e_1 + s_2e_2, N_1e_1 + N_2e_2]$ is calculated by the formula $[s, N] = (s_1N_2 - s_2N_1)e_3$, where e_3 is a unit vector, orthogonal to the plane domain D, and the triple $\{e_j\}_{j=1}^3$ is a standard orthonormal basis in $\mathbb{R}^3$.

Let us state the following characterization of spheres.

Theorem 2. *If* S *is a smooth surface homeomorphic to a sphere and* $[s, N] = 0$ *on* S*, then* S *is a sphere.*

The proof of Theorem 2 will be given in the coordinate system in which the condition $[s, N] = 0$ on S is valid. This condition is not invariant with respect to translations, although the geometrical conclusion, namely, that S is a sphere, is invariant with respect to translations.

It follows from Theorem 2 that *the conclusion of Conjecture 1 will be established if one proves that* $[s, N] = 0$ *on* S.

Let us start by proving Theorem 2, then Theorem 1 is proved, and, finally, we prove Theorem 3. It is assumed throughout that $n = 3$, but most of our arguments can be used for any $n \geq 2$.

Proof of Theorem 2. Let $n = 3$ and assume that $s(p, q)$ is the parametric equation of the surface S. The normal N to S is a vector directed along the vector $[s_p, s_q]$, where s_p is partial derivative with respect to the parameter p. The assumption $[s, N] = 0$ on S, yields

$$[s, [s_p, s_q]] = s_p s \cdot s_q - s_q s \cdot s_p = 0. \tag{5}$$

Since the vectors s_p and s_q are linearly independent on a smooth surface, equation (5) implies

$$\frac{\partial s^2}{\partial p} = 0, \quad \frac{\partial s^2}{\partial q} = 0. \tag{6}$$

Therefore

$$s^2 = const. \tag{7}$$

This is an equation of a sphere in the coordinate system with the origin at the center of the sphere.

Theorem 2 is proved. □

Proof of Theorem 1. Let $n = 3$ and $\mathcal{N}$ denote the set of all smooth solutions to (4) in a ball B, containing D. Multiply (2) by and arbitrary solution h to equation (4) in a ball B, containing D, integrate by parts, take into account the boundary conditions in (2), and get the relation

$$\int_D h(x)dx = 0 \qquad \forall h \in \mathcal{N}.$$

Since $h \in \mathcal{N}$ implies $h(gx) \in \mathcal{N}$ $\forall g$, where g is an arbitrary rotation in $\mathbb{R}^3$ about the origin O, one obtains

$$\int_D h(gx)dx = 0 \qquad \forall h \in \mathcal{N}, \ \forall g. \tag{8}$$

Let $O \in D$ be arbitrary, and take an arbitrary straight line ℓ passing through O and directed along a unit vector α. Let $g = g(\phi)$ be the rotation about ℓ by an angle ϕ

counterclockwise. Differentiate (8) with respect to ϕ and then set $\phi = 0$, see [9], cf [11]. The result is

$$\int_D \nabla h(x) \cdot [\alpha, x] dx = 0, \tag{9}$$

where $[\alpha, x]$ is the cross product and $\cdot$ stands for the inner product in $\mathbb{R}^3$. Equation (9) is invariant with respect to translations because $\int_D \nabla h(x) dx = 0 \; \forall h \in \mathcal{N}$. Indeed, $h \in \mathcal{N}$ implies $\nabla h \in \mathcal{N}$. Using the divergence theorem, the relation $\nabla \cdot [\alpha, x] = 0$, valid for any constant vector α, and the arbitrariness of α, one derives from (9) the following relation

$$\int_S h(s)[s, N] ds = 0, \quad \forall h \in \mathcal{N}, \tag{10}$$

which is also invariant with respect to translations. Indeed, N is invariant under translations beacause $[s_p, s_q]$ is, and $\int_S h(s)[a, N] ds = 0$ for any constant vector a because $\int_S h(s) N ds = \int_D \nabla h dx = 0$, as was pointed out above. Let us derive from (10) equation (3). We need the following result.

Lemma 1. *The orthogonal complement of the set M of the restrictions of all $h \in \mathcal{N}$ to S is a finite-dimensional space spanned by the functions u_{jN}, where $\{u_j\}_{j=1}^J$ is the basis of the eigenspace of the Dirichlet Laplacian in D, corresponding to the eigenvalue k^2.*

Remark 1. It follows from (10) that $[s, N]$ is orthogonal in $L^2(S)$ to the set M. Therefore, by Proposition 1, each of the three components of $[s, N]$ must be linear combinations of the functions u_{jN}, $1 \leq j \leq J$. In other words, equation (3) holds.

Proof of Lemma 1. Note that the result of Lemma 1 is equivalent to the assertion that the boundary value problem

$$(\nabla^2 + k^2) h = 0 \quad in \quad D, \quad h|_S = f \tag{11}$$

is solvable if and only if

$$\int_S f u_{jN} ds = 0, \quad 1 \leq j \leq J. \tag{12}$$

where u_j, $1 \leq j \leq J$, is a basis of the solutions to problem (4). The *necessity* of conditions (12) is proved by the relation

$$0 = \int_D u_j (\nabla^2 + k^2) h dx = -\int_S f u_{jN} ds,$$

where an integration by parts and the boundary condition $u_j = 0$ on S were used, and equation (4) for h was taken into account.

The *sufficiency* of conditions (12) is proved as follows. Denote by $H^m(D)$ the usual Sobolev spaces. Given an $f \in H^{3/2}(S)$, construct an arbitrary $F \in H^2(D)$, such that $F|_S = f|_S$, and define $h := w + F$, where

$$(\nabla^2 + k^2)w = -(\nabla^2 + k^2)F \quad in \quad D, \quad w|_S = 0. \tag{13}$$

If such w exists, then $h = w+F$ solves problem (11). For the existence of w it is necessary and sufficient that

$$\int_D (\nabla^2 + k^2)Fu_j dx = 0, \quad 1 \leq j \leq J.$$

An integration by parts shows that these conditions are equivalent to conditions (12) because u_j solve problem (4). Thus, Lemma 1 is proved. □

Equation (10) says that $[s, N]$ is orthogonal to the set M, that is, to the restrictions of all $h \in \mathcal{N}$ to S.

Lemma 2. *The set M is dense in $L^2(S)$ in the set of all $\psi \in H^{3/2}(S)$ for which the boundary problem* (13) *is solvable.*

Therefore equation (10) and Lemma 1 imply (3).

Proof of Lemma 2. Assume the contrary. Then for some $f \in H^{3/2}(S)$, $f \neq 0$, problem (11) is solvable and

$$\eta(y) := \int_S f(s)\psi(s, y)ds = 0 \quad \forall y \in D' := \mathbb{R}^3 \setminus D,$$

where $\psi(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|} \in \mathcal{N}$ for $y \in B'$, that is, outside a ball containing D. The function η is a simple-layer potential which vanishes in B', and by the unique continuation property for solutions of the homogeneous helmholtz equation, $\eta = 0$ everywhere in D'. Thus, it vanishes on S. Therefore, η solves problem (4). By the jump relation for the normal derivative of η across S, one has $f = \eta_N$, where η_N is the limiting value of the normal derivative of η on S from inside D. If problem (10) is solvable, then, as we have proved, f is orthogonal to all functions u_{jN}. The function η_N is a linear combination of these functions. This and the relation $f = \eta_N$ prove that $f = 0$. Consequently, we have proved the claimed density of M in the set of all $H^{3/2}(S)-$functions f for which problem (11) is solvable. Lemma 2 is proved. □

This completes the proof of Theorem 2. □

Theorem 3. *If (1) holds, then (2) holds.*

Proof of Theorem 3. Write (1) as

$$\int_{\mathbb{R}^3} f(gx + y)\chi(x)dx = 0 \qquad \forall y \in \mathbb{R}^3, \quad \forall g \in G,$$

where $\chi(x)$ is the characteristic function of D. Applying the Fourier transform and the convolution theorem one gets

$$\tilde{f}(\xi)\overline{\tilde{\chi}(g^{-1}\xi)} = 0, \tag{14}$$

where $\tilde{f}$ and $\tilde{\chi}$ are the Fourier transforms of f and χ, respectively, and the overbar stands for the complex conjugate. The Fourier transform of f is understood in the sense of distributions. The Fourier transform $\tilde{\chi}$ is an entire function of exponential type because function χ has support $\overline{D}$, which is a bounded set. Moreover, $\tilde{\chi}$ is a uniformly bounded function of $\xi \in \mathbb{R}^n$. Therefore, the product of the tempered distribution $\tilde{f}$ and the function $\tilde{\chi}$ is a tempered distribution also.

Since g^{-1} runs through all the rotations, one can replace g^{-1} by g. It follows from (14) that

$$\mathit{supp}\,\tilde{f} = \cup_k C_k, \text{ where } C_k := \{\xi : \tilde{\chi}(\xi) = 0 \ \ \forall \xi : \xi^2 - k^2 = 0\}. \tag{15}$$

In other words, the support of the distribution $\tilde{f}$ is a subset of the union of spherical surfaces of zeros of $\tilde{\chi}$, the Fourier transform of the characteristic function of the bounded domain D. Since $\tilde{\chi}(\xi)$ is an entire function of exponential type, vanishing on an irreducible algebraic variety $\xi^2 - k^2 = 0$ in $\mathbb{C}^3$, one concludes, using the division lemma, that

$$\tilde{\chi}(\xi) = (\xi^2 - k^2)\tilde{u}(\xi), \tag{16}$$

where $\tilde{u}$ is an entire function of the same exponential type as $\tilde{\chi}$ (see [3]). Therefore, by the Paley-Wiener theorem, the corresponding u has compact support. Taking the inverse Fourier transform of equation (16), one gets:

$$(-\nabla^2 - k^2)u(x) = \chi(x) \qquad in \quad \mathbb{R}^3, \quad u = 0 \,\text{if} \quad |x| > R, \tag{17}$$

where $R > 0$ is sufficiently large. By the elliptic regularity results, one concludes that $u \in H^2_{loc}(\mathbb{R}^3)$. Since u solves the Helmholtz elliptic equation and vanishes near infinity, that is, in the region $|x| > R$, the uniqueness of the solution to the Cauchy problem to the equation (17) and the path connectedness of the complement $D_1 := \bar{D}'$ of the closure $\bar{D}$ of D allow one to conclude that $u = 0$ in D_1. The connectedness and path connectedness of D_1 follow from our Assumptions A and from the isotopy extension theorem ([4]). If $u = 0$ in D_1 and $u \in H^2_{loc}(\mathbb{R}^3)$, it follows from the Sobolev embedding theorem that the boundary conditions (2) hold. Since $\chi(x) = 1$ in D, equation (2) holds. Theorem 3 is proved. $\Box$.

3 Relation to analyticity

The classical Morera theorem in complex analysis says that if $\int_C f(z)dz = 0$ for any closed polygon C in a domain D of the complex plane, and if f is continuously differentiable in D, then f is analytic in D. A simple proof is based on a version of Green's formula: $0 = \int_C f(z)dz = 2i\int_\Delta \bar{\partial} f dxdy$. Here Δ is the plane domain with the boundary C and $\bar{\partial} f := \frac{f_x + if_y}{2}$. If $\int_\Delta \bar{\partial} f dxdy = 0$ for any polygon Δ, then one easily derives that $\bar{\partial} f = 0$ evrywhere in D. This implies that f is analytic in D.

One may ask if the assumption that f is continuously differentiable can be replaced by a weaker assumption, and if the set of polygons can be replaced by some other sets. The answer to the first question is easy: if $f \in L^1_{loc}(D)$, then one considers a mollified function $f_\epsilon(z) := \int_{\zeta:|z-\zeta|\leq\epsilon} \omega_\epsilon(z-\zeta) f(\zeta) dudv$, where $\zeta = u + iv$ and $\omega_\epsilon(z)$ is the standard mollifying kernel (citeHo, p.14). It is known that $f_\epsilon \to f$ in $L^1(D)$ as $\epsilon \to 0$, and one can select a subsequence $\epsilon_j \to 0$, such that $f_{\epsilon_j} \to f$ almost everywhere in D. If $\int_C f(z)dz = 0$ for any closed polygon C, then $\int_C f_\epsilon(z)dz = 0$ for any closed polygon C, and the above argument for a C^1-smooth f leads to the conclusion that f_ϵ is analytic in D for all sufficiently small ϵ. Since a sequence f_{ϵ_j} of analytic functions converges to f in $L^1(D)$ and almost everywhere in D, one concludes that f is analytic in D. The second question: can one replace the set of polygons by other sets is less simple. For example, one cannot replace polygons by the set $\sigma(B)$, where B is a ball. Indeed, using the above argument one arrives at the relation $\int_{\sigma(B)} \bar{\partial} f dxdy = 0$, and this does not imply that $\bar{\partial} f = 0$.

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