classical fields

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Here is a collected notes on physics readings.

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§1 Basic Equations

§1.1 Scalar fields

Legendre transform from hamiltonian to lagrangian

$$ML[\phi, \dot{\phi}] = \pi[\phi, \dot{\phi}]\dot{\phi} - \mathcal{H}[\phi, \pi[\phi, \dot{\phi}]]$$

Euler Lagrange equation

$$S = \int \mathrm{d}^4 x \mathcal{L}$$

$$\delta S = \int \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right]$$

$$\int \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\delta\phi(x) + \underbrace{}\text{surface term}$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \textbf{ Euler Lagrange}$$

Noether's theorem:

$$\vec{\phi} \rightarrow U(\underbrace{\alpha}_{\text{continuous param}})\vec{\phi}$$

$$J_\mu = \underbrace{\sum_n}_{\text{field component sum}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}$$

SHO/wave equation

$$E = \dot{\phi}^2 + m^2\phi^2 - \underbrace{(\nabla\phi)^2}$$

$$L = -\phi(\underbrace{\partial_t^2 - \partial_x^2}_{\partial_\mu\partial^\mu \equiv \partial^2} + m^2)\phi$$

§1.2 Gauge Fields

Maxwell's equations, tensor notation:

$$
\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
\partial^\mu &= (\frac{\partial}{\partial t}, -\nabla) \\
F &= dA \\
g_{\alpha\beta} &= \mathrm{diag}(+1,-1,-1,-1) = g^{\alpha\beta} \\
F^{\mu\nu} &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \\
F_{\mu\nu} &= g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu} \\
&= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \\
\text{Faraday Tensor : } G^{\alpha\beta} &= \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \\
G^{\alpha\beta} &= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \\
&\text{EM duality: } (E,B) \to (B,-E) \\
G &= *F
\end{aligned}
$$

$F_{\mu\nu}$ is a covariant 2-tensor. It transforms accordingly:

$$F_{\mu\nu} = \Lambda^\sigma_\mu \Lambda^\rho_\nu F_{\rho\sigma} \tag{1}$$

A natural lagrangian density is:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
&= - \underbrace{2}_{\text{symmetric sum}} \times (\frac{1}{4}) \left(E \times (-E) - +B \times B \right) \\
&= \frac{1}{2} \left(E^2 - B^2 \right)
\end{aligned}
$$

Furthermore one can compute the energy momentum tensor:

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \left(\partial^\mu \phi \right)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu \tag{2}$$

Note the energy-momentum tensor is not unique. One can always add a completely anti-symmetric divergence (see landau lifshitz vol. 2, 33):

$$\partial_\mu T^{\mu\nu} = 0 \to \partial_\mu \left(T^{\mu\nu} + \partial_\rho \underbrace{U^{\rho\mu\nu}}_{\text{anti-symmetric}} \right) = 0 \tag{3}$$

We can rewrite the equations compactly

$$\nabla \times \mathbf{B} = -\frac{\partial}{\partial t}\mathbf{E} \qquad \nabla \cdot \mathbf{E} = 0 \tag{4}$$
$$\partial_\mu F^{\mu\nu} = J^\nu \qquad \mathrm{d} * F = *J \tag{5}$$
$$\nabla \times \mathbf{E} = \frac{\partial}{\partial t}\mathbf{B} \qquad \nabla \cdot B = 0 \tag{6}$$
$$\partial_{[\alpha} F_{\beta\rho]} = 0 \qquad \mathrm{d}F = 0 \tag{7}$$

Generalized charged conservation:

$$\mathrm{d}(\mathrm{d} * F) = 0 \to \int_{\partial\mathcal{R}} *J = 0 \tag{8}$$

Example 1.1
Duality symmetries.

$$\tilde{F} \equiv F + i * F \tag{9}$$
$$d\tilde{F} = 0. \tag{10}$$

Note 9 only make sense in 4 dimensions where the F is a (0,2) tensor and *F is a (2, 0) tensor. There's a continous set of duality transformation, of which hodge is 1 special case for $\theta = \frac{\pi}{2}$ (Zee, Penrose)

$$\tilde{F} \to e^{i\theta}\tilde{F} \tag{11}$$

E, B rotate into each other (literally) under this transformation.

§2 Solving the Equations

§2.1 Klein gordon

Solution:

$$e^{ikx} = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

In fourier space:

$$k_\mu k^\mu - m^2 = 0 \to -\omega^2 + |\mathbf{k}|^2 + m^2 = 0$$

Green function:

$$G(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon} = \frac{1}{(2\pi)^4} \int \mathrm{d}^3 k \mathrm{d}\omega \frac{e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}}{\omega^2 - |\mathbf{k}|^2 - m^2} \tag{12}$$

$$\text{poles at } \omega = \pm\omega_k \to = \frac{1}{(2\pi)^d} \int d^{d-1}k \frac{1}{2 \underbrace{\omega_k}_{\sqrt{|\mathbf{k}|^2 + \mathbf{m}^2}}} \left(e^{-i\omega_k t}\theta(t) + e^{i\omega_k t}\theta(-t)\right) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{13}$$

$$= \frac{1}{(2\pi)^d} \int d^{d-1}k \frac{1}{2\omega_k} \exp(-i\omega_k |t| - i\mathbf{k}\cdot\mathbf{r}) \tag{14}$$

Volume element in spherical coordinate:

$$dV_d = r^{d-1} \underbrace{\sin(\theta_1) ... \sin(\theta_{d-2})}_{d-2\text{terms}} \cos(\phi) dr d\phi d\theta_1 ... d\theta_{d-2}$$

$$G(x) = \frac{S_d}{(2\pi)^d} \int_0^\infty dk k^{d-1} \int_{-\pi}^{\pi} d\theta \sin(\theta)^{d-2} \exp\left(i\omega_k |t| - ikr\cos(\theta)\right) \tag{15}$$

$$= \frac{1}{2(2\pi)^{\frac{d}{2}} r^{\frac{d}{2}-1}} \int_0^\infty dk k^{d-1} \underbrace{J^1_{\frac{d}{2}-1}}_{\text{bessel}} \exp\left(-i\omega_k |t| - ikr\right) \tag{16}$$

For the full general derivation, see `https://arxiv.org/pdf/0811.1261.pdf`

Point source. If the point source is $J(r,t) = \delta^{d-1}(r)$ and time independent, it reduces to the hemholtz equation.

$$(-\nabla^2 + m^2)\phi = \delta^{d-1}(r) \tag{17}$$

2 ways to solve:

- Direct fourier:

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2+m^2} \tag{18}$$

  in $d = 3$, it's simple

$$\phi(x) = \frac{2\pi}{(2\pi)^3} \int k^2 dk \int_{-1}^{1} d(\cos(\theta)) \frac{e^{-ikr\cos(\theta)}}{k^2+m^2} \tag{19}$$

$$= \int_0^\infty \frac{kdk}{(2\pi^2)} \frac{\sin(kr)}{k^2+m^2} \tag{20}$$

$$\text{residue at } \pm im = \frac{e^{-mr}}{4\pi r} \tag{21}$$

- The laplace operator in d dimension is in arbitrary curvilinear coordinate:

$$\text{laplace beltrami: } \Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial \xi^j} \right) \tag{22}$$

  For spherical symmetric:

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left(r^{d-1} \frac{\partial}{\partial r} \right) + \text{angular part} \rightarrow \tag{23}$$