QFT for gifted amateur probelms

2023

9 Problems

9.1 Problem 2.3

$$
\begin{aligned}
x_j &\equiv \frac{1}{\sqrt{N}} \sum_k e^{ijka} \tilde{x}_k && (1) \\
&= \frac{1}{\sqrt{N}} \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} e^{ijka} \left(a_k + a_k^\dagger \right) && (2) \\
&= \frac{1}{\sqrt{N}} \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} \left(a_k e^{ijka} + \underbrace{a_{-k}^\dagger e^{-ijka}}_{\text{swap k} \to \text{-k}} \right) && (3)
\end{aligned}
$$

9.2 Problem 3.3

First we will calculate $\frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2m}p_1^2$:

$$
\begin{aligned}
x_1 &\equiv \sqrt{\frac{\hbar}{2m\omega}}(a_1 + a_1^\dagger) && (4) \\
p_1 &\equiv -i\sqrt{\frac{m\omega\hbar}{2}}(a_1 - a_1^\dagger) && (5) \\
\frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2m}p_1^2 &= \hbar\omega(\frac{1}{2}(a_1 a_1^\dagger + a_1^\dagger a_1)) && (6) \\
&= \hbar\omega \left(a_1^\dagger a_1 + \frac{1}{2} \right) && (7)
\end{aligned}
$$

Since the "1" above is just label, it follows this must be true for "2" and "3" as well so that:

$$
\hat{H} = \hbar\omega \sum_{n=1}^{3} \left(a_n^\dagger a_n + \frac{1}{2} \right) \tag{8}
$$

To prove the next step, first note that $[b_0, b_{\text{anything else}}] = 0$ because a_3 commutes with a_2, a_1. Also note $[b_0, b_0^\dagger] = 1$.

Therefore, $[b_i, b_j^\dagger] = \delta_{ij}$ holds for i = 0.

For $i = -1, +1$, we only have to check, the following 3 calculations:

Calculation 1:

$$[b_{-1}, b_{+1}^\dagger] = -\frac{1}{2}[a_1 + ia_2, a_1^\dagger + ia_2^\dagger] \tag{9}$$

$$= \frac{1}{2}\left([a_1, a_1^\dagger] - [a_2, a_2^\dagger]\right) = 0 \tag{10}$$

Calculation 2:

$$[b_{-1}, b_{-1}^\dagger] = \frac{1}{2}[a_1 + ia_2, a_1^\dagger - ia_2^\dagger] \tag{11}$$

$$= \frac{1}{2}\left([a_1, a_1^\dagger] + [a_2, a_2^\dagger]\right) = 1 \tag{12}$$

Calculation 3:

$$[b_{+1}, b_{+1}^\dagger] = \frac{1}{2}[a_1 - ia_2, a_1^\dagger + ia_2^\dagger] \tag{13}$$

$$= \frac{1}{2}\left([a_1, a_1^\dagger] + [a_2, a_2^\dagger]\right) = 1 \tag{14}$$

We now calculate the claimed hamiltonian:

$$\hbar\omega \sum_m \left(b_m^\dagger b_m + \frac{1}{2}\right) = \hbar\omega(a_3^\dagger a_3) + \frac{3}{2}\hbar\omega \times \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + \text{cross terms}) + \hbar\omega \times \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + \text{cross terms}) \tag{15}$$

$$= \hbar\omega \sum_n (a_n^\dagger a_n + \frac{1}{2}) \tag{16}$$

To check the 2nd claim, we note that the $m = 0$ term goes away, while the $m = 1$ and $m = -1$ term subtract. This means we keep the cross terms from the previous calculation:

$$\hbar \sum_{m=-1}^{+1} m b_m^\dagger b_m = \hbar(b_{+1}^\dagger b_{+1} - b_{-1}^\dagger b_{-1}) \tag{17}$$

$$= -i\hbar\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) = \hat{L}^3 \tag{18}$$

Commentary

This problem illustrates a special case of the more general symmetry of the 3-d harmonic oscillator hamiltonian under U(3) group:

$$\mathbf{a} \equiv (a_1, a_2, a_3) \tag{19}$$

$$\to \mathbf{Ua} \to \hat{H} \to \hat{H} \tag{20}$$

9.3 Problem 5.4

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} \approx -mc^2\left(1-\frac{v^2}{2c^2}\right) \approx -mc^2+\frac{1}{2}mv^2 \tag{21}$$

$$\begin{aligned} p &\equiv \frac{\partial L}{\partial \dot{q}} && (22)\\ &= mv && (23)\end{aligned}$$

$$\begin{aligned} H &\equiv p\dot{q} - L && (24)\\ &= mv^2 - (-mc^2+\frac{1}{2}mv^2) = mc^2+\frac{1}{2}mv^2 && (25)\end{aligned}$$

9.4 Problem 5.8

Note that the electromagnetic field tensor, doing $\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ basically swaps $E \to -B$ and $B \to -E$.

Then we are asked to compute $F_{\alpha\beta}\left(\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\right)$ which is just the sum of all the matrix entries element wise multiplied, which gives: $\propto \sum_{i=1}^{3} E_i(-B_i) \propto E \cdot B$

This $E \cdot B$ term is topological in nature. The way to see it is in form notation, it can be written as $F \wedge F$. Its integral over the manifold gives the winding number of the gauge field configuration and is a topological invariant. It's integral over a closed manifold gives an integer (up to factors of pi and 2).

Another comment: this term obviously violates parity (by nature of having the ϵ tensor).

9.5 Problem 5.9

Consider the equation: $\partial_\mu F^{\mu\nu} = J^\nu$ for $\nu = 0$

This gives:

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = \rho \text{ (Gauss's Law)} \tag{26}$$

Let's work out the same equation for $\nu = 1$ (x) component:

$$\partial_t E_x + \partial_y B_z - \partial_z B_y = J_x \tag{27}$$
$$\partial_t E_y - \partial_x B_z + \partial_z B_x = J_y \tag{28}$$
$$\text{et cetera} \tag{29}$$
$$\rightarrow \nabla \times \mathbf{B} + \partial_t \mathbf{E} = \mathbf{J} \tag{30}$$

The bianchi's identity below:

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \tag{31}$$

follows straight from the definition of F:

$$F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu \tag{32}$$

A way to see it is that an exact form is necessarily closed: $F = dA \rightarrow dF = 0$.

In particular for the spatial components $\mu, \nu, \lambda = 1, 2, 3$ it implies:

$$\nabla \cdot \mathbf{B} = 0 \tag{33}$$

which is one of maxwell's equations (no magnetic monopoles).

For the combination $\mu, \nu, \lambda = 0, (1 \rightarrow 3), (1 \rightarrow 3)$ where $\nu \neq \lambda$, we can for example use the case $0, 1, 3$ to get the following equation:

$$\partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = 0 \tag{34}$$
$$\partial_t B_y - \partial_x E_z + \partial_z E_x = 0 \tag{35}$$
$$\tag{36}$$

We can work out the 2 other cases for B_x, B_z to get Faraday's law:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0. \tag{37}$$

9.6 Problem 5.10

F is a anti-symmetric object. It is contracted with symmetric tensor $\partial_\alpha \partial_\beta$ which implies that the result is 0 (this is because for every term, there is a term of opposite sign).

The continuity equation is a statement about conservation of charge:

$$\partial_t \rho = -\nabla \cdot \mathbf{J} \rightarrow \frac{d}{dt} \underbrace{\int_M \rho dV}_{\text{charge in region "M"}} = - \underbrace{\int_{\partial M} \mathbf{J} \cdot d\mathbf{A}}_{\text{current flowing out of the boundary of } M} \tag{38}$$

9.7 Problem 8.2

Set $\hbar = 1$ for simplicity.

$$a_k^\dagger(t) = U^{-1} a_k^\dagger(0) U \tag{39}$$

$$= \exp(iHt) a_k^\dagger \exp(-iHt) \tag{40}$$

$$= \exp(i \sum_{k'} E_{k'} \hat{N}_{k'} t) a^\dagger \exp(-i \sum_{k'} E_{k'} \hat{N}_{k'} t) \tag{41}$$

Since the operators commute for $k' \neq k$, we only need to consider the term where k' = k.

We invoke the BCH formula to evaluate:

$$a_k^\dagger(t) = a_k^\dagger(0) + [itE_k N_k, a_k(0)] + \frac{1}{2}[itE_k N_k, [itE_k N_k, a_k(0)]] + ... \tag{42}$$

$$= a_k^\dagger(0) \times \left(\sum_{n=0}^{\infty} \frac{(itE_k)^n}{n!} \right) \tag{43}$$

$$= a_k^\dagger(0) \times \exp(iE_k t) \tag{44}$$

It follows

$$a_k(t) = a_k(0) \exp(-iE_k t) \tag{45}$$

9.8 Problem 9.1

$$\hat{U} = e^{-i\hat{p}a} \rightarrow \frac{\partial}{\partial a} \hat{U}|_{a=0} = i\hat{p} \underbrace{U(0)}_{1} \rightarrow \hat{p} = \frac{-1}{i} \frac{\partial \hat{U}}{\partial a} \tag{46}$$

9.9 Problem 9.2

$$\Lambda(\phi_1) = \begin{pmatrix} \cosh(\phi) & \sinh(\phi) & 0 & 0 \\ \sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial \phi_1}|_{\phi_1=0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{47}$$

The other generators are obtained trivially by plugging "1" into the location Λ^0_j and Λ^j_0 with $j = 1, 2, 3$. Add the $\frac{1}{i}$ to get the answer. Note how the generator are anti-hermitian (which makes the lie group non-compact).

9.10 Problem 9.4

$$\begin{aligned}
\omega^\mu_\nu x^\nu \partial_\mu &= \omega_{\mu\nu} x^\nu \partial^\mu &(48)\\
&= \underbrace{\frac{1}{2}\omega_{\mu\nu} x^\nu \partial^\mu + \frac{1}{2}\omega_{\mu\nu} x^\nu \partial^\mu}_{\text{split in 2}} &(49)\\
&= \frac{1}{2}\omega_{\mu\nu} x^\nu \partial^\mu \underbrace{-\frac{1}{2}\omega_{\nu\mu}}_{\text{anti-symmetry}} x^\nu \partial^\mu &(50)\\
&= \frac{1}{2}\omega_{\mu\nu}\left(x^\nu \partial^\mu - \underbrace{x^\mu \partial^\nu}_{\text{swap index}}\right) &(51)
\end{aligned}$$

$$\begin{aligned}
f(x') &= 1 + a^\mu \partial_\mu f(x) + \frac{1}{2}\omega_{\mu\nu}\left(x^\nu \partial^\mu - x^\mu \partial^\nu\right) &(52)\\
&= 1 + ia^\mu \underbrace{(i\partial_\mu)}_{p_\mu} f(x) + \frac{i}{2}\omega_{\mu\nu} \times \underbrace{-i\left(x^\nu \partial^\mu - x^\mu \partial^\nu\right)}_{M^{\mu\nu}} f(x) &(53)
\end{aligned}$$

Because $\frac{1}{2}M^{\mu\nu}$ is the generator of lorentz transformation, the equation follows:

$$\Lambda = \exp(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}) \tag{54}$$

9.11 10.1