# Physics problems, Part I

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November 2021

Here is a collected notes on physics readings.

## Contents

# 1 Basic Calculations

### 1.0.1 How to compute the effective action (Zee, Peskin)

We'll compute Γ in 2 step, first compute W(J). For this calculation, we will do it in minkowski time and re-introduce $\hbar$ to make clear expansion around a classical solution. From the definition,

$$e^{\frac{i}{\hbar}W(J)} = \int d\phi \exp(S[\phi]] + \int dx J\phi) \tag{1}$$

We first expand via the saddle point (classical solution) called $\phi_0$, which satisfies the classical equations of motion $\frac{\delta S}{\delta\phi}|_{\phi_0} = -J$. Shifting the field variables as a fluctuation around the classical solution, we define $\tilde{\phi} = \phi - \phi_0$:

$$e^{\frac{i}{\hbar}W(J)} \approx e^{\frac{i}{\hbar}(S[\phi_0]+\int dx J\phi_0)} \int d\tilde{\phi} e^{\tilde{\phi}D^{-1}\tilde{\phi}} \tag{2}$$

$$\tag{3}$$

Taking the log of both sides:

$$W(J) \approx S[\phi_0] + \int dx J\phi_0 + \frac{\hbar}{2i}\mathrm{Tr}\ln(D^{-1}) \tag{4}$$

Γ just removes the extraneous current (tadpoles!), giving

$$\Gamma = \underbrace{S[\phi_0]}_{\text{Classical Action}} + \frac{\hbar}{2i}\underbrace{\mathrm{Tr}\ln(D^{-1})}_{\text{Quantum correction}} + ... \tag{5}$$

We expect the quantum action to be extensive. For the case of translation invariant arguments, we expect:

$$\Gamma[\phi] \equiv VT \times V_{eff}(\phi) \tag{6}$$

**Example 1.1**

*Compute to first order the correction to the $V_{eff}$ for the following lagrangian (Zee section IV.3)*

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4$$

To 0th order, the quantum action is just the classical action. For constant field, the effective potential is just the classical potential:

$$V_{eff} = -\frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$$

To first order in, we need to compute the quantum correction. First, the potential energy density can be rewritten as:

$$\int dx\mathcal{L} = \int dxdy\phi(x)\underbrace{[-\partial^2 - V''(\phi)]\delta(x-y)}_{D^{-1}}\phi(y)$$

The operator D is just the propagator for a free theory with a different mass, and is diagonal in k space:

$$\mathrm{Tr}\ln(D^{-1}) = \int d^4x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\log(-k^2 + m'^2) = VTI(d) \tag{7}$$

This integral is divergent, but we can evaluate up to a cutoff Λ:

$$I(d) = \frac{\Lambda^2}{32\pi^2}V''(\phi) - \frac{V''(\phi)^2}{64\pi^2}\log\left(\frac{e^{\frac{1}{2}}\Lambda^2}{V''(\phi)}\right) \tag{8}$$

The cutoff dependent terms will be absorbed by renormalization conditions. However, the part that is not cutoff dependent, shows as a log correction to the potential:

$$V_{eff}(\phi) = A\phi^2 + B\phi^4(D + E\log(\frac{\phi^2}{\Lambda^2})) \tag{9}$$

*Comments:* We see the effective potential will be deeper near the classical minima (see Peskin 378). Note peskin does the higher dimensional version of this problem, which is why it looks a bit obscure.

The way the coefficients A, B, D, E are set are via renormalization conditions (lab measurements). There are 2 renormalization conditions (2 coupling constants in the bare lagrangian, the mass $m$ and the interaction $g$)

- We measure in lab the mass of the particle to be $m_P$ where P stands for physical. This sets:

$$\left.\frac{d^2V_{eff}}{d\phi^2}\right|_{\phi=0} = m_P^2$$

- We measure the strength of the interaction at a given scale $M$ to be some coupling $g_P$. This sets:

$$\left.\frac{d^4V_{eff}}{d\phi^4}\right|_{\phi=M} = g_P$$

  Question: Why can't we measure it at $\phi = 0$? What is the meaning of this scale $M$ we picked?

## 2 The Renormalization Group

### 2.1 Emergent symmetry (Peskin 12.3)

**Problem 2.1.** Consider a lagrangian with 2 massless scalar fields $\phi_1$, $\phi_2$:

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + (\partial\phi_2)^2 + \frac{\lambda}{4!}\left(\phi_1^2 + \phi_2^2\right) + \frac{\lambda}{4!}2\rho\phi_1^2\phi_2^2$$

a) Compute $\beta_\lambda, \beta_\rho$ b) How does the ratio $\frac{\rho}{\lambda}$ flow under RG? c) Investigate the stability of the flow in $4-\epsilon$ dimensions.

First let's recall some feynman diagram combinatoric. The fundamental rules comes from wick contracting an expression of the form:

$$\langle ...\phi\phi\phi\phi...\rangle$$

Naively, we'd say each $\phi^n$ interaction vertex has $n!$ permutation that are indistinguishable, hence we associate a $\frac{1}{n!}\phi^n$ term to cancel this combinatorial factor. However some feynman diagrams with symmetry represent fewer permutations by a **symmetry** factor. The symmetry factor is the order of the group of automorphisms of the feynman graph.

Note when $\lambda = \rho$, this lagrangian has $O(2)$ symmetry Rewrite the lagrangian with the correct combinatorial factor:

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + (\partial\phi_2)^2 + \frac{\lambda}{4!}\left(\phi_1^2 + \phi_2^2\right) + \frac{\lambda}{2!2!}\frac{\rho}{3}\phi_1^2\phi_2^2 \tag{10}$$

We can therefore see the vertex are $-i\Lambda$ and $-i\frac{\rho}{3}$. The wavefunction renormalization is 0 for both $\phi_1$ and $\phi_2$ (by symmetry). The interaction renormalizes via the 4 point 1PI.

- divergent part of $\Gamma^{(4)}_\lambda$:

$$\frac{i}{(4\pi)^2}\underbrace{\frac{1}{2}}_{\text{symmetry factor}} \times \underbrace{3}_{s,t,u\text{channels}} \times \left(\lambda^2 + (\frac{\rho}{3})^2\right) \times \underbrace{\frac{2}{\epsilon}}_{d=4-\epsilon}$$

$$\to \delta_\lambda = \frac{3\lambda^2 + \frac{\rho^2}{3}}{(4\pi)^1}\frac{2}{\epsilon}$$

- divergent part of $\Gamma^{(4)}_\rho$: first diagram:

$$\frac{i}{(4\pi)^2}\underbrace{\frac{1}{2}}_{\text{symmetry}} \times \underbrace{2}_{\text{diagrams}} \times \frac{\rho\lambda}{3} \times \frac{2}{\epsilon}$$

  2nd diagram:

$$\frac{i}{(4\pi)^2}\underbrace{1}_{\text{symmetry}} \times \underbrace{2}_{\text{diagrams}} \times \frac{\rho^2}{9} \times \frac{2}{\epsilon}$$

  Because the $\delta_\rho$ counter term is divided by 3, we need to multiply these amplitudes by 3X to obtain the counterterm value.

$$\to \delta_\rho = \frac{2\lambda\rho + \frac{4}{3}\rho^2}{(4\pi)^2}\frac{2}{\epsilon}$$

This gives

$$\beta_\lambda = \frac{3\lambda^2 + \rho^2}{(4\pi)^2} \tag{11}$$

$$\beta_\rho = \frac{2\lambda\rho + \frac{4}{3}\rho^2}{(4\pi)^2} \tag{12}$$

Define $x \equiv \frac{\rho}{\lambda}$, denote $\frac{d}{d\ln\mu}x \equiv x'$

$$x' = \frac{\rho\lambda' - \lambda\rho'}{\lambda^2} \tag{13}$$

$$= \rho(3 + \frac{1}{3}x^2) - \lambda(2x + \frac{4}{3}x^2) \tag{14}$$

$$= \frac{\lambda x}{3}(x-3)(x-1) \tag{15}$$

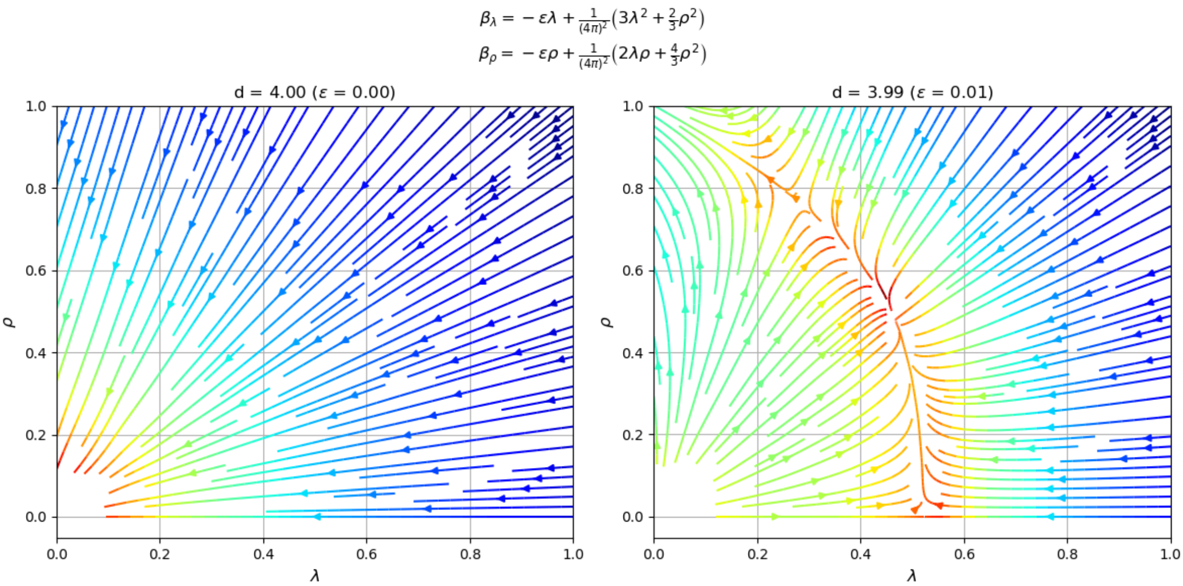
The flow is stable at $x = 1$ ($\beta_x$ has a negative crossing) which means that there is an **emergent** $O(2)$ symmetry with flowing towards $\lambda = \rho$.

If we use $d = 4-\epsilon$, the beta functions just take a $(d-D)$coupling term in dim-reg from $\mu^{4-\epsilon}$.

$$\beta_\lambda = -\epsilon\lambda + \frac{3\lambda^2 + \rho^2}{(4\pi)^2} \tag{16}$$

$$\beta_\rho = -\epsilon\rho + \frac{2\lambda\rho + \frac{4}{3}\rho^2}{(4\pi)^2} \tag{17}$$

We plot below the RG flow 4 dimensions and slightly less. Note that the flow direction vector is $-(\beta_\lambda, \beta_\rho)$ since we are interested in flowing towards the IR ($\mu \to 0$)!




Solving for $\beta_\lambda = \beta_\rho = 0$ shows there are now 4 solutions, but only the $\rho = \lambda$ solution is an attractive fixed point.

### 2.2 Covariance of beta function

*Zinn Justin, section 10.11, MIT 8.324 pset 6.* For classically scale invariant theories, the beta function only flows at second order of the coupling. This is because only loop diagrams lead to a broken scale invariance.

$$\beta(\lambda) = b_2\lambda^2 + b_3\lambda^3 + .... \tag{18}$$

We can show that under smooth change $\lambda' = \lambda + a_2\lambda^2 + ...$, the 2nd and 3rd order coefficients of the beta function are univeral. In general under coupling rescaling:

$$\beta'_j(\lambda') = \beta_i \frac{\partial \lambda'_j}{\partial \lambda_i} \text{ general reparametrization} \tag{19}$$

$$\lambda(\lambda') \approx \lambda' - a_2\lambda'^2 + \mathcal{O}(\lambda'^3) \tag{20}$$

$$\beta'(\lambda') = (b_2(\lambda' - a_2\lambda'^2 + ...)^2 + b_3(\lambda' - a_2\lambda' + ...)^2)(1 + 2a_2\lambda' + ...) \tag{21}$$

$$= b_2\lambda'^2 + b_3\lambda'^3 + \lambda'^3(-2a_2b_2 - 2b_3a_2 + 2a_2(b_2 + b_3)) + \mathcal{O}(\lambda'^4) \tag{22}$$

$$= b_2\lambda'^2 + b_3\lambda'^3 + \mathcal{O}(\lambda'^4) \tag{23}$$

For a marginally relevant interaction like QCD, we have

$$\beta = -b_2g^2 - b_3g^3 + ... \tag{24}$$

$$\to \ln\left(\frac{\mu}{\Lambda}\right) = \frac{1}{g(\mu)} + \frac{c}{b^2}\ln\left(bg(\mu)\right) + \mathcal{O}(g) \tag{25}$$

$$\tag{26}$$

This is an example of **dimensional transmutation**, where a new scale $\Lambda$ appears in a classically scale invariant theory. For QED, it's the landau pole scale where the theory blows up. For QCD, it's a low energy scale where perturbation theory breaks down in the IR.

### 2.3 $\phi^3$ RG, D=6

**Problem 2.2.** (MIT 8.324, pset 4 problem 2) Compute the vertex correction to $\phi^3$ theory at 1 loop using dimensional regularization. Evaluate it in the limit $p_3^2 >> m^2, p_1^2, p_2^2$

One question is why is $\phi^4$ marginally irrelevant in 4d while $\phi^3$ in 6d is marginally relevant?

The key to this is to recognize the key quantum correction to the vertex at 1 loop. $\phi^4$ at 1 loop has 2 propagators and 2 vertices, giving a net positive sign. This makes the coupling grow stronger due to quantum correction. $\phi^3$ has 3 propagators and 3 vertices, giving a net negative sign. This makes the quantum correction grow weaker.

To illustrate this let's compute the vertex correction of $\phi^3$:

$$\Gamma^3(p_1,p_2,p_3) = \underbrace{-ig}_{\text{tree level}} \underbrace{-i\mathcal{C}}_{\text{counterterm}} + \underbrace{(-ig)^3}_{\text{vertices}} \underbrace{(i)^3}_{\text{propagators}} \mu^{6-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k-p_1)^2-m^2} \frac{1}{(k+p_2)^2-m^2} \frac{1}{k^2+m^2} \tag{27}$$

Wick rotate $k_0 \to ik_0$ and we get (keep track of the factors of i's and signs!)

$$\Gamma^3(p_1,p_2,p_3) = -ig - i\mathcal{C} - i\mu^{6-d}g^3 \underbrace{\int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{1}{(k-p_1)^2+m^2} \frac{1}{(k+p_2)^2+m^2} \frac{1}{k^2+m^2}}_{\mathcal{A}} \tag{28}$$

- Then do feynman parameter

$$\frac{1}{ABC} = 2\int dxdydz\delta(x+y+z-1)\frac{1}{x(A+yB+zC)^3} \tag{29}$$

$$\mathcal{A} = \int dxdydz\delta(...) \times 2\int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{1}{(xA+yB+zC)^3} \tag{30}$$

- Simplify the denominator by completing square:

$$xA+yB+zC = l^2+\Delta \tag{31}$$

$$l \equiv k + p_2z - p_1y \text{ (shift integration variable)} \tag{32}$$

$$\Delta \equiv m^2 + yp_1^2 + zp_2^2 - (p_2z-p_1y)^2 \tag{33}$$

$$= m^2 + y(1-y)p_1^2 + z(1-z)p_2^2 + 2\underbrace{p_{12}}_{p_1\cdot p_2}yz \tag{34}$$

$$\Delta = m^2 + xyp_1^2 + xzp_2^2 + yzp_3^3 \text{ (use momentum conservation): } 2p_{12} \quad = p_3^2 - p_2^2 - p_1^2 \tag{35}$$

$$\tag{36}$$

- Integrate the momenta, giving you divergent term (the physical stuff) in terms of gamma functions

$$\mathcal{A} = 2\int_0^1 dxdydz\delta(x+y+z-1)\frac{1}{2(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{(3-\frac{d}{2})}}\Gamma\left(\frac{6-d}{2}\right) \tag{37}$$

$$\to d = 6-\epsilon \tag{38}$$

$$\Gamma^{(3)}(p_1,p_2,p_3) = -ig - i\mathcal{C} - ig^3\frac{1}{(4\pi)^3}\underbrace{\left(1+\frac{\epsilon}{2}\ln\mu^2\right)}_{\mu^\epsilon}\int dxdydz\delta(..)\underbrace{\left(1-\frac{\epsilon}{2}\ln\Delta\right)}_{\Delta^{\frac{-\epsilon}{2}}}\underbrace{\left(\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)\right)}_{\Gamma(\frac{\epsilon}{2})} \tag{39}$$

$$\text{Collect terms of order } \frac{1}{\epsilon} \text{ and order 1} \tag{40}$$

$$= ... - ig^3\frac{1}{(4\pi)^3}\int dxdydz\delta(...)\left(\frac{2}{\epsilon}+\ln\left(\frac{4\pi\mu^2e^{-\gamma_E}}{\Delta}\right)\right) \tag{41}$$

We can already see several important physics things. Quantum corrections give $-\ln(\Delta)$ to the coupling, which means the theory is marginally irrelevant.

This is clearer by imposing lab measurement of $\Gamma(0,0,0) = -ig$. This fixes the counter term and gives a vertex of:

$$\Gamma^{(3)}(p_1,p_2,p_3) = -ig\left[1\underbrace{-}_{\text{asymptotic freedom}}g^2\frac{1}{(4\pi)^3}\int_0^1 dxdydz\delta(x+y+z-1)\ln\left(\frac{\Delta}{m^2}\right)\right] \tag{42}$$

In the large momentum limit where $p_3^2 >> m^2, p_1^2, p_2^2$ we can make some simplifications to $\Delta \approx yzp_3^2$:

$$\Gamma^{(3)}(p_3^2) \approx -ig\left[1-\frac{g^2}{2(4\pi)^3}\left(\ln\left(\frac{p_3^2}{m^2}+O(1)\right)\right)\right] \tag{43}$$

## 2.4 Asymptotic Freedom, $\phi^3$

**Problem 2.3.** (MIT 8.324, pset 7 prob 2) Using previous result, compute the beta function for the coupling for $\phi^3$ theory in $d = 6-\epsilon$

### 2.5 Srednicki Problem 29.2

The propagator at different scales must agree with each other.

$$\frac{1}{Z(\Lambda_0) - \underbrace{\Sigma(p^2, \Lambda_0, \Lambda)}_{\text{self energy integrated over}\delta\Lambda}} = \frac{1}{Z(\Lambda)} \tag{44}$$

Simplifying this gives:

$$Z(\Lambda) = Z(\Lambda_0)(1 - \frac{\Sigma(p^2)}{Z(\Lambda_0)}) \tag{45}$$

$\Sigma(p^2)$ is the 1 loop correction to the propagator from fast modes between $\Lambda \to \Lambda_0$. It admits an expansion in the momentum

$$\frac{\Sigma(p^2)}{Z(\Lambda_0)} = \frac{1}{2}(ig)^2 \int_{\Lambda,\Lambda_0} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{i}{k^2} \frac{i}{(k-p)^2} \tag{46}$$

$$\equiv \frac{g^2}{2} F(p^2, \Lambda, \Lambda_0) \tag{47}$$

$$F(p^2) \approx \cancel{A} + \frac{d}{d(p^2)} F(p^2) p^2 + O(p^4) \tag{48}$$

The constant value A is a mass shift and cancelled by counterterms (since the theory is renormalized to be massless). The remaining terms give a series expansion in $p^2$ to $Z(\Lambda)$

$$Z(\Lambda) = Z(\Lambda_0)\left(1 - \frac{g^2}{2} F'(p^2, \Lambda, \Lambda_0)\right) \tag{49}$$

The correction to the vertex includes both the wavefunction renormalization and the 3 point vertex loop.

$$Z(\Lambda)^{\frac{3}{2}} g(\Lambda) = Z(\Lambda_0)^{\frac{3}{2}} g(\Lambda_0)\left(1 + (-ig(\Lambda))^3 i^3 \int_{\delta\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2(k-p_1)^2(k+p_2)^2}\right) \tag{50}$$

$$g(\Lambda) \approx \left(\frac{Z(\Lambda_0)}{Z(\Lambda)}\right)^{\frac{3}{2}} g(\Lambda_0)\left(1 + g(\Lambda_0)^2 \int_{\Lambda}^{\Lambda_0} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^6} + \ldots\right) \tag{51}$$

b) To compute the beta function, we can evaluation equation (49) directly:

$$F(p^2, \Lambda, \Lambda_0) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{(x(k+p)^2 + (1-x)k^2)^2} \tag{52}$$

$$= \int \frac{d^d q}{(2\pi)^d} \int dx \frac{1}{(q^2 + x(1-x)p^2)^2} \tag{53}$$

$$\frac{d}{d(p)^2} F(p^2 = 0) = -2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^6} \int_0^1 dx x(1-x) \tag{54}$$

$$= -\frac{1}{3} \frac{\Omega_d}{(2\pi)^d} \int_{\Lambda}^{\Lambda_0} dq q^{d-7} \tag{55}$$

$$=_{d=6} -\frac{1}{3} \frac{1}{(4\pi)^3} \ln \frac{\Lambda_0}{\Lambda} \tag{56}$$

$$Z(\Lambda) = Z(\Lambda_0)\left(1 + \frac{1}{6} \frac{g^2}{(4\pi)^3} \ln \frac{\Lambda_0}{\Lambda}\right) \tag{57}$$

$$\to -2\gamma_\phi \equiv \frac{d}{d\ln(\Lambda)} Z(\Lambda) = -\frac{1}{6} \frac{g^2}{(4\pi)^3} \tag{58}$$

The second equation (51) can also be evaluated directly:

$$g(\Lambda) \approx_{d=6} \left(\frac{Z(\Lambda_0)}{Z(\Lambda)}\right)^{\frac{3}{2}} g(\Lambda_0)\left(1 + g(\Lambda_0)^2 \frac{\Omega_d}{(2\pi)^d} \ln \frac{\Lambda_0}{\Lambda}\right) \tag{59}$$

$$\frac{d}{d\ln\Lambda} g(\Lambda) = g\left(\underbrace{-\frac{3}{2}\frac{dZ}{d\ln\Lambda}}_{=3\gamma_\phi} - g^2 \frac{\Omega_d}{(2\pi)^d}\right) \tag{60}$$

$$\beta_g = -\frac{3}{4} \frac{g^3}{(4\pi)^3} \tag{61}$$

Note how in $\phi^3$, quantum correction to the anomalous dimension $\gamma_\phi$ and the vertex interaction compete to give asymptotic freedom. This cancellation is highly non-trivial so it doesn't seem like one can easily see the asymptotic freedom behavior. This is unlike $\phi^4$ where at one loop the anomalous dimension is 0 so one can directly see the main contributor to the vertex is the 1 loop 4 point vertex 1PI diagram (which increases the coupling).

## 2.6 CPN model (Peskin 13.3)

The **complex projective space** $\mathbb{CP}^N$ is basically the space of complex N-dimensional rays.
By that it means that:

- The space is made of vectors $z = (z_1, ..., z_N, z_{N+1})$ with constraint $\sum_k |z_k|^2 = 1$.
- gauge equivalence: $e^{i\alpha} z \to z$

a) We are asked to show the lagrangian proposed is invariant under gauge transformation $e^{i\alpha}$.
Note under gauge transformation, $\partial_\mu \to \partial_\mu + i\partial_\mu \alpha$

$$\tilde{z}_j = z_j e^{i\alpha} \tag{62}$$

$$\mathcal{L} = \frac{1}{g^2}\left[|\partial_\mu \tilde{z}_j|^2 - |z_j^* \partial_\mu \tilde{z}_j|^2\right] \tag{63}$$

$$= \frac{1}{g^2}\left[e^{i\alpha}\left(|\partial_\mu z_j + i\partial_\mu \alpha\right)|^2 - |z_j^* e^{-i\alpha} e^{i\alpha}\left(\partial_\mu z_j + i\partial_\mu \alpha\right)|^2\right] \tag{64}$$

$$= \frac{1}{g^2}\left[|\partial_\mu z|^2 + |\alpha'|^2 \cancelto{1}{|z|^2} + 2\mathrm{Re}(i\alpha' z^* \partial_\mu z) - |\alpha'|^2 \cancelto{1}{|z|^2} - |z^* \partial_\mu z|^2 - 2\mathrm{Re}(i\alpha' z^* \partial_\mu z)\right] \tag{65}$$

$$= \frac{1}{g^2}\left[|\partial_\mu \tilde{z}_j|^2 - |z_j^* \partial_\mu \tilde{z}_j|^2\right] \tag{66}$$

We are asked to show that in N = 1 (2-d complex space), we can map to the O(3) nonlinear sigma model with $n^i = z^* \sigma^i z$ First note that $n^i$ is real because $\sigma_i$ is hermitian.
To do so, expand the O(3) nonlinear sigma model:

$$\mathcal{L} = \frac{1}{2g^2}(\partial_\mu (z^* \sigma^i z))^2 \tag{67}$$

$$= \frac{1}{2g^2}|z^{*\prime}\sigma_i z + z^* \sigma_i z'|^2 \tag{68}$$

Furthermore, we'll need an identity about SU(N) generators:

$$\sigma^i_{\alpha\beta}\sigma^i_{\mu\nu} = 2(\delta_{\alpha\mu}\delta_{\beta\nu}) - \delta_{\alpha\beta}\delta_{\mu\nu}$$

Horrible index math ensues...
b) We want to show that we can map this problem to one with gauge field $A_\mu$ and lagrange constraint field $\lambda$.

$$Z = \int \mathcal{D}z\mathcal{D}z^*\mathcal{D}A\mathcal{D}\lambda \exp\left[\frac{1}{g^2}\int \mathrm{d}^d x \left(|\partial_\mu z + iA_\mu z|^2 - \lambda\left(|z|^2 - 1\right)\right)\right] \tag{69}$$

$$\underbrace{=}_{\text{integrate out }\lambda} \int \mathcal{D}z\mathcal{D}z^*\mathcal{D}A \prod_x \delta(|z|^2 - 1)\exp\left[\frac{1}{g^2}\int \mathrm{d}^d x\left(|\partial_\mu z|^2 + A_\mu A^\mu |z|^2 + 2iA_\mu z\partial^\mu z^*\right)\right] \tag{70}$$

Integrate $A^\mu$ out gives a gaussian integral with the form $J^\mu = 2iz\partial^\mu z^*$ and kernel $|z|^2$. Integrating out this gives:

$$\int dx e^{-xMx - Jx} = \exp(\frac{1}{2}\mathrm{tr}\ln(M) - \frac{1}{2}JM^{-1}J) \tag{71}$$

$$\to Z = \int \mathcal{D}z\mathcal{D}z^* \prod_x \delta(|z|^2 - 1)\exp\left[-\frac{1}{2g^2}\int \mathrm{d}^d x \ln \cancelto{0}{\frac{1}{|z|^2}} - 2iz\partial_\mu z \cancelto{1}{\frac{1}{|z|^2}}\, 2iz\partial_\mu z^*\right] \tag{72}$$

$$= \int \mathcal{D}z\mathcal{D}z^* \exp\left[\frac{1}{g^2}\int \mathrm{d}^d x(\partial z)^2\right]\prod_x \delta(|z|^2 - 1)\exp\left[-\frac{1}{g^2}\int \mathrm{d}^d x|z\partial_\mu z^*|^2\right] \tag{73}$$

d) The operator that affect $z$ is $-\mathcal{D}^2 - \lambda$ integrating out the field z is just a gaussian integral. Note because $z$ is a complex field, the factor is $N+1$ and not $\frac{N+1}{2}$ (also note in the large N limit $N \approx N+1$)
We get

$$Z = \int \mathcal{D}A^\mu \mathcal{D}\lambda \exp\left[-N\underbrace{\int \mathrm{d}^d x \langle x|\ln D^2 - i\lambda|x\rangle}_{\mathrm{Tr}\ln(D^2 - i\lambda)} + \frac{1}{g^2}\int \mathrm{d}^d x\lambda\right] \equiv e^{-S} \tag{74}$$

In the large N limit on evaluates the integral via saddle point. The saddle point is when both functional derivates with respec to $\lambda, A^\mu$ vanish:

$$\frac{\delta}{\delta\lambda}S = 0 \tag{75}$$

$$\to N\langle x|\frac{1}{-D^2 - i\lambda}|x\rangle = \frac{i}{g^2} \tag{76}$$

$$\tag{77}$$

The LHS of the equation must be constant and complex. This implies that $A, \lambda$ must be constant. Using this ansatz, let's find the minimum for A:

$$\frac{\delta}{\delta A} S = 0 \tag{78}$$

$$\rightarrow \frac{1}{-D^2 - i\lambda} \frac{\delta}{\delta A} (\partial^2 + A^2) = 0 \tag{79}$$

The operator over the top is a positive definite operator that is quadratic in A. It is therefore minimized at A = 0.

With that assumption $D^2 = k^2 + A^2$. We can evaluate the RHS with a cutoff $\Lambda$:

$$\frac{N}{2\pi} \ln \frac{\Lambda}{\sqrt{-\lambda}} = \frac{i}{g_0^2} \tag{80}$$

$$\text{Impose renormalization condition} \rightarrow \frac{1}{g_0^2} = \frac{1}{g^2} + \frac{N}{2\pi} \ln \frac{\Lambda}{M} \tag{81}$$

$$\text{define: } \lambda = m^2 \tag{82}$$

$$\rightarrow m = M \exp(-\frac{2\pi}{g^2 N}) \tag{83}$$