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Twelve Articles on Giuseppe Peano

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All but one of these articles originally appeared in various journals in the years 1963–1984.

Comments and suggestions are welcome. Please write to hubertk@pacbell.net.

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Introduction

For many years my research in the history of mathematics was focused on Giuseppe Peano (1858–1932), the foremost Italian mathematician at the turn of the twentieth century. This culminated in the biography *Peano: Life and Works of Giuseppe Peano* (Dordrecht: D. Reidel, 1980), which I have re-published in a "definitive" edition as an ebook (2002). But there were a number of other shorter publications. Most of them are collected here. Not included are the 31-page booklet *Giuseppe Peano*, published by Birkhäuser Verlag Basel in their series "Kurze Mathematiker-Biographien," which appeared in a German translation by Ruth Amsler in 1974, and—also in 1974—the entry "Giuseppe Peano" in the *Dictionary of Scientific Biography*, edited by C. C. Gillispie (New York: Charles Scribner's Sons), 10: 441–444. The latter will be in a collection of the seven articles I wrote for the *DSB*.

The present articles appeared in a variety of journals and so are inevitably repetitious. The final article in this collection, written in 1983, appears here for the first time. It was written for a Festschrift in honor of Ludovico Geymonat (whose lectures Corrado Mangione and I attended together in 1957–58 at the University of Milan), but was rejected by the editors, Corrado Mangione and Umberto Bottazzini, who spend a complete evening trying to pressure me into writing something else. The article reflects my recent enthusiasm for Max Stirner, which may be what they objected to.

Two of the articles are in Italian. The first was written in English for translation into Italian. The second was originally written in Italian.

For this ebook I have had to use various fonts and to manufacture several symbols that were not in the fonts available to me; this last was especially necessary for the article "Nine Letters from Giuseppe Peano to Bertrand Russell." I was unable to make all of them fit as neatly on the line as they should—and some fonts do not convert well to the PDF format—but I hope their somewhat strange appearance will not make the formulas too difficult to read.

Much of this material was, of course, used in the biography, but in the articles I was able to emphasize certain points. "Peano's concept of number," for example, goes into great detail in one aspect of Peano’s thought. “Nine Letters from Giuseppe Peano to Bertrand Russell” points up the historical importance of the contact between the two. All give evidence that Peano was one of the great mathematicians of the late nineteenth and early twentieth centuries—a figure well worth recalling.

Giuseppe Peano (1858–1932)

Philosophy of Science
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THE MATHEMATICAL PHILOSOPHY OF GIUSEPPE PEANO

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Because Bertrand Russell adopted much of the logical symbolism of Peano, because Russell always had a high regard for the great Italian mathematician, and because Russell held the logicist thesis so strongly, many English-speaking mathematicians have been led to classify Peano as a logicist, or at least as a forerunner of the logicist school. An attempt is made here to deny this by showing that Peano's primary interest was in axiomatics, that he never used the mathematical logic developed by him for the reduction of mathematical concepts to logical concepts, and that, instead, he denied the validity of such a reduction.

An attempt will be made in this brief note to partially answer the question: Did Peano have a philosophy of mathematics? This question is less easy to answer than one might think. On one hand, despite his evident interest in the philosophy of mathematics (shown by attendance at philosophical congresses, etc.) he showed extreme reserve and modesty when questioned about the subject; on the other hand, his express aim was not the development of a philosophy of mathematics, despite the obvious influence of his work on that subject. This influence is first seen in the group of Italian mathematicians who collaborated with him in the publication of the several editions of the *Formulaire de mathématiques*, among them his assistants at the University of Turin, F. Castellano, G. Vailati, and C. Burali-Forti. The last is remembered as the discoverer in 1897 of the first important antinomy in Cantor's set theory [2], though the English-speaking mathematicians remained generally unaware of such a contradiction until B. Russell's publication in 1903 of similar antinomies [21]. It is interesting to note that Russell's own interest in this subject, an interest that consequently led, with the collaboration of A. N. Whitehead, to the publication of the famous *Principia Mathematica*, was aroused by Peano in 1897 at the First International Congress of Mathematicians (Zurich), where the young Russell heard Peano present Section I from Tome II of the *Formulaire de mathématiques* (i.e. the part pertaining to mathematical logic.)[1]

Peano is perhaps best known for his Postulates for the Natural Numbers. These postulates mark the completion of the process of “arithmetization” which began in the last century when H. Grassmann, in his *Lehrbuch der Arithmetik* (1861), showed that the commutative law can be derived from the associative law by means of the principle of complete induction. They may be stated:

1. Zero is a number.
2. The successor of any number is another number.
3. There are no two numbers with the same successor.
4. Zero is not the successor of a number.
5. Every property of zero, which belongs to the successor of every number with this property, belongs to all numbers.

B. Russell objected that these postulates do not characterize the natural numbers. They characterize a much more general concept, that of a progression of any objects whatsoever which has a first member, contains no repetitions, and for each member of which there is an immediate successor. Peano made no reply to this “objection” and even advised his students to seriously consider it. Russell’s objection, however, had little force for Peano. R. Dedekind had said in his essay, *Was sind und was sollen die Zahlen?*, of 1888, after defining an “ordered simply infinite system” to be a set of elements which, as we would now say, satisfies Peano’s postulates, that

> “if we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed ... then are these elements called natural numbers or ordinal numbers or simply

[1] [Added in 2002] This statement is false. Russell first met Peano at the International Congress of Philosophy, Paris, 1900.

> numbers. ... With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind" ([6], p. 68).

Peano introduced his postulates in *Arithmetices Principia, Nova Methodo Exposita* (1889). He first read Dedekind's essay as this little book was going to press. Indeed, he notes in the preface that he had found Dedekind's essay useful and remarks that "in it, questions pertaining to the foundations of numbers are keenly examined."[2] We should note, however, that he arrived at the postulates independently of Dedekind, and that perhaps the "usefulness" was in finding "moral proof of the independence of the primitive propositions from which I started, in their substantial coincidence with the definitions of Dedekind" ([20], p. 243).

Two years later, in "Sul concetto di numero" (1891), we find explicit recognition of the fact that the postulates do not characterize the natural numbers: "The propositions express the necessary and sufficient conditions that the entities of a system can be made to correspond univocally to the series of natural numbers" ([18], p. 87). It seems, then, that Peano was aware from the beginning that the postulates do not characterize the natural numbers. They received their definitive statement in *Formulaire de mathématiques*, t. II, Sec. 2 (1898), where 'one' is replaced by 'zero' among the primitive notions, and where Peano remarks, in a note immediately following their statement:

> "These primitive propositions . . . suffice to deduce all the properties of the numbers that we shall meet in the sequel. There is, however, an infinity of systems which satisfy the five primitive propositions. . . . All systems which satisfy the five primitive propositions are in one-to-one correspondence with the natural numbers. The natural numbers are what one obtains by abstraction from all these systems; in other words, the natural numbers are the system which has all the properties and only those properties listed in the five primitive propositions" ([14], p. 218).

That Peano considered his work no more than axiomatization, and not an answer to the basic question—What is a number?, may be seen in "Sul concetto di numero." After discussing Dedekind's work and his own postulates, Peano says:

[2] [12], p. 22. All translations are mine.

> “From what precedes, and what Dedekind says, there is here an apparent contradiction, which should be pointed out at once. Here number is not defined, but the fundamental properties of it are given. Instead, Dedekind defines number, and calls number precisely that which satisfies the conditions stated above. Evidently the two things coincide” ([18], p. 88).

This seems to be a clear, if not too sophisticated, statement of the axiomatic method (and, perhaps, gives us the correct interpretation of Dedekind’s thought). With typical generosity, Peano remarks: “The primitive propositions which precede are due to Dedekind” ([18], p. 86).

How far did Peano agree with the logicist thesis of B. Russell? Can we say, as some have said: “Much of Russell’s work, like that of his collaborator, Professor Whitehead, and his great Predecessors, Frege and Peano, was devoted to performing the reduction of mathematical concepts to logical concepts”? ([1], p. 8). Peano’s reserve in discussing this question is known. L. Geymonat has recorded that whenever he tried to get a judgment from Peano regarding the effective value of Russell’s objections to the postulates, Peano “preferred to take refuge in joking or evasive words, saying that these were philosophical questions about which he was ‘absolutely incompetent’” ([9], p. 56). Nevertheless, he did not entirely avoid the question of whether the concept of number can be defined. In “Sul concetto di numero” he wrote: “The first numbers presented and with which we form all the others are the positive integers. And the first question is: Can we define unity, number, the sum of two numbers?” ([18], p. 84). He answers this question in two ways. From the practical side, it is useless to define “number” since the notion is already clear to the students, the words “one”, “two”, “three”, etc. having been in their vocabulary from early childhood. From the theoretical side we must be told first which notions we may use. If we use only the notions of logic developed in *Mathematices Principia*, “then *number cannot be defined*, seeing that it is evident that however these words are combined, there will never result an expression equivalent to number” ([18], p. 84). When someone, not content with this answer, asked whether one could define the idea of number, using ideas which are more simple, Peano replied:

> "To this question one can give different answers from various authors, seeing that simplicity can be understood differently. For my part, the answer is that number (positive integer) cannot be defined (seeing that the ideas of order, succession, aggregate, etc., are as complex as that of number)" ([9], p. 58).

Prof. Geymonat has carefully considered Peano's relation to Russell in "I fondamenti dell'aritmetica secondo Peano e le obiezioni 'filosofiche' di B. Russell." From the above and similar quotations, and from personal discussions with Peano, he concludes that "despite his extreme reserve in this regard, he had no doubts about the artificial character of Russell's reduction of number to a class of classes" ([9], p. 60). Typically, we find a definitive statement of Peano's intention in a parenthetical comment: "Mathematical logic, useful in mathematical reasoning (and it is in this sense alone that I made use of it), is of interest also to philosophy" ([15], p. 396).

Thus, despite Peano's denial of philosophical competence, we find a rejection of the logicist thesis of the reduction of mathematics to logic. Nor, as Prof. U. Cassina has so clearly pointed out, was he a formalist: "G. Peano—whether in Logic or in Mathematics—never worked with pure symbolism; i.e. he always required that the primitive symbols introduced represent intuitive ideas to be explained with ordinary language" ([5], p. 334). With regard to these intuitive ideas, Peano says:

> "In every science, after having analysed the ideas, expressing the more complicated by means of the more simple, one finds a certain number that cannot be reduced among them, and that one can define no further. These are the primitive ideas of the science; it is necessary to acquire them through experience, or through induction; it is impossible to explain them by deduction" ([16], p. 173).
>
> "These concepts [number, unity, successor of a number] cannot be obtained by deduction; it is necessary to obtain them by induction (abstraction)" ([18], p. 85).

But what does Peano mean by "abstraction"? The term is certainly ambiguous. With regard to number, it is possible that he meant it in the technical sense it has in the realist philosophy of Aristotle and St. Thomas Aquinas. This interpretation is suggested by a similar remark on the foundations of geometry:

> "Certainly it is permitted to anyone to put forward whatever hypotheses he wishes, and to develop the logical consequences contained in those hypotheses. But in order that this work merit the name of Geometry, it is necessary that these hypotheses or postulates express the result of the more simple and elementary observations of physical figures" ([19], p. 141).

The evidence, however, is far too little for us to pass judgment, and we cannot presume to settle Peano with this "answer" to the basic question.

We must recognize, also, Peano's concern for the historical continuity of mathematics. (A briefest glance into the *Formulario Mathematico* will convince one of the wealth of historical information there.) The importance of this direction of research for the philosophy of mathematics has been pointed out, among others, by G. Vailati, F. Enriques, L. Geymonat, and U. Cassina.[3]

While it is true of Peano that, "at the hint of anything philosophical, he quickly withdrew, one might almost say for fear of compromising himself" ([9], p. 62), as Prof. Geymonat says, it would be superficial "to attribute his uncertainty to an incapacity to perceive the philosophical problem underlying the mathematical one" ([9], p. 63). We may conclude, with Prof. Geymonat:

> "This caution could appear discouraging in 1931–32; it no longer appears so today, when we are accustomed to distrust all those theories (such as that of Russell and Frege) which are presented as definitive solutions of some philosophical problem. In other words: the position of Peano can result sterile if considered as a point of arrival; it can, however, result extremely efficacious if considered as a point of departure, i.e., as an earnest call not to hide the enormous philosophical complexity of any serious inquiry into the foundations of mathematics" ([9], p. 63).

We have seen that Peano's primary interest was in axiomatics, that he never used the mathematical logic developed by him for the reduction of mathematical concepts to logical concepts, and that, instead, he denied the validity of such a reduction. Some sugges-

[3] See [3], [4], [7], [8], [22].

tion has been given of his ideas on the definability of the number concept. Now that the major writings of Peano are again readily available, ([13], [17]), it is to be expected that there will be a renewed appreciation of the man who, as Bertrand Russell said, “has a rare immunity from error” [21].

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GIUSEPPE PEANO at the University of Turin

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GIUSEPPE Peano visited America only once, as a delegate to the International Mathematical Congress of 1924 in Toronto. A group photograph was made and published in the Proceedings. Peano is near the center of the last row, a balding man with gray hair and beard. His vest is buttoned, but his coat is not, and his tie is skewed to his left. He was the official delegate of the University of Turin, the Academy of Sciences of Turin, and the Academia pro Interlingua. At the Congress Peano gave a paper entitled "De Aequalitate," which appears as two pages of the *Proceedings* in the section containing abstracts. However, it is probably the complete paper. One wonders what impression Peano made on the younger delegates at the congress. Here was a man who had emerged in the last decades of the nineteenth century as an acute mathematician and logician, now reading a distressingly simple paper in a strange language (his own invention!), speaking with a gruff voice that must have contrasted sharply with his childish-sounding pronunciation. (He was unable to say the letter "r" and said "l" in its place.)

These qualities are stressed because we wish to present Peano as a person. In this day of the French Nicolas Bourbaki and the Swedish H. Gask, when the latest writers cavalierly strip Peano of all credit for the postulates for the natural numbers and whenever mention is made of the space-filling curve the reference is usually to Hilbert's explanation, it would be no surprise if some people wondered whether there actually was a Giuseppe Peano, or at least whether he was more than a spokesman for another collective.

The title of this note, "Giuseppe Peano at the University of Turin," hardly limits a discussion of Peano's activities—they mostly centered around the University of Turin. But what we would like to do is suggest something of the makeup of the Mathematics Department during the fifty-six years that Peano was associated with it and to suggest some of the reasons why his reputation as a mathematician suffered from the length of that association. If Peano had died at a much younger age or at least had fought a duel or something, he would probably be better known. But what can be said of a man who lived long enough to give years of ineffectual lectures while younger, more capable teachers waited for the chair they felt they deserved, and who died at night of a heart attack after spending the evening watching a Maurice Chevalier movie?

Peano was eighteen years old when he entered the University of Turin as a student in 1876. He had spent his early childhood in Cuneo (fifty miles south of Turin) and in the little crossroads village of Spinetta (three miles out of Cuneo), where Peano's father worked a small farm with the elegant name of "Tetti Galant." Giuseppe was the second of four sons and a daughter. When he began school, he walked the distance to Guneo with his older brother, but later the family moved into Cuneo so that the children would not have to walk so far to school. Peano was a good student from the beginning and at age twelve or thirteen was sent to Turin, on the invitation of his mother's brother, to complete his studies there. This uncle, who was a priest and lawyer, tutored him in some subjects, and he had private lessons in others or taught himself. He successfully passed the lower secondary exam of the Cavour School in 1873 and completed the regular program of the upper secondary school in 1876. He was a very promising student and had, in fact, won a room-and-board scholarship at a *collegio* set up to assist university students from the provinces. Peano had to pay enrollment and examination fees at the university during his first two years, but at the end of the second year he won a cash prize that made up more than half this amount, and thereafter he was dispensed from all fees.

To give some idea of the size of the university and its mathematics program: During Peano's first year at the university there were 1,558 students enrolled in all subjects and all years. In mathematics there were 103 freshmen, 94 sophomores, 1 junior, and 3 seniors. The drop in the last two years is explained by the fact that most mathematics students went into the engineering program (which had 77, 91, and 127 students, respectively, enrolled in its three-year program). Peano had at first intended to go into the engineering program, but before he enrolled for his junior year he decided to stay in mathematics. When he took his degree in 1880, he was 1 of only 2 students who graduated in mathematics that year.

Peano's first year courses were algebra, analytic geometry, projective geometry, chemistry, and ornamental design—all required. His design teacher, Carlo Ceppi, among other reasons for teaching at the university, was probably trying to escape military service. The others seem to have been competent teachers, and Enrico D'Ovidio, who taught both algebra and analytic geometry, was a capable mathematician.

The second year Peano registered for experimental physics, descriptive geometry, infinitesimal calculus, ornamental and architectural design, and mineralogy and geology—all required of preengineering students—and he elected zoology. Angelo Genocchi held the chair of infinitesimal calculus, but it appears from the university register that Peano's instructor was Genocchi's assistant. At any rate, Peano was probably familiar with Genocchi's lectures. In his final two years as a student, Peano studied mechanics for two years and had one year each of higher analysis, higher geometry, mathematical physics, and geodesy. Astronomy was also listed as a required course, but Peano omitted this and registered for a series of teachers' conferences. Under Genocchi this included a discussion of selected topics in analysis. His instructors included D'Ovidio again, Francesco Siacci, and Francesco Faà di Bruno, a graduate of the Sorbonne (where he had studied with Cauchy), who was later ordained a priest and became known also for charitable activities. (There is now a cause for his beatification.) Peano's final exam covered this two years' work. It was held in July 1880 and Peano was graduated "with high honors." We should also note here that although modern languages were not a part of his regular program, Peano added knowledge of English, German, and French to the Latin and Greek studied in secondary school.

Peano's career continued to advance smoothly. To the Chair of Algebra and Analytic Geometry had been assigned an assistant two years earlier, and D'Ovidio used this to give a university position to his best graduates. Peano was named for the year 1880–81. (D'Ovidio had been instrumental in persuading the Minister of Public Instruction to establish the position of university assistant.)

The following year Peano became assistant to Genocchi in infinitesimal calculus, and he was assistant, and later substitute, until Genocchi's death in 1889. Peano then assumed full responsibility for the calculus course and after regular competition was given the chair the following year. He had in the meantime been appointed, in 1886, professor at the Military Academy, next door to the university, and the financial help this gave allowed him to marry the following year. His wife was the daughter of Luigi Crosio, a successful Torinese genre painter of Pompeian and seventeenth-century scenes. Crosio's daughter, Carola, shared some of his interests in the arts. Turin had a very good opera house at this time; the first performances of Puccini's *Manon Lescaut* and *La Bohème* were given at the Royal Theater of Turin in 1893 and 1896, respectively. Mrs. Peano enjoyed opera, but Peano seems to have sometimes slept during a performance. At any rate, his musical taste ran to lighter things, as we have seen.

Thus Peano's life and career advanced without disturbance. It is almost typical that the one serious illness Peano had—an attack of smallpox that lasted almost a month in 1889—occurred during the university's summer holiday. In his early life be had to live quite frugally, but he had parents, relatives, and friends who recognized his worth and were able to help him. To the end he retained rather frugal habits of living, and he returned his parents' devotion with affection and gratitude. As a university student he returned to Cuneo for the summer holidays and helped out on the farm. Later he was able to assist his parents financially.

One incident almost disturbed the advance of Peano's career. A publisher had been trying to get Genocchi to write up his calculus course for publication. When he substituted for Genocchi, Peano was approached about this. Peano obtained permission from Genocchi to make up a text from his course, and this was published in 1884. Genocchi's name was on the title page, and the title was *Differential Calculus and Fundamentals of Integral Calculus*, "published with additions by Dr. Giuseppe Peano." The text was probably better than Genocchi's lessons, and, of course, the additions were the best part of all. Naturally this irritated the quick-tempered Genocchi, and he had published in mathematical journals of Italy, France, and Belgium a declaration that he had had nothing to do with the book. Peano managed to weather the embarrassment caused by this, and the book made him an immediate reputation. Why? Pringsheim, in the *Encyklopädie der Mathematischen Wissenschaften*, lists this as one of the nineteen most important calculus texts since the time of Euler and Cauchy (Peano's calculus text of 1893 is also one of the nineteen) and cites the following results contained in it: theorems and remarks on limits of indeterminate expressions, pointing out errors in the better texts then in use; a generalization of the mean-value theorem for derivatives; a theorem on uniform continuity of functions of several variables; theorems on the existence and differentiability of implicit functions; an example of a function whose partial derivatives do not commute; conditions for expressing a function of several variables with a Taylor's formula; a counterexample to the current theory of minima; and rules for integrating rational functions when roots of the denominator are not known.

This is just a sample—and we cannot go into this much detail with Peano's other works—but so much is worth noting. It is typical. If producing counterexamples has become such a mathematical pastime, it is partly because Peano helped popularize it. "There is no need to prove every theorem in class," Peano would say, "but let us at least have precise concepts and correct definitions. Rigor does not consist in proving everything. It consists in saying what is true and not saying what is not true." The counterexample, if it may be called that, that brought Peano most fame was the example of a space-filling curve published in 1890. This curve is given by continuous parametric functions and goes through every point in a square as the parameter ranges over some interval. It can be defined as the limit of a sequence of more "ordinary" curves. Peano was so proud of this discovery that he had one of the curves in the sequence put on the terrace of his home, in black tiles on white.

By this time Peano had also discovered the method of successive approximations for the solution of differential equations and had published a book clarifying and popularizing Grassmann's *Ausdehnungslehre*, or "Geometrical Calculus," as Peano called it. He had also discovered, independently of the work of Dedekind, the postulates for the natural numbers. This last appeared as a booklet of thirty-six pages, mainly written in the symbolism of mathematical logic which by then Peano had begun to adapt, invent, and apply in his study of mathematics. The introduction and notes are in classical Latin. This and the title, *Arithmetices principia*, are probably a romantic tribute to the memory of Newton.

The next decade saw the gathering together of Peano's "school," the collaborators in the *Formulario* project. None of these were professors at the university. Some of the group were university assistants, some were professors at the Military Academy, and others were teachers in secondary schools and technical institutes. Their goal was to gather together in one book all branches of mathematics, with definitions, theorems, and proofs all completely written in the symbolism of mathematical logic. This appeared in five editions from 1895 to 1908, the last containing some 4,200 theorems.

A paper on mathematical logic as developed by him up to 1900 was presented by Peano to the International Philosophical Congress in Paris that year. That was when, according to Bertrand Russell, he first met Peano—“one of the two men,” he has said, “who most influenced my philosophical development.”

Meanwhile, back at the university, the students were restless. From 1894 to 1900, for example, the university had to be closed at least five times, for periods of up to a month, due to various demonstrations. We mention this, not as something extraordinary, but as a fact of life at the university. This cannot help influencing the quantity and quality of instruction given, as well as the professor’s general attitude toward his profession.

Peano soon began to lose his students for other reasons. With nineteenth-century calculus fossilized in “the symbols,” as his students called it, Peano’s “school” drifted on to other interests, while Peano himself spent precious classroom hours deciphering painfully precise formulas for the benefit of classes made up mostly of preengineering students. Nor did the fact that he began to pass everyone who registered for his courses regain his popularity. Because of objections to this method of teaching, Peano resigned from the Military Academy in 1901 and a few years later stopped lecturing at the Polytechnic.

Part cause of this decline was Peano’s interest in the idea of an international auxiliary language, already foreshadowed by the publication of the last edition of the *Formulario*, not in French as the earlier ones had been, but in *Latino sine flexione*, or Latin without grammar, Peano’s own proposal for an international language. Paradoxically, what was intended to extend the usefulness of the Formulario probably hindered it.

Perhaps the time has come for a renewal of appreciation for the very real accomplishments of Peano. In 1915 he printed a list of his publications and gave his own evaluation.

My works refer especially to infinitesimal calculus, and they have not been entirely useless, seeing that, in the judgment of competent persons, they contributed to the constitution of this science as we have it today.

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IS THERE AN ELEMENTARY PROOF OF PEANO'S EXISTENCE THEOREM FOR FIRST ORDER DIFFERENTIAL EQUATIONS?

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In 1886 Giuseppe Peano stated [8] that the initial value problem: $y' = f(x, y)$, $y(a) = b$, has a solution on the sole condition that f is continuous, and he gave an elementary proof of this. His theorem is correct, but a historical investigation into the work of Peano has led to the conclusion that his proof is invalid. This raises the question: Is there an elementary proof of Peano's Theorem? It is clear that f must satisfy some condition if the existence of a solution of the differential equation $y' = f(x, y)$ satisfying the initial value $y(a) = b$ is to be guaranteed. Cauchy proved that there exists a solution of the initial value problem if f and f_y are continuous [6], or if f is synectic (continuous, monodrome, and monogenic). Charles Briot and Jean Bouquet improved the proof of the latter theorem [l]. Rudolf Lipschitz gave an existence theorem which imposed a less restrictive condition on f [3]. (The reference usually given for this is [4], which is essentially a French translation of the Italian article [3].)

In 1886 Peano first stated and proved [8] the theorem that the given initial value problem has a solution on the sole condition that f is continuous. In 1890 he extended this theorem to systems of first order differential equations [9], using an entirely different method of proof (successive approximations). This method of proof became well known especially through the efforts of G. Mie, who gave a popular reworking [5] of Peano's elaborately symbolic exposition. Peano's original proof of 1886, generally ignored, was rediscovered by Oskar Perron in 1915 [11]. W. F. Osgood, independently of the work of Cesare Arzelà, discovered a proof of Peano's Theorem [7] in which the solution of the differential equation is presented as the limit of a sequence of functions. (A very readable proof of Peano's Theorem as a direct consequence of the Arzelà-Ascoli Theorem may be found in [2].)

Peano’s original proof of 1886 (and its rediscovery by Perron in 1915) is elementary in the sense that a function is given by defining its functional value at each point of its domain as, say, the least upper bound of a set of numbers, and an attempt is then made to show that the function so defined is a solution of the differential equation by the direct application of elementary inequalities. Neither Peano nor Perron, however, succeeds in doing this! Peano’s proof is, for him, surprisingly unrigorous and fails completely at “one of these functions may be selected” (*si potrà determinare una di queste funzioni*), where the possibility of the selection is not certain, at least after one has cleared up previous difficulties. (Peano’s article is perhaps most easily consulted in [10], where the difficulty mentioned occurs in Vol. 1, page 79, line 14.) Perron’s proof, while beginning with more rigor than Peano’s, flounders similarly at an “obviously” (*offenbar*), which is also followed by an inexact inequality. (See [11], page 475, lines 4 and 6.) This raises the question: Can a rigorous proof be given, using this method?

Since Peano gave a proof of this theorem in 1890 [9], using a method which is undoubtedly correct (successive approximations), he certainly has priority. But it would be interesting to know whether his proof of 1886 must be discarded entirely, or whether it can be ‘cleaned up’ using only the method suggested by him, namely, the direct application of elementary inequalities to a function defined in a way similar to that given above.

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THE ORIGINS OF MODERN AXIOMATICS: PASCH TO PEANO

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The modern attitude toward the undefined terms of an axiomatic mathematical system is that popularized by Hilbert's remark: "One must be able to say at all times —instead of points, straight lines, and planes—tables, chairs, and beer mugs" [20, p. 57]. This view was not widely accepted before the twentieth century, and even in 1959 the well-known James and James *Mathematics Dictionary* gave "A self-evident and generally accepted principle" as first meaning of the term "axiom," although this may only be meant as a reflection of the view universally accepted before the developments in geometry in the nineteenth century. The change in attitude appears to be due to internal pressures within mathematics (what R. L. Wilder has called "hereditary stress" [22, p. 170]). These include the flowering of projective geometry and, especially, the discovery of the non-euclidean geometries, i.e., of the possibility of a geometry based on axioms, one of which is the negation of one of Euclid's axioms. The transition from viewing an axiom as "a self-evident and generally accepted principle" to the modern view took place in the second half of the nineteenth century and can be found in the very brief period from 1882 to 1889, from Pasch's *Vorlesungen über neuere Geometrie* [13], to Peano's *I principii di Geometria logicamente esposti* [15].

Already in 1882, Pasch showed a shift in interest from the theorems to the axioms from which the theorems are derived, when he insisted that everything necessary to deduce the theorems must be found among the axioms [13, p. 5]. Pasch was concerned that his axiom set be complete, i.e., that it furnish a basis for rigorous proofs of the theorems. ("The father of rigor in geometry is Pasch", wrote Hans Freudenthal [5, p. 619].) There is also a strong hint of the modern attitude, as expressed in Hilbert's remark about "tables, chairs, beer mugs" in his statement: "In fact, provided the geometry is to be truly deductive, the process of inference must be entirely independent of the meaning of the geometrical terms, just as it must be independent of the figures" [13, p. 98].

Hilbert's remark was made to a few friends in the waiting room of a railway station in 1891 but was not published until 1935 [10, p. 403]. The exposition of the axioms in his famous *Grundla-*

gen der Geometrie [7] begins: “Let us consider three distinct systems of things. The things composing the first system, we will call *points*, and designate them by the letters *A*, *B*, *C*, . . .” [8, p. 3]. The viewpoint is quite clear—but he was not the first to publish this view. Pasch has already been mentioned, and Hans Freudenthal, in a study of geometrical trends at the turn of the century says: “Hilbert had in this view too at least one forerunner, namely G. Fano, . . .” [4, p. 14]. He refers to Fano’s statement: “As basis for our study we assume an arbitrary *collection* of entities of an arbitrary nature; entities which, for brevity, we shall call *points*, and this quite independently of their nature” [3, p. 108].

Somewhat surprisingly Freudenthal overlooks Peano’s monograph of 1889, even though it is cited in Fano’s article, perhaps because Fano says that Peano’s work was based on that of Pasch. Peano’s work was indeed based on his reading of Pasch, but there are important innovations, and one of them is the explicit statement of the modern attitude toward the undefined terms of an axiomatic mathematical system. The first line of his exposition is: “The sign **1** is read **point**,” and in his commentary he says: “We thus have a category of entities, called points. These entities are not defined. Also, given three points, we consider a relation among them, indicated by *c* ε *ab*, and this relation is likewise undefined. The reader may understand by the sign **1** any category whatever of entities, and by *c* ε *ab* any relation whatever among three entities of that category, . . .” [15; 18, p. 77]. We find in this statement explicit acceptance of the axiomatic view. (It should be noted that Peano’s view was purely methodical. As we have indicated elsewhere [12, p. 264], he was not a member of what came to be called the ‘formalist’ school.)

E. W. Beth has noted [2, p. 82]: “Since the publication of D. Hilbert’s *Grundlagen der Geometrie* (1899), it has become customary to require every set of axioms to be (1) *complete*, (2) *independent*, and (3) *consistent*.” Again, it was Hilbert who popularized this ‘custom’, but that these properties of an axiom set are desirable was already accepted by Peano and others. The property of consistency is indeed a *sine qua non*, but as the consistency of Euclid’s axioms was never doubted, it was only with the advent of noneuclidean geometry that attention was focused on this property, and it was not until 1868 that a consistency proof was found by E. Beltrami [1]. The property of independence can be reduced to that of consistency; we often say that Beltrami proved the independence of Euclid’s “parallel postulate”, but this reflects a later view, that of Peano who developed this technique into a general method.

Peano's acceptance of the goal of an independent set of axioms is indicated in his *I principii di Geometria*: "This ordering of the propositions clearly shows the value of the axioms, and we are morally certain of their independence" [15; 18, p. 57]. In a similar remark about his axioms for the natural numbers, published earlier that year, Peano later wrote: "I had moral proof of the independence of the primitive propositions from which I started, in their substantial coincidence with the definitions of Dedekind" [17; 19, p. 243]. It was only in 1891, however, after he had separated the 'famous five' from the postulates dealing with the symbol =, that he showed their absolute independence [16; 19, p. 87].

Hermann Weyl wrote of Hilbert [20, p. 264]: "It is one thing to build up geometry on sure foundations, another to inquire into the logical structure of the edifice thus erected. If I am not mistaken, Hilbert is the first who moves freely on this higher 'metageometric' level: systematically he studies the mutual independence of his axioms and settles the question of independence from certain limited groups of axioms for some of the most fundamental geometric theorems. His method is the construction of models: the model is shown to disagree with one and to satisfy all other axioms; hence the one cannot be a consequence of the others." This method was, as we have seen, already used systematically by Peano, although one would not learn this from reading Hilbert. In the *Grundlagen der Geometrie* there is no mention of Peano. The only Italian mentioned is G. Veronese, and the reference is to a German translation of his work. Nor does Hilbert mention Peano even in his presentation of postulates for the real numbers [9]. Indeed (without naming him) he labels Peano's development of the real numbers the "genetic method," while reserving the label "axiomatic method" for his own presentation!

A word more may be said about the originality of Peano's work. In contrast with Hilbert, Peano always tried to place his work in the historical evolution of mathematics, to see it as a continuation and development of the work of others. Furthermore he was scrupulously honest (although sometimes mistaken) in assigning priority of discovery. Thus in *Principii di Geometria* he praises Pasch's book and indicates precisely to what extent his treatment coincides with that of Pasch, and where it differs. On the other hand, Peano's discovery of the postulates for the natural numbers was entirely independent of the work of Dedekind, contrary to what is often supposed. Jean van Heijenoort says [6, p. 83]: "Peano acknowledges that his axioms come from Dedekind," referring the reader to the statement of Peano: "The preceding primitive propositions are due to Dedekind" [16, 19, p. 86]. Hao Wang says [21, p. 145]: "It is rather well known, through Peano's own acknowledgement . . . that Peano borrowed his axioms from Dedekind. . . .," and he gives a reference to Jourdain [11, p. 273], which in turn refers to the same passage of Peano just quoted. Since Peano had already written in *Arithmetices Principia*: "Also quite useful to me was a recent work: R. Dedekind, *Was sind and was sollen die Zahlen?*, Braunschweig, 1888" [14; 18, p. 22], the conclusion of these authors would seem justified. In fact, Peano was only acknowledging Dedekind's priority of publication.

The exact story was given in 1898 when Peano wrote: "The composition of my work of 1889 was still independent of the publication of Dedekind just mentioned; before it was printed I had moral proof of the independence of the primitive propositions from which I started, in their substantial coincidence with the definitions of Dedekind. Later I succeeded in proving their independence" [17; 19, p. 243]. We see from this that the reference to Dedekind's work was added to the preface of *Arithmetices Principia* just before the pamphlet went to press, and we have an explanation of how Dedekind's work was "useful".

Ironically, the very modesty of Peano and his desire to see his work as in the mainstream of the evolution of mathematics have contributed to the lack of recognition of his originality. As for clarity, while giving much credit to Peano, Constance Reid says of Hilbert that in the *Grundlagen der Geometrie* he [20, p. 60] "attempted to present the modern point of view with even greater clarity than either Pasch or Peano." What could be clearer than: "The reader may understand by the sign **1** any category whatever of entities"? Let the reader compare for himself the clarity of Dedekind's presentation of the foundations of arithmetic with that of Peano. There can be no doubt that the famous five axioms for the natural numbers are rightly called Peano's Postulates.

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WHAT RUSSELL LEARNED FROM PEANO

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On a quiz in a "Study of the History of Mathematics" course, the question was asked: "When Russell met Peano, what language did they speak to one another?" One student answered "symbolic logic." The student was clever, but wrong—and not just wrong: he had missed completely the historical importance of that meeting, for it was precisely symbolic logic that Russell learned as a result of meeting Peano, and he learned it from Peano. The evidence comes from Russell himself in, among other places, the description of the International Congress of Philosophy in Paris, 1900, in his *Autobiography* ([21], p. 217–219):

> The Congress was a turning point in my intellectual life, because I there met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussions at the Congress I observed that he was more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. I therefore got him to give me all his works, and as soon as the Congress was over I retired to Fernhurst to study quietly every word written by him and his disciples. It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring a new powerful technique for the work that I had long wanted to do.

But what, specifically, did Russell learn from Peano? According to Russell, the enlightenment he received came mainly from two purely technical advances. He notes, by the way, that: "Both these advances had been made at an earlier date by Frege, but I doubt whether Peano knew this, and I did not know it until somewhat later" ([20], p. 66). "The first advance consisted in separating propositions of the form 'Socrates is mortal' from propositions of the form ‘All Greeks are mortal’” ([20], p. 66). In the symbolism of Peano, adopted by Russell, this distinction is between $s \mathrm{\varepsilon} M$ and $x \mathrm{\varepsilon} G \supset_x s \mathrm{\varepsilon} M$. “The second important advance I learnt from Peano was that a class consisting of one member is not identical with that one member” ([20], p. 67). That is, $s \mathrm{\varepsilon} M$ is not the same as $x \subset M$. Both the Greek epsilon for set membership and the subscript for universal quantification were introduced by Peano in 1889 [8]. The letter ε is the initial of ἐστί, *is*. The symbol ⊃ was substituted in 1898 for the upside-down letter C of 1889. The symbol C, the initial of the Latin *consequentia*, was introduced in 1889, but was used only to define its inverse, symbolized by the same letter upside-down. (Inverted symbols for inverse relations and operations are typical of Peano’s notation.) Peano also used $A \supset B$ to symbolize “A is a subset of B,” noting that this gave the same symbolic statement of theorems in the calculus of classes and the calculus propositions. He considered this a great practical advantage. It was Russell who, feeling it would be advantageous to have distinct symbols, re-introduced ⊂ for set inclusion. “It is convenient in mathematics to think of ‘classes,’ and for a long time I thought it necessary to distinguish between classes and prepositional functions” ([20], p. 69). This may be seen as an example of something Russell did not learn from Peano, for Peano considered it of the greatest advantage to memory and ease of manipulation to have parallel symbols. (For Russell’s criticism, see [18], §13.) He likewise considered it advantageous to have symbols printed on a single line, something that Russell, in the article he submitted in the fall of 1900 for publication in Peano’s journal, admitted that he had not succeeded in doing ([17], p. 116). (Frege remarked in this connection that the convenience of the typesetter was not the highest good! ([3], p. 364).)

Apart from the things specifically mentioned by Russell, it is difficult to know just what of Peano’s work he did learn from him and what he discovered for himself. He mentioned that the older theories of number always got into difficulties over 0 and 1, and it was Peano’s capacity of dealing with these difficulties that first impressed him. Russell was probably influenced, also, by Peano’s stress on the distinction between real and apparent variables (see, for example [9] and [11]). Along with the symbolism mentioned above, Russell adopted Peano’s symbol ∃ for existential quantification, which had been introduced in 1897 [11], as well as the decimal ordering of propositions, which was introduced by Peano in 1898 [12]. A copy of the “*Formules de logique mathématique*”

(July 20, 1899) [13] would certainly have been furnished Russell by Peano, and should be consulted for a systematic exposition of what Russell could have learned from Peano.

Last, but certainly not least, Russell learned about Frege from Peano. Indeed, as Peter Nidditch has pointed out [7], Peano was one of the few before 1900 who took note of Frege's work, and in answer to a direct question, Russell replied: "I know, quite definitely, that it was through Peano that I first became aware of Frege's existence." Nidditch also called attention to the passage in Russell's *Portraits from Memory* ([19], p. 22), which says he learned of Frege from Peano's review of *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet* [10]. Ironically, Russell had been given a copy of Frege's *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* by his philosophy teacher, James Ward, who had not read it. and Russell did not read it until 1901. Reporting this, Russell remarked: "I rather suspect that I was its first reader" ([19], p. 22).

Russell, of course, did not accept everything Peano was doing. To illustrate this, we consider an important instance in an area of most concern to Peano: definitions in mathematics. (This was the title of his paper at the International Congress of Philosophy, Paris, 1900, and he published two more articles with the same title.) Peano accepted definition by postulates and definition by abstraction, as well as nominal definitions. Russell rejected the first two ([18], §108), and it was possibly his rejection of definition by abstraction that led him to the definition of a cardinal number as a class of classes. He published this definition (for the first time, I believe) in Peano's journal ([17], p. 121). This issue is dated July 15, 1901, but the manuscript was submitted the previous year. (The one reference in it to the *Formulaire* of 1901 was probably added by Peano.) We have, in this delayed publication, the explanation of the curious passage in §32 of the *Formulaire* of 1901 in which Peano considers the class of equinumerous classes—and rejects it as a possible definition of cardinal number.

Section 32 of the *Formulaire,* both of 1899 and of 1901, presents the definition, by abstraction, of cardinal number, symbolized Num by Peano. In F1901 ([14], p. 70) we read: "This proposition defines the quality 'Num a = Num b,' which holds if a reciprocal correspondence can be established between a and b. We do not write an equality of the form

Num a = (expression composed of the preceding symbols)."

(In F1899 ([13], p. 61) he had said, "we are unable to write . . .") Peano then adds: "Given a class a, we may consider the class of classes [similar to a (Peano gives a symbolic definition)] . . .; but we cannot identify Num a with the class of classes considered, for these objects have different properties." Russell later commented on this passage ([18], §111): "He does not tell us what these properties are, and for my part I am unable to discover them." (Although the publication of F1901 preceded that of Russell's article by several months, Peano's remark was almost certainly prompted by Russell's definition of cardinal number as a class of classes, for he would have had the manuscript of Russell's article since October 1900.)

J. J. A. Mooij also finds curious, in §32, the statement following his definition of cardinal number: "This definition is expressed only by signs of logic. We can begin arithmetic here: we shall define directly the signs > 0 N_0 + × $\hat{/}$ without going through the primitive ideas of §20" (which contains the postulates for the natural numbers). "This statement," Mooij remarks ([6], p. 46), "seems to be leading up to logicism, which is all the more curious, since earlier in the same edition he had remarked: 'Can number be defined? The answer depends on the set of ideas that we suppose known. If we assume only those represented by the logical signs Cls, ε, ⊃, ∩, =, of §1, then the answer is negative.'" The contradiction that Mooij finds between this statement of Peano and the fact that he defines Num independently of §20 (as Peano himself admits in §20) disappears if we assume that Peano was asking about the possibility of a nominal definition of number. Since, according to Peano, number cannot be given a nominal definition, the choice was arbitrary as to whether to begin arithmetic with a definition by postulates or a definition by abstraction. Peano chose the former.

That this is the correct interpretation is shown by Peano's constant insistence on the *form* of a definition. In F1901, §1, for example, in paragraph 2 "Definitions," we find: "A *possible definition* is an equality that contains in one member a sign which does not occur in the other, or which occurs there in a different position," and a bit further: "The primitive ideas are explained here by ordinary language, and are determined by the primitive propositions; the latter play the role of definitions with respect to the primitive ideas, but they do not have their form.” This last sentence is copied from F1899, §2, where we also find: “Let us suppose that the signs which represent the ideas of a science have been ordered. The symbolic definition of a simple sign x has the form

$$x = \text{(expression composed of preceding signs).”}$$

To say, as Russell has ([18], §108), that there were “three kinds of definitions admitted by Peano” obscures this distinction. Peano “admitted” more than three (e.g., what Peano called “definition by induction”), but the nominal definition was always pre-eminent. The idea of the class of classes considered in F1901, §32, may or may not have occurred to him before it was suggested by Russell (although not mentioned in Peano’s review, it was in Frege’s *Grundgesetze* ([1], p. 56), but he clearly rejected it. It was only later, after Russell had shown how to reduce definitions by abstraction to nominal definitions, that Peano gave some degree of acceptance to the class of classes definition of number. That was in 1913, in a review of the *Principia Mathematica* [15]. There, Peano seems to accept it, in context, as a valid technical device. That he did not accept it as a final answer to the question, “Can number be defined?” is seen clearly in a statement in 1915 [16]: “If a and b are two classes (sets, groups), we write $\mathsf{Num}\ a = \mathsf{Num}\ b$, and read it ‘the number (cardinal number or power of G. Cantor) of the a is equal (or identical) to that of the b’ when we can establish a one-to-one correspondence between the a and b. We thus define the equality of two numbers, not number itself; and this because this definition may be placed before arithmetic, and also because the number that results is not the finite number of arithmetic.” (For a critique of the views of Peano and Russell regarding definition by abstraction, see [22], Chapter V.)

Corrado Mangione has remarked [5, pp. 66–67]: “The original and fundamental observation of Frege consists basically in having recognized the possibility of expressing the equality of numbers without bringing in the concept of number itself; . . . Now, let us consider the extension of this concept; it is obviously *the class of all classes similar to ...*” Peano did not find all this so “obvious,” but his final agreement with Frege is remarkable, for Frege also distinguished between cardinal numbers and the numbers of arithmetic ([4], p. 155, as translated in [2], p. liv):

Since the Numbers (Anzahlen) are not proportions, we must distinguish them from the positive whole numbers (Zahlen). Hence it is not possible to enlarge the realm of Numbers to that of real numbers; they are wholly disjoint realms. The Numbers give the answer to the question, "How many objects are there of a given kind?", whereas the real numbers may be regarded as numbers giving a measure, stating how great a magnitude is as compared with a unit magnitude.

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A PROSPECTIVE BIOGRAPHY OF GIUSEPPE PEANO (1858–1952)

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Peano's name is well known, having been attached to several things in mathematics, e.g., the Peano theorem in differential equations, Peano's postulates for the natural numbers, the Peano curve, and even (though he would not have recognized it) Peano space. He was perhaps at the height of his fame in 1900, but after that his reputation among mathematicians and logicians suffered a decline. There were many reasons for this, and first among them was his shift of interest to linguistics and his dedication to the international auxiliary language movement. Among other causes for this decline were Peano's own personality and a certain factionalism that developed at the University of Turin. As a result, for almost 30 years after Peano's death he seemed to be appreciated fully by only one man, Ugo Cassina. But although Cassina wrote several articles about Peano's work, he never really enlarged on the three "biographical" pages in his *elogia* [Cassina 1928]. Furthermore, no translation into English of a complete article by him appeared prior to my 1975 volume containing translations of some 20 articles, a chronological list of Peano's publications, a bibliography on Peano's life and work, and a brief biographical sketch. A 24-page biography will shortly be published by Birkhäuser Verlag, Basel, in the series "Kurze Mathematiker-Biographien."

Peano had many students and disciples. The late Ugo Cassina once published a list of 45 Italians who, at one time or another, belonged to the "school" of Peano. His two-volume collection of reprints (1961) contain 10 articles treating Peano's work, and there were others. His devotion was more strongly shown by his editing of the three-volume selected works (Peano 1957–1959) and the facsimile reprint of the *Formulario* (Peano

1960). Cassina's interest in making Peano's work known is evident. So it is surprising that the biographical data in these volumes is limited to one page of the introduction to the *Opere Scelte*. One expected that a biography of Peano would follow.

On this assumption, I wrote to Cassina (under whom I had studied during the academic year 1957–58 in Milan) on 29 October 1963, enquired whether he planned to write a biography of Peano, and offered my assistance. (I thought at the time that I could return to Italy in 1965.) Cassina replied on 28 November 1963 with a suggestion that we collaborate. He wrote: "Io penso che potremmo fare un lavoro di collaborazione: cioè lei potrebbe scrivere la biografia di Peano in inglese servendosi del materiale da me raccolto (sia riguardante la vita che le opere) e di altro che potrei eventualmente procurarle." [I think that we could do a work of collaboration: that is, you could write the biography of Peano in English using material collected by me (whether concerning his life or his works) and other material that I could eventually provide.] But Cassina died on 5 October 1964, and his widow also died before my arrival in Italy in the fall of 1966. In the meantime, I tried in vain to learn the contents of the "materiale da me raccolto." In the spring of 1967, I finally traced it to a shoe box in the possession of a relative. There was nothing of biographical interest in it beyond a postcard from Peano to Cassina, which the somewhat embarrassed owner of the shoe box gave me.

Another possible source of biographical information might be the correspondence in Peano's possession when he died. The remains of this material was eventually traced to another shoe box, this time in the Biblioteca Municipale di Cuneo. This contained a few items of correspondence with minor figures in the artificial language movement, and no correspondence with mathematicians.

A few of Peano's letters have been published [Cassina 1952, Gabba 1957. Note also Kennedy 1974?] I. Grattan-Guinness has noted that there are letters from Peano in the Institut Mittag-Leffler, and he has furnished me a copy of a letter to P.E.B. Jourdain. A few others have come to my attention: there is a postcard to Tullio Levi-Civita at the California Institute of Technology, a letter to D. E. Smith at Columbia University, three letters to Felix Klein at the University of Göttingen, two letters and six postcards to Gottlob Frege and a postcard to Jakob Lüroth in Berlin (Staatsbibliothek, Preussischer Kulturbesitz), and eight letters to Frege at the University of Konstanz.

Besides collecting material for a biography of Peano, I have been studying his published works, but I believe that a study of Peano's correspondence will shed light on the origin and dissemination of some of his ideas. Hence, given the apparent disappearance of letters to Peano, it becomes even more important to locate letters from him. For example, one would hope to gain insight into the relation between Peano's interest in mathematics and his interest in the international auxiliary language movement, and of the mutual influences of those (Louis Couturat, Samuel Dickstein, and others) who shared these two interests with Peano. I plan to spend the academic year 1974–75 in Europe and would be grateful to anyone who furnished information of possible locations of Peano letters. During that year I hope to complete the biography, proposed to me a decade ago by Ugo Cassina.

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PEANO'S CONCEPT OF NUMBER

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SUMMARIES

Giuseppe Peano's development of the real number system from his postulates for the natural numbers and some of his views on definitions in mathematics are presented in order to clarify his concept of number. They show that his use of the axiomatic method was intended to make mathematical theory clearer, more precise, and easier to learn. They further reveal some of his reasons for not accepting the contemporary "philosophies" of logicism and formalism, thus showing that he never tried to found mathematics on anything beyond our experience of the material world.

Lo sviluppo dei numeri reali dai numeri naturali di Giuseppe Peano è qui tracciato, ed alcune sue vedute sulle definizioni matematiche sono presentate allo scopo di chiarire il suo concetto di numero. Esse dimostrano ch'egli adoperò il metodo assiomatico a fin di rendere la teoria della matematica più chiara, più precisa e più facile ad imparare. Esse rivelano, inoltre, alcune sue ragioni per non accettare le "filosofie" contemporanee del logicismo e del formalismo. Cosí è dimostrato ch'egli non ha mai cercato di fondare la sua teoria matematica su altro che la nostra esperienza del mondo materiale.

Развитие системы действительных чисел от натуральных уисел и несколько взглядов у Дж. Пеано представляются об определениях его в математике, чтобы выяснить его понятие о числе. Употреблением аксиоматического метода, Пеано хотел сделать математическую теорию яснее, точнее, и легче учиться.

Обнаруживается также несколько причин зачем он не принимал современных "философий" логицизма и формализма, доказывая, таким образом, что он никогда не старался обосновать математику ни на чем кроме опыта у нас материального мира.

1. Introduction

In 1891, two years after the publication of his now famous postulates for the natural numbers, Giuseppe Peano published, in the journal founded by him that year, an article with the title "Sul concetto di numero" ["On the concept of number"] [Peano 1959, 80–109].[1] In it he simplified his system by eliminating the undefined term symbolized by = , and the axioms relating to it. This system then consisted of three undefined terms: *N* (number), *1* (one), and *a+* (the successor of *a*, where *a* is a number). The five axioms were [Peano 1958, 84]:

(1) $1 \; \varepsilon \; N$

(2) $+ \; \varepsilon \; N \backslash N$

(3) $a, b \; \varepsilon \; N \,.\, a+ = b+ : \supset . \; a = b$

(4) $1 - \varepsilon \; N+$

(5) $s \; \varepsilon \; K \,.\, 1 \; \varepsilon \; s \,.\, s+ \supset s : \supset . \; N \supset s$

These may be read [Peano 1958, 85]:[2]

(1) One is a number.

(2) The sign + placed after a number produces a number.

(3) If *a* and *b* are two numbers, and if their successors are equal, then they are also equal.

(4) One is not the successor of any number.

[1] Since for most readers the volumes of the *Opere scelte* [Peano 1957, 1958, 1959] will be the most available source of Peano's writings, I have given page references to them wherever possible.

[2] I have translated into English all quotations that were originally in Italian.

(5) If *s* is a class containing one, and if the class made up of the successors of *s* is contained in *s*, then every number is contained in the class *s*.
These postulates were not further reduced in number, although their form was changed, due to modifications in Peano's notation.

Because Peano was able to build all of arithmetic on the basis of this set of axioms, his work in this field has become justly famous. (Evidence of continued interest in Peano's ideas is shown by the recent translation of "Sul concetto di numero" into Japanese [Peano 1969].) It is therefore of interest to know just what his conception of number was at that time and whether it later changed. His ideas did indeed evolve in this regard, and even in 1891 he was not as explicit as one would like, so that it is difficult to say exactly what his views were at any one moment. In the present article two lines of development in his work will be traced – his technical presentation of the real number system and his repeated discussion of definitions in mathematics – in the hope of exhibiting some constant factors in his mathematical thought and philosophical commitment. Further, although Peano's derivation of the laws of arithmetic from his postulates for the natural numbers has been called "superb" [Birkhoff 1973, 769], I believe there still exist misconceptions about the manner in which the various types of numbers were introduced by him, misconceptions that may be cleared up by this presentation.

Peano first presented his postulates for the natural numbers in 1889 in *Arithmetices principia, nova methodo exposita* [Peano 1958, 20–55; English translation in Peano 1973, 101–134]. With the simplification of 1891 he was able to prove that the five postulates were mutually independent (of which he was "morally certain" in 1889, because of their substantial agreement with Dedekind's analysis in *Was sind und was sollen die Zahlen?* [Dedekind 1888], a work that Peano read just as his own was going to press). In 1891 Peano also constructed a more complete system than in 1889, for although he often claimed to have been the first, in *Arithmetices principia*, to develop a complete system in symbols (the *nova methodo* of the title), in fact negative numbers were not defined there, rather only positive integers, positive rationals, and positive reals – and few operations were defined for these. In later developments, the various classes of numbers were defined in the order: natural numbers *N*, integers *n*, positive rationals *R*, rationals *r*, positive reals *Q*, reals *q*. Each of these will be treated separately.

2. Development of the Real Number System

2.1 Natural numbers N

In 1889 and 1891 the sequence of natural numbers began with *1*, and the set of natural numbers was designated by N. This was modified in 1898 [Peano 1959, 216] so that the sequence began with *0*, the set being designated by N_0. The set of five postulates for N, *1*, $a+1$ (or *0*, N_0, $a+$) was increased to six in 1901 with the addition of: $N_0 \varepsilon Cls$, i.e., the natural numbers form a class. With the addition of this last, the postulates have received their final form, as follows (where $a+1$ is identified with $a+$) [Peano 1901, 41–43]:

(0) $N_0 \ \varepsilon \ Cls$

(1) $0 \ \varepsilon \ N_0$

(2) $a \ \varepsilon \ N_0 \ . \supset . \ a+ \ \varepsilon \ N_0$

(3) $s \ \varepsilon \ Cls \ . \ 0 \ \varepsilon \ s : x \ \varepsilon \ s \ . \supset_x . \ x+ \ \varepsilon \ s : \supset . \ N_0 \supset \mathrm{s}$

(4) $a, b \ . \ \varepsilon \ N_0 \ . \ a + l = b + 1 \ . \supset . \ a = b$

(5) $a \ \varepsilon \ N_0 \ . \supset . \ a + l - = 0$

The change to N_0 from N (or N_1 as it was now designated) brought other changes with it. For example, in 1891 (as already in 1889) addition of natural numbers was defined by the single equation: $a + (b + 1) = (a + b) + 1$. (This is to be understood, as Peano explained, in the sense that if a and b are numbers and the right hand member of the equation has meaning, but the left has to this point been undefined, then the expression on the left has the meaning of the expression on the right.) By the mathematical induction postulate, addition is then defined for all pairs of natural numbers. In 1898 the recursive definition of addition required two equations: $a + 0 = a$, $a + (b+) = (a + b)+$. (This definition was early criticized by K. Grandjot and others, who believed that these equations should be taken as postulates. Peano has been ably defended against these criticisms by Ugo Cassina [1961, 291–298]. The remarks of later critics, such as J. van Heijenoort [1967, 83] have faulted this as part of Peano's overall lack of a deductive scheme. These criticisms are more to the point, although Peano's pioneer role in the use of recursive definitions should be emphasized.)

2. 2 Integers *n*

With the natural numbers available, it is a relatively simple matter to define the integers and rational numbers in terms of ordered pairs. W.O. Quine, for example, writes [Quine 1963, 120]:

Another way would be to take 2/3 as the class of all pairs ⟨2,3⟩, ⟨4,6⟩, ⟨6,9⟩, etc.; this would mean defining x/y in general as

$$\{ \langle z,w\rangle : z,w \in N \,.\, x\cdot w = z\cdot y\}.$$

Such was Peano's version (1901).

But such was, in fact, *not* Peano's version. Neither for rationals nor for integers did he use ordered pairs. Rather, in each case these were defined as operations (or somewhat ambiguously as result of operations). In 1891, after defining $\alpha-$ as the operation inverse to α, he wrote [Peano 1959, 94]:

The combination of the sign of the inverse − and of the positive number b is what is called negative number. Hence the sign −5 has the meaning "invert, and then repeat five times." Thus, if xp indicates "the father of x," the expression xp−2 means "the son of the son of x"; and if x is a [natural] number, x+−5 means "that which is obtained by carrying out the operation inverse to successor of x five times" or "the number preceding x by five places." We shall have no need to introduce any new symbol to indicate "negative number," since the notation −N is sufficient.

This is then followed by the definition: $n = N \cup 0 \cup -N$. (In which, as Cassina has observed [Peano 1959, 94], the symbol *0* stands for ι*0*, i.e. the set composed of the single element *0*. This lapse is surprising here, since Peano uses the symbol ι, introduced the previous year [Peano 1957, 130], later in this same article.)

In 1898 (the next time he treated negative numbers), Peano commented more explicitly on this way of introducing negative numbers, without, however, removing the ambiguities inherent in his definition. He wrote [Peano 1959, 227]:

> *P·7. 'Let a be a [natural] number; then −a is an operation that, applied to a number not less than a, produces a number.'*
>
> *This is only different in form from P·2. It may be compared with P015·4, which says that +a, i.e. the operation + repeated a times, is an* N_0 ɟ N_0 *[i.e. a function that maps* N_0 *into* N_0*].*
>
> *The symbols* $+N_0$ *and* $-N_0$ *(P024) correspond, more or less, to the words 'positive numbers' and 'negative numbers'; they appear as operations* ɟ*. The symbol +a is equivalent to the expression 'add a' and −a means 'subtract a'. Following common usage, however, we call these operations 'whole numbers' (P030) and by definitions P025, 037, and 039·1, we have the formal coincidence of the numbers* N_0 *and the positive numbers* $+N_0$*.*

In an article commenting on this section of the *Formulaire*, he wrote [Peano 1959, 245]:

> *The so-called negative numbers and fractions are defined as the operations of subtraction and division. They thus go back to remotest antiquity, and well represent the use that we make of them; the resulting theory is very simple. They are considered this way in a number of texts, in a more or less clear manner.*

If all this seems inadequate as *definition*, Peano does at least give a definition of the equality of two integers [Peano 1959, 227]:

> *P031. 'Two whole numbers x and y are equal, by definition, when for each positive number u, we have u + x = u + y, so long as these operations are possible on positive numbers.'*

$$031. \quad x,y\,\varepsilon\, n. \ \supset :. \ x = y \,.=: u\,\varepsilon\, N_0 \,.\, u + x,\, u + y\,\varepsilon\, N_0 \,.\, \supset_u .\, u + x = u + y \qquad Df.$$

The above explanations were repeated the following year, with the additional comment [Peano 1899, 58]: "This manner of considering positive and negative numbers is found more or less clearly in several authors," and Peano quotes from Maclaurin and Cauchy. This is again repeated in the 1901 *Formulaire de Mathématiques* [Peano 1901, 49], but the explanation is greatly reduced in the last two editions of the *Formulaire* [Peano 1905, Peano 1908]. Integers continued, however, to be defined as operations, and this process was continued in the case of rational numbers.

2.3 Positive Rationals *R*

Peano defined positive rational numbers in 1889 as follows [Peano 1958, 46; Peano 1975, 126]:

$$R = :: [x\,\varepsilon] :. \ p,q\,\varepsilon\, N,\ \ p/q = x : - =_{p,q} \Lambda$$

This may be read: R is the non-empty set of all x such that p and q are positive integers and $x = p/q$. But what is something of the form p/q? This has not been directly defined, but Peano did define the operation of the "ratio" p/q on a positive integer as:

$$m,p,q\,\varepsilon\, N \,.\ \supset .\ m(p/q) = mp/q.$$

This must be understood in the sense that, if mp/q has meaning for some m,p,q then $m(p/q)$ has the same meaning. But again, what is mp/q? This symbolism was previously defined by:

$$a,b\,\varepsilon\, N \,.\ \supset .\ b/a = N[x\,\varepsilon]\,(xa = b).$$

This may be read: If a and b are positive integers, then b/a is that positive integer that when multiplied by a equals b. Thus $m(p/q)$ is a positive integer.

This ambiguous state of affairs was clarified only slightly in “Sul concetto di numero” where, after defining p/q only if p is a multiple of q, Peano notes that if m and p are integers and q is a positive integer (or at least not 0), then $m \times (p/q) = (m \times p)/q$. Then he comments that if p is not a multiple of q, the expression $m \times (p/q)$ has no meaning. The expression $(m \times p)/q$ may, however, have meaning [i.e., if $m \times p$ is a multiple of q]. In this case he simply proposed to define $m \times (p/q)$ as $(m \times p)/q$. This still leaves open the question of whether he is thinking in this case of p/q merely as an ordered pair. I believe the most likely explanation is given in his next treatment of this subject, in 1898 [Peano 1898, 16] and repeated the following year [Peano 1899, 43], in which he gives the definition:

$$R = x \ni \exists\, (a;b) \ni [a,b\ \varepsilon\ N_1\ .\ x = (\times\ b)(/a) \qquad Df.$$

His understanding of this is explained by the comment [Peano 1899, 42]:

> *According to ordinary language, b/a precedes the number, or magnitude, on which one operates, and means ‘divide by a and multiply by b’. E.g., ‘3/5 of 15 francs’ means ‘that which is obtained by dividing 15 francs by 5 and multiplying the result by 3’. But we prefer to give to b/a, which follows the number on which one operates, the meaning ‘multiply by b and divide by a’, in order to make the operation possible in a larger number of cases.*

It seems clear from this that Peano was already thinking of a rational number as a (double) operation, and he confirmed this explicitly in 1901, in a footnote added to the publication of the paper he read at the International Congress of Philosophy in Paris (1900) [Peano 1958, 367]:

> *Now, the fraction a/b is introduced ‘by abstraction’ [by others]. They have no equality of the form :*
>
> *a/b = (expression composed of the preceding ideas),*

> *but they give only a nominal definition of the relation a/b = c/d.*
>
> *We prefer to consider a/b as representing the composite operation ×a/b, i.e. 'multiply by a and divide by b'. The two operators a/b and c/d are equal whenever, applied to the same number (one that makes the two operations possible), they give equal results . . . .*
>
> *One also finds this way of considering fractions as operators in Méray,* Leçons sur l'Analyse Infinitésimale *(1894), p. 2, and already published by him in* Nouvelles Annales de Mathématiques *(1889), p. 421. This idea is the most natural.*

Peano did not, however, insist on being followed in this way of defining rationals. Indeed, with regard to those, just mentioned, who defined the equality a/b = c/d and so introduced the rationals "by abstraction", he wrote in 1902 [Peano 1902, 64]: "The choice of one or the other of the possible definitions to be the actual definition depends on reasons of convenience."

2.4 Rationals *r*

Peano did not define the set of rational numbers in *Arithmetices principia* (1889), and in "Sul concetto di numero" (1891) there is only the abbreviated definition: $R = N/N$; $r = n/N$. This follows the introduction of positive rationals, and the first half of this is meant as a definition of the symbol *R*. However one understands *R*, then, *r* is to be understood similarly, replacing *N* in the "numerator" by *n*. This article contains no development of the rationals. This definition of *r* is repeated in all editions of the *Formulaire de Mathématiques* from 1898 on, except in the last edition (1908), where it is given as a "possible definition" and *r* is defined by: $r = +R \cup -R \cup \iota\, 0$ (which was given as a possible definition in 1901 and 1903, and as *the* definition in the *Aritmetica generale e algebra elementare* of 1902.)

2.5 Positive real numbers Q

Peano defined the set of positive real numbers in *Arithmetices principia* by:

$$Q = [x\ \varepsilon]\ (a\ \varepsilon\ KR : a - = \Lambda : R \ni > Ta\ .\ - = \Lambda : Ta = x :.\ - = \Lambda)\ .$$

This may be read: Q is the non-empty set of all x such that x is a least upper bound of a non-empty set of positive rational numbers and there are rational numbers larger than this least upper bound. Thus, a positive real number *is* a "least upper bound", which Peano had just introduced *without* a definition! He merely said [Peano 1958, 49; Peano 1973, 128]:

> *If $a\ \varepsilon\ KR$ [i.e., a is a set of positive rational numbers], the symbol Ta is read* upper boundary *or* upper limit of the class *a [i.e., least upper bound]. We shall define only a few relations and operations on this new entity.*

[In fact, he defines only the relations $x < Ta$, $x = Ta$, $x > Ta$, for $x\ \varepsilon\ R$.]

This time the explanation of his views came only a decade later in 1899 in "Sui numeri irrazionali", in which he reviews the above definition, transcribing it into his current symbolism. He then discusses the method of introducing irrationals of no less than eleven authors, quoting the following passage of Dedekind with especial approval [Peano 1959, 264]:

> *Jedesmal nun, wenn ein Schnitt (A_1, A_2) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so erschaffen wir eine neue, eine irrationale Zahl α, welche wir als durch diesen Schnitt (A_1, A_2) vollständig definiert ansehen.*

Peano notes: "In this 'erschaffen' (create) is precisely indicated that the real number is considered as an entity different from section, or segment."

Near the end of the article, Peano gives four statements, which include:

(a) Every class of (rational) numbers, all less than a given number, has a real number as least upper bound.

(c) Every sequence (of rationals) that satisfies the criterion of convergence, effectively converges toward a limit.

He concludes the article by commenting on these statements [Peano 1959, 267]:

These propositions, all true, even if one suppresses the condition 'of rationals' written between parentheses, are easily deduced one from the other. One of them may be transformed into the definition of irrationals. Our authors' opinions vary on this choice.

In the applications, especially to analysis, proposition (a) is used more often than (c). Indeed, in most texts, to recognize the convergence of series, or of integrals, etc., almost constant use is made of (c). Hence the opinion of Cantor [Mathematische Annalen 21 (1883), 567], *is justified, that his method is 'die einfachste und natürlichste von allen' and speaking of that of Dedekind, he says that 'die Zahlen in der Analysis* niemals *in der Form von "Schnitten" darbieten, in welche sie erst mit grosser Kunst und Umständlichkeit gebracht werden müssen.'*

But in several of my works I have shown that the definition of integral depends on the sole concept of least upper bound, and not on that of the limit toward which a function converges. Similarly, in geometrical applications we define, for example, the length of an arc of a curve as the least upper bound of the lengths of the inscribed polygons, rather than as the limit toward which their lengths converge (see Enzyclopädie der Mathematischen Wissenschaften, *p. 72). Thus the most appropriate form of the principle and the one that is applied continually in analysis appears to me to be (a), to which the definition of the irrational can therefore be directly attached.*

2.6 Real numbers *q*

Negative real numbers were not defined by Peano in *Arithmetices principia* (1889). In “Sul concetto di numero” (1891) the set of real numbers *q* was defined directly, without first defining *Q*, as [Peano 1959, 108]:

$q = (l'Kr)\ (-\iota\ \infty)\ (-\iota - \infty)$

“Real numbers are the least upper bounds of the classes of rational numbers, excluding $+\infty$ and $-\infty$.” This definition arises as an alternative to Dedekind’s “cut,” which Peano had just analysed, since, he says:

> *. . . instead of considering both classes A_1, and A_2, it is sufficient, as Dedekind has already observed, to consider only the first, A_1, seeing that the other, A_2, is the set of rational numbers not in A_1.*

This least upper bound, *l'A* where *A* is a set of rational numbers, is to be distinguished from the set *A*. In this regard Peano criticized Pasch for confusing the two, remarking [Peano 1959, 107]:

> *Hence, to remove this difficulty it is necessary to make correspond to every segment A a new entity, that I shall indicate by l'A (least upper bound of A); and to indicate the relation in question one writes l'A = a. Real numbers are therefore least upper bounds of segments.*

The difficulty that Peano is trying to avoid occurs, of course, in the case where the least upper bound of a set *A* is rational. Then the identification of the set *A* with its real (in this case rational) least upper bound a requires $a = A$, whereas the appropriate relation is $a\ \varepsilon\ A$.

Peano repeated the above definition of reals in 1899, but that same year in the *Formulaire de Mathématiques*, after having introduced the positive reals *Q*, he gave the definition: $q = Q \cup - Q \cup \iota\, 0$, and this definition was repeated in all later editions of the *Formulaire*.

3. Mathematical Definitions

We have just surveyed Peano's development of the real number system, starting from the natural numbers. Let us now return to the natural numbers to ask: To what extent did Peano believe the postulates *defined* the natural numbers? Peano himself raised the question of the possibility of defining the natural numbers in "Sul concetto di numero." After stating the five "primitive propositions" and some "immediate consequences," he observed [Peano 1959, 84–85]:

> *The first numbers presented, with which we form all the others, are the positive integers. And the first question is: Can we define one, number, sum of two numbers? The common, Euclidean, definition of number, 'number is the collection of several units', may serve as a clarification, but is not satisfactory as a definition. Indeed, very young children use the words one, two, three, etc. They later adopt the word number, and only much later does the word collection appear in their vocabulary. Indeed, philology teaches that these words appear in this same order in the development of the Indo-European languages. Hence, from the practical side, the question appears to me to be settled, or rather, there is no need for the teacher to give any definition of number, seeing that this idea is very clear to the pupils, and any definition would only have the effect of confusing them. The majority of authors also share this opinion.*
>
> *From the theoretical side, to decide the question of the definition of number, one should be told first what ideas he may use. Here we suppose known only the ideas represented by the signs* $\cap$ *(and),* $\cup$ *(or) ,* $-$ *(not) ,* ε *(is), etc., which have been treated in the preceding note. Therefore,* number cannot be defined, *since it is evident that however these words are combined among themselves, we can never have an expression equivalent to number. If number cannot be defined, however, we can*

still state those properties from which the many other well known properties of the numbers are derived.

The concepts, then, that we do not define are those of number N, of one 1, and of successor of a number a, which we indicate for the moment by a+. These concepts may not be obtained by deduction; it is necessary to obtain them by induction (abstraction). The successor of a is here indicated by a+, instead of the customary a + 1, and this is done so as to indicate by a single sign, +, the fundamental operation 'successor of'. Besides, in the following sections, having defined the sum a + b of two numbers, we shall see that a + 1 has precisely the value of a+, i.e., the successor of a, and thus we return to the customary notation.

A bit further on he adds [Peano 1959, 88]:

Between the preceding and what Dedekind says there is an apparent contradiction that should immediately be pointed out. Here, number is not defined, but its principal properties are stated. Instead, Dedekind defines number as precisely that which satisfies the preceding conditions. Evidently, the two coincide.

With regard to this quotation, two remarks may be made. First, we should notice that Peano answers his questions from two viewpoints: didactic and theoretical. His "theoretical" view is of more interest here, but we must always realize that it was colored by his views on teaching, and seldom did he make the distinction so clearly as here. Indeed, in his later years he discussed the didactic question more often. Second, Peano's notion of definition in mathematics varied a great deal, and his answers must be interpreted in the light of his acceptance of the various types of definitions at the time. In 1891 his view was rather narrow. It is clear that he is asking for a nominal definition of the form

x = (expression composed of preceding symbols).

This he has not given for the natural numbers, and he asserts its relative impossibility. Yet, his acceptance of Dedekind's use of the term "definition" ("Evidently the two coincide") is probably more than just conciliatory, for he shortly came closer to what he here described as Dedekind's position.

But what did Peano mean when he said that an idea was obtained "by abstraction"? He next discussed this in 1894 [Peano 1958, 167–168]:

> *There are some ideas obtained by abstraction and with which the mathematical sciences are constantly being enriched that cannot be defined in the form stated. Let u be an object; by abstraction, one deduces a new object φu. We cannot form an equality*
>
> $$\varphi u = \textit{known expression},$$
>
> *for φu is an object of a nature different from all those that we have considered up to the present. Rather, we define the equality $\varphi u = \varphi v$ by setting*
>
> $$h_{u,v} \,.\, \supset : \varphi u = \varphi v \,.\, =. \; P_{u,v} \qquad \textit{Def.}$$
>
> *where $h_{u,v}$ is the hypothesis on the objects u and v. Thus $\varphi u = \varphi v$ means the same as $P_{u,v}$, which is a condition, or relation, between u and v, having a previously known meaning. This relation must satisfy the three conditions of equality that follow: . . .* [Here Peano describes the reflexive, symmetric, and transitive properties of an equivalence relation.]
>
> *The object indicated by φu is therefore what one obtains by considering in u all and only those properties that it has in common with the other objects v such that $\varphi u = \varphi v$.*

Peano gives as an example of this Euclid's definition of ratio of two magnitudes in Book V of the *Elements*. He is still unwilling to admit that objects are truly *defined* "by abstraction," but he moved a bit closer to this view in his statement near the end of this article [Peano 1958, 175]:

Whatever the manner of reasoning, if a science does not contain primitive ideas, as happens in every advanced theory, one can define and prove everything in it. But if the science touches its very elements, and if there are ideas that cannot be defined, one will also find propositions that cannot be proved, and from which all the others follow. We shall call these primitive *propositions, abbreviated by Pp; they are also called axioms, postulates, and sometimes hypotheses, experimental laws, etc. These propositions determine or, if you like, define the primitive ideas that have not been given a direct definition.*

In 1897 Peano was still insisting on this form of definition [Peano 1958, 208]:

By symbolic definition of a new symbol x we understand the convention of calling x a group of symbols already having a known meaning; and we indicate this by

$$x = a \qquad \textit{Def.}$$

In 1898, with regard to the postulates for the natural numbers, he wrote [Peano 1959, 217]:

Les idées primitives sont déterminées par les 5 propositions primitives 002 desquelles découlent toutes les P[ropositions] de l'Arithmétique.

Peano meant that the postulates "determine" the natural numbers to the extent that the theorems of arithmetic follow. He was aware from the beginning that they do not uniquely characterize the natural numbers, as he explicitly pointed out in "Sul concetto di numero" [Peano 1959, 87]:

These propositions express the necessary and sufficient conditions that the entities of a system can be put into one-to-one correspondence with the series of natural numbers.

In 1898, again with regard to the postulates for the natural numbers, he wrote [Peano 1959, 243]:

> *The analysis of the ideas of arithmetic contained in F_2§2* [i. e. the *Formulaire* of 1898] *is the only one in existence today. To this group of propositions 002·1–5, which may be called a definition of the positive integers, using the word definition in a wider sense than that given in F_2§1 P7* [i.e. in the *Formulaire* of 1897, as quoted above]*, the nearest work is that of Dedekind of 1888.*

By 1899 Peano was willing to admit [Peano 1959, 261]: “The word ‘definition’, even in books of mathematics, has several meanings.” He went on to say:

> *Prof. C. Burali-Forti, in his text* Logica Matematica*, Milan, 1898, pp. 120–148, has classified the definitions that are met in the theories already expressed in ideographic symbols, distinguishing them by the names : nominal definition, definition by induction, by abstraction, etc.* [The “etc.” seems superfluous, since Burali-Forti only discusses the three types mentioned.] *Of these various types of definitions, the nominal appears to be the most satisfactory. Many definitions of the other types contained in the early works of mathematical logic could be transformed into nominal definitions. Of definitions by abstraction, in F_2N2 (Arithmetic)* [i.e. the *Formulaire* of 1898] *use is made only once, in P210·1, to define the cardinal number, or power, of a set.*

Since Peano says that he used definition by abstraction only once in the *Formulaire* of 1898, we must understand that he is using the term “definition by abstraction” as a technical term, since the expression “by abstraction” is used in that book in referring to the postulates for the natural numbers, presumably used in a wider sense. After noting that there are an infinity of systems that satisfy the postulates, he wrote [Peano 1959, 218]:

> *Tous les systèmes qui satisfont aux 5 Pp sont en correspondence réciproque avec les nombres. Le nombre, N_0, est ce qu’on obtient par abstraction de tous ces*

systèmes; autrement dit, le nombre N_0 est le système qui a toutes et seules les propriétés énoncées par les 5 P primitives.

This sentence, slightly altered, was repeated in the *Formulaire* of 1899, of 1901, and of 1903, but was omitted from the final edition of 1908.

By 1900 the problem of what constitutes a definition in mathematics had become of greater interest to Peano, and he made this the subject of his talk at the International Congress of Philosophy in Paris that summer. There he said [Peano 1958, 362]:

Une définition est réductible à une égalité, dont un membre (le premier) est le nom qu'on définit, et l'autre en exprime la valeur. Example :

(dérivé d'une fonction) = (limite du rapport des accroissement de la function et de la variable).

En consequence, un proposition qui n'est pas une égalité ne pourra pas être une definition.

The problem posed by Peano does not concern the *form* that a definition must have – this he consistently affirms must be an equality – but rather is the question of which equalities are (or could be) taken as definitions. Of special interest, because of its immediate influence, is the example $0 = a - a$, as an equality that may not be taken as a definition of zero. First of all, according to Peano, it is incomplete, since we are not told what value to give to the letter *a*. Second, even if we say: "Let *a* be a number; then $0 = a - a$," the equality is still not homogeneous, for the first member is the constant symbol *0*, while the second is a function of the variable letter *a*. Peano then shows how to properly phrase the definition [Peano 1959, 366]:

La proposition:

0 = (la valeur constante de l'expression a – a, quel que soit le nombre a) *est une égalité homogène, car, bien que dans le second membre figure la lettre a, elle n'y figure qu'en apparence, puisque la valeur de ce second membre n'est pas une fonction de a. Cette proposition est une définition possible.*

After the talk, Schröder objected that Peano's condition of homogeneity was too restrictive, but Peano (backed up by his disciple Alessandro Padoa) defended it. Ivor Grattan-Guinness has discovered from manuscript sources (to be described in detail in an article in preparation by him) that it was their discussion on this topic that convinced Bertrand Russell of Peano's superiority over Schröder.

In 1901, as part of a projected dictionary of mathematics, Peano wrote a brief dictionary of mathematical logic. There he again insisted on the "equality" form of a definition, but we find under the heading "abstraction" the following [Peano 1958, 373]:

In mathematical logic, what is called "definition by abstraction" is the definition of a function φx, having the form:

$$\varphi x = \varphi y \ . = . \ \text{(expression composed of the preceding symbols)},$$

that is, the isolated symbol φx is not defined, but only the equality $\varphi x = \varphi y$.

At this time, although Peano recognized definitions by abstraction as valid, he felt that their use should be avoided where possible. This is shown in a letter of May 1902 to L. Gérard, editor of the *Bulletin des sciences mathématiques et physiques*, and published in that journal the following month. After recalling that Burali-Forti had classified the definitions used in mathematics and had called "definition of the first type" those of the form [Peano 1959, 371]:

$$x = a \qquad\qquad Df.$$

ou x est le signe simple qu'on definit, et a est un groupement. des signes connus. Le signe = accompagné de Df signifie « est égal par definition » ou « nous nommons »,

he gives several examples of definitions by abstraction, concluding with [Peano 1959, 572]:

Plusieurs analystes introduisent les nombres rationnels par abstraction, en posant

$$a,b,c,d \ \varepsilon \ (\textit{nombres naturels}) \ . \supset : a/b = c/d \ . = . \ ad = bc,$$

mais cela n'est pas necessaire, car on peut en donner une Df de première espèce.

D'une façon analogue, notre Df 1 est une Df par abstraction, car elle definit une égalité.

Or, il est bon de remarquer qu'on peut définir les nombres imaginaires par des Df de première espèce, et qu'il n'est pas necessaire de recourir à des Df par abstraction.

Peano continued to be interested in the question of definitions in mathematics (and in the last year of his life assigned this as a thesis topic to the young Ludovico Geymonat, now Professor of the Philosophy of Science at the University of Milan, Italy), but his next publication on this topic was not until 1911 and by that time he had ceased to be the innovator in this field, and although he was still capable of incisive comments, he was usually content to describe the views of others. This resulted from two major events, both dating from 1903. One was the introduction of *Latino sine flexione* as an international auxiliary language. The following years saw his almost complete dedication to the international auxiliary language movement, confirmed by his election in 1908 to the directorship of the Academia pro Interlingua, a position he held until his death in 1932.

The second event of 1903 that influenced Peano was the publication of Bertrand Russell's *Principles of Mathematics*, which, as Peano wrote to Russell [Kennedy 1974, 31], “marked an epoch in the field of philosophy of mathematics.” He had already written Russell in 1901 [Kennedy 1974?]:

> *Permettez-moi de me féliciter avec vous de la facilité et de la precision avec lesquelles vous maniez les symboles de Logique.*

Now he was content to let Russell take over leadership in this field, and in 1910 could only write him [Kennedy 1974?]:

> *Je vous remercie du livre* Principia Mathematica*, que je me propose de lire avec attention. Maintenant, mes heures libres de l’école sont occupées dans la question de l’Interlingua, ...*

Peano did not, however, accept everything that Russell was doing. In particular, he rejected Russell’s “class of classes” definition of cardinal number, published by Russell in Peano’s journal in July 1901 [Russell 1901, 121]. Curiously, Peano’s rejection was published several months earlier in *Formulaire de mathématiques*, vol. 3, dated 1 January 1901. There Peano defined the cardinal number of a set *a*, symbolized by *Num a*, “by abstraction,” adding [Peano 1901, 70]: “mais on ne peut pas identifier Num a avec la Cls de Cls considérée, car ces objets ont des propriétés différentes.” Although this publication preceded that of Russell, Peano would have had Russell’s manuscript since October 1900 and so was probably prompted by it to consider the “class of classes” definition. At any rate, Russell did not alter his manuscript, but commented in *Principles of Mathematics* [Russell 1905, section 111]: “He does not tell us what these properties are, and for my part I am unable to discover them.” Significantly, however, Peano does identify the finite cardinal numbers with the natural numbers. Indeed, he had already done this in “Sul concetto di numero” where, although Peano was aware that the postulates do not characterize the concept of natural numbers, nevertheless, as Ugo Cassina has noted [letter to the author dated 4 June 1961], “the way in which he attacks the problem of numeration [in Peano 1959, 100], i.e., his inductive definition of ‘number of objects of a [finite] class’ results in giving to the primitive entities 0, N_0, + (or ‘suc’) the intuitive meaning.”

What, then, was Peano's objection to the "class of classes" definition? Most probably it was the artificiality of this concept as opposed to what Peano saw as the "natural" concept of number. Consider, for example, the passage of 1906 [Peano 1957, 343]:

That is, we have deduced theorems identical to the postulates of arithmetic. Therefore, for the symbols of arithmetic 0, N, +, there exists an interpretation that satisfies the system of postulates. Thus it has been proved (if proof were necessary), that the postulates of arithmetic, which the collaborators of the Formulaire *have shown to be necessary and sufficient, do not involve a self-contradiction.*

Other examples of entities that satisfy the system of postulates have been given by Burali-Forti and by Russell. But a proof that a system of postulates of arithmetic, or of geometry, does not involve a self-contradiction is not, I think, necessary. For we do not create postulates at will, but we assume as postulates the simplest statements that, either written in an explicit way or implicitly, are in every treatise of arithmetic or of geometry. Our analysis of the principles of these sciences is the reduction of the ordinary statements to a necessary and sufficient minimum. Systems of postulates of arithmetic and of geometry are satisfied by the ideas of number and point that every writer of arithmetic and geometry has.

With the publication of Whitehead and Russell's *Principia Mathematica*, Peano seems willing to concede to them the "class of classes" definition of number. He wrote in 1913 in his review of Vol. 1 [Peano 1958, 597–398]:

Page 363 begins the treatment of cardinal numbers. The authors eliminate definitions by abstraction. In many cases mathematicians introduce a new entity φx, not by a definition of the form

$$\varphi x = \text{expression composed of } x \text{ and known symbols,}$$

but they define only the equality:

$\varphi x = \varphi y \; . = . \; \text{relation } P_{x,y} \text{ composed of } x,y, \text{ and known elements.}$

The authors prove that definitions by abstraction can be reduced to nominal definitions; it suffices to set

$$x = y \ \exists \ (P_{x,y}).$$

Peano's acceptance of all this, however, is not complete, and in 1915 he again defends definitions by abstraction in an article with the title "Le definizioni per astrazione" [Peano 1958, 402– 416]. Perhaps he felt, in some way, the necessity of defending what was his, for although the recognition of the importance of the procedure that leads to it goes back at least to H. Grassmann's *Ausdehnungslehre* of 1844, nevertheless it was Peano who introduced the name "definition by abstraction" in 1894 [Peano 1958, 167]. But his objections are on practical grounds; he seems to accept the logical validity of replacing definitions by abstraction by nominal definitions. In this same article there is an interesting comment apropos definitions by abstraction and the cardinal numbers (of Cantor). After noting that one says the cardinal numbers of two sets are the same if the sets can be put into a one-to-one correspondence, he concludes [Peano 1958, 404]:

> *One thus defines the equality of two numbers, and not number itself; and this because this definition may be placed before arithmetic, and also because the number that results is not the finite number of arithmetic.*

It is not clear what distinction he intends here, since in the *Formulaire* he identified the finite cardinals with the natural numbers – but perhaps he only means that some cardinals are infinite. Peano next describes Padoa's theory of equivalence relations (relations that are reflexive, symmetric, and transitive – Padoa called them "egualiforme") and concludes [Peano 1958, 415]; "Between the two theories, definition by abstraction and equivalence, I see only a difference of language."

Finally, Peano comments on Russell's practice of replacing definitions by abstraction by nominal definitions. But by 1915 he does not want to oppose the authority of Russell, and he defends his neutrality. He wrote [Peano 1958, 414]:

> *The question of abstraction pertains to pure logic, and we can give non-mathematical examples. Are the following equations true or not?*
>
> | | |
> |---|---|
> | *whiteness* | *= white things,* |
> | *sickness* | *= sick people,* |
> | *youth* | *= young people,* |
> | *Italy* | *= the Italians,* |
> | *justice* | *= judges, police, jail.* |
>
> *The theory of Russell answers in the affirmative. I, investing myself with the authority of Euclid (like a live ass covered with the hide of a dead lion), neither affirm nor deny. This identity is denied by the doctor who says 'there are no sicknesses, but only sick people', as well as by the opposite theory that says 'I conquered the sickness and killed the sick person'.*

The question of definitions in mathematics continued to interest Peano, and he returned to this topic in 1921 in an article discussing the various types of definitions. He even included a discussion of "primitive ideas" and their use in setting up an axiom system [Peano 1958, 435; 1973, 244]:

> *The fundamental properties of the primitive ideas are determined by the "primitive propositions", or the propositions that are not proven, and from which are deduced all the other properties of the entities considered. The primitive propositions function in a certain fashion as definitions of the primitive ideas.*

With this last remark, we apparently have Peano's complete acceptance of the Postulates for the Natural Numbers as a definition, something he appeared reluctant to accept in 1891. Having in the meantime searched for another definition, and not finding it, he ends by adopting a naive axiomatic viewpoint, while still equivocating with the restriction "in a certain fashion."

4. Conclusion

Peano made no pretense of being a philosopher and, indeed, denied competence in this field. Nevertheless several things stand out in his development of, and comment on, the real number system, as it has been traced above. Perhaps the most important aspect of his work in the foundations of mathematics is that he made no attempt to found mathematics on any prior discipline. In particular, he denied the validity of Russell's reduction of mathematics to pure logic. Peano was concerned with reducing arithmetic, say, to the minimum number of undefined terms and axioms, from which all the rest could be defined and proved. For him the goal of the axiomatic method was to make the theory clearer, more precise, and *easier to learn*.

Peano saw mathematical logic as contributing to this goal. Already in 1894 he wrote Felix Klein [Peano 1894]:

> *Mathematical logic, with a very limited number of signs (actually seven, but further reducible), has succeeded in expressing all the logical relations imaginable among classes and propositions; or rather, the analysis of these relations has led to the use of these signs, with which everything can be expressed, even the most complicated relations that are expressed in ordinary language with fatigue and difficulty. But the advantage is not limited to simplifying the writing; its usefulness lies especially in the analysis of the ideas and reasoning that make up mathematics* .

As an example of this usefulness, he continues:

> *I could cite many so-called theories that evaporate when translated into symbols; they exist only in appearance, by exchanging a new name for an old idea. I limit myself to mentioning that several parts of the theory of fields and moduli of*

Dedekind are merely propositions of logic, and as such are included in Part I of the Formulaire.

It seems clear, too, that Peano was not a formalist, in the sense of Hilbert. This point has already been emphasized by his ardent disciple Ugo Cassina (1894–1964) [see Cassina 1961]. Mathematics, then, has content and is not merely a game. In 1923 Peano wrote [Peano 1923, 383]:

> *Mathematics has a place between logic and the experimental sciences. It is pure logic; all its propositions are of the form: 'If one supposes A, then B is true.' But these logical constructions must not be made for the mere pleasure of reasoning about them. The object studied by them is given by the experimental sciences; they must have a practical goal.*

Thus Peano believed that mathematics can, and must, be based on experience, as we have already seen above in his comment on consistency proofs of an axiom system.

Many of Peano's views on the concept of number were echoed in 1939, in an article that has recently been reprinted, by A. Ya. Khinchin (Александр Яковлевич Хинчин, 1894–1959). For example [Khinchin 1968, 3]:

> *The concept of number is distinguished from many other concepts of the school course by its* primarity. *This means that in the overwhelming majority of ways in which mathematics can be developed as a logical system, the idea of number belongs to the set of those concepts which are not defined in terms of other concepts, but together with the axioms enter into the ranks of the initial data. It means that mathematics does not contain within itself an answer to the question 'what is a number?' – an answer, that is, which would consist of a definition of this concept in terms of concepts that had been introduced at an earlier stage; mathematics gives this answer in a different form, by listing the properties of a number as axioms.*

Further, I believe that Peano, with his great concern for the teaching of mathematics, would have approved Khinchin's advice to the teacher whose pupil asked the question 'what is a number?' [Khinchin 1968, 4]:

> *Tell him that the question he has posed is one of the most difficult questions in the philosophy of science, one to which we are still far from having a complete answer; that a number, in the same way as any mathematical concept, is the reflection in our own consciousness of certain relations in the real world; but that the question, precisely which relations of the world find their reflection in the concept of number, which relations are quantitative, is a deep and difficult philosophical problem; mathematics itself can only point out to those who study it what types of numbers there are, what are their properties, and how they can and should be manipulated.*

That Peano himself recognized the seriousness of this problem and was not misled by the facile "philosophical" solutions current in his lifetime (e.g. logicism and formalism) is confirmed by the conclusion of L. Geymonat with regard to the difficulty in determining Peano's views. He wrote [Geymonat 1955, 62–63]:

> *If we must recognize this hesitation of Peano, it would nevertheless be superficial to attribute it to his incapacity to detect the philosophical problem underlying the mathematical one. It seems to me more exact to recognize, on the contrary, the* critical depth *of his position, inspired by a somewhat excessive caution. The slightly, but constantly, ironic tone with which he talked about philosophical discussions – was it not perhaps the trench behind which he wished to defend himself from the temptations of the rash theories of the 'philosophers'?*

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Nine Letters from Giuseppe Peano to Bertrand Russell

H. C. KENNEDY

INTRODUCTION

PRESERVED IN THE BERTRAND RUSSELL ARCHIVES at McMaster University (Hamilton, Ontario) are nine letters from Giuseppe Peano to Bertrand Russell, covering the period from 1901 to 1912.[1] The first was written only a few months after they met at the International Congress of Philosophy, Paris, 1900, and the last was written at the Fifth International Congress of Mathematicians, Cambridge, 1912. This was a period of great change in the interests of Peano, a transition from the development and promotion of mathematical logic to his dedication to the ideal of an international auxiliary language. In 1900 he was the most prominent figure in the field of mathematical logic, but his accomplishments were in the preceding period. While Peano's interests waned, Russell's grew, and from the inspiration of meeting Peano (Russell wrote that the event "was a turning point in my intellectual development"[2]), Russell became, by 1912, the leader in this field.

The correspondence presented here is one-sided, since we have only the letters Peano wrote to Russell. (The letters from Russell to Peano have apparently been lost.) Hence, we learn little of Russell, except for additional confirmation of his generosity. Nor, indeed, do the letters reveal a great deal about Peano, but this little is precious, since the documentary evidence of his life is sparse. It shows that, far from seeing Russell as a rival, Peano gladly recognized his accomplishments. We also get an echo of the opposition Peano met in his efforts to introduce his own language for scientific communication, when he mentions, in 1910 and again in 1912, that most people think it absurd. This opposition becomes a direct confrontation in the last letter.

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[1] I wish to thank Mr. Kenneth Blackwell, Archivist of the Bertrand Russell Archives, McMaster University, Hamilton, Ontario, for very kindly furnishing copies of these letters, and Peano's nieces, Carola, Maria, and Caterina Peano, for permission to publish the writings of Peano.

[2] B. Russell, *The Autobiography of Bertrand Russell: 1872–1914* (Boston: Little, Brown and Company, 1967), p. 217.

We present the letters first in English translation, with corresponding footnotes, and then in the original language text. (Only obvious corrections of grammar and spelling have been made.)

THE PEANO-RUSSELL LETTERS

Turin, 19 March 1901[3]

Dear Sir:

I shall publish directly your interesting memoir, which fills a gap between the work of Peirce and Schröder on the one hand and the *Formulaire* on the other. Let me congratulate you on the facility and precision with which you manage the symbols of logic. Here are a few typographical remarks.

The notations $\breve{R}$ $\breve{\rho}$ do not pose serious typographical difficulties. Still, if you wish to write all the signs on the same line, you could replace them by Inv(erse) R, IR. The notation R | of the *Introduction au Formulaire* (1894), p. 39, would now produce some confusion with the sign for substitution. The notation ρ to indicate the domain of R is clear, so long as the relation is indicated by a single letter. I am afraid there will be difficulties in indicating the domain of RS, R ∪ S, $\breve{R}$, i.e. relations indicated by a composite expression.

Classes of pairs correspond to relations.

Your P 1·4 $\quad R \,\varepsilon\, \mathrm{rel} \,.\, \supset : \exists\, \rho \,.\, = .\, \exists\, \breve{\rho}$

corresponds to 2·2 of § ∃ (p. 28 of the *Formulaire*)

[3] Russell's memoir, referred to in this letter, was submitted in the fall of 1900 and was published in the next issue of Peano's *Rivista di Malematica* with the title, "Sur la logique des relations avec des applications à la théorie des series." (This issue, however, did not appear until 15 July 1901.) Peano's delay in writing to Russell is partly explained by final preparations for the publication of the third edition of the *Formulaire de Mathématiques* (preface dated 1 January 1901), and then by a trip south that Peano took in February and March, visiting Rome and going as far as Tunis.

We see in this letter that Peano immediately recognized the importance of Russell's theory of relations, "which fills a gap between the work of Schröder on the one hand and the *Formulaire* on the other." He points out, however, that a relation corresponds to a class of pairs, and that this had already been symbolized in the *Formulaire*. (His reference is to the 1901 *Formulaire*, but the formula was already in the 1899 edition.)

$$\exists\, x \ni \exists\, y \ni [(x;y)\ \varepsilon\ a]\ .=.\ \exists\, y \ni \exists\, x \ni [(x;y)\ \varepsilon\ a]$$

May we represent your sign by the letter Z turned 90°, and the sign by T?

The introduction of signs that are not in the typesetters’ stock of ordinary or mathematical symbols causes long delays for the manufacture of new type.

Sincerely yours,
G. Peano
Via Barbaroux 6.

Turin, 27 May 1903[4]

Esteemed colleague:

Thank you for your book *The Principles of Mathematics*, which I have read in part with great interest. It will surely interest every reader. I sent to Professors Pieri and Vailati the copy addressed to them.

I would very much like to talk at length about your book, which marks an epoch in the field of philosophy of mathematics, but I have always had to travel around Italy. This evening I leave for Rome; hence I can only express my deepest thanks to you.

Yours sincerely,
G. Peano

[4] This is the only one of Peano’s letters that was written in Italian. It is hardly more than a “thank you” note for Russell’s gift of *Principles of Mathematics*, but Peano’s remark that the book “marks an epoch in the field of philosophy of mathematics” is high praise indeed!

Rome, 7 March 1905[5]

Dear Sir:

I received your money order for 128 francs, which I negotiated without difficulty. Thank you very much for your cooperation in the publication of the *Revue de Mathématiques*.

My health is not very good. I am traveling about Italy, and the publication of the *Revue* is late, as is this letter. I received from Mr Vacca the first part of an article dealing with your work, Vol. 1, which I see mentioned in Italy, too, in several publications.

I shall be back in Turin within a few days.

Best wishes, and sincere thanks.

G. Peano

Turin, 16 February 1906[6]

[5] This is another "thank you" note, this time for financial support of Peano's journal. As the subscription price of the current issue of the *Rivista di Matematica* was only 11 francs, Russell's contribution of 128 francs was very generous. "Volume 1" is again, of course, *Principles of Mathematics.* As we learn from the next letter, Giovanni Vacca did not finish his review of this book and, consequently, no review was published in the *Rivista.*

This letter is especially interesting for Peano's rare complaint that his health was "not very good." Except for an attack of smallpox in August 1890 he enjoyed generally good health until his unexpected death in 1932.

[6] Peano's first publication in logic was in 1888. He had indeed begun to retire from the field. The publication of the last volume (vol. 8) of the *Rivista di Matematica*, for example, was begun in 1902, but not completed until 1906.

We see from Peano's remark regarding Pieri that he did not see a way out of Russell's paradox. (Peano was probably among the first to whom Russell communicated the paradox, since in his famous letter of 22 June 1902 to Frege, Russell complains that he had written to Peano about it, but that Peano still owed him an answer.) We also find Peano defending "definitions by abstraction" against their rejection by Russell.

The article of Poincaré referred to is "Les mathématiques et la logique" (*Revue de Métaphysique et de Morale*, 13 [1905], 815–835). The last part of this article was in the January 1906 issue of the *Revue de Métaphysique et de Morale* (14 [1906], 17–34). (Poincaré would reply to criticism in another article with the same title.) Peano's brief discussion in this letter of the Cantor-Bernstein theorem was elaborated in an article "Super theorema de Cantor-Bernstein," dated 31 March 1906, in the *Rendiconti del Circolo Matematico di Palermo*, 21 (1906), 360–366 (reprinted in *Rivista di Matematica*, 8 [1902–1906], 136–143).

Dear Sir:

Thank you for your contribution to the promotion of symbolic logic. I worked toward this goal for seventeen years almost alone. Now, thanks to you and Messrs Couturat, Whitehead, and others, these theories have entered the public domain and I can take a rest, as I began to do some time ago.

You once told me that you were a subscriber to the *Revue de Mathématiques*. Consequently you received the *Formulario Mathematico*, vol. 5, part 1. If you have not received it regularly, please let me know, and I will take care of sending it to you.

Dr. Vacca, who was an assistant at the University of Turin, wrote, but did not complete, the review of your *Principles of Mathematics*. Subsequently he entered politics, was municipal councilor for Genoa, and is now in Florence. His incomplete article could not be published, and I regret that, the appropriate time for it having passed, the *Revue de Mathématiques* was unable to publish a review of your book. However, it is read and well received in Italy. I have received an article from Pieri, "Sopra una definizione aritmetica degli irrazionali," that studies your definition, and the definitions by means of classes, as you make them, which correspond to definitions by abstraction. He distinguishes classes of classes from classes, and thus it seems to me that he is getting nearer to the solution of the celebrated paradox that bears your name.

Some analogous questions have been the occasion of debates among Zermelo, Borel, Lebesgue, Jourdain, and others in the *Mathematische Annalen* and in the *Bulletin de la societé mathématique de France*.

You have read the critical article of Mr. Poincaré concerning logistic logic. I am not answering him because Mr. Couturat is going to do so. Mr. Burali-Forti has replied to several lines that concern him. It appears that Mr. Poincaré was not *au courant* with the progress of this science. His challenge to logicians, concerning the proof of the Cantor-Bernstein Theorem, can be accepted, and the idea of number, N_0, can be eliminated, replacing it by its definition according to the primitive propositions.

In fact, the Cantor-Bernstein Theorem is:

$$a,b \;\varepsilon\; \mathrm{Cls} \;.\; g \;\varepsilon\; (bFa)\mathrm{sim} \;.\; h \;\varepsilon\; (aFb)\mathrm{sim} \;.\; \supset . \; \exists \; (bFa)\mathrm{rcp}$$

It can be reduced to the particular case considered by Borel:

$$a \,\varepsilon\, \mathrm{Cls} \,.\, f \,\varepsilon\, (aFa)\mathrm{sim} \,.\, b \,\varepsilon\, \mathrm{Cls} \,.\, f\,‘a \,.\, \supset b \supset a \,.\, \supset \,.\, \exists\, (bFa)\mathrm{rcp}.$$

The proof of Bernstein, reproduced by Borel, and then by Poincaré, depends finally on S_x, which is defined as follows:

$$x \,\varepsilon\, a \,.\, \supset \,.\, S_x = f^{N_0}\, x \qquad \mathrm{Df}$$

S_x, a series determined by the element x, is the set of individuals x, fx, f^2x, f^3x, ... Hence the proof is:

$$[f, S(a\sim b)] \cup [\mathrm{idem}, a\sim S(a\sim b)] \,\varepsilon\, (bFa)\mathrm{rcp}$$

“the function equal to f in the field of the series determined by the class a$\sim$b, and equal to the identity in the remaining field, is in fact a reciprocal correspondence (one-to-one) between the members of a and the members of b.”

It is a question now of eliminating the symbol for number, N_0, in the definition of S. This definition could be given as follows:

$$x \,\varepsilon\, a \,.\, \supset \,.\, S_x = \cap\, \mathrm{Cls} \cap s \ni (x \,\varepsilon\, s : y \,\varepsilon\, s \,.\, \supset_y .\, fy \,\varepsilon\, s) \qquad \mathrm{Df}$$

S_x is the intersection of the classes that contain x and contain the successor fy of every member y of the class.

If I could be useful to you by furnishing information on the articles I have mentioned (if, by chance, you have not seen them), I would be only too happy to do so.

Sincerely yours,

G. Peano.

Turin, 24 July 1906[7]

Dear Sir:

Thank you for the reprint of your article in the *American Journal of Mathematics*. I had already written you a post card when I received it.

Preoccupied with my professional duties, I had forgotten Mr. Poincaré and his latest article. Rereading it, I would make the following remarks.

In 1890 (*Mathematische Annalen*, vol. 37, p. 210) I encountered the form of reasoning that constitutes the principle of Zermelo: “One may not apply an infinite number of times an arbitrary law by which to a class is made to correspond an individual of that class.”

Indeed, this form of reasoning is not reducible to the usual forms (for example, to those contained in pages 1–14 of the *Formulaire*, vol. 5); and to prove a proposition means to deduce it from known propositions by the usual forms of reasoning, without adding new principles.

The form of reasoning in which one takes an arbitrary element is:

$\exists\, a$	(1)
$x \,\varepsilon\, a \,.\, Ↄ \,.\, p$	(2) where p is a proposition that does not contain x.

[7] The article that Russell sent Peano was “The theory of implication” (*American Journal of Mathematics*, 28 [1906], 159–202). The post card mentioned by Peano is missing.

Peano was the first to state explicitly the Axiom of Choice, or Zermelo’s Postulate. He begins his remarks concerning Poincaré’s article (apparently requested by Russell) by elaborating his reasons for rejecting that principle. (The reference to the *Formulaire,* vol. 2, is to no. 1, of 1897.) We are making no attempt here to analyze his ideas, but perhaps it should be noted that, in Peano’s notation, “∃ a” means “the set a is not empty.” In his definition of “difference,” Peano underscored (in italics here) the words corresponding to Poincaré’s statement. It appears from Peano’s concluding remark in this letter that he did not plan an article on the subject of the antinomies. Nevertheless, he did elaborate these ideas and publish them in the last issue of the *Rivista di Matematica* as an “Additione” to a reprinting of the earlier article on the Cantor-Bernstein theorem. It concludes with comments on the Richard paradox that Jean van Heijenoort says are “perhaps the most penetrating commentaries on the paradox” (J. van Heijenoort, ed., *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931* [Cambridge, Mass.: Harvard University Press, 1967], p. 142). This “Additione” is dated 23 August 1906. The issue of the *Rivista* is dated 24 August. Peano left the next day for Geneva.

$(1) . (2) . \supset . p.$

This is called "elimination" (*Formulario*, vol. 5, p. 12, Prop. 3·1). It is reducible to the rule of "importation," as results from the *Formulario*, vol. 2, Prop. 405. I find this last in your "Theory of implication," but I do not find the rule of "elimination."

If we take two arbitrary elements, the proof contains one more proposition. For example, see *Formulario*, vol. 5, p. 139, Prop. 1·2, where the two letters y and z are eliminated. The elimination of n elements leads to an argument with n + 2 propositions. Consequently, one may not make $n = \infty$.

The statement of Mr. Poincaré on page 313 does not seem to me to be satisfactory. It is not a question of evidence, but of absolute truths.

The scholastics said that "every form of reasoning is reducible to syllogisms." This means that they consider as syllogisms several forms of reasoning, such as elimination, which contains three propositions, and three terms a, x, p.

Consequently, the answer is (I believe) the following: "The assumption of a finite number of choices is reducible to syllogisms. The application of the principle of Zermelo in many cases is reducible to syllogisms. Thus several proofs, in which Mr. Borel implicitly adopts the principle of Zermelo, are given in the *Formulaire* without it. If, however, an author states that he has reduced to syllogisms what he has not so reduced, his statement is false."

The antinomy of Richard has been given, by its author, the solution: "N is defined by a finite number of words and by E. We may not conclude: N is defined by a finite number of words." If, however, we replace E by its definition, which is composed of a finite number of words, we will have: "N is defined by a finite number of words," and the antinomy remains.

The "true solution" of Mr. Poincaré introduces the condition that in the definition of E may not be introduced the *notion* of E. If he means the letter, or symbol, E, or composite symbols containing E, then the matter cannot be argued. However, every definition contains in its second member *exactly* the notion of the symbol defined, for it is equivalent to it.

For example, we could apply the criticism of Mr. Poincaré (page 319) to the definition of difference:

$$a \,\varepsilon\, N_0 \,.\, b \,\varepsilon\, a + N_0 \,.\, \supset .\, b - a = N_0 \cap x \ni (a + x = b) \qquad \text{Def} -.$$

The defect remains the same: b – a is, among *all* numbers, that which satisfies the condition. *Under penalty of a vicious circle, this must mean* among *all* numbers, *in whose definition the notion of* difference *does not enter. That excludes* b – a, *which depends on the sign* – .

The solution of the Richard antinomy is, I think, the following: If n is the place that the phrase G, the definition of N, has in the set E, then assuming it pertains to E, the author sets:

$$\mathrm{Gr}_{-n}\, N \sim\, = \mathrm{Gr}_{-n}\, N,$$

that is, the absurd condition: $x \sim\, = x$. The phrase G does not express a number (although it has the appearance of the definition of a number). This is an expression, like max Np [maximum prime number], which has no value.

I am not sure whether you will find these explanations satisfactory. Mr. Couturat has told me that you will reply in the *Revue de Métaphysique et de Morale*. You may take these remarks into account, if you wish.

Sincerely yours,
G. Peano
Via Barbaroux 4, Turin.

Turin, 9 September 1906[8]

[8] Peano returned to Turin on 5 September, having in the meantime read Russell's article "Les paradoxes de la logique." The "kind (but unmerited) remarks" were a defense of Peano against Poincaré, who had written in his second article with the title "Les mathématiques et la logique": "I have the greatest esteem for Mr. Peano, who has done some very nice things (for example, his curve that fills a planar region); but in the end he has gone neither farther, nor higher, nor faster than most wingless mathematicians, and he could have done just as well on his legs" (*Revue de Métaphysique et de Morale*, 14 [1906], 295). (This sarcastic remark is aimed directly at Couturat's statement that "logistic" lends "stilts and wings" to discovery, and perhaps indirectly at Peano's often repeated saying that "symbols give wings to the mind of man.") Russell replied: "I believe that this is Mr Poincaré's way of saying that Peano's principal work does not interest him. Mr Peano has forged a tool of great power for certain kinds of research. Some of us are interested in these researches and, consequently, honor Mr Peano, who has, according to us, gone so much farther and higher than 'wingless' mathematicians that they have lost sight of him and do not know how much he is ahead of them. One specialist is certainly not obliged to interest himself in the work of another specialist; but he should, by courtesy, admit that the sub-

Dear Sir:

I hasten to reply to your letter of the 6th. Not having received news from you, I wrote and published in the *Revue de Mathématiques*, of which I sent a copy to you at Friday's Hill, Haslemere, my opinions on the interesting questions of antinomies. When my article was completed, I went to Switzerland, and as I was leaving I received from Mr. Couturat the page proofs of your article. If I had received this sooner, I could very well have kept silent. I thank you for the kind (but unmerited) remarks in my regard. Your article is quite clear, and I believe Mr. Poincaré will have nothing further to say.

We are in agreement on your criticism of the "True Solution" of Mr. Poincaré. There is, however, a verbal contradiction between us: you say, with Mr. Poincaré, that the key to Richard's paradox is to be found in the idea of a vicious circle; I maintain that there is no vicious circle, but a contradiction, or absurd condition. At the end of my article I remark that my solution cannot be definitive, for there always remains a weak point in the argument of Mr. Richard—linguistics. Still, I am afraid that this verbal contradiction will be pointed out by our opponents. All the contradictions that you discuss are reducible to the following form (I am writing w in place of φ):

$$w \,\varepsilon\, \mathrm{Cls}.\ f \,\varepsilon\, w\mathrm{F}\mathrm{Cls}'w : u \,\varepsilon\, \mathrm{Cls}'w\ .\ \supset_u .\ fu \sim \varepsilon\, u : \supset\ .\ fw \,\varepsilon\, w\ .\ fw \sim \varepsilon\, w.$$

If I have correctly translated your proposition, the hypothesis is absurd, for among the classes of w there is a w, and one may deduce fw ε w, in contradiction with the other hypothesis: fw ~ ε w. From an absurd hypothesis may be deduced an absurd theorem.

The intriguing paradox of Mr. Berry may be resolved as follows: "the smallest integer not definable in less than seventeen syllables" expresses a contradiction. I am not entirely certain, however.

Please excuse my disorganized remarks. I leave tomorrow for Modena, then for Naples (Military College), where I shall be about the 20th of this month, then for Rome (Military College), and I shall return to Turin at the beginning of October.

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jects he does not himself study also have their importance; and, in fact, Mr Poincaré recognizes this importance by discussing it" ("Les paradoxes de la logique," *Revue de Métaphysique et de Morale*, 14 [1906],628).

Sincerely yours,

G. Peano.

Turin, 16 December 1910[9]

Dear Sir:

Thank you for the book *Principia Mathematica*, which I mean to read carefully. At present, the time I have free from school is occupied with the question of Interlingua, which I do not believe so absurd as the majority of people tend to believe. I am sending you some of my articles on this subject.

Sincerely, yours

G. Peano.

Please thank Mr Whitehead also.

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[9] This "thank you" note for volume 1 of *Principia Mathematica* was written on a 'propaganda post card' of the Academia pro Interlingua. Peano had by now abandoned all active promotion of mathematical logic and had become an ardent apostle of the idea of an international auxiliary language. He had invented *Latino sine flexione* (or "Latin without grammar") in 1903 and used it for many publications after that, including the fifth and last edition of the *Formulaire*, called *Formulario Mathematico*. Seeking a wider influence, he was elected a member and director, in 1908, of the *Akademi internasional de lingu universel*, successor to the *Kadem volapüka*. (This last was founded in 1887 to promote Volapük, the language invented in 1879 by Johann Martin Schleyer.) Peano then transformed the *Akademi internasional* into a free discussion society, the transformation being completed in 1910 with the dissolution of the *Akademi internasional* and the assignment of all rights to the newly formed *Academia pro lnterlingua*, of which Peano remained director until his death in 1932.

Turin, 20 March 1912[10]

Dear Sir:

Thank you for your letter of the 16th. I wish to go to London to the Congress, and to present a short communication to the section which is also worthily represented by you. I am inviting my friends to do the same. I hope that nothing will prevent me.

I have just received Volume II of the *Principia Mathematica*. I read Volume I, and wrote a review-notice for the *Atti de l'Accademia dei Lincei*. I believe I sent you a copy, but I am sending you another, to be sure.

I marvel at your activity, which has allowed you in so short a time to complete the second volume, which I shall read little by little, for the subject requires a great deal of thought.

Some time ago I withdrew from the tiring part of teaching. Consequently, I have more time for myself, but I have less occasion to cultivate students. My former students are scattered throughout Italy.

I am now occupied with another question, apparently quite different, and judged to be absurd by a great many persons, but in reality closely connected with mathematical logic. I mean Interlingua; for three years ago I was named director of the former Volapük Academy. I am sending you the latest number of the *Discussiones de Academia pro Interlingua*, in which one of my articles is on pages 20–43, where you will recognize mathematical logic.

If this matter entertains you a bit, and is a diversion among more serious studies, such as the publication of the third volume, I will send you some other publications. And if you wish to join

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[10] The "review-notice" of volume 1 of *Principia Mathematica* is a very brief note ("Sulla definizione de funzione," *Atti dell'Accademia dei Lincei*, *Rendiconti Cl. Sci. Fis. Mat. Nat.*, 5th ser., 20-I [1911], 3–5) in which Peano shows that the notion of function, as expressed in *Principia Mathematica*, can be defined in the symbols of the *Formulario Mathematico.*

The article in which Peano expects Russell to "recognize mathematical logic" discusses the derivation of one grammatical form from another ("De derivatione," *Academia pro Interlingua Discussiones*, 3 [1912], 20–43). There is little mathematical logic in it, but there is some use of set theory. More importantly, there is a mathematical treatment of grammar that would be developed later into a full-fledged "algebra of grammar" (G. Peano, "Algebra de grammatica," *Schola et Vita,* 5 [1930], 323–336). This "algebra of grammar" bears a curious resemblance to some ideas of the "transformational grammar" of Noam Chomsky and others.

our international Academy, which includes several English philosophers, you would do me a great honor.

Please thank Mr Whitehead for me.

Cordially,
G. Peano

[Fifth International Congress of Mathematicians]
[Cambridge, 1912], 22 August[11]

[Reception Room,]
[Examination Hall,]
[Cambridge]

Dear Sir:

Yesterday evening at the reception desk I gave the title of my communication: *Propositiones existentiales*, and they made the following remark with regard to the language. The regulation allows only four languages. Now, by no means do I wish to produce a confusion of languages, quite the contrary.

My *Latino sine flexione* is *Italian*, for it is intelligible at first sight to every Italian. It is more Italian than the *Second circular*. I could have the Italians present at the congress be witnesses that it is Italian. Its advantage over Italian, however, is that it is also intelligible to non-Italians. I could make a declaration that it agrees with the rule; my communication itself is very short, and I believe the experience would be profitable.

[11] This remarkable note shows evidence of being written in haste, but it is none the less out of character in its peevishness. No doubt his friends Giovanni Vacca and Alessandro Padoa, present at the congress, would have supported him, but in the end the rule of “English, French, German, or Italian” prevailed, and Peano read his paper in Italian. This did not stop his attempts to use *Latino sine flexione,* but it must have damped his enthusiasm and perhaps partly explains his disappointment with the way the English pronounced it (see Alessandro Terracini, *Ricordi di un matematico* [Rome: Cremonese, 1968], p. 69).

Sincerely,

G. Peano

ORIGINAL TEXT OF THE PEANO-RUSSELL LETTERS

Turin, 19.3.1901

Monsieur et cher collègue,

Je publierai aussitôt votre intéressante Mémoire, qui comble une lacune entre les travaux de MM. Peirce—Schröder et ceux du Formulaire. Permettez-moi de me féliciter avec vous de la facilité et de la précision avec lesquelles vous maniez les symboles de Logique. Voici quelques remarques typographiques.

Les notations $\breve{R}$ $\breve{\rho}$ n'apportent pas de difficulté typographique grave. Toutefois, si l'on veut écrire tous les signes sur la même ligne, on pourra les remplacer par Inv(erse) R, IR. La notation R | de l'*Introduction au Formulaire* a. 1894 p. 39 produirait maintenant confusion avec le signe de substitution. La notation ρ pour indiquer le domaine de R est claire, lorsque la relation est indiquée par une lettre seule. Je crains qu'on rencontrera des difficultés pour indiquer le domaine de RS, $R \cup S$, $\breve{R}$, c'est-à-dire des relations indiquées par une expression composée.

Les classes de couples correspondent aux relations.

Votre P 1·4 $\quad R \varepsilon \mathrm{rel} \,.\, \supset : \exists \rho \,.\, = .\, \exists \breve{\rho}$

correspond à la 2·2 du § ∃ (p. 28 du Formul).

$$\exists x \ni \exists y \ni [(x;y) \varepsilon a] \,.\, = .\, \exists y \ni \exists x \ni [(x;y) \varepsilon a]$$

Peut-on représenter votre signe ϟ par la letter Z tournée de 90°, et le signe ɜ par T?

L'introduction de signes n'appartenant pas à la caisse typographique commune, ou à la caisse mathématique, produit des longs retards, pour la fabrication de nouveau type.

Veuillez agréer, Monsieur et cher collègue, les sentiments de ma considération la plus dévouée.

G. Peano

Via Barbaroux 6.

Torino, 27 maggio 1903

Pregiatissimo Collega,

Grazie del suo libro *The Principles of Mathematics*, che ho letto in parte col più vivo interesse, e che desterà pari interesse in tutti i lettori. Ho spedito ai prof. Pieri e Vailati la copia loro indirizzata.

Ho vivo desiderio di parlarle diffusamente di questo suo libro, che fa epoca nel campo della filosofia matematica. Ma fui sempre costretto a girare per l'Italia. Stassera parto per Roma; quindi mi limito per ora ad esprimerle i più vivi ringraziamenti.

Ho l'onore di professarmi colla massima stima

dev*mo*

G. Peano

Rome, 7.3.05

Monsieur et cher collègue,

J'ai reçu votre mandat de 128 fr, que j'ai exigé sans difficulté, et je vous remercie vivement de votre concours à la publication de la R.d.M.

Ma santé n'est pas très bonne. Je voyage en Italie, et la publication de la Revue est en retard, comme aussi cette lettre. J'ai reçu de M. Vacca la première partie d'un travail à propos de votre oeuvre, tome 1, que je vois mentionné aussi en Italie, en plusieurs publications.

En quelques jours je serai à Turin.

Veuillez agréer, Monsieur et cher collègue, les sentiments les plus cordiaux, et mes vifs remerciements.

G. Peano

Turin, 16.2.06

Monsieur et cher collègue,

Je vous remercie de votre contribution à la propagande pour la logique symbolique. J'ai travaillé pour ce but, pendant 17 années, presque seul. Maintenant grâce à vous, et à MM. Couturat, Whitehead, et autres, ces théories sont entrées dans le domaine publique, et je peux me reposer, comme j'ai commencé à faire, il y a quelque temps.

Vous m'avez dit une fois, que vous étiez associé à la Revue d. M., et en conséquence vous avez reçu le *Formulario Mathematico*, t.5, partie 1ère. Si vous ne l'avez pas reçu régulièrement, je vous prie de m'aviser, et je curerai l'expédition.

Le dr. Vacca, qui était assistant à l'Université de Turin, a écrit, mais non terminé, la recension de vos *The Principles of Mathematics*. Ensuite il s'est donné à la politique, a été conseiller municipal à Gênes; maintenant est à Florence. Son travail non terminé, n'a pas pu être publié, et je regrette que, étant passé le temps utile, la R.d.M. ne put publier une recension de votre livre. Mais il est lu, et bien jugé en Italie. Je reçois un article de Pieri, *Sopra una definizione aritmetica degli irrazionali*, qui étudie votre définition, et les définitions par classes, comme vous faites, qui correspondent aux définitions par abstraction. Il distingue les classes des classes de classes, et ainsi il me semble qu'il l'approche à la solution du célèbre paradoxe qui porte votre nom.

Des questions analogues ont donné lieu à des débats entre Zermelo, Borel, Lebesgue, Jourdain, etc. dans les *Mathematische Annalen*, et dans le *Bulletin de la société mathématique de France*.

Vous avez lu l'article critique de M. Poincaré à la Logique logistique. Je ne réponds pas, car il y a M. Couturat qui va répondre. M. Burali-Forti a répondu à quelques lignes, qui le regardent. Il me semble que M. Poincaré n'était pas au courant des progrès de cette science. Le défi aux Logiciens, sur la démonstration du Théorème de Cantor-Bernstein, peut être accepté, et on peut éliminer l'idée de nombre N_0, en la remplaçant par sa définition, selon les Propositions primitives.

En effet, le théorème de Cantor-Bernstein est:

$$a,b \varepsilon \mathrm{Cls} . g \varepsilon (bFa)\mathrm{sim} . h \varepsilon (aFb)\mathrm{sim} . \supset . \exists (bFa)\mathrm{rcp}$$

Il est réductible au cas particulier, considéré par Borel:

$$a \varepsilon \mathrm{Cls} . f \varepsilon (aFa)\mathrm{sim} . b \varepsilon \mathrm{Cls} . f \text{‘}a . \supset b \supset a . \supset . \exists (bFa)\mathrm{rcp}.$$

La démonstration de Bernstein, reproduite par Borel, et puis par Poincaré, dépend en définitive de S_x défini comme suit:

$$x \varepsilon a . \supset . S_x = f^{N_0} x \qquad \mathrm{Df}$$

S_x, série déterminée par l'élément x, est l'ensemble des individus x, fx, f^2x, f^3x, . . . alors la démonstration est:

$$[f, S(a{\sim}b)] \cup [\mathrm{idem}, a{\sim}S(a{\sim}b)] \varepsilon (bFa)\mathrm{rcp}$$

'la fonction égale à f dans le champ de la série déterminée par la classe a~b, et égale à l'identité dans le champ restant, est effectivement une correspondance réciproque (un—un) entre les a et les b'.

Il s'agit maintenant d'éliminer le symbole de nombre N_0 dans la Df de S. On peut donner cette définition comme suit:

$$x \varepsilon a . \supset . S_x = \cap \mathrm{Cls} \cap s \ni (x \varepsilon s : y \varepsilon s . \supset_y . fy \varepsilon s) \qquad \mathrm{Df}$$

S_x est la partie commune aux classes qui contiennent x, et contient le successif fy de tout individu y de la classe.

Si je puis vous être utile dans quelques informations sur les travaux que je viens de citer, si par hasard vous ne les avez pas vus, je me ferai un grand plaisir de le faire.

Veuillez agréer l'assurance de ma considération la plus dévouée,

G. Peano

Turin, 24.7.06

Monsieur et cher collègue,

Je vous remercie de l'extrait de votre article de l'*American Journal*. Je vous avais déjà écrit une carte postale lorsque j'ai reçu le journal.

Pressé par les occupations de ma profession, j'avais oublié M. Poincaré, et son dernier article. En le relisant, je vous adresse les remarques suivantes.

J'avais rencontré en 1890 (*Mathematische Annalen* t. 37 p. 210) la forme de raisonnement que constitue le principe de Zermelo: *On ne peut pas appliquer une infinité de fois une loi arbitraire avec laquelle à une classe on fait correspondre un individu de cette classe.*

En effet cette forme de raisonnement n'est pas réductible aux formes communes (p. ex. à celles contenues dans les pages 1–14 du *Formulaire* t. V); et démontrer une proposition signifie la déduire des propositions connues par les formes communes de raisonnement, sans y ajouter de nouveaux principes.

Le forme de raisonnement, dans laquelle on prend un élément arbitraire, est:

$\exists\, a$	(1)
$x \,\varepsilon\, a \,.\, Ɔ \,.\, p$	(2) où p est une proposition ne contenant pas x
$(1) \,.\, (2) \,.\, Ɔ \,.\, p.$	

Est dite *éliminer* (*Formul*. V, p. 12 Prop. 3 ·1). Est réductible à la règle *importer* comme résulte du *Formul*, t. 2 Prop. 405. Je trouve cette dernière dans votre *Theory of implication*, mais je ne trouve pas la règle *éliminer*.

Si l'on prend deux éléments arbitraires, la démonstration contient une proposition de plus. P. ex. *Formul*. V p. 139 Prop. 1·2, où l'on élimine 2 lettres y et z. L'élimination de n éléments conduit à un raisonnement de n + 2 propositions. En conséquence, on ne peut pas faire $n = \infty$.

L'affirmation de M. Poincaré p. 313 ne me semble pas satisfaisante. Il ne s'agit pas d'une question d'évidence, mais bien de vérités absolues.

Les scholastiques ont dit que 'toute forme de raisonnement est réductible aux syllogismes'. Cela signifie qu'ils considèrent comme des syllogismes plusieurs formes de raisonnements, comme l'élimination, qui contient 3 propositions, et 3 termini a, x, p.

En conséquence la réponse est (je crois) la suivante: L'assomption d'un nombre fini de choix est réductible aux syllogismes. L'application du principe de Zermelo *en plusieurs cas* est réductible aux syllogismes. Ainsi plusieurs démonstrations, dans lesquelles M. Borel adopte implicitement le principe de Zermelo, sont données dans le *Formulaire*, sans y faire recours. Mais si quelque Auteur affirme d'avoir réduit aux syllogismes ce qu'il n'a pas réduit, il affirme le faux.

L'antinomie de Richard a reçu par l'Auteur la solution: 'N est défini par un nombre fini de mots, et par E, on ne peut pas conclure que N est défini par un nombre fini de mots'. Mais si l'on remplace E par sa définition, qui est composée par un nombre fini de mots, on aura 'N est défini par un nombre fini de mots', et l'antinomie reste.

La *vraie solution* de M. Poincaré introduit une condition, que dans la définition de E on ne peut pas introduire la *notion* de E. Si c'est la lettre, ou le symbole, E, ou des symboles composées par E, la chose n'est pas controverse. Mais toute définition contient dans le second membre *exactement* la notion du symbole défini, car il lui est égal.

Par exemple, on peut appliquer la critique de M. P. p. 319 à la déf. de la différence:

$$a \varepsilon N_0 . b \varepsilon a + N_0 . \supset . b - a = N_0 \cap x \ni (a + x = b) \qquad \text{Def} -.$$

Le défaut est encore le même; b – a est parmi *tous les* nombres celui qui satisfait à la condition. *Sous peine de cercle vicieux cela doit vouloir dire* parmi *tous les* nombres *dans la définition desquels n'entre pas la notion* de la différence. *Cela exclut* b – a, *qui dépend du signe* – .

La solution de l'antinomie R, est, je pense, la suivante. Si n est le rang que la phrase G, définition de N, a dans l'ensemble E, en supposant qu'elle appartienne à E, l'Auteur pose:

$$\mathrm{Gr}_{-n}\, N \sim\, = \mathrm{Gr}_{-n}\, N,$$

c'est-à-dire, une condition est absurde: x ~ = x. La phrase G n'exprime pas un nombre (bien qu'elle semble une définition de nombre). C'est une écriture, comme max Np, qui n'a pas de valeur.

Je ne sais pas si ces explications vous satisfairont. M. Couturat m'a dit que vous répondez dans la *Rev. de Métaphys*. Vous pourrez tenir compte, si vous voulez, de ces remarques.

Votre tout dévoué
G. Peano
Via Barbaroux 4, Torino

Turin, 9.9.06

Monsieur et cher collègue,

Je m'empresse de répondre à votre lettre du 6. N'ayant pas reçu de nouvelles de vous, j'ai écrit et publié dans la R.d.M., dont je vous ai envoyé un exemplaire, à Friday's Hill, Haslemere, mes opinions sur la question intéressante des antinomies. Ayant terminé mon travail, je suis allé en Suisse, et en partant j'ai reçu de M. Couturat les épreuves typographiques de votre article. Si je l'avais reçu plutôt, j'aurais bien pu me taire. Je vous remercie d'abord des expressions gentiles (et imméritées) à mon regard. Votre article est bien clair, et je crois que M. P. n'aura plus rien à répondre.

Nous sommes d'accord sur la critique à la *Vraie Solution* de M. P. Mais il y a entre nous une contradiction de mots: vous dites, avec M. P. que la clef du paradoxe de Richard doit se trouver dans l'idée d'un cercle vicieux; je soutiens qu'il n'y a pas de cercle vicieux, mais une contradiction, ou condition absurde. A la fin de mon article je remarque que ma solution ne peut pas être définitive, car il y a toujours un point faible dans l'argument de M. R., la *linguistique*. Toutefois je crains que cette contradiction de mots sera relevé par nos contradicteurs. Toutes les contradictions que vous discutez sont réductibles à la forme suivante: J’écris w au lieu de φ:

$$w \,\varepsilon\, \mathrm{Cls}.\ f \,\varepsilon\, w\mathrm{F}\mathrm{Cls}'w : u \,\varepsilon\, \mathrm{Cls}'w\ .\ \supset_u .\ fu \sim\varepsilon\, u : \supset\ .\ fw \,\varepsilon\, w\ .\ fw \sim\varepsilon\, w.$$

Si j’ai bien traduit votre proposition, l’Hyp. est absurde, car parmi les classes de w il y a w, et l’on déduit fw ε w, en contradiction avec l’autre Hyp. fw ~ ε w. De l’Hyp. absurde on peut déduire une Ths. absurde.

Le paradoxe très curieux de M. Berry peut être résolu comme suit: *le plus petit entier non nommable en moins de 17 syllabes* exprime une contradiction. Mais je ne suis pas très sûr.

Excusez-moi de ces remarques désordonnées. Je pars demain pour Modène, puis pour Naples (Collegio Militare), où je serai vers le 20 de ce mois, puis pour Rome (Collegio Militare), et je reviendrai à Turin au commencement d’octobre.

Veuillez agréer, Monsieur et cher collègue, l’assurance de ma considération la plus dévouée.

G. Peano

Turin, 16.XII.1910

Cher Monsieur,

Je vous remercie du livre *Principia Mathematica*, que je me propose de lire avec attention. Maintenant, mes heures libres de l’école sont occupées dans la question de 1’Interlingua, que je ne crois pas aussi absurde, comme la majorité des personnes le peuvent croire. Je vous envoyé quelques-uns de mes articles sur ce sujet.

Tout dévoué
G. Peano.

Je vous prie aussi de remerçier M. Whitehead.

Turin, 20.III.1912

Cher collègue,

Merci de votre lettre du 16. Je désire aller à Londres au Congrès, et d'y faire une petite communication, dans la section aussi dignement représentée par vous, et j'inviterai mes amis à faire autant; et j'espère qu'aucun empêchement ne se présentera.

Je viens de recevoir le volume II des *Principia Mathematica*. J'ai lu le tome I, et j'en ai fait une recension-annonce dans les *Atti de l'Accademia dei Lincei*, dont je crois vous avoir envoyé un exemplaire. Je vous en envoyé un autre, pour assurance.

J'admire votre activité, qui vous a permis en aussi peu de temps, de conduire à terme le Second volume, que je lirai peu à peu; car le sujet exige beaucoup de méditation.

Il y a quelque temps que je me suis retiré de la partie fatigante de l'enseignement. En conséquence, j'ai plus de temps pour moi, mais je n'ai moins d'occasion de faire des élèves. Mes anciens élèves se sont répandus dans l'Italie.

Je m'occupe maintenant d'une autre question, apparemment tout à fait différente, et jugée absurde par beaucoup de personnes. Mais en realité, très liée à la logique mathématique. Je veux dire de l'Interlingua; car il y a trois ans, j'ai été nommé directeur de l'ancienne Académie de Volapük. Je vous envoyé le dernier numéro des *Discussiones de Academia pro Interlingua*, où il y a p. 20–43 un de mes articles, où vous reconnaissez la logique mathématique.

Si ces questions vous amusent un peu, et sont de distraction parmi les études plus sérieuses, comme la publication du troisième volume, je vous en enverrai d'autres publications. Et si vous voulez entrer dans notre Académie internationale, qui contient plusieurs philosophes anglais, vous me ferez un grand honneur.

Je vous prie de mes remerciements à M. Whitehead; et je vous prie d'accepter mes salutations cordiales.

Tout dévoué

G. Peano

[Fifth International Congress of Mathematicians]
[Cambridge, 1912], 22 août
[Reception Room,]
[Examination Hall,]
[Cambridge]

Cher Monsieur,

J’ai présenté hier soir à la réception le titre de ma communication: *Propositiones existentielles*, et l’on m’a fait cette remarque à propos de la langue. Le règlement permet seulement les 4 langues. Or je ne désire nullement produire la confusion des langues, bien au contraire.

Mon *latino sine ftexione* est *italien*, car il est intelligible à première vue, par tout italien. Il est plus italien que le *Second circular*. Je pourrais faire attester qu’il est italien, par les italiens présents au congrès. Mais son avantage sur l’italien, est qu’il est aussi intelligible aux non italiens. Je pourrais faire une déclaration qui accorde le règlement, et ma communication celle-ci est très courte et je crois qu’il y aurait avantage à cette expérience.

Tout dévoué
G. Peano

Nominazione; collana-rivista internazionale di logica 1 (1980), no. 1 ("La sfida di Peano"), 21–26.

Contributi di Peano alla matematica

HUBERT C. KENNEDY

Le oltre duecentosettanta pubblicazioni di Peano ci sorprendono per il numero e la varietà[1]. Peano fornì contributi in vari campi della matematica, della logica e della linguistica. In alcuni di questi campi, per esempio nella logica matematica, gli fu presto riconosciuto il ruolo di pioniere, in altri come nella linguistica matematica, il suo lavoro è stato rivalutato solo di recente[2]. Questo articolo è un tentativo di esaminare i più importanti risultati di Peano nel campo della matematica pura. Per questo lavoro esiste un precedente dello stesso Peano[3], benché sia arbitrario limitarsi alla matematica in quanto i suoi contributi in altri campi sono senza dubbio altrettanto considerevoli.

Forse il contributo più noto di Peano è l'insieme dei suoi assiomi relativi ai numeri naturali formulato nell'articolo *Sul concetto di numero* del 1891:

1. L'unità è un numero.
2. Il segno + messo dopo un numero produce un numero.
3. Se a e b sono due numeri, e se i loro successivi sono eguali, anche essi sono uguali.
4. L'unità non segue alcun numero.

[1] L'elenco più completo dclle pubblicazioni di Peano si trova in *Selected Works of Giuseppe Peano* edito e tradotto da H. C. Kenncdy, University of Toronto Press, Toronto 1973. Cfr. pp. 11–29.

[2] Cfr., per esempio, P. FREGUGLIA, *Giuseppe Peano ed i prodromi della linguistica matematica*, in "Physis. Rivista Internazionale di Storia della Scienza", 19, 1977, pp. 303–317. Questo articolo va letto tuttavia con una certa prudenza.

[3] Nel 1915 Peano fece stampare un elenco delle sue pubblicazioni cui appose questa nota: "I miei lavori si riferiscono specialmente al Calcolo infinitesimale; e non furono del tutto inutili, poiché a giudizio dei competenti, essi contribuirono alla costituzione attuale di questa scienza".

5. Se *s* è una classe, contenente l'unità, e se la classe formata dai successivi di *s* è contenuta in *s*, allora ogni numero è contenuto nella classe *s* [4].

Questi assiomi sono sufficienti per derivare tutti i teoremi dell'aritmetica e furono introdotti da Peano nel 1889 nell'importante opuscolo dal titolo *Arithmetices principia nova methodo exposita*[5]. Dato che in quest'opera Peano fa riferimento a altri matematici, la sua originalità è messa in dubbio. Ci fu subito chi pensò che avesse tratto gli assiomi da Dedekind ma Peano dichiarò in seguito di averli scoperti indipendentemente da Dedekind[6]. Più recentemente si è asserito che fossero tratti da Grassmann[7], ma il genio di Peano stava nella sua capacità di semplificare, di ridurre una teoria matematica ai suoi elementi essenziali. In effetti la formulazione degli assiomi qui riportata intcndeva essere una spiegazione; la loro prima formulazione fu data da Peano in forma simbolica – secondo il ''nova methodo" del titolo. Forse l'introduzione della matematica originale era per Peano non meno importante dell'introduzione della sua pasigrafia, campo in cui fu presto riconosciuto come maestro[8].

Dopo la formulazione degli assiomi dei numeri naturali, Peano applicò il suo metodo alla geometria di posizione. Il risultato[9] non fu molto originale (il suo debito verso Moritz Pasch fu puntualmente riconosciuto) ma segna una tappa importante nello sviluppo dell'assiomatica e completa il processo di astrazione da oggetti matematici particolari iniziato da Pasch – un risultato che spesso viene attribuito alle successive *Grundlagen der*

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[4] La fonte più facilmente consultabile delle pubblicazioni di Peano sono i tre volumi delle *Opere scelte*, a cura di U. Cassina, Cremonese, Roma 1957–1959. Gli assiomi citati sono in *op. cit.*, III, Cremonese, Roma 1959, p. 85.

[5] PEANO, *Arithmetices principia nova methodo exposita*, in *Opere scelte*, II, Cremonese, Roma 1958, pp. 20–55. L'opuscolo fu scritto in latino scolastico e il titolo è una curiosa commistione di latino e greco.

[6] Cfr. *Sul § 2 del Formulario, t. II: Aritmetica*, in *Opere scelte*, III, cit., p. 243.

[7] Cfr. D. FEARNLY-SANDER, *Hermann Grassmann and the creation of linear algebra*, in "American Mathematical Monthly", 86, 1979, pp. 809–817, in particolare p. 810.

[8] Questo fu notato per esempio da E. Schröder al Primo Congresso Internazionale di Matematica, tenuto a Zurigo nel 1897, nella sua relazione dal titolo *Über Pasigraphie, ihren gegenwärtigen Stand und die pasigraphische Bewegung in Italien.*

[9] PEANO, *I principii di geometria logicamente esposti*, in *Opere scelte*, II, cit., pp. 56–91.

Geometrie di Hilbert (1899)[10]. Sembra che Hilbert ignorasse deliberatamente i matematici italiani[11] (eccetto quelli che pubblicavano in tedesco).

Fin dal 1888 Peano aveva dato una definizione assiomatica di uno spazio lineare[12], ma questo risultato restò pressoché ignorato per molti anni[13]. Nella sua trattazione assiomatica di varie teorie matematiche Peano usò la "definizione per astrazione", non fu il primo ma contribuì alla diffusione di questo concetto e sembra che sia stato lui a dare questa formulazione[14]. Va notato anche che nello stesso contesto Peano usa definizioni ricorsive.

All'inizio della sua lunga carriera accademica (insegnò all'Università di Torino per quasi cinquant'anni) Peano mostrò un notevole talento nello scoprire errori e ambiguità nei testi matematici allora in uso. I suoi due testi di calcolo, il *Calcolo differenziale e principi di calcolo integrale* del 1884 e i due volumi delle *Lezioni di analisi infinitesimale* del 1893, sono modelli di chiarezza. Il primo testo che in genere viene chiamato "Genocchi-Peano"[15] dimostrò l'abilità di Peano nel costruire convincenti controesempi a nozioni e definizioni comunemente accettate in matematica, molti dei quali, come

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[10] Per il ruolo di Peano nel movimento assiomatico cfr. H. KENNEDY, *The origins of modern axiomatics: Pasch to Peano*, in "American Mathematical Monthly", 79, 1972, pp. 133–136, e M. PEDRAZZI, *Sistemi assiomatici nell'opera di Giuseppe Peano e della sua scuola*, in "Physis. Rivista Internazionale di Storia della Scienza", 19, 1977, pp. 319–335.

[11] Per esempio, al Congresso Internazionale di Matematica a Parigi nel 1900, dopo che Hilbert ebbe letto il suo famoso elenco di problemi, Peano disse all'assemblea che uno di questi problemi era stato risolto da Alessandro Padoa, che era presente al Congresso. Hilbcrt non fece nessun tentativo per conoscere la soluzione di Padoa.

[12] PEANO, *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva*, Fratelli Bocca Editori, Torino 1888, pp. 141, sgg.

[13] È stato notato di recente da A. F. MONNA, *Functional analysis in historical perspective*, Wiley, New York 1973, p. 114.

[14] Questa asserzione è di G. VAILATI, *La grammatica dell'algebra*, in *Scritti*, Barth-Seeber, Leipzig-Firenze 1911, p. 884.

[15] Il nome di Angelo Genocchi compare sul frontespizio in cui viene indicato anche che il libro fu "pubblicato con aggiunte del Dr. Giuseppe Peano". In seguito Genocchi declinò ogni responsibilità. Questa curiosa storia si trova in U. CASSINA, *Alcune lettere e documenti inediti sul tratto di Calcolo di Genocchi-Peano*, in "Rendiconti Ist. Lomb., Cl. sc.", 85, 1952; ristampato nell'opera *Dalla geometria egiziana alla matematica moderna*, Cremonese, Roma 1961, pp. 375–396.

l’esempio di una funzione le cui derivate parziali non sono commutabili, sono stati adottati dalla maggior parte dei testi di calcolo, benché raramente vengano attribuiti a lui. Come esempio dell’abilità di Peano può essere citata la funzione che è uguale a 0 se x è razionale, e uguale a 1 se x è irrazionale. Tale funzione era già stata presa in esame da Dirichlet nel 1829, ma Peano fu il primo a darne un’espressione analitica – cosa che ancora nel 1891 Gottlob Frege continuava a ritenere impossibile[16]. Il testo di Genocchi-Peano esercitò una notevole influenza a livello internazionale. Fu tradotto in tedesco da Bohlmann e da Schepp nel 1899 e fu tradotto due volte in russo prima da Sineokov, nel 1903, e poi da Posse, nel 1922.

Giustamente Peano è stato chiamato “maestro del controesempio”. Il controesempio più brillante è la famosa curva che riempie la superficie[17]. Georg Cantor aveva dimostrato nel 1878 che i punti di un intervallo unità potevano essere messi in corrispondenza biunivoca con un quadrato unità. Nel 1890 Peano diede l’éspressione analitica di due funzioni parametriche continue, $x = x\ (t)$ e $y = y\ (t)$ che fanno corrispondere in maniera continua a un intervallo unità un quadrato unità. Questo passo nello sviluppo della teoria della dimensione fu definito da Felix Hausdorf “uno dei fatti più importanti della teoria degli insiemi”[18].

Nella seconda metà del XIX secolo ci fu un impulso generale verso un maggior rigore in matematica. Peano, con i suoi molti controesempi, fu fra coloro che diedero un contributo di primaria importanza a questo processo. Riguardo al rigore nell’insegnamento dichiarò:

> Il rigore matematico è molto semplice. Esso sta nell’affermare tutte cose vere, e nel non affermare cose che sappiamo non vere[19].

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[16] G. FREGE, *Funktion und Begriff, Vortrag gehalten in der Sitzung vom 9. Januar 1891 der Jenaischen Gesellschaft für Medizin und Naturwissenschaft*, H. Pohle, Jena 1891; la traduzione italiana di questo importante passo si trova in G. FREGE, *Logica e aritmetica*, a cura di C. Mangione, Boringhieri, Torino 1965, p. 51.

[17] PEANO, *Sur une courbe qui remplit toute une aire plane*, in *Opere scelte*, I, Cremonese, Roma 1957, pp. 110–114.

[18] F. HAUSDORFF, *Grundzüge der Mengenlehre*, Leipzig 1914, p. 369.

[19] PEANO, *Sui fondamenti dell’analisi*, in *Opere scelte*, III, cit., p. 275; cfr. inoltre pp. 273–280.

La fama di Peano come insegnante diminuì nettamente negli ultimi anni della sua vita ma i suoi testi riflettono il suo genio iniziale e esercitarono un effetto salutare sull'insegnamento del calcolo.

La teoria delle eqazioni differenziali fu avviata da Leibniz nel 1676; è sorprendente che il teorema fondamentale – ossia che $y' = f(x, y)$ ha una soluzione solo a condizione che f sia continua – non sia stato scoperto per più di duecento anni. Fu formulato e dimostrato da Peano nel 1886. Oggi questo teorema viene generalmente considerato un caso particolare del più generale teorema del punto fisso, ma la dimostrazione "elementare" di Peano continua a destare l'interesse degli insegnanti. Va dato rilievo anche alla sua scoperta indipendente del metodo delle approssimazioni successive per la soluzione delle equazioni differenziali, anche se in questa scoperta Peano è stato preceduto da Schwartz.

Anche Bourbaki fa ora riferimento alla misura di "Jordan-Peano" benché sia stato a lungo misconosciuto che Peano ne aveva compiuto una scoperta indipendente. Lebesgue comunque riconobbe l'influenza di Peano sullo sviluppo delle proprie idee. A proposito dell'uso fatto da Peano dei limiti superiori e dei limiti inferiori per indicare l'integrale definito, è stato detto che "Peano compì l'ultimo passo in questo ordine di idee liberando la definizione di integrale dal concetto di limite"[20].

Nel 1887, nello stesso libro in cui introdusse la sua nozione di misura[21], Peano riprese anche una precedente idea di Cauchy e la estese fino a includere la differenziazione e l'integrazione di una funzione di un insieme per mezzo di un'altra. Ma quest'aspetto del lavoro fu largamente ignorato e quando Peano lo riprese nel 1915 vi erano giunti anche Lebesgue, Young e altri. Si può dire tuttavia che "per la generalità e la profondità delle sue idee, questo capitolo del libro di Peano è più notevole del saggio di Lebesgue del 1910 generalmente considerato come la fonte da cui si sviluppò l'illimitato fiume della ricerca moderna nella teoria delle funzioni degli insiemi"[22].

[20] F. A. MEDVEDEV, *Razvitie ponyatiya integrala*, Nauka, Moskva 1974, p. 214.

[21] PEANO, *Applicazioni geometriche del calcolo infinitesimale*, Bocca, Torino 1887.

[22] F. A. MEDVEDEV , *Ocherki istorii teorii funktsii deistvitel'nogo peremennogo*, Nauka, Moskva 1975, p. 68.

Si può notare che i due decenni precedenti il 1900 furono i più produttivi per la ricerca matematica di Peano; tutti i contributi citati sono di quel periodo. Dopo il 1900 rivolse molte energie alla costruzione e alla promozione di una lingua ausiliare internazionale. Nondimeno ci furono ulteriori validi risultati relativi in particolare alle formule di quadratura in cui si allontanò dai consueti metodi di interpolazione. In un articolo del 1913[23]:

[...] fu il primo a fare un tentativo in questa direzione e riuscì a ottenere la formula di Cavalieri-Simpson con un'espressione integrale del resto, avvalendosi soltanto dell'integrazione per parti. Questo metodo veniva usato sistematicamente da R. von Mises, che [...] mostrò come sia possibile ottenere qualsiasi formula di quadratura facendo uso soltanto dell'integrazione per parti[24].

Tuttavia, nonostante gli ultimi risultati, i più brillanti contributi matematici furono dati da Peano nel XIX secolo e, benché molti di essi non siano stati menzionati qui[25], quanto abbiamo discusso è certamente sufficiente per concludere che Peano fu uno tra i più influenti matematici italiani vissuti alla fine del XIX secolo, anzi uno dei maggiorì scienziati di quel secolo.*

* Traduzione dall'inglese di Alessandro Atti e Umberto Bottazzini.

[23] PEANO, *Resto nelle formule di quadratura, espresso con un integrale definito*, in *Opere scelte*, I, cit., pp. 410–418.

[24] A. GHIZZETTI e A. OSSICINI, *Quadrature Formulae*, Academic Press, New York 1970, p. 5.

[25] In particolare sono stati omessi i riferimenti ai contributi di Peano allo sviluppo della teoria degli insiemi; questi dovrebbero più propriamente essere discussi in connessione con i suoi contributi allo sviluppo della logica.

Archimede. Rivista per gli insegnanti e i cultori di matematiche pure e applicate 32 (1980): 56–58.

UNA LETTERA INEDITA DI G. PEANO SULLA PREPARAZIONE MATEMATICA DEI SUOI ALLIEVI

La seguente lettera, finora inedita ([1]), di Giuseppe Peano è probabilmente l'unica che è rimasta della sua corrispondenza con matematici americani, coi quali del resto i suoi contatti furono scarsi. Peano pubblicò due soli articoli in riviste americane ([2]), ma era presente al Congresso Internazionale dei Matematici a Toronto (Canada) nel 1924, dove tenne una breve comunicazione ([3]). La lettera qui pubblicata fu scritta tre anni dopo che Peano era stato nominato professore (straordinario) di calcolo infinitesimale nella R. Università di Torino, ed è la risposta alla lettera con cui David Eugene Smith ([4]) gli chiedeva informazioni sia sulle sue pubblicazioni matematiche e didattiche che sulla preparazione matematica dei suoi allievi.

Nel 1893 Smith non aveva ancora cominciato a pubbilcare la lunga serie di libri di testo e di storia della matematica per cui è diventato famoso. ([5]) A quell'epoca Peano, più

([1]) La lettera originale è di proprietà della Columbia University Libraries (New York City), che qui ringrazio vivemente per aver acconsentito alla pubblicazione.

([2]) *Sur la definition de la limite d'une fonction. Exercise de logique mathématique*, « American Journal of Mathemetics » 17 (1894), ristampata in PEANO, *Opere Scelte*, a cura di Ugo Cassina (Roma, Edizioni Cremonese, 1957–59), vol. 1, pp. 228–257; *Praepositiones internationale,* « World-Speech » (Marietta, Ohio, 1915), n. 37, pp. 2–3.

([3]) *De aequalitate*, in *Proceedings of the International Congress of Mathematicians*, Toronto, 1924 (Toronto, University of Toronto Press, 1928), vol. 2, pp. 988–989.

([4]) David Eugene Smith (21 gennaio 1860 - 29 luglio 1944) era Professor of Mathematics presso la Michigen State Normal School at Ypsilanti (Michigan) 1891–1898, direttore della New York State Normal School at Brockport (New York) 1898–1901, e Professor of Mathematics presso il Teachers College, Columbia University (New York City), dal 1901 fino al suo ritiro dall'insegnamento, nel 1926.

([5]) Smith ha pubblicato poco meno di 600 fra libri, articoli e recensioni, fra cui i due volumi della *History of Mathematics* (1923–1925). I suoi primi libri sono: *Plane and Solid Geometry* (1895); *History of Modern Mathematics* (1896).

anziano di Smith di quasi due anni, era già autore di più di 60 pubblicazioni, fra cui quattro libri ([6]), ed aveva fondato la sua « Rivista di Matematica » nel 1891. Non è quindi sorprendente che Smith si sia rivolto a Peano per avere informazioni sull'insegnamento matematico in Italia. Ecco la risposta di Peano.

Torino 4-11-93

Pregiatissimo signore,

Oltre ai miei opuscoli *Arithmetices principia* e *Principi di geometria*, pubblicati dalla libreria F.lli Bocca, di Torino, io ho pubblicati alcuni articoli didattici nella « Rivista di Matematica », la quale contiene pure alcuni articoli didattici del prof. Burali, e di vari autori. Ho pure pubblicato, sempre nello stesso giornale alcune recensioni. I miei lavori d'indole scientifica li ho invece sparsi nei vari giornali scientifici.

Io insegno Calcolo infinitesimale all'Università, ed alla Accademia militare.

Gli allievi dei due corsi sono assai diversi. Gli universitari, tolto qualcuno che è buono, ed anche ottimo, sono in generale scadenti nella matematica elementare, specialmente nella trigonometria. Questo proviene dalle leggi nostre che permettono l'ingresso all'Università senza alcun nuovo esame, dei giovani che hanno la licenza liceale. Questa licenza liceale dà adito a tutte le carriere giurisprudenza, medica, lettere, ecc. e non prova nel giovane una attitudine alla matematica. Invece gli allievi dell'Accademia militare devono subire un rigoroso esame di ammissione, tutto sulla matematica. Essi si presentano perciò franchi a seguire il loro corso; e non saprei dire in qual punto siano difettosi; poichè se la commissione esaminatrice si accorge di un punto in cui i giovani siano deboli, subito interroga su quel punto, ed obbliga gli insegnanti a ben prepararli. Però, per accennare ad un punto, dirò che dove sono deboli si è nelle unità di misura, e nella loro conversione. Così qualche volta la formula

([6]) È forse il primo di questi, il celebre *Calcolo differenziale e principi di calcolo integrale* (Torino 1884), che ha attirato l'attenzione di Smith. In copertina compare come autore Angelo Genocchi e sotto il titolo la dicitura: « Pubblicato con aggiunte dal Dr. Giuseppe Peano ». In seguito Genocchi dichierò che l'opera era dovuta a Peano, che allora ne assunse completamente le responsabilità. Il libro fu tradotto in tedesco da G. Bohlmen e A. Schlepp (1899) e due volte tredotto in russo, da, N. S. Sineokov (1903) e da K. A. Posse (1922).

$$x - \sin x < x^3/4$$

tradotta in numeri, diventa:

$$10° - \sin 10° < 1000°/4 \quad (^7)$$

I rimandati all'Università sono il 50/100, all'Accademia circa il 2/100.

Colla massima stima ho l'onore di professarmi

Suo devotissimo

G. Peano

Questa lettera è interessante per quanto ci dice sia sulla preparazione degli studenti che incominciavano l'università che sulle conseguenze, secondo Peano, degli esami d'ammissione. Sebbene Peano non abbia qui espresso con forza il suo punto di vista, egli era chiaramente in favore degli esami. Più tardi la sua opinione mutò radicalmente, tanto che nel 1912 egli voleva abolire gli esami di promozione: « È un vero delitto contro l'umanità il tormentare i poveri alunni con esami, per assicurarsi che essi sappiano cose che la generalità del pubblico istruito ignora. Così nelle scuole superiori, e nell'Università » ([8]).

Inoltre è interessante notare che l'esempio scelto da Peano per illustrare le debolezze matematiche degli studenti tratta di grandezze, un argomento su cui egli è tornato più volte. Nel 1915 scrisse ([9]) infatti:

Il prodotto di metri per metri, o in generele, di due grandezze, era di uso generale, fino a 30 anni fa; poi venne la moda di chiamarlo inesatto, erroneo, perchè non definito, ma ora ritorna in uso, premesse definizioni convenienti.... Facendo uso delle abbreviazioni comuni, con valore legale, in virtù della circolare del ministro d'I. P., 22 luglio 1896, la definizione è

$$m \times m = m^2.$$

([7]) Questa formula, si ricava per $x > 0$ dallo sviluppo di Taylor di sen x. Ora D sen $x = \cos x$ solo se x è misurato in radianti.

([8]) *Contro gli esami*, « Torino Nuova » (17 agosto 1913); in *Opere Scelte* (U.M.I.), vol. 3, p. 383.

([9]) *Sul prodotto di grandezze*, « Il Bollettino di matematiche e di scienze fisiche e naturali » [Conti-Tenca], 16 (1915), p. 99.

Nel 1921 egli si schierò ancora a favore del calcolo con le grandezze ([10]), e nel 1922 presentò una comunicazione sull'argomento alla R. Accademia delle Scienze di Torino ([11]). Nel 1924 egli tenne anche una pubblica conferenza presso l'Università di Torino, nella quale osservò che: « Or sono alcuni anni si iniziò una vera crociata contro le grandezze, dicendo che i calcoli si fanno sui numeri e non sulle grandezze » ([12]) ed ancora una volta si dichiarò in favore delle operazioni con grandezze.

È da notare che Peano si è lungamente interessato all'insegnamento della matematica elemenare partecipando ad esempio, sin dal 1897 alle adunanze della sezione della « Mathesis » di Torino, una associazione di insegnanti di matematica fondata nel 1895. Peano prese viva parte anche al primo Congresso nazionale della « Mathesis » che si tenne a Firenze nel 1906.

D'altra parte, per ironia, proprio negli anni in cui Peano si interessava all'insegnamento di matematica nelle scuole, il suo insegnamento universitario era oggetto d'una severa critica. Ma non vi è dubbio che i suoi libri di testo, in particolare il « Genocchi-Peano » ([13]), abbiano avuto un effetto salutare sull'insegnamento universitario, non solo in Italia, ma anche in altri paesi europei e perfino in America.

HUBERT KENNEDY

([10]) *Area de rectangulo*, « Rassegna, di matemetica e fisica » [Istituto fisico-matematico G. Ferraris, Roma], 1 (1921); in *Opere Scelte*, vol. 3, pp. 410–414.

([11]) Operazioni sulle grandezze, « Atti dell'Accademia delle Scienze di Torino », 57 (1922); in *Opere Scelte*, vol. 3, pp. 422–440. Quest'articolo, tradotto in *latino sine flexione* da Peano stesso (*Operationes super magnitudines*) è stato pubblicato anche nella, « Rassegna di matematica e fisica », 2 (1922), pp. 269–283.

([12]) *Sui libri di testo per l'aritmetica nelle scuole elementari*, « Periodico di Matematiche » (4), 4 (1924); in *Opere Scelte*, vol. 3, p. 445.

([13]) Cfr. nota. 6, p. 56.

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[Letters to the Editor]

Axiom of Choice

Questions of priority are notoriously difficult. With regard to the *Axiom of Choice*, I prefer not to enter further into a controversy, except to point out that in his letter (*Notices*, January 1984, page 17) Irving H. Anellis has misrepresented my position, apparently due to a misreading of a footnote on page 80 of Gregory H. Moore's *Zermelo's Axiom of Choice*. There Moore correctly states that I have "pressed a claim for Peano."

I do agree with Professor Anellis that historically there has been a "denigration of the contributions of Peano," but his suggestion that this was due to anti-Semitism cannot be correct: Peano was not Jewish, and I know of no evidence that anyone who knew him thought he was. (I suspect that Professor Anellis's unnamed Milanese colleague confused Peano with his friend, the logician Alessandro Padoa, who was Jewish.)

The causes of the neglect of Peano are rather more complicated. I discussed some of them in my biography of Peano; my views have been confirmed in part by a grandniece of Peano in her *Prezentazone* to the Italian translation of my book (*Peano: Storia di un Matematico*, Torino: Editore Boringhieri, 1983).

Hubert Kennedy
Providence College

(Received January 19, 1984)

Hubert Kennedy

PEANO—THE UNIQUE

A half century after the death of Giuseppe Peano, with both his devoted disciple (Ugo Cassina) and his persistent detractor (F. G. Tricomi) gone from the scene, we are better able to dispassionately examine and evaluate the life and works of this unique man. Born on 27 August 1858, the second child of Bartolomeo and Rosa (Cavallo) Peano, Giuseppe spent his earliest childhood in Spinetta, *frazione di Cuneo*, and later also attended school in Cuneo. Sometime in 1870 or 1871 (the date is uncertain) he was taken by his uncle, the priest-lawyer G. Michele Cavallo, to Torino to complete his preparation for university study. There he became a student at the University of Torino in 1876, thus beginning an association with that university that was to continue until his death 56 years later.

In 1880, as "dottore in matematica", Peano became assistant to Enrico D'Ovidio for one year and then assistant to Angelo Genocchi, professor of infinitesimal calculus, until the latter's death in 1889. During this time Peano abundantly demonstrated his quite extraordinary mathematical ability. Indeed, his 20 publications (including 3 books) would be considered a proud lifetime accomplishment by a lesser mathematician. No doubt Peano was proud of them, too, although this did not prevent him from worrying about his eventual appointment to the calculus chair of the late Genocchi.

Why should he have worried? Consider only some of the mathematical accomplishments of these years: a new definition of the definite integral (that freed it from the concept of limit) an analytic definition of the Dirichlet function (that equals 0 for rational x and 1 for irrational x), the first proof of the basic existence theorem of differential equations (that $y' = f(x,y)$ has a solution on the sole condition that f be continuous), the independent discovery of the method of successive approximations for the solution of linear differential equations, the definition of what is now called Peano-Jordan measure, the first axiomatic definition of a vector space, the famous postulates for the natural numbers, and the introduction of his mathematical logic.

In fact, he was appointed to the chair of infinitesimal calculus in 1890, and we may assume that he had earlier felt fairly secure when he was appointed in 1886 as professor at the Military Academy, secure enough to marry Carola Crosio in 1887. She was the daughter of the genre painter Luigi Crosio, a not unsuccessful artist (his paintings were in several exhibitions of the Società Promotrice delle Belle Arti), but known for leading a somewhat ‘Bohemian’ life. This marriage outside of academic circles is, I think, indicative of Peano’s position as ‘outsider’ at the university.

From his origins (“sottoproletario”, to use Tricomi’s description) Peano was already an ‘outsider’, and he remained such. His egalitarian and socialist views, although somewhat idealistic and romantic, alienated him from the other professors. Thus we discover that, when a ‘school of Peano’ developed, Peano’s assistants were almost the only ones in it who were directly connected with the university.

Peano was also a religious ‘outsider’. Although of a pious family (besides the uncle who brought him to Torino, a younger brother also became a priest), he abandoned his Catholic faith. One wonders whether he influenced or was influenced by his student and later assistant, the affectionate (and probably homosexual) Giovanni Vailati, who is known to have lost his faith during his early university years.

Of course, Peano’s promotion of his *latino sine flexione* after 1903 merely strengthened his position as ‘outsider’. Only his legal claim to the university chair allowed him to retain an influential base of operation. Is it any wonder that he stubbornly clung to it, despite concerted efforts to remove him? Indeed, he did lose his position at the Military Academy in 1901.

Whether Peano’s marriage in 1887 was a result of the security he felt in his career or an attempt to give security to his life is not clear. At any rate, the brilliant mathematical achievements of those early years must have been convincing to the authorities, and no doubt Peano also made efforts at that time to ‘fit in’. With the security of his appointment as ordinary professor in 1890, however, his independent path soon became apparent, especially with the founding of his own journal, the *Rivista di matematica*, in 1891.

With the freedom to publish which the journal gave him (he contributed no less than 5 articles, 4 reviews, and 3 notes to its first year) he was able to quickly make public the developments of his mathematical logic, becoming the leading European exponent of this field. In 1897 he was selected as one of the 4 principal speakers at the First International Congress of Mathematicians in Zurich, where, of course, he spoke about mathematical logic, and in 1900 he led the Italian delegation to the First International Congress of Philosophy in Paris, where, as Hans Freudenthal has reported, “the Italian phalanx was supreme”.

Not that mathematics as such was neglected by him during this last decade of the century. Already in 1890 he had astonished the mathematical world with his curve that fills a planar region, as well as generalizing his fundamental theorem of differential equations. If at the time of Genocchi’s death in 1889 Peano’s 20 publications were impressive, by the end of the century, 11 years later, he could list more than 100 publications, many of them of first-rank importance. Peano was at the height of his fame. And yet, one year later he lost his position at the Military Academy and opposition to him had developed at the university. Why? The ‘official’ reason was given as his poor teaching. In fact, I believe, his independence was resented. While his brilliant mathematical discoveries could not be denied, he had never really ‘fit in’; he was never accepted by the ‘establishment’. No doubt Peano was also a mathematical opportunist, turning out articles on whatever interested him at the moment, and these could be very diverse: calendar studies, interpolation formulas, finite differences, definitions of probability. This, too, was resented.

Above all, the university establishment never forgave him for his advocacy of his interlingua, introduced as *latino sine flexione* in 1903. But this was **his** interest, and if the stated goals of that movement have not been realized, that should not hinder us from appreciating Peano’s efforts and accomplishments, however ephemeral they have been. Just consider: *If* there had not been two world wars and the goal of a world language had been accomplished, as then seemed not entirely unlikely, then there is no doubt that Peano would be known around the world as one of the great apostles of this noble idea.

But apart from such contrary-to-fact considerations, the ingenuity of his actions compels our admiration. His first publication introducing *latino sine flexione* was a stroke of genius. It begins in classical Latin and, one by one, as suggestions are made for simplifying the language, they are immediately adopted, so that the article ends in *latino sine flexione*. No less ingenious was the manner with which he appropriated the old Volapük Academy.

Volapük was the first successful artificial language. First published in 1879, it had a phenomenal growth; ten years later it had over a million adherents. After that, the movement declined, due in part to internal factions, so that in January 1904 a five-year moratorium on changes in the language was voted by the Volapük Academy, which then ceased activity. When the time came for its revival, Peano presented himself as a candidate for membership and, at the some time, a candidate for the directorship. It was a bold move—and it worked. He was elected member and director in December 1908. By 1910 the old Volapük academy had been transformed into **his** Academia pro Interlingua.

Over 40 of Peano's publications were concerned in one way or another with the international auxiliary language movement. The most remarkable of these was the book *Vocabulario commune ad latino-italiano-français-english-deutsch* (2nd ed., 1915), a volume of 352 pages and some 14,000 entries, for which not only are the named languages given, but often Greek, Spanish, Portuguese, Russian, or Sanskrit as well. An impressive achievement, indeed, for one man.

The famous *Formulario matematico*, although begun as a collective effort, ended as a one-man project as well. It was initiated in 1894 as an auxiliary project of the *Rivista di matematica*, with the goal of collecting all the known theorems in various branches of mathematics and stating them precisely—for this the ideography Peano was then developing was to be used—together with complete proofs. No doubt work on this project was useful to the younger mathematicians who collaborated. In the end, however, a passion for collecting took over, so that Peano could boast that the fifth edition of 1908 continued some 4200 formulas and theorems in its 516 pages. This final edition was prepared almost single-handedly by Peano; think of seeing such a volume through the press—it has remarkably few misprints. The fact that this work was **his** was underscored by the change of the language of explanation in it from French to *latino sine flexione*. I am suggesting here a change in Peano's attitude toward his own quite extraordinary ability. In the beginning he was trying to 'make a career', to impress (and please) others with his accomplishments, and if there were, here and there, controversies over priority to discoveries or disagreements about the statements of others, well, that is how the 'academic game' is played. Around the turn-of-the-century, however, began to appear signs that he was now acting to please himself, and this is evident throughout his involvement in the international auxiliary language movement.

The curve that fills a planar region was **his**, so he had a symbol of it constructed on the terrace of his villa in Cavoretto, a hilly section of Torino, There, too, in 1906 he allowed himself the grand gesture of inviting the striking cotton mill workers for an outing, an action, as *La Stampa* reported, “entirely original and without precedent in the annals of Torino strikes”. Having been elected a member of the Academy of Sciences of Torino in 1891, he used his membership privilege to fill the pages of its journal with the articles of his friends, even during the paper shortage of the war years: in 1918 he submitted no less than 12 such works.

Peano did not ‘play the game’: he no longer concerned himself with priority questions; he no longer worried whether he was doing what others though he should be doing: Inside the classroom he discussed calculus with the use of his symbols; outside the classroom he did not hesitate to invite students to join him for an ice cream. In neither case was the ‘norm’ observed; and what most angered his detractors was that he did not seem in the least concerned that he was thereby an ‘outsider’. The ultimate expression of this rage—and lack of comprehension—was Tricomi’s description of Peano as “scemo” [idiot].

When Peano, at an Esperanto congress in Geneva in 1906, met L. L. Zamenhof, the creator of that language, the latter said to him with a smile, “If my disciples see me now, they will excommunicate me.” Peano replied: “I have few disciples, but they are all tolerant: it is one compensation.” This illustrates the other side of Peano’s egoism. If on the one hand he demanded the freedom to pursue his own interests, on the other hand he allowed the same freedom to others. Those who united with him for the pursuit of common goals were to do so voluntarily, of their own free choice. This principle was to guide him to the end of his life. Indeed his last publication was an exhortation in 1931 to the members of the Academia pro Interlingua to continue in “freedom and unity”.

The resentment of Peano by other members of the faculty of mathematics at the University of Torino during his last years hardly disturbed him. He took pleasure in his accomplishments, and there were always a few who appreciated him. How many men ever have such a devoted disciple as Ugo Cassina? The respect was mutual; when one of have such a devoted disciple as Ugo Cassina? The respect was mutual; when one of his visiting nieces made too much noise, she was told: “Quiet, Cassina is working.”

His last years were serene. Despite all urging he continued his classes at the university because he enjoyed them. He likewise enjoyed his contacts throughout the world in the international auxiliary language movement. In Cavoretto he had his “Eden”. Peano died early in the morning of 20 April 1932, after having stopped on his way home the evening before to see Maurice Chevalier in *Il Tenente Allegro* (*The Smiling Lieutenant*), the film version of the Oscar Straus operetta *The Waltz Dream*. He had enjoyed the film immensely.

Instead of complaining, as some have done, that Peano did not make this or that improvement in, for example, his mathematical logic, we will gain a truer, and more useful, appreciation of this extraordinary man if we do not separate his mathematical accomplishments from his supreme accomplishment of leading a truly human life, a life of free choice, a life that was his own—and unique.

(1983)