Charge Pump Derivation

Ryan J. Billing

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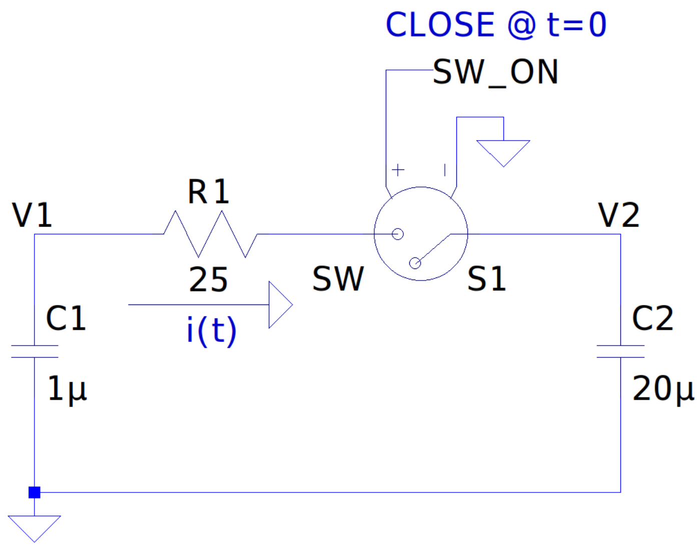

Figure 1: Charge Pump discharge cycle operation

0.1 Initial observations

Figure 1 depicts part of a charge pump circuit during the discharging cycle. C_1 is the flying capacitor that was charged to a voltage of V_1. At time $t = 0$ the switch S_1 will close and C_1 will begin to discharge into C_2.

The goal of this exercise is to determine how much energy is dissipated in R1 and determine the final voltage after the circuit has stabilized. A closed-form solution for $i(t)$ is determined, it can be used to quantify how much charge is transferred from C_1 to C_2 in this scenario. The final voltages, V_1 and V_2, can be determined from the integral of i_t by subtracting this charge from C_1 or adding it to C_2. In the final settled state it is apparent that $V_1 = V_2$.

0.2 Useful relationships

This section presents a set of fundamental relationships that may be used to determine the function for $i(t)$.

$$i(t) = -C_1 \frac{dV_1}{dt} \tag{1}$$
$$i(t) = C_2 \frac{dV_2}{dt} \tag{2}$$
$$i(t) = \frac{V_1(t) - V_2(t)}{R_1} \tag{3}$$

0.3 Derivation of i(t)

First find V_1 and V_2 from equations (1) and (2).

$$V_1(t) = -\frac{1}{C_1} \int i(t)dt \text{ (Negative sign because current is going out of } C_1\text{)}$$
$$V_2(t) = \frac{1}{C_2} \int i(t)dt \tag{4}$$

Plug these into (3) and then solve the resulting differential equation for $i(t)$.

$$i(t) = -\frac{1}{R_1}\left(\frac{1}{C_1}\int i(t)dt - \frac{1}{C_2}\int i(t)dt\right) \tag{5}$$

Next differentiate and collect like terms.

$$\frac{di(t)}{dt} = -\frac{\frac{1}{C_1} + \frac{1}{C_2}}{R_1} i(t)$$
$$R_1 \frac{C_1 C_2}{C_1 + C_2}\left(\frac{1}{i(t)}\right)\frac{di(t)}{dt} = -1 \tag{6}$$

Integrating both sides.

$$R_1 \frac{C_1 C_2}{C_1 + C_2} \int \frac{1}{i(t)} dt = \int -1 dt \tag{7}$$
$$R_1 \frac{C_1 C_2}{C_1 + C_2}\Big(ln\big(i(t)\big) + C\Big) = -t \tag{8}$$
$$ln\big(i(t)\big) + C = -t\frac{C_1 + C_2}{R_1 C_1 C_2} \tag{9}$$

Exponentiate to solve for $i(t)$,

$$
\begin{aligned}
e^{ln\left(i(t)\right)}e^{C} &= e^{-t\frac{C_1+C_2}{R_1C_1C_2}} \\
i(t) &= e^{-C}e^{-t\frac{C_1+C_2}{R_1C_1C_2}} \qquad (10)
\end{aligned}
$$

Finally the integration constant can be determined by considering the initial conditions,

$$i(0) = \frac{V_1(0) - V_2(0)}{R_1} \qquad (11)$$

The expression (10) is solved for the known initial conditions at $t = 0$ to determine the yet unknown constant of integration. In this case the constant C itself is a "don't care" quantity as we really want to know e^{-C} directly so the entire expression may be expressed in terns of initial voltages and the resistor value.

$$
\begin{aligned}
\frac{V_1(0) - V_2(0)}{R_1} &= e^{-C}e^{-0\frac{C_1+C_2}{R_1C_1C_2}} \\
\frac{V_1(0) - V_2(0)}{R_1} &= e^{-C}1 \\
e^{-C} &= \frac{V_1(0) - V_2(0)}{R_1} \qquad (12)
\end{aligned}
$$

Plugging (12) into(10), a closed-form expression for the resistor current, $i(t)$, is obtained.

$$i(t) = \frac{V_1(0) - V_2(0)}{R_1}e^{-t\frac{(C_1+C_2)}{R_1C_1C_2}} \qquad (13)$$

Then to help tidy the appearance of things, let

$$
\begin{aligned}
V_1(0) &= V_{i1} \qquad (14) \\
V_2(0) &= V_{i2} \qquad (15) \\
\tau &= \frac{1}{\frac{C_1+C_2}{R_1C_1C_2}} = R_1\frac{C_1C_2}{C_1+C_2} \qquad (16)
\end{aligned}
$$

And then the final expression reveals a familiar time domain response. In fact, the circuit could be redrawn as a single capacitor and resistor network driven by a step response with amplitude $V_{i1} - V_{i2}$, and where the capacitor value is equivalent to the series combination of C_1 and C_2.

$$i(t) = \frac{V_{i1} - V_{i2}}{R_1} e^{-\frac{t}{\tau}} \tag{17}$$

0.4 Determine final settled voltage

Either one of two approaches may be used to determine the final voltage measured at nodes $V_1 = V_2$. One approach integrates the current, $i(t)$, to determine the charge transferred and then using the formula $Q = CV$ to solve for the final voltage.

A second approach looks directly to energy lost during the stabilization, or,

$$E_{R1} = R_1 \int_0^\infty i(t)^2 dt \tag{18}$$

0.4.1 Charge based approach

The total charge transfer is the following.

$$\begin{aligned} Q_{R1} &= \frac{V_{i1} - V_{i2}}{R_1} \int_0^\infty e^{-\frac{t}{\tau}} dt \\ &= \frac{V_{i1} - V_{i2}}{R_1} \Big[-\tau e^{-\frac{t}{\tau}} \Big]_0^\infty \\ &= \frac{V_{i1} - V_{i2}}{R_1} \Big[0 - \big(-\tau e^{-\frac{0}{\tau}} \big) \Big] \\ &= \tau \frac{V_{i1} - V_{i2}}{R_1} \end{aligned} \tag{19}$$

Next it is observed that the same amount of charge leaves C_1 and is added to C_2. The final charge, Q_f, can be expressed in terms of either V_1 or V_2 as follows.

$$\begin{aligned} Q_f &= C_1 V_{i1} - Q_{R1} \\ &= C_2 V_{i2} + Q_{R1} \\ &= (C_1 + C_2) V_f \end{aligned} \tag{20}$$

Picking one,

$$Q_f = C_2 V_{i2} + \tau \frac{V_{i1} - V_{i2}}{R_1} \tag{21}$$

Since the quantity V_f is wanted, the expression (21) is solved.

$$\begin{aligned} (C_1 + C_2) V_f &= C_2 V_{i2} + \tau \frac{V_{i1} - V_{i2}}{R_1} \\ V_f &= \frac{C_2 V_{i2} + \tau \frac{V_{i1} - V_{i2}}{R_1}}{C_1 + C_2} \end{aligned} \tag{22}$$

Final manipulation yields the following.

$$V_f = \frac{C_2 V_{i2} + \tau (V_{i1} - V_{i2})}{R_1 (C_1 + C_2)} \tag{23}$$

0.4.2 Energy based approach

Combined with knowledge of energy stored in capacitors,

$$\begin{aligned} E_{FINAL} &= E_{C1} + E_{C2} - E_{R1} \\ &= \frac{1}{2}(C_1 + C_2) V_{FINAL}^2 \end{aligned} \tag{24}$$

where,

$$\begin{aligned} E_{C1} &= \frac{1}{2} C_1 V_1^2 \\ E_{C2} &= \frac{1}{2} C_2 V_2^2 \end{aligned} \tag{25}$$

And then the expression for E_{R1} is determined by the following,

$$
\begin{aligned}
E_{R1} &= R_1 \int_0^\infty \left(\frac{V_{i1} - V_{i2}}{R_1} e^{-\frac{t}{\tau}} \right)^2 dt \\
&= R_1 \left(\frac{V_{i1} - V_{i2}}{R_1} \right)^2 \int_0^\infty e^{-\frac{2t}{\tau}} dt \\
&= \frac{\left(V_{i1} - V_{i2}\right)^2}{R_1} \left[\frac{\tau}{2} e^{-\frac{2t}{\tau}} \right]_0^\infty \\
&= \frac{\left(V_{i1} - V_{i2}\right)^2}{R_1} \left[\frac{\tau}{2} e^{-\frac{2(0)}{\tau}} - \frac{\tau}{2} e^{-\frac{2(\infty)}{\tau}} \right] \\
&= \frac{\left(V_{i1} - V_{i2}\right)^2}{R_1} \left[\frac{\tau}{2} - 0 \right] \\
&= \frac{\tau}{2R_1} \left(V_{i1} - V_{i2}\right)^2 \qquad (26)
\end{aligned}
$$

Then (24) is expanded.

$$
\begin{aligned}
\frac{1}{2}(C_1 + C_2) V_{FINAL}^2 &= \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2 - \frac{\tau}{2R_1} \left(V_{i1} - V_{i2}\right)^2 \\
V_{FINAL}^2 &= \frac{C_1 V_1^2 + C_2 V_2^2 - \frac{\tau}{R_1} \left(V_{i1} - V_{i2}\right)^2}{C_1 + C_2} \\
V_{FINAL} &= \sqrt{\frac{C_1 V_1^2 + C_2 V_2^2 - \frac{\tau}{R_1} \left(V_{i1} - V_{i2}\right)^2}{C_1 + C_2}} \qquad (27)
\end{aligned}
$$

With sufficient algebraic manipulation of (27) the expression stated in (23) is obtained, so it will not re-stated here.