Charge Pump Derivation

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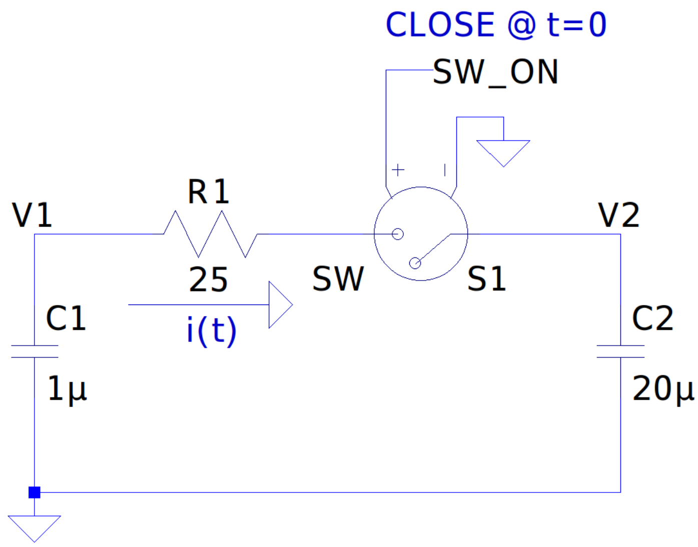

Figure 1: Charge Pump discharge cycle operation

0.1 Initial observations

Figure 1 depicts part of a charge pump circuit during the discharging cycle. C_1 is the flying capacitor that was charged to a voltage of V_1. At time $t = 0$ the switch S_1 will close and C_1 will begin to discharge into C_2.

The goal of this exercise is to determine how much energy is dissipated in R1 and determine the final voltage after the circuit has stabilized. A closed-form solution for $i(t)$ is determined, it can be used to quantify how much charge is transferred from C_1 to C_2 in this scenario. The final voltages, V_1 and V_2, can be determined from the integral of $i(t)$ by subtracting this charge from C_1 or adding it to C_2. In the final settled state it is apparent that $V_1 = V_2$.

0.2 Useful relationships

This section presents a set of fundamental relationships that may be used to determine the function for $i(t)$.

$$i(t) = -C_1 \frac{dV_1}{dt} \tag{1}$$

$$i(t) = C_2 \frac{dV_2}{dt} \tag{2}$$

$$i(t) = \frac{V_1(t) - V_2(t)}{R_1} \tag{3}$$

0.3 Derivation of i(t)

First find V_1 and V_2 from equations (1) and (2).

$$V_1(t) = -\frac{1}{C_1} \int i(t)dt \text{ (Negative sign because current is going out of } C_1\text{)}$$

$$V_2(t) = \frac{1}{C_2} \int i(t)dt \tag{4}$$

Plug these into (3) and then solve the resulting differential equation for $i(t)$.

$$i(t) = -\frac{1}{R_1}\left(\frac{1}{C_1}\int i(t)dt - \frac{1}{C_2}\int i(t)dt\right) \tag{5}$$

Next differentiate and collect like terms.

$$\frac{di(t)}{dt} = -\frac{\frac{1}{C_1} + \frac{1}{C_2}}{R_1} i(t)$$

$$R_1 \frac{C_1 C_2}{C_1 + C_2}\left(\frac{1}{i(t)}\right)\frac{di(t)}{dt} = -1 \tag{6}$$

Integrating both sides.

$$R_1 \frac{C_1 C_2}{C_1 + C_2} \int \frac{1}{i(t)} dt = \int -1 dt \tag{7}$$

$$R_1 \frac{C_1 C_2}{C_1 + C_2}\Big(ln\big(i(t)\big) + C\Big) = -t \tag{8}$$

$$ln\big(i(t)\big) + C = -t\frac{C_1 + C_2}{R_1 C_1 C_2} \tag{9}$$

Exponentiate to solve for $i(t)$,

$$e^{ln\left(i(t)\right)}e^{C} = e^{-t\frac{C_1+C_2}{R_1C_1C_2}}$$
$$i(t) = e^{-C}e^{-t\frac{C_1+C_2}{R_1C_1C_2}} \tag{10}$$

Finally the integration constant can be determined by considering the initial conditions,

$$i(0) = \frac{V_1(0) - V_2(0)}{R_1} \tag{11}$$

The expression (10) is solved for the known initial conditions at $t = 0$ to determine the yet unknown constant of integration. In this case the constant C itself is a "don't care" quantity as we really want to know e^{-C} directly.

$$\frac{V_1(0) - V_2(0)}{R_1} = e^{-C}e^{-0\frac{C_1+C_2}{R_1C_1C_2}}$$
$$\frac{V_1(0) - V_2(0)}{R_1} = e^{-C}1$$
$$e^{-C} = \frac{V_1(0) - V_2(0)}{R_1} \tag{12}$$

Plugging (12) into(10), a closed-form expression for the resistor current, $i(t)$, is obtained.

$$i(t) = \frac{V_1(0) - V_2(0)}{R_1}e^{-t\frac{(C_1+C_2)}{R_1C_1C_2}} \tag{13}$$

Then to help tidy the appearance of things, let

$$V_{i1} = V_1(0) \tag{14}$$
$$V_{i2} = V_2(0) \tag{15}$$
$$\tau = \frac{1}{\frac{C_1+C_2}{R_1C_1C_2}} = R_1\frac{C_1C_2}{C_1 + C_2} \tag{16}$$

And then the final expression reveals a familiar time domain response. In fact, the circuit could be redrawn as a single capacitor and resistor network driven by a step response with amplitude $V_{i1} - V_{i2}$, and where the capacitor value is equivalent to the series combination of C_1 and C_2.

$$i(t) = \frac{V_{i1} - V_{i2}}{R_1} e^{-\frac{t}{\tau}} \tag{17}$$

0.4 Determine final settled voltage

Either one of two approaches may be used to determine the final voltage measured at nodes $V_1 = V_2$. One approach integrates the current, $i(t)$, to determine the charge transferred and then using the formula $Q = CV$ to solve for the final voltage.

A second approach looks directly to energy lost during the stabilization, or,

$$E_{R1} = R_1 \int_0^\infty i(t)^2 dt \tag{18}$$

0.4.1 Charge based approach

The total charge transfer is the following.

$$\begin{aligned} Q_{R1} &= \frac{V_{i1} - V_{i2}}{R_1} \int_0^\infty e^{-\frac{t}{\tau}} dt \\ &= \frac{V_{i1} - V_{i2}}{R_1} \Big[-\tau e^{-\frac{t}{\tau}} \Big]_0^\infty \\ &= \frac{V_{i1} - V_{i2}}{R_1} \Big[0 - \big(-\tau e^{-\frac{0}{\tau}} \big) \Big] \\ &= \tau \frac{V_{i1} - V_{i2}}{R_1} \end{aligned} \tag{19}$$

Next it is observed that the same amount of charge leaves C_1 and is added to C_2. The final charge, Q_f, can be expressed in terms of either V_1 or V_2 as follows.

$$\begin{aligned} Q_{f1} &= C_1 V_{i1} - Q_{R1} = C_1 V_f \\ Q_{f2} &= C_2 V_{i2} + Q_{R1} = C_2 V_f \end{aligned} \tag{20}$$

Picking one,

$$Q_{f2} = C_2 V_{i2} + \tau \frac{V_{i1} - V_{i2}}{R_1} \tag{21}$$

Since the quantity V_f is wanted, the expression (21) is solved.

$$\begin{aligned} C_2 V_f &= C_2 V_{i2} + \tau \frac{V_{i1} - V_{i2}}{R_1} \\ V_f &= V_{i2} + \tau \frac{V_{i1} - V_{i2}}{R_1 C_2} \end{aligned} \tag{22}$$

0.4.2 Energy based approach

Combined with knowledge of energy stored in capacitors,

$$\begin{aligned} E_{FINAL} &= E_{C1} + E_{C2} - E_{R1} \\ &= \frac{1}{2}(C_1 + C_2) V_{FINAL}^2 \end{aligned} \tag{23}$$

where,

$$\begin{aligned} E_{C1} &= \frac{1}{2} C_1 V_1^2 \\ E_{C2} &= \frac{1}{2} C_2 V_2^2 \end{aligned} \tag{24}$$

And then the expression for E_{R1} is determined by the following,

$$\begin{aligned} E_{R1} &= R_1 \int_0^\infty \left(\frac{V_{i1} - V_{i2}}{R_1} e^{-\frac{t}{\tau}} \right)^2 dt \\ &= R_1 \left(\frac{V_{i1} - V_{i2}}{R_1} \right)^2 \int_0^\infty e^{-\frac{2t}{\tau}} dt \\ &= \frac{\left(V_{i1} - V_{i2}\right)^2}{R_1} \left[\frac{\tau}{2} e^{-\frac{2t}{\tau}} \right]_0^\infty \\ &= \frac{\left(V_{i1} - V_{i2}\right)^2}{R_1} \left[\frac{\tau}{2} e^{-\frac{2(0)}{\tau}} - \frac{\tau}{2} e^{-\frac{2(\infty)}{\tau}} \right] \\ &= \frac{\left(V_{i1} - V_{i2}\right)^2}{R_1} \left[\frac{\tau}{2} - 0 \right] \\ &= \frac{\tau}{2R_1} \left(V_{i1} - V_{i2}\right)^2 \end{aligned} \tag{25}$$

Then (23) is expanded.

$$\begin{aligned} \frac{1}{2}(C_1 + C_2) V_{FINAL}^2 &= \frac{1}{2} C_1 V_{i1}^2 + \frac{1}{2} C_2 V_{i2}^2 - \frac{\tau}{2R_1} \left(V_{i1} - V_{i2}\right)^2 \\ V_{FINAL}^2 &= \frac{C_1 V_{i1}^2 + C_2 V_{i2}^2 - \frac{\tau}{R_1} \left(V_{i1} - V_{i2}\right)^2}{C_1 + C_2} \\ V_{FINAL} &= \sqrt{\frac{C_1 V_{i1}^2 + C_2 V_{i2}^2 - \frac{\tau}{R_1} \left(V_{i1} - V_{i2}\right)^2}{C_1 + C_2}} \end{aligned} \tag{26}$$

With sufficient substitution and algebraic manipulation of (26) the expression stated in (22) can be obtained even though equivalence is not immediately apparent. The expression for final voltage from the energy-based approach shows how the charge-based approach is less work.