Charge Pump Derivation

Ryan J. Billing

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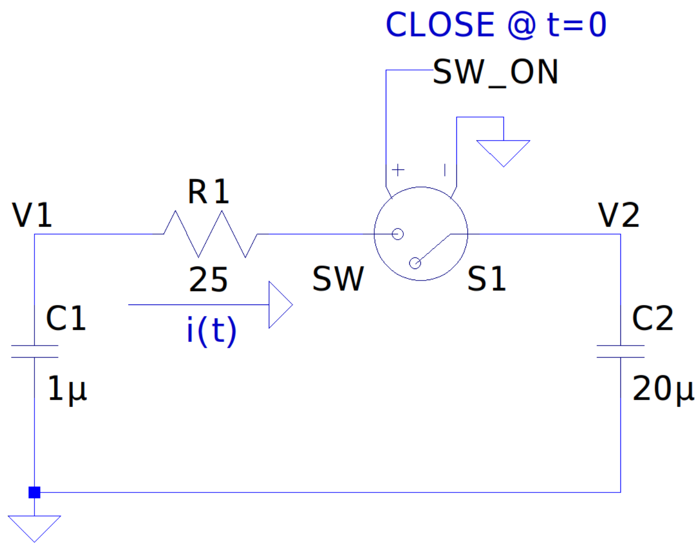

Figure 1: Charge Pump discharge cycle operation

0.1 Initial observations

Figure 1 depicts part of a charge pump circuit during the discharging cycle. C_1 is the flying capacitor that was charged to a voltage of V_1. At time $t = 0$ the switch S_1 will close and C_1 will begin to discharge into C_2.

The goal of this exercise is to determine how much energy is dissipated in R1 and determine the final voltage after the circuit has stabilized. A closed-form solution for $i(t)$ is determined, it can be used to quantify how much charge is transferred from C_1 to C_2 in this scenario. The final voltages, V_1 and V_2, can be determined from the integral of i_t by subtracting this charge from C_1 or adding it to C_2. In the final settled state it is apparent that $V_1 = V_2$.

0.2 Useful relationships

This section presents a set of fundamental relationships that may be used to determine the function for $i(t)$.

$$i(t) = -C_1 \frac{dV_1}{dt} \tag{1}$$
$$i(t) = C_2 \frac{dV_2}{dt} \tag{2}$$
$$i(t) = \frac{V_1(t) - V_2(t)}{R_1} \tag{3}$$

0.3 Derivation of i(t)

First find V_1 and V_2 from equations (1) and (2).

$$V_1(t) = -\frac{1}{C_1} \int i(t)dt \text{ (Negative sign because current is going out of } C_1\text{)}$$
$$V_2(t) = \frac{1}{C_2} \int i(t)dt \tag{4}$$

Plug these into (3) and then solve the resulting differential equation for $i(t)$.

$$i(t) = -\frac{1}{R_1}\left(\frac{1}{C_1}\int i(t)dt - \frac{1}{C_2}\int i(t)dt\right) \tag{5}$$

Next differentiate and collect like terms.

$$\frac{di(t)}{dt} = -\frac{\frac{1}{C_1} + \frac{1}{C_2}}{R_1} i(t)$$
$$R_1 \frac{C_1 C_2}{C_1 + C_2}\left(\frac{1}{i(t)}\right)\frac{di(t)}{dt} = -1 \tag{6}$$

Integrating both sides.

$$R_1 \frac{C_1 C_2}{C_1 + C_2} \int \frac{1}{i(t)} dt = \int -1 dt \tag{7}$$
$$R_1 \frac{C_1 C_2}{C_1 + C_2}\left(ln\big(i(t)\big) + C\right) = -t \tag{8}$$
$$ln\big(i(t)\big) + C = -t\frac{C_1 + C_2}{R_1 C_1 C_2} \tag{9}$$

Exponentiate to solve for $i(t)$,

$$
\begin{aligned}
e^{ln\left(i(t)\right)}e^{C} &= e^{-t\frac{C_1+C_2}{R_1C_1C_2}} \\
i(t) &= e^{-C}e^{-t\frac{C_1+C_2}{R_1C_1C_2}} \qquad (10)
\end{aligned}
$$

Finally the integration constant can be determined by considering the initial conditions,

$$i(0) = \frac{V_1(0) - V_2(0)}{R_1} \qquad (11)$$

The expression (10) is solved for the known initial conditions at $t = 0$ to determine the yet unknown constant of integration. In this case the constant C itself is a "don't care" quantity as we really want to know e^{-C} directly so the entire expression may be expressed in terns of initial voltages and the resistor value.

$$
\begin{aligned}
\frac{V_1(0) - V_2(0)}{R_1} &= e^{-C}e^{-0\frac{C_1+C_2}{R_1C_1C_2}} \\
\frac{V_1(0) - V_2(0)}{R_1} &= e^{-C}1 \\
e^{-C} &= \frac{V_1(0) - V_2(0)}{R_1} \qquad (12)
\end{aligned}
$$

Plugging (12) into(10), a closed-form expression for the resistor current, $i(t)$, is obtained.

$$i(t) = \frac{V_1(0) - V_2(0)}{R_1}e^{-t\frac{(C_1+C_2)}{R_1C_1C_2}} \qquad (13)$$

Then to help tidy the appearance of things, let

$$
\begin{aligned}
V_1(0) &= V_{i1} \qquad (14) \\
V_2(0) &= V_{i2} \qquad (15) \\
\tau &= \frac{1}{\frac{C_1+C_2}{R_1C_1C_2}} = R_1\frac{C_1C_2}{C_1+C_2} \qquad (16)
\end{aligned}
$$

And then the final expression reveals a familiar time domain response. In fact, the circuit could be redrawn as a single capacitor and resistor network driven by a step response with amplitude $V_{i1} - V_{i2}$, and where the capacitor value is equivalent to the series combination of C_1 and C_2.

$$i(t) = \frac{V_{i1} - V_{i2}}{R_1} e^{-\frac{t}{\tau}} \tag{17}$$