Design Reference Sheet
Switched Inductor Power Supplies

Ryan J Billing

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1 System quantities and definitions

1.1 Converter types considered

The following can be modeled using information contained in this reference sheet.

1. Boost

2. Buck

3. Buck-Boost

4. Flyback (Isolated Buck-Boost)

5. Forward (Buck)

1.2 Control modes

The following controls schemes are considered.

1. Current Programmed Mode (CPM) Control (a.k.a Peak Current) Mode Control

2. Duty Control (a.k.a Voltage Mode Control)

Others such as Hysteretic, Constant On Time (COT), Quasi-resonant (QR) can be characterized for steady-state values but dynamic behaviors need to be derived and modified to describe the elements unique to the control scheme.

1.3 Operational modes

A converter can operate in a mode where energy stored in the inductor during the charging cycle is entirely discharged to the load within each cycle, or it can partially discharge and maintain a certain amount of energy stored between cycles.

1. **Continuous Conduction Mode (CCM):** Inductor does not fully discharge

State Variable	Steady State Value	Small Signal Value	Description
i_C	I_c	$\hat{i}_c$	Control set point for Current Programmed Mode (CPM) control
d	D	$\hat{d}$	Duty cycle control set point for Voltage Mode (Duty) Control
i_V	I_v	$\hat{i}_v$	Inductor valley current
v_{CG}	V_{cg}	$\hat{v_c}g$	Voltage across inductor during charging cycle
v_{DG}	V_{dg}	$\hat{v_d}g$	Voltage across inductor during discharging cycl
i_{CG}	I_{cg}	$\hat{i_c}g$	Charging cycle inductor current
i_{DG}	I_{dg}	$\hat{i_d}g$	Discharging cycle inductor current

2. **Boundary Conduction Mode (BCM):** Inductor fully discharges

3. **Discontinuous Conduction Mode (DCM):** Inductor fully discharges

In BCM the inductor finishes discharging immediately at termination of the switching cycle. During DCM operation the inductor discharges and remains idle for a finite period of time between cycles.

1.4 Definitions for Variables

From Tan/Middlebrook, "Notation conventions adopted here are as follows. Capital letters are used to indicate quantities associated with steady- state, hatted letters are quantities associated with small-signal perturbations, and small letters represent quantities associated with perturbed state, i.e., quantities which are the sum of capital and hatted letters."

The following parameters are generally used to express system state variables,

Items generally used as constants,

$F_{SW} = $ Modulator switching frequency

$T_{SW} = \dfrac{1}{F_{SW}}$, Modulator switching (sampling) period

$L_m = $ Converter magnetizing inductance

$m_c = \dfrac{V_{CG}}{L_m}$, Slope of current during charge cycle (A/s)

$m_d = \dfrac{V_{DG}}{L_m}$, Slope of current during discharge cycle (A/s)

$m_{cmp} = $ Artificial ramp added to primary current sense signal (slope compensation) (A/s)

$\alpha = \dfrac{m_c + m_d}{m_c + m_{cmp}} = \dfrac{V_{CG} + V_{DG}}{V_{CG} + m_{cmp}L_m}$

$D = $ Modulator charging cycle duty

$D' = (1 - D)$ Modulator discharging cycle duty

1.5 Useful identities

$$D = \frac{m_d}{m_c + m_d} = \frac{V_{DG}}{V_{CG} + V_{DG}} \tag{1}$$

$$I_{pk} = I_c \frac{m_c}{m_c + m_{cmp}} \tag{2}$$

2 Continuous Conduction Mode

2.1 Control-to-Valley-Current Recursive Filter Structures

The average currents can be computed as a function of inductor valley current and charge/discharge slopes. Valley current is easily related to the control input through a linear IIR filter.

2.1.1 Peak current control input to valley current

$$i_v[n] = \alpha i_c[n] + (1 - \alpha)i_v[n-1] - m_d T_{SW} \tag{3}$$

2.1.2 Duty control input to valley current

$$i_v[n] = i_v[n-1] + D[n](m_c + m_d)T_{SW} - m_d T_{SW} \tag{4}$$

2.2 Steady State Quantities related to Valley Current

This section presents interesting quantities as a function of the valley currents.

2.2.1 Steady State Valley Current

$$I_v = I_c - \frac{T_{SW}m_d}{\alpha} = I_c - \left(\frac{T_{SW}}{L_m}\right)\frac{(V_{CG} + L_m m_{cmp})V_{DG}}{V_{CG} + V_{DG}} \tag{5}$$

2.2.2 Steady State Inductor Current Ripple

$$\Delta_i = \frac{m_c m_d}{m_c + m_d}T_{SW} = \left(\frac{T_{SW}}{L_m}\right)\frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} \tag{6}$$

2.2.3 Steady State Peak Current

$$I_{pk} = I_v + \Delta_i \tag{7}$$

2.2.4 Steady State Cycle Average Currents

Average current during the inductor charging cycle.

$$I_{CG} = \frac{L_m}{2V_{CG}T_{SW}}(I_{pk}^2 - I_v^2) \tag{8}$$

Average current during the inductor discharging cycle.

$$I_{DG} = \frac{L_m}{2V_{DG}T_{SW}}(I_{pk}^2 - I_v^2) \tag{9}$$

Sanity check (by inspection), input power is equal to output power. Solve (8) for $I_{CG}V_{CG}$ and (9) for $I_{DG}V_{DG}$. We see both are equal to the same quantity, so the following is also true.

$$I_{CG}V_{CG} = I_{DG}V_{DG} \tag{10}$$

2.3 Steady State Quantities related to cycle average currents

In some cases it is more convenient to calculate converter state variables as a function of input or output current. For example, if the load current is known, one would want to work backwards through the modulator to determine the valley current, inductor current ripple, and control set points.

2.3.1 Valley current as function of charge cycle current

$$I_v = \frac{V_{CG} + V_{DG}}{V_{DG}}I_{CG} - \left(\frac{T_{SW}}{2L_m}\right)\frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} \tag{11}$$

From (7),

$$I_{pk} = \frac{V_{CG} + V_{DG}}{V_{DG}}I_{CG} + \left(\frac{T_{SW}}{2L_m}\right)\frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} \tag{12}$$

From (5),

$$I_c = \frac{V_{CG} + V_{DG}}{V_{DG}}I_{CG} - \left(\frac{T_{SW}}{2L_m}\right)\frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} + \frac{T_{SW}m_d}{\alpha} \tag{13}$$

2.3.2 Valley current as function of discharge cycle current

$$I_v = \frac{V_{CG} + V_{DG}}{V_{CG}}I_{DG} - \left(\frac{T_{SW}}{2L_m}\right)\frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} \tag{14}$$

From (7),

$$I_{pk} = \frac{V_{CG} + V_{DG}}{V_{CG}} I_{DG} + \left(\frac{T_{SW}}{2L_m}\right) \frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} \tag{15}$$

From (5),

$$I_c = \frac{V_{CG} + V_{DG}}{V_{CG}} I_{DG} - \left(\frac{T_{SW}}{2L_m}\right) \frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} + \frac{T_{SW}m_d}{\alpha} \tag{16}$$

3 Boundary Conduction Mode (BCM)

Formulas in this section help characterize where the converter transitions from Discontinuous Conduction Mode (DCM) to Continuous Conduction Mode (CCM). The point at which a converter charges and completely discharges the inductor energy concurrent to termination of the switching period is called "Boundary Conduction Mode" (BCM).

3.1 Peak current during BCM

$$I_{pk} = \left(\frac{T_{SW}}{L_m}\right) \frac{V_{CG}V_{DG}}{V_{CG} + V_{DG}} \tag{17}$$

3.2 Steady state average currents during BCM

Average inductor currents during charge cycle:

$$I_{CG} = \left(\frac{T_{SW}}{2L_m}\right) \frac{V_{DG}^2 V_{CG}}{(V_{CG} + V_{DG})^2} \tag{18}$$

Average inductor currents during discharge cycle:

$$I_{DG} = \left(\frac{T_{SW}}{2L_m}\right) \frac{V_{CG}^2 V_{DG}}{(V_{CG} + V_{DG})^2} \tag{19}$$

4 Discontinuous Conduction Mode (DCM)

4.1 Steady State Peak Current in DCM

$$I_{pk} = \sqrt{\frac{2T_{SW}V_{DG}}{L_m} I_{DG}} \tag{20}$$

or referenced to discharge cycle average current,

$$I_{pk} = \sqrt{\frac{2T_{SW}V_{CG}}{L_m} I_{CG}} \tag{21}$$

4.2 Duty cycle and switch idle time

Duty cycle expressed in (1), but the quantity D' no longer represents discharge cycle time. There is a switch conduction interval followed by an interval while the switch is idle (no conduction) during the discharging cycle.

$$D = \frac{I_{pk}}{T_{SW}m_c} = \frac{1}{T_{SW}m_c}\sqrt{\frac{2T_{SW}V_{CG}}{L_m}I_{CG}} = \sqrt{\frac{2L_m}{T_{SW}V_{CG}}I_{CG}} = \sqrt{\frac{2L_mV_{DG}}{T_{SW}V_{CG}^2}I_{DG}} \quad (22)$$

$$D' = \frac{I_{pk}}{T_{SW}m_d} = \frac{1}{T_{SW}m_d}\sqrt{\frac{2T_{SW}V_{DG}}{L_m}I_{DG}} = \sqrt{\frac{2L_m}{T_{SW}V_{DG}}I_{DG}} \quad (23)$$

$$D_{idle} = 1 - D - D' \quad (24)$$

4.3 Steady State Control-to-Average Current in DCM

$$I_{CG} = \frac{L_m}{2T_{SW}V_{CG}}I_{pk}^2 \quad (25)$$

$$I_{DG} = \frac{L_m}{2T_{SW}V_{DG}}I_{pk}^2 \quad (26)$$

5 RMS Current

5.1 CCM

Computation for RMS currents during CCM operation, based on the steady state peak and valley current and duty cycle.

$$I_{CG.RMS} = \sqrt{D}\sqrt{I_v^2 + I_v(I_{pk} - I_v) + \frac{1}{3}(I_{pk} - I_v)^2} \quad (27)$$

$$I_{DG.RMS} = \sqrt{D'}\sqrt{I_v^2 + I_v(I_{pk} - I_v) + \frac{1}{3}(I_{pk} - I_v)^2} \quad (28)$$

It is possible to show how (27) and (28) reduce to the formulas for DCM at BCM, where $\sqrt{D} = \sqrt{\frac{1}{T_{SW}m_c}}$ and $I_v = 0$. The corollary is shown by the same process for discharge cycle currents.

5.2 DCM and BCM

Valid for both DCM and BCM RMS current calculations.

$$I_{CG.RMS} = I_{pk}\frac{1}{\sqrt{3T_{SW}m_c}} \quad (29)$$

$$I_{DG.RMS} = I_{pk}\frac{1}{\sqrt{3T_{SW}m_d}} \quad (30)$$

6 CCM Small signal dynamic behavior

For behavior 2 decades below the switching frequency DC values can be used to express the gain of the modulator with a reasonable degree of accuracy. Often the designer seeks to design a control loop crossover frequency much higher than $F_{SW}/100$ where the modulator dynamic behavior becomes progressively more interesting.

Behaviors of primary interest:

- $L_m C_{LOAD}$ resonance for duty (voltge-mode) control (see (4))

- $\dfrac{F_{SW}}{2}$ peaking and phase response for CPM control (see (3))

- Right-Half-Plane-Zero (RHPZ) for boost, buck-boost and flyback topologies

6.1 CPM CCM Control to Valley Current Transfer Function

The first stage in the system dynamic response is the relationship between peak current control input and the valley current.

$$H_v(z) = \frac{I_v(z)}{I_c(z)} = \frac{\alpha z^{-1}}{1 - (1 - \alpha)z^{-1}} \tag{31}$$

Evaluating ROC and other stability yields the condition for "academic stability":

$$m_{cmp} > \frac{m_d - m_c}{2} \tag{32}$$

The term "academic stability" means the system response is guaranteed to settle within a noise-free environment. Such a designation emphasizes the likelihood that a switched-mode converter system is perpetually perturbed at high frequencies and the observed behavior near academic stability will be indistinguishable from instability.

A useful quantity is the magnitude response at the Nyquist frequency (where peaking occurs):

$$\left| H_V(z) \right|_{z = \frac{\omega_{SW}}{2}} = \frac{\alpha}{2 - \alpha} \tag{33}$$

It is usually more useful to pick a desired maximum peaking response (recommended < 6dB) and then determine what amount of slope compensation is needed to meet the specification.

$$m_{cmp} = (m_c + m_d)\frac{1 + |H_v(z)|}{2|H_v(z)|} - m_c \bigg|_{z = \frac{1}{2}\omega_{SW}} \tag{34}$$

Then of course, to express values in dB,

$$G_{dB} = 20log10\left(\frac{\alpha}{2-\alpha}\right) \tag{35}$$

$$m_{cmp} = \frac{1}{2}(m_c + m_d)\frac{1 + 10^{\frac{G_{dB}}{20}}}{10^{\frac{G_{dB}}{20}}} - m_c \tag{36}$$

A reasonable design rule is given below by solving (36) for $G_{dB} = 6dB$.

$$m_{cmp} = \frac{1}{4}(3m_d - m_c) \tag{37}$$

A further interesting observation is if (37) is solved for the case $m_c = 3m_d$, the peak current mode controller has 6dB peaking at 25% duty cycle with no slope compensation. Industry wisdom that slope compensation is needed at 37% duty corresponds to 12 dB valley current peaking.

6.2 CCM Valley to Average Current Transfer Function

The charging cycle current gain and left-half plane zero (LHPZ).

$$H_{cg}(s) = \frac{\hat{I}_{cg}(s)}{\hat{I}_v(s)} = \frac{I_v}{m_c + m_d}\left(\frac{m_d}{I_v} + s\right) \tag{38}$$

The discharging cycle current gain and right-half plane zero (RHPZ).

$$H_{dg}(s) = \frac{\hat{I}_{dg}(s)}{\hat{I}_v(s)} = \frac{I_v}{m_c + m_d}\left(\frac{m_c}{I_v} - s\right) \tag{39}$$

6.3 CCM Average to Pulse Current Transfer Function

The following is derived by computing the Laplace Transform of the steady-state current pulse shape. This forms a low-pass FIR filter with magnitude response rolling off above the Nyquist frequency. The most interesting information given by these filters is the group delay at higher frequencies.

6.3.1 Laplace Transform of Charge and Discharge cycle current pulses

$$F_{CG}(s) = I_v\left(\frac{1 - e^{-sDT_{SW}}}{s}\right) + m_c\left(\frac{1 - (1 + sDT_{SW})e^{-sDT_{SW}}}{s^2}\right) \tag{40}$$

$$F_{DG}(s) = (I_v + m_dD'T_{SW})\left(\frac{1 - e^{-sD'T_{SW}}}{s}\right)e^{-sDT_{SW}}$$
$$- m_d\left(\frac{1 - (1 + sD'T_{SW})e^{-sD'T_{SW}}}{s^2}\right)e^{-sDT_{SW}} \tag{41}$$

6.3.2 DC gain of pulse transfer function

The expressions in (39) and (38) as well as the control-to-valley current functions give information about the system small-signal gain while (40) and (41) include redundancy. Evaluating the limits for F_{CG} and F_{DG} yield the DC gain, the inverse of which can be used to set the overall transfer function DC gain to unity.

$$\lim_{s \to 0} F_{CG}(s) = \frac{1}{2} D^2 T_{SW}^2 m_c + DT_{SW} I_v = \frac{D^2 T_{SW}^2 m_c + 2DT_{SW} I_v}{2} \tag{42}$$

$$\lim_{s \to 0} F_{DG}(s) = \frac{1}{2} D'^2 T_{SW}^2 m_d + D'T_{SW} I_v = \frac{D'^2 T_{SW}^2 m_d + 2D'T_{SW} I_v}{2} \tag{43}$$

6.3.3 Normalized pulse transfer function

The following will generate the system group delay and high frequency response without affecting the system's small-signal gain that has been captured within the control-to-valley current function and valley-to-average current functions.

$$F_{CG.NORM}(s) = \frac{2}{D^2 T_{SW}^2 m_c + 2DT_{SW} I_v} F_{CG}(s) \tag{44}$$

$$F_{DG.NORM}(s) = \frac{2}{D'^2 T_{SW}^2 m_d + 2D'T_{SW} I_v} F_{DG}(s) \tag{45}$$

7 DCM small signal dynamic behavior

The transfer functions expressed in this section cover DCM converter operation. In most instances the CCM model is better suited for operation near BCM, however, results will be very similar using either model for characterizing BCM operation.

7.1 CPM DCM control-to-average-current transfer functions

Steady state values from (25) and (26) are evaluated for small-signal response by partial derivative, yielding the following small-signal gain quantities.

$$\hat{i}_{CG} = \frac{L_m I_{pk}}{T_{SW} V_{CG}} \hat{i}_{pk} = \frac{m_c}{m_c + m_{cmp}} \frac{L_m I_c}{T_{SW} V_{CG}} \hat{i}_c \tag{46}$$

$$\hat{i}_{DG} = \frac{L_m I_{pk}}{T_{SW} V_{DG}} \hat{i}_{pk} = \frac{m_c}{m_c + m_{cmp}} \frac{L_m I_c}{T_{SW} V_{DG}} \hat{i}_c \tag{47}$$

Notice control-to-output gain is expressed in a single step whereas CCM uses control-to-valley current, then valley-to-output current in its transfer function.

During DCM operation there is no memory from cycle-to-cycle so its transfer function is captured as a single FIR system.

7.2 Voltage Mode DCM control-to-average-current transfer functions

When the modulator input is duty control, the transfer functions are the following.

By using the relationship between duty cycle and peak current, (??) and (??) can be used with the following substitution.

$$I_{pk} = DT_{SW}m_c = D\frac{T_{SW}V_{CG}}{L_m} \tag{48}$$

$$\hat{i}_{CG} = \frac{L_m DT_{SW}m_c}{T_{SW}V_{CG}}\hat{d}T_{SW}m_c = \frac{DT_{SW}V_{CG}}{L_m}\hat{d} \tag{49}$$

$$\hat{i}_{DG} = \frac{L_m DT_{SW}m_c}{T_{SW}V_{DG}}\hat{d}T_{SW}m_c = \frac{DT_{SW}V_{CG}^2}{L_m V_{DG}}\hat{d} \tag{50}$$

7.3 DCM Average to Pulse Current Transfer Function

The following is derived by computing the Laplace Transform of the steady-state current pulse shape. This forms a low-pass FIR filter with magnitude response rolling off above the Nyquist frequency. The most interesting information given by these filters is the group delay at higher frequencies.

7.3.1 Laplace Transform of Charge and Discharge cycle current pulses

$$F_{CG}(s) = m_c\left(\frac{1 - (1 + sT_{CG})e^{-sT_{CG}}}{s^2}\right) \tag{51}$$

$$F_{DG}(s) = m_d T_{DG}\left(\frac{1 - e^{-sT_{DG}}}{s}\right)e^{-sT_{CG}}$$
$$- m_d\left(\frac{1 - (1 + sT_{DG})e^{-sT_{DG}}}{s^2}\right)e^{-sT_{CG}} \tag{52}$$

where,

$$T_{CG} = \frac{I_{pk}}{m_c} \text{ , time to charge inductor during charging cycle} \tag{53}$$

$$T_{DG} = \frac{I_{pk}}{m_d} \text{ , time to discharge inductor during discharging cycle} \tag{54}$$

7.3.2 Normalized pulse transfer function

The pulse transfer function normalization uses the same formulae as (44) and (45), setting $I_v = 0$.

$$F_{CG.NORM}(s) = \frac{2}{T_{CG}^2 m_c} F_{CG}(s) \tag{55}$$

$$F_{DG.NORM}(s) = \frac{2}{T_{DG}^2 m_d} F_{DG}(s) \tag{56}$$

8 Error Amplifier

Abstracting impedances connected within the error amplifier, the gain can be expressed by means of basic circuit analysis while preventing the multiplicity of terms becoming unruly.

$$H_{EA} = \frac{i_o}{v_o} = \frac{Z_i + G_{thev}(Z_f - \frac{1}{g_m}) + \frac{1}{g_m}}{Z_i Z_o + (Z_f + Z_i + Z_o)\frac{1}{g_m}} \tag{57}$$

$$= \frac{1}{Z_o} \frac{Z_i + G_{thev}(Z_f - \frac{1}{g_m}) + \frac{1}{g_m}}{Z_i + (\frac{Z_f + Z_i}{Z_o} + 1)\frac{1}{g_m}} \tag{58}$$

The value in such analysis is to see more clearly how this strange device (TL431 or TLV431) actually behaves in this circuit. The relative role of the device's amplification factor defines how closely it emulates the response of an ideal amplifier. Below the transfer function is expressed in the limit as transconductance approaches infinity.

$$\lim_{g_m \to \infty} H_{EA} = \frac{1}{Z_o}(G_{thev}\frac{Z_f}{Z_i} + 1) \tag{59}$$

$$\tag{60}$$

Finally, the input resistor divider is expanded back out in the expression as $\lim_{g_m \to \infty}$, and we see the DC bias resistor, R_b simply drops out of the formula.

$$G_{thev} = \frac{R_b}{R_b + R_i} \tag{61}$$

$$Z_i = \frac{R_b R_i}{R_b + R_i} \tag{62}$$

$$H_{EA} = \frac{1}{Z_o}(\frac{R_b}{R_b + R_i}\frac{Z_f}{\frac{R_b R_i}{R_b + R_i}} + 1) = \frac{1}{Z_o}(G_{thev}\frac{Z_f}{R_i} + 1) \tag{63}$$

The transfer function for current through Z_o, so only a simple current divider network needs to be considered for expressing final gain to the optocoupler LED.

$$g_{LED} = \frac{i_{LED}}{i_o} = \frac{R_o}{R_{LED} + R_o} \tag{64}$$

Finally,

$$H_{LED} = \frac{i_{LED}}{v_o} = g_{LED}H_{EA} = \frac{R_o}{R_{LED} + R_o}\frac{1}{Z_o}(G_{thev}\frac{Z_f}{R_i} + 1) \tag{65}$$

Then expanding Z_o the expression becomes much more tidy,

$$H_{LED} = \frac{1}{R_{LED}}(G_{thev}\frac{Z_f}{R_i} + 1) \tag{66}$$

The full transfer function with finite transconductance included below. It is apparent by inspection that (67) reduces to (66) in the limit as $g_m \to \infty$.

$$H_{LED} = \frac{1}{R_{LED}}\frac{Z_i + G_{thev}(Z_f - \frac{1}{g_m}) + \frac{1}{g_m}}{Z_i + (\frac{Z_f + Z_i}{Z_o} + 1)\frac{1}{g_m}} \tag{67}$$