Probability and Statistics 2021
Assignment 5

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Exercise 1

Lightening strikes occur as a Poisson process with rate λ per hour.
Let X, Y and Z be the number of strikes between the time 9-11, 11-3, 3-7pm respectively.

Join pmf of X,Y,Z

$$X \sim exp(2\lambda) \Rightarrow P[X=x] = \frac{(2\lambda)^x}{x!}e^{-2\lambda}$$
$$Y \sim exp(4\lambda) \Rightarrow P[Y=y] = \frac{(4\lambda)^y}{y!}e^{-4\lambda}$$
$$Z \sim exp(4\lambda) \Rightarrow P[Z=z] = \frac{(4\lambda)^z}{z!}e^{-4\lambda}$$

Since X,Y,Z are independent

$$\begin{aligned} P[X=x, Y=y, Z=z] &= P[X=x] * P[Y=y] * P[Z=z] \\ &= \frac{(2\lambda)^x}{x!}e^{-2\lambda}\frac{(4\lambda)^y}{y!}e^{-4\lambda}\frac{(4\lambda)^z}{z!}e^{-4\lambda} \\ &= e^{-10\lambda}\frac{(2\lambda)^x(4\lambda)^y(4\lambda)^z}{x!y!z!} \end{aligned}$$

Conditional joint pmf of X,Y,Z given that $X+Y+Z=20$

Previously we have prove that if $X \sim exp(2\lambda)$ and $Y \sim exp(4\lambda)$ and $Z \sim exp(4\lambda)$, than $X + Y + Z \sim exp(10\lambda)$

$$
\begin{aligned}
P[X=x|X+Y+Z=20] &= \frac{P[X=x, X+Y+Z=20]}{P[X+Y+Z=20]} \\
&= \frac{P[X=x]*P[Y+Z=20-x]}{P[X+Y+Z=20]} \\
&= \frac{\frac{(2\lambda)^x}{x!}e^{-2\lambda} * \frac{(8\lambda)^{20-x}}{(20-x)!}e^{-8\lambda}}{\frac{(10\lambda)^{20}}{20!}e^{-10\lambda}} \\
&= \frac{20!}{(10\lambda)^{20}} \frac{2^x\lambda^x 8^{20-x}\lambda^{20-x}}{x!(20-x)!} \\
&= \frac{20!\lambda^{20}}{(10\lambda)^{20}} \frac{2^x 8^{20-x}}{x!(20-x)!} \\
&= \frac{20!}{x!(20-x)!}\left(\frac{2}{10}\right)^x\left(\frac{8}{10}\right)^{20-x} \\
&= C_x^{20}\left(\frac{2}{10}\right)^x\left(\frac{8}{10}\right)^{20-x}
\end{aligned}
$$

Based on its conditional joint pmf, $X=x|X+Y+Z=20$ follows binominal distribution with $n=20, p=0.2$

Similarly, we have following results:

- $Y=y|X+Y+Z=20$ follows binominal distribution with $n=20, p=0.4$
- $Z=z|X+Y+Z=20$ follows binominal distribution with $n=20, p=0.4$

Exercise 2

An eight-sided die is rolled n times. Let X_1 be the number of 1s that are observed, X_4 be the number of 4s.

We also have $X_i \to$ numbers of i, where $1 \leq i \leq 8$

Find $cov(X_1, X_4)$

X has following properties:

- n identical trials
- there're 8 outcomes for each trials
- probability for each outcome is the same $p_i = \frac{1}{8}$
- trials are independent

$\Rightarrow X$ follows multinomial distribution with $p = \frac{1}{8}$

$$cov(X_1, X_4) = -np_1p_4 = \frac{-n}{64}$$

Find $corr(X_1, X_4)$

$$\begin{aligned} var(X_1) = var(X_4) = np(1-p) &= \frac{7n}{64} \\ corr(X_1, X_4) &= \frac{cov(X_1, X_4)}{sd(X_1)sd(X_4)} \\ &= \frac{\frac{-n}{64}}{\sqrt{var(X_1)var(X_4)}} \\ &= \frac{-n}{64\frac{7n}{64}} \\ &= -\frac{1}{7} \end{aligned}$$

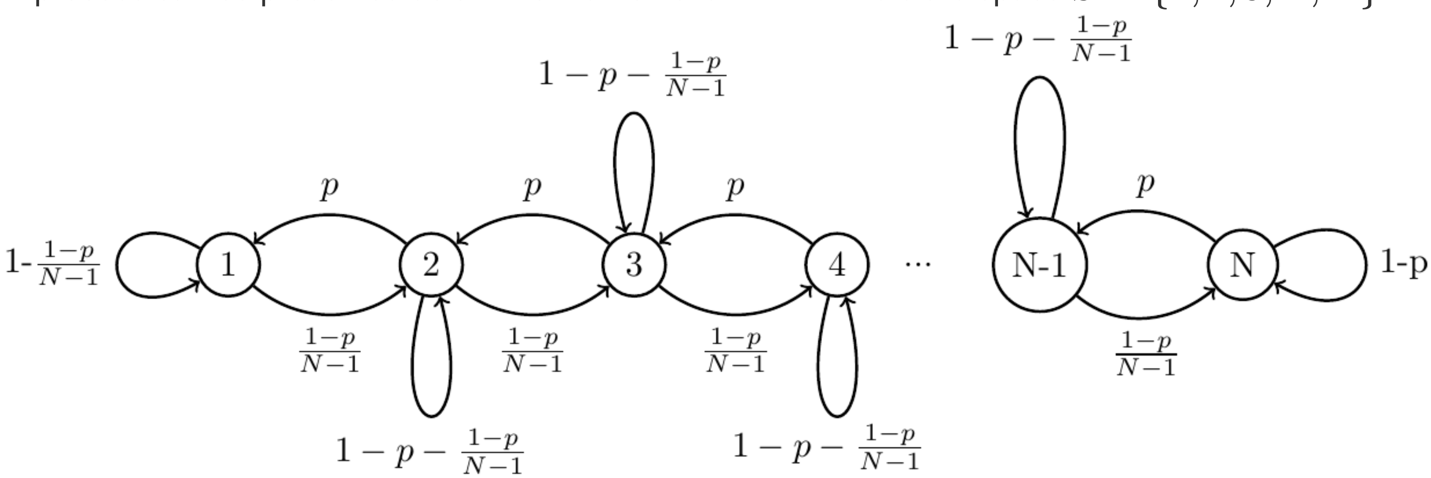
Exercise 3

Bookshelf have N books, including:
- 1 book with red cover and has selecting probability of p
- $N-1$ books with plain covers and has selecting probability of $\frac{1-p}{N-1}$

When selected, a book and its neighbour to the left swap position. If the leftmost is selected, it is returned there
X_n denotes the position of the red books after n unit of time.

Show that X_n is a Markov chain with non-zero transition probabilites

X process can be presented as a finite Markov chain with finite state space $S = \{1, 2, 3, ..., N\}$:



At every moment of n, 1 of following case will happen:

1. X_n = i, X_{n+1} = $i-1$, where $2 < i < N$
   This case happen only when:
   - red book is not the leftmost

- a red book is seletected

$$p_{i,i-1} = p$$

2. X_n = i, X_{n+1} = i, where $2 < i < N - 1$
   This case happen when:
   - red book is not the leftmost
   - neither a red book nor its right neighbor is selected
     - probability of selecting a red book nor its right neighbor is $p + \frac{1-p}{N-1}$

$$p_{i,i} = 1 - p - \frac{1-p}{N-1}$$

3. X_n = 1, X_{n+1} = 1
   This case happen when:
   - red book is the leftmost
   - selected book is any but the red book's right neighbor
     - probability of selecting a red book nor its right neighbor is $\frac{1-p}{N-1}$

$$p_{1,1} = 1 - \frac{1-p}{N-1}$$

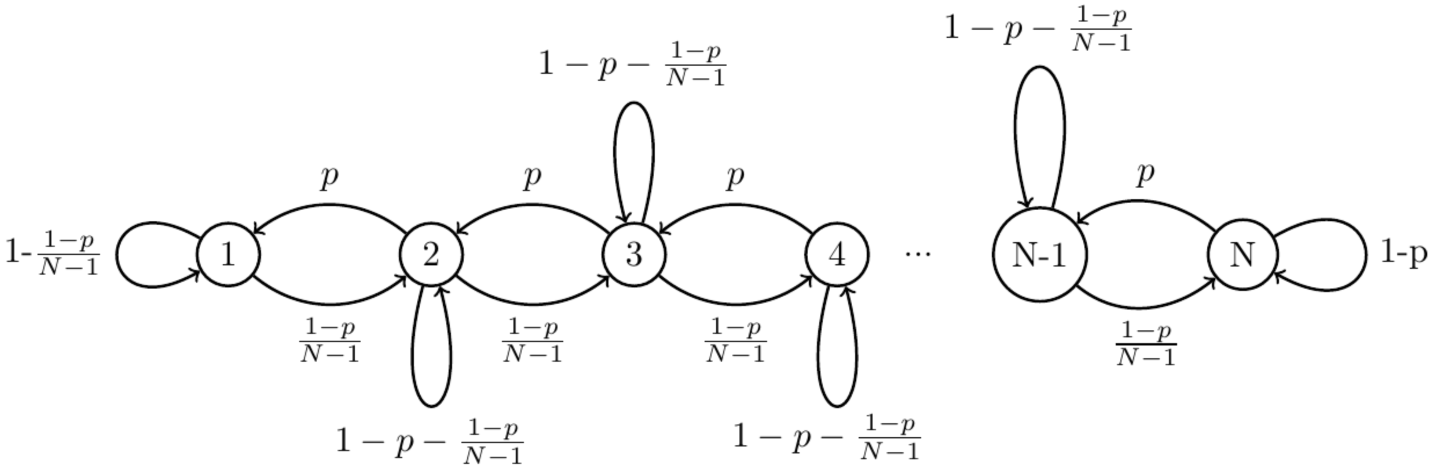
4. X_n = i, X_{n+1} = $i + 1$, where $1 < i < N - 1$
   This case happen when neither a red book's right neighbor is selected

$$p_{i,i+1} = \frac{1-p}{N-1}$$

5. X_n = N, X_{n+1} = N
   This case happen when:
   - red book is the rightmost
   - selected book is any but the red book

$$p_{N,N} = 1 - p$$

Let π_i is limiting probability of the system being in start i



$$
\begin{aligned}
\pi_i &= \sum P_{ij}\pi_j \\
\pi_1 &= P_{2,1}\pi_2 + P_{1,1}\pi_1 \\
&= p\pi_2 + \left(1 - \frac{1-p}{N-1}\right)\pi_1 \\
\pi_2 &= P_{1,2}\pi_1 + P_{2,2}\pi_2 + P_{3,2}\pi_3 \\
&= \frac{1-p}{N-1}\pi_1 + \left(1 - p - \frac{1-p}{N-1}\right)\pi_2 + p\pi_3 \\
\ldots
\end{aligned}
$$

Caclulate π_2 based on π_1

$$
\begin{aligned}
\pi_1 &= p\pi_2 + \left(1 - \frac{1-p}{N-1}\right)\pi_1 \\
\Leftrightarrow p\pi_2 &= \pi_1 - \left(1 - \frac{1-p}{N-1}\right)\pi_1 \\
\Leftrightarrow p\pi_2 &= \frac{1-p}{N-1}\pi_1 \\
\Leftrightarrow \pi_2 &= \frac{1-p}{p(N-1)}\pi_1
\end{aligned}
$$

We also have

$$
\pi_1 = \frac{p(N-1)}{1-p}\pi_2
$$

Caclulate π_3 based on π_2

$$
\begin{aligned}
\pi_2 &= \frac{1-p}{N-1}\pi_1 + \left(1 - p - \frac{1-p}{N-1}\right)\pi_2 + p\pi_3 \\
&= \frac{1-p}{N-1}\frac{p(N-1)}{1-p}\pi_2 + \left(1 - p - \frac{1-p}{N-1}\right)\pi_2 + p\pi_3 \\
&= p\pi_2 + \left(1 - p - \frac{1-p}{N-1}\right)\pi_2 + p\pi_3 \\
&= \left(1 - \frac{1-p}{N-1}\right)\pi_2 + p\pi_3 \\
\Rightarrow p\pi_3 &= \frac{1-p}{N-1}\pi_2 \\
\Leftrightarrow \pi_3 &= \frac{1-p}{p(N-1)}\pi_2
\end{aligned}
$$

By extending π_2, we also have

$$
\pi_3 = \left(\frac{1-p}{p(N-1)}\right)^2 \pi_1
$$

Find general solution for π_i for $1 \leq i \leq N$

Based on previous findings, we have:

$$
\pi_i = \left(\frac{1-p}{p(N-1)}\right)^{i-1} \pi_1, 2 \leq i \leq N
$$

As a finite Markov chain,

$$
\begin{aligned}
&\sum_{i=1}^{N} \pi_i = 1 \\
\Leftrightarrow &\pi_1 \sum_{i=1}^{N} \left(\frac{1-p}{p(N-1)} \right)^{i-1} = 1 \\
\Leftrightarrow &\pi_1 \frac{1 - \left(\frac{1-p}{p(N-1)} \right)^N}{1 - \left(\frac{1-p}{p(N-1)} \right)} = 1 \\
\Leftrightarrow &\pi_1 = \frac{1 - \frac{1-p}{p(N-1)}}{1 - \left(\frac{1-p}{p(N-1)} \right)^N} \\
\Rightarrow &\pi_i = \left(\frac{1-p}{p(N-1)} \right)^{i-1} \frac{1 - \frac{1-p}{p(N-1)}}{1 - \left(\frac{1-p}{p(N-1)} \right)^N}
\end{aligned}
$$