Probability and Statistics 2021
Assignment 4

Student name: Thien Toan Tran
Student ID: A1808080

Exericse 1

Given the random variable X with pdf

$$f(x) = \begin{cases} x/2, & \text{if } 0 \leq \text{x} \leq 2 \\ b, & \text{otherwise} \end{cases}$$

(a) Find the pdf of $Y = g(X) = 1 - \sqrt{4 - X^2}/2$.

calculate Y's CDF:

$$\begin{aligned} P(Y \leq y) &= P(1 - \sqrt{4 - x^2}/2 \leq y) \\ &= P(1 - y \leq \sqrt{4 - x^2}/2) \\ &= P((2 - 2y)^2 \leq 4 - x^2) \\ &= P(x^2 \leq 4 - (2 - 2y)^2) \\ &= P(x \leq \sqrt{4 - (2 - 2y)^2}) \\ &= \int_{x=0}^{\sqrt{4-(2-2y)^2}} \frac{x}{2} dx \\ &= \frac{x^2}{4} \Big|_0^{\sqrt{4-(2-2y)^2}} \\ &= \frac{4 - (2 - 2y)^2}{4} \end{aligned}$$

Therefore

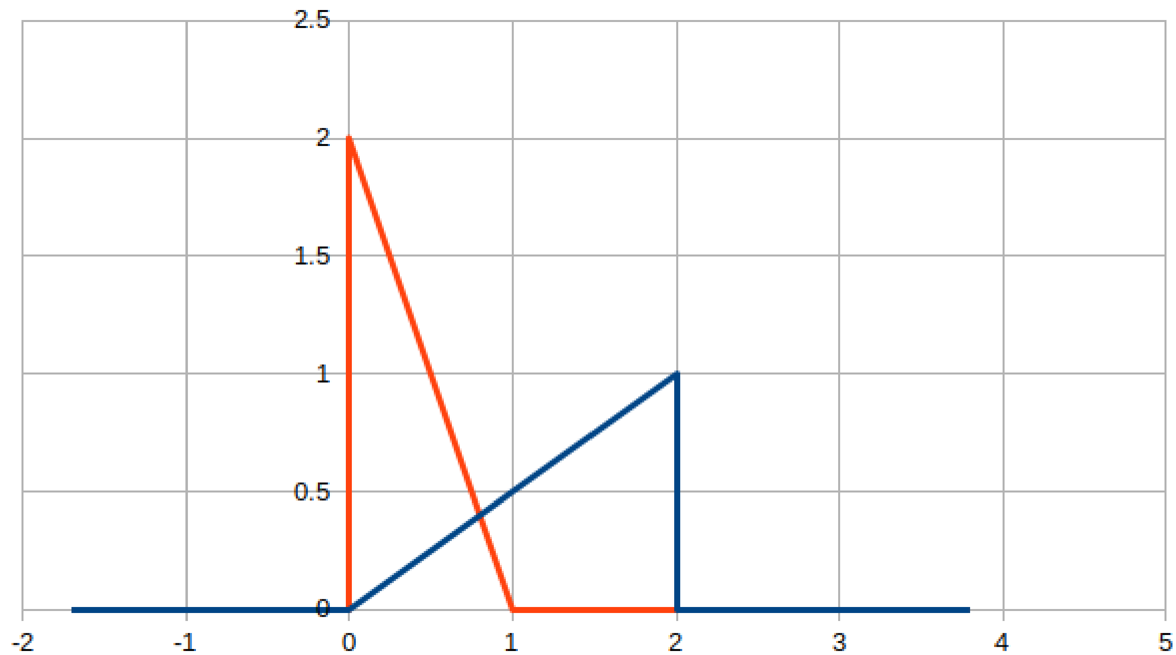
$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{4-(2-2y)^2}{4}, & 0 \le y \le 2 \\ 1, & y > 2 \end{cases}$$

calculate Y's PDF:

$\frac{d}{dy}(\frac{4-(2-2y)^2}{4}) = \frac{1}{4} * 2(2-2y) * 2 = 2 - 2y$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 2-2y, & 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) Sketch the probability density functions of X and Y .



- $f_X(x)$: displayed by the blue line
  The support $f_X(x)$ is

$$0 \le x \le 2$$

- $f_Y(y)$: displayed by the orange line
  The support $f_Y(y)$ is

$$0 \le y \le 1$$

Exericse 2

Suppose X_1 , . . . , X_n are i.i.d. $exp(\lambda)$ for some λ > 0 and some integer n > 0. Prove that X_1 , the first order statistic, is $exp(n\lambda)$.

Density of the minimum:

$$\begin{aligned} f_{X_1}(x) &= nf_X(x)(1-F_X(x))^{n-1} \\ &= n(\lambda e^{-\lambda x})(1-(1-e^{-\lambda x}))^{n-1} \\ &= n(\lambda e^{-\lambda x})(e^{-\lambda x})^{n-1} \\ &= n(\lambda e^{-\lambda x})e^{-\lambda x(n-1)} \\ &= (n\lambda)e^{-(n\lambda)x} \\ &= exp(n\lambda) \end{aligned}$$

Exericse 3

Let n be the total amount of console

Let x_0 be a proportion of console that have no defect

Let x_1 be a proportion of console that have exactly 1 defect

Let x_2 be a proportion of console that have more than 2 defect

$$\begin{cases} x_0 + x_1 + x_2 = 1 \\ x_1 + x_2 < 1 \end{cases}$$
$$\Rightarrow x_0 > 0$$

(a) What is the probability of randomly choosing a working console, i.e. one with no defects?

$$P_0 = \frac{C^1_{x_0 * n}}{C^1_n} = \frac{x_0 n}{n} = x_0 = 1 - x_1 - x_2$$

(b) If 5 are chosen at random, what is the probability of getting at least one with no defects?

Let p_{pass} be the probability for getting a console with no defect

Let p_{fail} be the probability for getting a console with at least 1 defect

$$\begin{aligned} p_{pass} &= 1 - p_1 - p_2 \\ p_{fail} &= p_1 + p_2 \end{aligned}$$

Apply binominal distribution, probability for getting 5 consoles, all have at least 1 defect is

$$P_{allFail} = \binom{5}{5} * (p_{fail})^5 * (p_{pass})^0 = (p_{fail})^5 = (p_1 + p_2)^5$$

Probability for getting at least 1 working consold is:

$$P = 1 - P_{allFail} = 1 - (p_1 + p_2)^5$$

(c) Keith buys n Gameboys selected at random from the box. The cost of repairing the broken Gameboys is $C = Y_1 + 3Y_2$, where Y_1 is the number with one defect and Y_2 is the number with more than one defect. Find the expected value and variance of C.

$$\begin{aligned} C &= Y_1 + 3Y_2 \\ E(C) &= E(Y_1) + E(3Y_2) \end{aligned}$$

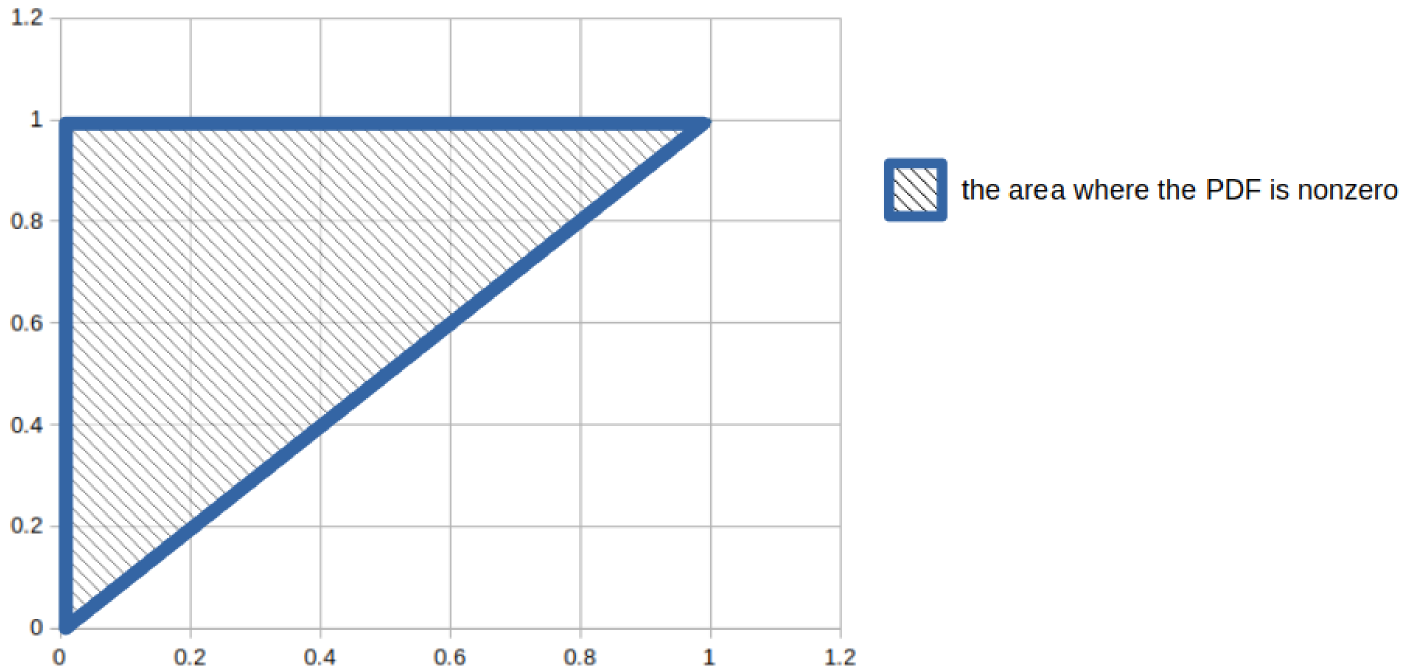
Exericse 4

$$f(y_1, y_2) = \begin{cases} k(1 - y_2) & 0 \leq y_1 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Draw a sketch of the area where the probability density function is nonzero

The support of $f(y_1, y_2)$ is

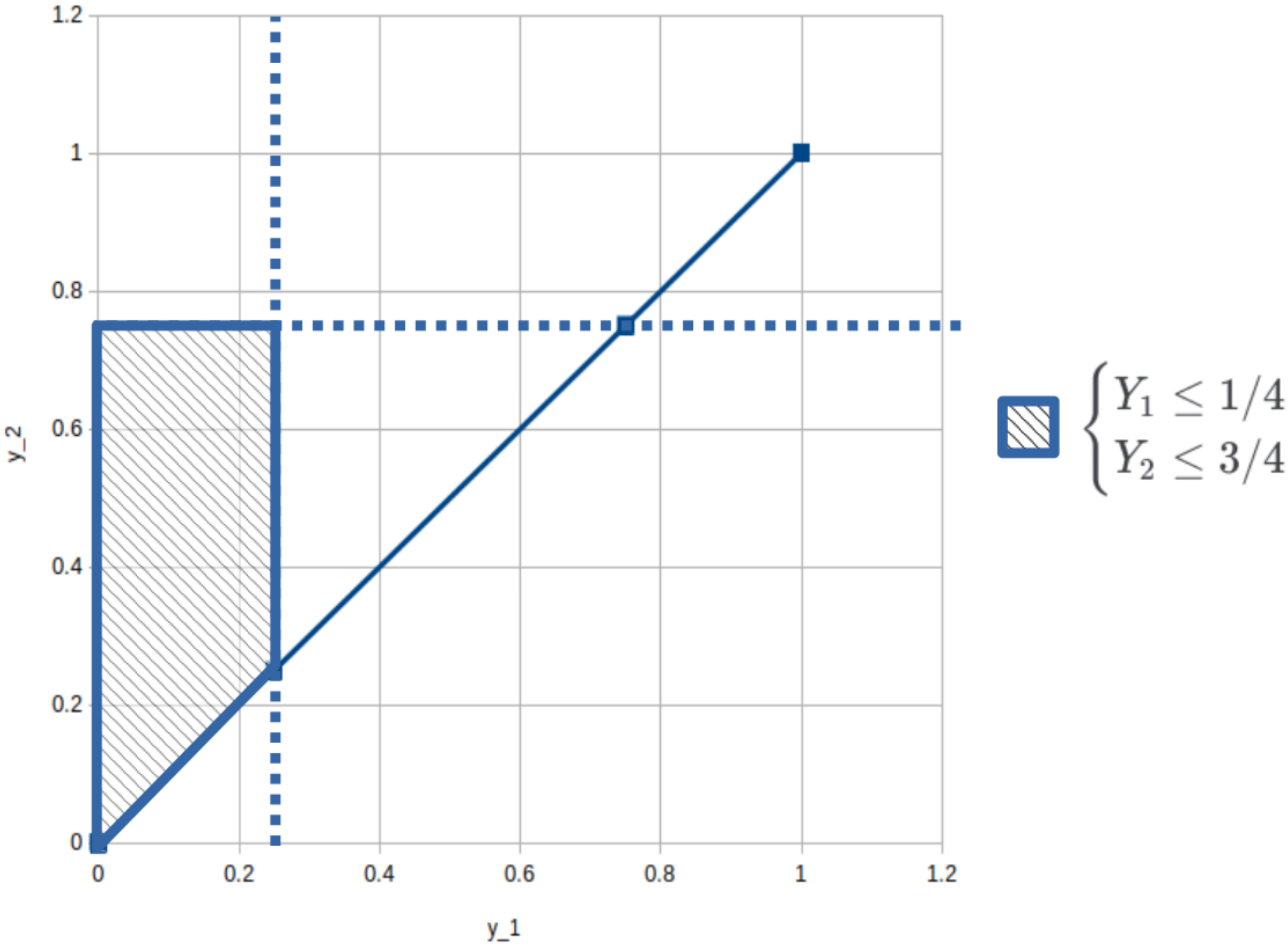
$$\begin{cases} 0 \leq y_1 \leq y_2 \\ 0 \leq y_2 \leq 1 \end{cases}$$



(b) What are the conditions for f to be a valid pdf? Show that for these to be true we need k = 6

$$
\begin{aligned}
\int_0^1 \int_0^{y_2} k(1-y_2)dy_1 dy_2 &= 1 \\
\Leftrightarrow \int_0^1 k(1-y_2) \int_0^{y_2} dy_1 dy_2 &= 1 \\
\Leftrightarrow \int_0^1 k(1-y_2)\ y_1\,|_0^{y_2}\ dy_2 &= 1 \\
\Leftrightarrow \int_0^1 k(1-y_2)y_2 dy_2 &= 1 \\
\Leftrightarrow \int_0^1 k(y_2-y_2^2)dy_2 &= 1 \\
\Leftrightarrow k\ (\frac{y_2^2}{2}-\frac{y_2^3}{3})\Big|_0^1 &= 1 \\
\Leftrightarrow k(\frac{1}{2}-\frac{1}{3}) &= 1 \\
\Leftrightarrow k\frac{1}{6} &= 1 \\
\Leftrightarrow k &= 6
\end{aligned}
$$

(c) Draw the appropriate integration region and find $P(Y_1 \leq 1/4, Y_2 \leq 3/4)$



$$
\begin{aligned}
P(Y_1 \leq 1/4, Y_2 \leq 3/4) &= P(0 \leq Y_1 \leq 1/4, Y_1 \leq Y_2 \leq 3/4) \\
&= \int_0^{1/4} \int_{y_1}^{3/4} 6(1-y_2) dy_2 dy_1 \\
&= \int_0^{1/4} 6(y_2 - \frac{y_2^2}{2})|_{y_1}^{3/4} dy_1 \\
&= \int_0^{1/4} 6(\frac{15}{32} - y_1 + \frac{y_1^2}{2}) dy_1 \\
&= 6\ (\frac{15y_1}{32} - \frac{y_1^2}{2} + \frac{y_1^3}{6})\Big|_0^{1/4} \\
&= \frac{17}{32} \\
&= 0.531\,25
\end{aligned}
$$

(d) Find the marginal density function of Y_1

$$
\begin{aligned}
f(y_1) &= \int_{y_1}^{1} 6(1-y_2)dy_2 \\
&= 6(y_2 - \frac{y_2^2}{2})|_{y_1}^{1} \\
&= 6(\frac{1}{2} - y_1 + \frac{y_1^2}{2})|_{y_1}^{1} \\
&= 3 - 6y_1 + 3y_1^2
\end{aligned}
$$

(e) Find the marginal density function of Y_2

$$
\begin{aligned}
f(y_2) &= \int_{0}^{y_2} 6(1-y_2)dy_1 \\
&= 6(1-y_2)y_1\,|_{0}^{y_2} \\
&= 6(1-y_2)y_2
\end{aligned}
$$