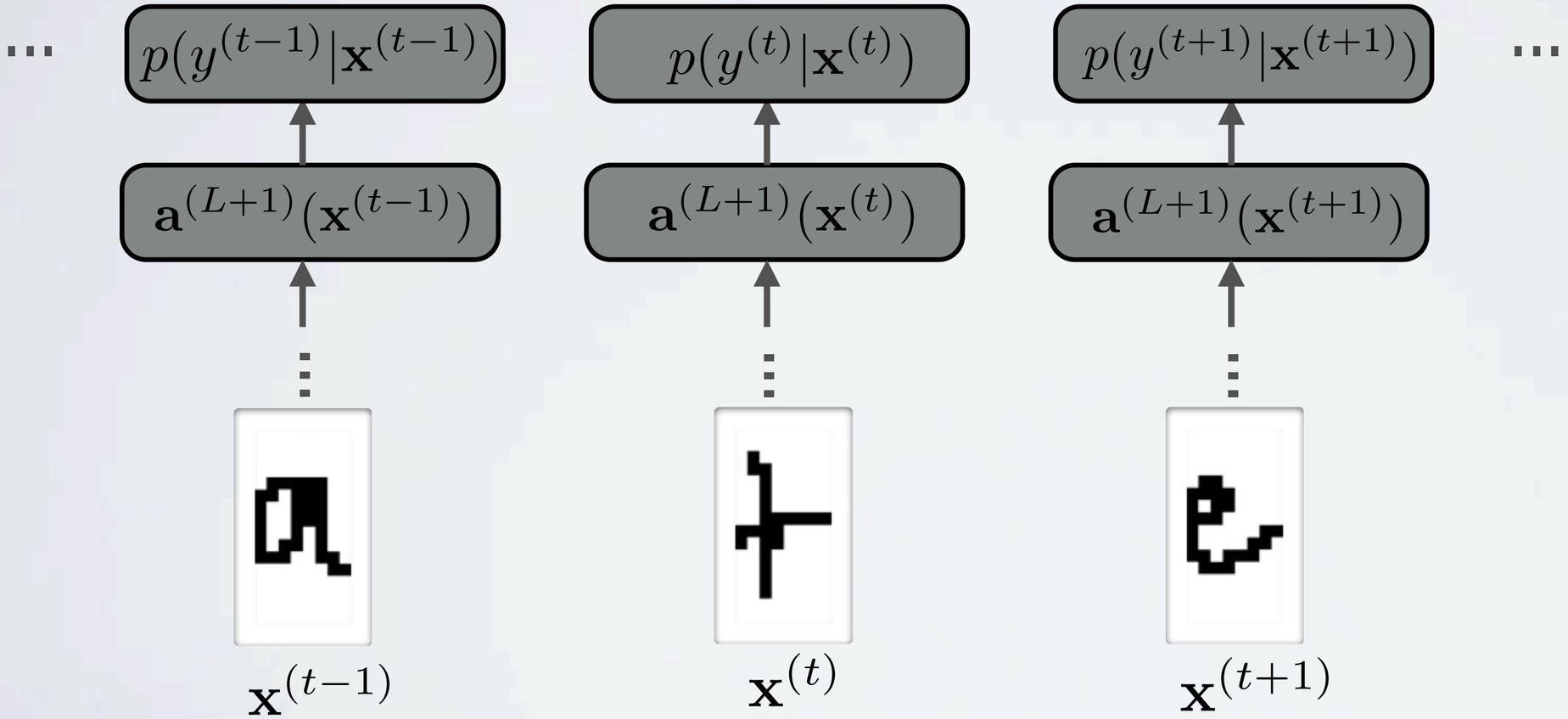
Neural networks

Conditional random fields - motivation

CONDITIONAL RANDOM FIELD

Topics: sequence classification

- What if examples organized in a sequence



CONDITIONAL RANDOM FIELD

Topics: sequence classification

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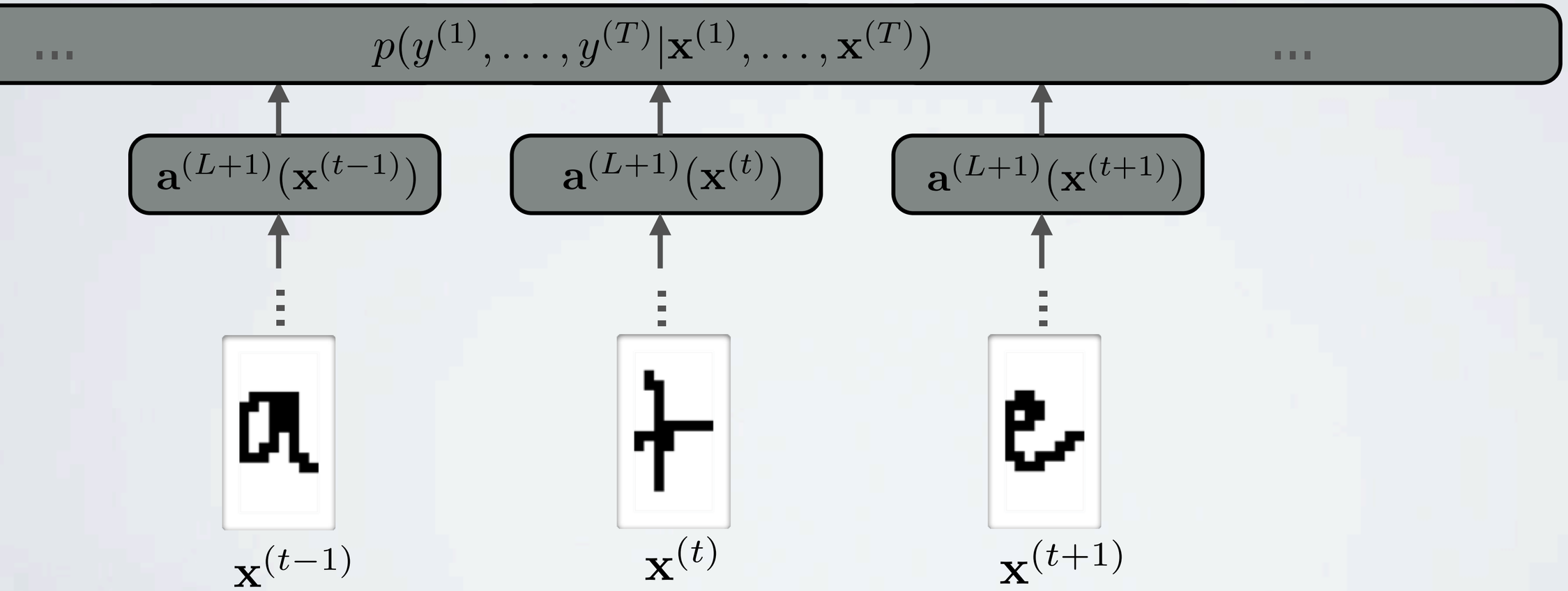


NOTATION

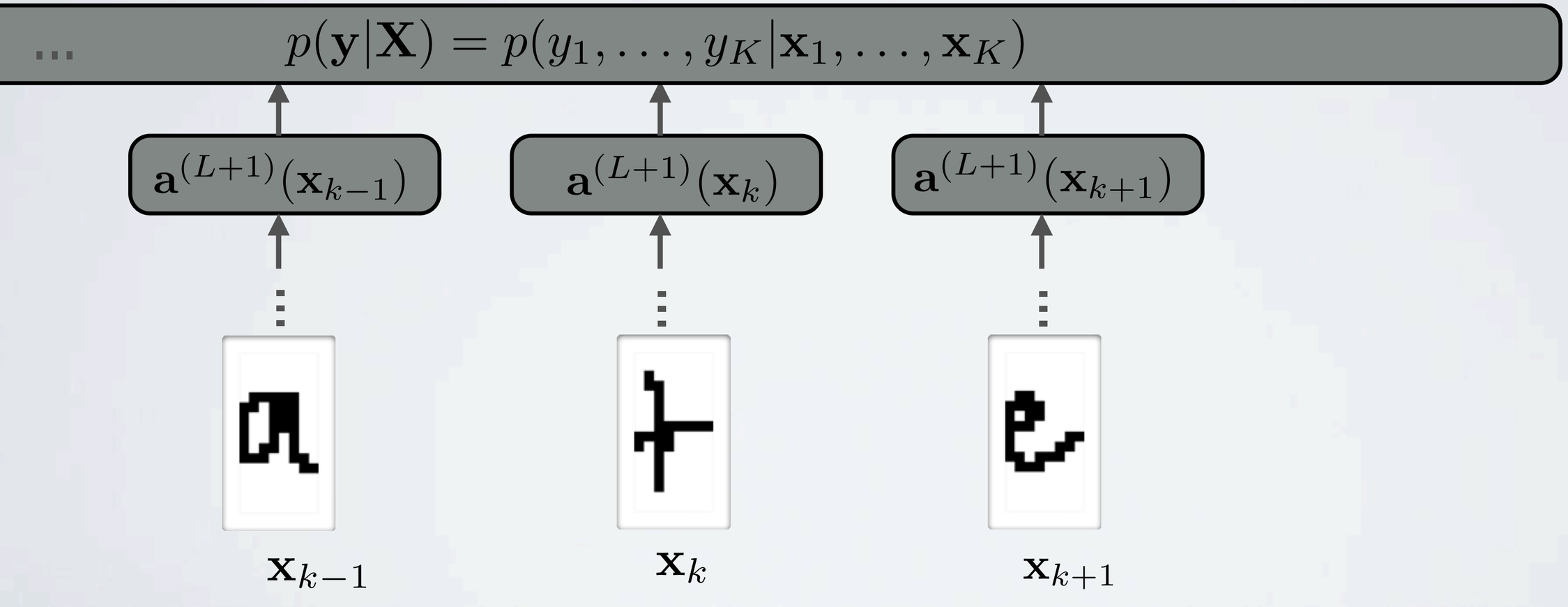
Topics: notation for inputs and targets

- Training set $\{(\mathbf{X}^{(t)}, \mathbf{y}^{(t)})\}$ is a set of input and target sequences pairs:
  - inputs are $\mathbf{X}^{(t)} = [\mathbf{x}_1^{(t)}, \ldots, \mathbf{x}_{K_t}^{(t)}]$
  - targets are $\mathbf{y}^{(t)} = [y_1^{(t)}, \ldots, y_{K_t}^{(t)}]$
  - K_t is the length of the t^{th} sequence

CONDITIONAL RANDOM FIELD

Topics: sequence classification

- For a given example $(\mathbf{X}, \mathbf{y})$:



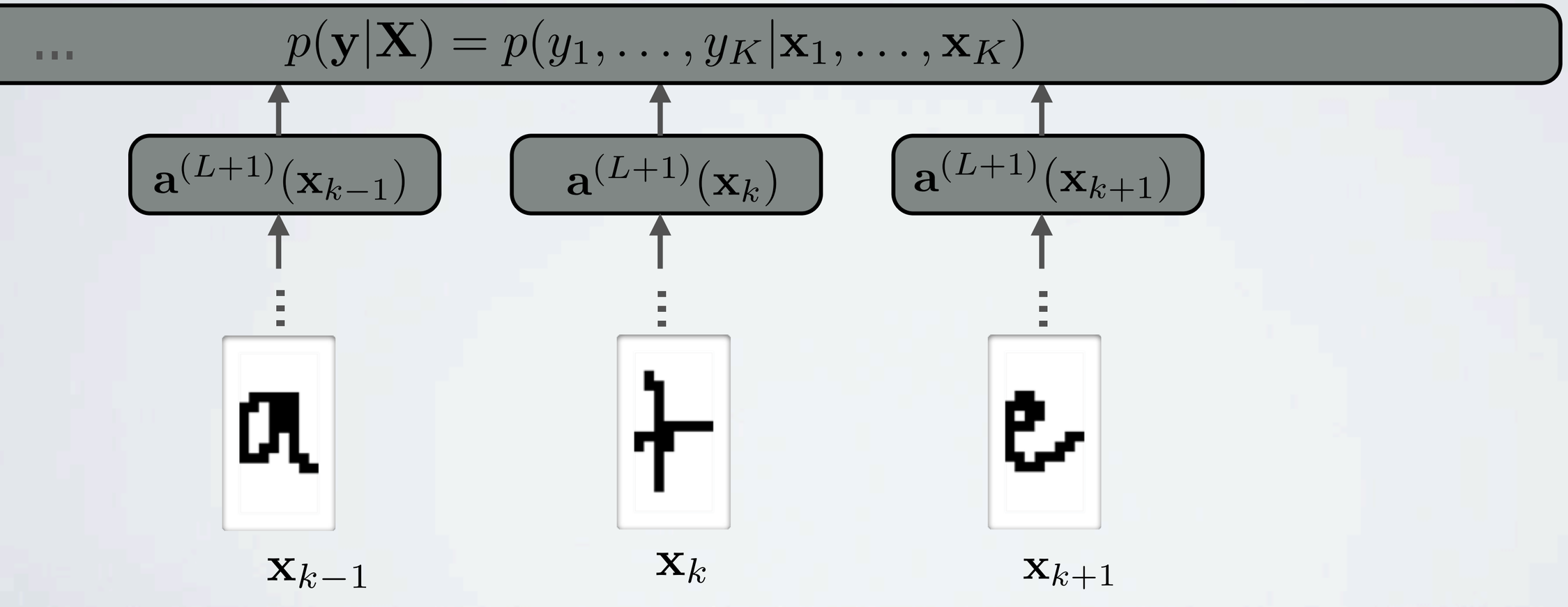
Neural networks

Conditional random fields - linear chain CRF

CONDITIONAL RANDOM FIELD

Topics: sequence classification

- For a given example $(\mathbf{X}, \mathbf{y})$:



LINEAR CHAIN CRF

Topics: lateral weights

- Regular classification:

$$p(\mathbf{y}|\mathbf{X}) = \prod_k p(y_k|\mathbf{x}_k) = \prod_k \exp(a^{(L+1)}(\mathbf{x}_k)_{y_k})/Z(\mathbf{x}_k)$$

$$= \exp\left(\sum_k a^{(L+1)}(\mathbf{x}_k)_{y_k}\right) / \left(\prod_k Z(\mathbf{x}_k)\right)$$

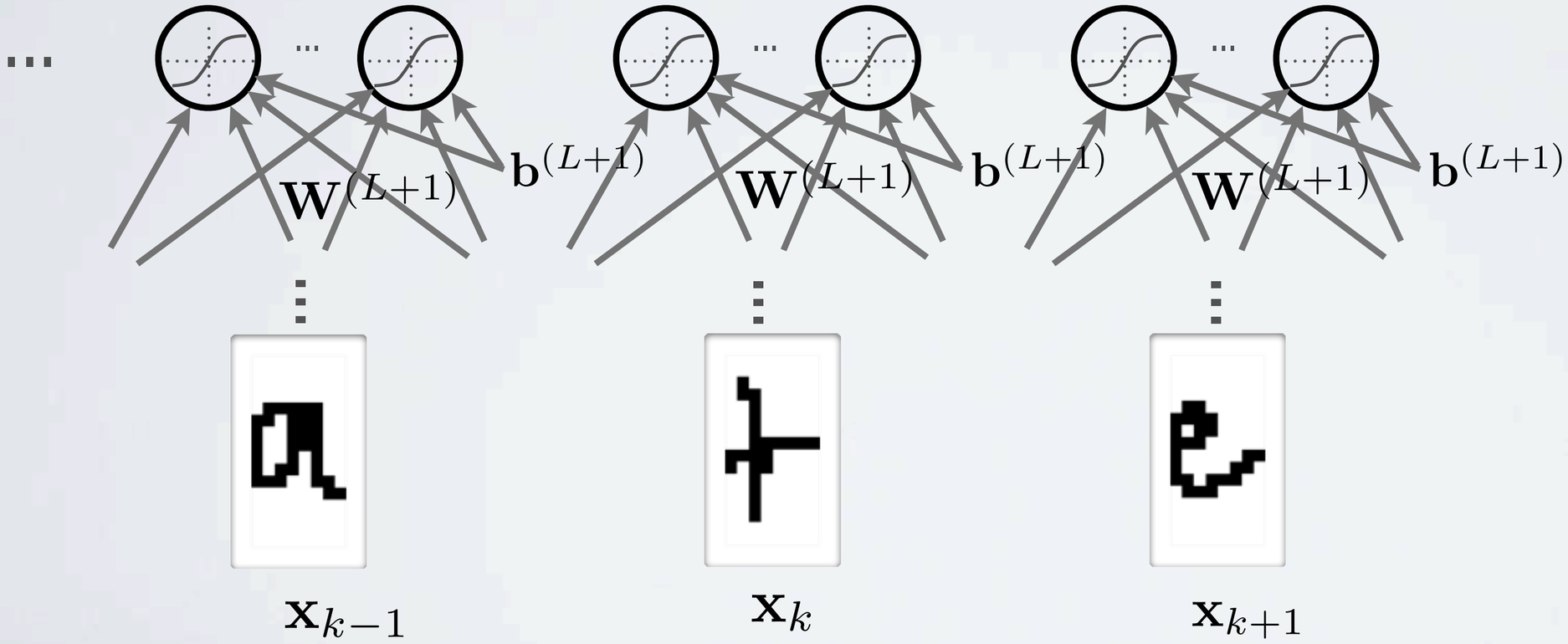
- Sequence classification with linear chain:

$$p(\mathbf{y}|\mathbf{X}) = \exp\left(\underbrace{\sum_{k=1}^{K} a^{(L+1)}(\mathbf{x}_k)_{y_k}}_{\text{is } y_k \text{ likely given input ?}} + \underbrace{\sum_{k=1}^{K-1} V_{y_k,y_{k+1}}}_{\text{is } y_k \text{ followed by } y_{k+1} \text{ likely?}}\right) \underbrace{/Z(\mathbf{X})}_{\text{partition function}}$$

LINEAR CHAIN CRF

Topics: lateral weights

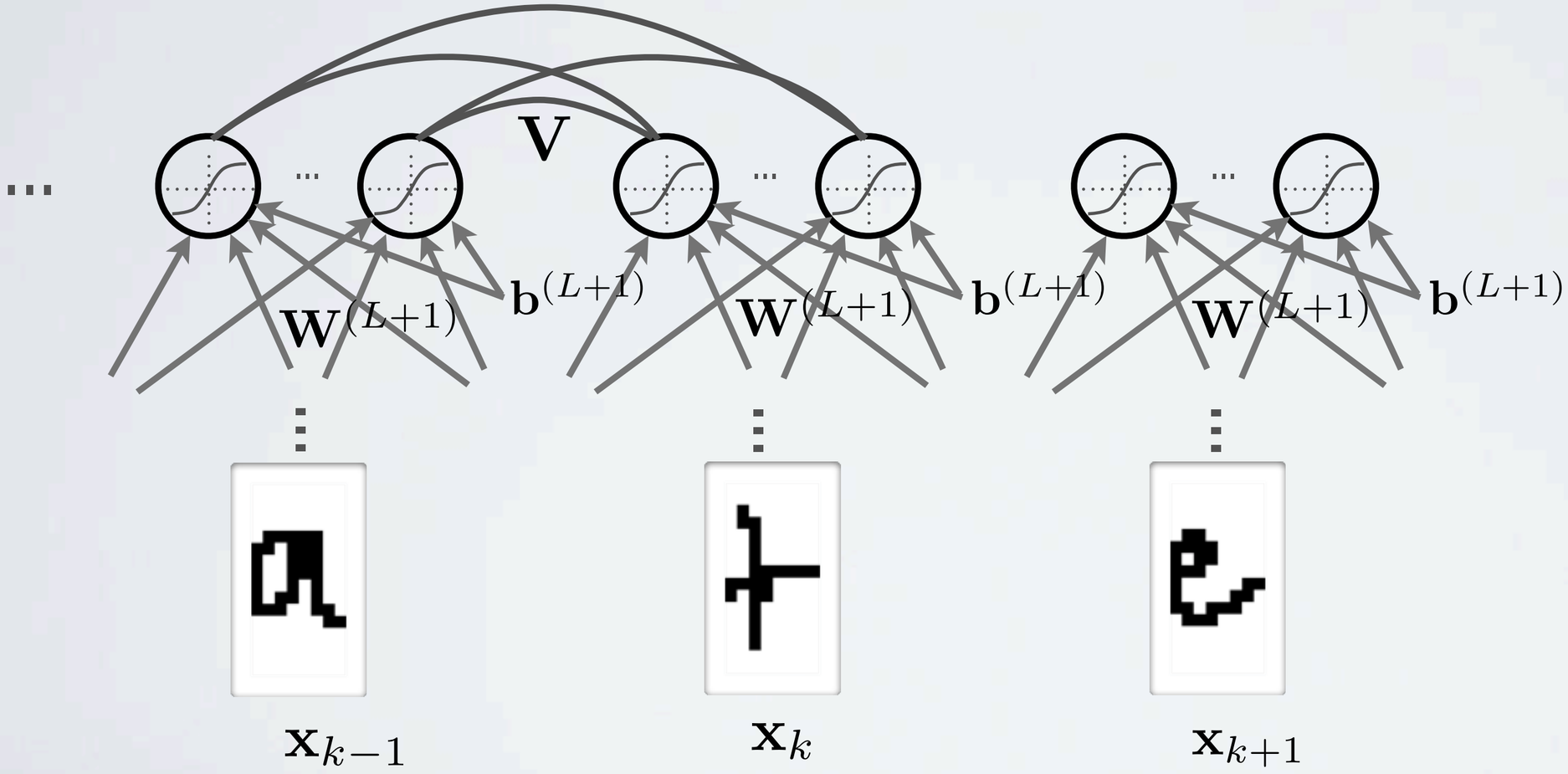
- Sequence classification with linear chain:



LINEAR CHAIN CRF

Topics: lateral weights

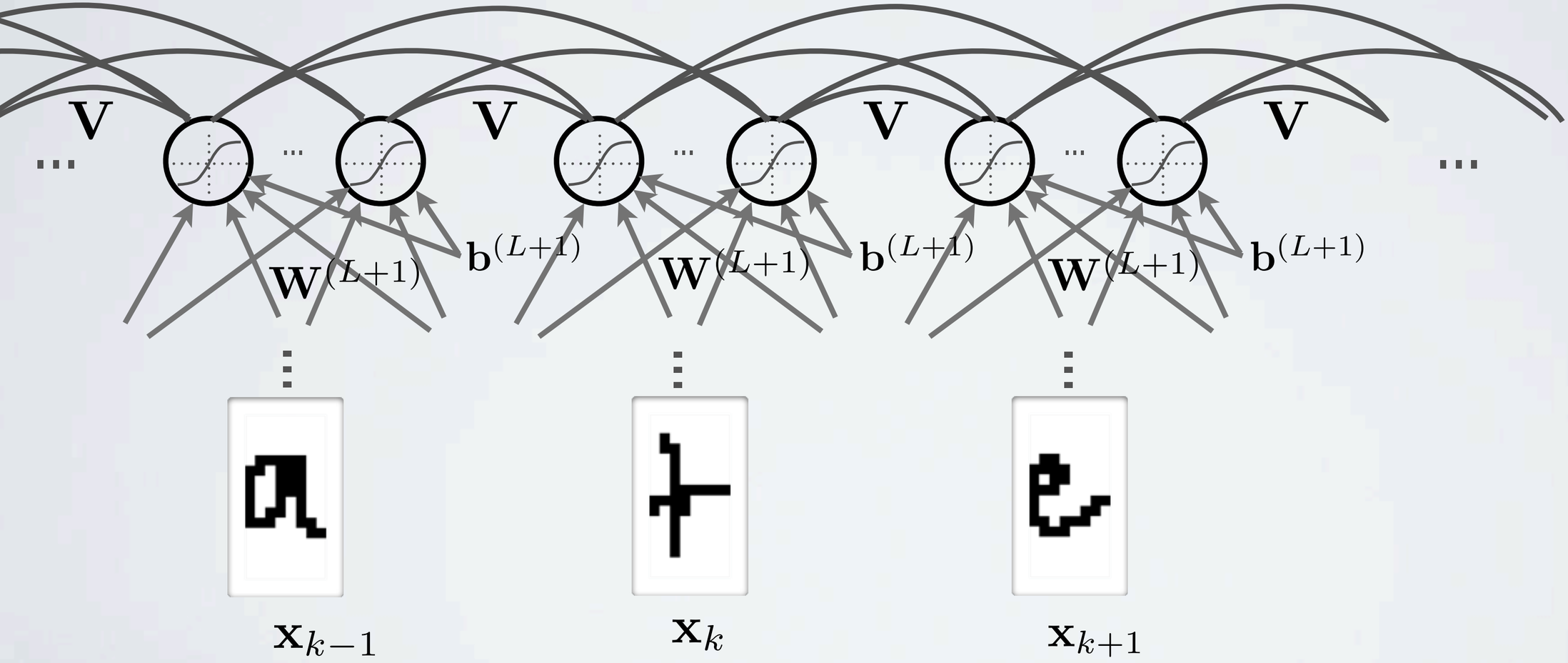
- Sequence classification with linear chain:



LINEAR CHAIN CRF

Topics: lateral weights

- Sequence classification with linear chain:



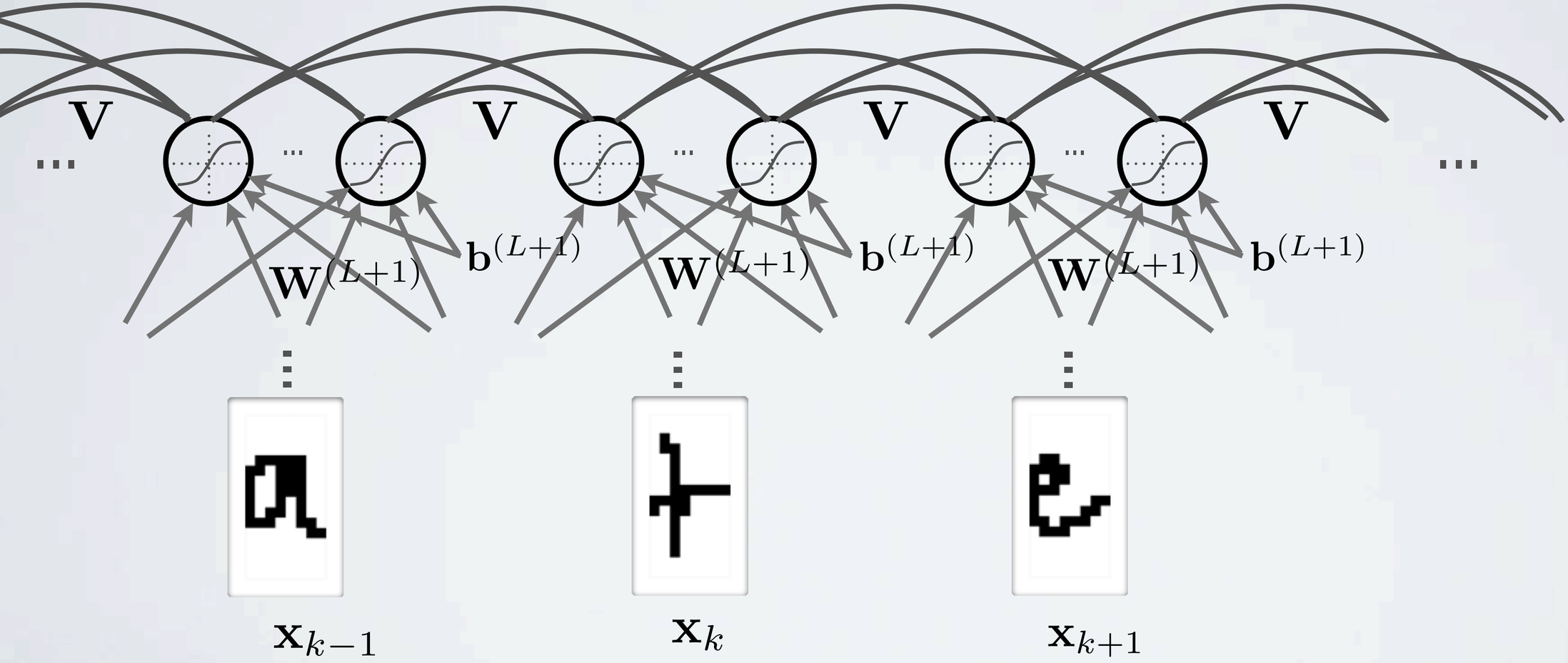
Neural networks

Conditional random fields - context window

LINEAR CHAIN CRF

Topics: lateral weights

- Sequence classification with linear chain:



LINEAR CHAIN CRF

Topics: context window

- Could incorporate a context window to the prediction at each position
  - e.g. context window of radius 1

$$p(\mathbf{y}|\mathbf{X}) = \exp\left(\sum_{k=1}^{K} a^{(L+1,0)}(\mathbf{x}_k)_{y_k} + \sum_{k=1}^{K-1} V_{y_k,y_{k+1}} + \right) / Z(\mathbf{X})$$

LINEAR CHAIN CRF

Topics: context window

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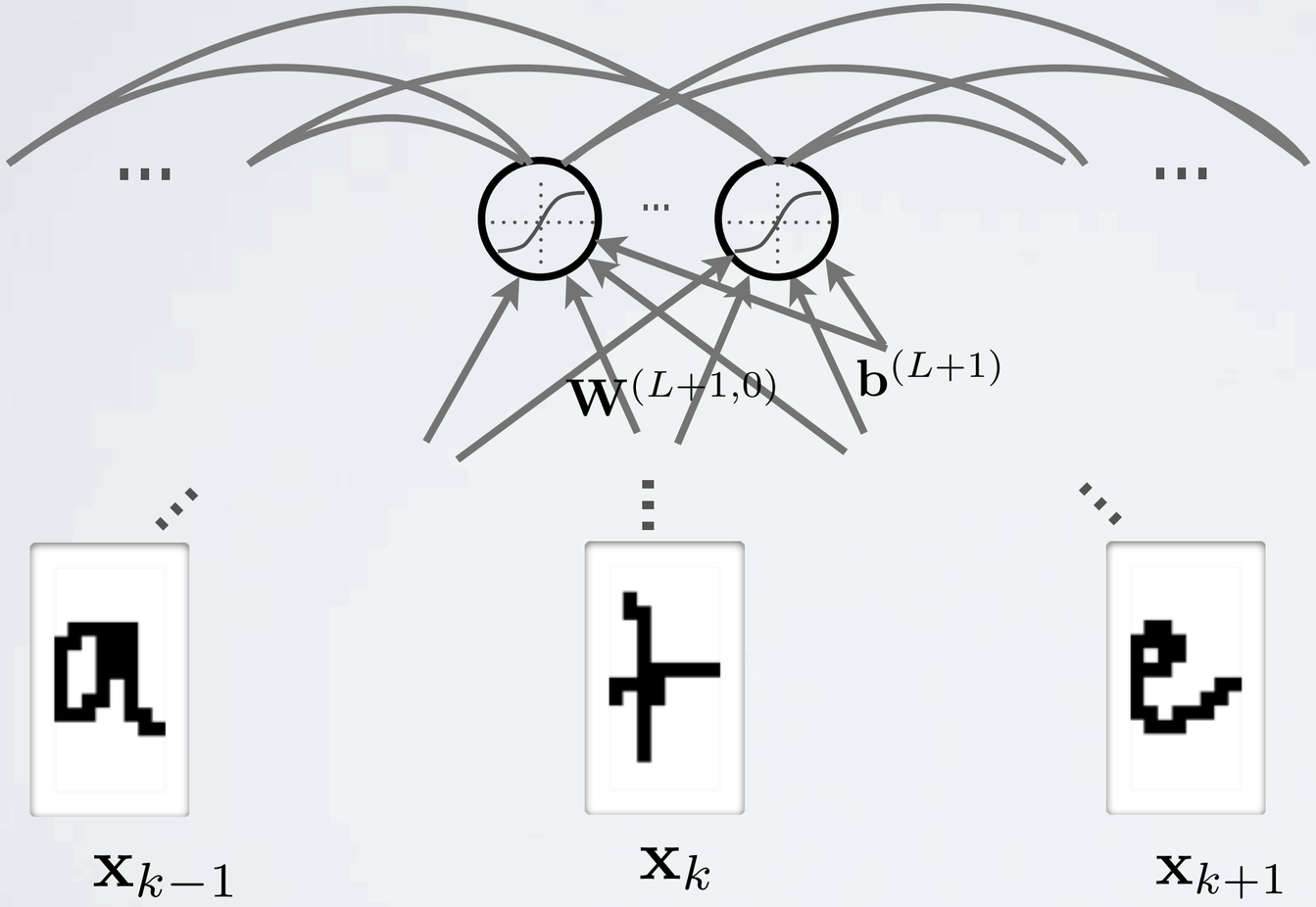
LINEAR CHAIN CRF

Topics: context window

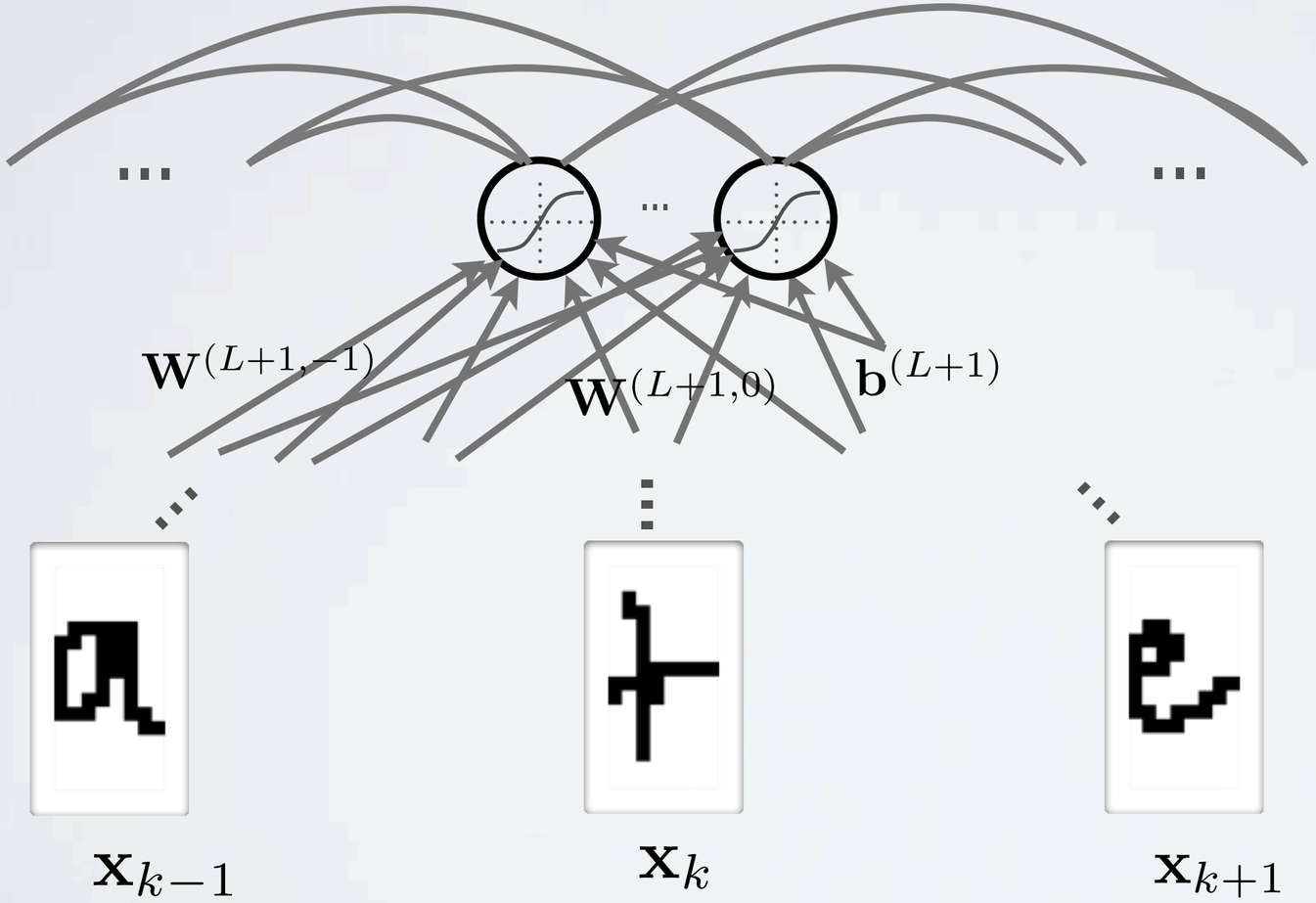
- Sequence classification with linear chain:



LINEAR CHAIN CRF

Topics: context window

- Sequence classification with linear chain:



LINEAR CHAIN CRF

Topics: context window

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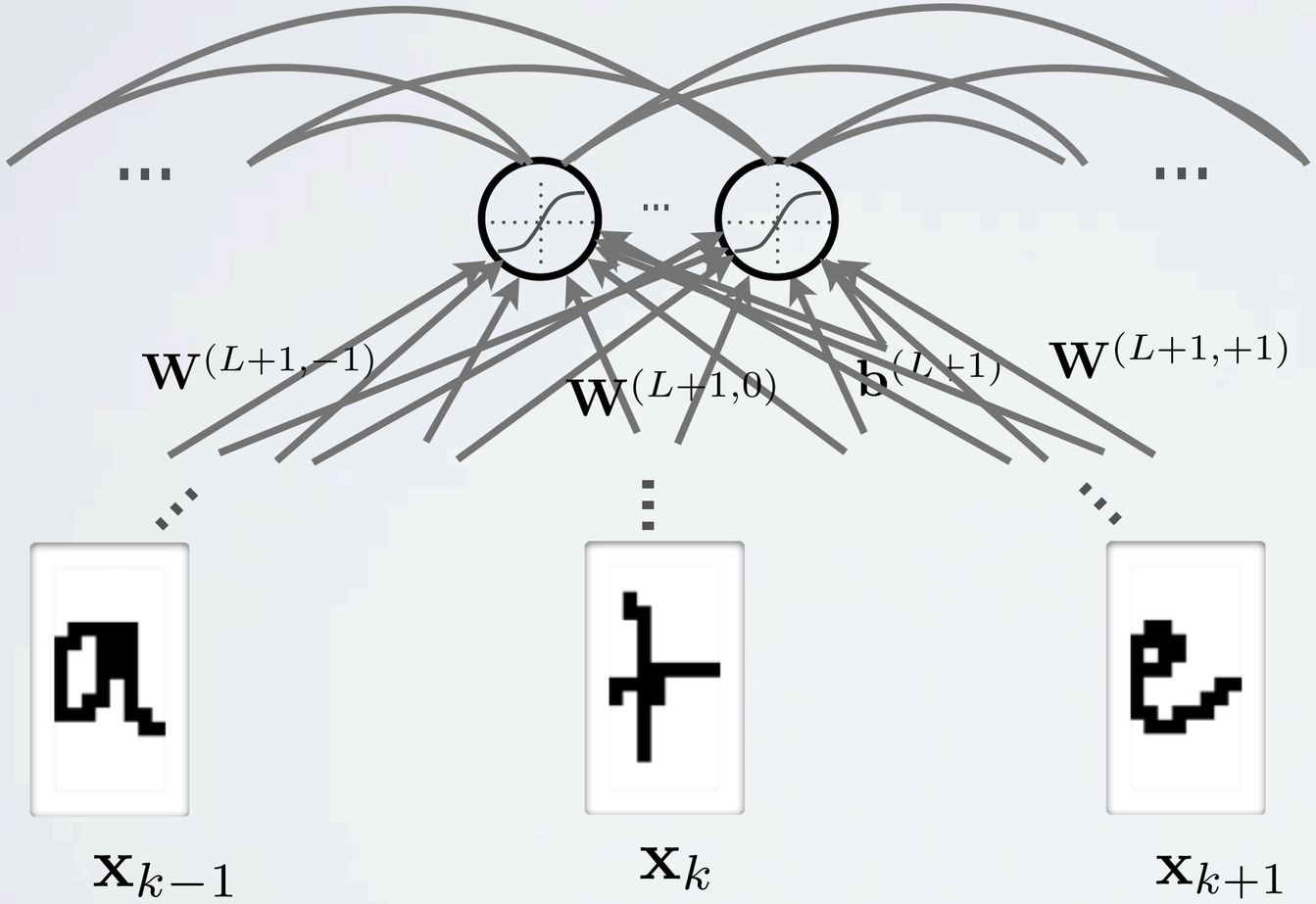


LINEAR CHAIN CRF

Topics: context window

- Could instead feed the window to a single neural network
  - neural network can learn about the whole context jointly

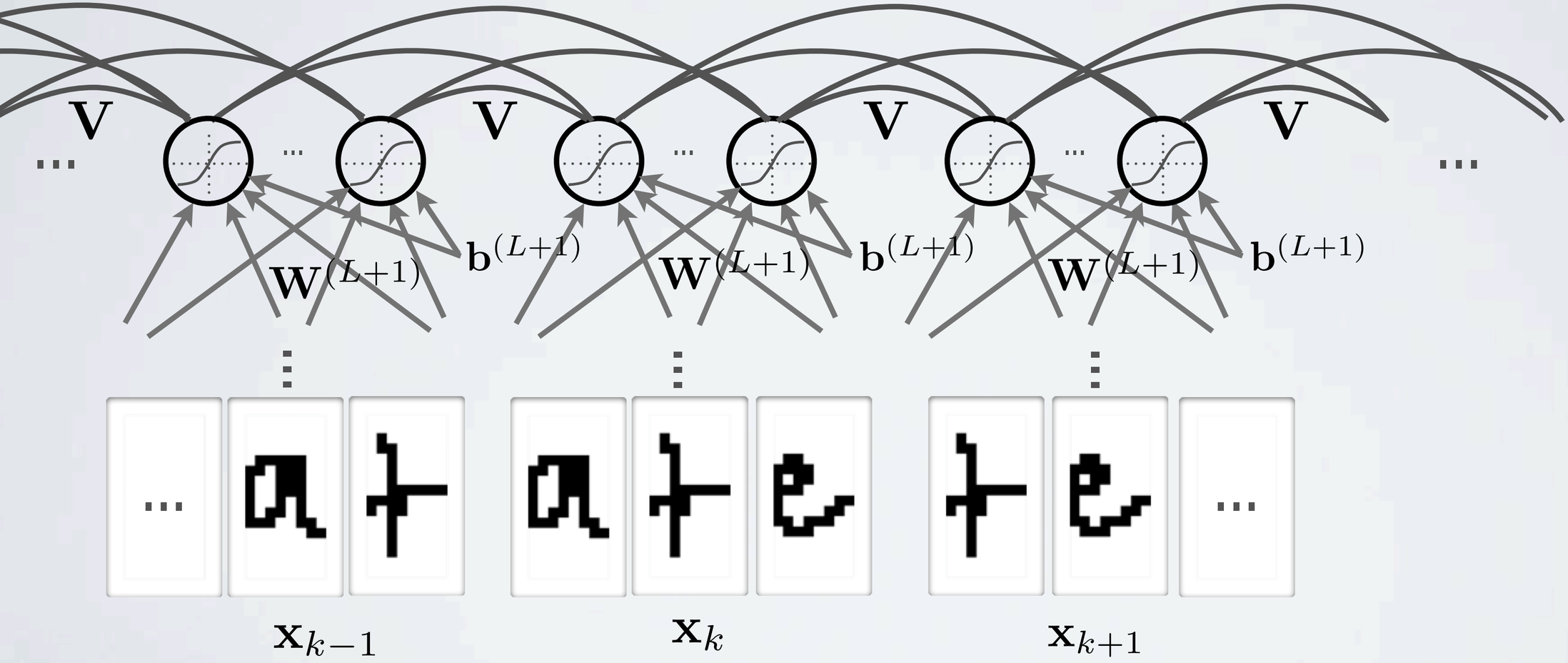
$$p(\mathbf{y}|\mathbf{X}) = \exp\left(\sum_{k=1}^{K} a^{(L+1)}(\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1})_{y_k} + \sum_{k=1}^{K-1} V_{y_k, y_{k+1}}\right) / Z(\mathbf{X})$$

where $\mathbf{x}_0 = 0$ and $\mathbf{x}_{K+1} = 0$ (or some chosen special vectors that indicate beginning/end of sequences)

LINEAR CHAIN CRF

Topics: context window

- Sequence classification with linear chain:



LINEAR CHAIN CRF

Topics: unary and pairwise log-factors

- For brevity, let's assume this notation:
  - unary log-factors

$$a_u(y_k) = a^{(L+1,0)}(\mathbf{x}_k)_{y_k} + 1_{k>1}\, a^{(L+1,-1)}(\mathbf{x}_{k-1})_{y_k} + 1_{k<K}\, a^{(L+1,+1)}(\mathbf{x}_{k+1})_{y_k}$$

or

$$a_u(y_k) = a^{(L+1)}(\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1})_{y_k}$$

  - pairwise log-factors

$$a_p(y_k, y_{k+1}) = 1_{1 \le k < K}\, V_{y_k, y_{k+1}}$$

- Then we have:

$$p(\mathbf{y}|\mathbf{X}) = \exp\left(\sum_{k=1}^{K} a_u(y_k) + \sum_{k=1}^{K-1} a_p(y_k, y_{k+1})\right) / Z(\mathbf{X})$$

Neural networks

Conditional random fields - computing the partition function

LINEAR CHAIN CRF

Topics: unary and pairwise log-factors

- For brevity, let's assume this notation:
  - unary log-factors

$$a_u(y_k) = a^{(L+1,0)}(\mathbf{x}_k)_{y_k} + 1_{k>1}\, a^{(L+1,-1)}(\mathbf{x}_{k-1})_{y_k} + 1_{k<K}\, a^{(L+1,+1)}(\mathbf{x}_{k+1})_{y_k}$$

or

$$a_u(y_k) = a^{(L+1)}(\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1})_{y_k}$$

  - pairwise log-factors

$$a_p(y_k, y_{k+1}) = 1_{1 \le k < K}\, V_{y_k, y_{k+1}}$$

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INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

- Then we have:

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where

$$Z(\mathbf{X}) = \sum_{y'_1} \sum_{y'_2} \cdots \sum_{y'_K} \exp\left(\sum_{k=1}^{K} a_u(y'_k) + \sum_{k=1}^{K-1} a_p(y'_k, y'_{k+1})\right)$$

hard to compute

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$
\begin{aligned}
Z(\mathbf{X}) \quad = \quad & \sum_{y'_K} \exp(a_u(y'_K)) \\
& \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-1}, y'_K)) \right. \\
& \qquad \cdots \\
& \qquad \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_2, y'_3)) \right. \\
& \qquad\quad \left. \left(\sum_{y'_1} \exp(a_u(y'_1) + a_p(y'_1, y'_2)) \right) \right) \cdots \left. \right)
\end{aligned}
$$

INFERENCE

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& \qquad \cdots \\
& \qquad \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_2, y'_3)) \right. \\
& \qquad\quad \alpha_1(y'_2) \left. \left(\sum_{y'_1} \exp(a_u(y'_1) + a_p(y'_1, y'_2)) \right) \right) \cdots \left. \right)
\end{aligned}
$$

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$Z(\mathbf{X}) = \sum_{y'_K} \exp(a_u(y'_K)) \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-1}, y'_K)) \right) \cdots$$

$$\alpha_2(y'_3) \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_2, y'_3)) \right)$$

$$\alpha_1(y'_2) \left(\sum_{y'_1} \exp(a_u(y'_1) + a_p(y'_1, y'_2)) \right) \cdots \Bigg)$$

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$Z(\mathbf{X}) = \sum_{y'_K} \exp(a_u(y'_K))$$

$$\alpha_{K-1}(y'_K) \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-1}, y'_K)) \right)$$

$$\cdots$$

$$\alpha_2(y'_3) \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_2, y'_3)) \right)$$

$$\alpha_1(y'_2) \left(\sum_{y'_1} \exp(a_u(y'_1) + a_p(y'_1, y'_2)) \right) \cdots \Bigg)$$

INFERENCE

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$$Z(\mathbf{X}) = \sum_{y'_K} \exp(a_u(y'_K))$$

$$\alpha_{K-1}(y'_K) \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-1}, y'_K)) \right)$$

$$\dots$$

$$\alpha_2(y'_3) \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_2, y'_3)) \right)$$

$$\alpha_1(y'_2) \left(\sum_{y'_1} \exp(a_u(y'_1) + a_p(y'_1, y'_2)) \right) \dots \Bigg)$$

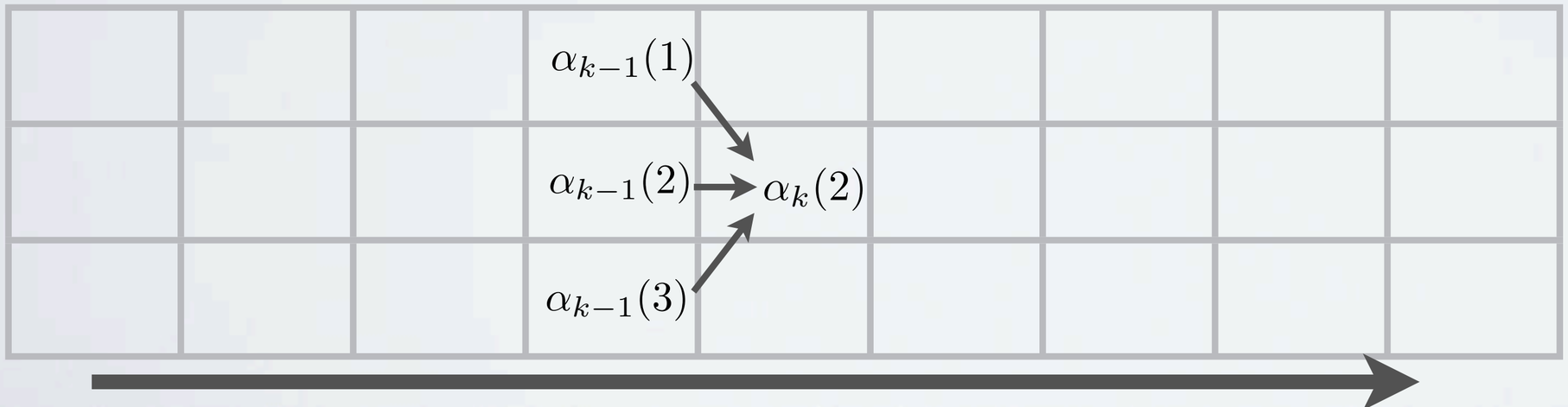
INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

nb. of classes

Complexity in $\mathrm{O}(KC^2)$

- Algorithm goes as follows:
  - initialize, for all values of y'_2 : $\alpha_1(y'_2) \Longleftarrow \sum_{y'_1} \exp(a_u(y'_1) + a_p(y'_1, y'_2))$
  - for $k = 2$ to K-1, for all values of y'_{k+1} :
    - $\alpha_k(y'_{k+1}) \Longleftarrow \sum_{y'_k} \exp(a_u(y'_k) + a_p(y'_k, y'_{k+1}))\ \alpha_{k-1}(y'_k)$
  - $Z(\mathbf{X}) \Longleftarrow \sum_{y'_K} \exp(a_u(y'_K))\ \alpha_{K-1}(y'_K)$



INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$
\begin{aligned}
Z(\mathbf{X}) \quad = \quad & \sum_{y'_1} \exp(a_u(y'_1)) \\
& \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_1, y'_2)) \right. \\
& \qquad \cdots \\
& \qquad \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-2}, y'_{K-1})) \right. \\
& \qquad\quad \left. \left. \left(\sum_{y'_K} \exp(a_u(y'_K) + a_p(y'_{K-1}, y'_K)) \right) \right) \cdots \right)
\end{aligned}
$$

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$
\begin{aligned}
Z(\mathbf{X}) \quad = \quad & \sum_{y'_1} \exp(a_u(y'_1)) \\
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& \qquad \cdots \\
& \qquad \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-2}, y'_{K-1})) \right. \\
& \qquad \left. \beta_K(y'_{K-1}) \left(\sum_{y'_K} \exp(a_u(y'_K) + a_p(y'_{K-1}, y'_K)) \right) \right) \cdots \left. \right)
\end{aligned}
$$

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$Z(\mathbf{X}) = \sum_{y'_1} \exp(a_u(y'_1)) \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_1, y'_2)) \right) \cdots$$

$$\beta_{K-1}(y'_{K-2}) \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-2}, y'_{K-1})) \right)$$

$$\beta_K(y'_{K-1}) \left(\sum_{y'_K} \exp(a_u(y'_K) + a_p(y'_{K-1}, y'_K)) \right) \cdots \Bigg)$$

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$Z(\mathbf{X}) = \sum_{y_1'} \exp(a_u(y_1'))$$

$$\beta_2(y_1') \left(\sum_{y_2'} \exp(a_u(y_2') + a_p(y_1', y_2')) \right.$$

$$\cdots$$

$$\beta_{K-1}(y_{K-2}') \left(\sum_{y_{K-1}'} \exp(a_u(y_{K-1}') + a_p(y_{K-2}', y_{K-1}')) \right.$$

$$\left. \beta_K(y_{K-1}') \left(\sum_{y_K'} \exp(a_u(y_K') + a_p(y_{K-1}', y_K')) \right) \right) \cdots \Bigg)$$

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

$$Z(\mathbf{X}) = \sum_{y'_1} \exp(a_u(y'_1))$$

$$\beta_2(y'_1) \left(\sum_{y'_2} \exp(a_u(y'_2) + a_p(y'_1, y'_2)) \right.$$

$$\dots$$

$$\beta_{K-1}(y'_{K-2}) \left(\sum_{y'_{K-1}} \exp(a_u(y'_{K-1}) + a_p(y'_{K-2}, y'_{K-1})) \right.$$

$$\beta_K(y'_{K-1}) \left(\sum_{y'_K} \exp(a_u(y'_K) + a_p(y'_{K-1}, y'_K)) \right) \left. \right) \dots \left. \right)$$

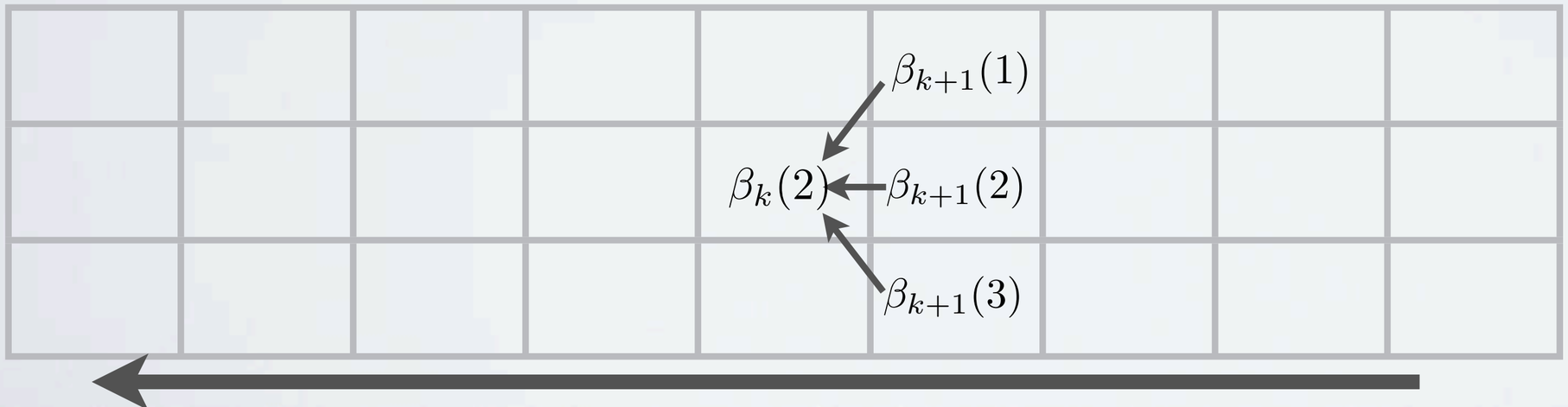
INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

nb. of classes

Complexity in $\mathrm{O}(KC^2)$

- Algorithm goes as follows:
  - initialize, for all values of y'_{K-1} : $\beta_K(y'_{K-1}) \Longleftarrow \sum_{y'_K} \exp(a_u(y'_K) + a_p(y'_{K-1}, y'_K))$
  - for $k = K$-1 to 2, for all values of y'_{k-1} :
    - $\beta_k(y'_{k-1}) \Longleftarrow \sum_{y'_k} \exp(a_u(y'_k) + a_p(y'_{k-1}, y'_k))\ \beta_{k+1}(y'_k)$
  - $Z(\mathbf{X}) \Longleftarrow \sum_{y'_1} \exp(a_u(y'_1))\beta_2(y'_1)$



INFERENCE

Topics: forward/backward or belief propagation

- Computing both tables is often referred to as the forward/backward algorithm for CRFs
  - α is computed with a forward pass
  - β is computed with a backward pass

- It has other names
  - belief propagation
  - sum-product
- α gives the summation from the left
- β gives the summation from the right

INFERENCE

Topics: stable implementation of belief propagation

- For a stable implementation, should work in log space

$$\log \alpha_k(y'_{k+1}) \Longleftarrow \log \sum_{y'_k} \exp\left(a_u(y'_k) + a_p(y'_k, y'_{k+1}) + \log \alpha_{k-1}(y'_k)\right)$$

$$\log \beta_k(y'_{k-1}) \Longleftarrow \log \sum_{y'_k} \exp\left(a_u(y'_k) + a_p(y'_{k-1}, y'_k) + \log \beta_{k+1}(y'_k)\right)$$

- Log-sum-exp operations are more stable if computed like this:

$$\log \sum_i \exp(z_i) = \underbrace{\max_i(z_i) + \log \sum_i \exp(z_i - \max_i(z_i))}_{\text{numerically stable}}$$

Neural networks

Conditional random fields - computing marginals

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

- Then we have:

$$p(\mathbf{y}|\mathbf{X}) = \exp\left(\sum_{k=1}^{K} a_u(y_k) + \sum_{k=1}^{K-1} a_p(y_k, y_{k+1})\right) / Z(\mathbf{X})$$

where

$$Z(\mathbf{X}) = \sum_{y'_1} \sum_{y'_2} \cdots \sum_{y'_K} \exp\left(\sum_{k=1}^{K} a_u(y'_k) + \sum_{k=1}^{K-1} a_p(y'_k, y'_{k+1})\right)$$

hard to compute

INFERENCE

Topics: forward/backward or belief propagation

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INFERENCE

Topics: computing $p(y_k|\mathbf{X}),\ p(y_k,\ y_{k+1}|\mathbf{X})$

- The α / β tables can be used to compute marginals

$$p(y_k|\mathbf{X}) = \frac{\exp(a_u(y_k) + \log \alpha_{k-1}(y_k) + \log \beta_{k+1}(y_k))}{\sum_{y'_k} \exp(a_u(y'_k) + \log \alpha_{k-1}(y'_k) + \log \beta_{k+1}(y'_k))}$$

$$p(y_k, y_{k+1}|\mathbf{X}) = \frac{\exp\left(\begin{array}{c} a_u(y_k) + a_p(y_k, y_{k+1}) + a_u(y_{k+1}) \\ + \log \alpha_{k-1}(y_k) + \log \beta_{k+2}(y_{k+1}) \end{array}\right)}{\sum_{y'_k} \sum_{y'_{k+1}} \exp\left(\begin{array}{c} a_u(y'_k) + a_p(y'_k, y'_{k+1}) + a_u(y'_{k+1}) \\ + \log \alpha_{k-1}(y'_k) + \log \beta_{k+2}(y'_{k+1}) \end{array}\right)}$$

Neural networks

Conditional random fields - performing classification

INFERENCE

Topics: computing $p(\mathbf{y}|\mathbf{X})$

- Then we have:

$$p(\mathbf{y}|\mathbf{X}) = \exp\left(\sum_{k=1}^{K} a_u(y_k) + \sum_{k=1}^{K-1} a_p(y_k, y_{k+1})\right) / Z(\mathbf{X})$$

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hard to compute

INFERENCE

Topics: computing $p(y_k|\mathbf{X}),\ p(y_k,\ y_{k+1}|\mathbf{X})$

- The α / β tables can be used to compute marginals

$$p(y_k|\mathbf{X}) = \frac{\exp(a_u(y_k) + \log \alpha_{k-1}(y_k) + \log \beta_{k+1}(y_k))}{\sum_{y'_k} \exp(a_u(y'_k) + \log \alpha_{k-1}(y'_k) + \log \beta_{k+1}(y'_k))}$$

$$p(y_k, y_{k+1}|\mathbf{X}) = \frac{\exp\left(\begin{array}{c} a_u(y_k) + a_p(y_k, y_{k+1}) + a_u(y_{k+1}) \\ + \log \alpha_{k-1}(y_k) + \log \beta_{k+2}(y_{k+1}) \end{array}\right)}{\sum_{y'_k} \sum_{y'_{k+1}} \exp\left(\begin{array}{c} a_u(y'_k) + a_p(y'_k, y'_{k+1}) + a_u(y'_{k+1}) \\ + \log \alpha_{k-1}(y'_k) + \log \beta_{k+2}(y'_{k+1}) \end{array}\right)}$$

CLASSIFICATION

Topics: making a prediction (option 1)

- At each position k, pick label y_k with highest marginal probability $p(y_k|\mathbf{X})$
  - this choice is the one that minimizes the sum of the classification errors over the whole sequence, assuming the CRF is the true distribution

$$\min_{\mathbf{y}^*} \mathrm{E}\left[\sum_k 1_{y_k \neq y_k^*}\right]$$

$$\min_{\mathbf{y}^*} \mathrm{E}\left[\sum_k 1_{y_k \neq y_k^*}\right]$$

$$= \min_{\mathbf{y}^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right) \; p(\mathbf{y}|\mathbf{X})$$

$$\min_{\mathbf{y}^*} \mathrm{E}\left[\sum_k 1_{y_k \neq y_k^*}\right]$$

$$= \min_{\mathbf{y}^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right) \; p(\mathbf{y}|\mathbf{X})$$

$$= \min_{y_1^*} \ldots \min_{y_K^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right) \; p(\mathbf{y}|\mathbf{X})$$

$$
\begin{aligned}
& \min_{\mathbf{y}^*} \mathrm{E}\left[\sum_k 1_{y_k \neq y_k^*}\right] \\
= \quad & \min_{\mathbf{y}^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right) \; p(\mathbf{y}|\mathbf{X}) \\
= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right) \; p(\mathbf{y}|\mathbf{X}) \\
= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_k \sum_{y_1} \cdots \sum_{y_K} 1_{y_k \neq y_k^*} \; p(\mathbf{y}|\mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
& \min_{\mathbf{y}^*} \mathrm{E}\left[\sum_k 1_{y_k \neq y_k^*}\right] \\
= \quad & \min_{\mathbf{y}^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right) \; p(\mathbf{y}|\mathbf{X}) \\
= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right) \; p(\mathbf{y}|\mathbf{X}) \\
= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_k \sum_{y_1} \cdots \sum_{y_K} 1_{y_k \neq y_k^*} \; p(\mathbf{y}|\mathbf{X}) \\
= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_k \sum_{y_k} 1_{y_k \neq y_k^*} \sum_{y_1} \cdots \sum_{y_{k-1}} \sum_{y_{k+1}} \sum_{y_K} p(\mathbf{y}|\mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
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= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_k \sum_{y_k} 1_{y_k \neq y_k^*} \sum_{y_1} \cdots \sum_{y_{k-1}} \sum_{y_{k+1}} \sum_{y_K} p(\mathbf{y}|\mathbf{X}) \\
= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_k \sum_{y_k} 1_{y_k \neq y_k^*} \; p(y_k|\mathbf{X})
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$$

$$
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= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right)\ p(\mathbf{y}|\mathbf{X}) \\
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= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_k \sum_{y_k} 1_{y_k \neq y_k^*}\ p(y_k|\mathbf{X}) \\
= \quad & \min_{y_1^*} \left(\sum_{y_1} 1_{y_1 \neq y_1^*}\ p(y_1|\mathbf{X})\right) + \cdots + \min_{y_K^*} \left(\sum_{y_K} 1_{y_K \neq y_K^*}\ p(y_K|\mathbf{X})\right)
\end{aligned}
$$

$$
\begin{aligned}
& \min_{\mathbf{y}^*} \mathrm{E}\left[\sum_k 1_{y_k \neq y_k^*}\right] \\
= \quad & \min_{\mathbf{y}^*} \sum_{y_1} \cdots \sum_{y_K} \left(\sum_k 1_{y_k \neq y_k^*}\right)\ p(\mathbf{y}|\mathbf{X}) \\
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= \quad & \min_{y_1^*} \ldots \min_{y_K^*} \sum_k \sum_{y_k} 1_{y_k \neq y_k^*}\ p(y_k|\mathbf{X}) \\
= \quad & \min_{y_1^*} \left(\sum_{y_1} 1_{y_1 \neq y_1^*}\ p(y_1|\mathbf{X})\right) + \cdots + \min_{y_K^*} \left(\sum_{y_K} 1_{y_K \neq y_K^*}\ p(y_K|\mathbf{X})\right) \\
= \quad & \min_{y_1^*}(1 - p(y_1^*|\mathbf{X})) + \cdots + \min_{y_K^*}(1 - p(y_K^*|\mathbf{X}))
\end{aligned}
$$

$$
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= \quad & \min_{y_1^*} \left(\sum_{y_1} 1_{y_1 \neq y_1^*} \; p(y_1|\mathbf{X})\right) + \cdots + \min_{y_K^*} \left(\sum_{y_K} 1_{y_K \neq y_K^*} \; p(y_K|\mathbf{X})\right) \\
= \quad & \min_{y_1^*} (1 - p(y_1^*|\mathbf{X})) + \cdots + \min_{y_K^*} (1 - p(y_K^*|\mathbf{X})) \\
= \quad & 1 - \max_{y_1^*} p(y_1^*|\mathbf{X}) + \cdots + 1 - \max_{y_K^*} p(y_K^*|\mathbf{X})
\end{aligned}
$$

CLASSIFICATION

Topics: making a prediction (option 2)

- Find most probable prediction:

$$\arg\max_{\mathbf{y}^*} p(\mathbf{y}|\mathbf{X})$$

- A Viterbi decoding algorithm can be used for that
  - forward pass: replace $\log \sum_{y'_k} \exp(\cdot)$ by $\max_{y'_k}(\cdot)$ and fill the table
  - backward pass: starting from the maximal value y^*_K at the last position, follow the backward trace of the max operations to decode the maximizing sequence

CLASSIFICATION

Topics: making a prediction (option 2)

nb. of classes

Complexity in $\mathrm{O}(KC^2)$

- Algorithm goes as follows:
  - initialize, for all values of y'_2 :
    - $\alpha_1^*(y'_2) \Longleftarrow \max_{y'_1} a_u(y'_1) + a_p(y'_1, y'_2)$
    - $\alpha_1^{\leftarrow}(y'_2) \Longleftarrow \arg\max_{y'_1} a_u(y'_1) + a_p(y'_1, y'_2)$
  - for $k = 2$ to K-1, for all values of y'_{k+1} :
    - $\alpha_k^*(y'_{k+1}) \Longleftarrow \max_{y'_k} a_u(y'_k) + a_p(y'_k, y'_{k+1}) + \alpha_{k-1}^*(y'_k)$
    - $\alpha_k^{\leftarrow}(y'_{k+1}) \Longleftarrow \arg\max_{y'_k} a_u(y'_k) + a_p(y'_k, y'_{k+1}) + \alpha_{k-1}^*(y'_k)$
  - $\max_{\mathbf{y}^*} p(\mathbf{y}|\mathbf{X}) \Longleftarrow \max_{y'_K} a_u(y'_K) + \alpha_{K-1}^*(y'_K)$
  - $y_K^* \Longleftarrow \arg\max_{y'_K} a_u(y'_K) + \alpha_{K-1}^*(y'_K)$
  - for $k = K$-1 to 1: $y_k^* \Longleftarrow \alpha_k^{\leftarrow}(y_{k+1}^*)$

} backward decoding

Neural networks

Conditional random fields - factors, sufficient statistics and linear CRF

LINEAR CHAIN CRF

Topics: factor, sufficient statistic

- CRFs are often written in the standard form:

$$p(\mathbf{y}|\mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_f \underbrace{\Psi_f(\mathbf{y}, \mathbf{X})}$$

factors or compatibility functions

- A parametrization of the factor that is often used is:

$$\Psi_f(\mathbf{y}, \mathbf{X}) = \exp\left(\sum_s \theta_{f,s} \underbrace{t_{f,s}(\mathbf{y}, \mathbf{X})}\right)$$

sufficient statistics

LINEAR CHAIN CRF

Topics: factor, sufficient statistic

- With hidden units, the CRF factors could be:

$$\Psi_f(\mathbf{y}, \mathbf{X}) = \begin{cases} \phi_f(y_k, \mathbf{x}_{k-1}) = \exp\left(a^{(L+1,-1)}(\mathbf{x}_{k-1})_{y_k}\right) \\ \phi_f(y_k, \mathbf{x}_k) = \exp\left(a^{(L+1,0)}(\mathbf{x}_k)_{y_k}\right) \\ \phi_f(y_k, \mathbf{x}_{k+1}) = \exp\left(a^{(L+1,+1)}(\mathbf{x}_{k+1})_{y_k}\right) \\ \phi_f(y_k, y_{k+1}) = \exp\left(V_{y_k, y_{k+1}}\right) \end{cases}$$

- There is no simple form for the sufficient statistics

LINEAR CHAIN CRF

Topics: factor, sufficient statistic

- With hidden units, the CRF factors could be:

unary factors

$$\Psi_f(\mathbf{y}, \mathbf{X}) = \begin{cases} \phi_f(y_k, \mathbf{x}_{k-1}) = \exp\left(a^{(L+1,-1)}(\mathbf{x}_{k-1})_{y_k}\right) \\ \phi_f(y_k, \mathbf{x}_k) = \exp\left(a^{(L+1,0)}(\mathbf{x}_k)_{y_k}\right) \\ \phi_f(y_k, \mathbf{x}_{k+1}) = \exp\left(a^{(L+1,+1)}(\mathbf{x}_{k+1})_{y_k}\right) \\ \phi_f(y_k, y_{k+1}) = \exp\left(V_{y_k, y_{k+1}}\right) \end{cases}$$

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pairwise factors

- There is no simple form for the sufficient statistics

LINEAR CHAIN CRF

Topics: linear CRF

- In a linear (linear chain) CRF (i.e. without hidden units), the CRF factors can be written as:

$$\Psi_f(\mathbf{y}, \mathbf{X}) = \begin{cases} \phi_f(y_k) = \exp\left(b_c^{(L+1)} 1_{y_k=c}\right) \\ \phi_f(y_k, x_{k-1,i}) = \exp\left(W_{c,i}^{(L+1,-1)} x_{k-1,i} 1_{y_k=c}\right) \\ \phi_f(y_k, x_{k,i}) = \exp\left(W_{c,i}^{(L+1,0)} x_{k,i} 1_{y_k=c}\right) \\ \phi_f(y_k, x_{k+1,i}) = \exp\left(W_{c,i}^{(L+1,+1)} x_{k+1,i} 1_{y_k=c}\right) \\ \phi_f(y_k, y_{k+1}) = \exp\left(V_{c,c'} 1_{y_k=c} 1_{y_{k+1}=c'}\right) \end{cases}$$

LINEAR CHAIN CRF

Topics: linear CRF

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sufficient statistics

LINEAR CHAIN CRF

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parameters $\theta_{f,s}$ — sufficient statistics

Neural networks

Conditional random fields - Markov network

LINEAR CHAIN CRF

Topics: factor, sufficient statistic

- With hidden units, the CRF factors could be:

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- There is no simple form for the sufficient statistics

LINEAR CHAIN CRF

Topics: factor, sufficient statistic

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unary factors

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LINEAR CHAIN CRF

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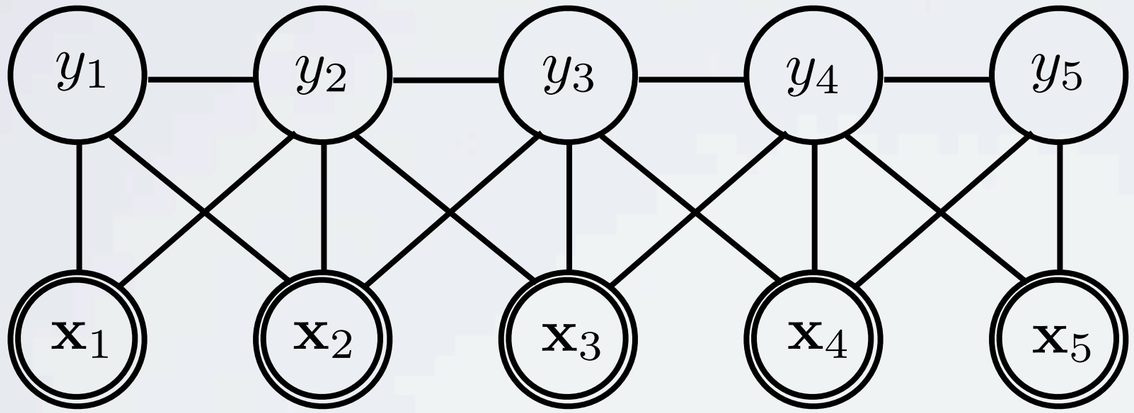
pairwise factors

- There is no simple form for the sufficient statistics

MARKOV NETWORK VISUALIZATION

Topics: Markov network

- Illustration for $K=5$

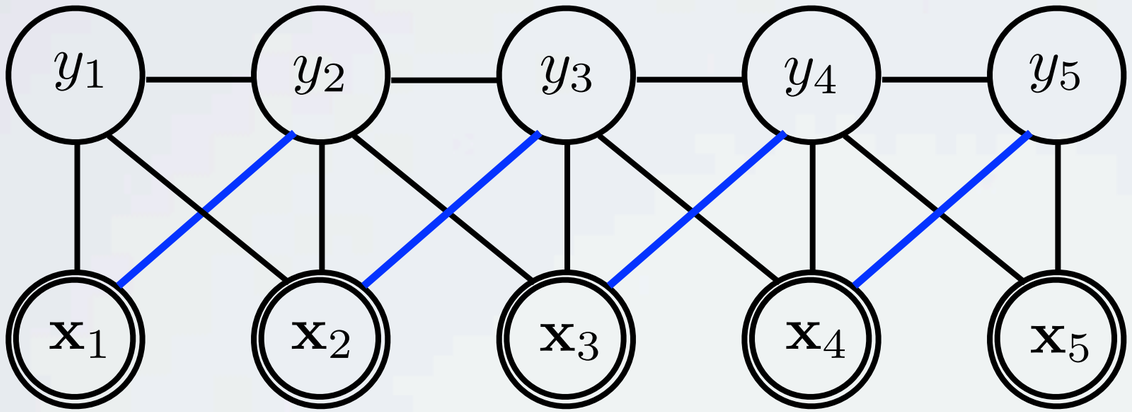


- Network with an edge between each node (random variable) that shares a factor

MARKOV NETWORK VISUALIZATION

Topics: Markov network

- Illustration for $K=5$



= observed

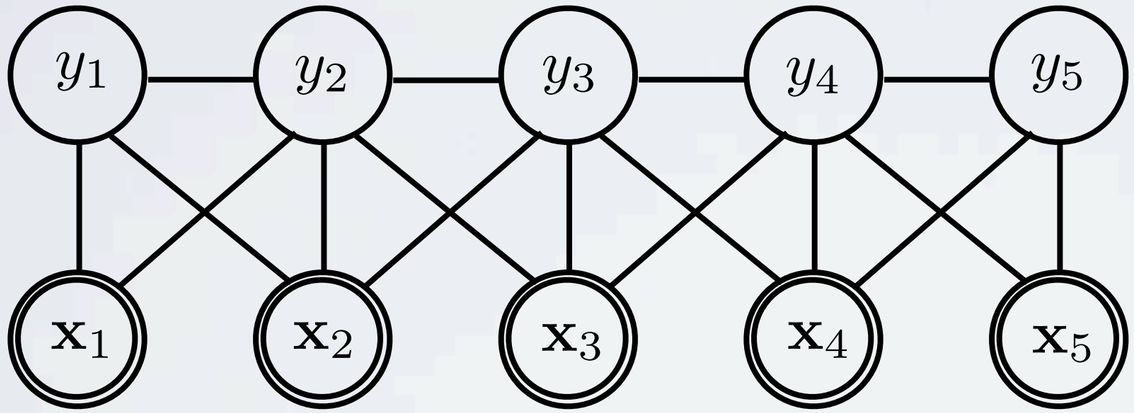
$\phi_f(y_k, \mathbf{x}_{k-1})$

- Network with an edge between each node (random variable) that shares a factor

MARKOV NETWORK VISUALIZATION

Topics: Markov network

- Illustration for $K=5$



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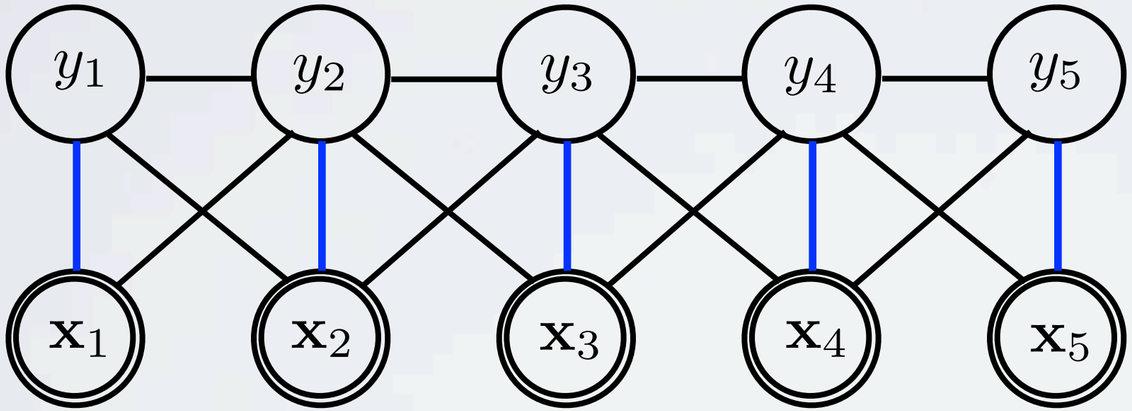

= observed

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MARKOV NETWORK VISUALIZATION

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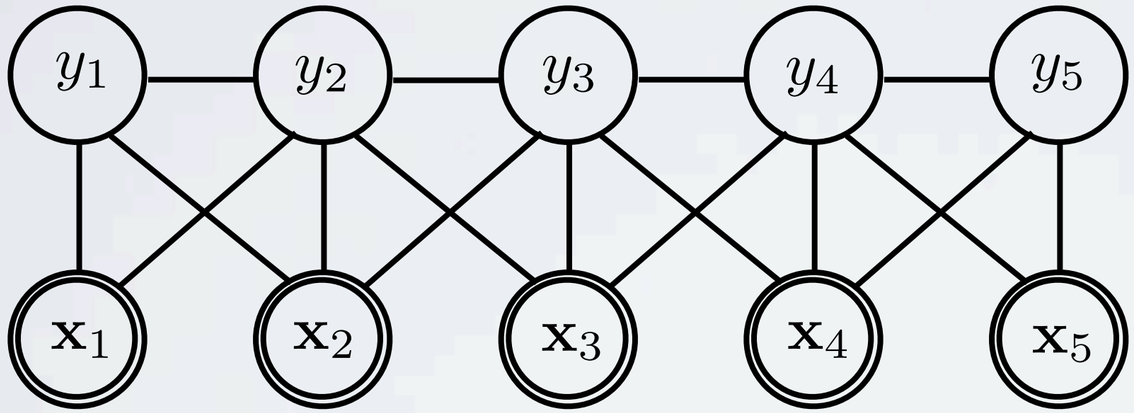

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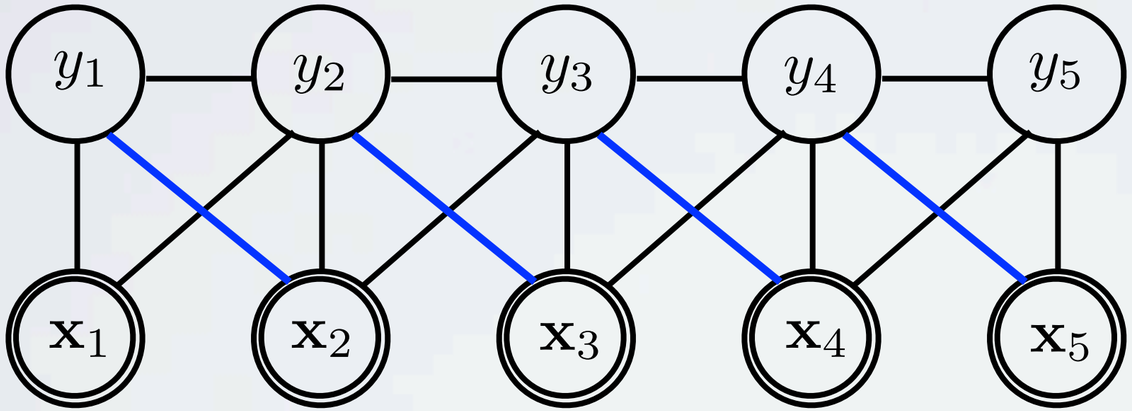

= observed

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MARKOV NETWORK VISUALIZATION

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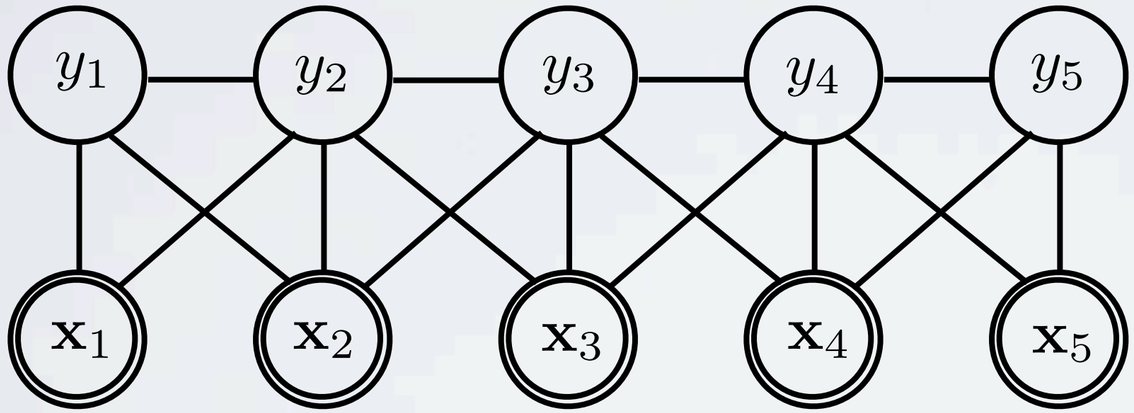

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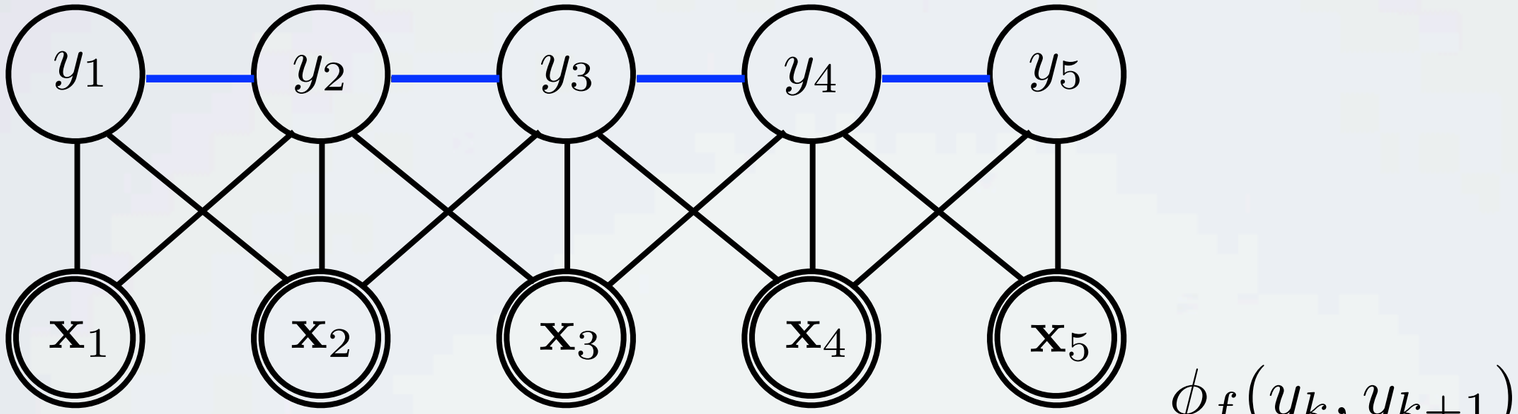
Topics: Markov network

- Illustration for $K=5$

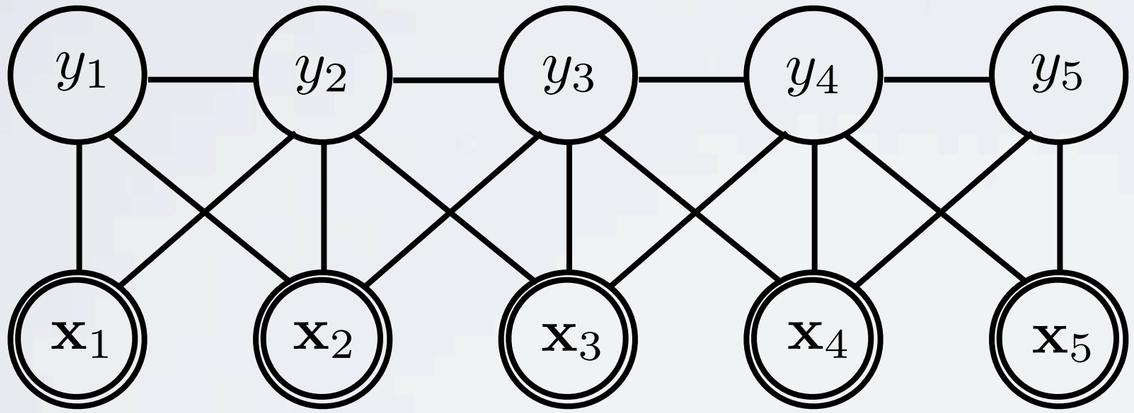


- Network with an edge between each node (random variable) that shares a factor

MARKOV NETWORK VISUALIZATION

Topics: Markov network

- Illustration for $K=5$



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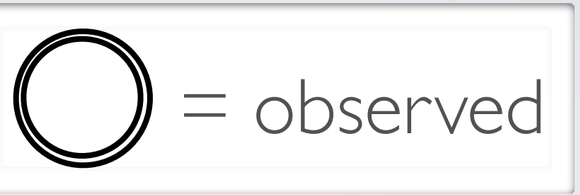
MARKOV NETWORK VISUALIZATION

Topics: Markov network



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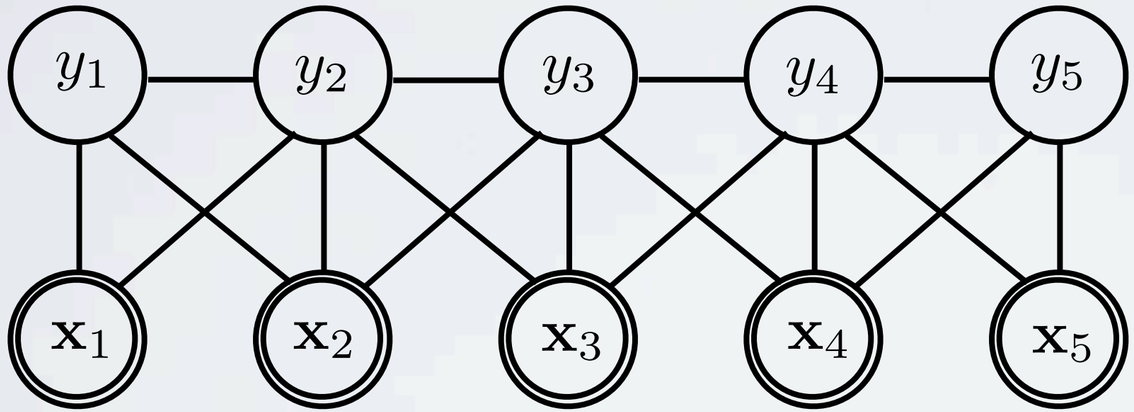

- Local Markov property
  - each node is independent of other nodes **given** its neighbors

MARKOV NETWORK VISUALIZATION

Topics: Markov network

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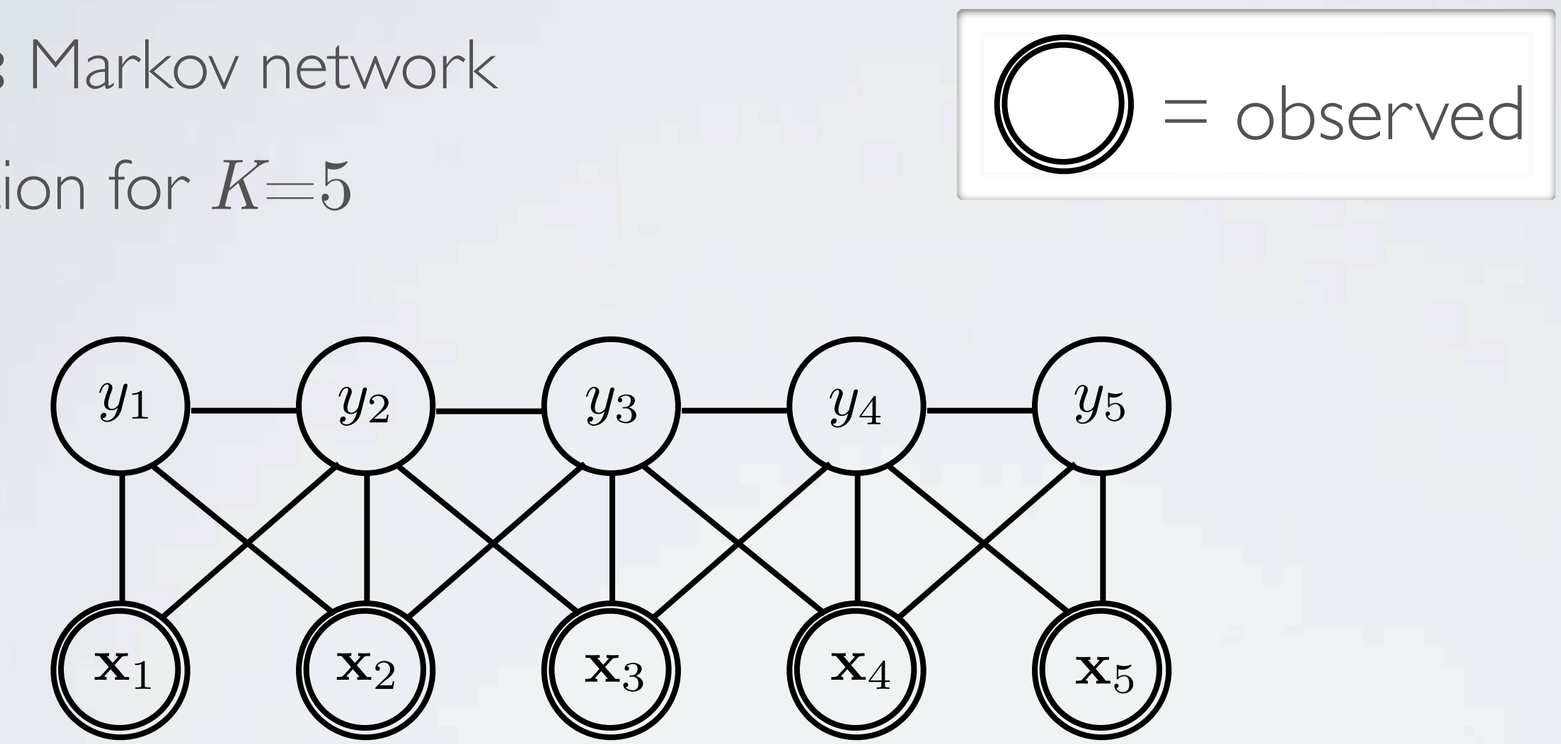
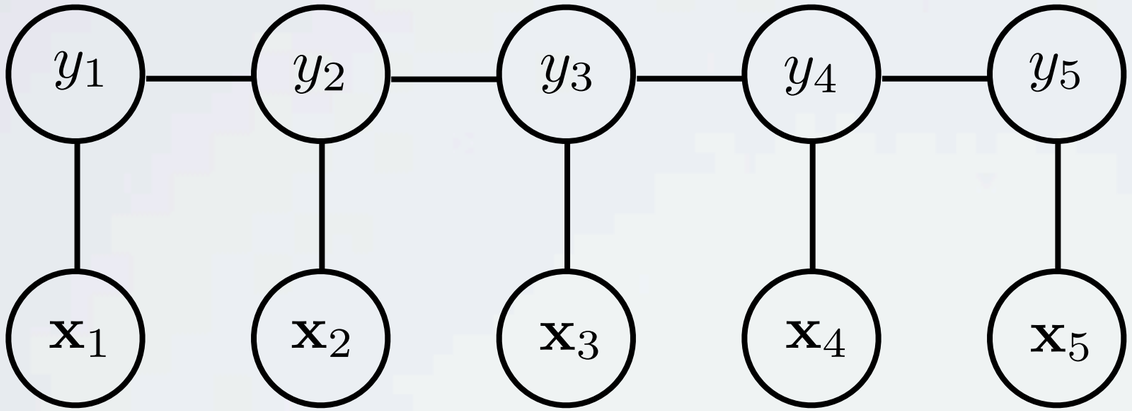


- General conditional independence
  - two nodes are conditionally independent if all paths between them contain at least one of the conditioning node

DIRECTED VS. UNDIRECTED

Topics: undirected graphical model, directed graphical model

- CRF is an undirected graphical model



$\phi_f(y_k, \mathbf{x}_k)$

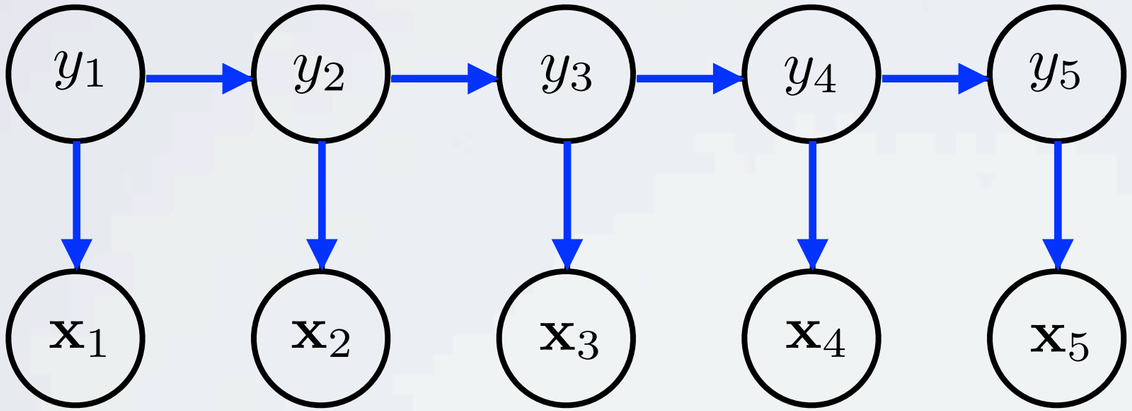
$\phi_f(y_k, y_{k+1})$

} only need to be non-negative

DIRECTED VS. UNDIRECTED

Topics: undirected graphical model, directed graphical model

- HMM is a directed graphical model



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Topics: undirected graphical model, directed graphical model

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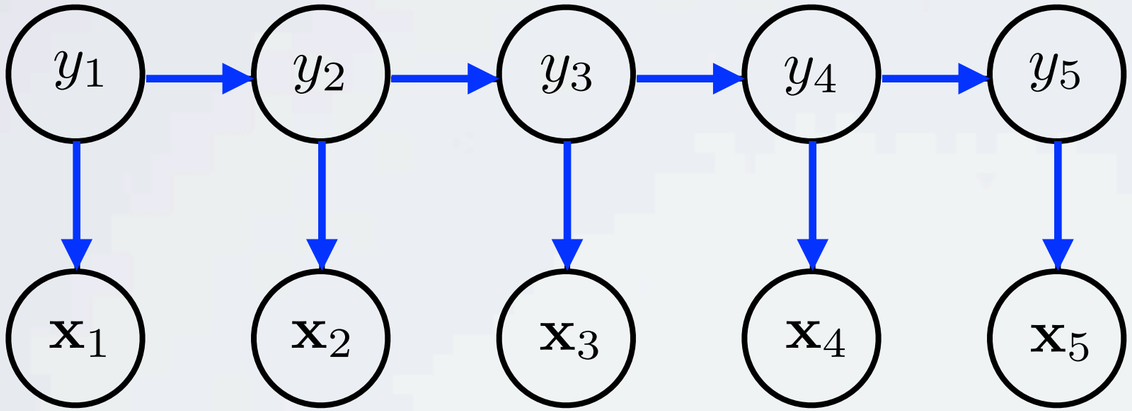

$$\phi_f(y_k, \mathbf{x}_k) = p(\mathbf{x}_k|y_k)$$

$$\phi_f(y_k, y_{k+1}) = p(y_{k+1}|y_k)$$

Neural networks

Conditional random fields - factor graph

MARKOV NETWORK VISUALIZATION

Topics: Markov network

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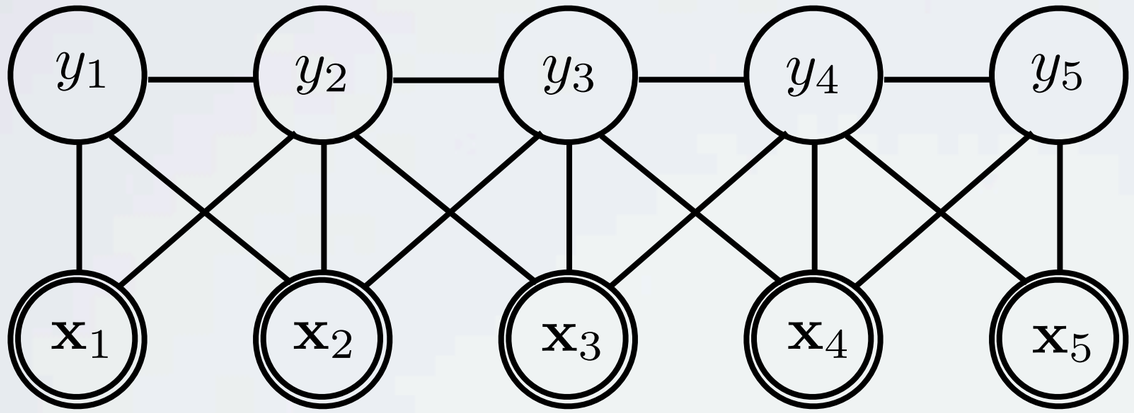
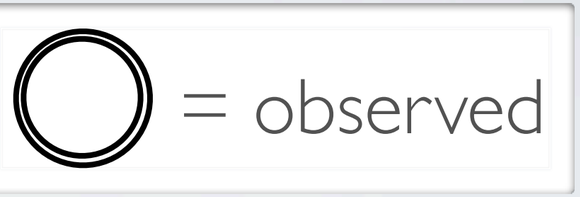


- Each edge is associated with one of more factors

MARKOV NETWORK VISUALIZATION

Topics: Markov network



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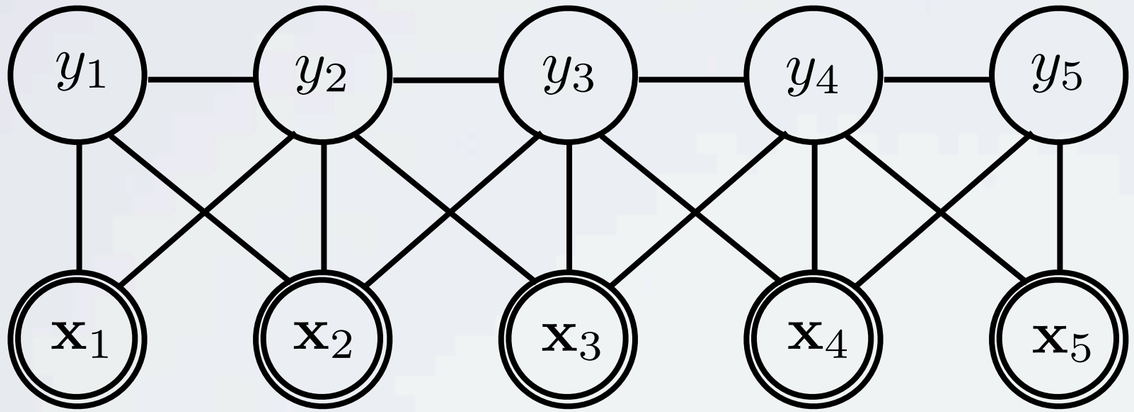


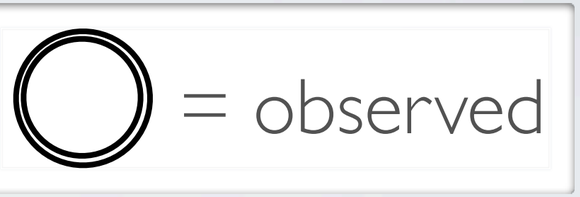
- Representation is ambiguous:
  - clique between y_1, y_2 and $\mathbf{x}_2$, but we didn't explicitly define a joint factor $\phi_f(y_1, y_2, \mathbf{x}_2)$
  - in fact, we defined $\phi_f(y_1, y_2)$, $\phi_f(y_2, \mathbf{x}_2)$ and $\phi_f(y_1, \mathbf{x}_2)$, which is equivalent to having:

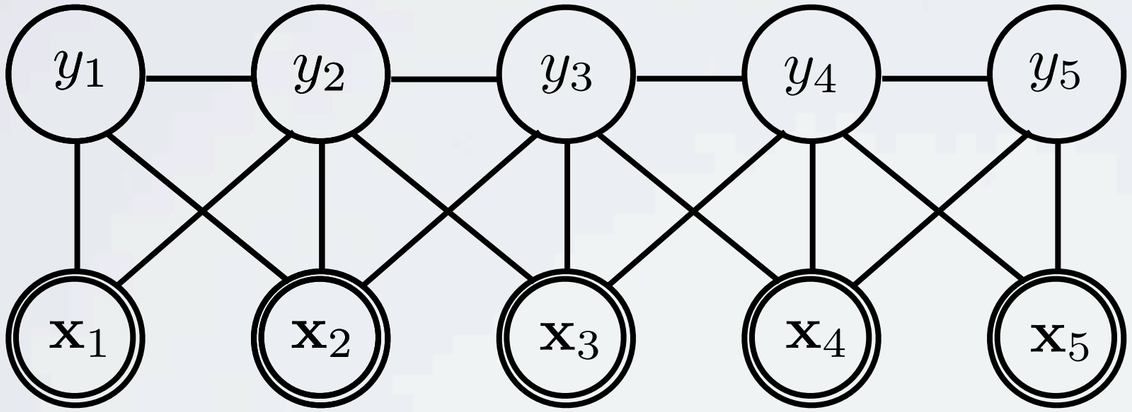
$$\phi_f(y_1, y_2, \mathbf{x}_2) = \phi_f(y_1, y_2)\phi_f(y_2, \mathbf{x}_2)\phi_f(y_1, \mathbf{x}_2)$$

FACTOR GRAPH VISUALIZATION

Topics: factor graph

- Factor graphs better represent factors





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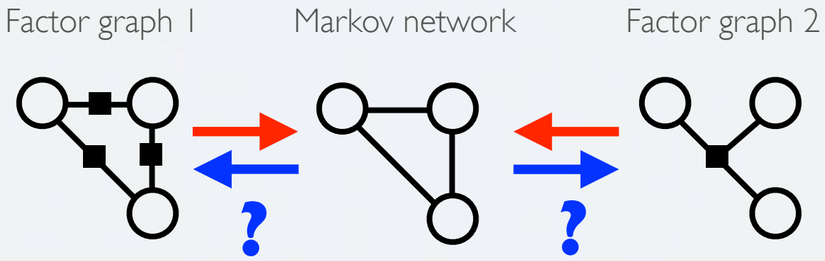

FACTOR GRAPH VISUALIZATION

Topics: factor graph

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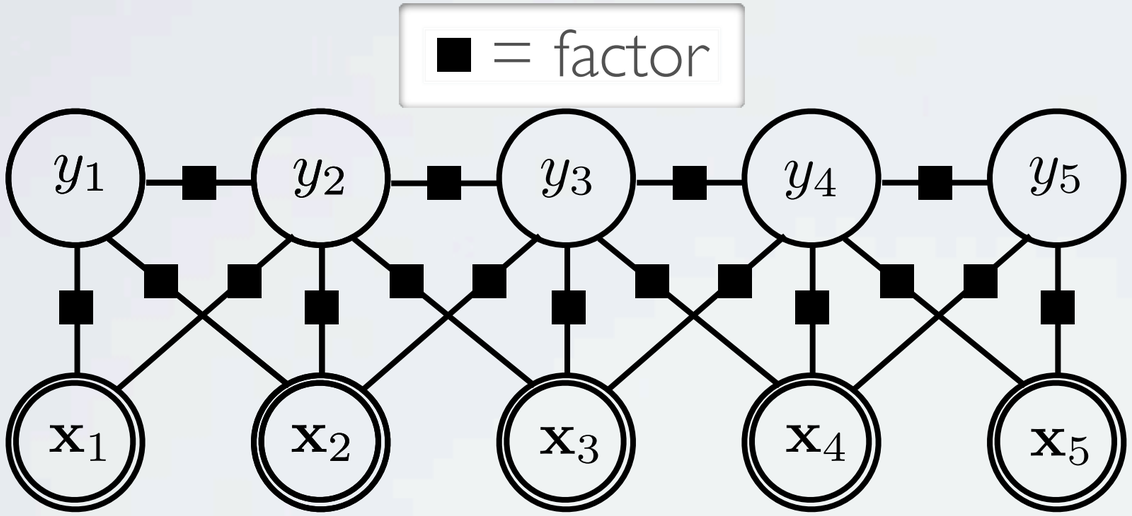

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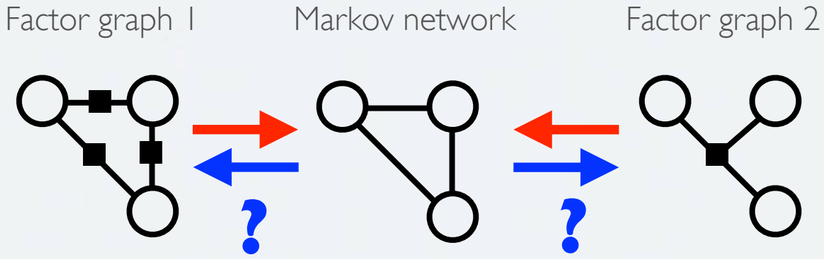

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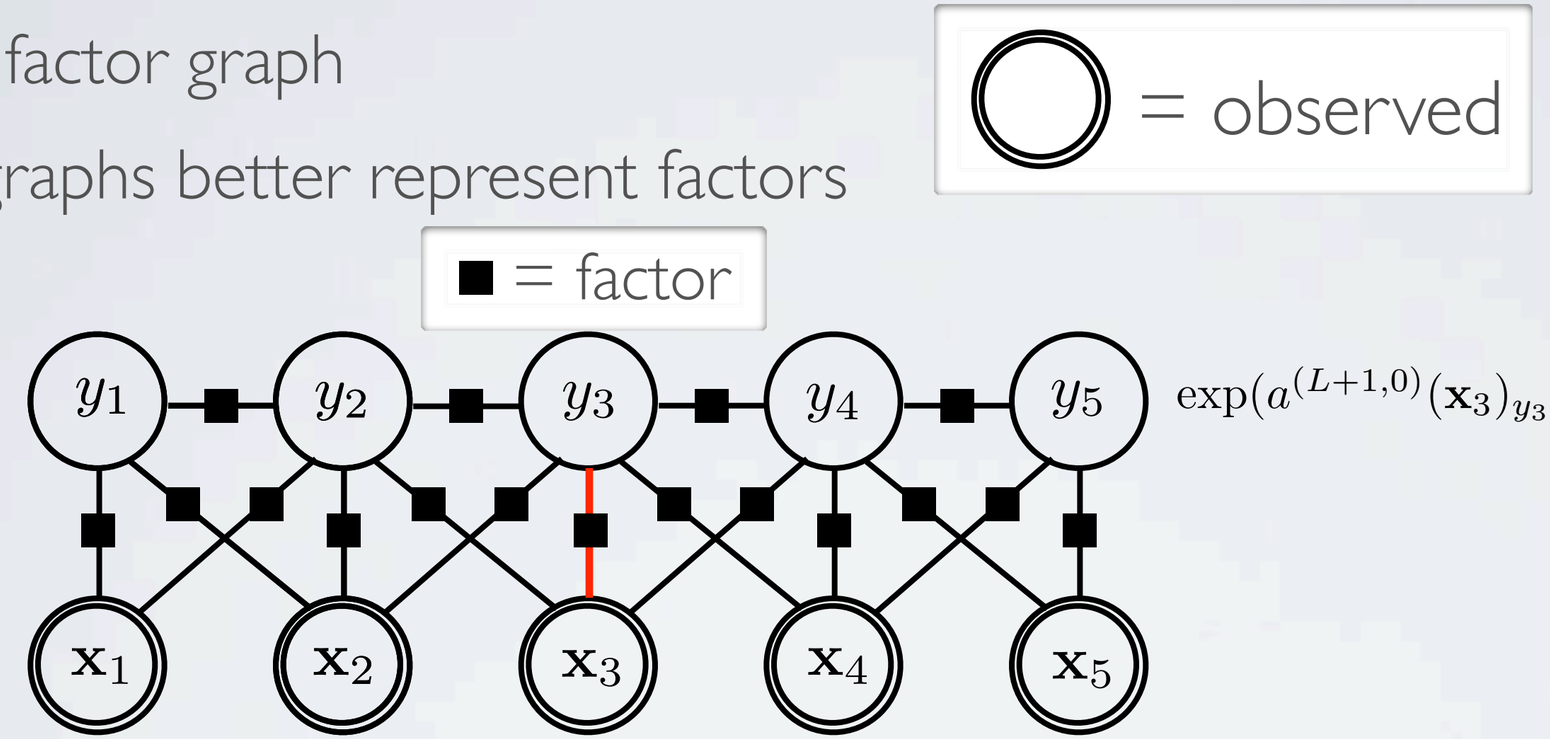

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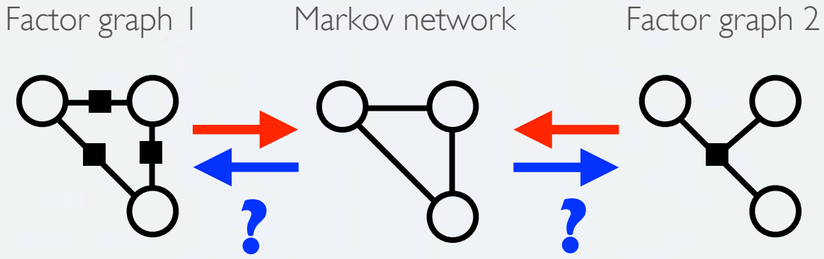

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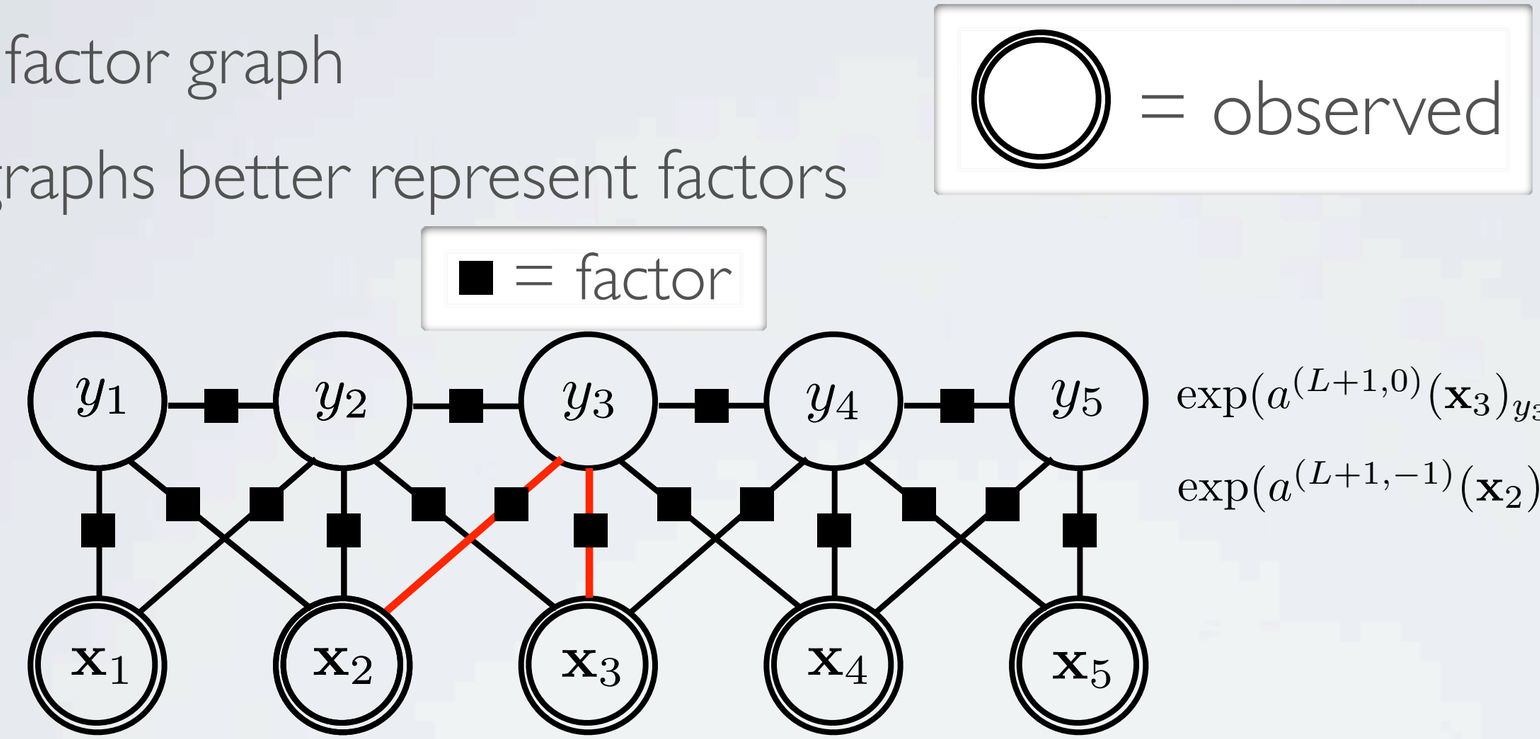

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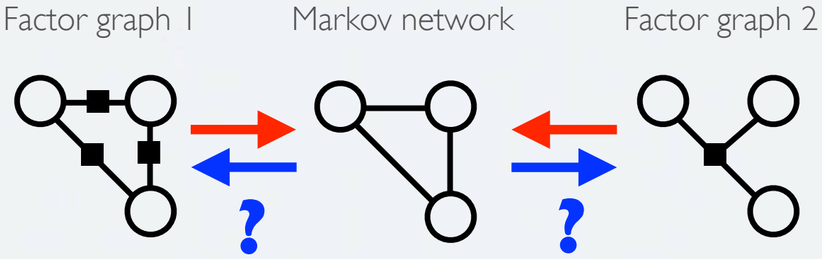

FACTOR GRAPH VISUALIZATION

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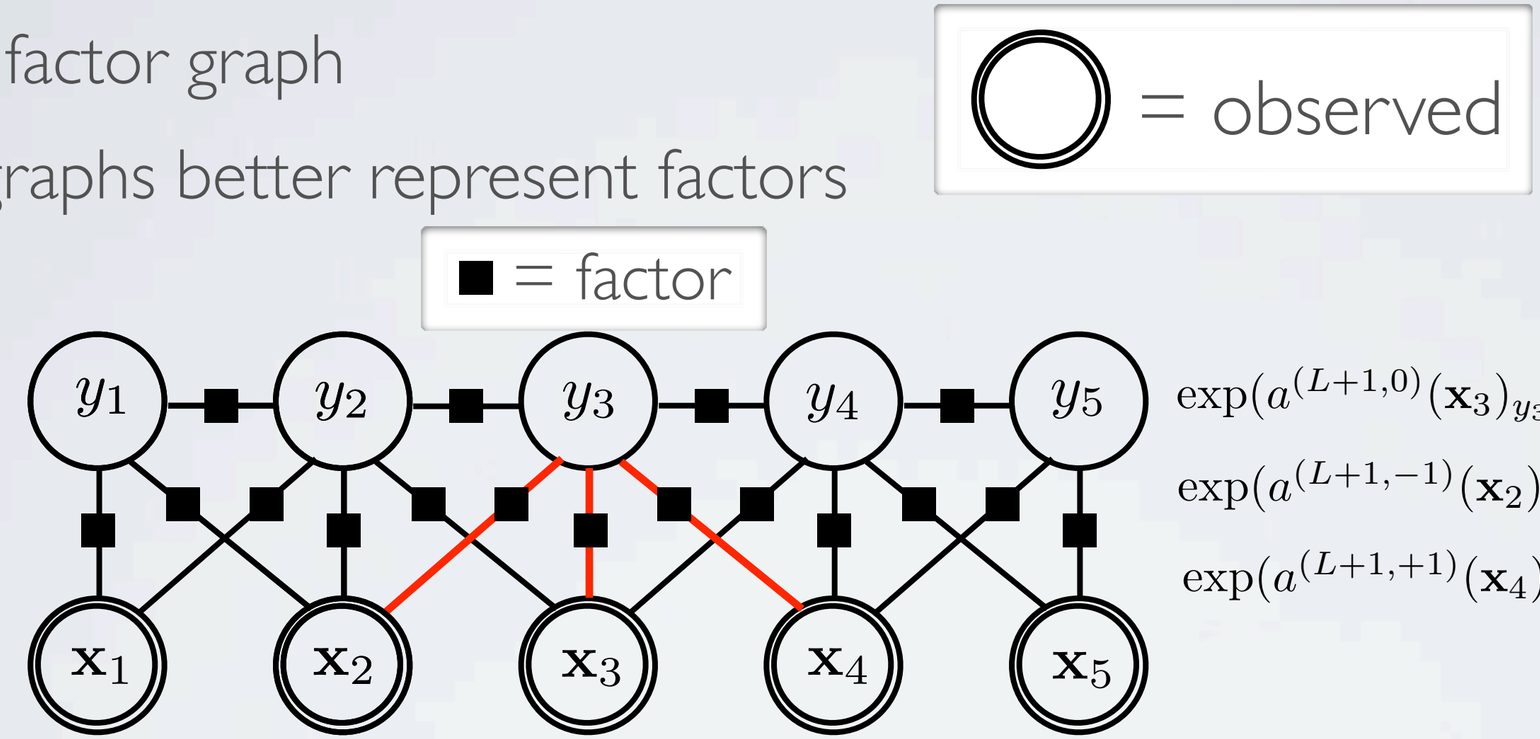

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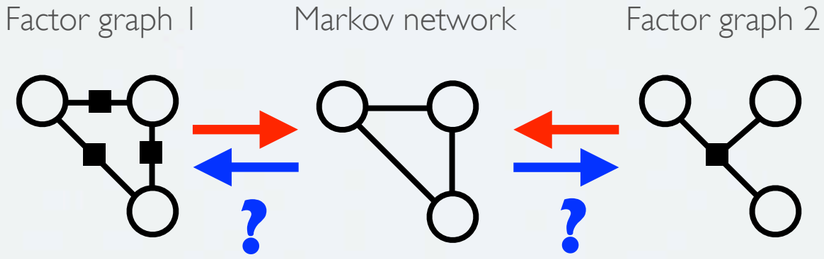

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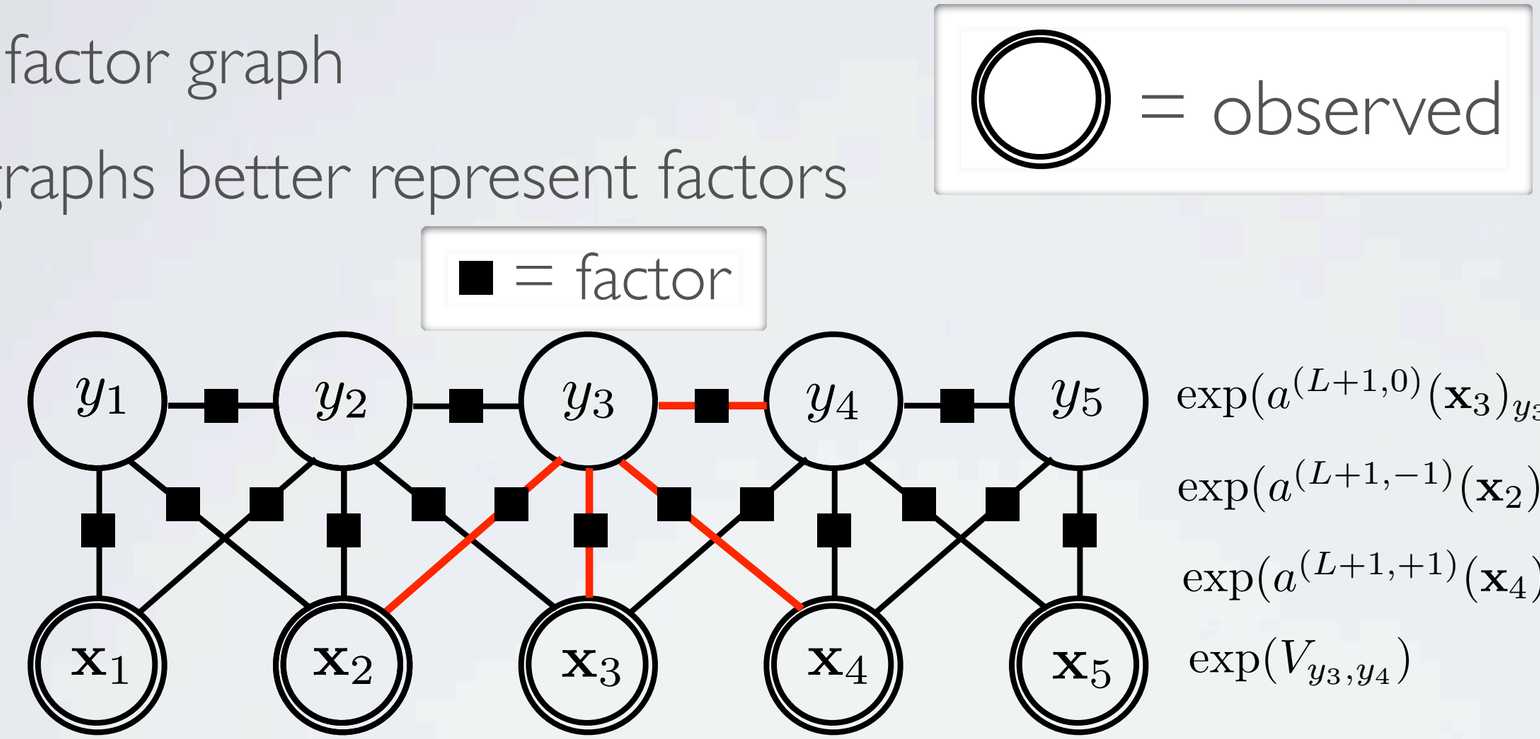

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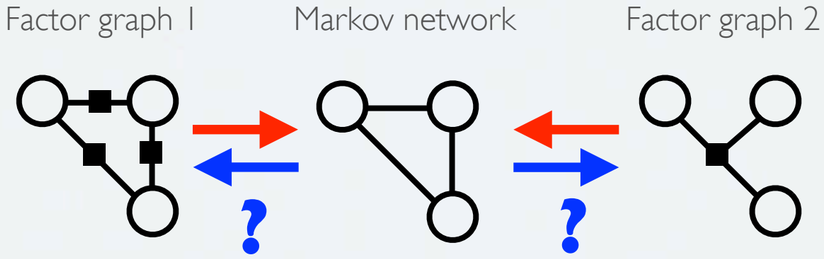

Neural networks

Conditional random fields - belief propagation

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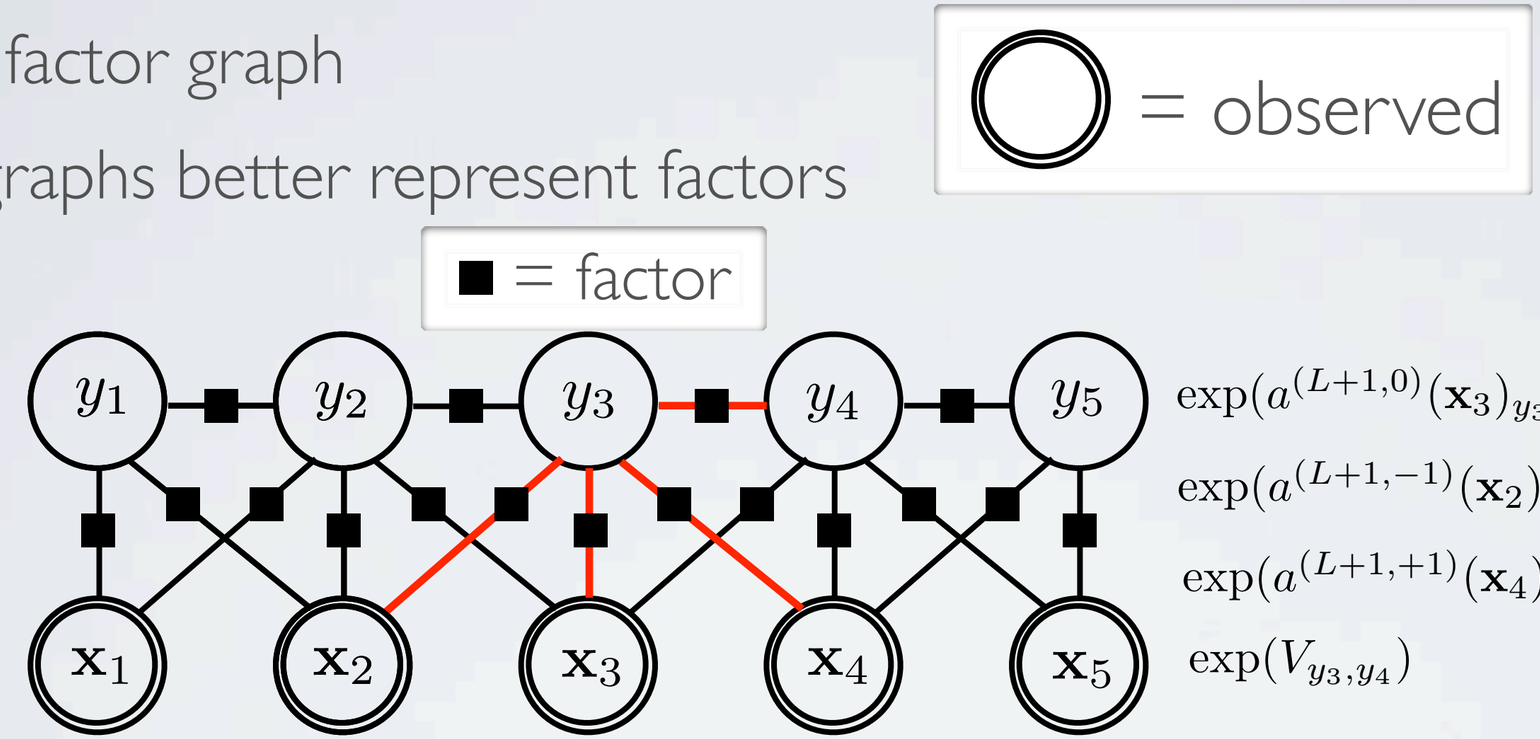

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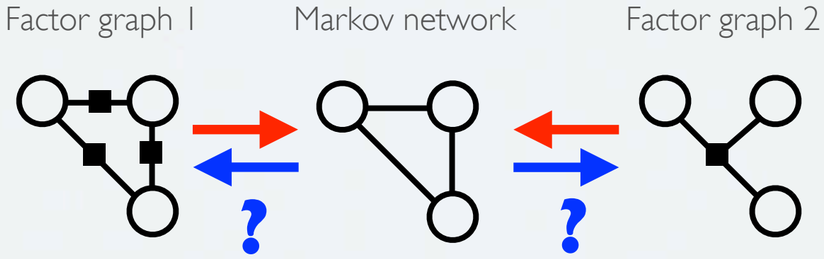
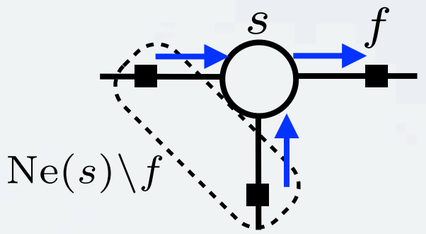


BELIEF PROPAGATION

Topics: belief propagation, message passing

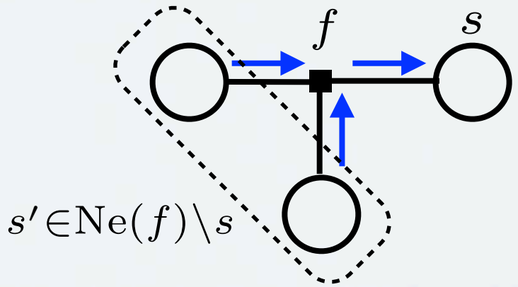
- Factor graphs better represent the computations needed to do inference
  - we can write the forward-backward algorithm seen before into a general message passing form
  - there are two types of message:
    - from a variable node (◯) to its neighbor factor nodes (■):

$$\mu_{s\to f}(i) = \prod_{f'\in \mathrm{Ne}(s)\setminus f} \mu_{f'\to s}(i)$$



    - from a factor node (■) to its neighbor variable nodes (◯):

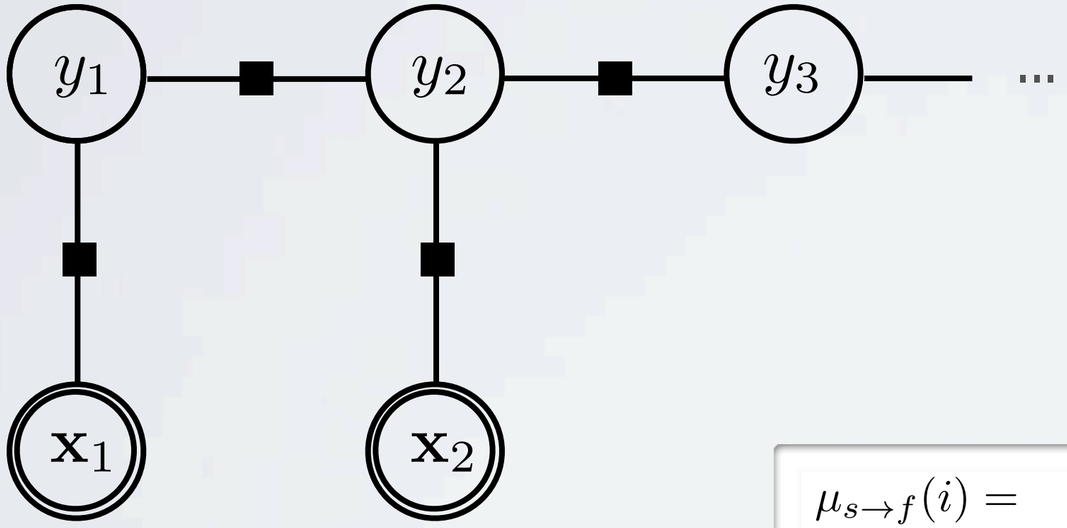
$$\mu_{f\to s}(i) = \sum_{\substack{\mathbf{z} \text{ is unobs.} \\ \text{and } z_s=i}} \phi_f(\mathbf{z}) \prod_{s'\in \mathrm{Ne}(f)\setminus s} \mu_{s'\to f}(z_{s'})$$



BELIEF PROPAGATION

Topics: belief propagation, message passing

- Example: our (simplified) linear chain CRF



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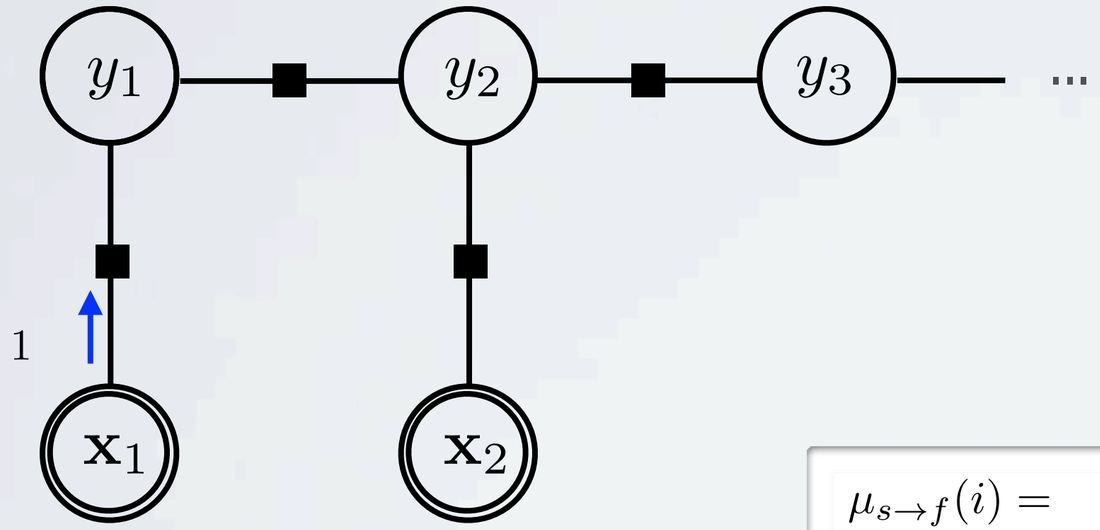

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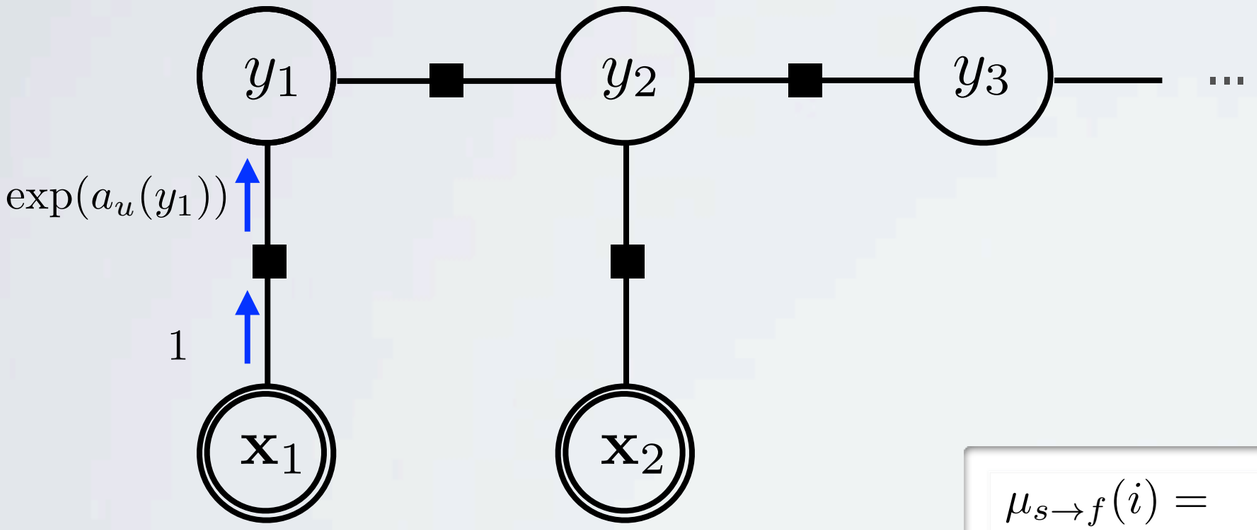

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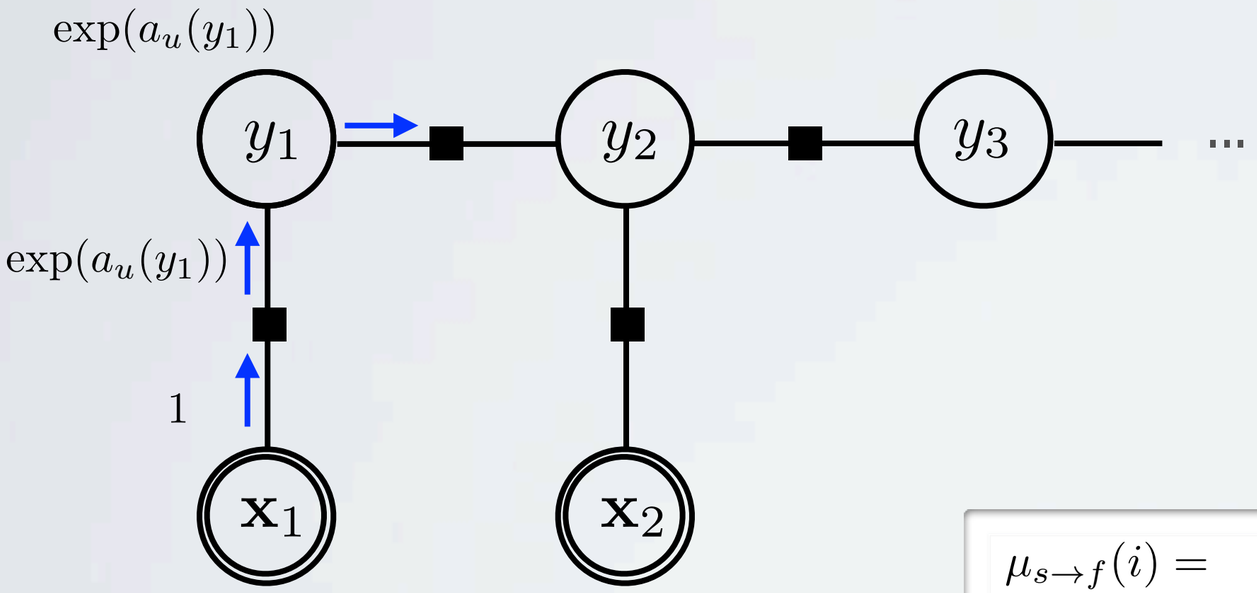

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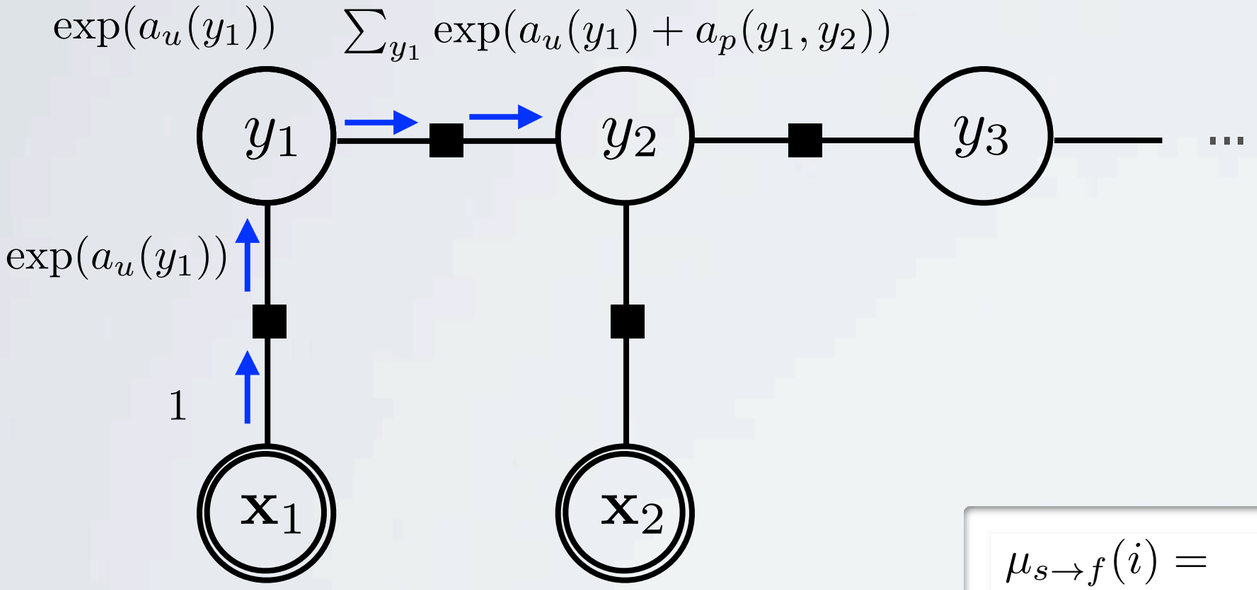

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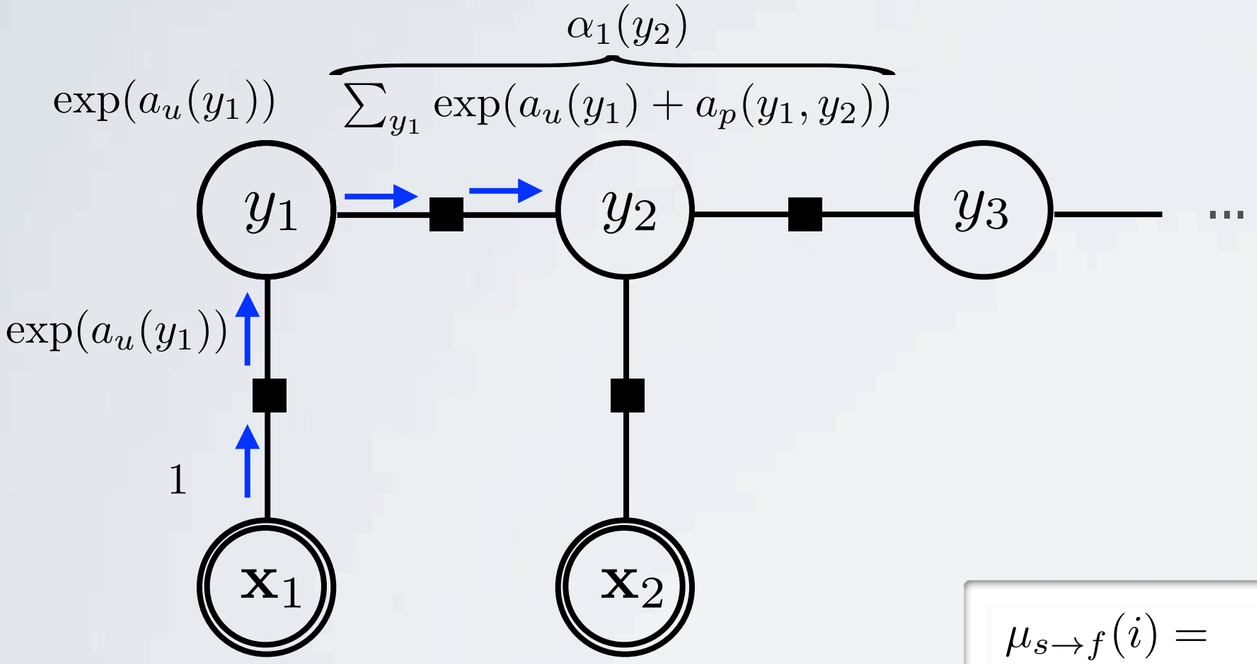

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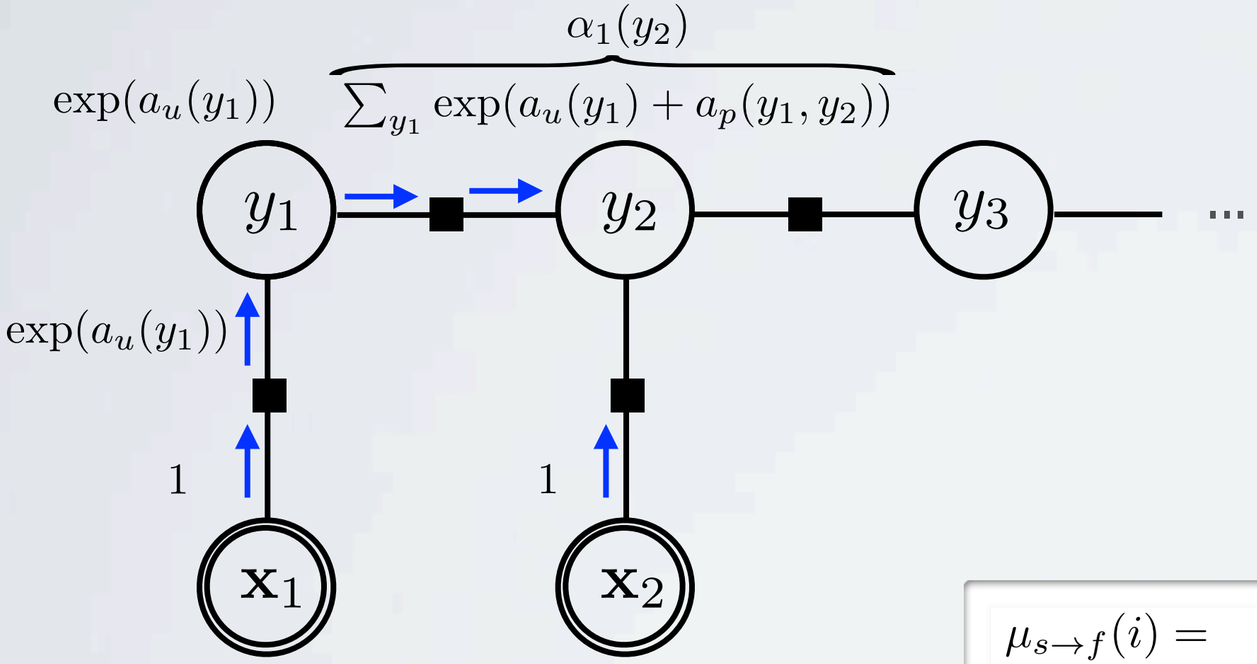

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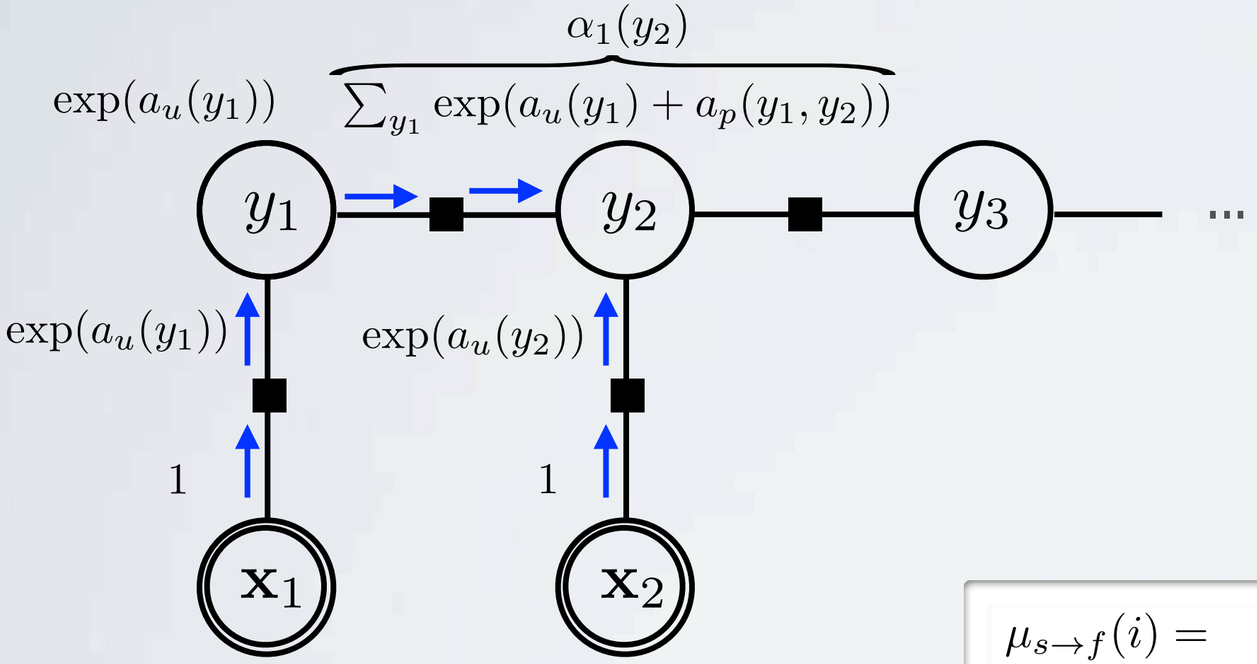

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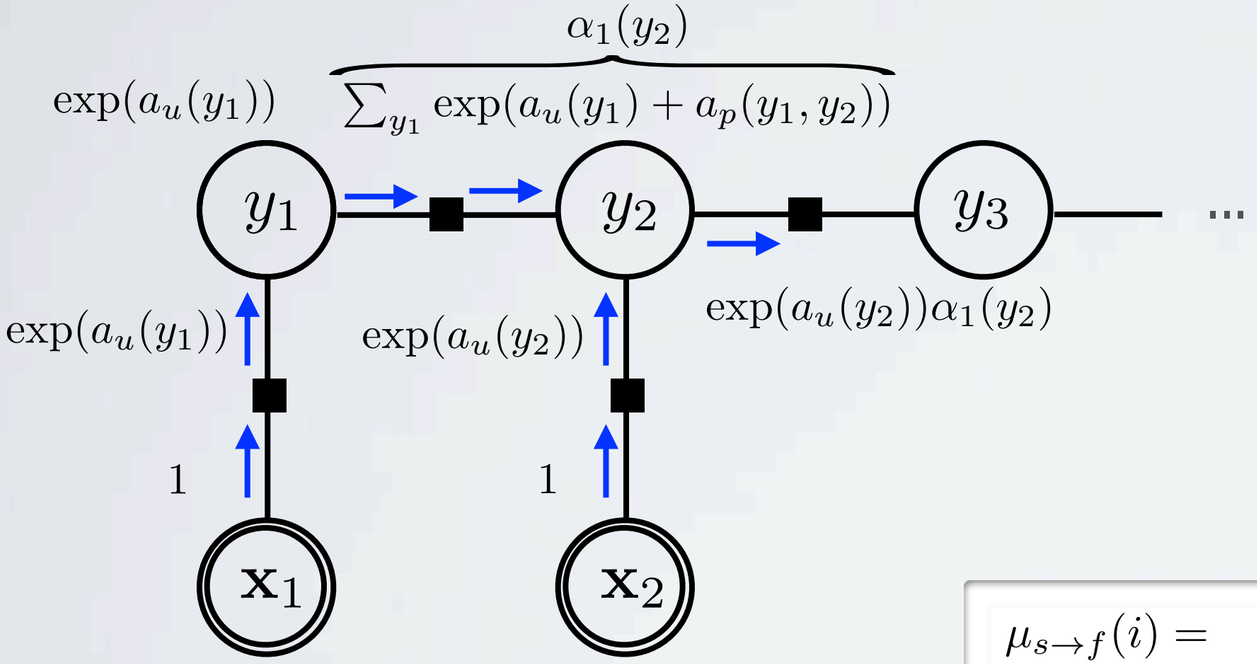

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Topics: belief propagation, message passing

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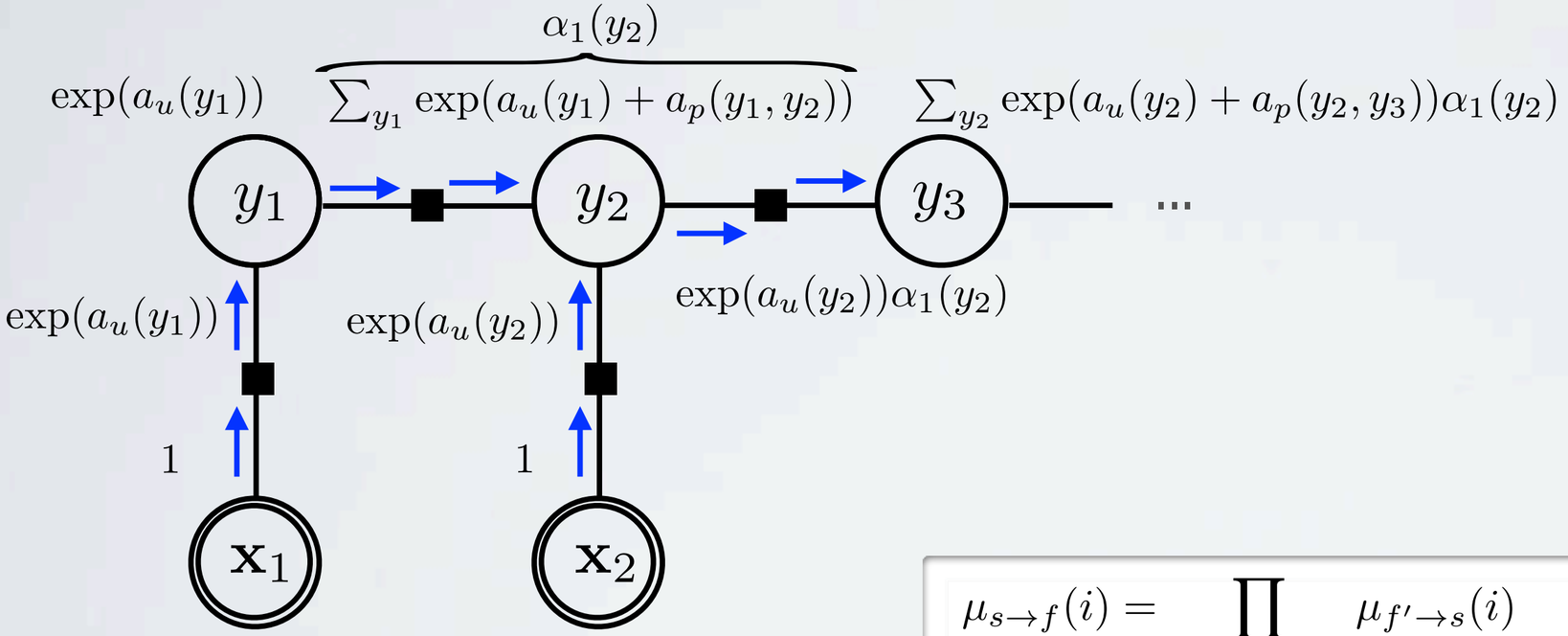

$$\mu_{s \to f}(i) = \prod_{f' \in \text{Ne}(s) \setminus f} \mu_{f' \to s}(i)$$

$$\mu_{f \to s}(i) = \sum_{\substack{\mathbf{z} \text{ is unobs.} \\ \text{and } z_s = i}} \phi_f(\mathbf{z}) \prod_{s' \in \text{Ne}(f) \setminus s} \mu_{s' \to f}(z_{s'})$$

BELIEF PROPAGATION

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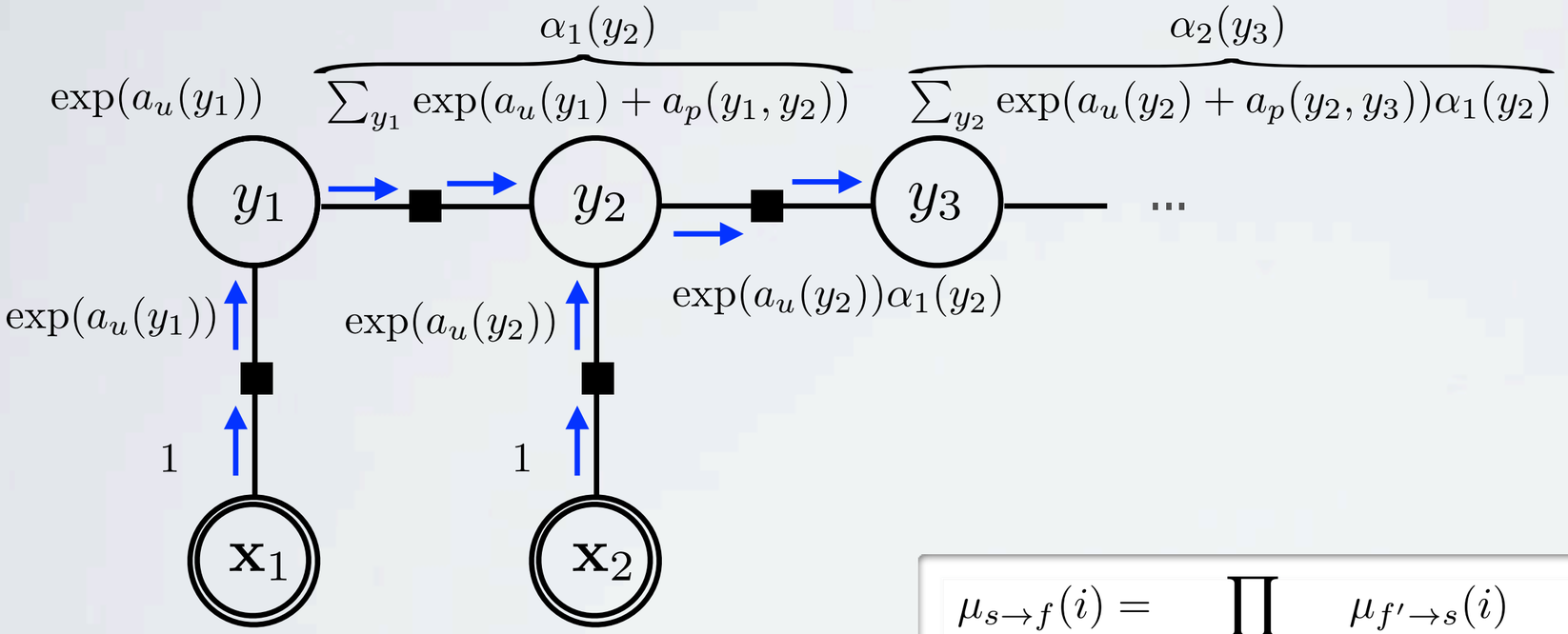

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(LOOPY) BELIEF PROPAGATION

Topics: belief propagation, message passing

- On a linear chain graph, belief propagation is the same a forward-backward
  - forward pass of message passing computes the $\alpha_k(y_{k+1})$
  - backward pass of message passing computes the $\beta_k(y_{k-1})$
- For numerical stability, passing log-messages is preferred
- Can do inference on other types of structures
  - belief propagation is also exact on arbitrary trees
  - on a graph with loops, (loopy-)belief propagation can be used to do approximate inference (but can divergence, if not careful)
  - many general purpose libraries are publicly available

LINEAR CHAIN CRF

Topics: other variations on linear chain CRF

- We could add lateral connections between labels that are at 2 positions away $\phi_f(y_k, y_{k+2})$

- We could add lateral connections for triplets of labels $\phi_f(y_k, y_{k+1}, y_{k+2})$

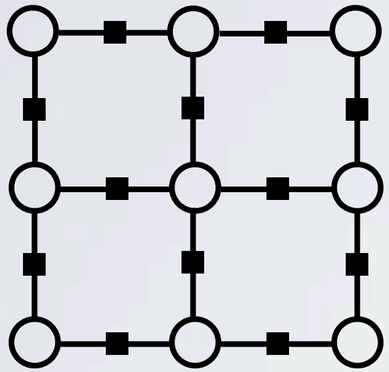
- The idea is to add connections between things to model their dependency more directly

- The connectivity can even change between examples

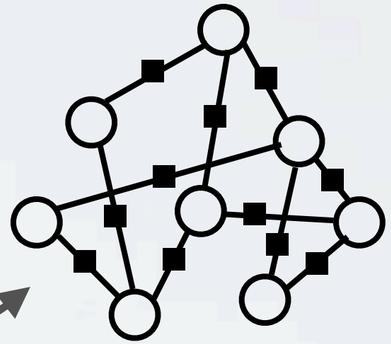
GENERAL CRF

Topics: CRFs in general

- We don't have to restrict the CRF structure to linear chains



$$p(\mathbf{y}|\mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_f \Psi_f(\mathbf{y}, \mathbf{X})$$



Grid structure
(pixels in image)

General pair-wise structure
(webpages sharing a link)

- We could also have n-ary factors, with $n>2$

(LOOPY) BELIEF PROPAGATION

Topics: CRFs in general

- Marginals can be approximated with:

$$p(y_k|\mathbf{X}) = \frac{\exp(\log \phi_f(y_k) + \sum_{f' \in \mathrm{Ne}(k) \setminus f} \log \mu_{f' \to k}(y_k))}{\sum_{y'_k} \exp(\log \phi_f(y'_k) + \sum_{f' \in \mathrm{Ne}(k) \setminus f} \log \mu_{f' \to k}(y'_k))}$$

- In general, an approximated marginal is computed by

1. summing all the log-factors that involve only the y_k variables of interest
2. summing all the log-messages coming into the y_k variables from other factors
3. exponentiating
4. renormalizing