Review of fundamentals

IFT 725 - Réseaux neuronaux

LINEAR ALGEBRA

Topics: matrix, vector, norms, products

- Vector: $\mathbf{x} = [x_1, \ldots, x_d]^\top = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$
  - product: $< \mathbf{x}^{(1)}, \mathbf{x}^{(2)} >= {\mathbf{x}^{(1)}}^\top \mathbf{x}^{(2)} = \sum_{i=1}^{d} x_i^{(1)} x_i^{(2)}$
  - norm: $||\mathbf{x}||_2 = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_i x_i^2}$ (Euclidean)
- Matrix: $\mathbf{X} = \begin{bmatrix} X_{1,1} & \ldots & X_{1,m} \\ \vdots & \vdots & \vdots \\ X_{n,1} & \ldots & X_{n,m} \end{bmatrix}$
  - product: $(\mathbf{X}^{(1)} \mathbf{X}^{(2)})_{i,j} = \mathbf{X}_{i,\cdot}^{(1)} \mathbf{X}_{\cdot,j}^{(2)} = \sum_k \mathbf{X}_{i,k}^{(1)} \mathbf{X}_{k,j}^{(2)}$
  - norm: $||\mathbf{X}||_F = \sqrt{\mathrm{trace}(\mathbf{X}^\top \mathbf{X})} = \sqrt{\sum_i \sum_j X_{i,j}^2}$ (Frobenius)

LINEAR ALGEBRA

Topics: special matrices

- Identity matrix $\mathbf{I}$: $\mathrm{I}_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

- Diagonal matrix $\mathbf{X}$: $X_{i,j} = 0 \text{ if } i \neq j$

- Lower triangular matrix $\mathbf{X}$: $X_{i,j} = 0 \text{ if } i < j$

- Symmetric matrix $\mathbf{X}$: $X_{i,j} = X_{j,i}$ (i.e. $\mathbf{X}^\top = \mathbf{X}$)

- Square matrix: matrix with same number of rows and columns

LINEAR ALGEBRA

Topics: operations on matrices

- Trace of matrix: $\mathrm{trace}(\mathbf{X}) = \sum_i X_{i,i}$
    - trace of products:

$$\mathrm{trace}(\mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)}) = \mathrm{trace}(\mathbf{X}^{(3)}\mathbf{X}^{(1)}\mathbf{X}^{(2)}) = \mathrm{trace}(\mathbf{X}^{(2)}\mathbf{X}^{(3)}\mathbf{X}^{(1)})$$

- Inverse of matrix: $\mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\,\mathbf{X}^{-1} = \mathrm{I}$
    - doesn't exist if determinant is 0
    - inverse of product: $(\mathbf{X}^{(1)}\mathbf{X}^{(2)})^{-1} = \mathbf{X}^{(2)}{}^{-1}\mathbf{X}^{(1)}{}^{-1}$

- Transpose of matrix: $(\mathbf{X}^\top)_{i,j} = \mathbf{X}_{j,i}$
    - transpose of product: $(\mathbf{X}^{(1)}\mathbf{X}^{(2)})^\top = \mathbf{X}^{(2)}{}^\top\mathbf{X}^{(1)}{}^\top$

LINEAR ALGEBRA

Topics: operations on matrices

- Determinant
  - of triangular matrix: $\det(\mathbf{X}) = \prod_i \mathbf{X}_{i,i}$
  - of transpose of matrix: $\det(\mathbf{X}^\top) = \det(\mathbf{X})$
  - of inverse of matrix: $\det(\mathbf{X}^{-1}) = \det(\mathbf{X})^{-1}$
  - of product of matrix: $\det(\mathbf{X}^{(1)}\mathbf{X}^{(2)}) = \det(\mathbf{X}^{(1)})\det(\mathbf{X}^{(2)})$

LINEAR ALGEBRA

Topics: properties of matrices

- Orthogonal matrix: $\mathbf{X}^\top = \mathbf{X}^{-1}$

- Positive definite matrix: $\mathbf{v}^\top \mathbf{X} \mathbf{v} > 0 \quad \forall \mathbf{v} \in \mathbb{R}$
  - if « $\geq$ » , then positive semi-definite

LINEAR ALGEBRA

Topics: linear dependence, rank, range and nullspace

- Set of linearly dependent vectors $\{\mathbf{x}^{(t)}\}$:

$$\exists \mathbf{w}, t^* \text{ such that } \mathbf{x}^{(t^*)} = \sum_{t \neq t^*} w_t \mathbf{x}^{(t)}$$

- Rank of matrix: number of linear independent columns

- Range of a matrix:

$$\mathcal{R}(\mathbf{X}) = \{\mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{w} \text{ such that } \mathbf{x} = \sum_j w_j \mathbf{X}_{\cdot,j}\}$$

- Nullspace of a matrix:

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \notin \mathcal{R}(\mathbf{X})\}$$

LINEAR ALGEBRA

Topics: eigenvalues and eigenvectors of a matrix

- Eigenvalues and eigenvectors

$$\{\lambda_i, \mathbf{u}_i \mid \mathbf{X}\mathbf{u}_i = \lambda_i \mathbf{u}_i \text{ and } \mathbf{u}_i^\top \mathbf{u}_j = 1_{i=j}\}$$

- Properties
  - can write $\mathbf{X} = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^\top$
  - determinant of **any** matrix: $\det(\mathbf{X}) = \prod_i \lambda_i$
  - positive definite if $\lambda_i > 0 \quad \forall i$
  - rank of matrix is the number of non-zero eigenvalues

DIFFERENTIAL CALCULUS

Topics: derivative, partial derivative

- Derivative:

$$\frac{d}{dx} f(x) = \lim_{\Delta \to 0} \frac{f(x + \Delta) - f(x)}{\Delta}$$

  - direction and rate of increase of function

- Partial derivative:

$$\frac{\partial}{\partial x} f(x, y) = \lim_{\Delta \to 0} \frac{f(x + \Delta, y) - f(x, y)}{\Delta}$$

$$\frac{\partial}{\partial y} f(x, y) = \lim_{\Delta \to 0} \frac{f(x, y + \Delta) - f(x, y)}{\Delta}$$

  - direction and rate of increase for variable assuming others are fixed

DIFFERENTIAL CALCULUS

Topics: derivative, partial derivative

- Example:

$$f(x, y) = \frac{x^2}{y}$$

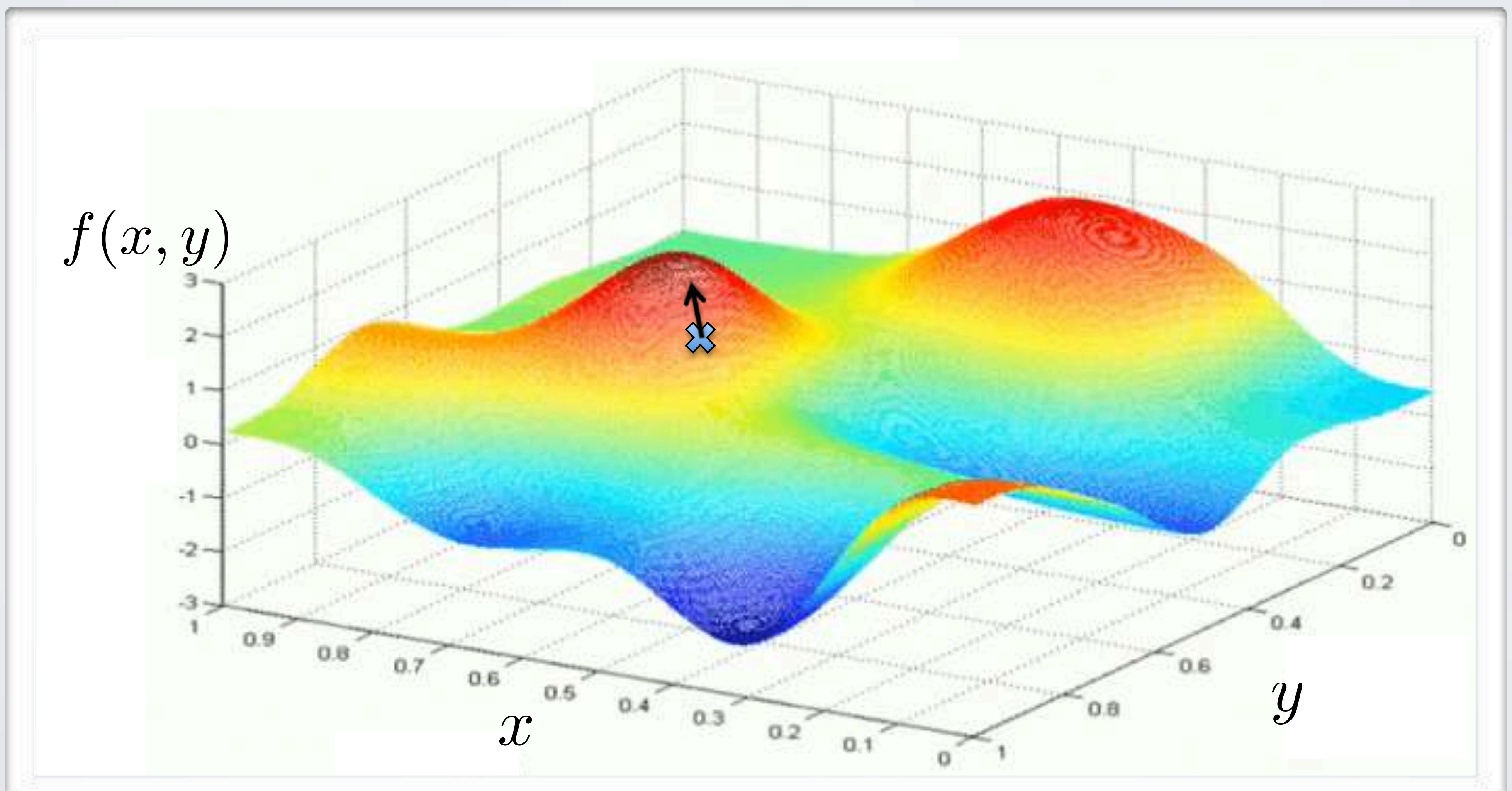
$$\frac{\partial f(x, y)}{\partial x} = \underbrace{\frac{2x}{y}}_{\substack{\text{treat } y \\ \text{as constant}}} \qquad \frac{\partial f(x, y)}{\partial y} = \underbrace{\frac{-x^2}{y^2}}_{\substack{\text{treat } x \\ \text{as constant}}}$$

DIFFERENTIAL CALCULUS

Topics: gradient

- Gradient:

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \left[\frac{\partial}{\partial x_1} f(\mathbf{x}), \ldots, \frac{\partial}{\partial x_d} f(\mathbf{x})\right]^\top = \begin{bmatrix} \frac{\partial}{\partial x_1} f(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_d} f(\mathbf{x}) \end{bmatrix}$$



DIFFERENTIAL CALCULUS

Topics: Jacobian, Hessian

- Hessian:

$$\nabla^2_{\mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f(\mathbf{x}) & \dots & \frac{\partial^2}{\partial x_1 \partial x_d} f(\mathbf{x}) \\ \vdots & \dots & \vdots \\ \frac{\partial^2}{\partial x_d \partial x_1} f(\mathbf{x}) & \dots & \frac{\partial^2}{\partial x_d^2} f(\mathbf{X}) \end{bmatrix}$$

- If $\mathbf{f}(\mathbf{x}) = [f(\mathbf{x})_1, \dots, f(\mathbf{x})_k]^\top$ is a vector, the Jacobian is:

$$\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(\mathbf{x})_1 & \dots & \frac{\partial}{\partial x_d} f(\mathbf{x})_1 \\ \vdots & \dots & \vdots \\ \frac{\partial}{\partial x_1} f(\mathbf{x})_k & \dots & \frac{\partial}{\partial x_d} f(\mathbf{x})_k \end{bmatrix}$$

DIFFERENTIAL CALCULUS

Topics: gradient for matrices

- If scalar function $f(\mathbf{X})$ takes a matrix $\mathbf{X}$ as input

$$\nabla_{\mathbf{X}} f(\mathbf{X}) = \begin{bmatrix} \frac{\partial}{\partial X_{1,1}} f(\mathbf{X}) & \dots & \frac{\partial}{\partial X_{1,m}} f(\mathbf{X}) \\ \vdots & \dots & \vdots \\ \frac{\partial}{\partial X_{n,1}} f(\mathbf{X}) & \dots & \frac{\partial}{\partial X_{n,m}} f(\mathbf{X}) \end{bmatrix}$$

- For functions that output functions and take matrices as input, we organize into 3D tensors

PROBABILITY

Topics: probability space

- Probability space: triplet $(\Omega, \mathcal{F}, P)$
  - Ω is the space of possible outcomes
  - $\mathcal{F}$ is the space of possible events
  - P is a probability measure mapping an outcome to its probability [0,1]
  - example: throwing a die
    - $\Omega = \{1, 2, 3, 4, 5, 6\}$
    - $e = \{1, 5\} \in \mathcal{F}$ (i.e. die is either 1 or 5)
    - $P(\{1, 5\}) = \frac{2}{6}$
- Properties:

1. $P(\{\omega\}) \geq= 0 \quad \forall \omega \in \Omega$ 2. $\sum_{\omega \in \Omega} P(\{\omega\}) = 1$

PROBABILITY

Topics: random variable

- Random variable: a function on outcomes
- Examples:
  - X is the value of the outcome
  - O is 1 if the outcome is 1, 3 or 5, otherwise it's 0
  - S is 1 if the outcome is smaller than 4, otherwise it's 0

PROBABILITY

Topics: distributions (joint, marginal, conditional)

- Joint distribution: $p(X = x, O = o, S = s)$ ($p(x, s, o)$ for short)
  - the probability of a complete assignment of all random variables
  - example: $p(X = 1, O = 1, S = 0) = 0$
- Marginal distribution: $p(o, s) = \sum_x p(x, o, s)$
  - the probability of a partial assignment
  - example: $p(O = 1, S = 0) = \frac{1}{6}$
- Conditional distribution: $p(S = s | O = o)$
  - the probability of some variables, assuming an assignment of other variables
  - example: $p(S = 1 | O = 1) = \frac{2}{3}$

PROBABILITY

Topics: probability chain rule, Bayes rule

- Probability chain rule: $p(s,o) = p(s|o)p(o) = p(o|s)p(s)$
  - in general:

$$p(\mathbf{x}) = \prod_i p(x_i|x_1,\ldots,x_{i-1})$$

- Bayes rule:

$$p(O=o|S=s) = \frac{p(S{=}s|O{=}o)p(O{=}o)}{\sum_{o'} p(S{=}s|O{=}o')p(O{=}o')}$$

PROBABILITY

Topics: independence between variables

- Independence: variables X_1 and X_2 are independent if

$$p(x_1, x_2) = p(x_1)p(x_2)$$

or $p(x_1|x_2) = p(x_1)$

or $p(x_2|x_1) = p(x_2)$

- Conditional independence: variables X_1 and X_2 are independent given X_3 if

$$p(x_1, x_2|x_3) = p(x_1|x_3)p(x_2|x_3)$$

or $p(x_1|x_2, x_3) = p(x_1|x_3)$

or $p(x_2|x_1, x_3) = p(x_2|x_3)$

PROBABILITY

Topics: expectation, variance

- Expectation: $\mathrm{E}[X] = \sum_x x\, p(X = x)$
  - properties:
    - $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$
    - $\mathrm{E}[f(X)] = \sum_x f(x)\, p(X = x)$
    - if independent, $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$

- Variance: $\mathrm{Var}[X] = \sum_x (x - \mathrm{E}(X))^2\, p(X = x)$
  - properties:
    - $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$
    - if independent, $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

PROBABILITY

Topics: covariance matrix

- Covariance:

$$\begin{aligned}\mathrm{Cov}(X_1, X_2) &= \mathrm{E}[(X_1 - \mathrm{E}[X_1])(X_2 - \mathrm{E}[X_2])] \\ &= \sum_{x_1}\sum_{x_2}(x_1 - \mathrm{E}[X_1])(x_2 - \mathrm{E}[X_2])\, p(x_1, x_2)\end{aligned}$$

  - if independent $\mathrm{Cov}(X_1, X_2) = 0$
  - $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$

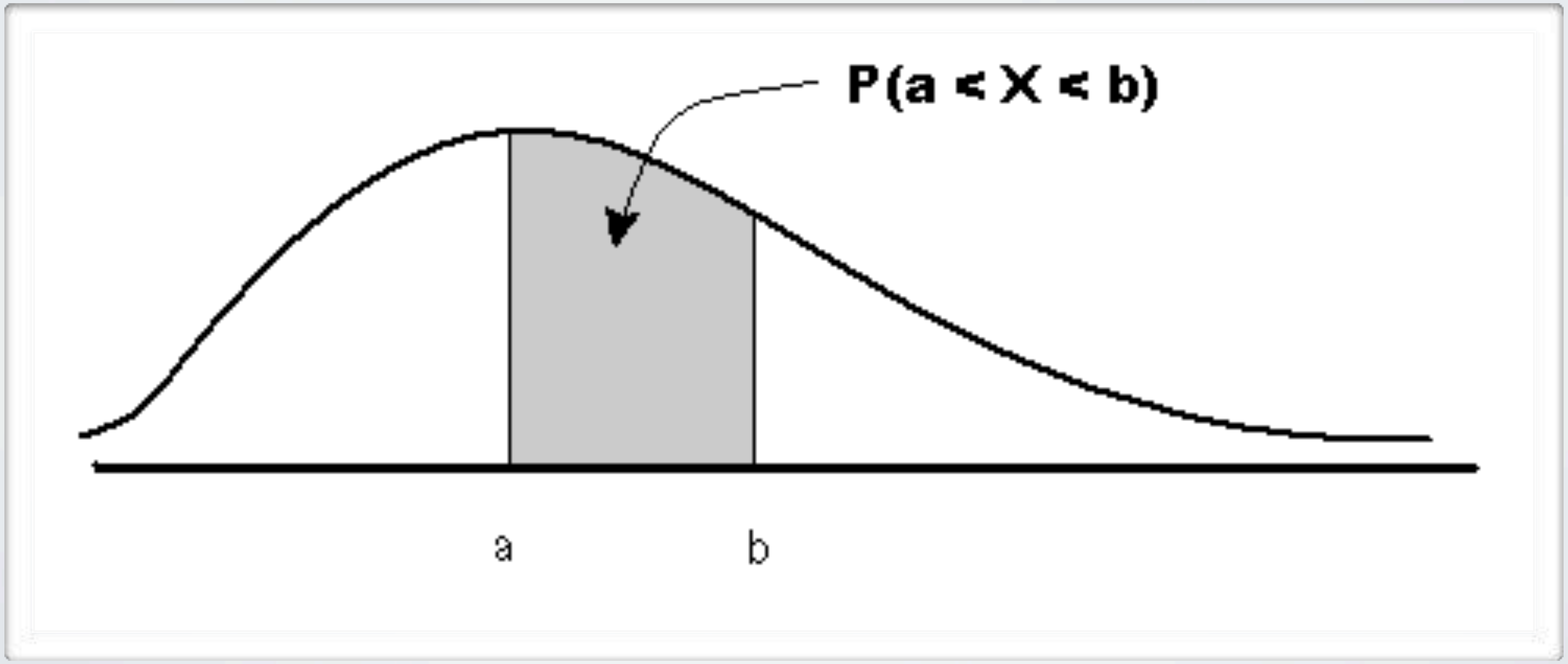
- Covariance matrix:

$$\mathrm{Cov}(\mathbf{X}) = \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \dots & \mathrm{Cov}(X_1, X_d) \\ \vdots & \vdots & \vdots \\ \mathrm{Cov}(X_d, X_1) & \dots & \mathrm{Cov}(X_d, X_d) \end{bmatrix}$$

PROBABILITY

Topics: continuous variables

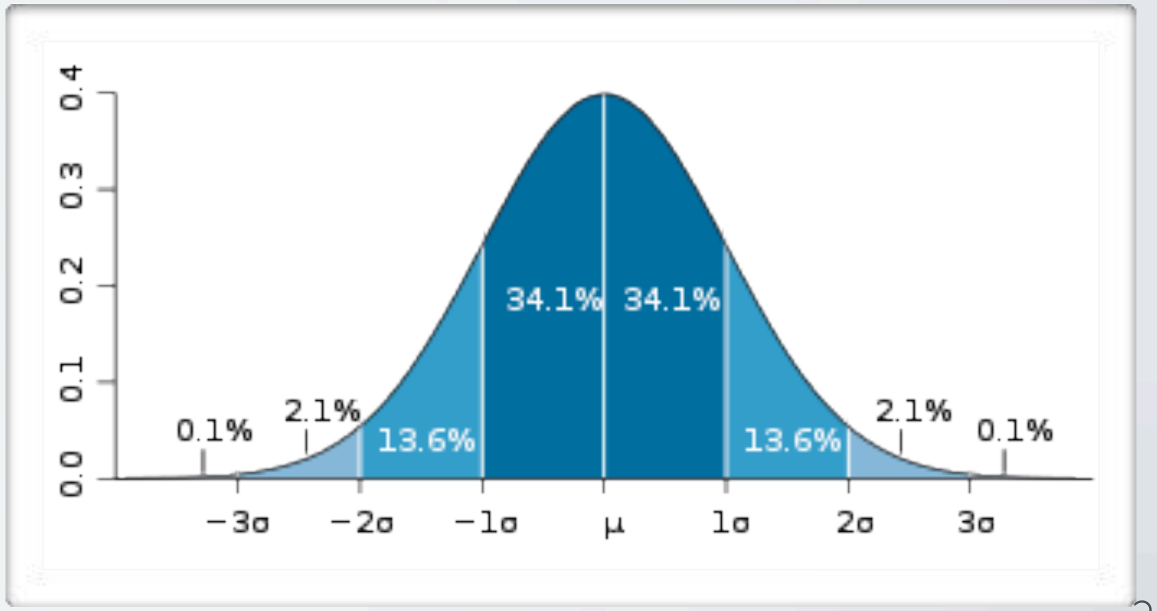
- for continuous variable X , $p(x)$ is a density function
  - $P(X \in A) = \int_{x \in A} p(x)dx$
  - the probability $P(X = x)$ is zero for continuous variables
  - in previous equations, summations would be replaced by integrals



PROBABILITY

Topics: Bernoulli, Gaussian distributions

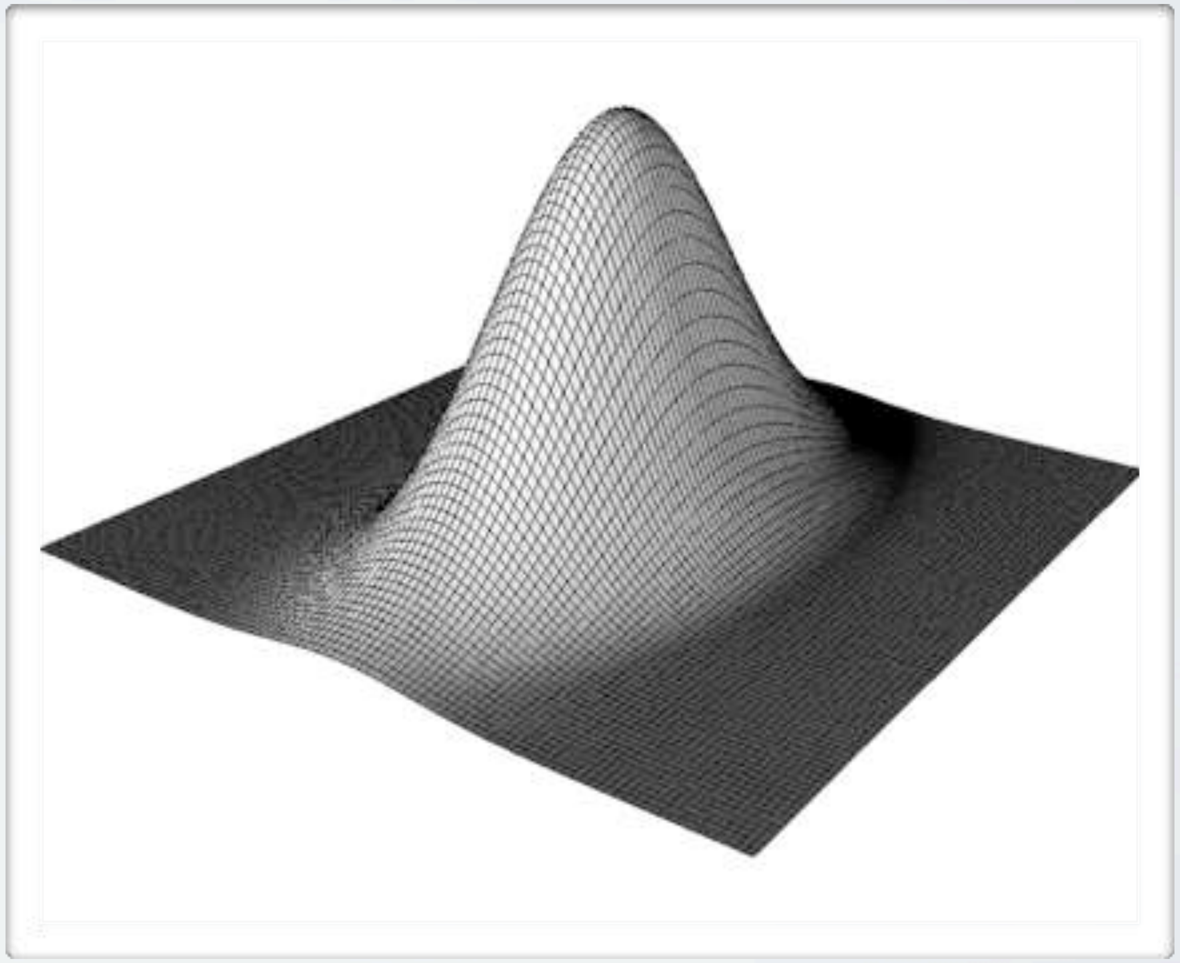
- Bernoulli variable: $X \in \{0, 1\}$
  - $p(X = 1) = \mu$
  - $p(X = 0) = 1 - \mu$
  - $\mathrm{E}[X] = \mu$
  - $\mathrm{Var}[X] = \mu(1 - \mu)$
- Gaussian variable: $X \in \mathbb{R}$
  - $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
  - $\mathrm{E}[X] = \mu$
  - $\mathrm{Var}[X] = \sigma^2$



PROBABILITY

Topics: Multivariate Gaussian distributions

- Gaussian variable: $\mathbf{X} \in \mathbb{R}^d$
  - $p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$
  - $\mathrm{E}[\mathbf{X}] = \boldsymbol{\mu}$
  - $\mathrm{Cov}[\mathbf{X}] = \Sigma$



STATISTICS

Topics: estimate of the expectation and covariance matrix

- Sample mean:

$$\widehat{\boldsymbol{\mu}} = \frac{1}{T}\sum_t \mathbf{x}^{(t)}$$

- Sample variance:

$$\widehat{\boldsymbol{\sigma}}^2 = \frac{1}{T-1}\sum_t (\mathbf{x}^{(t)} - \widehat{\boldsymbol{\mu}})^2$$

- Sample covariance matrix:

$$\widehat{\Sigma} = \frac{1}{T-1}\sum_t (\mathbf{x}^{(t)} - \widehat{\boldsymbol{\mu}})(\mathbf{x}^{(t)} - \widehat{\boldsymbol{\mu}})^\top$$

- These estimators are unbiased, i.e.:

$$\mathrm{E}[\widehat{\boldsymbol{\mu}}] = \boldsymbol{\mu} \quad \mathrm{E}[\widehat{\sigma}^2] = \sigma^2 \quad \mathrm{E}\left[\widehat{\Sigma}\right] = \Sigma$$

STATISTICS

Topics: confidence interval

- Confidence interval of the sample mean (1D):
  - if T is big, the following estimator is approx. Gaussian with mean 0 and variance 1

$$\frac{\widehat{\mu}-\mu}{\sqrt{\widehat{\sigma}^2/T}}$$

  - then we have that, with 95% probability, that

$$\mu \in \widehat{\mu} \pm -1.96 \sqrt{\widehat{\sigma}^2/T}$$

STATISTICS

Topics: maximum likelihood, I.I.D. hypothesis

- maximum likelihood estimator (MLE):

$$\widehat{\theta} = \arg\max_{\theta} p(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(T)})$$

  - the sample mean is the MLE for a Gaussian distribution
  - the sample covariance matrix isnt, but this is

$$\tfrac{T-1}{T}\widehat{\Sigma} = \tfrac{1}{T}\textstyle\sum_t (\mathbf{x}^{(t)} - \widehat{\boldsymbol{\mu}})(\mathbf{x}^{(t)} - \widehat{\boldsymbol{\mu}})^\top$$

- Independent and identically distributed variables

$$p(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(T)}) = \prod_t p(\mathbf{x}^{(t)})$$

SAMPLING

Topics: Monte Carlo estimate

- Monte Carlo estimate:
  - a method to approximate an expensive expectation

$$\mathrm{E}[f(\mathbf{X})] = \sum_{\mathbf{x}} f(\mathbf{x})\, p(\mathbf{x}) \approx \frac{1}{K} \sum_k f(\mathbf{x}^{(k)})$$

  - the $\mathbf{x}^{(k)}$ must be sampled from $p(\mathbf{x})$

SAMPLING

Topics: importance sampling

- Importance sampling:
  - a sampling method for when $p(\mathbf{x})$ is expensive to sample from

$$\mathrm{E}[f(\mathbf{X})] = \sum_{\mathbf{x}} f(\mathbf{x}) \, \frac{p(\mathbf{x})}{q(\mathbf{x})} q(\mathbf{x}) \approx \frac{1}{K} \sum_k f(\mathbf{x}^{(k)}) \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

  - $q(\mathbf{x})$ is easier to sample from and should be as similar as possible to $p(\mathbf{x})$
    - designing a good $q(\mathbf{x})$ is often hard to do

SAMPLING

Topics: Markov Chain Monte Carlo (MCMC)

- MCMC:
  - iterative method to generate the sequence of $\mathbf{x}^{(k)}$
  - the set of $\mathbf{x}^{(k)}$ will be dependent of each other $\mathbf{x}^{(k)}$

$$\mathbf{x}^{(1)} \xrightarrow{T(\mathbf{x}' \leftarrow \mathbf{x})} \mathbf{x}^{(2)} \xrightarrow{T(\mathbf{x}' \leftarrow \mathbf{x})} \mathbf{x}^{(3)} \xrightarrow{T(\mathbf{x}' \leftarrow \mathbf{x})} \dots \xrightarrow{T(\mathbf{x}' \leftarrow \mathbf{x})} \mathbf{x}^{(K)}$$

  - $T(\mathbf{x}' \leftarrow \mathbf{x})$ is a transition operator, that must satisfy certain properties
  - K must be big for the set of samples be representative of distribution
  - usually, we drop the first samples, which are not reliable

SAMPLING

Topics: Gibbs sampling

- Gibbs sampling:
  - MCMC method which uses the following transition operator $T(\mathbf{x}' \leftarrow \mathbf{x})$
    - pick a variable x_i
    - obtain $\mathbf{x}'$ by only resampling this variable according to

$$p(x_i | x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$$

    - return $\mathbf{x}'$

  - often, we simply cycle through the variables, in random order

MACHINE LEARNING

Topics: supervised learning

- Learning example: $(\mathbf{x}, y)$
- Task to solve: predict target y from input $\mathbf{x}$
  - classification: target is a class ID (from 0 to nb. of class - 1)
  - regression: target is a real number

MACHINE LEARNING

Topics: unsupervised learning

- Learning example: $\mathbf{x}$
- No explicit target to predict
  - clustering: partition data into groups
  - feature extraction: learn meaningful features automatically
  - dimensionality reduction: learning a lower-dimensional representation of input

MACHINE LEARNING

Topics: learning algorithm, model, training set

- Learning algorithm
  - takes as input a training set $\mathcal{D}^{\text{train}} = \{(\mathbf{x}^{(t)}, y^{(t)})\}$
  - outputs a model $f(\mathbf{x}; \boldsymbol{\theta})$
- We then say the model $f(\mathbf{x}; \boldsymbol{\theta})$ was trained on $\mathcal{D}^{\text{train}}$
  - the model has learned the information present in $\mathcal{D}^{\text{train}}$
- We can now use the model $f(\mathbf{x}; \boldsymbol{\theta})$ on new inputs

MACHINE LEARNING

Topics: training, validation and test sets, generalization

- Training set $\mathcal{D}^{\mathrm{train}}$ serves to train a model
- Validation set $\mathcal{D}^{\mathrm{valid}}$ serves to select hyper-parameters
- Test set $\mathcal{D}^{\mathrm{test}}$ serves to estimate the generalization performance (error)

- Generalization is the behavior of the model on **unseen examples**
  - this is what we care about in machine learning!

MACHINE LEARNING

Topics: capacity of a model, underfitting, overfitting, hyper-parameter, model selection

- Capacity: flexibility of a model
- Hyper-parameter: a parameter of a model that is not trained (specified before training)
- Underfitting: state of model which could improve generalization with more training or capacity
- Overfitting: state of model which could improve generalization with more training or capacity
- Model selection: process of choosing the best hyper-parameters on validation set

MACHINE LEARNING

Topics: capacity of a model, underfitting, overfitting, hyper-parameter, model selection

MACHINE LEARNING

Topics: interaction between training set size/capacity/training time and training error/generalization error

- If capacity increases:
  - training error will ?
  - generalization error will ?
- If training time increases:
  - training error will ?
  - generalization error will ?
- If training set size increases:
  - generalization error will ?
  - difference between the training and generalization error will ?

MACHINE LEARNING

Topics: interaction between training set size/capacity/training time and training error/generalization error

- If capacity increases:
  - training error will decrease
  - generalization error will ?
- If training time increases:
  - training error will ?
  - generalization error will ?
- If training set size increases:
  - generalization error will ?
  - difference between the training and generalization error will ?

MACHINE LEARNING

Topics: interaction between training set size/capacity/training time and training error/generalization error

- If capacity increases:
  - training error will decrease
  - generalization error will increase or decrease
- If training time increases:
  - training error will ?
  - generalization error will ?
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MACHINE LEARNING

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MACHINE LEARNING

Topics: empirical risk minimization, regularization

- Empirical risk minimization
  - framework to design learning algorithms

$$\arg\min_{\boldsymbol{\theta}} \frac{1}{T} \sum_t l(f(\mathbf{x}^{(t)}; \boldsymbol{\theta}), y^{(t)}) + \lambda \Omega(\boldsymbol{\theta})$$

  - $l(f(\mathbf{x}^{(t)}; \boldsymbol{\theta}), y^{(t)})$ is a loss function
  - $\Omega(\boldsymbol{\theta})$ is a regularizer (penalizes certain values of $\boldsymbol{\theta}$)
- Learning is cast as optimization
  - ideally, we'd optimize classification error, but it's not smooth
  - loss function is a surrogate for what we truly should optimize (e.g. upper bound)

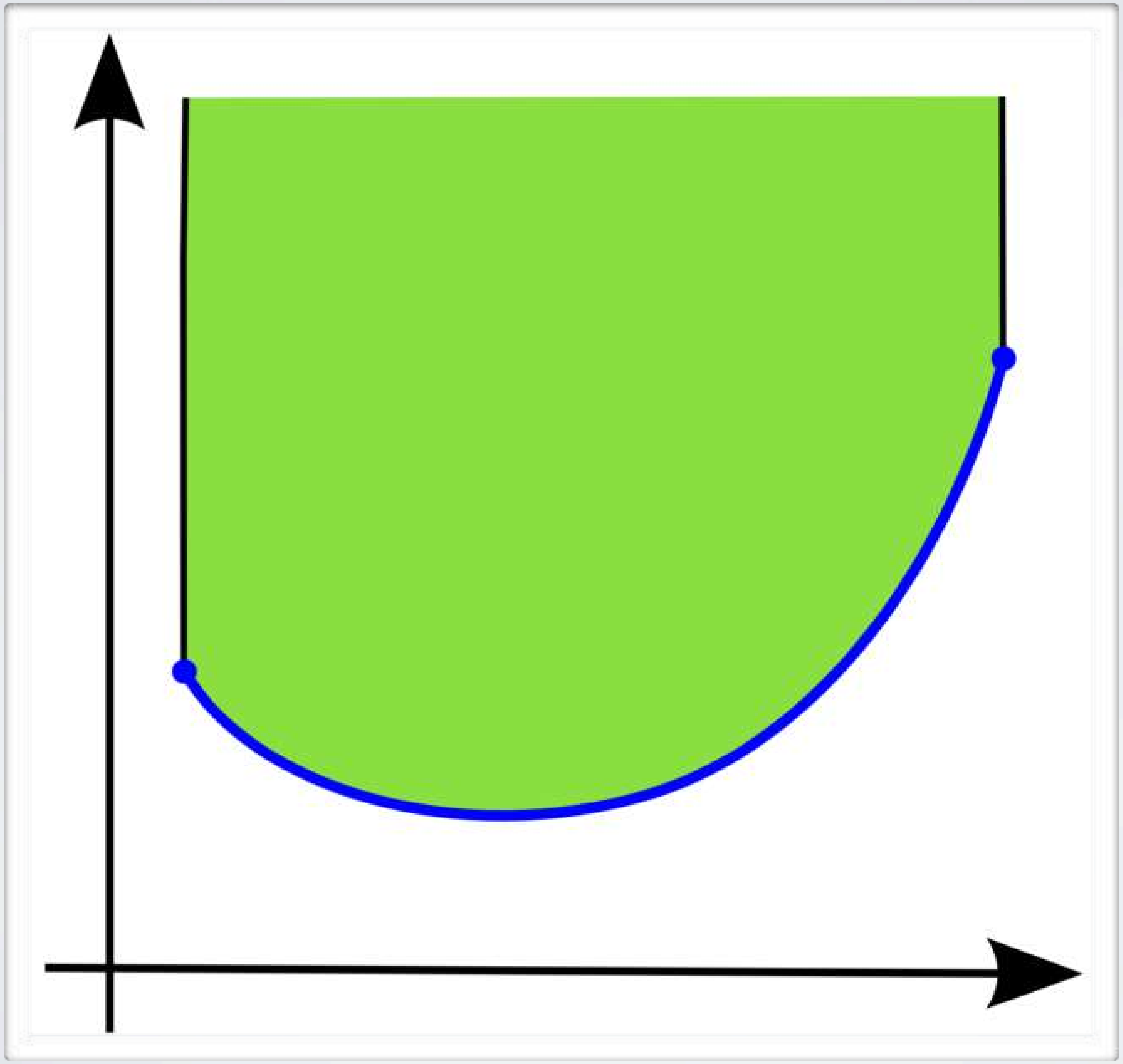
MACHINE LEARNING

Topics: gradient descent

- Gradient descent: procedure to minimize a function
  - compute gradient
  - take step in opposite direction

MACHINE LEARNING

Topics: gradient descent

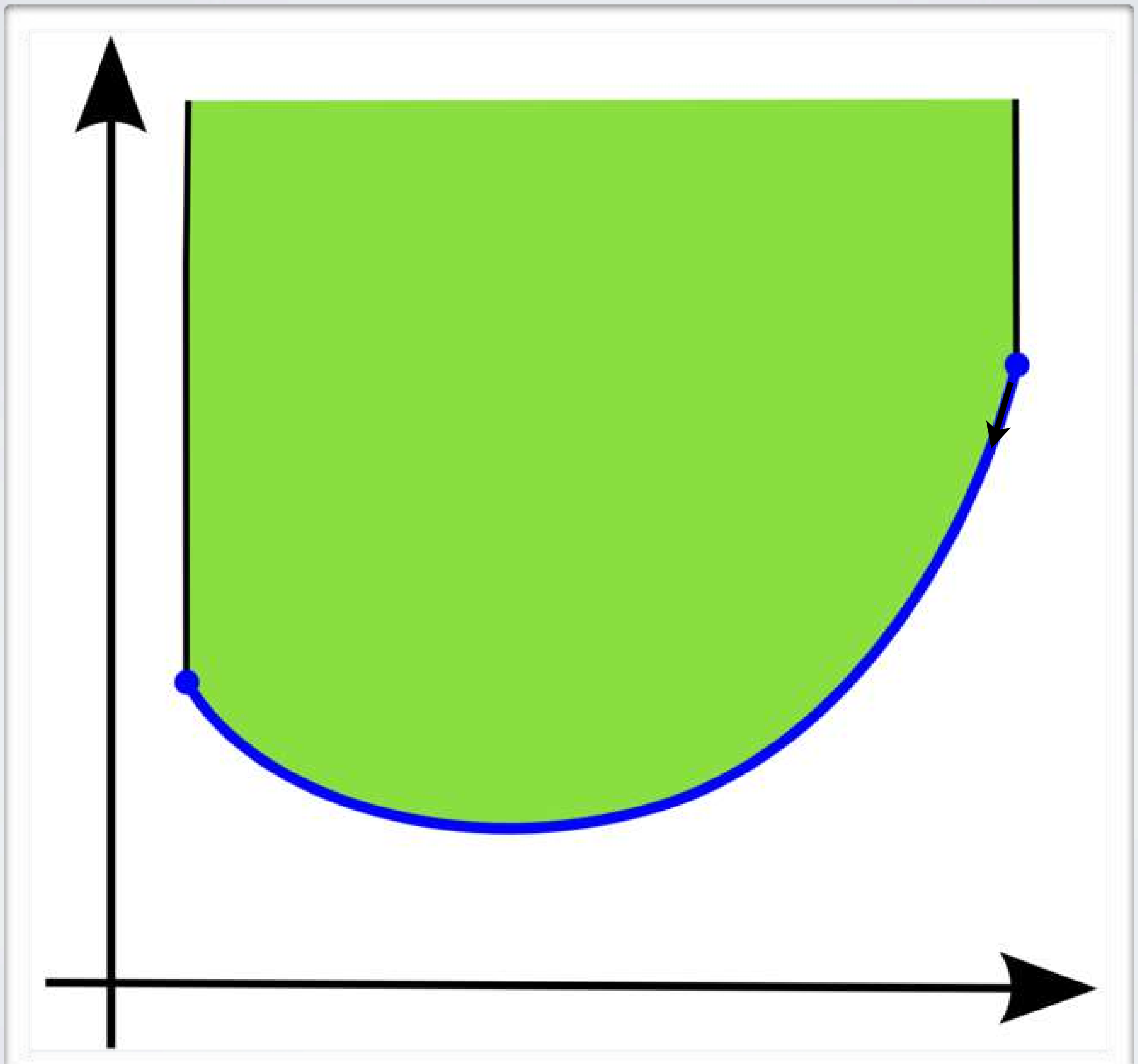


Descent direction

$$-\frac{\partial f(x)}{\partial x}$$

MACHINE LEARNING

Topics: gradient descent



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MACHINE LEARNING

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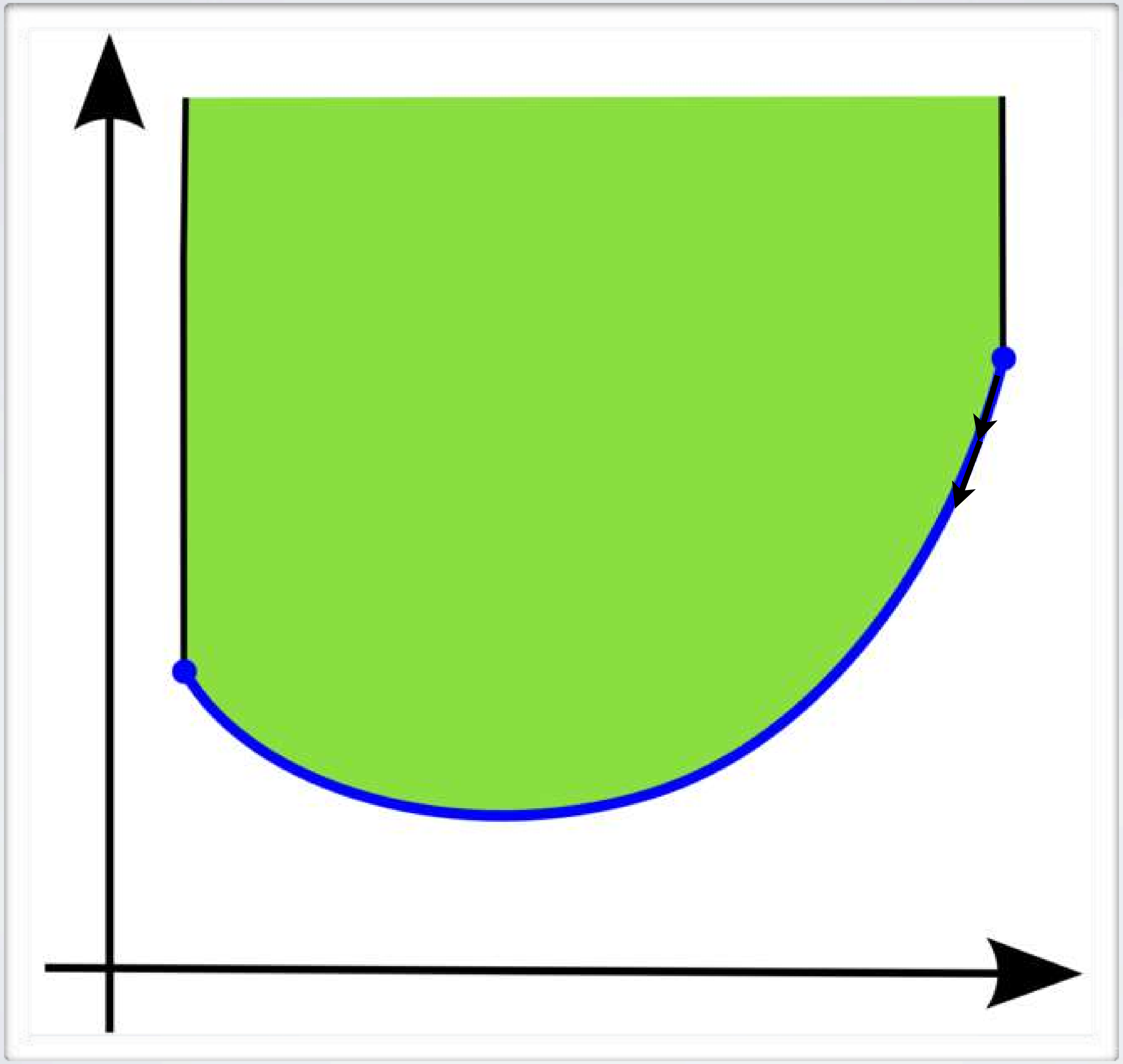

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MACHINE LEARNING

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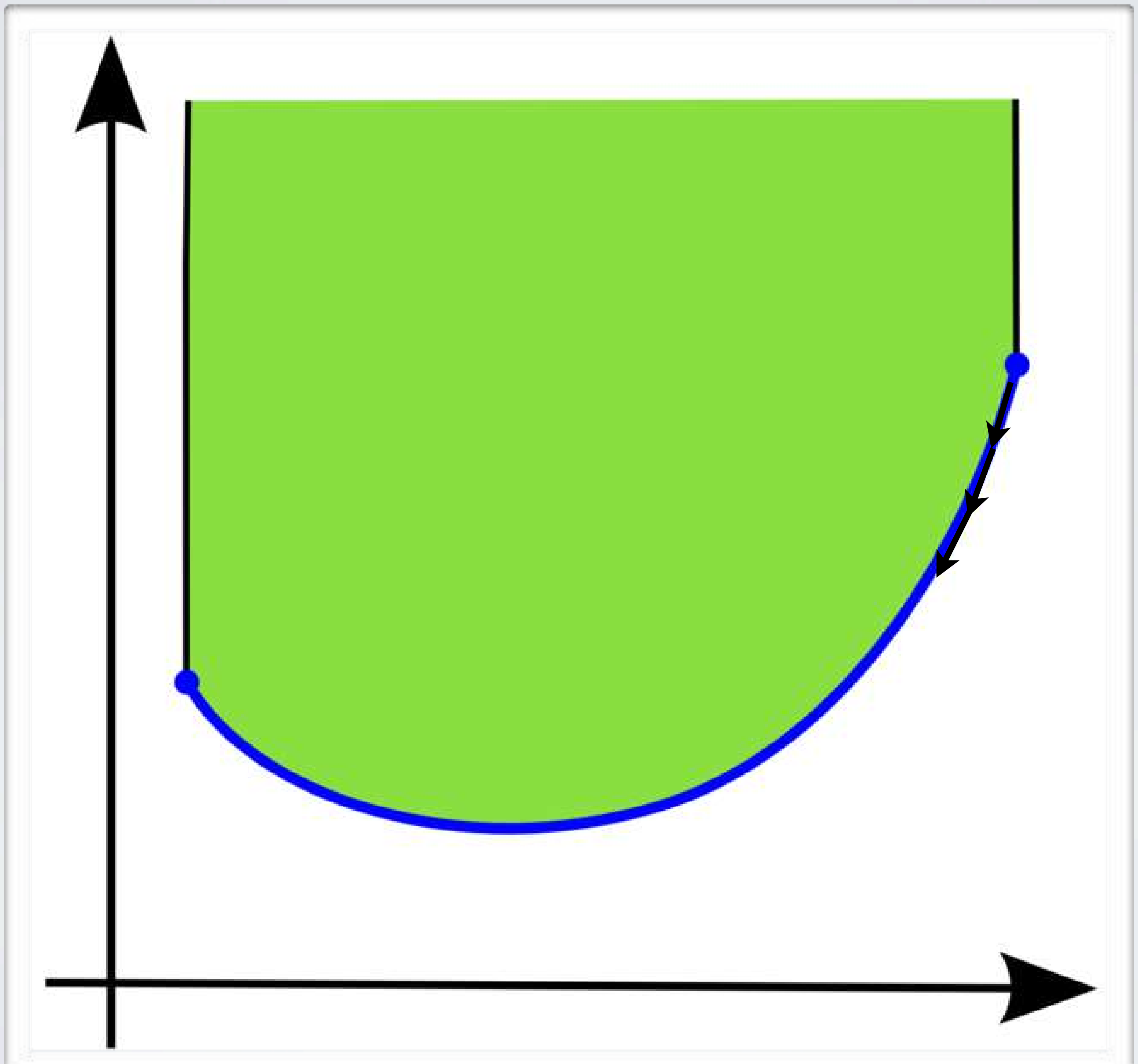

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MACHINE LEARNING

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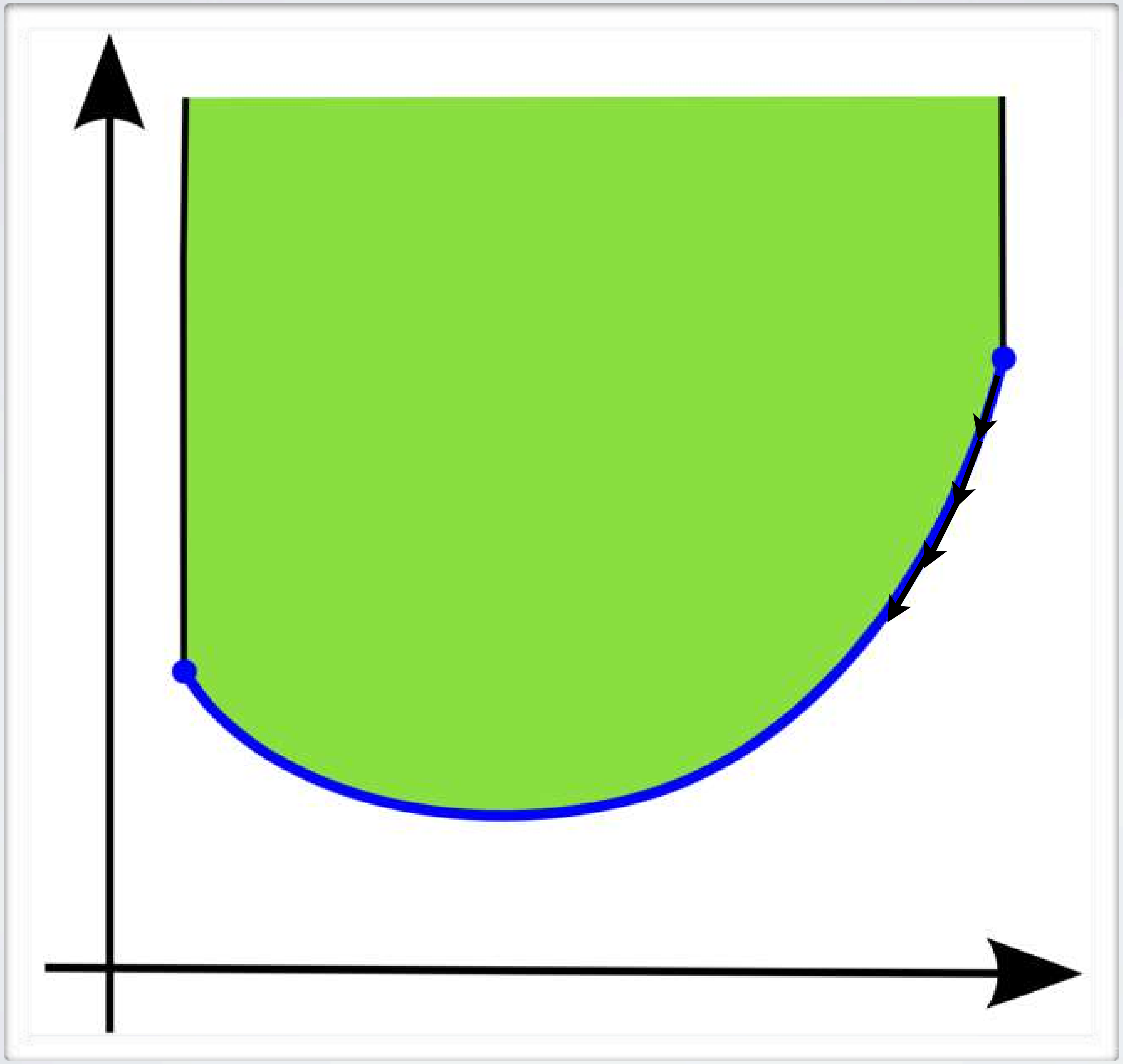

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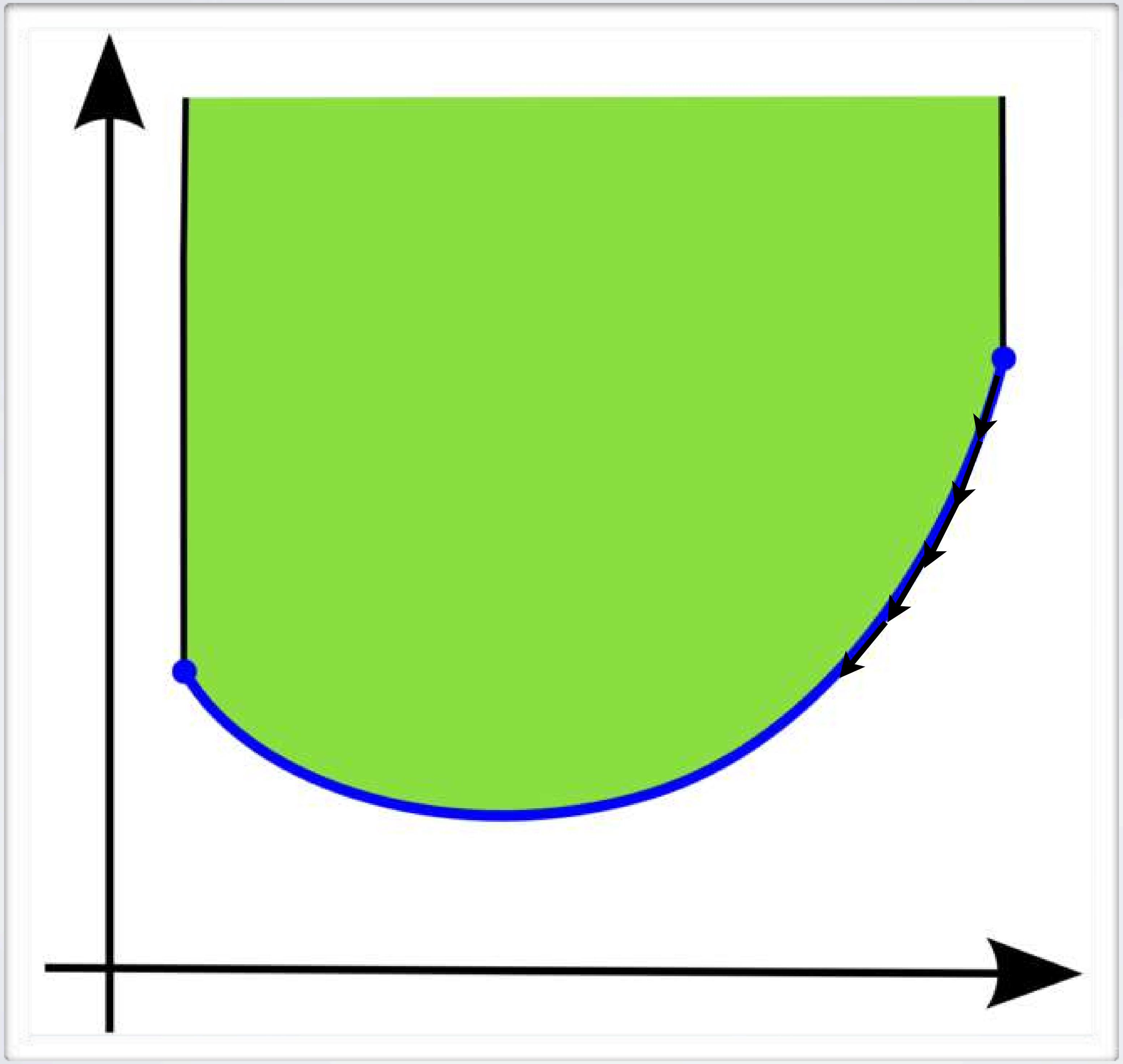
MACHINE LEARNING

Topics: gradient descent

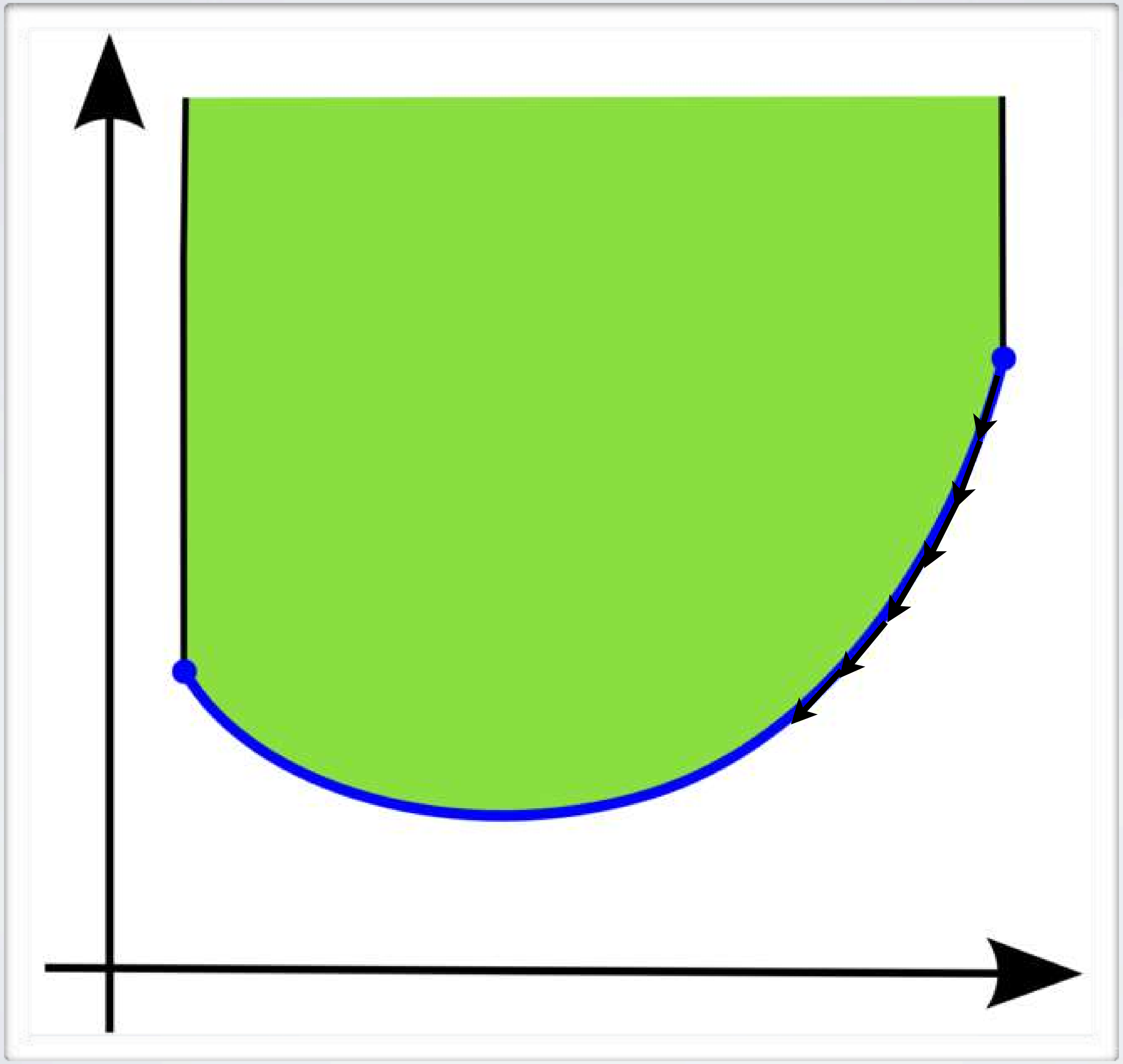


Descent direction

$$-\frac{\partial f(x)}{\partial x}$$

MACHINE LEARNING

Topics: gradient descent



Descent direction

$$-\frac{\partial f(x)}{\partial x}$$

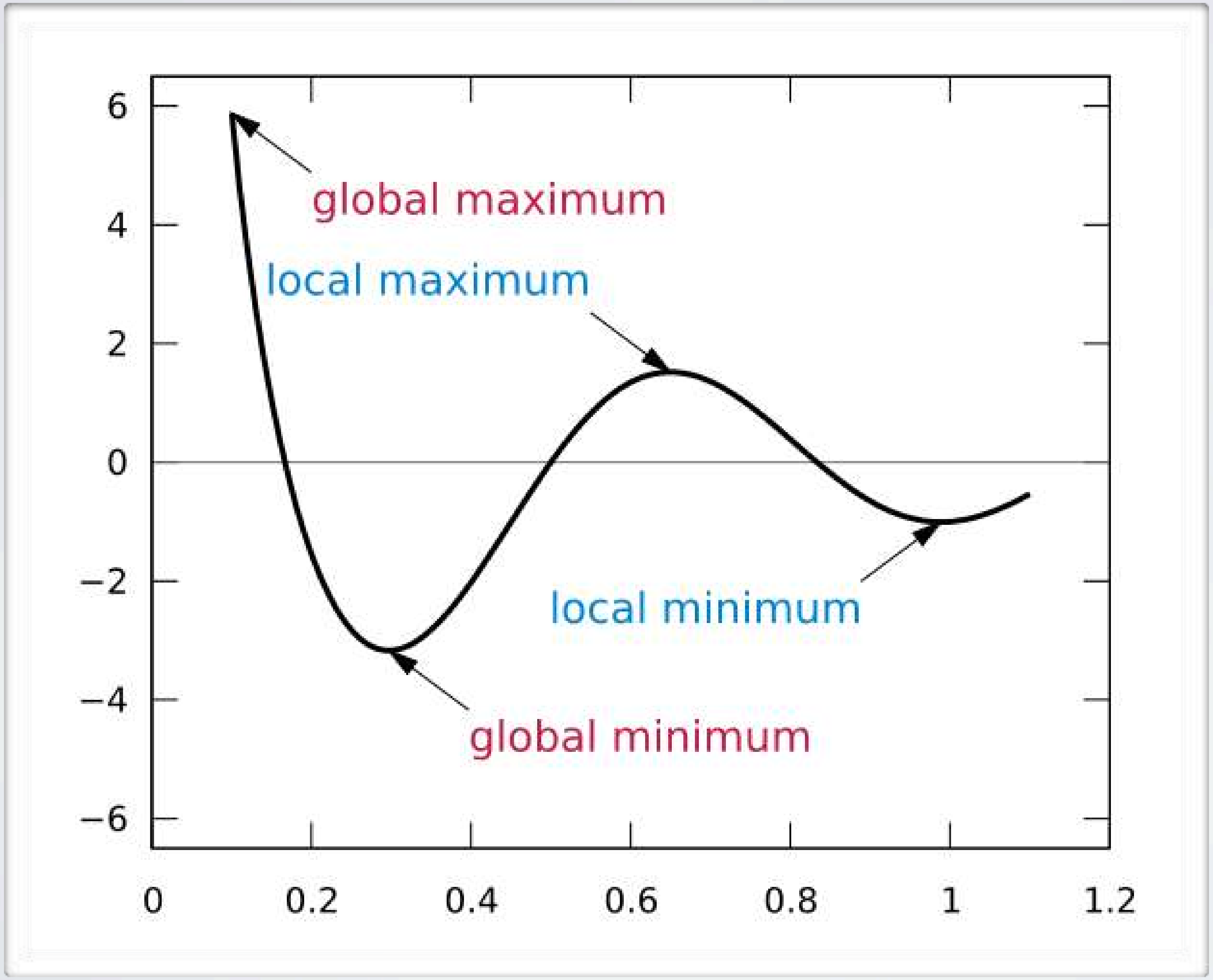
MACHINE LEARNING

Topics: gradient descent

- Gradient descent for empirical risk minimization
  - initialize $\boldsymbol{\theta}$
  - for N iterations
    - $\Delta = -\frac{1}{T}\sum_t \nabla_{\boldsymbol{\theta}} l(f(\mathbf{x}^{(t)};\boldsymbol{\theta}), y^{(t)}) - \lambda \nabla_{\boldsymbol{\theta}} \Omega(\boldsymbol{\theta})$
    - $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\,\Delta$

MACHINE LEARNING

Topics: local and global optima



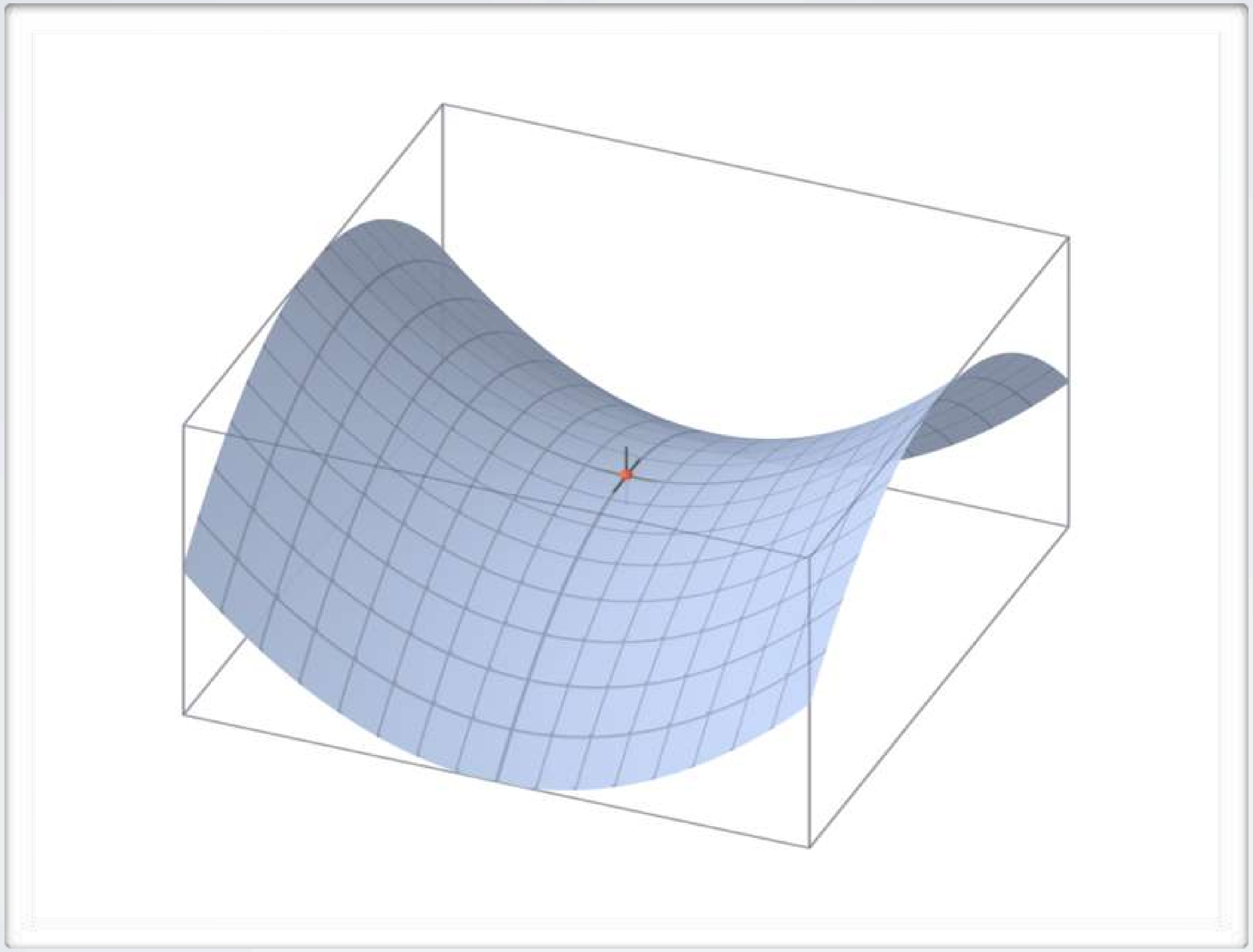
MACHINE LEARNING

Topics: critical points, local optima, saddle point, curvature

- Critical points: $\{\mathbf{x} \in \mathbb{R}^d \mid \nabla_{\mathbf{x}} f(\mathbf{x}) = 0\}$
- Curvature in direction $\mathbf{v}$: $\mathbf{v}^\top \nabla^2_{\mathbf{x}} f(\mathbf{x}) \mathbf{v}$
- Types of critical points:
  - local minima: $\mathbf{v}^\top \nabla^2_{\mathbf{x}} f(\mathbf{x}) \mathbf{v} > 0 \;\; \forall \mathbf{v}$ (i.e. $\nabla^2_{\mathbf{x}} f(\mathbf{x})$ positive definite)
  - local maxima: $\mathbf{v}^\top \nabla^2_{\mathbf{x}} f(\mathbf{x}) \mathbf{v} < 0 \;\; \forall \mathbf{v}$ (i.e. $\nabla^2_{\mathbf{x}} f(\mathbf{x})$ negative definite)
  - saddle point: curvature is positive in certain directions and negative in others

MACHINE LEARNING

Topics: saddle point



MACHINE LEARNING

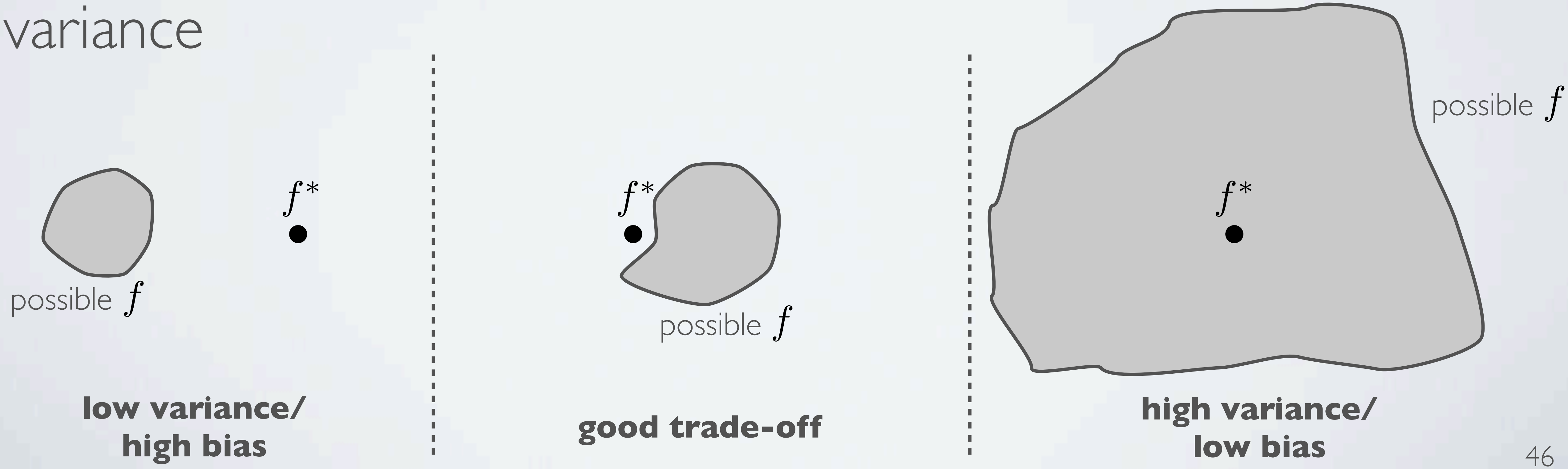
Topics: stochastic gradient descent

- Algorithm that performs updates after each example
  - initialize $\boldsymbol{\theta}$
  - for N iterations
    - for each training example $(\mathbf{x}^{(t)}, y^{(t)})$
      - ✓ $\Delta = -\nabla_{\boldsymbol{\theta}} l(f(\mathbf{x}^{(t)}; \boldsymbol{\theta}), y^{(t)}) - \lambda \nabla_{\boldsymbol{\theta}} \Omega(\boldsymbol{\theta})$
      - ✓ $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\, \Delta$

MACHINE LEARNING

Topics: bias-variance trade-off

- Variance of trained model: does it vary a lot if the training set changes
- Bias of trained model: is the average model close to the true solution
- Generalization error can be seen as the sum of bias and the variance



MACHINE LEARNING

Topics: parametric vs. non-parametric

- Parametric model: its capacity is fixed and does not increase with the amount of training data
  - examples: linear classifier, neural network with fixed number of hidden units, etc.
- Non-parametric model: the capacity increases with the amount of training data
  - examples: k nearest neighbors classifier, neural network with adaptable hidden layer size, etc.