Linear Algebra Done Right Chapter 2 - Solutions

FungusDesu

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Problem 2A.1. Find a list of four distinct vectors in $\mathbb{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}.$$

(1, -2, 1), (4, -3, -1), (2, 3, -5), (-3, 7, -4).

Problem 2A.2. Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

This statement is true. A proof follows.

Proof. Let S be the list v_1, v_2, v_3, v_4, S' be $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$. For any $a_1, a_2, a_3, a_4 \in \mathbb{F}$, choose $b_1 = a_1$, $b_2 = a_1 - a_2$, $b_3 = a_1 - a_2 - a_3$, $b_4 = a_1 - a_2 - a_3 - a_4$, then

$$\begin{aligned}
&b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\
&= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4 \\
&= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.
\end{aligned}$$

Thus $\operatorname{span}(S) \subseteq \operatorname{span}(S')$.

Conversely, for any $b_1, b_2, b_3, b_4 \in \mathbb{F}$, choose $a_1 = b_1$, $a_2 = -b_1 + b_2$, $a_3 = -b_2 + b_3$, $a_4 = -b_3 + b_4$, then

$$\begin{aligned}
a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 &= b_1 v_1 + (-b_1 + b_2)v_2 + (-b_2 + b_3)v_3 + (-b_3 + b_4)v_4 \\
&= b_1 v_1 - b_1 v_2 + b_2 v_2 - b_2 v_3 + b_3 v_3 - b_3 v_4 + b_4 v_4 \\
&= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4.
\end{aligned}$$

Thus $\operatorname{span}(S') \subseteq \operatorname{span}(S)$. Therefore $\operatorname{span}(S') = \operatorname{span}(S) = V$.

Problem 2A.3. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\operatorname{span}(v_1, \ldots, v_m) = \operatorname{span}(w_1, \ldots, w_m)$.

Proof. Let S be the list $v_1, \ldots, v_m$, S' be the list $w_1, \ldots, w_m$. By 2.6, the span of S is the smallest subspace of V, where V is the vector space containing every element of S; thus if every element of S is in the span of S', the subspace containing every element of S is a subset of that of S'. Note that every element of S is in $\operatorname{span}(S')$, as $v_1 = w_1$ and $v_k = w_k - w_{k-1}$ for $k \in \{2, \ldots, m\}$, thus $\operatorname{span}(S) \subseteq \operatorname{span}(S')$. Also note that every element of S' is in $\operatorname{span}(S)$, evident from the definition of w_k. Thus $\operatorname{span}(S') \subseteq \operatorname{span}(S)$, and so $\operatorname{span}(S) = \operatorname{span}(S')$.

Problem 2A.4a. Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

Proof. Let v be the vector in the list. Consider the linear combination av for some $a \in \mathbb{F}$. If $v = 0$, then we are done. If $v \neq 0$, then it must be that $a = 0$ by 1B.2.

Problem 2A.4b. Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

Proof. Let S be the list of vectors v_1, v_2. Suppose that S is linearly independent. Now suppose to the contrary that v_1 is a scalar multiple of v_2 or v_2 is a scalar multiple of v_1. Without loss of generality, we assume the former. Let there be $a \in \mathbb{F}$ for which $v_1 = av_2$; note that $v_1 - av_2 = 0$, which contradicts the fact that S is linearly independent.

Conversely, suppose to the contrapositive that S is linearly dependent. By 2.19, either $v_1 = 0 = 0v_2$ or $v_2 = av_1$, completing the proof.

Problem 2A.5. Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbb{R}^3$.

$t = 2$.

Problem 2A.6. Show that the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbb{F}^3$ if and only if $c = 8$.

Proof. Denote by S the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$. Suppose $c = 8$. Then there exists $a_1, a_2, a_3 \in \mathbb{F}$ that are not all 0 for which

$$a_1(2, 3, 1) + a_2(1, -1, 2) + a_3(7, 3, 8) = 0,$$

i.e. $a_1 = 2$, $a_2 = 3$ and $a_3 = -1$.

Conversely, suppose S is linearly dependent in $\mathbb{F}^3$. By linear dependence lemma, $(7, 3, c)$ is in the span of the list $(2, 3, 1), (1, -1, 2)$ (the first element is not a zero vector; the second element is not a scalar multiple of the first (2A.4B)). Thus there exists $a_1, a_2 \in \mathbb{F}$ for which

$$a_1(2, 3, 1) + a_2(1, -1, 2) = (7, 3, c). \tag{Γ}$$

We now seek such a_1, a_2. Solving the linear system of equation

$$\begin{cases} 2a_1 + a_2 = 7 \\ 3a_1 - a_2 = 3 \end{cases}$$

yields $a_1 = 2$, $a_2 = 3$. Substituting this into (Γ) yields $(7, 3, c) = (7, 3, 8)$, completing the proof.

Problem 2A.7a. Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $1 + i, 1 - i$ is linearly independent.

Proof. For $a_1, a_2 \in \mathbb{R}$, we solve the following equation

$$\begin{aligned} a_1(1+i) + a_2(1-i) &= 0 \\ a_1 + a_1 i + a_2 - a_2 i &= 0 \\ (a_1 + a_2) + (a_1 - a_2)i &= 0. \end{aligned}$$

Thus the equation has precisely one solution, namely $a_1 = 0$ and $a_2 = 0$. We note that this makes sense because $a_1, a_2 \in \mathbb{R}$.

> **Problem 2A.7b.** Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $1+i, 1-i$ is linearly dependent.

Proof. Observe that for $a_1, a_2 \in \mathbb{C}$, the equation

$$a_1(1+i) + a_2(1-i) = 0$$

has a solution $a_1 = i$, $a_2 = 1$.

> **Problem 2A.8.** Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list
>
> $$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$
>
> is also linearly independent.

Proof. Consider the following equation

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0.$$

The linear independence of the list $v_1, \ldots, v_m$ forces

$$a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0.$$

From $a_1 = 0$, we get $a_2 - a_1 = a_2 = 0$, and similarly $a_3 = 0$ and $a_4 = 0$. This shows linear independence of $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$.

> **Problem 2A.9.** Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vetors in V, then
>
> $$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$
>
> is linearly independent.

This statement is true. A proof follows.

Proof. For $a_1, \ldots, a_m \in \mathbb{F}$, consider the equation

$$a_1(5v_1 - 4v_2) + a_2 v_2 + \cdots + a_m v_m = 5a_1 v_1 + (a_2 - 4a_1)v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

The linear independence of the list $v_1, \ldots, v_m$ forces

$$5a_1 = a_2 - 4a_1 = a_3 = \cdots = a_m = 0.$$

From $5a_1 = 0$, we get $a_1 = 0$, and thus $a_2 - 4a_1 = a_2 = 0$. This shows linear independence of the list $5v_1 - 4v_2, v_2, \ldots, v_m$.

Problem 2A.10. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

This statement is true. A proof follows.

Proof. For $a_1, \ldots, a_m \in \mathbb{F}$, consider the equation

$$a_1(\lambda v_1) + \cdots + a_m(\lambda v_m) = (\lambda a_1)v_1 + \cdots + (\lambda a_m)v_m = 0.$$

The linear independence of the list $v_1, \ldots, v_m$ forces

$$\lambda a_1 = \cdots = \lambda a_m = 0.$$

From $\lambda a_1 = 0$, we get $a_1 = 0$, and similarly $a_2 = a_3 = \cdots = a_m = 0$. This shows linear independence of the list $\lambda v_1, \ldots, \lambda v_m$.

Problem 2A.11. Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

This statement is false by the following counterexample. The lists $(1, 2), (3, 1)$ and $(1, 2), (1, 7)$ are both linear independent in $\mathbb{R}^2$, however the list of sums of each list's respective elements $(2, 4), (4, 8)$ is not linear independent.

Problem 2A.12. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Proof. Suppose to the contrapositive that $w \notin \operatorname{span}(v_1, \ldots, v_m)$. Because of this and the linear independence of $v_1, \ldots, v_m$, the list $v_1, \ldots, v_m, w$ is also linearly independent by linear dependence lemma. For $a_1, \ldots, a_m \in \mathbb{F}$, consider the equation

$$a_1(v_1 + w) + \cdots + a_m(v_m + w) = a_1v_1 + \cdots + a_mv_m + (a_1 + \cdots + a_m)w = 0.$$

The linear independence of $v_1, \ldots, v_m, w$ forces $a_1 = \cdots = a_m = a_1 + \cdots + a_m = 0$, and therefore guarantees the linear independence of $v_1 + w, \ldots, v_m + w$.

Problem 2A.13. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \ldots, v_m, w \text{ is linearly independent} \iff w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Proof. Suppose $w \notin \operatorname{span}(v_1, \ldots, v_m)$. The linear independence of $v_1, \ldots, v_m$ tells us that none of the vectors belong to the span of the previous ones. Because of this and w also not belonging to its span, the list $v_1, \ldots, v_m, w$ is also linearly independent by the contrapositive of linear dependence lemma.

Conversely, suppose to the contrapositive that $w \in \operatorname{span}(v_1, \ldots, v_m)$. Then w can be written as a linear combination of $v_1, \ldots, v_m$, and therefore $v_1, \ldots, v_m, w$ is linearly dependent.

Problem 2A.14. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that the list $v_1, \ldots, v_m$ is linearly independent if and only if the list $w_1, \ldots, w_m$ is linearly independent.

Proof. Denote by S, S' the lists $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ respectively. Suppose S is linearly independent. For $a_1, \ldots, a_m \in \mathbb{F}$, consider the equation

$$\begin{aligned} a_1w_1 + \cdots + a_mw_m &= a_1v_1 + a_2(v_1 + v_2) + \cdots + a_m(v_1 + \cdots + v_m) \\ &= (a_1 + \cdots + a_m)v_1 + (a_2 + \cdots + a_m)v_2 + \cdots + a_mv_m \\ &= 0. \end{aligned}$$

Because S is linearly independent, we have $a_1 + \cdots + a_m = \cdots = a_m = 0$. The equation $a_m = 0$ forces $a_{m-1} = 0$, and repeats until $a_1 = 0$. Thus $a_1 = \cdots = a_m = 0$. This shows the linear independence of S'.

Conversely, suppose S' is linearly independent. For $a_1, \ldots, a_m \in \mathbb{F}$, consider the equation

$$\begin{aligned} a_1v_1 + \cdots + a_mv_m &= a_1w_1 + a_2(w_2 - w_1) + \cdots + a_m(w_m - w_{m-1}) \\ &= (a_1 - a_2)w_1 + \cdots + (a_{m-1} - a_m)w_{m-1} + a_mw_m. \end{aligned}$$

Because S is linearly independent, we have $a_1 + \cdots + a_m = \cdots = a_m = 0$. The equation $a_m = 0$ forces $a_{m-1} = 0$, and repeats until $a_1 = 0$. Thus $a_1 = \cdots = a_m = 0$. This shows the linear independence of S.

Problem 2A.15. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbb{F})$.

Because $\mathcal{P}_4(\mathbb{F}) = \text{span}(1, z, z^2, z^3, z^4)$, and no list of independent vectors in a vector space is longer than its spanning list (2.22).

Problem 2A.16. Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbb{F})$.

Because $1, z, z^2, z^3, z^4$ is linearly independent in $\mathcal{P}_4(\mathbb{F})$, and no list of independent vectors in a vector space is longer than its spanning list (2.22).

Problem 2A.17. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Proof. ($\Longleftarrow$) Suppose there exists a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m. Suppose to the contrary that V is finite-dimensional. Then there exists $v_1, \ldots, v_k$ for which $V = \text{span}(v_1, \ldots, v_k)$. But because there also exists linearly independent list $v_1, \ldots, v_{k+1}$, we arrived to a contradiction where the length of a linearly independent list in V is longer than the length of a list spanning V. This shows the infinite-dimensionality of V.

($\Longrightarrow$) Conversely, suppose that V is infinite-dimensional. Thus there does not exist a list of vectors in V spanning it. We consider an arbitrary list S consisting of vectors $v_1, \ldots, v_k$, which is linearly independent. Since this list does not span V, we choose a vector $v_{k+1} \notin \text{span}(S)$ and append it to S. Thus for each $k \in \mathbb{N}$, there exists a list of vectors $v_1, \ldots, v_k$ which is linearly independent.

Problem 2A.18. Prove that $\mathbb{F}^\infty$ is infinite-dimensional.

Proof. Suppose to the contrary $\mathbb{F}^\infty$ is finite-dimensional. Thus there exists a spanning list of $\mathbb{F}^\infty$. Let k be its length. Note that the list

$$(1, 0, 0, \ldots), (0, 1, 0, \ldots), \ldots, (\underbrace{0, \ldots}_{k \text{ times}}, 1, 0, \ldots)$$

is a linearly independent list in $\mathbb{F}^\infty$, but its length is longer than its spanning list. Thus we arrive to a contradiction (2.22).

Problem 2A.19. Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

Proof. By 2.14, we know that $\mathcal{P}(\mathbb{F})$ is infinite-dimensional. Note that $\mathcal{P}(\mathbb{F}) \subseteq \{f \in \mathbb{R}^{[0,1]}, f \text{ is continuous}\}$, thus $\{f \in \mathbb{R}^{[0,1]}, f \text{ is continuous}\}$ is also infinite-dimensional by the contrapositive of 2.25.

Problem 2A.20. Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$.

Proof. Suppose to the contrary that $p_0, p_1, \ldots, p_m$ is linearly independent in $\mathcal{P}_m(\mathbb{F})$. Denote by p_{-1} the polynomial 1. Let $a_i \in \mathbb{F}$ for $i \in \{-1, 0, \ldots, m\}$. Consider the sum

$$a_{-1}p_{-1} + a_0 p_0 + \cdots + a_m p_m = 0$$

Substituting $x = 2$, we get

$$a_{-1}p_{-1}(2) + a_0 p_0(2) + \cdots + a_m p_m(2) = a_{-1} \cdot 1 + a_1 \cdot 0 + \cdots + a_m \cdot 0 = a_{-1} = 0.$$

This makes sense because $p_k(2) = 0$ for $k \in \{0, \ldots, m\}$. Thus

$$a_{-1}p_{-1} + a_0 p_0 + \cdots + a_m p_m = a_0 p_0 + \cdots + a_m p_m = 0$$

The linear independence of $p_0, \ldots, p_m$ forces $a_0 = \cdots = a_m = 0$, and consequently the $(m+2)$-list $p_{-1}, p_0, \ldots, p_m$ is linearly independent in $\mathcal{P}_m(\mathbb{F})$. Note that $\mathcal{P}_m(\mathbb{F}) = \operatorname{span}(1, x, \ldots, x^m)$. Thus it is a contradiction that the length of a linearly independent list is longer than a spanning list in $\mathcal{P}_m(\mathbb{F})$.

Problem 2B.1. Find all vector spaces that have exactly one basis.

$\{0\}$

Problem 2B.2a. The list $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbb{F}^n$.

Proof. For $a_1, \ldots, a_n \in \mathbb{F}$, consider the sum

$$\begin{aligned} a_1(1, 0, \ldots, 0) + a_2(0, 1, 0, \ldots, 0) + \cdots + a_n(0, \ldots, 0, 1) &= (a_1, 0, \ldots, 0) + \cdots + (0, \ldots, 0, a_n) \\ &= (a_1, \ldots, a_n) = 0. \end{aligned}$$

For the sum to be true, $a_1 = \cdots = a_n = 0$ and thus the list $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is linearly independent. The above sum also shows that for $(a_1, \ldots, a_n) \in \mathbb{F}^n$,

$$(a_1, \ldots, a_n) = a_1(1, 0, \ldots, 0) + a_2(0, 1, 0, \ldots, 0) + \cdots + a_n(0, \ldots, 0, 1),$$

and so the list $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ spans $\mathbb{F}^n$. Therefore it is a basis of $\mathbb{F}^n$.

Problem 2B.2b. The list $(1, 2), (3, 5)$ is a basis of $\mathbb{F}^2$.

Proof. The vector $(1, 2)$ is not a scalar multiple of $(3, 5)$ and vice versa, thus the list $(1, 2), (3, 5)$ is linearly independent by 2A.4b. Suppose $(x, y) \in \mathbb{F}^2$. Observe that

$$(x, y) = (-5x + 3y)(1, 2) + (2x - y)(3, 5).$$

Thus $(1, 2), (3, 5)$ spans $\mathbb{F}^2$ and consequently is a basis of $\mathbb{F}^2$.

Problem 2B.2c. The list $(1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbb{F}^3$ but is not a basis of $\mathbb{F}^3$ because it does not span $\mathbb{F}^3$.

Proof. The nonzero vector $(1, 2, -4)$ is not a scalar multiple of the nonzero vector $(7, -5, 6)$, thus the list $(1, 2, -4), (7, -5, 6)$ is linearly independent by 2A.4b. Consider the vector $(1, 2, 1) \in \mathbb{F}^3$. Observe that

$$\left[\begin{array}{cc|c} 1 & 7 & 1 \\ 2 & -5 & 2 \\ -4 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 7 & 1 \\ 0 & -19 & 0 \\ 0 & 34 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{array}\right].$$

Thus there exists no linear combination of $(1, 2, -4), (7, -5, 6)$ to yield $(1, 2, 1)$, and so it does not span $\mathbb{F}^3$.

Problem 2B.2d. The list $(1, 2), (3, 5), (4, 13)$ spans $\mathbb{F}^2$ but is not a basis of $\mathbb{F}^2$ because it is not linearly independent.

Proof. Suppose $(x, y) \in \mathbb{F}^2$. Observe that

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & x \\ 2 & 5 & 13 & y \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & x \\ 0 & -1 & 5 & y - 2x \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 19 & -5x + 3y \\ 0 & 1 & -5 & 2x - y \end{array}\right].$$

This shows that for each $(x, y) \in \mathbb{F}^2$, there exists a linear combination that yields (x, y) and thus the list $(1, 2), (3, 5), (4, 13)$ spans $\mathbb{F}^2$. It however is not linearly independent, as there exists a nontrivial solution for

$$a(1, 2) + b(3, 5) + c(4, 13) = 0,$$

i.e. $a = -19, b = 5, c = 1$.

Problem 2B.2e. The list $(1, 1, 0), (0, 0, 1)$ is a basis of $\{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$.

Proof. The list $(1, 1, 0), (0, 0, 1)$ evidently has all elements belonging to $\{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$; also note that the list is also linearly independent as the sum

$$a(1, 1, 0) + b(0, 0, 1) = (a, a, b) = 0$$

is true if and only if $a = b = 0$. Suppose $(x, x, y) \in \{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$. Observe that

$$(x, x, y) = x(1, 1, 0) + y(0, 0, 1).$$

Thus the list $(1, 1, 0), (0, 0, 1)$ also spans $\{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$, and consequently is its basis.

Problem 2B.2f. The list $(1, -1, 0), (1, 0, -1)$ is a basis of

$$\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}.$$

Proof. The list $(1, -1, 0), (1, 0, -1)$ evidently has all elements belonging to $\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$. Consider the sum

$$a(1, -1, 0) + b(1, 0, -1) = (a + b, -a, -b) = 0.$$

The nonexistence of the sum's nontrivial solution shows the linear independence of $(1, -1, 0), (1, 0, -1)$. The list is also a spanning list, for each $(x, y, z) \in \{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$ can be written as

$$(x, y, z) = (x, y, -x - y) = (-y)(1, -1, 0) + (x + y)(1, 0, -1).$$

Thus $(1, -1, 0), (1, 0, -1)$ is a basis of $\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$.

Problem 2B.2g. The list $1, z, \dots, z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Proof. Note that any polynomial of degree m in $\mathbb{F}$ can be written as

$$a_0 + a_1 z + \cdots + a_m z^m = a_0(1) + a_1(z) + \cdots + a_m(z^m).$$

Thus $1, z, \dots, z^m$ is a spanning list of $\mathcal{P}_m(\mathbb{F})$. Also note that the solution to

$$a_0 + a_1 z + \cdots + a_m z^m = 0$$

is precisely $a_0 = a_1 = \cdots = a_m = 0$. Thus $1, z, \dots, z^m$ is linearly independent and is consequently a basis.

Problem 2B.3.

(a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

(a) Observe that the list $(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)$ is a basis of U. The solution to

$$a(3,1,0,0,0) + b(0,0,7,1,0) + c(0,0,0,0,1) = (3a, a, 7b, b, c) = 0$$

is precisely $a = b = c = 0$, thus shows the linear independence of $(1,3,0,0,0), (0,0,1,7,0), (0,0,0,0,1)$. It also spans U, for arbitrary $(3a, a, 7b, b, c) \in U$, we have

$$(3a, a, 7b, b, c) = a(3,1,0,0,0) + b(0,0,7,1,0) + c(0,0,0,0,1).$$

(b) This list can be extended to a basis of $\mathbb{R}^5$, i.e.

$$(3,1,0,0,0), (1,0,0,0,0), (0,0,7,1,0), (0,0,1,0,0), (0,0,0,0,1).$$

The list is linearly independent; to see why, we consider the sum

$$a(3,1,0,0,0) + b(1,0,0,0,0) + c(0,0,7,1,0) + d(0,0,1,0,0) + e(0,0,0,0,1) = (3a + b, a, 7c + d, c, e) = 0.$$

The fact that $a = 0$ implies $3a + b = b = 0$, $c = 0$ implies $7c + d = d = 0$. Thus the only solution to the sum is $a = b = c = d = e = 0$. The list also spans $\mathbb{R}^5$; given arbitrary $(a, b, c, d, e) \in \mathbb{R}^5$, we have

$$(a, b, c, d, e) = b(3,1,0,0,0) + (a - 3b)(1,0,0,0,0) + d(0,0,7,1,0) + (c - 7d)(0,0,1,0,0) + e(0,0,0,0,1).$$

(c) A subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$ is $\{(x, 0, y, 0, 0) \in \mathbb{R}^5\}$. To see why, observe that $(3,1,0,0,0)$, $(0,0,7,1,0)$, $(0,0,0,0,1)$ is a basis of U, and $(1,0,0,0,0)$, $(0,0,1,0,0)$ is a basis of W, thus $U + W$ contains the list $(3,1,0,0,0)$, $(0,0,7,1,0)$, $(0,0,0,0,1)$, $(1,0,0,0,0)$, $(0,0,1,0,0)$, whose span is $\mathbb{R}^5$, and so $\mathbb{R}^5 \subseteq U + W$. Therefore $U + W = \mathbb{R}^5$ ($U + W \subseteq \mathbb{R}^5$ is trivially true by 1.40). Now suppose $v \in U \cap W$. Then there exist u_1, u_2, u_3, v_1, v_2 such that

$$v = (3u_1, u_1, 7u_2, u_2, u_3) = (v_1, 0, v_2, 0, 0).$$

Since $u_1 = 0$, we get $3u_1 = 0 = v_1$; similarly, $u_2 = 0$ implies $7u_2 = 0 = v_2$. Thus $v = 0$ and so $\mathbb{R}^5 = U \oplus W$.

Problem 2B.4.

(a) Let U be the subspace of $\mathbb{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbb{C}^5$.

(c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

(a) A basis of U is $(1, 6, 0, 0, 0), (0, 0, 1, 1, -1), (0, 0, 1, -2, 1)$. To see why, observe that the list is linearly independent as the sum

$$a(1, 6, 0, 0, 0) + b(0, 0, 1, 1, -1) + c(0, 0, 1, -2, 1) = (a, 6a, b + c, b - 2c, -b + c) = 0$$

has no nontrivial solution. The list also spans U; for arbitrary element $(z_1, z_2, z_3, z_4, z_5) \in U$, we can rewrite it as

$$(z_1, z_2, z_3, z_4, z_5) = z_1(1, 6, 0, 0, 0) + \frac{2z_3 + z_4}{3}(0, 0, 1, 1, -1) + \frac{z_3 - z_4}{3}(0, 0, 1, -2, 1).$$

(b) The basis of U can be extended into a basis of $\mathbb{C}^5$, i.e. $(1, 6, 0, 0, 0)$, $(0, 0, 1, 1, -1)$, $(0, 0, 1, -2, 1)$, $(0, 1, 0, 0, 0)$, $(0, 0, 0, 0, 1)$. To see why, observe that the list is linearly independent as the sum

$$\begin{aligned} &a(1, 6, 0, 0, 0) + b(0, 0, 1, 1, -1) + c(0, 0, 1, -2, 1) + d(0, 1, 0, 0, 0) + e(0, 0, 0, 0, 1) \\ &= (a, 6a + d, b + c, b - 2c, -b + c + e) \end{aligned}$$

has no trivial solution. The list also spans $\mathbb{C}^5$; for arbitrary element $(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5$, we can rewrite it as

$$\begin{aligned} (z_1, z_2, z_3, z_4, z_5) &= z_1(1, 6, 0, 0, 0) + \frac{2z_3 + z_4}{3}(0, 0, 1, 1, -1) \\ &\quad + \frac{z_3 - z_4}{3}(0, 0, 1, -2, 1) + (z_2 - 6z_1)(0, 1, 0, 0, 0) + \frac{z_3 + 2z_4 + 3z_5}{3}(0, 0, 0, 0, 1). \end{aligned}$$

(c) A subspace W of $\mathbb{C}^5$ for which $\mathbb{C}^5 = U \oplus W$ is $W = \{(0, z_1, 0, 0, z_2) \in \mathbb{C}^5\}$ (we know this because by the proof of 2.33, we just need to set $W = \operatorname{span}((0, 1, 0, 0, 0), (0, 0, 0, 0, 1)) = \{(0, z_1, 0, 0, z_2) \in \mathbb{C}^5\}$).

Problem 2B.5. Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

Proof. Since V is finite-dimensional, so are its subspaces U and W by 2.25. Thus U and W have a basis by 2.31. Let the lists $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$ be a basis of U and W respectively. Because V is the sum of U and W, any vector in V can be written as a linear combination of bases of U and W; in particular, for arbitrary $v \in V$, there exists $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n.$$

Thus the list $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V. By 2.30, this list can be reduced to a basis of V, completing our proof.

Problem 2B.6. Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathcal{P}_3(\mathbb{F})$.

This statement is false due to the following counterexample. Consider the list $1, z, z^3 + z^2, z^3$ in $\mathcal{P}_3(\mathbb{F})$. It is linearly independent; the sum

$$a_0 \cdot 1 + a_1 z + a_2(z^3 + z^2) + a_3 z^3 = a_0 + a_1 z + a_2 z^2 + (a_2 + a_3)z^3 = 0$$

has no trivial solution. The list also spans $\mathcal{P}_3(\mathbb{F})$; for typical element $a_0 + a_1 z + a_2 z^2 + a_3 z_3$, we can rewrite it as

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 = a_0 \cdot 1 + a_1 z + a_2(z^3 + z^2) + (a_3 - a_2)z^3.$$

Problem 2B.7. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Proof. Observe that the list $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent; consider the sum

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4 = a_1 v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4 = 0.$$

The linear independence of v_1, v_2, v_3, v_4 forces $a_1 = 0$ and thus $a_1 + a_2 = 0 = a_2$. Similarly we can show that $a_3 = a_4 = 0$. Thus the sum has no nontrivial solution. The list also spans V; since v_1, v_2, v_3, v_4 does, we can rewrite a typical element $v \in V$ as

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2 + a_1)(v_3 + v_4) + (a_4 - a_3 + a_2 - a_1)v_4. \end{aligned}$$

Thus $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is a basis of V.

Problem 2B.8. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

This statement is false due to the following counterexample. Consider the list $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ in $\mathbb{R}^4$. The set $U = \{(x, y, z, z) \in \mathbb{R}^4\}$ is a subspace of $\mathbb{R}^4$ (whose proof is left as mental gymnastics for the reader). Note that $(1, 0, 0, 0), (0, 1, 0, 0) \in U$, but $(0, 0, 1, 0), (0, 0, 0, 1) \notin U$, yet $(1, 0, 0, 0), (0, 1, 0, 0) \in U$ is not a basis of U.

Problem 2B.9. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $v_1, \ldots, v_m$ is a basis of V if and only if $w_1, \ldots, w_m$ is a basis of V.

Proof. Suppose $v_1, \ldots, v_m$ is a basis of V. The list $w_1, \ldots, w_m$ is linearly independent in V by 2A.14 and also spans V by 2A.3 and thus is a basis of V. The converse can also be similarly proven, completing our proof.

Problem 2B.10. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof. By the proof of 2B.5, the list $u_1, \dots, u_m, w_1, \dots, w_n$ spans V. Since V is also a direct sum of U and W, no vectors of one subspace is a member of the another, save for the trivial vector; and so none of the basis vectors of one subspace can be constructed with a linear combination of basis vectors of the another. Thus the list is linearly independent by linear dependence lemma and consequently is a basis of V.

Problem 2C.1. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, all lines in $\mathbb{R}^2$ containing the origin, and $\mathbb{R}^2$.

Proof. We first show that $\{0\}$, $\{(ax, by) \in \mathbb{R}^2 : x, y \in \mathbb{R}\}$ and $\mathbb{R}^2$ are indeed subspaces of $\mathbb{R}^2$ for all $a, b \in \mathbb{R}$. The first and third set are trivial subspaces of $\mathbb{R}^2$. Observe that the set $= \{(ax, by) \in \mathbb{R}^2 : x, y \in \mathbb{R}\}$ satisfies these properties:

Additive identity. The vector $0 = (0, 0)$ is an element of , as $0 = (0, 0) = (a \cdot 0, b \cdot 0)$.

Closure under addition. Suppose $(a_1x, b_1y), (a_2x, b_2y) \in$. Observe that their sum

$$(ax_1, by_1) + (ax_2, by_2) = (a(x_1 + x_2), b(y_1 + y_2))$$

is also an element of .

Closure under scalar multiplication. Suppose $(ax, by) \in$ and $\lambda \in \mathbb{R}$. Observe that the scalar product

$$\lambda(ax, by) = (a(\lambda x), b(\lambda y))$$

is also an element of .

We now show that there exists no other set that is a subspace of $\mathbb{R}^2$. Let S be a subspace of $\mathbb{R}^2$. By 2.37, it must be that $\dim S \leq 2$. The only subspace with $\dim S = 0$ is the trivial subspace. Likewise, the only subspace with $\dim S = 2$ is $\mathbb{R}^2$ itself by 2.39. The set $\{\{(ax, by) \in \mathbb{R}^2 : x, y \in \mathbb{R}\} : a, b \in \mathbb{R},\}$ consists of all subspaces with dim $= 1$ because for each subspace, their basis is (a, b) (except for the case $a = b = 0$, but it does not matter since it is essentially the trivial subspace). Any other set with one dimension that is not a member of this set will not be a subspace of $\mathbb{R}^2$ since it will not have an additive identity, thus completing our proof.

Problem 2C.3.

(a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

(a) A basis of U is $z - 6, (z-6)^2, (z-6)^3, (z-6)^4$. To see why, observe the following sum:

$$a(z-6) + b(z-6)^2 + c(z-6)^3 + d(z-6)^4 = 0.$$

There is only one term of z with the fourth exponent (i.e. dz^4) and thus $d = 0$. There is also only one term of z with the third exponent (i.e. cz^3, the remaining third exponent is eliminated by the fact that $d = 0$) and thus $c = 0$. We can similarly show that $b = a = 0$ and thus the aforementioned list is linearly independent. We note that U is a strict subspace of $\mathcal{P}_4(\mathbb{F})$ (the polynomial 69 is a member of the latter but not of the former) and thus can have a dimension of at most $\dim \mathcal{P}_4(\mathbb{F}) - 1 = 4$ by 2.37 and 2.39. Since the linearly independent list $z - 6, (z-6)^2, (z-6)^3, (z-6)^4$ has a length of 4, it is therefore a basis of U by 2.38.

(b) We simply need to find an additional element that is a member of $\mathcal{P}_4(\mathbb{F})$ that does not belong in U. Thus $1, z - 6, (z-6)^2, (z-6)^3, (z-6)^4$ is a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) A subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$ is $\{a \in \mathbb{F}\}$ (we know this because by the proof of 2.33, we simply need to set $W = \text{span}(1) = \{a \in \mathbb{F}\}$).

Problem 2C.4.

(a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

(a) A basis of U is $1, z-6, (z-6)^3, (z-6)^4$. We can verify that the elements in this list are indeed members of U:

- $(1)'' = 0$.
- $(z-6)'' = 0$;
- $((z-6)^3)'' = 6(z-6)$; at $z = 6$, this polynomial equates to 0.
- $((z-6)^4)'' = 12(z-6)^2$; at $z = 6$, this polynomial equates to 0.

In order to prove this list is a basis of U, we first prove its linear independence. To this end, for $a, b, c, d \in \mathbb{R}$, observe the following sum:

$$a + b(z-6) + c(z-6)^3 + d(z-6)^4 = 0.$$

There is only one term of z with the fourth exponent (i.e. dz^4) and thus $d = 0$. There is also only one term of z with the third exponent (i.e. cz^3, the remaining third exponent is eliminated by the fact that $d = 0$) and thus $c = 0$. We can similarly show that $b = a = 0$ and thus the aforementioned list is linearly independent. We note that U is a strict subspace of $\mathcal{P}_4(\mathbb{R})$ (the polynomial z^4 is a member of the latter but not of the former) and thus can have a dimension of at most $\dim \mathcal{P}_4(\mathbb{R}) - 1 = 4$ by 2.37 and 2.39. Since the linearly independent list $1, z-6, (z-6)^3, (z-6)^4$ has a length of 4, it is therefore a basis of U by 2.38.

(b) We simply need to find an additional element of $\mathcal{P}_4(\mathbb{R})$ that does not belong in U. Thus $1, z-6, (z-6)^2, (z-6)^3, (z-6)^4$ is a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) By the proof of 2.33, we just need to set $W = \operatorname{span}((z-6)^2) = \{a(z-6)^2 : a \in \mathbb{R}\}$.

Problem 2C.6.

(a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

(a) A basis of U is $1, z^3 - 13z^2 + 52z - 60, z^4 - 13z^3 + 52z^2 - 60z$. We can verify the elements in this list are indeed members of U:

- $p(z) = 1$; then $p(2) = p(5) = p(6) = 1$.
- $p(z) = z^3 - 13z^2 + 52z - 60$; then $p(2) = p(5) = p(6) = 0$.
- $p(z) = z^4 - 13z^3 + 52z^2 - 60z$; then $p(2) = p(5) = p(6) = 0$.

In order to prove this list is a basis of U, we shall prove it is linearly independent and spans U. To this end, for $a, b, c \in \mathbb{F}$, observe the following sum:

$$a + b(z^3 - 13z^2 + 52z - 60) + c(z^4 - 13z^3 + 52z^2 - 60z) = 0.$$

There is only one z^4 term—namely cz^4—and thus $c = 0$. Similarly we can show that $b = a = 0$ and thus the aforementioned list is linearly independent. We now verify that $(z-2)(z-5)(z-6)(cz+b) + a$ is a typical element of U. Let $p \in U$. Then $p(2) = p(5) = p(6) = a$, where $a \in \mathbb{F}$. By the factor theorem, we have $p(z) = (z-2)q_1(z) + a$, where $q_1(z)$ is the quotient polynomial after dividing $p(z)$ by $z-2$. Note that $p(5) = 3q_1(5) + p(5)$ implies $q_1(5) = 0$ and so $q_1(z) = (x-5)q_2(z)$. Repeating this process again yields $p(z) = (z-2)(z-5)(z-6)q(z) + a$, where $q(z)$ is the quotient polynomial after dividing $p(z)$ by $(z-2)(z-5)(z-6)$. Since we divided $p(z)$ three times and $\deg p \leq 4$, we have $\deg q \leq 1$ and thus we can rewrite $p(z) = (z-2)(z-5)(z-6)(cz+b) + a$ for $c, b \in \mathbb{F}$. Observe that

$$(z-2)(z-5)(z-6)(cz+b) + a = a + b(z^3 - 13z^2 + 52z - 60) + c(z^4 - 13z^3 + 52z^2 - 60z)$$

and thus $1, z^3 - 13z^2 + 52z - 60, z^4 - 13z^3 + 52z^2 - 60z$ spans U, completing our proof.

(b) The extended basis is $1, z^3 - 13z^2 + 52z - 60, z^4 - 13z^3 + 52z^2 - 60z, z, z^2$. To see why, we first show the list is linearly independent. For $a, b, c, d, e \in \mathbb{F}$, observe the following sum

$$a + b(z^3 - 13z^2 + 52z - 60) + c(z^4 - 13z^3 + 52z^2 - 60z) + dz + ez^2 = 0.$$

There is only one z^4 term—namely cz^4—and thus $c = 0$. This forces $b = 0$ because $-13cz^3$ and bz^3 are the only z^3 terms, the former equates to 0. Similarly we can show that $e = d = a = 0$ and thus the list is linearly independent. Since $\dim \mathcal{P}_4(\mathbb{F}) = 5$, this shows the aforementioned list is a basis of $\mathcal{P}_4(\mathbb{F})$ by 2.38.

(c) By the proof of 2.33, we just need to set $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbb{F}\}$

Lemma 1. Let $p_1, \ldots, p_n$ be a list of polynomials in $\mathcal{P}_m(\mathbb{F})$ such that the degrees of the p_i's are pairwise distinct for $1 \leq i \leq n$. Then $p_1, \ldots, p_n$ is linearly independent.

Proof. Rearrange $p_1, \ldots, p_n$ in place such that $\deg p_1 > \deg p_2 > \cdots > \deg p_n$. For $a_1, \ldots, a_n \in \mathbb{F}$, observe the following sum

$$a_1 p_1 + a_2 p_2 + \cdots + a_n p_n = 0$$

The only term with the exponent $\deg p_1$ belongs to p_1 and thus $a_1 = 0$. This forces every term with lower exponent in p_1 to 0 and leaves the term with the exponent $\deg p_2$ to only belong to p_2. Repeating this process until p_n and we have shown $a_1 = a_2 = \cdots = a_n = 0$, completing the proof.

Problem 2C.7.

(a) Let $U = \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0 \right\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

(a) A basis of U is $z, 3z^2 - 1, z^3 + 3z^2 - 1, 5z^4 - 1$. We can verify these elements indeed belong to U:

- $\displaystyle\int_{-1}^{1} z \mathrm{d}z = \frac{1}{2}z^2 \bigg|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.$
- $\displaystyle\int_{-1}^{1} (3z^2 - 1) \mathrm{d}z = (z^3 - z) \bigg|_{-1}^{1} = (1+1) - (1+1) = 0.$
- $\displaystyle\int_{-1}^{1} (z^3 + 3z^2 - 1) \mathrm{d}z = \left(\frac{1}{4}z^4 + z^3 - z \right) \bigg|_{-1}^{1} = \left(\frac{1}{4} - 1 + 1 \right) - \left(\frac{1}{4} - 1 + 1 \right) = 0.$
- $\displaystyle\int_{-1}^{1} (5z^4 - 1) \mathrm{d}z = (z^5 - z) \bigg|_{-1}^{1} = (1+1) - (1+1) = 0.$

By Lemma 1, this list is linearly independent. Note that U is a strict subspace of $\mathcal{P}_4(\mathbb{R})$ (the polynomial 69 is a member of the latter but not of the former) and so $\dim U \leq 4$. Since we have a linearly independent list of length 4 in U, we can conclude $\dim U = 4$ and thus the list is a basis of U by 2.38.

(b) We simply need to find a polynomial that is not in U. Thus our extended basis is $1, z, 3z^2 - 1, z^3 + 3z^2 - 1, 5z^4 - 1$.

(c) By the proof of 2.33, we simply set $W = \operatorname{span}(1) = \{a \in \mathbb{R}\}$.

Problem 2C.8. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Proof. By 2.43, we have

$$\begin{aligned}
&\dim(\operatorname{span}(v_1 + w, \ldots, v_m + w) + \operatorname{span}(w)) \leq \dim \operatorname{span}(v_1 + w, \ldots, v_m + w) + \dim \operatorname{span}(w) \\
&\iff \dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq \dim(\operatorname{span}(v_1 + w, \ldots, v_m + w) + \operatorname{span}(w)) - \dim \operatorname{span}(w) \\
&\iff \dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq \dim \operatorname{span}(v_1 + w, \ldots, v_m + w, w) - \dim \operatorname{span}(w)
\end{aligned}$$

($\operatorname{span}(v_1 + w, \ldots, v_m + w) + \operatorname{span}(w)$ is the smallest subspace containing both spans and so $\operatorname{span}(v_1 + w, \ldots, v_m + w, w)$ is the smallest subspace containing $v_1 + w, \ldots, v_m + w, w$)

$$\iff \dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq \dim \operatorname{span}(v_1, \ldots, v_m, w) - \dim \operatorname{span}(w)$$

(for $a_1, \ldots, a_m, b$, any linear combination $a_1(v_1 + w) + \cdots + a_m(v_m + w) + bw$ can be rewritten as $a_1 v_1 + \cdots + a_m v_m + (a_1 + \cdots + a_m + b)w$). Note that $\dim \operatorname{span}(v_1, \ldots, v_m, w) - \dim \operatorname{span}(w) \geq m - 1$:

- If $w \notin \operatorname{span}(v_1, \ldots, v_m)$, then the list $v_1, \ldots, v_m, w$ is linearly independent and so

$$\dim \operatorname{span}(v_1, \ldots, v_m, w) - \dim \operatorname{span}(w) = m.$$

- If $w \in \operatorname{span}(v_1, \ldots, v_m)$, then $\dim \operatorname{span}(v_1, \ldots, v_m, w) = \dim \operatorname{span}(v_1, \ldots, v_m)$ and so

$$\dim \operatorname{span}(v_1, \ldots, v_m, w) - \dim \operatorname{span}(w) \geq m - 1$$

(equality occurs when $w \neq 0$).

Thus $\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$, and we are done.

Problem 2C.10. Suppose m is a positive integer. For $0 \leq k \leq m$, let

$$p_k(x) = x^k (1 - x)^{m-k}.$$

Show that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Proof. For $a_0, \ldots, a_m \in \mathbb{F}$, observe the following sum:

$$a_0 p_0(x) + \cdots + a_m p_m(x) = a_0(1 - x)^m + a_1 x (1 - x)^{m-1} + \cdots + a_m x^m = 0.$$

The only term containing the zeroth exponent is $a_0(1 - x)^m$—namely a_0—and thus forces $a_0 = 0$. This cancels out every term with higher exponent and so we can repeat this process m times for subsequent terms and obtain $a_0 = \cdots = a_m = 0$, showing the linear independence of $p_0, \ldots, p_m$. Since $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$, our list is also a basis of $\mathcal{P}_m(\mathbb{F})$ by 2.38.

Problem 2C.11. Suppose U and W are both four-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

Proof. Suppose to the contrary that for all pairs of vectors in $U \cap W$, one vector is a scalar multiple of the other. By 2A.4b, this shows every list of length two of vectors in $U \cap W$ is linearly dependent and thus $\dim(U \cap W) \leq 1$. It follows that $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) \geq 7$ by 2.43. But by 1.40, $U + W$ is still a subspace of $\mathbb{C}^6$ and thus $\dim(U + W)$ cannot exceed 6. Therefore this is a contradiction, completing the proof.

Problem 2C.12. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Proof. By 2.43, we see that

$$\dim(U + W) = 8 = \dim U + \dim W - \dim(U \cap W) = 8 - \dim(U \cap W)$$

and so $\dim(U \cap W) = 0$. By definition, this shows $U \cap W = \{0\}$ and implies $\mathbb{R}^8 = U \oplus W$.

Problem 2C.13. Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Proof. By 2.43, we see that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 10 - \dim(U \cap W) \leq 9,$$

showing $\dim(U \cap W) \geq 1$.

Problem 2C.14. Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Proof. By 2.43, we see that

$$7 \leq \dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) = 14 - \dim(V_1 \cap V_2) \leq 10$$

and thus $4 \leq \dim(V_1 \cap V_2) \leq 7$. Similarly

$$\begin{aligned} 7 \leq \dim((V_1 \cap V_2) + V_3) &= \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3) \\ &= \dim(V_1 \cap V_2) + 7 - \dim(V_1 \cap V_2 \cap V_3) \leq 10 \end{aligned}$$

Iterating though $4 \leq \dim(V_1 \cap V_2) \leq 7$, we observe that:

- $\dim(V_1 \cap V_2) = 4 \to 1 \leq \dim(V_1 \cap V_2 \cap V_3) \leq 4$.
- $\dim(V_1 \cap V_2) = 5 \to 2 \leq \dim(V_1 \cap V_2 \cap V_3) \leq 5$.
- $\dim(V_1 \cap V_2) = 6 \to 3 \leq \dim(V_1 \cap V_2 \cap V_3) \leq 6$.
- $\dim(V_1 \cap V_2) = 7 \to 4 \leq \dim(V_1 \cap V_2 \cap V_3) \leq 7$.

Thus there exists no circumstance where $\dim(V_1 \cap V_2 \cap V_3) = 0$, and we are done.

Problem 2C.15. Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Proof. Suppose to the contrapositive that $V_1 \cap V_2 \cap V_3 = \{0\}$. By 2.43, we see that

$$\dim((V_1 \cap V_2) + V_3) = \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3) = \dim(V_1 \cap V_2) + \dim V_3.$$

Also,

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Taking the sum of the equations' respective sides, we get

$$\dim V_1 + \dim V_2 + \dim V_3 = \dim((V_1 \cap V_2) + V_3) + \dim(V_1 + V_2) \leq \dim V + \dim V = 2\dim V.$$

Problem 2C.16. Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals U.

Proof. Let $u_1, \dots, u_m$ be a basis of U. Since $u_1, \dots, u_m$ is linearly independent in V, we can extend this list into a basis $u_1, \dots, u_m, u_{m+1}, \dots, u_n$ of V by 2.32 (this is consistent with the fact that $n = \dim V$ and $m = \dim U$). We now construct the set

$$S = \{u_1, \dots, u_m, u_{m+1}, \dots, \hat{u}_i \dots, u_n : m + 1 \leq i \leq n\}$$

which is the set of bases of all possible subspaces of V with dimension $n-1$ and has U as their subspace ($\hat{u}_i$ denotes the omission of the i-th vector from the basis). It is easily seen that $|S| = n - m$ and the intersection of all subspaces with these bases is precisely U (let v be a vector in the intersection; thus we can rewrite v as $v = a_1u_1 + \cdots + a_mu_m$. The fact that $u_1, \dots, u_m$ is a basis of U guarantees the uniqueness of $a_1, \dots, a_m$ and thus this vector also belongs to U. Conversely, let $v \in U$; then we can write v as a linear combination of $u_1, \dots, u_m$, which is also the intersection of every member of S), showing the existence of such subspaces of V.

Problem 2C.17. Suppose that $V_1, \dots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite-dimensional and

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

Proof. We shall prove this using induction. Denote by S_m the statement

$$S_m : \dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m,$$

we divide our proof into these steps.

Basis step. If $m = 1$, then $S_1 : \dim V_1 \leq \dim V_1$, which is true.

Inductive hypothesis. For $k > 1$, suppose S_k is true. We now wish to show that S_{k+1} is true.

Induction step. By 2.33, we see that

$$\begin{aligned}
\dim(V_1 + \cdots + V_k + V_{k+1}) &= \dim(V_1 + \cdots + V_k) + \dim V_{k+1} - \dim((V_1 + \cdots + V_k) \cap V_{k+1}) \\
&\leq \dim V_1 + \cdots + \dim V_k + \dim V_{k+1} - \dim((V_1 + \cdots + V_k) \cap V_{k+1}) \\
&\leq \dim V_1 + \cdots + \dim V_k + \dim V_{k+1},
\end{aligned}$$

as desired.

Problem 2C.18. Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, \dots, V_n$ of V such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

Proof. We shall prove this using induction. Denote by S_n the statement

$$S_n : V = V_1 \oplus \cdots \oplus V_n,$$

we divide our proof into these steps.

Basis step. If $n = 1$, then $S_1 : V = V_1$ is true ($\dim V = \dim V_1 = 1$ implies $V = V_1$ by 2.39).

Inductive hypothesis. For $k > 1$, suppose S_k is true. We now wish to show that S_{k+1} is true.

Induction step. Let $v_1, \ldots, v_k, v_{k+1}$ be a basis of V. Let $W = \operatorname{span}(v_1, \ldots, v_k)$ and thus $v_1, \ldots, v_k$ is a basis of W. Then $\dim W = k$ and so there exist one-dimensional subspaces $V_1, \ldots, V_k$ of W (and is also of V) such that

$$W = V_1 \oplus \cdots \oplus V_k$$

by the inductive hypothesis. Set $V_{k+1} = \operatorname{span}(v_{k+1})$ and thus by the proof of 2.33, we have

$$V = W \oplus V_{k+1} = V_1 \oplus \cdots \oplus V_k \oplus V_{k+1}$$

and thus proves S_{k+1} is true.

Problem 2C.19. Prove or give a counterexample: If V_1, V_2, V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}
&\dim(V_1 + V_2 + V_3) \\
&\qquad = \dim V_1 + \dim V_2 + \dim V_3 \\
&\qquad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\
&\qquad + \dim(V_1 \cap V_2 \cap V_3).
\end{aligned}$$

This statement is false due to the following counterexample. Let $V_1 = \{(x, 0) : x \in \mathbb{R}\}$, $V_2 = \{(x, y) : x, y \in \mathbb{R}, x + y = 0\}$ and $V_3 = \{(x, y) : x, y \in \mathbb{R}, x + 2y = 0\}$. Note that $\dim(V_1 + V_2 + V_3) = 2$, while $\dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3) = 3$ and so the equality does not hold.

Problem 2C.20. Prove that if V_1, V_2 and V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}
&\dim(V_1 + V_2 + V_3) \\
&\quad = \dim V_1 + \dim V_2 + \dim V_3 \\
&\quad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\
&\quad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.
\end{aligned}$$

Proof. By 2.43, we see that

$$\begin{aligned}
\dim((V_1 + V_2) + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim((V_1 + V_2) \cap V_3), \\
\dim(V_1 + (V_2 + V_3)) &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 \cap V_3) - \dim((V_2 + V_3) \cap V_1), \\
\dim((V_1 + V_3) + V_2) &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_3) - \dim((V_1 + V_3) \cap V_2).
\end{aligned}$$

Taking the sum of the equations' respective side, we get

$$\begin{aligned}
3\dim(V_1 + V_2 + V_3) = 3(&\dim V_1 + \dim V_2 + \dim V_3) \\
&- (\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3))
\end{aligned}$$

$$- \dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1).$$

Dividing both sides of the equation by 3, we get

$$\begin{aligned}
&\dim(V_1 + V_2 + V_3) \\
&\qquad = \dim V_1 + \dim V_2 + \dim V_3 \\
&\qquad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\
&\qquad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3},
\end{aligned}$$

as desired.