Linear Algebra Done Right Chapter 1 - Solutions

FungusDesu

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Problem 1A.1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Suppose $\alpha = a + bi$ and $\beta = c + di$ for some $a, b, c, d \in \mathbb{R}$. Then

$$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i = (c + a) + (d + b)i = (c + di) + (a + bi) = \beta + \alpha.$$

Problem 1A.2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. Suppose $\alpha = a + bi$, $\beta = c + di$ and $\lambda = e + fi$ for some $a, b, c, d, e, f \in \mathbb{R}$. Then

$$\begin{aligned}
(\alpha + \beta) + \lambda &= ((a + bi) + (c + di)) + (e + fi) = ((a + c) + (b + d)i) + (e + fi) \\
&= ((a + c) + e) + ((b + d) + f)i = (a + (c + e)) + (b + (d + f))i \\
&= (a + bi) + ((c + e) + (d + f)i) = (a + bi) + ((c + di) + (e + fi)) \\
&= \alpha + (\beta + \lambda).
\end{aligned}$$

Problem 1A.3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. Suppose $\alpha = a + bi$, $\beta = c + di$ and $\lambda = e + fi$ for some $a, b, c, d, e, f \in \mathbb{R}$. Then

$$\begin{aligned}
(\alpha\beta)\lambda &= ((a + bi)(c + di))(e + fi) \\
&= ((ac - bd) + (ad + bc)i)(e + fi) \\
&= ((ac - bd)e - (ad + bc)f) + ((ac - bd)f + (ad + bc)e)i \\
&= (ace - bde - adf - bcf) + (acf - bdf + ade + bce)i \\
&= (a(ce - df) - b(cf + de)) + (a(cf + de) + b(ce - df))i \\
&= (a + bi)((ce - df) + (cf + de)i) \\
&= (a + bi)((c + di)(e + fi)) \\
&= \alpha(\beta\lambda).
\end{aligned}$$

Problem 1A.4. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. Suppose $\alpha = a + bi$, $\beta = c + di$ and $\lambda = e + fi$ for some $a, b, c, d, e, f \in \mathbb{R}$. Then

$$\begin{aligned}
\lambda(\alpha + \beta) &= (e + fi)((a + bi) + (c + di)) \\
&= (e + fi)((a + c) + (b + d)i) \\
&= e(a + c) - f(b + d) + (e(b + d) + f(a + c))i \\
&= (ea + ec - fb - fd) + (e(b + d)i + f(a + c)i) \\
&= (ea - fb + ec - fd) + ((eb + ed + fa + fc)i)
\end{aligned}$$

$$
\begin{aligned}
&= (ea - fb + ec - fd) + (eb + fa)i + (ed + fc)i \\
&= (ea - fb) + (eb + fa)i + (ec - fd) + (ed - fc)i \\
&= (e + fi)(a + bi) + (e + fi)(c + di) \\
&= \lambda\alpha + \lambda\beta.
\end{aligned}
$$

Problem 1A.5. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Proof. Suppose $\alpha = a + bi$, then

$$-\alpha = (-1 + 0i)(a + bi) = (-a - 0) + (-b + 0)i = -a - bi.$$

Thus

$$\alpha + (-\alpha) = (a + bi) + (-a - bi) = (a - a) + (b - b)i = 0.$$

Thus $\beta = -\alpha$ is an additive inverse of α. Suppose β' is another complex number that also satisfies this property. Then

$$\alpha + \beta' = 0 \implies \alpha + \beta' + \beta = \beta \implies \alpha + \beta + \beta' = \beta \implies \beta' = \beta.$$

Thus β is unique.

Problem 1A.6. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Proof. Suppose $\alpha = a + bi$. Observe that

$$
\begin{aligned}
\left(\frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}\right)(a + bi) &= \left(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}\right) + \left(\frac{ab}{a^2+b^2} - \frac{ab}{a^2+b^2}\right)i \\
&= \frac{a^2+b^2}{a^2+b^2} + 0i \\
&= 1.
\end{aligned}
$$

Thus $\beta = a/(a^2 + b^2) - bi/(a^2 + b^2)$ is a multiplicative inverse of α. Suppose β' is another complex number that also satisfies this property. Then

$$\alpha\beta' = 1 \implies \alpha\beta'\beta = \beta \implies \alpha\beta\beta' = \beta \implies \beta' = \beta.$$

Thus β is unique.

Problem 1A.7. Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1.

Proof. Observe that

$$
\begin{aligned}
\left(\frac{-1 + \sqrt{3}i}{2}\right)^3 &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= \left(\left(\frac{1}{4} - \frac{3}{4}\right) + \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{4} + \frac{3}{4}\right) + \left(-\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right) i \\
&= 1.
\end{aligned}
$$

Problem 1A.8. Find two distinct square roots of i.

Let $z = a + bi$ for which $z^2 = i$. Then we have

$$i = z^2 = (a + bi)^2 = a^2 - b^2 + 2abi.$$

Thus we have $a^2 - b^2 = 0$ and $2ab = 1$. This system of equations have two solutions, i.e. $(a, b) = (\sqrt{2}/2, \sqrt{2}/2)$ and $(a, b) = (-\sqrt{2}/2, -\sqrt{2}/2)$. Thus the two distinct square roots of i is

$$z = \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} i \quad \text{and} \quad z = -\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i.$$

Problem 1A.9. Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Observe that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8) \implies 2x = (1, 12, -7, 1) \implies x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right).$$

Problem 1A.10. Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Suppose to the contrary there exists $\lambda \in \mathbb{C}$ for which $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$. By definition of scalar multiplication, we have

$$
\begin{aligned}
\lambda &= \frac{12 - 5i}{2 - 3i} = \frac{7 + 22i}{5 + 4i} = \frac{-32 - 9i}{-6 + 7i} \\
&= (12 - 5i)\left(\frac{2}{13} + \frac{3}{13} i\right) = (7 + 22i)\left(\frac{5}{41} - \frac{4}{41} i\right) = (-32 - 9i)\left(-\frac{6}{85} - \frac{7}{85} i\right) \\
&= 3 + 2i = 3 + 2i = 3 - 2i.
\end{aligned}
$$

Thus we have a contradiction, and no such λ exists.

Problem 1A.11. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}^n$.

Proof. Suppose $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{F}^n$. Then

$$
\begin{aligned}
(x + y) + z &= ((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) + (z_1, \ldots, z_n) \\
&= (x_1 + y_1, \ldots, x_n + y_n) + (z_1, \ldots, z_n) \\
&= ((x_1 + y_1) + z_1, \ldots, ((x_n + y_n) + z_n) \\
&= (x_1 + (y_1 + z_1), \ldots, (x_n + (y_n + z_n)) \\
&= (x_1, \ldots, x_n) + (y_1 + z_1, \ldots, y_n + z_n) \\
&= (x_1, \ldots, x_n) + ((y_1, \ldots, y_n) + (z_1, \ldots, z_n)) \\
&= x + (y + z).
\end{aligned}
$$

Problem 1A.12. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.

Proof. Suppose $a, b \in \mathbb{F}$ and $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$. Then

$$\begin{aligned}(ab)x &= (ab)(x_1, \ldots, x_n) = (abx_1, \ldots, abx_n) = (a(bx_1), \ldots, a(bx_n)) \\ &= a(bx_1, \ldots, bx_n) = a(b(x_1, \ldots, x_n)) = a(bx).\end{aligned}$$

Problem 1A.13. Show that $1x = x$ for all $x \in \mathbb{F}^n$.

Proof. Suppose $x = (x_1, \ldots x_n) \in \mathbb{F}^n$. Then

$$1x = 1(x_1, \ldots, x_n) = (1x_1, \ldots, 1x_n) = (x_1, \ldots, x_n) = x.$$

Problem 1A.14. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.

Proof. Suppose $\lambda \in \mathbb{F}$, $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$. Then

$$\begin{aligned}\lambda(x + y) &= \lambda((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) \\ &= \lambda(x_1 + y_1, \ldots, x_n + y_n) \\ &= (\lambda(x_1 + y_1), \ldots, \lambda(x_n + y_n)) \\ &= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n) \\ &= (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_n) \\ &= \lambda(x_1, \ldots, x_n) + \lambda(y_1, \ldots, y_n) \\ &= \lambda x + \lambda y.\end{aligned}$$

Problem 1A.15. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.

Proof. Suppose $a, b \in \mathbb{F}$ and $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$. Then

$$\begin{aligned}(a + b)x &= (a + b)(x_1, \ldots, x_n) \\ &= ((a + b)x_1, \ldots, (a + b)x_n) \\ &= (ax_1 + bx_1, \ldots, ax_n + bx_n) \\ &= (ax_1, \ldots, ax_n) + (bx_1, \ldots, bx_n) \\ &= a(x_1, \ldots, x_n) + b(x_1, \ldots, x_n) \\ &= ax + bx.\end{aligned}$$

Problem 1B.1. Prove that $-(-v) = v$ for every $v \in V$.

Proof. For $v \in V$, we have

$$-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1v = v.$$

Problem 1B.2. Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof. Suppose $a \in \mathbb{F}$, $v \in V$ and $av = 0$. If $a = 0$, then we are done. If $a \neq 0$, then there exists multiplicative inverse $b \in \mathbb{F}$ of a for which $ab = 1$. Thus we have the following

$$v = 1v = (ba)v = b(av) = b0 = 0.$$

Problem 1B.3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Proof. Observe that there exists $x \in V$ for which $v + 3x = w$ where $v, w \in V$, i.e. $x = (w - v)/3$:

$$v + 3\left(\frac{1}{3}(w - v)\right) = v + \left(3 \cdot \frac{1}{3}\right)(w - v) = v + 1(w - v) = v + (w + (-v)) = (v + (-v)) + w = 0 + w = w.$$

Suppose there exists $x' \in V$ that also satisfies this property. Then

$$v + 3x' = w \implies v + 3x' + 3x = w + 3x \implies v + 3x + 3x' = w + 3x \implies w + 3x' = w + 3x.$$

Adding the additive inverse of w to both sides of the equation yields $3x' = 3x$. Then

$$x' = 1x' = \left(\frac{1}{3} \cdot 3\right)x' = \frac{1}{3}(3x') = \frac{1}{3}(3x) = \left(\frac{1}{3} \cdot 3\right)x = 1x = x.$$

Problem 1B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space. Which one?

The empty set fails to satisfy the additive identity, since it does not contain 0.

Problem 1B.5. Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Proof. Suppose $0v = 0$ for all $v \in V$. Then

$$0 = 0v = (1 + (-1))v = 1v + (-1)v = v + (-1)v.$$

Thus there exists an additive inverse of v, namely $(-1)v$.

Conversely, suppose there exists an additive inverse for all vectors v in V. Observe that

$$0v = (0 + 0)v = 0v + 0v.$$

Let w be the additive inverse of $0v$. Adding the additive inverse of $0v$ to both sides of the equation, we get

$$0v + w = 0v + 0v + w \implies 0 = 0v + 0 = 0v.$$

Problem 1B.6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

The set $\mathbb{R} \cup \{\infty, -\infty\}$ is not a vector space over $\mathbb{R}$. Suppose non-zero $t \in \mathbb{R}$; we can see that associativity does not hold:

$$\begin{aligned} (t + \infty) + (-\infty) &= \infty + (-\infty) = 0 \\ t + (\infty + (-\infty)) &= t + 0 = t. \end{aligned}$$

Problem 1B.7. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Let f, g be functions mapping from S to V. Then addition on V^S is defined as a function $f + g \in V^S$ that takes $x \in V$ as an input and returns $f(x) + g(x)$, thus $(f + g)(x) = f(x) + g(x)$ (this formula makes sense because $f(x), g(x) \in V$, and thus can be added).

Let $\lambda \in \mathbb{F}$ and f be a function mapping from S to V. Then scalar multiplication on V^S is defined as $(\lambda f)(x) = \lambda \cdot f(x)$.

For the sake of convenience, denote f, g, h be functions mapping from S to V, and $\alpha, \beta \in \mathbb{F}$. Observe that with the operations we have defined, V^S is indeed a vector space, as these properties hold:

Commutativity.

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

Associativity.

$$\begin{aligned} ((f + g) + h)(x) &= (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = f(x) + (g + h)(x) = (f + (g + h))(x) \end{aligned}$$

and

$$(\alpha\beta f)(x) = (\alpha\beta)f(x) = \alpha(\beta f(x)) = \alpha((\beta f)(x)) = (\alpha(\beta f))(x).$$

Distributivity.

$$(\alpha(f + g))(x) = \alpha((f + g)(x)) = \alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x) = (\alpha f)(x) + (\alpha g)(x) = (\alpha f + \alpha g)(x)$$

and

$$((\alpha + \beta)f)(x) = (\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x) = (\alpha f + \beta f)(x).$$

Multiplicative identity.

$$(1f)(x) = 1(f(x)) = f(x).$$

Additive identity.

$$\text{There exists } (0f) \in V^S \text{ for which } (f + 0f)(x) = ((1+0)f)(x) = (1f)(x) = f(x).$$

Additive inverse.

$$\text{There exists } ((-1)f) \in V^S \text{ for which } (f + (-1)f)(x) = ((1 + (-1))f)(x) = (0f)(x) = 0(f(x)) = 0.$$

Problem 1B.8. Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbb{C}}$ is defined by

  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbb{C}}$ is defined by

  $$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbb{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space.

Proof. Observe that for $a, b, c, d, e, f \in V$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, these properties hold:

Commutativity.

$$(a + ib) + (c + id) = (a + c) + i(b + d) = (c + a) + i(d + b) = (c + id) + (a + ib).$$

Associativity.

$$\begin{aligned}
((a + ib) + (c + id)) + (e + if) &= ((a + c) + i(b + d)) + (e + if) \\
&= ((a + c) + e) + i((b + d) + f) \\
&= (a + (c + e)) + i(b + (d + f)) \\
&= (a + ib) + ((c + e) + i(d + f)) \\
&= (a + ib) + ((c + id) + (e + if))
\end{aligned}$$

and

$$\begin{aligned}
((\alpha + \beta i)(\gamma + \delta i))(a + ib) &= ((\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)i)(a + ib) \\
&= ((\alpha\gamma - \beta\delta)a - (\alpha\delta + \beta\gamma)b) + i((\alpha\gamma - \beta\delta)a + (\alpha\delta + \beta\gamma)b) \\
&= ((\alpha\gamma)a - (\beta\delta)b - (\alpha\delta)a - (\beta\gamma)b) + i((\alpha\gamma)a - (\beta\delta)b + (\alpha\delta)a + (\beta\gamma)b) \\
&= (\alpha(\gamma a) - \alpha(\delta b) - \beta(\delta a) - \beta(\gamma b)) + i(\alpha(\gamma b) + \alpha(\delta a) + \beta(\gamma a) - \beta(\delta b)) \\
&= (\alpha(\gamma a - \delta b) - \beta(\delta a + \gamma b)) + i(\alpha(\gamma b + \delta a) + \beta(\gamma a - \delta b)) \\
&= (\alpha + \beta i)((\gamma a - \delta b) + i(\gamma b + \delta a)) \\
&= (\alpha + \beta i)((\gamma + \delta i)(a + ib)).
\end{aligned}$$

Distributivity.

$$\begin{aligned}(\alpha+\beta i)((a+ib)+(c+id)) &= (\alpha+\beta i)((a+c)+i(b+d))\\ &= (\alpha(a+c)-\beta(b+d))+i(\alpha(b+d)+\beta(a+c))\\ &= (\alpha a+\alpha c-\beta b-\beta d)+i(\alpha b+\alpha d+\beta a+\beta c)\\ &= (\alpha a-\beta b+\alpha c-\beta d)+i(\alpha b+\beta a+\alpha d+\beta c)\\ &= ((\alpha a-\beta b)+i(\alpha b+\beta a))+((\alpha c-\beta d)+i(\alpha d+\beta c))\\ &= (\alpha+\beta i)(a+ib)+(\alpha+\beta i)(c+id)\end{aligned}$$

and

$$\begin{aligned}((\alpha+\beta i)+(\gamma+\delta i))(a+ib) &= ((\alpha+\gamma)+(\beta+\delta)i)(a+ib)\\ &= ((\alpha+\gamma)a-(\beta+\delta)b)+i((\alpha+\gamma)b+(\beta+\delta)a)\\ &= (\alpha a+\gamma a-\beta b-\delta b)+i(\alpha b+\gamma b+\beta a+\delta a)\\ &= (\alpha a-\beta b+\gamma a-\delta b)+i(\alpha b+\beta a+\gamma b+\delta a)\\ &= ((\alpha a-\beta b)+i(\alpha b+\beta a))+((\gamma a-\delta b)+i(\gamma b+\delta a))\\ &= (\alpha+\beta i)(a+ib)+(\gamma+\delta i)(a+ib).\end{aligned}$$

Multiplicative identity.

$$1(a+ib)=1a+i(1b)=a+ib.$$

Additive identity.

There exists $(0+i0)\in V_{\mathbb{C}}$ for which $(a+ib)+(0+i0)=(a+0)+i(b+0)=a+ib$.

Additive inverse.

There exists $(-a+(-b)i)\in V_{\mathbb{C}}$ for which $(a+ib)+(-a+i(-b))=(a+(-a))+i(b+(-b))=0+i0=0$.

We remark that the operations used in proving these properties make sense, since $a,b,c,d,e,f\in V$ and $\alpha,\beta,\gamma,\delta\in\mathbb{R}$.

Problem 1C.1. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$.

(a) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}$

(b) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=4\}$

(c) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1x_2x_3=0\}$

(d) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1=5x^3\}$

Proof. Let snow $=\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}$. We can see that snow is a subspace of $\mathbb{F}^3$, as these properties hold:

Additive identity. The element $0=(0,0,0)$ belongs to snow, as shown by $0+2\cdot 0+3\cdot 0=0$.

Closure under addition. Suppose $(x_1,x_2,x_3),(y_1,y_2,y_3)\in$ snow. Then $x_1+2x_2+3x_3=0$ and $y_1+2y_2+3y_3=0$. Observe that

$$(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3).$$

This is an element of snow, as shown by

$$(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3)=(x_1+2x_2+3x_3)+(y_1+2y_2+3y_3)=0+0=0.$$

Closure under scalar multiplication. Suppose $(x_1, x_2, x_3) \in$ snow and $\lambda \in \mathbb{F}$. Then $x_1 + 2x_2 + 3x_3 = 0$. Observe that

$$\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3).$$

This is an element of snow, as shown by

$$(\lambda x_1) + 2(\lambda x_2) + 3(\lambda x_3) = \lambda(x_1 + 2x_2 + 3x_3) = \lambda \cdot 0 = 0.$$

Let slayyla $= \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$. slayyla is not a subspace of $\mathbb{F}^3$, as it does not contain the additive identity. More specifically, the element $0 = (0, 0, 0)$ is not contained, as shown by $0 + 2 \cdot 0 + 3 \cdot 0 = 0 \neq 4$.

Let mikkel $= \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$. mikkel is not a subspace of $\mathbb{F}^3$, as it is not closed under addition. For $(-1, 0, 2), (0, 1, 0) \in$ mikkel, their sum is $(-1, 0, 2) + (0, 1, 0) = (-1, 1, 2)$, which is not an element in mikkel ($(-1) \cdot 1 \cdot 2 = -2 \neq 0$).

Let higher $= \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$. higher is a subspace of $\mathbb{F}^3$, as it satisfies these properties:

Additive identity. The element $0 = (0, 0, 0)$ is in higher, as shown by $0 = 5 \cdot 0$.

Closure under addition. Suppose $(x_1, x_2, x_3), (y_1, y_2, y_3) \in$ higher. Then $x_1 = 5x_3$ and $y_1 = 5y_3$. Observe that

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

This is an element of higher, as shown by

$$x_1 + y_1 = 5x_3 + 5y_3 = 5(x_3 + y_3).$$

Closure under scalar multiplication. Suppose $\lambda \in \mathbb{F}$ and $(x_1, x_2, x_3) \in$ higher. Then $x_1 = 5x_3$. Observe that

$$\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3).$$

This is an element of higher, as shown by

$$\lambda x_1 = \lambda(5x_3) = 5(\lambda x_3).$$

Problem 1C.2a. If $b \in \mathbb{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

Proof. Denote by steak the set $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$. Suppose steak is a subspace of $\mathbb{F}^4$. Then $0 = (0, 0, 0, 0)$ is an element of steak. Thus $0 = 5 \cdot 0 + b$ implies $b = 0$.

Conversely, suppose $b = 0$. Then steak $= \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4\}$ is a subspace of $\mathbb{F}^4$ as it satisfies the following properties:

Additive identity. The element $0 = (0, 0, 0, 0)$ is an element of steak, as shown by $0 = 5 \cdot 0$.

Closure under addition. Suppose $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in$ steak. Then $x_3 = 5x_4$ and $y_3 = 5y_4$. Observe that

$$(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4).$$

This is an element of steak, as shown by

$$x_3 + y_3 = 5x_4 + 5y_4 = 5(x_4 + y_4).$$

Closure under scalar multiplication. Suppose $\lambda \in \mathbb{F}$ and $(x_1, x_2, x_3, x_4) \in$ steak. Then $x_3 = 5x_4$. Observe that

$$\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4).$$

This is an element of steak, as shown by

$$\lambda x^3 = \lambda(5x_4) = 5(\lambda x_4).$$

Problem 1C.2b. The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Proof. Denote by S the set $\{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}$. Observe that S is a subspace of $\mathbb{R}^{[0,1]}$, as it satisfies these properties:

Additive identity. Consider the function $0 : [0, 1] \to \mathbb{R}$ defined by $0(x) = 0$. Choose $\varepsilon > 0$, then we can always choose $\delta = \varepsilon > 0$ for which $|x - x_0| < \delta$ implies $|0(x) - 0(x_0)| < |0 - 0| < \varepsilon$. Thus 0 is continuous and is an element of S. The function 0 is also the additive identity because for any $f \in S$:

$$(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).$$

Closure under addition. Suppose f, g be elements in S. Then f, g are continuous and both map from $[0, 1]$ to $\mathbb{R}$. Observe that for all $0 \leq c \leq 1$, we have

$$\lim_{x\to c} f(x) = f(c), \quad \lim_{x\to c} g(x) = g(c).$$

Thus we have

$$f(c) + g(c) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \lim_{x\to c}(f(x) + g(x)).$$

Therefore $f(x) + g(x)$ is continuous for $0 \leq x \leq 1$, and so $f + g$ is an element of S.

Closure under scalar multiplication. Suppose $\lambda \in \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ is an element of S. Then f is continuous. Observe that for all $0 \leq c \leq 1$, we have

$$\lim_{x\to c} f(x) = f(c).$$

Thus we have

$$\lambda f(c) = \lambda \lim_{x\to c} f(x) = \lim_{x\to c} \lambda f(x).$$

Thus $\lambda f(x)$ is continuous for $0 \leq x \leq 1$, and so λf is an element of S.

Problem 1C.2c. The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Proof. Denote by S the set $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}$. Observe that S is a subspace of $\mathbb{R}^{\mathbb{R}}$, as it satisfies the following properties:

Additive identity. Consider the function $0(x) = 0$ for all $x \in \mathbb{R}$. Observe the following limit

$$\lim_{h\to 0} \frac{0(x + h) - 0(x)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Thus 0 is differentiable. This is also an element of S because for every $f \in S$:

$$(0 + f)(x) = 0(x) + f(x) = 0 + f(x) = f(x).$$

Closure under addition. Suppose $f, g \in S$. Then f, g both map from $\mathbb{R}$ to $\mathbb{R}$ and are differentiable. Observe that for all $a \in \mathbb{R}$, there exist

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so their sum

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} &= \lim_{h \to 0} \frac{f(a+h) - f(a) + g(a+h) - g(a)}{h} \\ &= \lim_{h \to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \end{aligned}$$

exists. Thus $f + g$ is differentiable and is an element of S.

Closure under scalar multiplication. Suppose $f \in S$ and $\lambda \in \mathbb{R}$. Then f maps from $\mathbb{R}$ to $\mathbb{R}$ and is differentiable. Observe that for all $a \in \mathbb{R}$, there exists

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and so

$$\lambda \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\lambda f(a+h) - \lambda f(a)}{h} = \lim_{h \to 0} \frac{(\lambda f)(a+h) - (\lambda f)(a)}{h}.$$

exists. Thus λf is differentiable and is an element of S.

Problem 1C.2d. The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

Proof. Denote by S the set $\{f : (0, 3) \to \mathbb{R} \mid f \text{ is differentiable}, f'(2) = b\}$ for some real b. Suppose S is a subspace of $\mathbb{R}^{(0,3)}$. Then the element 0 defined by $0(x) = 0$ for which $f + 0 = f$ for all $f \in S$ is itself an element of S. Thus 0 is differentiable and so

$$f'(2) = \lim_{h \to 0} \frac{0(2+h) - 0(2)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Thus $b = 0$.

Conversely, suppose $b = 0$. Then $S = \{f : (0, 3) \to \mathbb{R} \mid f \text{ is differentiable}, f'(2) = 0\}$. Observe that S is a subspace as it satisfies these properties:

Additive identity. Consider the function $0 : (0, 3) \to \mathbb{R}$ defined by $0(x) = 0$. For $0 < x < 3$, observe the following limit:

$$\lim_{h \to 0} \frac{0(x+h) - 0(x)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Thus 0 is differentiable, and consequently $0'(2) = 0$. This shows 0 is an element of S, and is also the additive identity since for every $f \in S$:

$$(0 + f)(x) = 0(x) + f(x) = 0 + f(x) = 0.$$

Closure under addition. Suppose $f, g \in S$. Then there exist

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

for all $x \in \mathbb{R}$. This shows that $f + g$ is differentiable because their sum also exists:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}.$$

This is again an element of S, for if $x = 2$, then

$$\lim_{h\to 0} \frac{(f+g)(2+h)-(f+g)(2)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} + \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = f'(2)+g'(2) = 0.$$

Closure under scalar multiplication. Suppose $f \in S$ and $\lambda \in \mathbb{R}$. Then there exists

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

for all $x \in \mathbb{R}$. This shows that λf is differentiable because

$$\lambda \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\lambda f(x+h)-\lambda f(x)}{h} = \lim_{h\to 0} \frac{(\lambda f)(x+h)-(\lambda f)(x)}{h}$$

exists. This is again an element of S because if $x = 2$, then

$$\lim_{h\to 0} \frac{(\lambda f)(2+h)-(\lambda f)(2)}{h} = \lambda \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lambda \cdot 0 = 0.$$

Problem 1C.2e. The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^\infty$.

Proof. Denote by S the set

$$\{(x_1, x_2, \dots) \in \mathbb{C}^\infty : \lim_{n\to\infty} \{x_n\} = 0\}.$$

Observe that S is a subspace of C^∞ because it satisfies the following properties:

Additive identity. Consider the sequence $\{0_n\} = \{0, 0, \dots\}$. Choose some $\varepsilon > 0$, then there always exists $N > 0$ for which $n > N$ and $|0_n - 0| = |0 - 0| = 0 < \varepsilon$. Thus

$$\lim_{n\to\infty} \{0_n\} = 0.$$

Thus 0 is an element of S. Note that it is also the additive identity because for any $\{x_n\} = \{x_1, x_2, \dots\} \in S$:

$$\{x_n\} + \{0_n\} = \{x_1, x_2, \dots\} + \{0, 0, \dots\} = \{x_1 + 0, x_2 + 0, \dots\} = \{x_1, x_2, \dots\}.$$

Closure under addition. Suppose $\{x_n\} = \{x_1, x_2, \dots\}, \{y_n\} = \{y_1, y_2, \dots\} \in S$. Then we have

$$\lim_{n\to\infty} \{x_n\} = 0, \quad \lim_{n\to\infty} \{y_n\} = 0.$$

The sum of $\{x_n\}, \{y_n\}$ is also an element of S because

$$0 = \lim_{n\to\infty} \{x_n\} + \lim_{n\to\infty} \{y_n\} = \lim_{n\to\infty} (\{x_n\} + \{y_n\}) = \lim_{n\to\infty} (\{x_n + y_n\}).$$

Closure under scalar multiplication. Suppose $\{x_n\} = \{x_1, x_2, \dots\}$ and $\lambda \in \mathbb{C}$. Then we have

$$\lim_{n\to\infty} \{x_n\} = 0.$$

The product of $\{x_n\}$ and λ is also an element of S because

$$0 = \lambda \lim_{n\to\infty} \{x_n\} = \lim_{n\to\infty} \{\lambda x_n\}.$$

Problem 1C.3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Proof. Denote by S the set $\{f : (-4, 4) \to \mathbb{R} \mid f \text{ is differentiable}, f'(-1) = 3f(2)\}$. Observe that S is a subspace of $\mathbb{R}^{(-4,4)}$ as it satisfies these properties:

Additive identity. Consider the function $0 : (-4,4) \to \mathbb{R}$ defined by $0(x) = 0$. For $-4 < x < 4$, observe the following limit:

$$\lim_{h\to 0} \frac{0(x+h) - 0(x)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Thus 0 is differentiable, and consequently $f'(-1) = 0$. Note that S is also the additive identity of S because $0 = 3 \cdot 0$.

Closure under addition. Suppose $f, g \in S$. Then f, g is differentiable, $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$. Note that $f+g$ is also differentiable because

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}.$$

Observe that

$$3f(2) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h}, \quad 3g(2) = \lim_{h\to 0} \frac{g(-1+h) - g(-1)}{h}.$$

Thus $f+g$ is also an element of S, as shown by

$$\begin{aligned} 3(f+g)(2) = 3f(2) + 3g(2) &= \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} + \lim_{h\to 0} \frac{g(-1+h) - g(-1)}{h} \\ &= \lim_{h\to 0} \frac{(f+g)(-1+h) - (f+g)(-1)}{h} \\ &= (f+g)'(-1). \end{aligned}$$

Closure under scalar multiplication. Suppose $f \in S$ and $\lambda \in \mathbb{R}$. Then f is differentiable and $3f(2) = f'(-1)$. Note that λf is also differentiable because

$$\lambda \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(\lambda f)(x+h) - (\lambda f)(x)}{h}.$$

Observe that

$$3f(2) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h}$$

Thus λf is also an element of S, as shown by

$$3(\lambda f)(2) = \lambda \cdot (3f(2)) = \lambda \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{(\lambda f)(-1+h) - (\lambda f)(-1)}{h}.$$

Problem 1C.4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. Denote by S the set

$$\left\{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}, \int_0^1 f = b \right\}.$$

Suppose that S is a subspace of $\mathbb{R}^{[0,1]}$. Then the addditive identity $0 : [0,1] \to \mathbb{R}$ defined by $0(x) = 0$ is an element of S. Thus 0 is continuous and

$$\int_0^1 0 = 0.$$

Thus $b = 0$.

Conversely, suppose $b = 0$. Then

$$S = \left\{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}, \int_0^1 f = 0 \right\}.$$

Observe that S is a subspacce of $\mathbb{R}^{[0,1]}$, as it satisfies the following properties:

Additive identity. Consider the function $0 : [0,1] \to \mathbb{R}$ defined by $0(x) = 0$. Note that 0 is continuous ($1C.2b$) and

$$\int_0^1 0 = 0.$$

Thus 0 is an element of S. It is also the additive identity of S because for any $f \in S$, $0(x) + f(x) = f(x)$.

Closure under addition. Suppose $f, g \in S$. Then f, g are continuous and so is $f + g$ ($1C.2b$). Observe that

$$\int_0^1 f = 0, \quad \int_0^1 g = 0.$$

The function $f + g$ is again an element of S because

$$0 = \int_0^1 f = 0 + \int_0^1 g = \int_0^1 (f + g).$$

Closure under scalar multiplication. Suppose $f \in S$ and $\lambda \in \mathbb{R}$. Then f is continuous and so is λf ($1C.2b$). Observe that

$$\int_0^1 f = 0 \implies \int_0^1 \lambda f = 0.$$

Thus λf is an element of S. 

Problem 1C.5. Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

No. $\mathbb{R}^2$ is not a subspace of the complex vector space $\mathbb{C}^2$ because it is not closed under scalar multiplication. In particular, $i \cdot (69, 420) = (69i, 420i) \notin \mathbb{R}^2$.

Problem 1C.6a. Is $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

Proof. First note that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is injective. Suppose $a, b \in \mathbb{R}$ and $f(a) = f(b)$. Then

$$\begin{aligned} a^3 = b^3 &\implies (a-b)(a^2 + ab + b^2) = 0 \\ &\implies (a-b)\left(\left(\frac{a}{2} + b\right)^2 + \frac{3}{4}a^2\right) = 0 \\ &\implies a - b = 0 \\ &\implies a = b. \end{aligned}$$

Denote by S the set $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$. Observe that S is a subspace as it satisfies these properties:

Additive identity. The element $0 = (0,0,0)$ belongs to S, as shown by $0^3 = 0^3 = 0$. It is also the additive identity because for all $x = (x_1, x_2, x_3) \in S$, we have $0 + x = (0,0,0) + (x_1, x_2, x_3) = (x_1, x_2, x_3)$.

Closure under addition. Suppose $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S$. Then $x_3^3 = x_1^3$ and $y_3^3 = y_1^3$, which implies $x_3 = x_1$ and $y_3 = y_1$. The sum of these two vectors is

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

This is also an element of S because

$$(x_3 + y_3)^3 = (x_1 + y_1)^3.$$

Closure under scalar multiplication. Suppose $(x_1, x_2, x_3) \in S$ and $\lambda \in \mathbb{R}$; then their product is $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$. Since $(x_1, x_2, x_3) \in S$, we have $x_3^3 = x_1^3$, which implies $x_3 = x_1$. The product of λ and (x_1, x_2, x_3) is an element of S because

$$(\lambda x_3)^3 = (\lambda x_1)^3.$$

Problem 1C.6b. Is $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Denote by S is the set $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$. S is not a subspace of $\mathbb{C}^3$, as it is not closed under addition. In particular, the elements $\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right)$ and $\left(1, \frac{-1-\sqrt{3}i}{2}, 0\right)$ are in S, but their sum $(2, -1, 0)$ is not.

Problem 1C.7. Prove or give a counterexample: If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbb{R}^2$.

Disproof. This statement is false due to the following counterexample. Denote by U the set $\{(2n, 2n) \in \mathbb{R}^2 \mid n \in \mathbb{Z}\}$. Note that U is closed under addition (suppose $(2a, 2a), (2b, 2b) \in U$ for some $a, b \in \mathbb{Z}$, then their sum $(2(a+b), 2(a+b))$ is also an element of U because $a + b \in \mathbb{Z}$) and every element of U has an additive inverse (suppose $(2a, 2a) \in U$, then $(-2a, -2a)$ is its additive inverse). However U is not a subspace of $\mathbb{R}^2$, as it is not closed under scalar multiplication (consider the element $(2, 2)$; the product $0.1(2, 2)$ is not an element of U).

Problem 1C.8. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

The set $U = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ is closed under scalar multiplication; suppose $(x, y) \in U$ and $\lambda \in \mathbb{R}$. Then $xy = 0$. The product of (x, y) and λ, i.e. $(\lambda x, \lambda y)$, is also an element of U, as shown by $\lambda x \lambda y = \lambda^2 xy = 0$. The set u however is not closed under addition; the elements $(69, 0)$ and $(0, 420)$ are in S, but their sum $(69, 420)$ is not.

Problem 1C.9. A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

The set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ is not a subspace of $\mathbb{R}^{\mathbb{R}}$, as it is not closed under addition. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is integer} \\ 0 & \text{if } x \text{ is noninteger} \end{cases}$$

Then

$$f(\sqrt{2}x) = \begin{cases} 1 & \text{if } \sqrt{2}x \text{ is integer} \\ 0 & \text{if } \sqrt{2}x \text{ is noninteger} \end{cases}$$

Note that $f(x)$ and $f(\sqrt{2}x)$ are periodic, specifically the former is 1-periodic and the latter is $(1/\sqrt{2})$-periodic. Their sum however is aperiodic. Suppose to the contrary that $f(x) + f(\sqrt{2}x)$ is T-periodic. Since $f(x)$ and $f(\sqrt{2}x)$ are 1-periodic and $(1/\sqrt{2})$-periodic respectively, we have $f(x) = f(x + 1j)$ and $f(\sqrt{2}x) = f(\sqrt{2}x + k/\sqrt{2})$ for integer j, k. Then

$$T = 1j = \frac{1}{\sqrt{2}}k,$$

hence a contradiction that $\sqrt{2}$ is irrational.

Problem 1C.10. Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1 \cap V_2$ is a subspace of V.

Proof. Observe that $V_1 \cap V_2$ is a subspace of V as it satisfies these properties:

Additive identity. Since V_1 and V_2 are subspaces of V, they both have the additive identity 0, thus so is $V_1 \cap V_2$.

Closure under addition. Suppose $u, v \in V_1 \cap V_2$. Then the elements u and v are also the elements of V_1 and V_2. Thus $u + v \in V_1$ and $u + v \in V_2$ and so $u + v \in V_1 \cap V_2$.

Closure under scalar multiplication. Suppose $v \in V_1 \cap V_2$ and $\lambda \in \mathbb{F}$. Then the element v is also the element of V_1 and V_2. Thus $\lambda v \in V_1$ and $\lambda v \in V_2$ and so $\lambda v \in V_1 \cap V_2$.

Problem 1C.11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Proof. Denote by V_S the set of all possible subspaces of V. Suppose $S \in \mathscr{P}(V_S) \setminus \{\varnothing\}$. Denote by U the set

$$U = \bigcap_{s \in S} s.$$

Then observe that U is a subspace of V, as it satisfies the following properties:

Additive identity. Since the element 0 belongs to every element of S, 0 must also belong to U.

Closure under addition. Suppose $u, v \in U$. Then the elements u, v also belong to every element of S. For each subspace s of V in S, we have $u + v \in s$, and so $u + v$ is also an element of U.

Closure under scalar multiplication. Suppose $v \in U$ and $\lambda \in \mathbb{F}$. Then the element v also belongs to every element of S. For each subspace s of V in S, we have $\lambda v \in s$, and so λv is also an element of U.

Problem 1C.12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Denote by S_1 and S_2 the two subspaces of V. Suppose $S_1 \nsubseteq S_2$ and $S_2 \nsubseteq S_1$. Then there exists an element $u \in S_1$ for which $u \notin S_2$, and $v \in S_2$ for which $v \notin S_1$. Then $u, v \in S_1 \cup S_2$. Since $u \in S_1$ and $v \notin S_1$, we have $u + v \notin S_1$ (suppose to the contrapositive that $u + v \in S_1$; then $v = (v + u) + (-u)$ is an element of S_1). By the same line of reasoning we can also show that $u + v \notin S_2$. Thus $u + v \notin S_1 \cup S_2$, and so $S_1 \cup S_2$ is not a subspace.

Conversely, suppose $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. Without loss of generality, suppose the former. Then $S_1 \cup S_2 = S_1$ is a subspace of V.

Problem 1C.14. Suppose

$$U = \{(x, -x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

$U + W = \{(x, y, 2x) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$. The subspace $U + W$ of $\mathbb{F}^3$ is a set of elements in $\mathbb{F}^3$ whose third coordinate is twice the first coordinate.

Problem 1C.15. Suppose U is a subspace of V. What is $U + U$?

$U + U = U$, as U is the smallest subspace containing U and U.

Problem 1C.16. Is the operation of addition on the subspace of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Yes. A proof follows.

Proof. Denote by U, W the two subspaces of V. Then

$$U + W = \{v_1 + v_2 : v_1 \in U, v_2 \in W\} = \{v_2 + v_1 : v_2 \in W, v_1 \in U\} = W + U.$$

We note that this makes sense because addition between elements of vector space is commutative.

Problem 1C.17. Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Yes. A proof follows.

Proof. Denote by V_1, V_2, V_3 the subspaces of V. Then

$$\begin{aligned}(V_1 + V_2) + V_3 &= \{(v_1 + v_2) + v_3 : v_1 \in V_1, v_2 \in V_2, v_3 \in V_3\} \\ &= \{v_1 + (v_2 + v_3) : v_1 \in V_1, v_2 \in V_2, v_3 \in V_3\} \\ &= V_1 + (V_2 + V_3).\end{aligned}$$

We note that this makes sense because addition of elements of vector space is associative.

Problem 1C.18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

The operation of addition on the subspaces of V has an additive identity, namely the subspace $\{0\}$.

The subspace of V having additive inverses is precisely $\{0\}$. Suppose to the contrary that there exists another set W for which $V + W = \{0\}$. Then by 1.40, $V + W$ is the smallest set that contains V and W. Thus it is a contradiction that $\{0\}$ contains a set bigger than itself.

Problem 1C.19. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

This statement is false due to the following example. Let $V_1 = \{(x, 0) \in \mathbb{F}^2 : x \in \mathbb{F}\}, V_2 = \{(x, y) \in \mathbb{F}^2 : x, y \in \mathbb{F}\}, U = \{(0, x) \in \mathbb{F}^2 : x \in \mathbb{F}\}$. Note that V_1, V_2 and U are subspaces of V, and $V_1 + U = V_2 + U = \{(x, y) \in \mathbb{F}^2 : x, y \in \mathbb{F}\}$, however $V_1 \neq V_2$.

Problem 1C.20. Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

$W = \{(x, 0, y, 0) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$.

Problem 1C.21. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

$W = \{(0, 0, x, y, z) \in \mathbb{F}^5 : x, y, z \in \mathbb{F}\}$.

Problem 1C.22. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

$W_1 = \{(0, 0, x, 0, 0) \in \mathbb{F}^5 : x \in \mathbb{F}\}, W_2 = \{(0, 0, 0, x, 0) \in \mathbb{F}^5 : x \in \mathbb{F}\}, W_3 = \{(0, 0, 0, 0, x) \in \mathbb{F}^5 : x \in \mathbb{F}\}$.

Problem 1C.23. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

Then $V_1 = V_2$.

This statement is false due to the following counterexample. Let $V_1 = \{(x, 0) \in \mathbb{F}^2 : x \in \mathbb{F}\}$, $V_2 = \{(x, x) \in \mathbb{F}^2 : x \in \mathbb{F}\}$ and $U = \{(0, x) \in \mathbb{F}^2 : x \in \mathbb{F}\}$. Note that V_1, V_2 and F are subspaces of $\mathbb{F}^2$, and $V_1 \oplus U = V_2 \oplus U = \mathbb{F}^2$. However $V_1 \neq V_2$.

Problem 1C.24. A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let V_e denote the set of real-valued even functions on $\mathbb{R}$ and let V_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

Proof. We first show that any function f can be decomposed into a unique sum of odd and even functions. Denote by o and e those functions respectively, we wish to find o and e in terms of f. We thus have that $f(x) = o(x) + e(x)$, and therefore $f(-x) = e(x) - o(x)$. Solving the linear system of equations yields

$$o(x) = \frac{f(x) - f(-x)}{2}, \quad e(x) = \frac{f(x) + f(-x)}{2}.$$

We can verify that o is odd and e is even:

$$o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -o(x)$$
$$e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = e(x)$$

and in fact precisely sum to f:

$$o(x) + e(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) + f(-x)}{2} = \frac{2f(x)}{2} = f(x).$$

Thus we have shown that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.