Proceedings of 2016 IEEE
International Conference on Mechatronics and Automation
August 7 - 10, Harbin, China

Dynamic Modeling and Sliding Mode Controller Design of a Two-wheeled Self-balancing Robot*

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***Abstract*— A two-wheeled self-balancing robot is analyzed in this paper. The dynamic model of this robot is presented based on Appell Equations. The model is divided into two separate subsystems: balance subsystem and direction subsystem, and sliding mode controllers for each subsystem is designed separately. Computer simulation of the controller is taken with Matlab/Simulink. Simulation results show that the designed controller can track the ideal displacement and yaw angle effectively, and make the pitch angle stable at the meanwhile. It indicates that the designed controller for two-wheeled self-balancing robot have a fast convergence speed and a strong anti-interference ability, and is effective.**

***Index Terms*— Two-wheeled robot, Dynamic modeling, Appell Equation, Sliding mode, Self-balancing**

I. Introduction

Two-wheeled self-balancing robot has caught wide attentions all over the world. Users can control the robot's speed and direction though their tilt angle. This kind of robot has two parallel wheels which can be driven separately [1], thus it can change its direction by setting different speed of the two wheels, and can rotate around its center by setting different rotate direction of the two wheels [2]. Due to these characters, this kind of robot has some advantages that regular bicycles do not have. Two-wheeled self-balancing robot has multi-variable, nonlinear, strong coupling and parameter uncertainty characteristics, which make it to be an imagine platform to verify many kinds of control algorithms [3]. It can also be used in toy, education and service robot fields.

Sliding mode control is a nonlinear control method [4, 5] with simple algorithm, good robustness and high reliability. It is widely used in robot motion control. A stable sliding-mode method designed by Yu Xiuli [6] is taken to realize non-linear under-actuated system of bicycle robot stable. By using the sliding mode control method, the control law can guarantee that the equilibrium point is reachable in finite time and the system's state variables converge to equilibrium field. Kanghyun Nam [7] designs a robust yaw stability control system by using sliding mode control methodology to make vehicle yaw angle velocity to track its reference. A parameter adaptation law is applied to estimate varying vehicle parameters with respect to road conditions and is incorporated into sliding mode control framework.

In this paper, a dynamic model of the two-wheeled self-balancing robot is built based on Appell Equations [8, 9]. Gibbs Function is presented by Appell to simplify the computation process. The dynamic model is divided into two independent subsystems. Sliding-mode controllers of the two subsystems are designed to maintain the robot self-balancing during a given motion. It can be seen from the results of the simulation based on MATLAB Simulink that the controller has an excellent stability and robustness.

II. Dynamic Modeling of the Two-wheeled Self-balancing Robot

In this section, the schematic diagram of the two-wheeled self-balancing robot is shown in Fig.1. Compared to Lagrange Equation, Appell Equations have simpler form and easier to use for under-actuated systems. So the dynamic model was developed based on Appell Equations.

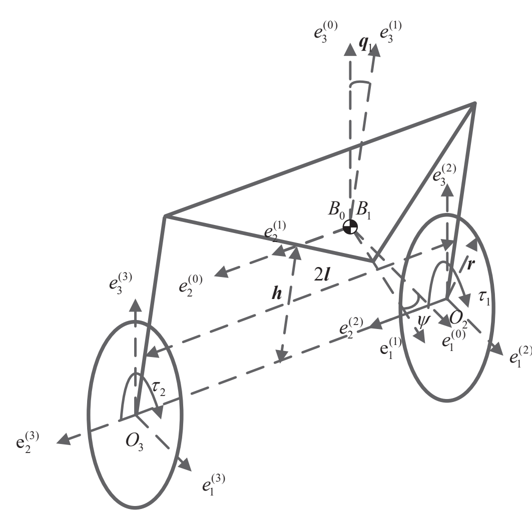

Fig. 1. Schematic diagram of a two-wheel self-balancing robot

*This work is supported by National Natural Science Foundation of China (51375059, 61105103), Beijing Higher Education Young Elite Teacher Project (YETP0452) and Beijing Natural Science Foundation (4132032).

Some assumptions are taken to simplify the analysis, which are shown as follows: 1) The frame which attaches the wheels is rigid.

2)The two wheels are seen as circles which the mass is concentrated towards the geometric centers. The faces of two wheels are always vertical to the ground.

3) The wheels' movement on the ground is pure rolling.

A. Kinetic Analysis of the Robot

In Fig.1, B_1 denotes the geometric center of the frame, $O_2(O_3)$ denotes the geometric center of the left wheel(right wheel). Table I gives the physical meaning of the notations given in Fig. 1 and the below analysis.

Fig.1 also shows the following coordinate systems of the robot:

- $B_0 - \boldsymbol{e}_1^{(0)}\boldsymbol{e}_2^{(0)}\boldsymbol{e}_3^{(0)}$ is the coordinate system fixed on the ground.
- $B_1 - \boldsymbol{e}_1^{(1)}\boldsymbol{e}_2^{(1)}\boldsymbol{e}_3^{(1)}$ represents the coordinate system of the frame, and B_1 denotes the mass center.
- $O_2 - \boldsymbol{e}_1^{(2)}\boldsymbol{e}_2^{(2)}\boldsymbol{e}_3^{(2)}$ represents the coordinate system fixed on the left wheel and its origin is located at the geometric center of the left wheel.
- $O_3 - \boldsymbol{e}_1^{(3)}\boldsymbol{e}_2^{(3)}\boldsymbol{e}_3^{(3)}$ represents the coordinate system fixed on the right wheel and its origin is also located at the geometric center of the right wheel.

First, the geometric relationship depicted in Fig. 1 is considered, and second, the movement relationship of the system is formulated. Since it is assumed that the wheels are are running on the ground without slipping, v_2 and v_3 can be represented as $r_2\dot{q}_2$, $r_3\dot{q}_3$. The yaw angle of the frame can be represented as $(v_2 - v_3)/(2l)$. The moving distance of the robot, the frame and two wheels can be represented as:

$$\begin{cases} d_2 = rq_2 \quad d_3 = rq_3 \\ d = \dfrac{1}{2}(d_2 + d_3) \\ d_1 = (d + h\sin q_1)\boldsymbol{e}_1 + (h\cos q_1)\boldsymbol{e}_3. \end{cases}$$

And the moving velocity of the robot, the frame and two wheels can be represented as:

$$\begin{cases} v_2 = r\dot{q}_2 \quad v_3 = r\dot{q}_3 \\ v = \dfrac{1}{2}(v_2 + v_3) \\ v_1 = (v + h\dot{q}_1\cos q_1)\boldsymbol{e}_1 + (-h\dot{q}_1\sin q_1)\boldsymbol{e}_3. \end{cases} \tag{1}$$

The yaw angle and angular velocity can be represented as:

$$\psi = \frac{d_2 - d_3}{2l}, \quad \dot{\psi} = \frac{v_2 - v_3}{2l}. \tag{2}$$

B. Appell Equations of the System

The standard Appell's equation is shown as (3)

$$\sum_{j=1}^{m}\left(Q_j - \frac{\partial G}{\partial \ddot{q}_j}\right)\delta q_j = 0 \quad (j = 1,2,\ldots,m), \tag{3}$$

in which Q_j denotes the generalized force of the system, G denotes the Gibbs function, which also known as "acceleration energy", q_j denotes the generalized variables. The Appell Equations can be represented as (4),

$$\frac{\partial G}{\partial \ddot{q}_j} - \frac{\partial P}{\partial q_j} = \tau_j \quad (j = 1,2,\ldots,m), \tag{4}$$

in which P denotes the potential energy of the system, τ_j denotes the external force.

To compute the Appell's equation, the Gibbs function must be first computed. It is simply the sum of the corresponding quantities for each individual body (two wheels and frame):

$$G = \sum_{i=1}^{n} G_i$$

1) Gibbs Functions of the robot: The movement of the left wheel is pure rolling on the ground, P_2 is the instantaneous center, so the linear velocity and accelerated velocity of the geometric center are v_2 and a_2, $a_2 = \dot{v}_2 = r\ddot{q}_2$. The angular velocity and angular acceleration of the left wheel around O_2 (The moment of inertia around $\boldsymbol{e}_2^{(2)}$ is $J = mr^2$) are $\dot{q}_2$ and $\ddot{q}_2$. The Gibbs function of the left wheel can be presented as

$$G_2 = \frac{1}{2}ma_2^2 + \frac{1}{2}J(\dot{q}_2^4 + \ddot{q}_2^2).$$

Similarly, the movement of the right wheel is pure rolling on the ground, P_3 is the instantaneous center, so the linear velocity and accelerated velocity of the geometric center are v_3 and a_3, $a_3 = \dot{v}_3 = r\ddot{q}_3$. The angular velocity and angular acceleration of the left wheel around O_3 (The moment of

TABLE I

NOMENCLATURE OF TWO-WHEELED ROBOT

Parameter	Definition
h	Height of B_1 above O_2O_3
r	Left and right wheels' radius
q_1, $\dot{q}_1$	Pitch angle, rate of the frame
q_2, $\dot{q}_2$	Rotational angle, rate of the left wheel
q_3, $\dot{q}_3$	Rotational angle, rate of the right wheel
ψ, $\dot{\psi}$	Yaw angle and rate of the frame
J	The wheel's moment of inertia around O_2O_3
J_{12}	The frame's moment of inertia around $\boldsymbol{e}_2^{(1)}$
J_{13}	The frame's moment of inertia around $\boldsymbol{e}_3^{(1)}$
m_1, m	Masses of the frame and the wheels
$\boldsymbol{v}, \boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$	Linear velocity of the robot, frame and wheels
$\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$	Accelerated velocity of the frame and wheels
l	Horizontal distance from O_2 to O_3
d, d_1, d_2, d_3	displacement of the robot, frame and wheels
τ_1, τ_2	Torques applied on the wheels

inertia around $\boldsymbol{e}_2^{(3)}$ is $J = mr^2$) are $\dot{q}_3$ and $\ddot{q}_3$. The Gibbs function of the right wheel can be presented as

$$G_3 = \frac{1}{2}ma_3^2 + \frac{1}{2}J(\dot{q}_3^4 + \ddot{q}_3^2).$$

The movement of the frame is translational motion horizontally, rotational movement around $\boldsymbol{e}_2^{(1)}$ and rotation movement around $\boldsymbol{e}_3^{(1)}$.

1) The horizontal component of translational velocity of the mass of the frame is $v_{1h} = v + h\dot{q}_1 \cos q_1$. The vertical component of the translational velocity is $v_{1v} = -h\dot{q}_1 \sin q_1$. The horizontal and vertical components of the translational accelerations are $a_{1h} = \dot{v}_{1h}$ and $a_{1v} = \dot{v}_{1v}$ respectively.

2) The angular velocity and angular acceleration of the frame around $\boldsymbol{e}_2^{(1)}$ (The moment of inertia around $\boldsymbol{e}_2^{(1)}$ is J_{12}) are $\dot{q}_1$ and $\ddot{q}_1$.

3) The angular velocity and angular acceleration of the frame around $\boldsymbol{e}_3^{(1)}$ (The moment of inertia around $\boldsymbol{e}_3^{(1)}$ is J_{13}) are $\dot{\psi}$ and $\ddot{\psi}$.

So the Gibbs function of the frame can be represented as

$$G_1 = \frac{1}{2}m_1(a_{1h}^2 + a_{1v}^2) + \frac{1}{2}J_{12}(\dot{q}_1^4 + \ddot{q}_1^2) + \frac{1}{2}J_{13}(\dot{\psi}^4 + \ddot{\psi}^2).$$

The total Gibbs function of the robot can be represented as

$$G = G_1 + G_2 + G_3.$$

Because the frame rotates around $\boldsymbol{e}_1^{(2)}$ with little angle, assumptions $\cos(q_1) = 1$ and $\sin(q_1) = q_1$ are taken, and neglect the higher order terms. Then take the partial derivative of G with the respect to the second derivative of generalized coordinates q_1, q_2, q_3.

The partial derivation of G with $\ddot{q}_1$ is represented as

$$\frac{\partial G}{\partial \ddot{q}_1} = (m_1h^2 + J)\ddot{q}_1 + \frac{1}{2}m_1hr\ddot{q}_2 + \frac{1}{2}m_1hr\ddot{q}_3. \quad (5)$$

Similarly, the partial derivations of G with $\ddot{q}_2$ and $\ddot{q}_3$ are represented as

$$\begin{aligned}\frac{\partial G}{\partial \ddot{q}_2} = {} & (mr^2 + J + \frac{1}{4}m_1r^2 + \frac{r^2}{4l^2}J_{13})\ddot{q}_2 \\ & + \frac{1}{2}m_1hr\ddot{q}_1 + (\frac{1}{4}m_1r^2 - \frac{r^2}{4l^2}J_{13})\ddot{q}_3,\end{aligned} \quad (6)$$

$$\begin{aligned}\frac{\partial G}{\partial \ddot{q}_3} = {} & (mr^2 + J + \frac{1}{4}m_1r^2 + \frac{r^2}{4l^2}J_{13})\ddot{q}_3 \\ & + \frac{1}{2}m_1hr\ddot{q}_1 + (\frac{1}{4}m_1r^2 - \frac{r^2}{4l^2}J_{13})\ddot{q}_2.\end{aligned} \quad (7)$$

2) Partial Derivatives for Potential Energy: Select horizontal plane though O_2O_3 as the zero potential energy surface, the potential energy of the system is shown as

$$P = m_1gh\cos q_1.$$

The partial derivations of P with q_1, q_2 and q_3 are represented as follows,

$$\frac{\partial P}{\partial q_1} = -m_1gh\sin q_1, \;\; \frac{\partial P}{\partial q_2} = 0, \;\; \frac{\partial P}{\partial q_3} = 0. \quad (8)$$

3) Appell's Equation of the Two-wheeled Robot: Two wheels are applied with torques τ_1 and τ_2. Hence the Appell's Equation (4) can be represented as

$$\begin{cases} \dfrac{\partial G}{\partial \ddot{q}_1} - \dfrac{\partial P}{\partial q_1} = -\tau_1 - \tau_2 \\ \dfrac{\partial G}{\partial \ddot{q}_2} - \dfrac{\partial P}{\partial q_2} = \tau_1 \\ \dfrac{\partial G}{\partial \ddot{q}_3} - \dfrac{\partial P}{\partial q_3} = \tau_2. \end{cases} \quad (9)$$

Substitute (5) ∼(8) to (9), and it can be rewritten as

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}'(\boldsymbol{q}) = \boldsymbol{\tau}, \quad (10)$$

in which

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}, \boldsymbol{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$$

$$\boldsymbol{C} = \boldsymbol{0}, \boldsymbol{G}' = \begin{bmatrix} -m_1ghq_1 & 0 & 0 \end{bmatrix}^T, \boldsymbol{\tau} = \begin{bmatrix} -\tau_1 - \tau_2 & \tau_1 & \tau_2 \end{bmatrix},$$

$$\begin{aligned} & d_{11} = m_1h^2 + J \quad d_{12} = d_{13} = d_{21} = d_{31} = \frac{1}{2}m_1hr \\ & d_{22} = d_{33} = mr^2 + J + \frac{1}{4}m_1r^2 + \frac{r^2}{4l^2}J_{13} \\ & d_{23} = d_{32} = \frac{1}{4}m_1r^2. \end{aligned}$$

Equation (10) is the dynamic model of the two-wheeled self-balancing robot. It can be represented as

$$\begin{cases} \ddot{q}_1 = a_1q_1 + b_{11}\tau_1 + b_{12}\tau_2 \\ \ddot{q}_2 = a_2q_1 + b_{21}\tau_1 + b_{22}\tau_2 \\ \ddot{q}_3 = a_3q_1 + b_{31}\tau_1 + b_{32}\tau_2, \end{cases} \quad (11)$$

in which,

$$\begin{aligned} a_1 = {} & \frac{1}{M_1}\left(ghm_1(2mr^2 + m_1r^2 + 2J)\right) \\ a_2 = a_3 = {} & \frac{1}{M_1}\left(-gh^2m_1^2r\right) \\ b_{11} = b_{12} = {} & \frac{1}{M_1}\left(-hm_1r - 2mr^2 - m_1r^2 - 2J\right) \\ b_{21} = b_{32} = {} & \frac{1}{M_2}\left(4h^2l^2mm_1r^2 + 2hl^2mm_1r^3 + 4Jh^2l^2m_1\right. \\ & + 2Jhl^2m_1r + 4Jl^2mr^2 + Jl^2m_1r^2 + J_{13}h^2m_1r^2 \\ & \left. + J_{13}hm_1r^3 + 4J^2l^2 + JJ_{13}r^2\right) \\ b_{22} = b_{31} = {} & \frac{r}{M_2}\left(2hl^2mm_1r^2 + 2Jhl^2m_1 - Jl^2m_1r\right. \\ & \left. + J_{13}h^2m_1r + J_{13}hm_1r^2 + JJ_{13}r\right) \end{aligned}$$

$$
\begin{aligned}
M_1 &= 2Mmh^2r^2 + 2JMh^2 + J_{12}Mr^2 + 2J_{12}mr^2 + 2JJ_{12} \\
M_2 &= 4Mh^2l^2m^2r^4 + 8JMh^2l^2mr^2 + 2J_{12}Ml^2mr^4 \\
&\quad + 4J_{12}l^2m^2r^4 + 2J_{13}Mh^2mr^4 + 4J^2Mh^2l^2 \\
&\quad + 2JJ_{12}Ml^2r^2 + 8JJ_{12}l^2mr^2 + 2JJ_{13}Mh^2r^2 \\
&\quad + J_{12}J_{13}Mr^4 + 2J_{12}J_{13}mr^4 + 4J^2J_{12}l^2 \\
&\quad + 2JJ_{12}J_{13}r^2
\end{aligned}
$$

4) Final Dynamic Model: Because the pitch angle, the yaw angle and the displacement of the robot are what we concerned, according to (1) and (2) and let $q_4 = d$, $q_5 = \psi$, $\ddot{q}_4$ and $\ddot{q}_5$ can be represented as

$$
\begin{aligned}
\ddot{q}_4 &= \ddot{d} = a_2rq_1 + \frac{r}{2}(b_{21}+b_{22})(\tau_1+\tau_2) \\
\ddot{q}_5 &= \ddot{\psi} = \frac{r}{2l}(b_{21}-b_{22})(\tau_1-\tau 2).
\end{aligned}
$$

To decouple the above equations, let $u_1 = \tau_1+\tau_2$, $u_2 = \tau_1-\tau_2$. The final dynamic model of the two-wheeled robot is shown as (12)

$$
\begin{cases}
\ddot{q}_1 = a_1q_1 + b_{11}u_1 \\
\ddot{q}_4 = a_2rq_1 + \dfrac{r}{2}(b_{21}+b_{22})u_1 \\
\ddot{q}_5 = \dfrac{r}{2l}(b_{21}-b_{22})u_2
\end{cases}. \tag{12}
$$

Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_4$, $x_4 = \dot{q}_4$, $x_5 = \dot{q}_5$, $x_6 = \dot{q}_5$, the dynamic model shown in (12) can be transformed into state space equations as shown in (13).

$$
\begin{cases}
\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu} \\
\boldsymbol{y} = \boldsymbol{Cx} = \boldsymbol{Ex}
\end{cases} \tag{13}
$$

in which

$$
\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_2r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad
\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ b_{11} & 0 \\ 0 & 0 \\ \frac{r}{2}(b_{21}+b_{22}) & 0 \\ 0 & 0 \\ 0 & \frac{r}{2l}(b_{21}-b_{22}) \end{bmatrix},
$$

$\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$.

III. Design of Sliding Mode Controller

The dynamic model of the robot shown in (13) can be divided into two independent subsystems: balance subsystem and direction subsystem. The balance subsystem includes the displacement and pitch angle, which are shown in (14). The control purpose is that the displacement tracking a sine signal and the pitch angle trending to zero. The direction angle subsystem includes the yaw angle, which shown in (15), its control purpose is that the yaw angle tracking a sine signal. Sliding mode controllers are designed for each of the subsystems.

$$
\begin{cases}
\ddot{q}_1 = a_1q_1 + b_{11}u_1 \\
\ddot{q}_4 = a_2rq_1 + \dfrac{r}{2}(b_{21}+b_{22})u_1
\end{cases} \tag{14}
$$

$$
\ddot{q}_5 = \frac{r}{2l}(b_{21}-b_{22})u_2 \tag{15}
$$

A. Sliding mode controller design for balance subsystem

Equation (14) can be transformed into state space equations as shown in (16). The control purpose is $x_1 \to 0$, $x_2 \to 0$, $x_3 \to x_{3d} = \sin t$, $x_4 \to \dot{x}_{3d} = \cos t$.

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = a_1x_1 + b_{11}u_1 \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = a_2rx_1 + \dfrac{r}{2}(b_{21}+b_{22})u_1
\end{cases} \tag{16}
$$

Based on the decoupling method for under-actuated system described in [5], take z_1, z_2, z_3 and z_4 as

$$
\begin{cases}
z_1 = x_3 - \dfrac{r}{2b_{11}}(b_{21}+b_{22})x_1 \\
z_2 = x_4 - \dfrac{r}{2b_{11}}(b_{21}+b_{22})x_4 \\
z_3 = x_1 \\
z_4 = x_2
\end{cases}. \tag{17}
$$

Let $v = a_1x_1 + b_{11}u_1$, the following equations can be obtained,

$$
\begin{cases}
\dot{z}_1 = z_2 \\
\dot{z}_2 = \left(a_2r - \dfrac{a_1r(b_{21}-b_{22})}{2b_{11}}\right)z_3 = Tz_3 \\
\dot{z}_3 = z_4 \\
\dot{z}_4 = v
\end{cases}. \tag{18}
$$

The tracking error of displacement of the robot is $e_1 = z_1 - z_{1d}$, the error equations can be obtained as

$$
\begin{cases}
e_1 = z_1 - z_{1d} \\
e_2 = \dot{e}_1 = z_2 - \dot{z}_{1d} \\
e_3 = \ddot{e}_1 = Tz_3 - \ddot{z}_{1d} \\
e_4 = \dddot{e}_1 = Tz_4 - \dddot{z}_{1d} \\
\dot{e}_4 = \ddddot{e}_1 = Tv - \ddddot{z}_{1d}
\end{cases}. \tag{19}
$$

The sliding surface is set as

$$
\begin{aligned}
&s_1 = c_1e_1 + c_2e_2 + c_3e_3 + e_4, \quad c_1, c_2, c_3 > 0, \\
&\dot{s}_1 = c_1e_2 + c_2e_3 + c_3e_4 + \dot{e}_4
\end{aligned}
$$

. Let $\dot{s}_1 = 0$, we can get the equivalent controlled quantity

$$
\begin{aligned}
u_{1eq} = &\frac{1}{T}(-c_1z_2 - c_2Tz_3 - c_3Tz_4 + +c_1\dot{z}_{1d} \\
&+ c_2\ddot{z}_{1d} + c_3\dddot{z}_{1d} + \ddddot{z}_{1d})
\end{aligned}
$$

Let $\dot{s}_1 = -w_1s_1 - k_1\operatorname{sign}(s_1)$, $w_1, k_1 > 0$, the switching controlled quantity can be obtained

$$
u_{1sw} = \frac{1}{T}(-w_1s_1 - k_1\operatorname{sign}(s_1)).
$$

The total controlled quantity is

$$
u_1 = u_{1eq} + u_{1sw}.
$$

Take the Lyapunov function as $V_1 = \frac{1}{2}s_1^2$,

$$\dot{V}_1 = s_1\dot{s}_1 = s_1(-w_1 s_1 - k_1 \mathrm{sign}(s_1)) = -w_1 s_1^2 - k_1|s_1| \leqslant 0.$$

It means that the total controlled quantity u_1 can ensure that the sliding surface s_1 exists, and it can converge to the equilibrium point in finite time.

In sliding surface s_1, parameters c_1, c_2, c_3 should ensure that the roots of polynomial $\lambda^3 + c_3\lambda^2 + c_2\lambda_3 + c_1 = 0$ are all negative. Take the eigenvalue as -2, $\lambda^3 + 6\lambda^2 + 12\lambda_3 + 8 = 0$, so we can take $c_1 = 8$, $c_2 = 12$, $c_3 = 6$.

B. Sliding mode controller design for direction subsystem

Equation (15) can be transformed into state space equations as shown in (20). The control purpose is $x_5 \to x_{5d} = \sin t$, $x_6 \to \dot{x}_{5d} = \cos t$.

$$\begin{cases} \dot{x}_5 = x_6 \\ \dot{x}_6 = \dfrac{r}{2l}(b_{21} - b_{22})u_2 \end{cases} \tag{20}$$

The tracking errors of yaw angle and angular rate are $e_5 = x_5 - x_{5d}$, $e_6 = \dot{e}_5 = x_6 - \dot{x}_{5d}$. The sliding surface is set as

$$s_2 = c_4 e_5 + e_6, \quad c_4 > 0$$
$$\dot{s}_2 = c_4\dot{e}_5 + \dot{e}_6$$

Let $\dot{s}_2 = -w_2 s_2 - k_2 \mathrm{sign}(s_2)$, $w_2, k_2 > 0$, we can get the total controlled quantity u_2

$$u_2 = \frac{2l\left(\ddot{x}_{5d} - c_4 e_6 - w_2 c_4 e_5 - w_2 e_6 - k_2 \mathrm{sign}(s_2)\right)}{r(b_{21} - b_{22})},$$

in which $b_{21} \neq b_{22}$.

Take the Lyapunov function as $V_2 = \frac{1}{2}s_2^2$,

$$\dot{V}_2 = s_2\dot{s}_2 = s_2(-w_2 s_2 - k_2 \mathrm{sign}(s_2)) = -w_2 s_2^2 - k_2|s_2| \leqslant 0$$

It means that the total controlled quantity u_2 can ensure that the sliding surface s_2 exists, and it can converge to the equilibrium point in finite time.

IV. Simulation and Analysis of the Control System

The physical parameters of the two-wheeled self-balancing robot are shown in Table II.

The coefficients of the controller are $c_1 = 8$, $c_2 = 12$, $c_3 = 6$, $c_4 = 5$, $k_1 = 1$, $k_2 = 1$, $w_1 = 1$, $w_2 = 0.5$. The initial conditions of the robot system are, $x_{10} = 0.5$rad, $x_{20} = 0.5\mathrm{rad/s}$, $x_{30} = 0.5$m, $x_{40} = 0.5$m/s, $x_{50} = 0.5$rad, $x_{60} = 0.5\mathrm{rad/s}$. The desired displacement is $x_{3d} = \sin t$, the desired yaw angular signal is $x_{5d} = \sin t$. The simulation time is set to 30s.

TABLE II
PHYSICAL PARAMETERS OF THE ROBOT

Parameter	Value	Parameter	Value
h	0.18 m	r	0.17 m
J	0.029 kg$\cdot$ m^2	J_{12}	0.742 kg$\cdot$m^2
J_{13}	2.408 kg$\cdot$m^2	m_1	21.34 kg
m	0.5 kg	l	0.255 m

The simulation results are shown in Fig.2$\sim$Fig.5. Fig.2 shows the response curves of the pitch angle and pitch angular rate, Fig.3 shows the tracking curves of the displacement and velocity of this robot, Fig.4 shows the tracking curves of the yaw angle and yaw angular rate. Fig.5 shows the torques applied on the wheels. The solid lines in above figures are the ideal curves, and the dotted lines are the practical curves.

At the beginning of the simulation, the ideal signals had large errors with the practical signals. After about 3s, the errors between ideal and practical signals tended to be zero. This shows that the controller had a fast convergence speed. The practical pitch angle and pitch angular velocity curves in Fig.2 are not divergent but equiamplitude oscillatory, the amplitudes of these curves are about 0.2rad and 0.2rad/s, which shows that the two-wheeled robot was stable during the whole simulation.

In Fig.3 and Fig.4 showed that the position, velocity, yaw angle and angular velocity of the robot converged towards the ideal curves at beginning of simulation and coincided with the ideal curves at about 5s.

The control inputs are the torques applied on the wheels, which are shown in Fig.5. The control inputs chattered in a high frequency. One of the reasons is the inertia of the robot. It takes only two seconds for the curves to stabilize during the control process. This shows that the controller is highly effective.

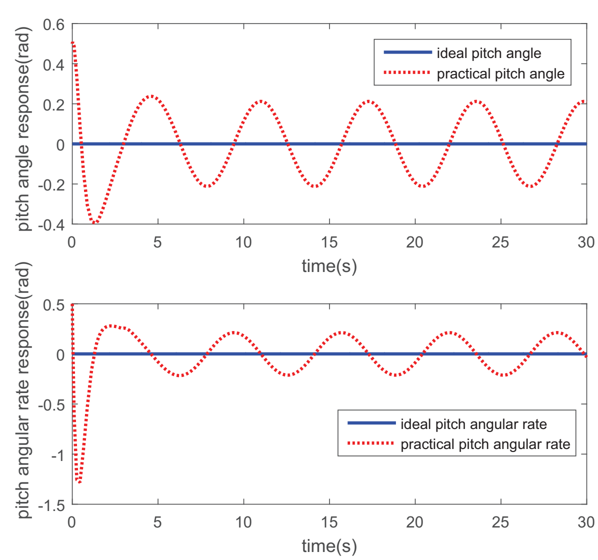

Fig. 2. The response curves of the pitch angle and angular rate

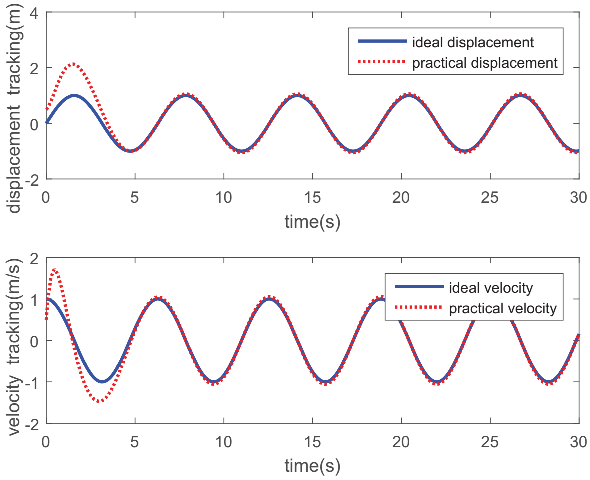

Fig. 3. The tracking curve of the displacement and velocity

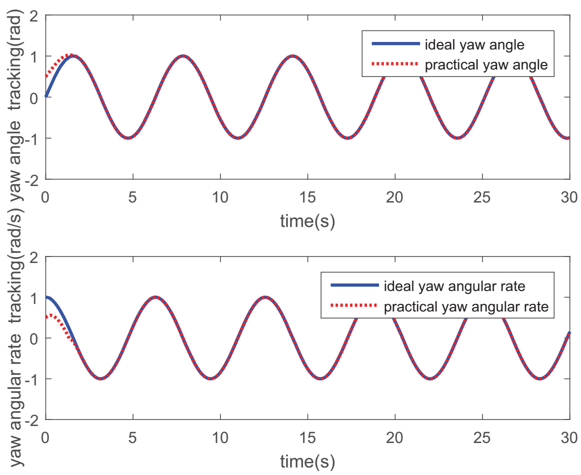

Fig. 4. The tracking curve of the yaw angle and angular rate

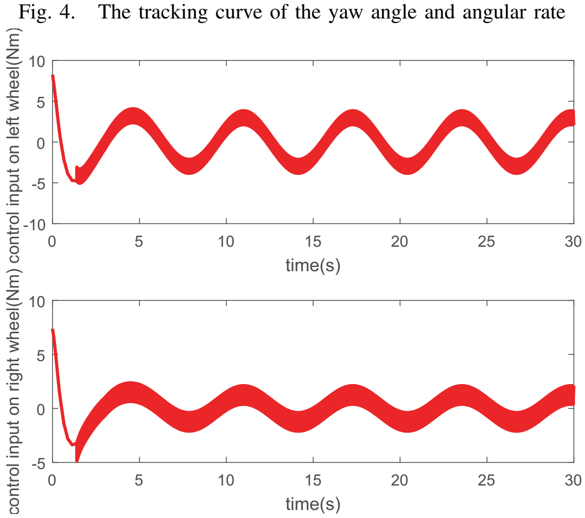

Fig. 5. Torques applied on the wheels

V. Conclusion

Two-wheeled self-balancing robot has caught an extensive attention all over the world, which provides a good experiment platform for various control methods. In this paper, the dynamic modeling of this robot based on Appell Equations is provided. The model is divided into two separate subsystems: balance subsystem and direction subsystem, and sliding mode controllers for each subsystems have accomplished based on the dynamic model. The designed controller can guarantee that the equilibrium point is reachable in finite time. And It can track the ideal displacement and yaw angle effectively and make the pitch angle not diverge but equiamplitude oscillate. The stability of the two-wheeled self-balancing robot by using the designed sliding mode method is verified in the simulation. It has shown that the designed controller has a fast convergence speed and a strong anti-interference ability, and is effective.

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