Sliding Mode Control of an Inverted Pendulum

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Chapter 6
Sliding Mode Control of an Inverted Pendulum

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Abstract This paper presents solutions to attenuate the chattering phenomenon raised by the classic sliding mode control. The first solution consists in approximating the discontinuity in the control law, origin of chatter effect, by using a continuous function. Another solution is to use the second order sliding mode control. Subsequently, these different algorithms will be applied to an inverted pendulum in order to achieve dynamic output tracking. Simulation results are presented to illustrate the efficiency of these algorithms.
Keywords – Sliding mode control · Chattering phenomenon · Saturation function · Twisting · Super-twisting · Inverted pendulum

6.1 Introduction

Since the late 1970s, the sliding mode control (SMC) have attracted a significant interest from the control research community, and this is due to its insensitivity to model parametric uncertainties and external disturbances (Bandyopadhyay et al., 2009; Utkin, 1977; Young et al., 1999). Indeed, the SMC design consists of two basic steps: selecting a stable sliding surface on the basis of control objectives and desired properties of the closed loop system, and synthesizing a discontinuous control law in such a way that the state trajectory of the system reaches the sliding surface in finite time and then remains there (Bandyopadhyay et al., 2009; Bregeaut, 2010). Yet, in practice and in the presence of switching imperfections, the control cannot switch at

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a very high frequency, and then the discontinuity in the control law yields to the so-called chattering phenomenon.
This phenomenon can actuate disregarded high-frequency dynamics, degrade system performances and even cause mechanical damages (Perruquetti and Barbot, 2002). In this regard, several approaches have been proposed in the literature in order to overcome this problem such as the one proposed by (Slotine and Sastry, 1983), which consists in substituting the signum function, origin of the discontinuity in the control law, by a smooth function like saturation function (Hung et al., 1993; Perruquetti and Barbot, 2002; Slotine, 1984; Slotine and Li, 1991). Actually, the state trajectory will evolve inside a thin boundary layer neighboring the switching surface and then the chatter effect will be attenuated close to this surface.
Another approach, introduced by (Levant, 2003), the higher-order sliding mode control (HOSMC), consists in constraining the state trajectory to reach in finite time the sliding set defined by:

$$S^r = \left\{ x \in \mathbb{R}^n : s = \dot{s} = \ddot{s} = ... = s^{(r-1)} = 0 \right\} \tag{6.1}$$

In addition to the chatter elimination, HOSMC ensure a better accuracy and resolve the problem of the restriction to a relative degree one, while preserving main features of the standard sliding mode (Levant, 2003; Perruquetti and Barbot, 2002). In this study, only second-order sliding mode control (SOSMC) and specially twisting and super-twisting algorithms (Emelyanov et al., 1996; Levant, 1993; Perruquetti and Barbot, 2002; Yuri Shtessel and Levant, 2014) will be demonstrated for the control of the inverted pendulum as a nonlinear and unstable system.
In section 6.2, we represent the first order sliding mode theory. Section 6.3 is devoted to second order sliding mode algorithms. In section 6.4, Simulations results for both first and second order sliding mode algorithms applied to the inverted pendulum will be shown. Finally, concluding remarks will be given in section 6.5.

6.2 First Order Sliding Mode Control

Well known for its high accuracy and robustness against parametric variations and external disturbances, the sliding mode control consists, by means of a discontinuous control, in constraining the system states to reach, in finite time, and then to evolve onto a sliding manifold defined by:

$$S = \{x \in \mathbb{R}^n : s = 0\} \tag{6.2}$$

Hence, the design of SMC involves two major steps: the first is to select a stable surface depending upon desired system dynamics, and the second is to

synthesize a control law so that the system states trajectory evolve onto the chosen surface (Bregeaut, 2010; Levant, 2003).
Consider a single input nonlinear system as

$$\dot{x} = f(x) + b(x)\,u \tag{6.3}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}$ is the control input. The control law of the sliding mode controller is expressed as follows:

$$u = u_{eq} + u_d \tag{6.4}$$

with u_{eq} is the equivalent control and u_d is the discontinuous control. Actually, using the equivalent control approach (DeCarlo and S.H. Zak, 1988; Gao and Hung, 1992; Utkin, 1992), u_{eq} is obtained by setting $\dot{s} = 0$ to ensure the maintain of the state trajectory onto the switching surface $s = 0$ during the sliding mode (Bandyopadhyay et al., 2009; Hung et al., 1993; Perruquetti and Barbot, 2002).
For the system (6.3), the first time derivative of the sliding variable $s = s(x)$ is given by:

$$\dot{s} = \frac{\partial s}{\partial x}\left(f(x) + b(x)\,u\right) \tag{6.5}$$

Assuming that $\left(\frac{\partial s}{\partial x} b(x)\right)$ is invertible, the equivalent control law is expressed by:

$$u_{eq} = -\left(\frac{\partial s}{\partial x} b(x)\right)^{-1} \frac{\partial s}{\partial x} f(x) \tag{6.6}$$

Yet, a η-reachability condition defined in (Bandyopadhyay et al., 2009; Edwards and K.Spurgeon, 1998; Perruquetti and Barbot, 2002) as follows:

$$s\dot{s} \leq -\eta\,|s|\,, \;\; \eta > 0 \tag{6.7}$$

must be met in order to ensure a finite time convergence to the sliding surface. Hence, the classic sliding mode control that satisfies the above condition is given by (Riachy, 2008):

$$u = -\left(\frac{\partial s}{\partial x} b(x)\right)^{-1} \left(\frac{\partial s}{\partial x} f(x) + k\,\mathrm{sign}(s)\right) \tag{6.8}$$

where k is a positive constant that verifies

$$s\dot{s} = s\left(-k\,\mathrm{sign}(s)\right) = -k\,|s| \leq -\eta\,|s| \Leftrightarrow k \geq \eta \tag{6.9}$$

and sign is the signum function defined by

$$\mathrm{sign}(s) = \left|\begin{array}{lll} 1 & \text{if} & s > 0 \\ 0 & \text{if} & s = 0 \\ -1 & \text{if} & s < 0 \end{array}\right. \tag{6.10}$$

Nevertheless, the discontinuity of the signum function in the vicinity of the sliding surface $s = 0$ involve an infinite frequency commutation of the control law which cannot exist in practice because of switching imperfections. This leads to the so-called chattering phenomenon (see Fig. 6.1) which can degrade performances of the controlled system, excite disregarded high-frequency dynamics and deteriorate the control member.

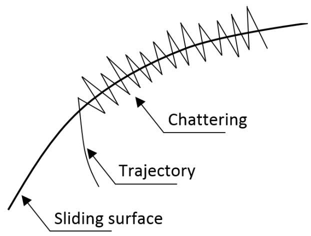


Fig. 6.1: Chattering phenomenon

To attenuate the chatter effect, several approaches were proposed in literature. The main idea of the first approach is to substitute signum function by a continuous one such as saturation function defined by:

$$\mathrm{sat}(s,\varphi) = \left| \begin{array}{ll} \dfrac{s}{\varphi} & \text{if} \quad \left|\dfrac{s}{\varphi}\right| \leq 1 \\ \mathrm{sign}(s) & \text{if} \quad \left|\dfrac{s}{\varphi}\right| > 1 \end{array} \right. \tag{6.11}$$

The continuous function will ensure the convergence of system state trajectory to a thin boundary layer in the vicinity of the sliding surface. Consequently, the chatter will be reduced to the detriment of optimum accuracy and robustness of sliding mode.

The above restrictions were removed with the higher order sliding mode control which satisfies a finite time convergence of not only the sliding variable to zero, but also of a finite number of its time derivatives. In other words, for a relative degree $r > 1$, i.e. $\frac{\partial}{\partial u} s^{(i)} = 0\,(i = [|1, r-1|])$ and $\frac{\partial}{\partial u} s^{(r)} \neq 0$, it is necessary to ensure, for stability reasons, the following equalities:

$$s = \dot{s} = \ddot{s} = ... = s^{(r-1)} = 0 \tag{6.12}$$

Hence, HOSMC remove another drawback of classic sliding mode: the restriction to a relative degree one, but this will require an additional information about the $(r-1)$ time derivatives of s. In what follows, only second order sliding mode algorithms will be treated.

6.3 Second Order Sliding Mode Control

The second order sliding mode controller consists on forcing the system state trajectories to evolve in finite time onto the second order sliding set defined by:

$$S^2 = \{x \in \mathbb{R}^n : s = \dot{s} = 0\} \tag{6.13}$$

Return to the system

$$\begin{aligned} \dot{x} &= f(x) + b(x)\,u \\ s &= s(x) \end{aligned} \tag{6.14}$$

with $x \in X = \{x : |x_i| \leq x_{i\,\max}, i \in [|1,n|]\}$ and $u \in U = \{u : |u| \leq u_{\max}\}$. Let the relative degree be equal to one, differentiating the sliding variable s twice yields to the following relation:

$$\ddot{s} = \frac{\partial \dot{s}}{\partial x}\left(f(x) + b(x)\,u\right) + \frac{\partial \dot{s}}{\partial u}\dot{u} = \alpha(x) + \beta(x)\,\dot{u} \tag{6.15}$$

It is assumed that if $|s(x)| < s_0$ then there are positive constants Φ , Γ_m and Γ_M such that

$$\begin{aligned} |\alpha(x)| &< \Phi \\ 0 < \Gamma_m \leq \beta(x) &\leq \Gamma_M \end{aligned} \tag{6.16}$$

Subsequently, Only Twisting and Super-Twisting algorithms will be studied.

6.3.1 Twisting algorithm

The twisting algorithm, one of the first known second order sliding mode algorithms, ensure a finite time convergence of the state trajectory to the origin of the phase plane $(s, \dot{s})$ after executing a certain number of rotations around it (Fig. 6.2), and this is due to the commutation of the control between two values.

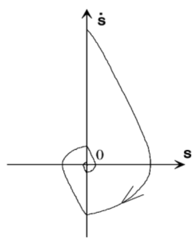


Fig. 6.2: Twisting Algorithm phase trajectory

For a relative degree 1, the control algorithm is defined by the following control law:

$$\dot{u}_{Tw} = \left| \begin{array}{lll} -u & \text{if} & |u| > u_{max} \\ -k_m \text{sign}(s) & \text{if} & s\dot{s} \leq 0 \text{ and } |u| \leq u_{max} \\ -k_M \text{sign}(s) & \text{if} & s\dot{s} > 0 \text{ and } |u| \leq u_{max} \end{array} \right. \tag{6.17}$$

The sufficient conditions that ensure a finite time convergence to the second order sliding set are:

$$k_M > k_m > 0,\ k_m > \frac{4\Gamma_M}{s_0},\ k_m > \frac{\Phi}{\Gamma_m},\ \Gamma_m k_M - \Phi > \Gamma_M k_m + \Phi \tag{6.18}$$

6.3.2 Super-twisting algorithm

Unlike twisting algorithm, this algorithm is only able to stabilize in finite time (Fig. 6.3) systems whose relative degree is equal to one and then it does not require information about the time derivative of the sliding variable.

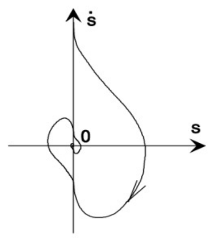


Fig. 6.3: Super-twisting algorithm phase trajectory

The control law is given by:

$$\begin{array}{c} u_{ST} = -\alpha |s|^{\rho} \text{sign}\,(s) + u_1 \\ \dot{u}_1 = -w\, \text{sign}\,(s) \end{array} \tag{6.19}$$

and the corresponding sufficient conditions are:

$$w > \frac{\Phi}{\Gamma_m},\ \ \alpha^2 \geq \frac{4\Phi}{{\Gamma_m}^2} \frac{\Gamma_M\,(w+\Phi)}{\Gamma_m\,(w-\Phi)},\ \ 0 < \rho \leq 0.5 \tag{6.20}$$

Choosing $\rho = 0.5$ ensures the achievement of real second-order sliding mode.

6.4 Application to an Inverted Pendulum

The inverted pendulum is an interesting classic system which consists of a pendulum attached by a rotation joint to a cart. This cart, driven by a DC motor, can move along a horizontal guide rail in order to maintain the pendulum in its vertical balance. This system has two degrees of freedom whose generalized coordinates are: x for the horizontal cart movement and θ for the pendulum rotation (Fig. 6.4).

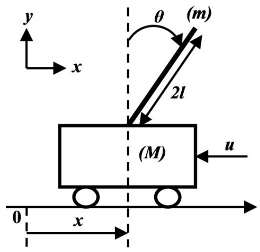


Fig. 6.4: Schematic of the inverted pendulum system

Let $x = \begin{bmatrix} x_1 \; x_2 \end{bmatrix}^\top = \begin{bmatrix} \theta \; \dot{\theta} \end{bmatrix}^\top$ be the state vector and u be the force applied to the cart, the dynamic equations of the inverted pendulum are:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) + b(x_1, x_2)u \end{cases} \tag{6.21}$$

with:

$$\begin{aligned} f(x_1, x_2) &= \frac{(M+m)\, g \sin x_1 - mlx_2^2 \sin x_1 \cos x_1}{\frac{4}{3}\,(M+m)\, l - ml \cos^2 x_1} \\ b(x_1, x_2) &= \frac{\cos x_1}{\frac{4}{3}\,(M+m)\, l - ml \cos^2 x_1} \end{aligned}$$

such that $M = 1\, kg$ is the cart mass, $m = 0.1\, kg$ is the pendulum mass, $l = 0.5\, m$ is the half length of the pendulum, $g = 9.8 m/s^2$ is the acceleration of gravity (Wang, 1994).

6.4.1 Application of first order sliding mode control

6.4.1.1 Classic sliding mode control

By referring to (Slotine and Li, 1991), the sliding function is chosen as:

$$s = \lambda\, e + \dot{e} \tag{6.22}$$

where $e = x_1 - x_{d1}$ is the tracking error and λ is a positive constant. The first time derivative of s is:

$$\begin{aligned} \dot{s} &= \lambda\,(x_2 - \dot{x}_{d1}) + (f - \ddot{x}_{d1}) + b\,u \\ &= h + b\,u \end{aligned} \tag{6.23}$$

Using (6.8) and (6.23), the control law of the classic sliding mode controller is expressed as:

$$u = -b^{-1}\,(h + k\,\mathrm{sign}\,(s)) \tag{6.24}$$

For simulation results, the design parameters are chosen as follows: $\lambda = 5$ and $k = 18$. The reference signal and the initial conditions are respectively $x_{d1} = \frac{\pi}{30}\sin t$ and $x_0 = [0.2 \quad 0]^\top$. Thus, by applying the classic sliding mode to the inverted pendulum, we obtain results shown in Fig. 6.5.

Fig. 6.5 shows clearly the chattering phenomenon present in the control input. Thus, in the next step, the signum function will be replaced by the saturation function in order to attenuate the chatter effect.

6.4.1.2 Sliding mode control with saturation

Substituting the signum function by the saturation function defined in (6.11) in the control law (6.24) yields to:

$$u = -b^{-1}\,(h + k\,\mathrm{sat}\,(s, \varphi)) \tag{6.25}$$

where φ is chosen to be equal to 0.5. Simulation results are shown in Fig. 6.6.

These results show the efficiency of this approach in reducing the chatter effect. However, the existence of the boundary layer neighboring the sliding surface may affect the main characteristics of the sliding mode control such as robustness.

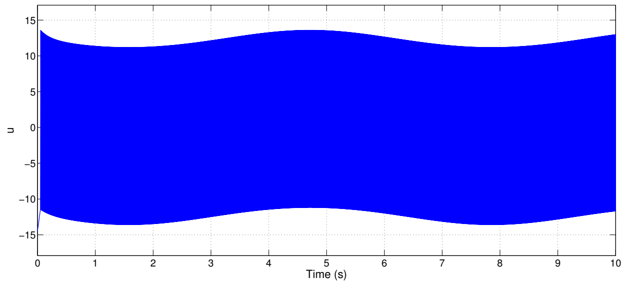
6.4.2 Application of Second Order Sliding Mode Control

6.4.2.1 Twisting algorithm

Using (6.23), the second time derivative of the sliding function is given by:

$$\ddot{s} = \lambda\,(f + b\,u - \ddot{x}_{d1}) + \left(\dot{f} + \dot{b}\,u - \dddot{x}_{d1}\right) + b\,\dot{u} = d + b\,\dot{u} \tag{6.26}$$

where d and b must verify relations given in (6.16).



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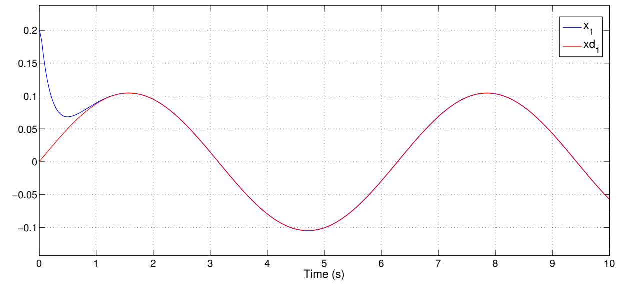

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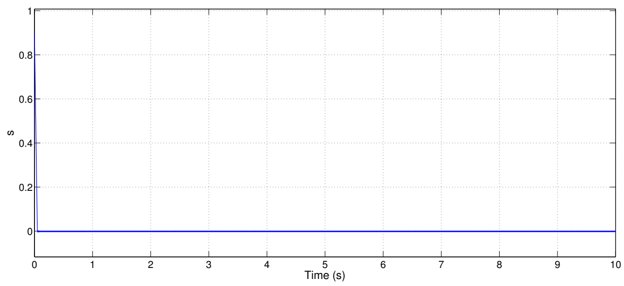

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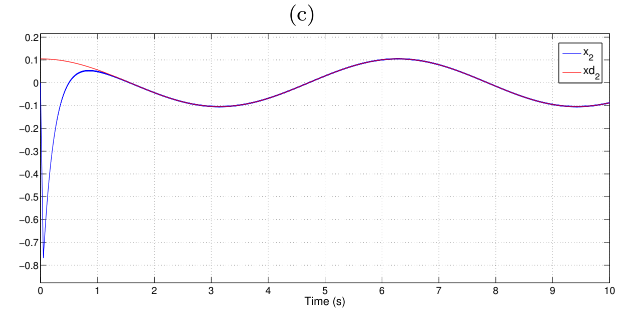

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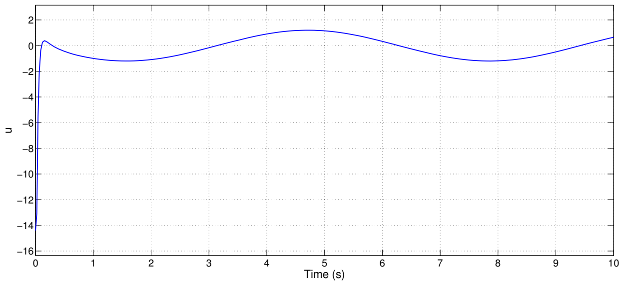
Fig. 6.5: (a) control input, (b) angular displacement, (c) sliding function, (d) angular velocity for classic SMC



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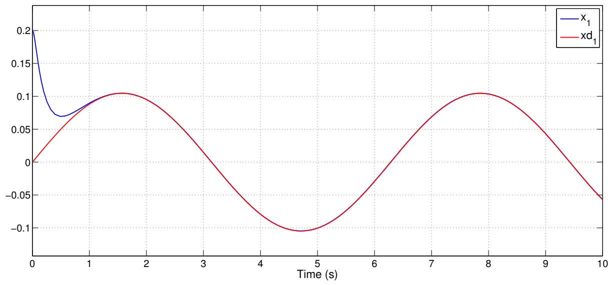

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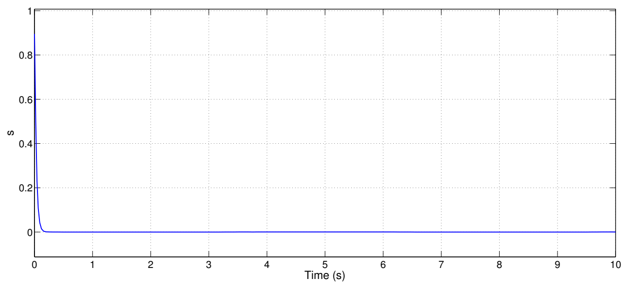

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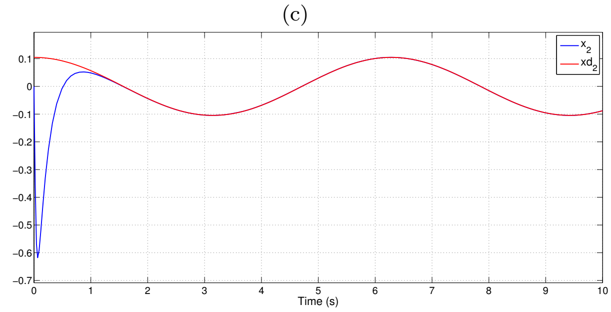

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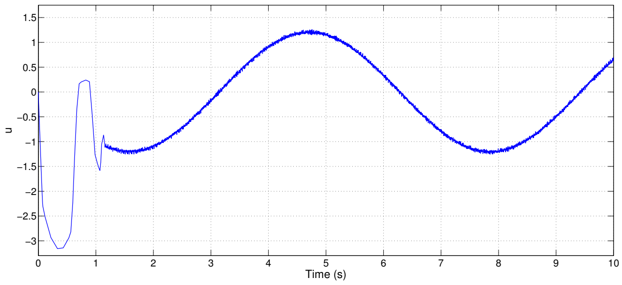
Fig. 6.6: (a) control input, (b) angular displacement, (c) sliding function, (d) angular velocity for SMC with saturation



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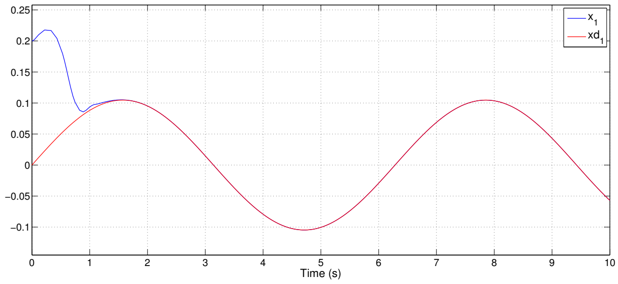

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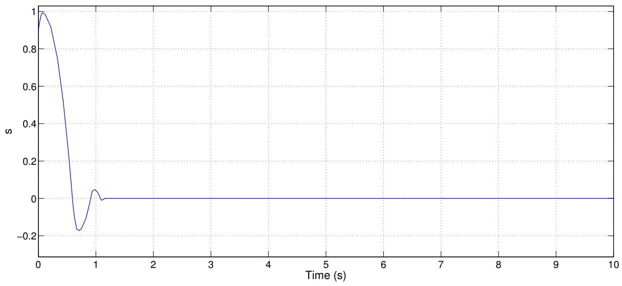

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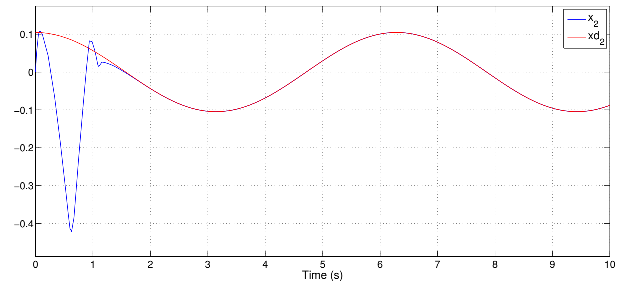


(d)

Fig. 6.7: (a) control input, (b) angular displacement, (c) sliding function, (d) angular velocity for twisting algorithm

Then, using (6.17) the control law of twisting algorithm is expressed as:

$$\dot{u} = b^{-1} \left(-d + \dot{u}_{Tw} \right) \tag{6.27}$$

By fulfilling the necessary conditions (6.16) and the sufficient conditions (6.18), the optimum value of k_m and k_M, chosen after some trials, are respectively 7 and 35.

Fig. 6.7 shows that the twisting Algorithm reduces the chattering phenomenon while Fig. 6.8 shows that the system state trajectory converges to the origin of the phase plane $(s, \dot{s})$.

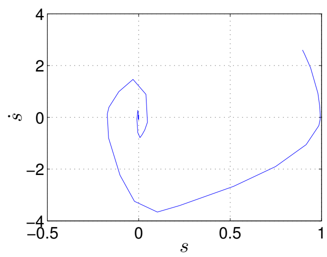


Fig. 6.8: Twisting Algorithm phase trajectory

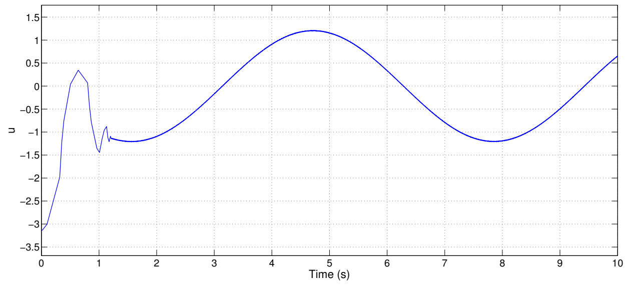
6.4.2.2 Super-twisting algorithm

Using (6.19) and (6.23), the control law for the super-twisting algorithm is given by:

$$u = b^{-1} \left(-h + u_{ST} \right) \tag{6.28}$$

In accordance with the conditions given in (6.16) and (6.20), and after performing some simulations, we choose and $w = 8$, $\alpha = 2$ and $\rho = 0.5$.

Both Fig. 6.9 and Fig. 6.10 show that the super-twisting algorithm ensures a finite time convergence of the system trajectory to the origin of the phase plane. In addition, by comparing to Fig. 6.7(a) and Fig. 6.8(a), we can conclude that super-twisting algorithm is better than twisting algorithm with regards to the reduction of chattering effect in the control input.



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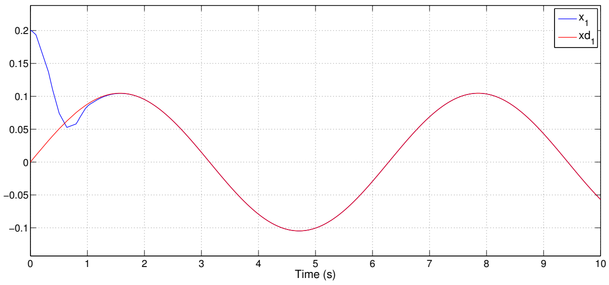

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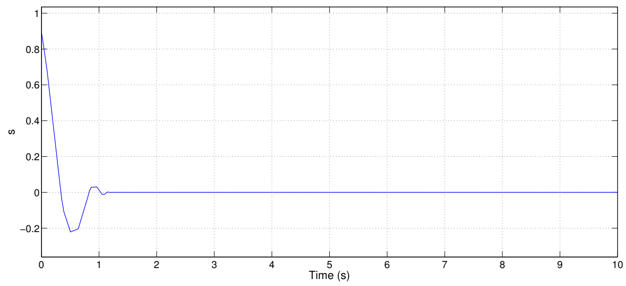

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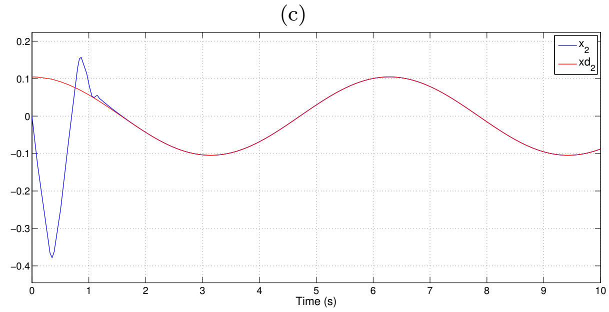

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Fig. 6.9: (a) control input, (b) angular displacement, (c) sliding function, (d) angular velocity for super-twisting algorithm

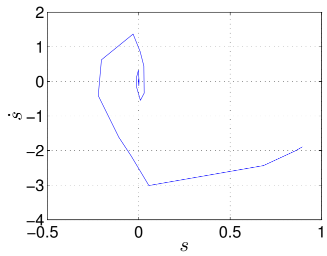

Fig. 6.10: Super-twisting algorithm phase trajectory

6.5 Conclusion

This paper has proposed some solutions to attenuate chattering phenomenon present in classic sliding mode control. At first, a continuous approximation has been made in the vicinity of the sliding surface and this by replacing the signum function by the saturation one. However, introducing a boundary layer may affect the main features of sliding mode such as robustness. Then, a second order sliding mode control algorithms: twisting and super-twisting algorithms, have been demonstrated on the inverted pendulum system. The simulations results have shown their efficiency in attenuating chatter effect while ensuring a finite time convergence and high accuracy.

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