Graph Theory Capstone

An Introduction to Undergraduate and Graduate Graph Theory with Projects by Nick and Cierra

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Introduction to Graph Theory: Douglass B. West
Welcome!
Königsberg Bridge
Coloring
Euler's Formula
Graphs and Matrices: R. B. Bapat
Linear Algebra
Adjacency Matrix
Eigenvalues
Bounds on Eigenvalues
Incidence Matrix
Graph and Laplacian Matrix
Friendship Theorem

Outline

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- Welcome!
- Königsberg Bridge
- Coloring
- Euler's Formula

Graphs and Matrices: R. B. Bapat
- Linear Algebra
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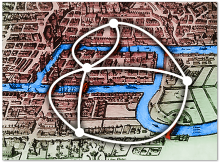
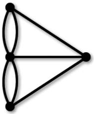
  - Eigenvalues
  - Bounds on Eigenvalues
- Incidence Matrix
- Graph and Laplacian Matrix
- Friendship Theorem

Welcome!

- Why a capstone?
- Outline
- Introduction

Königsbergs Seven Bridges





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Coloring

- Planar Graph

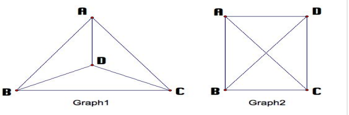

Figure: A Planar Graph

- Four Color Problem

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Non-Planar Graph

Non-planar Graph

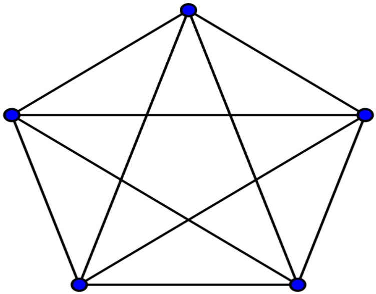

Figure: A Non-Planar Graph: K_5

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Euler's Formula

$$n - e + f = 2$$

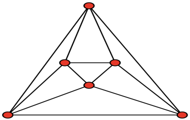

Figure: A Plane Graph

Vertices (n): 6
Faces (f): 8
Edges (e): 12
$6 - 12 + 8 = 2$

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GRAPH THEORY AND LINEAR ALGEBRA

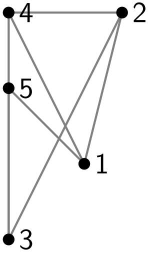
Adjacency Matrix

Let G be a graph with $V(G) = \{1, ..., n\}$ and $E(G) = \{e_1, ..., e_j\}$. The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix defined as follows.

Definition

If $i \neq j$ then the (i, j)-entry of $A(G)$ is 0 if vertices i and j are nonadjacent, and the (i, j)-entry of $A(G)$ is 1 if vertices i and j are adjacent. The (i, i)-entry of $A(G)$ is 0 for i=1,...,n.

Graph and Adjacency Matrix



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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Adjacency Matrix

Let G be a connected graph with vertices $1, ..., n$ and let A be the adjacency matrix of G. If i, j are vertices of G with $d(i, j) = m$ (minimum distance from i to j), then the matrices $I, A, ..., A^m$ are linearly independent.

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Adjacency Matrix

Let G be a connected graph with k distinct eigenvalues and let d be the diameter of G. Then $k > d$.

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Adjacency Matrix

Let G be a with vertices $V(G) = 1, ..., n$ and let A be the adjacency matrix of G. Then

$$detA = \sum(-1)^{n-c_1(H)-c(H)}2^{c(H)}$$

where the summation is over all spanning elementary subgraphs H of G.

- $c(H)$ is the number of components in a subgraph H which are cycles.
- $c_1(H)$ is the number of components in a subgraph H which are edges.

Adjacency Matrix

Let G be a with vertices $V(G) = 1, ..., n$ and let A be the adjacency matrix of G. Let

$$\phi_\lambda(A) = det(\lambda I - A) = \lambda^n + c_1\lambda_{n-1} + ... + c_n\lambda_n$$

be the characteristic polynomial of A. Suppose $c_3 = c_5 = ... = c_{2k-1} = 0$ Then G has no odd cycle of length i, $3 \leq i \leq 2k-1$. Furthermore, the number of $(2k-1)$-cycles in G is $-\frac{1}{2}c_{2k+1}$

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Eigenvalues of Complete Graphs

Definition

A complete graph is a graph in which every two vertices are adjacent. The symbol K_n is used for the complete graph with n vertices. A complete bipartite graph is a bipartite graph in which each pair of vertices (with one vertex in each partite set) are adjacent. For a graph with partite sets of size p and q, the symbol $K_{p,q}$ is used.

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Eigenvalues of Complete Graphs

The eigenvalues of K_n are $n-1$ with multiplicity 1, and -1 with multiplicity $n-1$. The eigenvalues of $K_{p,q}$ are $\sqrt{pq}, -\sqrt{pq}$ with multiplicity 1, and 0 with multiplicity $p+q-2$.

Eigenvalues of Complete Graphs

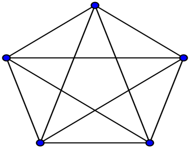

Figure: The complete graph K_5

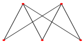

Figure: The complete bipartite graph $K_{3,2}$

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Eigenvalues of Path Graphs

Definition

A path graph is a tree with two end vertices of degree 1, and all other vertices of degree 2. A path graph can be drawn so that all of its vertices and edges lie on a single straight line.

Figure: The path graph P_6

The eigenvalues of P_n are $2\cos(\frac{\pi * k}{n+1}), k = 1, 2, ..., n$

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Calculating the Eigenvalues

Figure: P_6

Observe $n = 6, k = \{1, 2, 3, 4, 5, 6\}$

Eigenvalues:

$2\cos(\frac{\pi}{7}), 2\cos(\frac{2\pi}{7}), 2\cos(\frac{3\pi}{7}), 2\cos(\frac{4\pi}{7}), 2\cos(\frac{5\pi}{7}), 2\cos(\frac{6\pi}{7})$

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Bounds on Eigenvalues

Let G be a graph with n vertices and m edges, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\lambda_1 \leq \left(\frac{2m(n-1)}{n}\right)^{\frac{1}{2}}$$

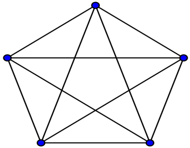

Figure: K_5

$\lambda_1 \leq \left(\frac{2*10(5-1)}{5}\right)^{\frac{1}{2}}$
$\lambda_1 \leq 16^{\frac{1}{2}}$
$\lambda_1 \leq 4$

Bounds on Eigenvalues

Recall coloring: $\chi(G)$ is the chromatic number. The minimum number of colors that is required to give a proper coloring.

For any graph G,

$$\chi(G) \leq 1 + \lambda_1(G)$$

Let G be a graph with n vertices and with at least one edge. Then,

$$\chi(G) \geq 1 - \frac{\lambda_1(G)}{\lambda_n(G)}$$

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Incidence Matrix

Let G be a graph with $V(G) = \{1, ..., n\}$ and $E(G) = \{e_1, ..., e_m\}$. Suppose each edge of G is assigned an orientation, which is arbitrary but fixed. The incidence matrix of G, denoted by $Q(G)$, is the $n \times m$ matrix defined as follows.

Definition

The (i, j)-entry of $Q(G)$ is 0 if vertex i and edge e_j are not incident. If e_j originates at i, then the (i, j)-entry of $Q(G)$ is 1, and if e_j terminates at i, then the entry is -1.

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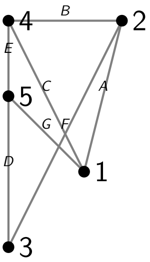
Graph and Incidence Matrix



$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Incidence Matrix

If G, with incidence matrix $Q(G)$, is a connected graph on n vertices and has k connected components, then $\text{rank}Q(G) = n - k$.

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Incidence Matrix

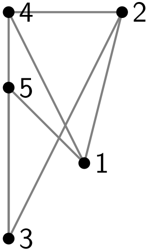
Let G be a graph with incidence matrix $Q(G)$. Then $Q(G)$ is totally unimodular.

- A matrix is said to be totally unimodular if the determinant of any square submatrix is either 0, -1, or $+1$.

Diagonal Matrix

Definition

Let $D(G)$ be the diagonal matrix of vertex degrees.



$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Laplacian Matrix

Let G be a graph with $V(G) = \{1, ..., n\}$ and $E(G) = \{e_1, ..., e_j\}$. The Laplacian matrix of G, denoted by $L(G)$, is the $n \times n$ matrix defined as follows.

Definition

The rows and columns of $L(G)$ are indexed by $V(G)$. If $i \neq j$ then the (i, j)-entry of $L(G)$ is zero if vertex i and j are not adjacent, and it is -1 if the vertices are adjacent. The (i, i)-entry of $L(G)$ is d_i, the degree of vertex i, $i = 1, 2, ..., n$.

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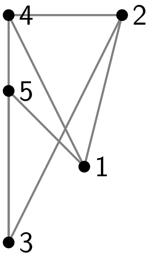
Graph and Laplacian Matrix



$$\begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix}$$

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Laplacian Matrix

If G has n vertices and k components, then $\text{rank} L(G) = n - k$, where k is the number of connected components.

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Laplacian Matrix

Let G be a graph with $V(G) = \{1, 2, ..., n\}$. Then the cofactor of any element of $L(G)$ equals the number of spanning trees of G.

- Tree
- Spanning tree

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Laplacian Matrix

The eigenvalues of $L(C_n) = 2 - 2\cos(\frac{2\pi j}{n})$ where $j = 1, ..., n$.

Laplacian Matrix

Let G be a graph with $V(G) = \{1, 2, ..., n\}$. Let the eigenvalues of $L(G) = \lambda_1 \geq ... \geq \lambda_{n-1} \geq \lambda_n = 0$ Then the number of spanning trees of G is $\frac{\lambda_1 ... \lambda_{n-1}}{n}$

Friendship Theorem

Theorem

Let G be a graph in which any two distinct vertices have exactly one common neighbor. Then G has a vertex that is adjacent to every other vertex, and, more precisely, G consists of a number of triangles with a common vertex.

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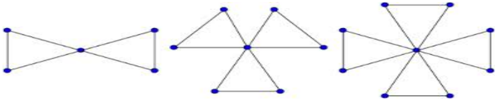

Friendship Theorem

Why is this called the Friendship Theorem?

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