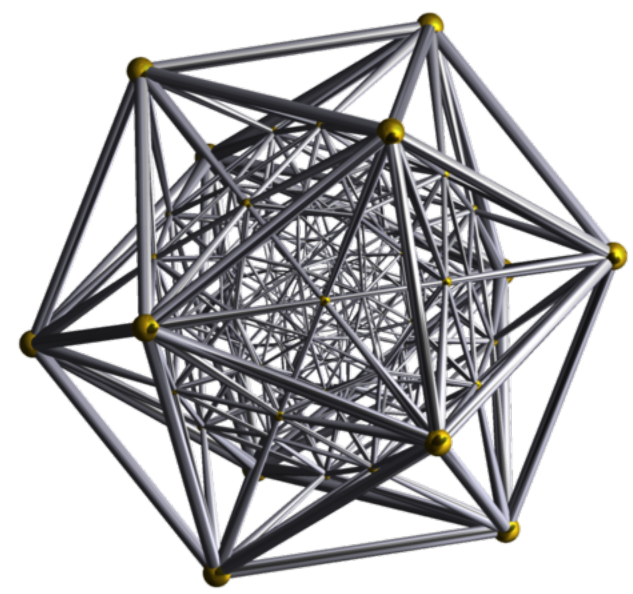
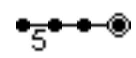
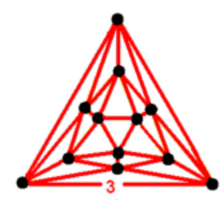
600-cell

In geometry, the **600-cell** is the convex regular 4-polytope (four-dimensional analogue of a Platonic solid) with Schläfli symbol {3,3,5}. It is also called a C_{600}, **hexacosichoron**[1] and **hexacosihedroid**.[2]

The 600-cell is regarded as the 4-dimensional analog of the icosahedron, since it has five tetrahedra meeting at every edge, just as the icosahedron has five triangles meeting at every vertex. It is also called a **tetraplex** (abbreviated from "tetrahedral complex") and **polytetrahedron**, being bounded by tetrahedral cells.

600-cell	
 Schlegel diagram, vertex-centered (vertices and edges)	
Type	Convex regular 4-polytope
Schläfli symbol	{3,3,5}
Coxeter diagram	
Cells	600 (*3.3.3*) 
Faces	1200 {3}
Edges	720
Vertices	120
Vertex figure	 icosahedron
Petrie polygon	30-gon
Coxeter group	H_4, [3,3,5], order 14400
Dual	120-cell
Properties	convex, isogonal, isotoxal, isohedral
Uniform index	35

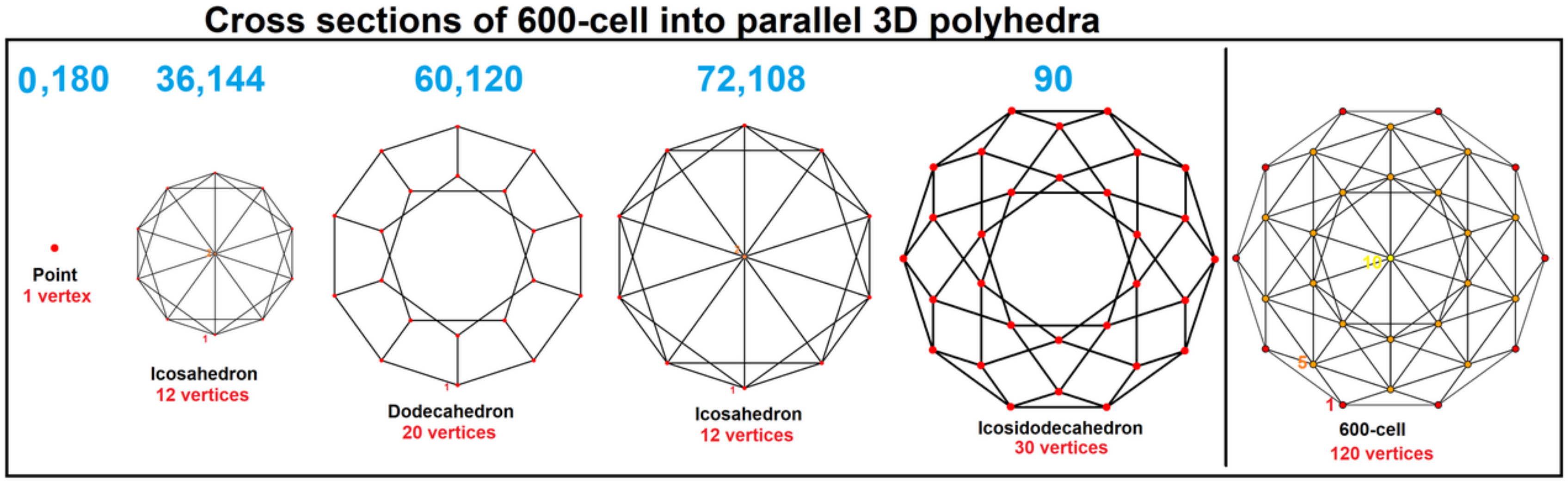
Contents

Geometry

Its boundary is composed of 600 tetrahedral cells with 20 meeting at each vertex. Together they form 1200 triangular faces, 720 edges, and 120 vertices. The edges form 72 flat regular decagons. Each vertex of the 600-cell is a vertex of six such decagons.

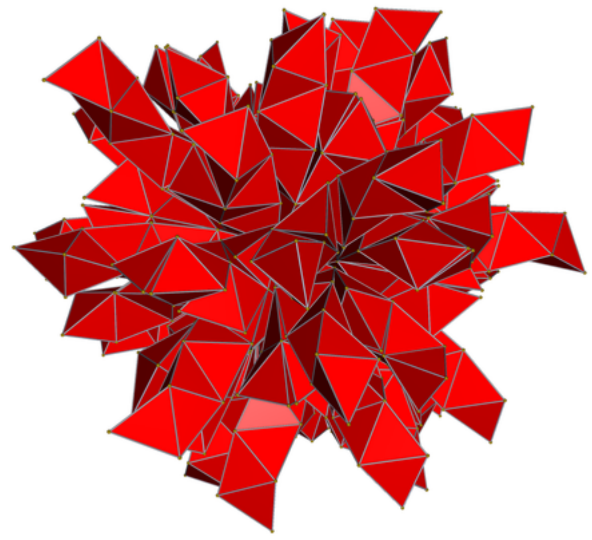
The mutual distances of the vertices, measured in degrees of arc on the circumscribed hypersphere, only have the values $36° = \frac{\pi}{5}$, $60° = \frac{\pi}{3}$, $72° = \frac{2\pi}{5}$, $90° = \frac{\pi}{2}$, $108° = \frac{3\pi}{5}$, $120° = \frac{2\pi}{3}$, $144° = \frac{4\pi}{5}$, and $180° = \pi$. Departing from an arbitrary vertex V one has at 36° and 144° the 12 vertices of an icosahedron, at 60° and 120° the 20 vertices of a dodecahedron, at 72° and 108° again the 12 vertices of an icosahedron, at 90° the 30 vertices of an icosidodecahedron, and finally at 180° the antipodal vertex of V.[3] These can be seen in the H3 Coxeter plane projections with overlapping vertices colored. Just like the icosidodecahedron can be partitioned into 6 central decagons (60 edge = 6×10), the 600-cell can be partitioned into 72 decagons (720 edges = 72 × 10).

Cross sections of 600-cell into parallel 3D polyhedra



Its vertex figure is an icosahedron, and its dual polytope is the 120-cell, with which it can form a compound. It has a dihedral angle of $\frac{\pi}{3} + \arccos(-\frac{1}{4}) \approx 164.4775°$.[4]

Each cell touches, in some manner, 56 other cells. One cell contacts each of the four faces; two cells contact each of the six edges, but not a face; and ten cells contact each of the four vertices, but not a face or edge.



Net

As a configuration

This configuration matrix represents the 600-cell. The rows and columns correspond to vertices, edges, faces, and cells. The diagonal numbers say how many of each element occur in the whole 600-cell. The nondiagonal numbers say how many of the column's element occur in or at the row's element.[5][6]

$$\begin{bmatrix} 120 & 12 & 30 & 20 \\ 2 & 720 & 5 & 5 \\ 3 & 3 & 1200 & 2 \\ 4 & 6 & 4 & 600 \end{bmatrix}$$

Here is the configuration expanded with *k*-face elements and *k*-figures. The diagonal element counts are the ratio of the full Coxeter group order, 14400, divided by the order of the subgroup with mirror removal.

H_4	⊙—●—●$_5$●	*k*-face	f_k	f_0	f_1	f_2	f_3	*k*-fig	Notes
H_3	× ●—●$_5$●	( )	f_0	120	12	30	20	{3,5}	H_4/H_3 = 14400/120 = 120
A_1H_2	⊙ × ●$_5$●	{ }	f_1	2	720	5	5	{5}	H_4/H_2A_1 = 14400/10/2 = 720
A_2A_1	⊙—● × ●	{3}	f_2	3	3	1200	2	{ }	H_4/A_2A_1 = 14400/6/2 = 1200
A_3	⊙—●—● ×	{3,3}	f_3	4	6	4	600	( )	H_4/A_3 = 14400/24 = 600

Coordinates

The vertices of a 600-cell centered at the origin of 4-space, with edges of length $\frac{1}{\varphi}$ (where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio), can be given as follows: 16 vertices of the form:[7]

$$(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}),$$

and 8 vertices obtained from

$$(0, 0, 0, \pm 1)$$

by permuting coordinates. The remaining 96 vertices are obtained by taking even permutations of

$$\tfrac{1}{2}(\pm\phi, \pm 1, \pm\tfrac{1}{\phi}, 0).$$

Note that the first 16 vertices are the vertices of a tesseract, the second eight are the vertices of a 16-cell, and that all 24 vertices together are vertices of a 24-cell. The final 96 vertices are the vertices of a snub 24-cell, which can be found by partitioning each of the 96 edges of another 24-cell (dual to the first) in the golden ratio in a consistent manner.

When interpreted as quaternions, the 120 vertices of the 600-cell form a group under quaternionic multiplication. This group is often called the binary icosahedral group and denoted by *2I* as it is the double cover of the ordinary icosahedral group *I*. It occurs twice in the rotational symmetry group *RSG* of the 600-cell as an invariant subgroup, namely as the subgroup $2I_L$ of quaternion left-multiplications and as the subgroup $2I_R$ of quaternion right-multiplications. Each rotational symmetry of the 600-cell is generated by specific elements of $2I_L$ and $2I_R$; the pair of opposite elements generate the same element of *RSG*. The centre of *RSG* consists of the non-rotation *Id* and the central inversion −*Id*. We have the isomorphism $RSG \cong (2I_L \times 2I_R) / \{Id, -Id\}$. The order of *RSG* equals $\frac{120 \times 120}{2} = 7200$.

The binary icosahedral group is isomorphic to SL(2,5).

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group. The rotational symmetry group was described by S.L. van Oss (1899); see References.

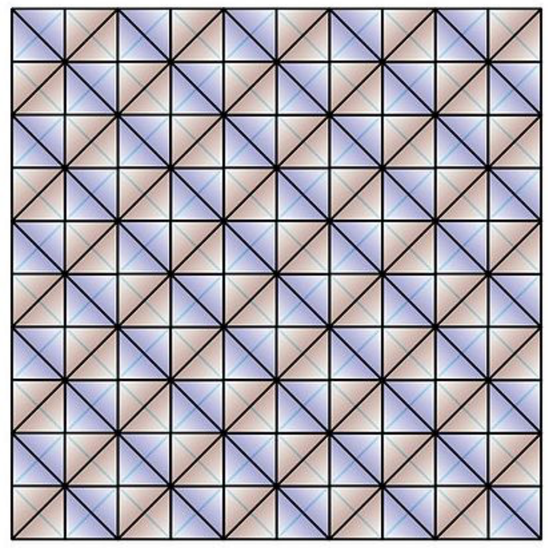
Visualization

The symmetries of the 3-D surface of the 600-cell are somewhat difficult to visualize due to both the large number of tetrahedral cells, and the fact that the tetrahedron has no opposing faces or vertices. One can start by realizing the 600-cell is the dual of the 120-cell. One may also notice that the 600-cell also contains the vertices of a dodecahedron, which with some effort can be seen in most of the below perspective projections.

A three-dimensional model of the 600-cell, in the collection of the Institut Henri Poincaré, was photographed in 1934–1935 by Man Ray, and formed part of two of his later "Shakesperean Equation" paintings.[8]

Union of two tori

The 120-cell can be decomposed into two disjoint tori. Since it is the dual of the 600-cell, this same dual tori structure exists in the 600-cell, although it is somewhat more complex. The 10-cell geodesic path in the 120-cell corresponds to a 10-vertex decagon path in the 600-cell. Start by assembling five tetrahedra around a common edge. This structure looks somewhat like an angular "flying saucer". Stack ten of these, vertex to vertex, "pancake" style. Fill in the annular ring between each "saucer" with 10 tetrahedra forming an icosahedron. You can view this as five, vertex stacked, icosahedral pyramids, with the five extra annular ring gaps also filled in. The surface is the same as that of ten stacked pentagonal antiprisms. You now have a torus consisting of 150 cells, ten edges long, with 100 exposed triangular faces, 150 exposed edges, and 50 exposed vertices. Stack another tetrahedron on each exposed face. This will give you a somewhat bumpy torus of 250 cells with 50 raised vertices, 50 valley vertices, and 100 valley edges. The valleys are 10 edge long closed paths and correspond to other instances of the 10-vertex decagon path mentioned above. These paths spiral around the center core path, but mathematically they are all equivalent. Build a second identical torus of 250 cells that interlinks with the first. This accounts for 500 cells. These two tori mate together with the valley vertices touching the raised vertices, leaving 100 tetrahedral voids that are filled with the remaining 100 tetrahedra that mate at the valley edges. This latter set of 100 tetrahedra are on the exact boundary of the duocylinder and form a clifford torus. They can be "unrolled" into a square 10x10 array. Incidentally this structure forms one tetrahedral layer in the tetrahedral-octahedral honeycomb.

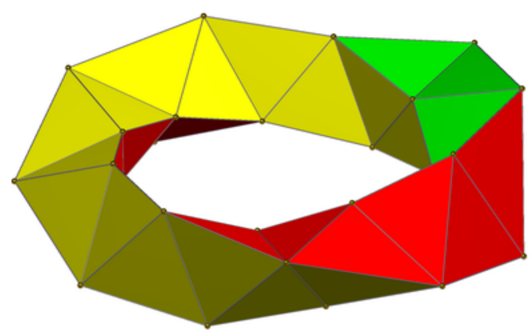


100 tetrahedra in a 10x10 array forming a clifford torus boundary in the 600 cell.

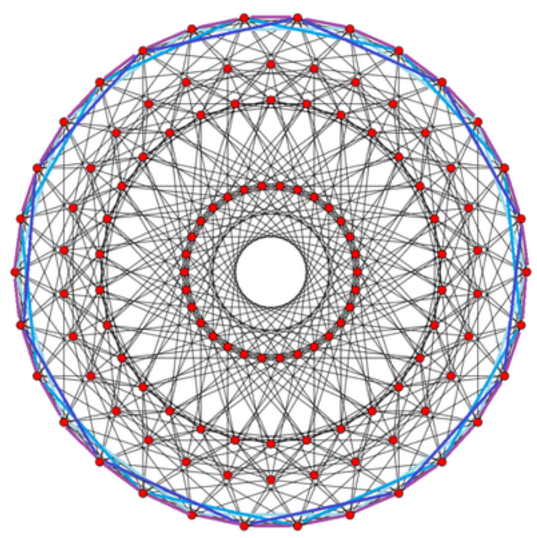
There are exactly 50 "egg crate" recesses and peaks on both sides that mate with the 250 cell tori. In this case into each recess, instead of an octahedron as in the honeycomb, fits a triangular bipyramid composed of two tetrahedra.

The 600-cell can be further partitioned into 20 disjoint intertwining rings of 30 cells and ten edges long each, forming a discrete Hopf fibration. These chains of 30 tetrahedra each form a Boerdijk–Coxeter helix. Five such helices nest and spiral around each of the 10-vertex decagon paths, forming the initial 150 cell torus mentioned above.

This decomposition of the 600-cell has symmetry $[[10,2^+,10]]$, order 400, the same symmetry as the grand antiprism. The grand antiprism is just the 600-cell with the two above 150-cell tori removed, leaving only the single middle layer of tetrahedra, similar to the belt of an icosahedron with the 5 top and 5 bottom triangles removed (pentagonal antiprism).



A single 30-tetrahedron ring Boerdijk–Coxeter helix within the 600-cell, seen stereographic projection



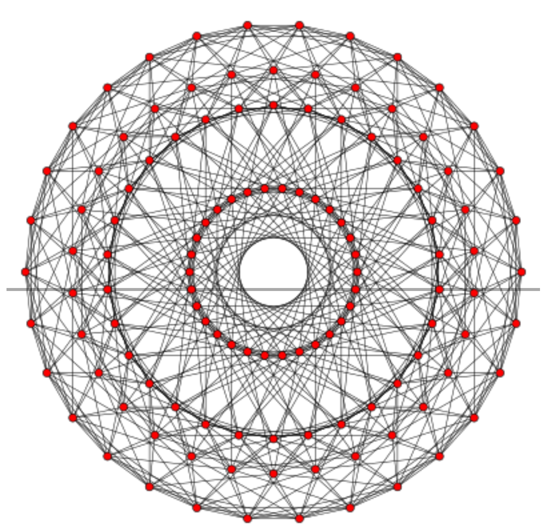
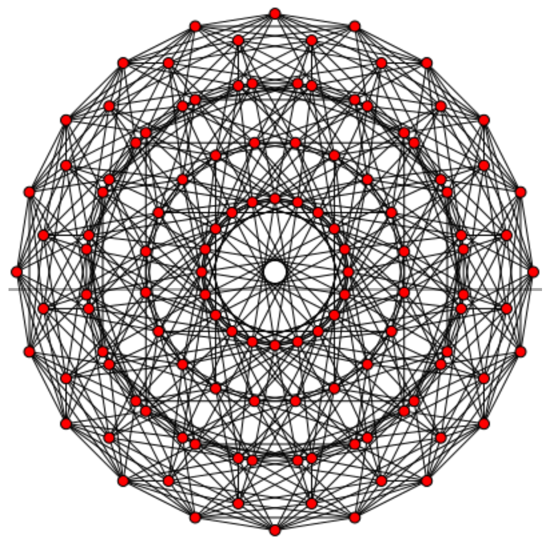
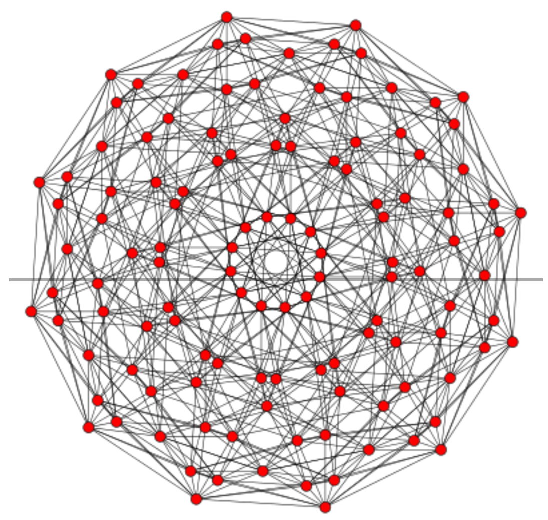
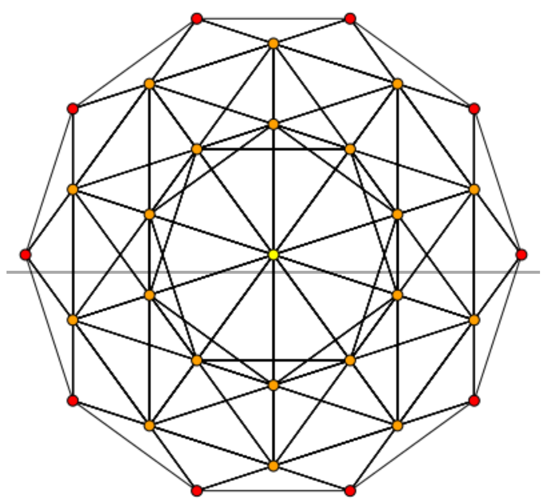
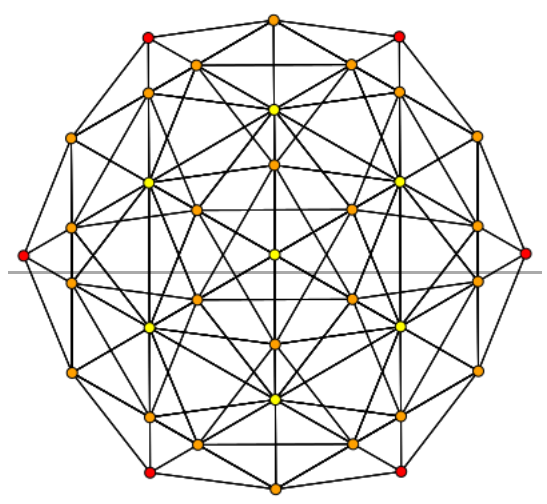
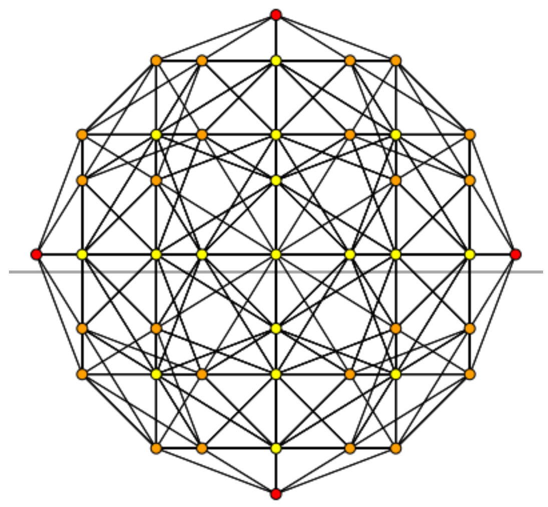
A 30-tetrahedron ring can be seen along the perimeter of this 30-gonal orthogonal projection

Images

2D projections

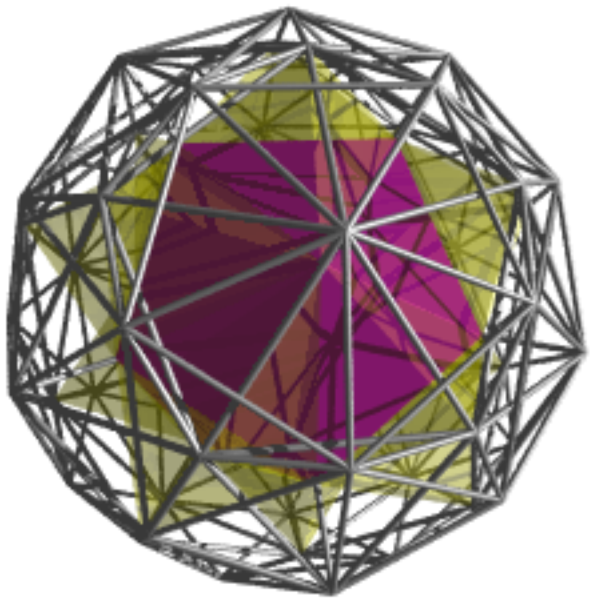
The H3 decagonal projection shows the plane of the van Oss polygon.

Orthographic projections by Coxeter planes

H_4	-	F_4
 [30]	 [20]	 [12]
H_3	**A_2 / B_3 / D_4**	**A_3 / B_2**
 [10]	 [6]	 [4]

3D projections

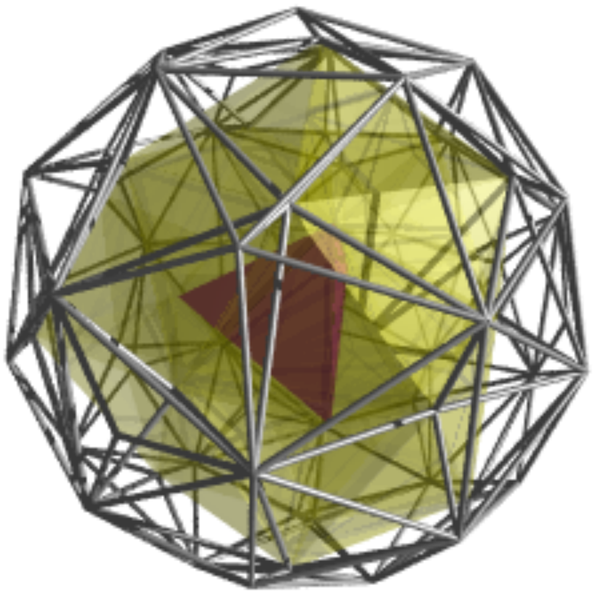
Vertex-first projection



This image shows a vertex-first perspective projection of the 600-cell into 3D. The 600-cell is scaled to a vertex-center radius of 1, and the 4D viewpoint is placed 5 units away. Then the following enhancements are applied:

- The 20 tetrahedra meeting at the vertex closest to the 4D viewpoint are rendered in solid color. Their icosahedral arrangement is clearly shown.
- The tetrahedra immediately adjoining these 20 cells are rendered in transparent yellow.
- The remaining cells are rendered in edge-outline.
- Cells facing away from the 4D viewpoint (those lying on the "far side" of the 600-cell) have been culled, to reduce visual clutter in the final image.

Cell-first projection.

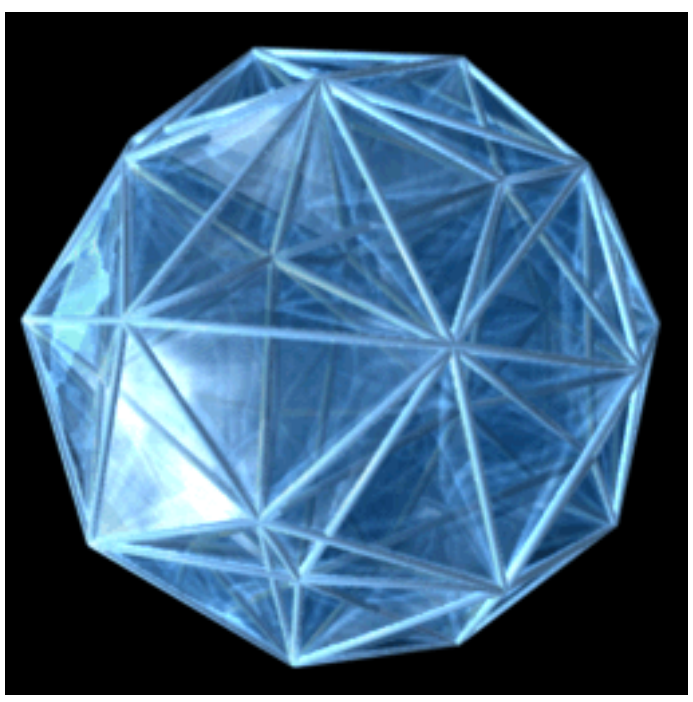


This image shows the 600-cell in cell-first perspective projection into 3D. Again, the 600-cell to a vertex-center radius of 1 and the 4D viewpoint is placed 5 units away. The following enhancements are then applied:

- The nearest cell to the 4d viewpoint is rendered in solid color, lying at the center of the projection image.
- The cells surrounding it (sharing at least 1 vertex) are rendered in transparent yellow.
- The remaining cells are rendered in edge-outline.
- Cells facing away from the 4D viewpoint have been culled for clarity.

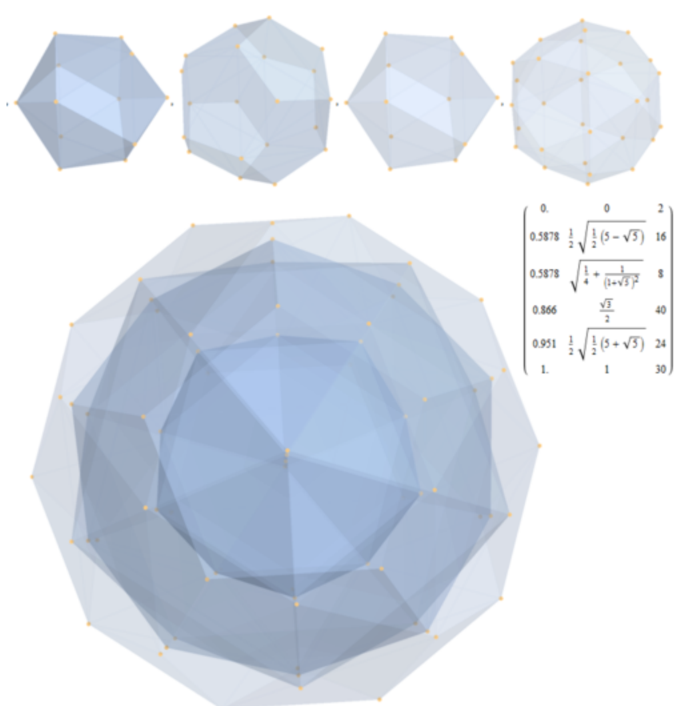
This particular viewpoint shows a nice outline of 5 tetrahedra sharing an edge, towards the front of the 3D image.

Simple Rotation



A 3D projection of a 600-cell performing a simple rotation.

Concentric Hulls



The 600-cell is projected to 3D using an orthonormal basis. The vertices are sorted and tallied by their 3D norm. Generating the increasingly transparent hull of each set of tallied norms shows pairs of:

1) points at the origin

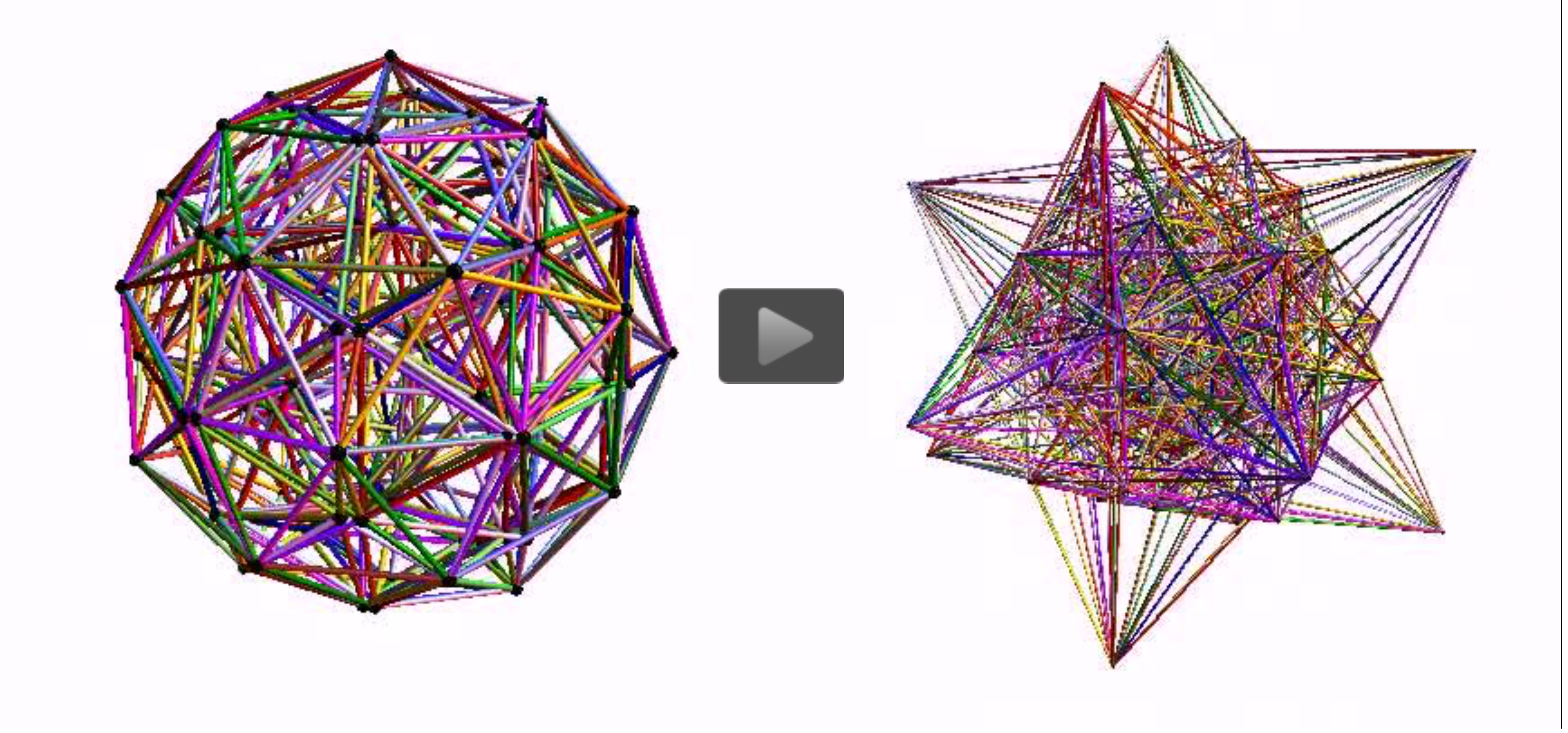
2) icosahedrons

3) dodecahedrons

4) icosahedrons

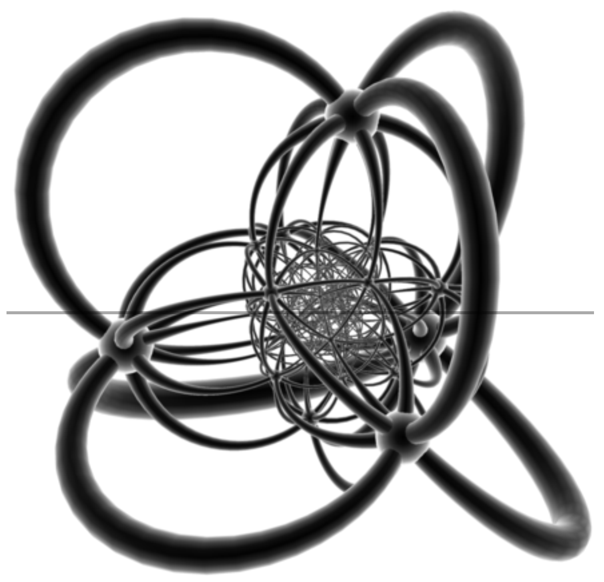
5) and a single icosadodecahedron

for a total of 120 vertices.



Frame synchronized animated comparison of the 600 cell using orthogonal isometric (left) and perspective (right) projections.

Stereographic

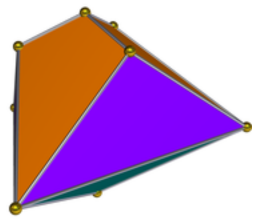
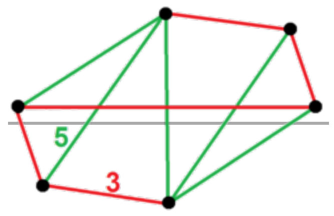
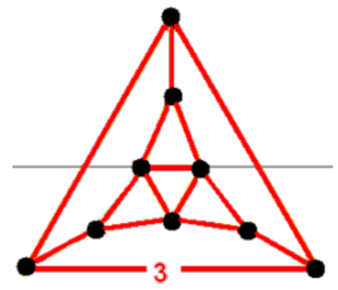
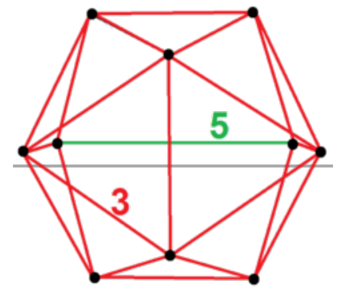
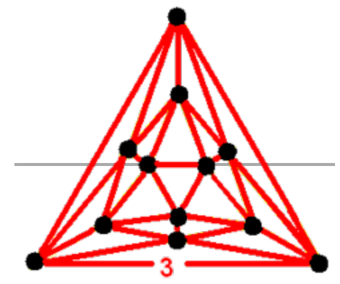
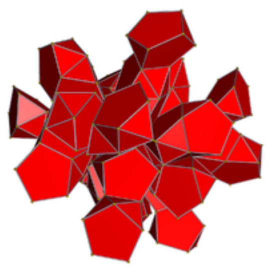
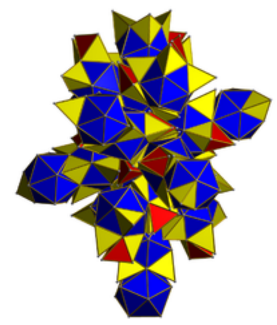
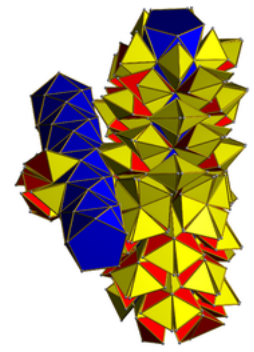
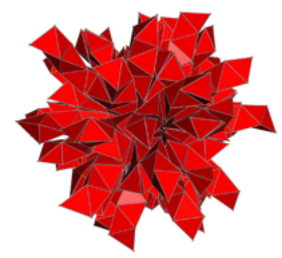
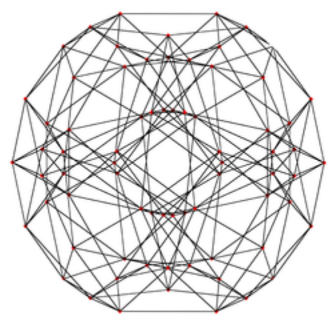
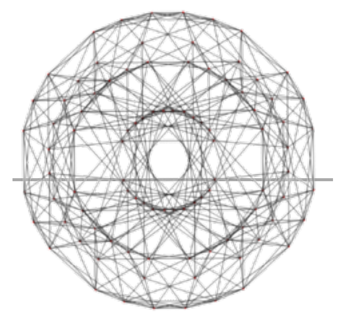
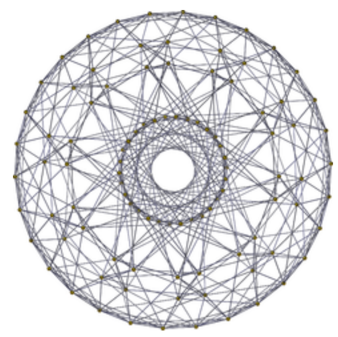
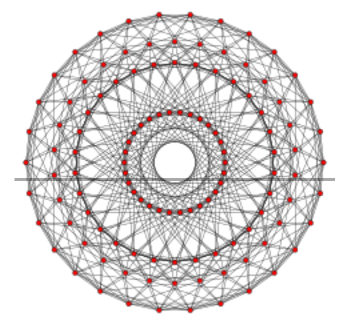
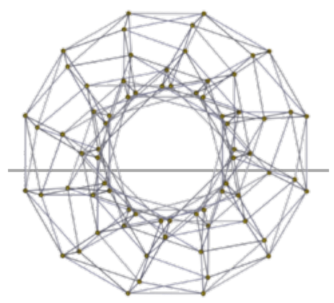
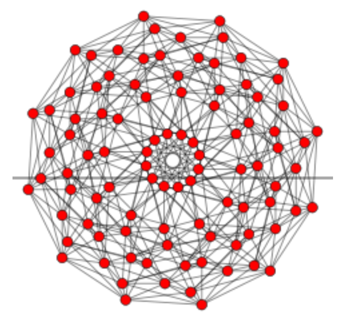
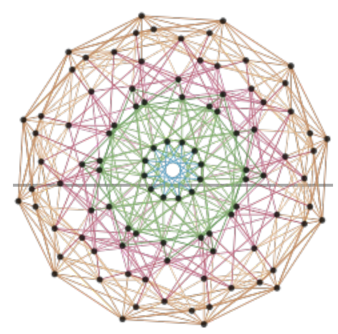
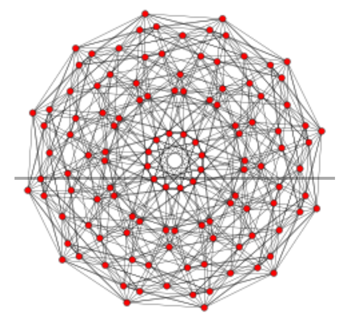
Stereographic projection (on 3-sphere)	
	Cell-Centered. The 720 edges of the 600-cell can be seen here as 72 circles, each divided into 10 arc-edges at the intersections. Each vertex has 6 circles intersecting.

Diminished 600-cells

The snub 24-cell may be obtained from the 600-cell by removing the vertices of an inscribed 24-cell and taking the convex hull of the remaining vertices. This process is a *diminishing* of the 600-cell.

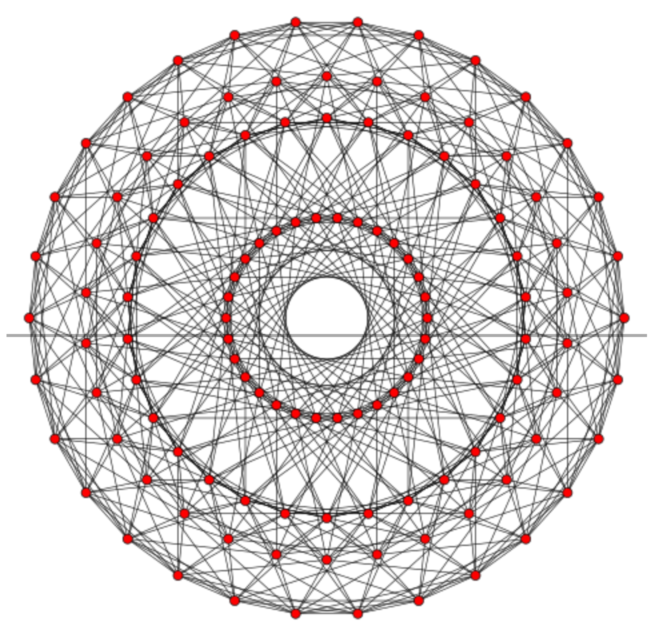
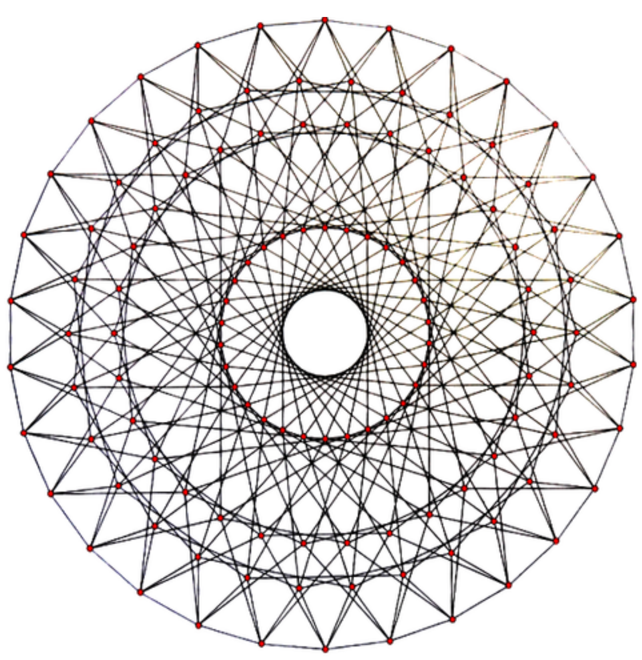
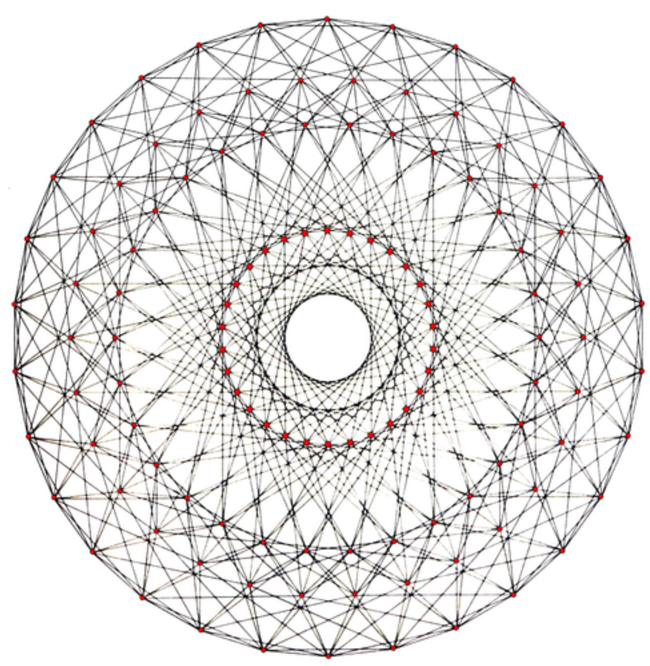
The grand antiprism may be obtained by another diminishing of the 600-cell: removing 20 vertices that lie on two mutually orthogonal rings and taking the convex hull of the remaining vertices.

A bi-24-diminished 600-cell, with all tridiminished icosahedron cells has 48 vertices removed, leaving 72 of 120 vertices of the 600-cell. The dual of a bi-24-diminished 600-cell, is a tri-24-diminished 600-cell, with 48 vertices and 72 hexahedron cells.

Diminished 600-cells					
Name	**Tri-24-diminished 600-cell**	**Bi-24-diminished 600-cell**	**Snub 24-cell (24-diminished 600-cell)**	**Grand antiprism (20-diminished 600-cell)**	**600-cell**
Vertices	48	72	96	100	120
Vertex figure (Symmetry)	 Dual of tridiminished icosahedron ([3], order 6)	 bi-tridiminished icosahedron ($[2]^+$, order 2)	 tridiminished icosahedron ([3], order 6)	 2-diminished icosahedron ([2], order 4)	 Icosahedron ([5,3], order 120)
Symmetry	Order 144 (48×3 or 72×2)		$[3^+,4,3]$ Order 576 (96×6)	$[[10,2^+,10]]$ Order 400 (100×4)	[5,3,3] Order 14400 (120×120)
Net					
Ortho H_4 plane					
Ortho F_4 plane					

Related complex polygons

The regular complex polytopes $_3\{5\}_3$, and $_5\{3\}_5$, , in $\mathbb{C}^2$ have a real representation as *600-cell* in 4-dimensional space. Both have 120 vertices, and 120 edges. The first has Complex reflection group $_3[5]_3$, order 360, and the second has symmetry $_5[3]_5$, order 600.[9]

Regular complex polytope in orthogonal projection of H_4 Coxeter plane		
 {3,3,5} Order 14400	 $_3\{5\}_3$ Order 360	 $_5\{3\}_5$ Order 600

Related polytopes and honeycombs

The 600-cell is one of 15 regular and uniform polytopes with the same symmetry [3,3,5]:

H₄ family polytopes							
120-cell	rectified 120-cell	truncated 120-cell	cantellated 120-cell	runcinated 120-cell	cantitruncated 120-cell	runcitruncated 120-cell	omnitruncated 120-cell
{5,3,3}	r{5,3,3}	t{5,3,3}	rr{5,3,3}	$t_{0,3}${5,3,3}	tr{5,3,3}	$t_{0,1,3}${5,3,3}	$t_{0,1,2,3}${5,3,3}
600-cell	rectified 600-cell	truncated 600-cell	cantellated 600-cell	bitruncated 600-cell	cantitruncated 600-cell	runcitruncated 600-cell	omnitruncated 600-cell
{3,3,5}	r{3,3,5}	t{3,3,5}	rr{3,3,5}	2t{3,3,5}	tr{3,3,5}	$t_{0,1,3}${3,3,5}	$t_{0,1,2,3}${3,3,5}

It is similar to three regular 4-polytopes: the 5-cell {3,3,3}, 16-cell {3,3,4} of Euclidean 4-space, and the order-6 tetrahedral honeycomb {3,3,6} of hyperbolic space. All of these have a tetrahedral cells.

{3,3,p} polytopes							
Space	S^3			H^3			
Form	Finite			Paracompact	Noncompact		
Name	{3,3,3}	{3,3,4}	{3,3,5}	{3,3,6}	{3,3,7}	{3,3,8}	... {3,3,∞}
Image							
Vertex figure	{3,3}	{3,4}	{3,5}	{3,6}	{3,7}	{3,8}	{3,∞}

This 4-polytope is a part of a sequence of 4-polytope and honeycombs with icosahedron vertex figures:

{p,3,5} polytopes							
Space	S^3	H^3					
Form	Finite	Compact		Paracompact	Noncompact		
Name	{3,3,5}	{4,3,5}	{5,3,5}	{6,3,5}	{7,3,5}	{8,3,5}	... {∞,3,5}
Image							
Cells	{3,3}	{4,3}	{5,3}	{6,3}	{7,3}	{8,3}	{∞,3}

See also

- Uniform 4-polytope family with [5,3,3] symmetry
- regular 4-polytope
- 120-cell, the dual 4-polytope to the 600-cell

Notes

1. N.W. Johnson: *Geometries and Transformations*, (2018) ISBN 978-1-107-10340-5 Chapter 11: *Finite Symmetry Groups*, 11.5 *Spherical Coxeter groups*, p.249
2. Matila Ghyka, *The Geometry of Art and Life* (1977), p.68
3. S.L. van Oss (1899); F. Buekenhout and M. Parker (1998)
4. Coxeter, Regular polygons, 3rd edition, Dover Publications, p.293, ≈164°29′
5. Coxeter, Regular Polytopes, sec 1.8 Configurations
6. Coxeter, Complex Regular Polytopes, p.117
7. Weisstein, Eric W. "600-cell" (http://mathworld.wolfram.com/600-Cell.html). *MathWorld*.
8. Grossman, Wendy A.; Sebline, Edouard, eds. (2015), *Man Ray Human Equations: A journey from mathematics to Shakespeare*, Hatje Cantz. See in particular *mathematical object mo-6.2*, p. 58; *Antony and Cleopatra*, SE-6, p. 59; *mathematical object mo-9*, p. 64; *Merchant of Venice*, SE-9, p. 65, and "The Hexacosichoron", Philip Ordning, p. 96.
9. Coxeter, H. S. M., *Regular Complex Polytopes*, second edition, Cambridge University Press, (1991). pp.48-49

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- Oss, Salomon Levi van: Das regelmässige 600-Zell und seine selbstdeckenden Bewegungen. *Verhandelingen der Koninklijke (Nederlandse) Akademie van Wetenschappen, Sectie 1 Deel 7 Nummer 1 (Afdeeling Natuurkunde). Amsterdam: 1899.* Online at URL [3] (http://www.dwc.knaw.nl/english/academy/digital-library/?pagetype=publDetail&pId=PU00011478), reachable from the home page of the KNAW Digital Library at URL [4] (http://www.dwc.knaw.nl/english/academy/digital-library/). REMARK: Van Oss does not mention the arc distances between vertices of the 600-cell.
- F. Buekenhout, M. Parker: The number of nets of the regular convex polytopes in dimension <= 4. *Discrete Mathematics, Volume 186, Issues 1-3, 15 May 1998, Pages 69-94.* REMARK: The authors do mention the arc distances between vertices of the 600-cell.

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  - Convex uniform polychora based on the hecatonicosachoron (120-cell) and hexacosichoron (600-cell) - Model 35 (https://web.archive.org/web/20070204075028/members.aol.com/Polycell/section4.html), George Olshevsky.
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- The 600-Cell (http://eusebeia.dyndns.org/4d/600-cell) Vertex centered expansion of the 600-cell

Fundamental convex regular and uniform polytopes in dimensions 2–10

Family	A_n	B_n	$I_2(p)$ / D_n	E_6 / E_7 / E_8 / F_4 / G_2	H_n
Regular polygon	Triangle	Square	p-gon	Hexagon	Pentagon
Uniform polyhedron	Tetrahedron	Octahedron • Cube	Demicube		Dodecahedron • Icosahedron
Uniform 4-polytope	5-cell	16-cell • Tesseract	Demitesseract	24-cell	120-cell • 600-cell
Uniform 5-polytope	5-simplex	5-orthoplex • 5-cube	5-demicube		
Uniform 6-polytope	6-simplex	6-orthoplex • 6-cube	6-demicube	1_{22} • 2_{21}	
Uniform 7-polytope	7-simplex	7-orthoplex • 7-cube	7-demicube	1_{32} • 2_{31} • 3_{21}	
Uniform 8-polytope	8-simplex	8-orthoplex • 8-cube	8-demicube	1_{42} • 2_{41} • 4_{21}	
Uniform 9-polytope	9-simplex	9-orthoplex • 9-cube	9-demicube		
Uniform 10-polytope	10-simplex	10-orthoplex • 10-cube	10-demicube		
Uniform *n*-polytope	*n*-simplex	*n*-orthoplex • *n*-cube	*n*-demicube	1_{k2} • 2_{k1} • k_{21}	*n*-pentagonal polytope

Topics: Polytope families • Regular polytope • List of regular polytopes and compounds