Unified Lab 5: Bottle Rocket

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Contents

1 The Lab

1.1 Goals

Through the theoretically-optimized design and practically-engineered launch of a bottle rocket, the authors aim to achieve a thorough understanding of the key parameters involved, to the extent that the best possible rocket configuration may be found. Use of the the Bernoulli Equation and other physical models enable prediction of maximum apogee, and experimental results show how the process may be improved for future research.

1.2 Description

Brainstorming of possible rocket assets enables creative solutions and improvements to rudimentary rocket designs, such as additional nozzles or bottle insets to help the water outflow become increasingly laminar. These ideas are evaluated in terms of feasibility and time cost to determine final design. Before construction, theoretical MATLAB models of the rocket's flight test and confirm initial thoughts on optimal designs. Independent experimental testing enables prototype evaluation before the official launch date. After rocket launch, during which electronic altimeter and optical triangulation measurements enable accurate data collection, the theoretical models are evaluated and assessed. Improvements for future work are recommended in theoretical, experimental, and design aspects.

2 The Theory

2.1 Assumptions

A number of assumptions simplify the modeling of the rocket launch without significantly affecting the accuracy of the predictions. The water, which acts as a propellant, is assumed to be incompressible. The gaseous expansion is assumed adiabatic and reversible, since it can be thought of as an ideal piston system while the air pushes the neck of the stand and subsequently the water down to move the body of the rocket upwards. Air is assumed to be an ideal gas. Gravity is assumed to act on the structural mass of the rocket and the interior mass of the rocket system, since the air and water ejected from the throat of the bottle can be thought of as in free fall, while the substances in contact with the bottle's interior surfaces exert some pressure and therefore weight on the rocket as it ascends. Drag coefficients used are estimates based on various known geometries and approximations but may be calculated through experimental testing. Variations in the liquid flow leaving the rocket are neglected, such as vortices induced by the throat of the bottle. Instead, as stated earlier, the piston approximation for the liquid as a solid moving boundary on the control mass of air is used. Throughout the flight and especially when the rocket begins descent, variations in latitudinal or longitudinal direction with respect to the earth are ignored, so only upwards radial direction is considered. This means that the rocket may flip around in a complicated maneuver which may provide modeling challenges, but this section treats the rocket as a simple point mass with forces acting on it; for instance, drag is proportional to the nose surface throughout, even though the nose may not always be coincident with the velocity vector. Other reasonable and similar approximations may be used, but this section should provide sufficient coverage of the main assumptions necessary to begin analysis. Some assumptions, such as those made with the Tsiolkovsky rocket equation where fuel exit velocity is approximately constant, are perhaps too simplifying and will not be used here. Further research could attempt to model more involved phenomena, such as heat transfer from the compressed air to the atmosphere through the thin plastic of the rocket, which may non-trivially decrease pressure in the rocket. In the following analysis, certain quantities trivially calculated from known information, such as mass of compressed air as found through ideal gas relations, are assumed to be known.

2.2 Math Models

The rocket flight is composed of multiple stages, wherein the physical influences change significantly. A piecewise model is necessary to accurately represent phenomena throughout the flight. In each stage, the component factors of drag, thrust, and gravitational forces change. These forces are set equal to the time-variant rocket system mass times acceleration to solve for acceleration, which may be integrated twice to find position. This method is the general solution to the rocket problem.

$$F_{\text{Total}} = \int \dot{m}_{\text{sys}}\, a\, \mathrm{dt} = \int dF_T - dF_d - dF_g \tag{1}$$

Note that for each instant in the rocket's flight, assuming sufficiently high Reynolds number, $F_d = \frac{1}{2}\rho_\infty v^2 C_d S_d$ where ρ_∞ is density of surrounding fluid, v is instantaneous velocity, C_d is the rocket's characteristic drag coefficient, and S_d is the characteristic drag surface, usually approximated as cross-sectional area but in reality dependent on geometry and instantaneous velocity. Also, F_g changes only based on $\dot{m}_{\text{sys}}$ so the two quantities that must be calculated for each stage are $\dot{m}_{\text{sys}}$ and F_T. To do so will require thermodynamic state equations for each stage.

2.2.1 Rocket On Stand

Assuming the stand is a cylindrical solid of radius r_s and height h_s intruding into the interior of the rocket such that the propulsive force results from the Newtonian equal and opposite shaft work done by the control mass of compressed air adiabatically and reversibly moving the shaft down in the rocket's reference frame from h_s to h_i, the initial height of the stand, find the thermodynamic states of the compressed air control mass. Volume

$$V(t) = V_i + \pi r_s^2 h(t) \tag{2}$$

where V_i is initial (constant) volume of air control mass and setting $h(t)$ and moving down the shaft.

Temperature

$$T(t) = T_i \left(\frac{V_i}{V(t)} \right)^{\gamma - 1} \tag{3}$$

Pressure

$$p(t) = \frac{mRT(t)}{V(t)} \tag{4}$$

Now

$$W_{shaft} = \int_{h_i}^{h_s} F(t)\,\mathrm{dh} = \int p(t)\,\mathrm{dv} = \int_{h_i}^{h_s} \frac{m\,RT(t)\,\pi\,r_s^2}{V(t)}\,\mathrm{dh} \tag{5}$$

The above implies

$$F_T(t) = \frac{mRT(t)\pi r_s^2}{V_i + \pi r_s^2 h(t)} \tag{6}$$

Since $\dot{m}_{\mathrm{sys}}$ is zero when no mass is expelled during this stage, the gravitational force only depends on the system mass including compressed gas, total mass of water, and structural mass of the rocket body. Stage transition occurs at time $t_{1\to 2}$ when

$$h_s - h_i = \iint \frac{F_T(t) - F_d(t) - F_g}{m_{\mathrm{sys}}} \mathrm{dt}^2 \tag{7}$$

2.2.2 Water Power

After the rocket leaves the stand, the pressurized air continues to expand, expelling water against atmospheric pressure to propel the system via conservation of momentum. With $r(t)$ representing the radius of the top water surface solid boundary in the bottle and r_t as the constant throat radius of the bottle exit nozzle, the change in the water height h_w (measured from the exit of the bottle to the fill level of the water and beginning with 0 at time $t_{1\to 2}$) is derived from the conservation of mass relation

$$\rho_w \frac{dh_w(t)}{dt} \pi r(t)^2 = \rho_w v_e \pi r_t^2 \tag{8}$$

Then the air control mass volume is given by

$$V(t) = V(t_{1\to 2}) + \pi r(t)^2 h_w(t) \tag{9}$$

Temperature

$$T(t) = T_{1\to 2} \left(\frac{V(t_{1\to 2})}{V(t)} \right)^{\gamma - 1} \tag{10}$$

Pressure

$$p(t) = \frac{mRT(t)}{V(t)} \tag{11}$$

Now the unknown quantities are $h_w(t)$ and the exit velocity v_e. Assume the water travels in a vertical channel downward such that the atmospheric pressure p_∞ propagates upstream to the exit, so that the displaced water essentially acts just like the piston approximation used in stage 1. Then the interface of the water and compressed air accelerates approximately following

$$\frac{d^2 h_w(t)}{dt^2} = \frac{p(t)\pi r(t)^2 - p_\infty \pi r_t^2}{m_w} \tag{12}$$

Where $m_w(t)$ is the mass of water expelled. This neglects gravity for the moment and assumes that the only opposing force comes from the exit surface and not the bottle sides with some resistance component parallel to the downward flow, viscosity, etc. Using conservation of momentum and the conservation of mass relation to find the exit velocity in terms of $\frac{dh_w(t)}{dt}$, the above analysis yields the time-variant function for thrust

$$F_T(t) = \rho_w v_e^2 \pi r_t^2 \tag{13}$$

which may now be solved.

The time-variant mass of the rocket system changes as water is expelled, as described in the relation

$$m_{\text{sys}}(t) = m_{str} + m_{air} + m_{w0} - \rho_w \pi r_t^2 \int_{t_{1\to 2}}^{t} v_e \text{dt} \tag{14}$$

Where structural mass, initial air mass, initial water mass, and expelled fluid mass are all calculable quantities. This process of water expulsion occurs from $t_{1\to 2}$ until $h_w(t) = h_e$ at $t_{2\to 3}$, or in other words, until all the water is expelled and the above system mass is comprised entirely of the structure and compressed air.

2.2.3 Air Power

Once only air is left in the rocket, the pressure difference between the rocket's air and atmosphere causes continued expulsion of mass from the system. Three possible assumptions may be made; one probably overestimates and one likely underestimates the effect of the compressed air expansion. If the gas expands quickly enough, meaning the pressure difference between the interior and ambient air is virtually that of between a full vessel and a vacuum, then since no work is done, no kinetic energy is gained and the air rushing out of the chamber does not increase the velocity of the rocket at all. Although the effect may be small, it is unlikely that this assumption accurately captures the physics of the situation. Alternatively, we may consider the non-adiabatic reversible expansion of the gas, since the atmosphere may be approximated as a heat reservoir which

exerts a pressure on the control volume of the rocket equal to atmospheric pressure p_∞. However, since the flow of gas exiting the rocket is presumably fast enough to essentially shove aside atmosphere, the process may be considered adiabatic since minimal heat is lost prior to leaving the exit, which is where the conservation of momentum principle applies to provide thrust for the rocket system. This assumption is used, knowing that it may slightly overestimate the propulsive force of the expelled air.
Consider the control volume of the rocket, but use an exit surface just inside of the nozzle. Then the exiting air flows out of the CV in a controlled pipe-like flow. For this control mass

$$V(t) = V(t_{2\to3}) + h(t)\pi r_t^2 \tag{15}$$

where h is the height function of the imaginary cylinder of air projected out of the nozzle. The volume at atmospheric pressure, once this stage reaches equilibrium, is easily found using ideal gas relations described earlier in the analysis, as is the pressure and temperature at time t. By conservation of momentum, the quantity needed to solve for thrust is $\frac{dh(t)}{dt}$. The force with which the air is forced out of the system is related via pressure, so the acceleration of the gas is found similarly to the water in stage 2, where the control mass of expulsion is simply the mass comprising the volume of the air at atmospheric pressure less the volume of the container. The interface between the working fluid and the piston-like gas ejected is treated just like in the analysis of the water, so the relation is the same for exit velocity and change in height. Just like with the water, conservation of momentum may then be applied to the system where the reactionary velocity of the rocket is related to the momentum of the ejected gas. Once more, the transition between stages occurs after all the air is expelled to equilibrium at $t_{3\to4}$.

2.2.4 Coasting and Descent

Now the rocket system may be treated as a simple projectile with initial speed $v(t_{3\to4})$, height $h(t_{3\to4})$, and weight of the structure plus air remaining in the system. Of course, drag still applies, but without thrust it is trivial to solve for the time until the system reaches ground level. Recall that this analysis assumes nothing exceptional occurs when the rocket changes orientation nor throughout the rest of the flight.

2.3 MATLAB

Eulerian numerical integration techniques step through small time iterations to approximate the solutions to the differential equations enumerated in the previous section, one for the interior water propellant height, and one for the altitude of the rocket system. First, global variables are defined. Then, known parameters such as surface areas, atmospheric pressure, masses of various fluids, etc. are estimated and input. Calculated quantities such as interior temperature of the compressed air are found using standard thermodynamic techniques as outlined in the preceding section. States are defined for each stage of the rocket cycle. Within each stage, functions calculating the thrust force, gravitational force, and drag force are computed along with the internal states to find the acceleration of the rocket system. Arrays of all the state variables and forces are created for visualization purposes and to help iterate through assumed and estimated values to test various cases. Upon implementation, it is found that the simulation has numerous problems, including mismatched array dimensions and complex heights resulting from incorrect volume ratios and continuously accelerating rocket systems which never return to Earth. With significant effort, the code is fixed and values are found ranging from 1 to 10m in maximum height. To further rectify the simulation and obtain reasonable numbers, new formulae are implemented to redo the defunct portions. The most notable of these changes is the substitution of the Bernoulli equation (assuming one-dimensional continuity) for the more convoluted exit velocity calculation described above in stage 3.

$$p_{air} - p_{\infty} = \frac{1}{2}\rho v_e^2 \tag{16}$$

enables quick and simple calculation of exit velocity, which in turn allows easy calculation of the internal volume changes by numerical integration of the flow velocity times the surface area of the exit, which then yields thermodynamic ratios of states, including pressure. This simplifying equation allows for more manageable code while still finding optimal values for water height, etc. Dimensional analysis and peer code review is used for quality assurance. Johannes helped significantly with the review and correction of key equation implementation.

2.3.1 Assumptions

Using a drag coefficient of 0.2, slightly less than that of a bullet shape since the frontal ellipsoid shape is longer and more streamlined, throat and stand radius of 14mm, bottle radius of 36mm, atmospheric pressure of 101325Pa, atmospheric density of approximately 1.25 $\frac{\text{kg}}{\text{m}^3}$ assuming temperature of 283K. The radius of the bottle is assumed to vary with height. For the first 3cm, assume that radius is simply the radius of the throat. Then the bottle curves following an approximately square root function until the radius becomes constant after 5cm at 36.5mm. Commonly accepted precise values are used such as 999.97 $\frac{\text{kg}}{\text{m}^3}$ for density of water and 9.80665$\frac{\text{m}}{\text{s}^2}$ for gravitational acceleration. Note that the predictions include initial temperature rise of over 200K for the compressed air chamber, which would boil the water and possibly melt the plastic. This clearly will not happen in practice, so heat must be transferred into the water and dissipated in order for the rocket to remain intact. For the purposes of the simulation, assume that the effects of these temperature changes are negligible with respect to fill level and maximum height.

As stated in Section 2.1, every stage, save for the pressurizing of the rocket, is assumed to be adiabatic and reversible. Under that assumption, pressure and temperature are calculated from the volume of air in the rocket. The graph of how pressure and temperature vary with volume can be found in Figure 1.

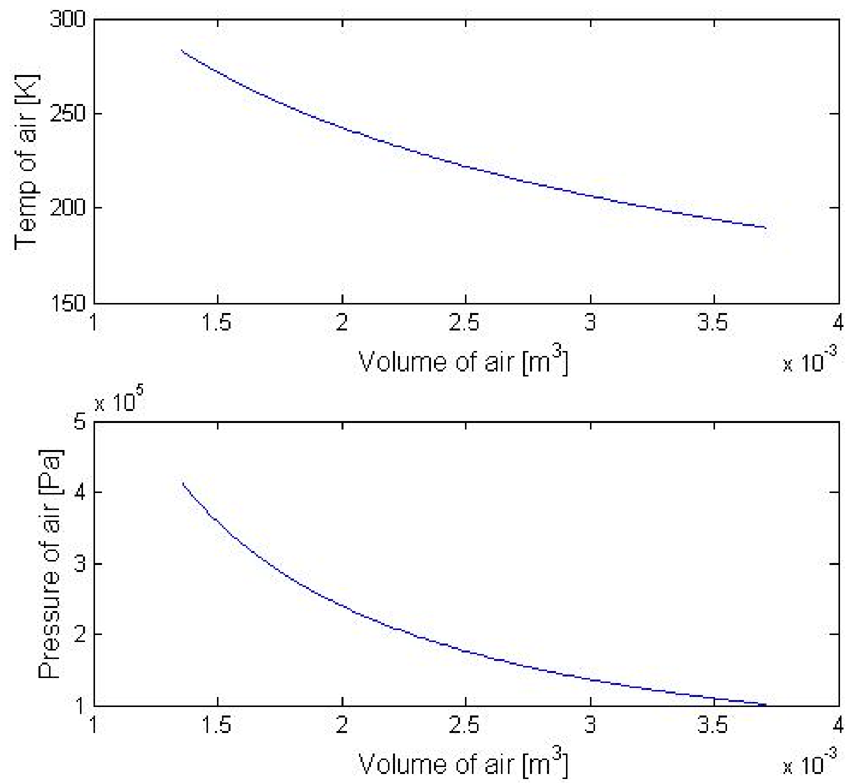

Figure 1: Pressure and Temperature varying with Volume of Air

2.3.2 Predictions

0.45 L is the optimal fill level as determined by the simulation. 110 m is the corresponding height, which is approximately 361 feet. Because of the temperature assumption discussed above, the maximum height is most likely less than the idealized prediction. Below is the plot of Max Height(m) as varies with Fill Level of Water(m^3).

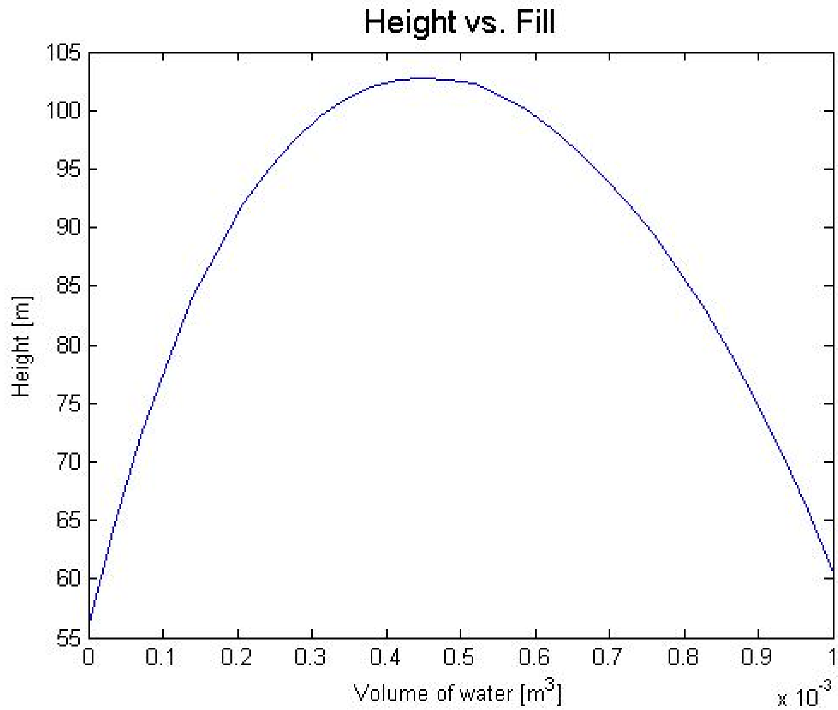

Figure 2: Max Height as varies with Fill Level of Water

3 The Build

3.1 Brainstorming

Initially, thoughts are gathered as to design optimization. Categories include bottle size/shape, weight optimization, fins, nose cone, and miscellaneous improvements. Bottle size, if increased without sacrificing internal pressure could be changed to 3 liters and keep the same nozzle for the stand locking mechanism given. Weight is minimized while structural integrity is maximized through lightweight 3-D printed ABS components and balsa wood. Smart Water bottles use potentially thinner plastic than soda bottles, so weight is minimized. Structural integrity may be maintained through strong tapes, glues, or epoxies.

3.2 Further Thoughts

When asked, the lab instructors said no to the larger 3L bottles. Apparently, 30-40% fill is the consensus on the best fill amount by experience from previous years. Simulation results suggest otherwise. These discrepancies between predicted and tested values is likely due to the temperature effects described above and would need further research to confirm which is more valid. Wind tunnel testing could refine the drag coefficient estimate, but time limitations preclude such testing.

3.3 Final Design Details

The final design incorporates a 3-D printed nose cone, sanded to a very smooth surface. This is attached via snug fit onto a three liter Smart Water body, where the two liter chamber is reinforced with epoxy on the joint and end cap. Fins are $\frac{1}{8}$ balsa, evenly spaced every 90 degrees toward the base of the rocket on a 3-D printed custom bracket mount. Epoxy reinforcements perpendicular to the grain of the balsa are added for support. The chamber is measured to be a volume of 2115 milliliters using a graduated cylinder to measure volumetric fill. The mass is 118.6g. The altimeter is mounted inside the nose of the rocket using a small amount of clay for wadding and support.

3.4 Applications of Theory and Sources of Error

Considerations influencing the design of the rocket include minimizing drag, minimizing weight, and maximizing structural integrity. To minimize drag, fins are sanded to a rounded balsa leading edge and the nose cone is smooth. A hemi-ellipsoid is chosen as the minimal drag geometry for the speeds expected to be encountered during flight. For weight considerations, minimal adhesives are used in the construction of the rocket. The smart water bottles are connected such that the heavier plastic at the bottom faces of the bottles are cut away for all of the three bottles. Judicious sanding of various components and careful balancing of the rocket such that the nose cone provides sufficient mass to bring the center of gravity far enough above the center of pressure that free flight should be aerodynamically stable ensure minimal weight while optimizing flight robustness. Of course, weight and structural integrity are a trade-off. Epoxy reinforcement of the air chamber and joint are applied with priority on structure over weight considerations, for obvious explosive reasons. Similarly, although epoxy reinforcements of fins, etc. add weight, it is judged that the possibility for multiple flight trials is substantially more important, given that the simulation optimal water height is potentially different from the real value for this particular rocket, since the drag coefficient and other estimated quantities are not fully characterized. The largest problem for the first launch is the fill level of water being unoptimized. Using standard practices and experience gathered orally from previous Unified students, as well as peers, the fill level is 700ml for the first attempt. Upon subsequent flight, the simulation finally is run correctly to yield the appropriate theoretical fill level of 450ml.

4 Results

4.1 Launch

The first launch opportunity consisted of one flight on Tuesday December 2 2014. The altitude was measured at 203ft by the altimeter, a measurement verified by the average altitudes found via triangulation at 58 and 48 degrees, with corresponding altitudes of 235 and 175ft, respectively. The offset between the measurements is caused by the wind-induced drift in the flight path, where the rocket traveled towards the person measuring the higher altitude and away from the lower measurement marker. Upon landing, the rocket remained intact except for the balsa fins, which fractured along the mounting bracket. Hasty re-gluing of the fins enabled a second launch attempt, but the epoxy bond between the two halves of the air vessel could not handle the 60psi pressure and ruptured on the stand before launch. Future improvements include further reduction of weight, better epoxy seal bond, improved fin structure, and use appropriate water fill level.

4.2 Rebuild

New materials are purchased, a second rocket is fabricated similarly to the first, and fill level is found via working simulation at 0.45 L. The new rocket is 179.3g in mass. Sanding of the bottles to increase bond surface adhesion on the pressure vessel bottles, 3-D printed fins, and increased sanding of the nose cone via a drill-powered hand lathe mechanism to ensure even surfacing all aid in the second iteration design. During the launch of this rocket, the nosecone became dislodged so the rocket failed mid-flight. The altimeter malfunctioned during the violent landing process, but the triangulation measures gave a height of approximately 130ft. The rocket was then patched for two additional flights, which attained altitudes of approximately 170ft and then 210ft without the third bottle, damaged in the second flight. This demonstrates the robustness and reusability of the rocket, proving the modular design and epoxied body capable of handling multiple loads and quick fixes. Even these respectable heights, attained with crippled rockets using needlessly heavy store-bought fins, were less than the potential maximum apogee since the fill level was incorrectly estimated at 0.5L for these flights, when the true optimal value was 0.45.

4.3 Conclusion and Future Improvements

Many improvements can be made to the process, design, and experimental procedures. For instance, with a pressurized rocket ready to be launched, a wind speed sensor integrated with the trigger mechanism could find a local minimum in the wind conditions during which to launch the rocket with minimal wind interference. Better nozzles and custom launchers could be made to reduce losses during launch. Specifically, the exit of the bottle could be optimized in surface area and the geometry of the throat leading to the exit could be optimized for inviscid flow. The launcher necessitates loss of water when mounting, so improving the attachment mechanism would yield better results. Water could be substituted with hydrogen and oxygen gas in an extreme case. Most importantly, the system could be better integrated through further application of adhesives to the nose cone and third bottle on the body, which failed during the second flight. Through further modeling it was found that minimizing the total rocket mass by potentially using only a 1 liter pressure vessel could allow for a higher max height than those achieved experimentally during the launches. The best way to optimize the rocket design is to have the smallest structural mass posible. If further time were available a 1 liter bottle would be constructed to test this hypothesis. Hindsight is 20/20.

Despite the large room for improvement, the design constructed proved extremely robust (despite the fragile fins) as it allowed for many flights and was easy to rebuild on short notice (thanks to the 3d printed parts).

4.4 Team Reflections

The lab's open-ended structure provided significant room for innovation and optimization of the rocket system. While the design implemented seemed significantly better, or at least more innovative than the conventional soda bottle design, the performance was not substantially increased due to improper fill levels and system integrity failures. The authors enjoyed building and testing the rocket and code, which both turned out well in the end despite the arguably sub-perfect results (did not attain class maximum apogee). In the future, less time debugging code and rebuilding rockets is key. To accomplish this, either less effort and meticulousness is necessary or better consideration of the key factors is necessary. Although structural integrity is a priority, the system fails to remain intact for multiple launches. Although prediction for fill level is obligatory, reliable numbers were not obtained efficiently. This caused the authors to spend an estimated 50 hours cumulatively on this lab, far above what was necessary for a successful launch. Although it is difficult to see where the process could have been improved without prior experience, future experiments should only take an estimated half of the time in the future. Teamwork is difficult to improve, since the lab went smoothly and the team supported one another fully. The circumstances simply did not allow for sufficient iterative processes on the ambitious design the team chose.

5 Gallery

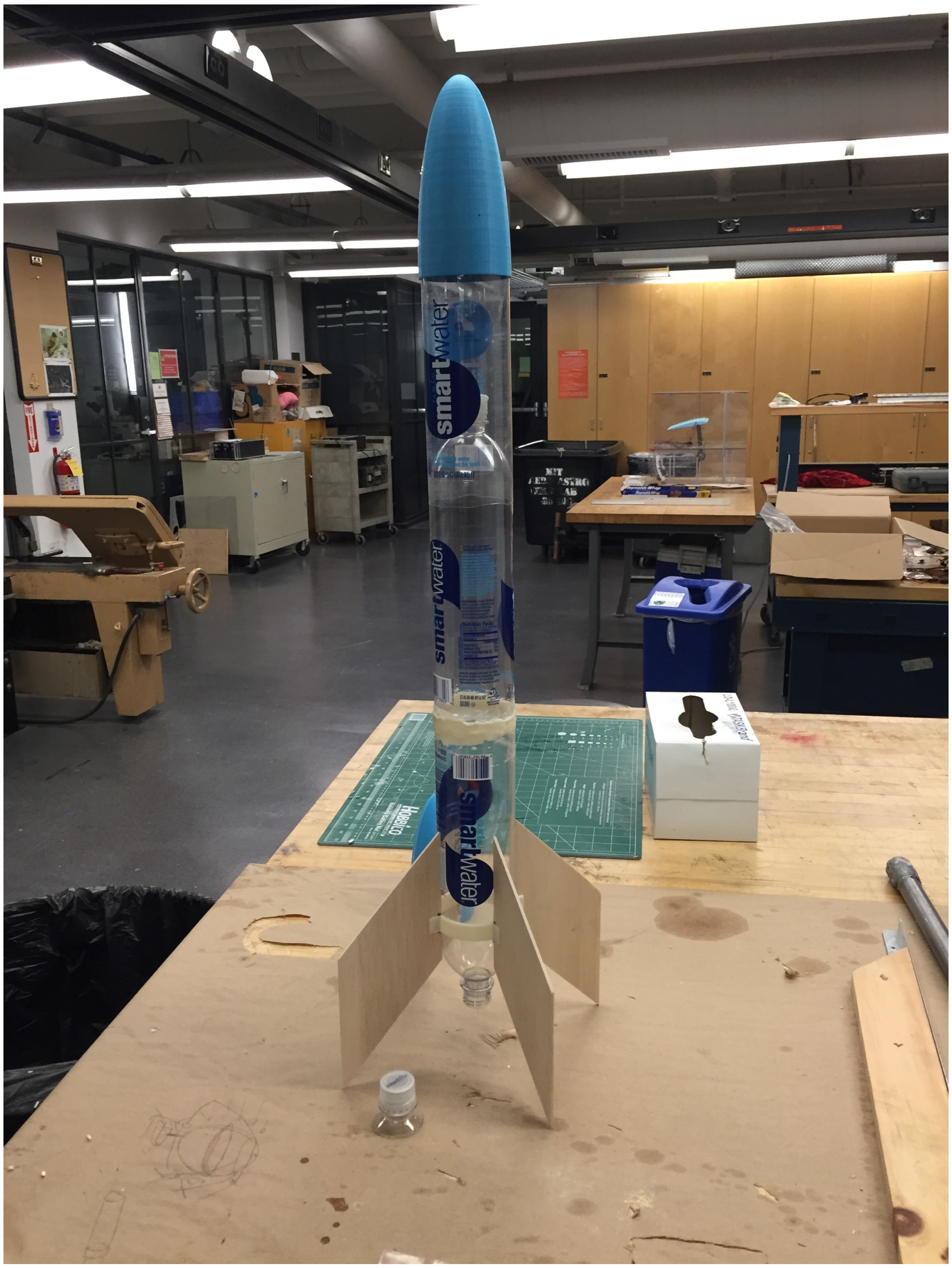

Figure 3: Complete rocket assembly

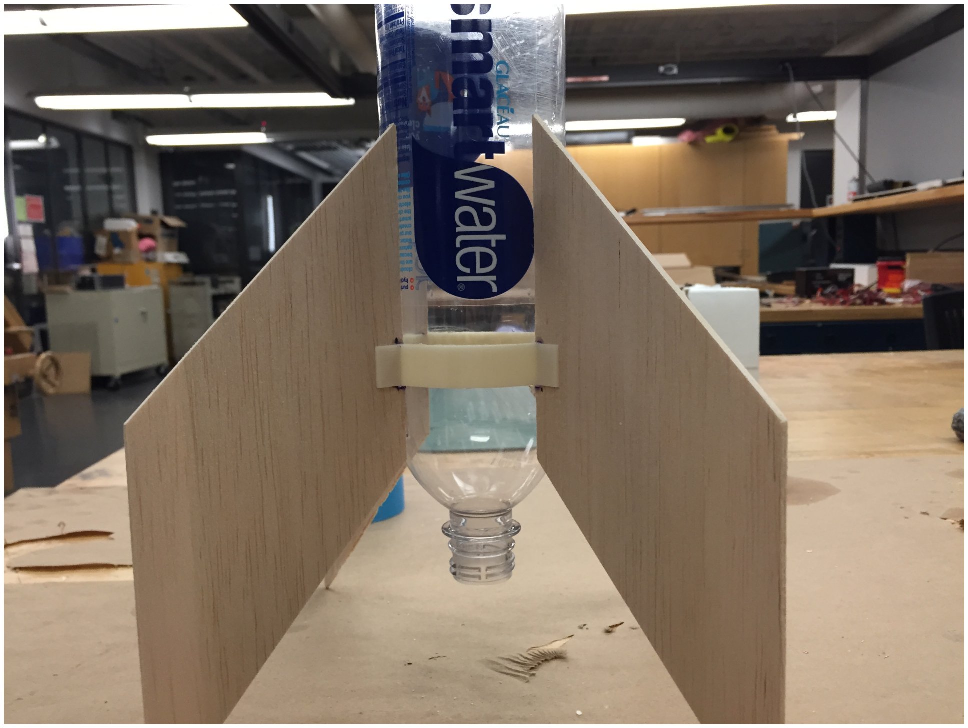

Figure 4: 3D printed fin holder

Figure 5: 3D printed nose cone

Figure 6: Fin strengthened with epoxy