2.1 Signals and Systems

a) z^t is an eigensignal of a LTI system. It can be shown using the input/output relation of an LTI system (convolution).

An Eigensignal has to satisfy the following condition:

$$y(t) = H\{x(t)\} = \underbrace{\lambda}_{Eigenvalue} \cdot \underbrace{x(t)}_{Eigensignal} \tag{1}$$

Setting $x(t)$ to z^t leads to the following:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) d\tau = \int_{-\infty}^{\infty} z^{t-\tau} \cdot h(\tau) d\tau = \underbrace{z^t}_{Eigensignal} \cdot \underbrace{\int_{-\infty}^{\infty} z^{-\tau} \cdot h(\tau) d\tau}_{Eigenvalue} \tag{2}$$

You can see at the equation above, that an eigensignal z^t at the input of a LTI, leads to the scaled Eigensignal at the ouput. The Eigensignal will be scaled by the Eigenvalue.

$$z^t = e^{ln(z) \cdot t} \tag{3}$$

Relation between s and z:

$$s = ln(z) \tag{4}$$

b) To find the impulse response $h[n]$ of the system given by the difference equation, we perform a z transformation of the given difference equation.

$$a_1 \cdot Y(z) = b_2 \cdot X(z) \cdot z^{-1} + b_1 \cdot X(z) + b_0 + a_2 \cdot Y(z) z^{-1} \tag{5}$$

$$Y(z) \cdot [a_1 - a_2 \cdot z^{-1}] = X(z) \cdot [b_2 \cdot z^{-1} + b_1] + b_0 \tag{6}$$

As you can see, at the equation above, you cannot find an expression for the relation $H(z) = \frac{Y(z)}{X(z)}$. I claim that the given system is time-variant and non linear, and therefore we can not find an expression for $h[n]$ (i.e $H(Z)$).

The system is non-linear, because it does not meet the homogeneity condition ($f(a \cdot x) = a \cdot f(x)$). Multiplying the input $X(z)$ by a factor a does not lead to the output $a \cdot Y(z)$

To proof that the system is time-variant, we have to show that the condition for time-invariance will not be fulfilled for any signal $x(t)$.

Time Invariance: (for any signal $x[n]$)

$$y[n] = H\{x[n]\} => y[n-k] = H\{x[n-k]\} \tag{7}$$

So let's proof it:

$$x[n] \circ\!\!-\!\!\bullet X(z)$$

$$x[n-k] \circ\!\!-\!\!\bullet z^{-k} X(z)$$

$$y[n] \circ\!\!-\!\!\bullet Y(z)$$
$$y[n-k] \circ\!\!-\!\!\bullet z^{-k}Y(z)$$

Replacing $x[n]$ by $x[n-k]$ in Equation 6 should lead to $y[n-k]$, But this is a different equation:

$$z^{-k}Y(z) \cdot [a_1 - a_2 \cdot z^{-1}] = z^{-k}X(z) \cdot [b_2 \cdot z^{-1} + b_1] + b_0 \tag{8}$$

But as you can see, this equation is different than the Equation 6.

$$Y(z) \cdot [a_1 - a_2 \cdot z^{-1}] = X(z) \cdot [b_2 \cdot z^{-1} + b_1] + z^{-k}b_0 \tag{9}$$

Because the system ist time-variant and non-linear, the system cannot be described by an impulse response $h[n]$. We could of course calculate the output $y[n]$, if $x[n] = \delta[n]$, but this would not be an overall valid impulse response.

c)

$$y(t) = \int_{-\infty}^{\infty} h(t,\hat{\tau}) \cdot x(t-\hat{\tau}) d\hat{\tau} \tag{10}$$

Let's calculate the output $\hat{h}(t,\tau)$ of the system, by inserting $x(t) = \delta(t-\tau)$ into the equation above.

$$\hat{h}(t,\tau) = \int_{-\infty}^{\infty} h(t,\hat{\tau}) \cdot \delta(t-\tau-\hat{\tau}) d\hat{\tau} \tag{11}$$

The dirac-delta $\delta(t-\tau-\hat{\tau})$ is nonzero for $\hat{\tau} = t - \tau$:

$$\hat{h}(t,\tau) = h(t, t-\tau) \tag{12}$$

d) The first impulse response is time-invariant and the second impulse response is time-variant.

Proof that the second impulse response is time-variant.

$$h(t,\hat{\tau}) = e^{\frac{t}{3}} u(t) \tag{13}$$

In order that a system is a System is time-invariant it has to fulfill the following condition:

For any signal $x(t)$:

$$y(t) = H\{x(t)\} \Rightarrow y(t-t_0) = y'(t) = H\{x(t-t_0)\} \tag{14}$$

Let $x(t)$ be an dirac impulse $\delta(t)$.

$$y(t) = \int_{-\infty}^{\infty} h(t,\hat{\tau}) \cdot \delta(t-\hat{\tau}) d\hat{\tau} = h(t,t) = e^{\frac{t}{3}} u(t) \tag{15}$$

If we now shift the input $x(t)$ by t_0, we get:

$$x(t - t_0) = \delta(t - t_0)$$

$$y'(t) = \int_{-\infty}^{\infty} h(t, \hat{\tau}) \cdot \delta(t - t_0 - \hat{\tau}) d\hat{\tau} = h(t, t - t_0) = e^{\frac{t}{3}} u(t) \tag{16}$$

The system is time-variant since, $y'(t) \neq y(t - t_0)$.

2.2 Analog Filters

a) The matlab function *ellipord* was used to find the minimum order of a filter that meets the specification. The minimum order of the filter is 7. A plot of the magnitude, phase, group delay and the s-plane can be found in Figures 2.2 - 2.

b) The filter was designed to have a minimum stop-band attenuation of 60dB and the stop band starts at 24kHz. Therefore, we can be sure that all frequencies above 24kHz will be attenuated by at least 60dB. To make sure that all aliasing-components are below 60dB, we have to use a sampling frequency of at least $2 * 24kHz = 48kHz$.

NOTE: From the bode diagram, you can see that the actual frequency response, is "better" than the specification. The 60dB stop-band attenuation is already reached at a frequency below 24 kHz. In the calculation above, we assumed that the stop-band "starts" at 24 kHz. For this reason we have some kind of reserve. For instance the frequency response may change a little bit due to quantization of the coefficients.

c) Like in task **a**, matlab was used to find the filter coefficients for the same specification, but with a filter order fixed to 4. From the bode diagram in Figure 2.2, you can see that all frequencies above approximately $51.5kHz$ have an attenuation of more the 60dB. So, for this filter, we have to use a sampling frequency of $2 \cdot 51.5kHz = 103kHz$ to achieve an attenuation of 60 dB for out-of-band distortion.

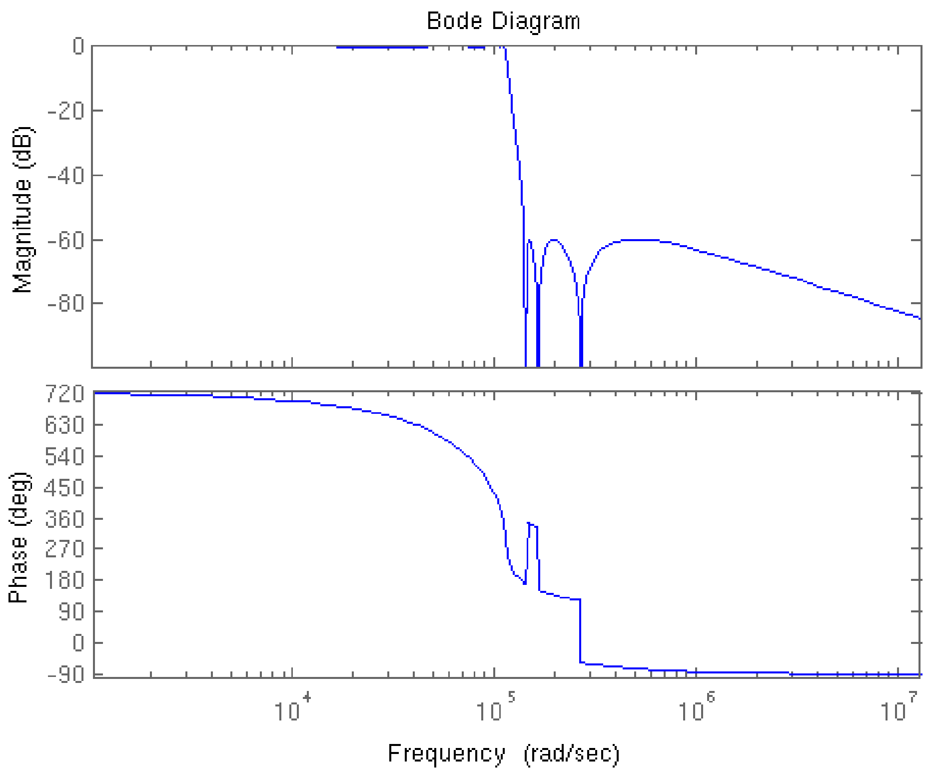
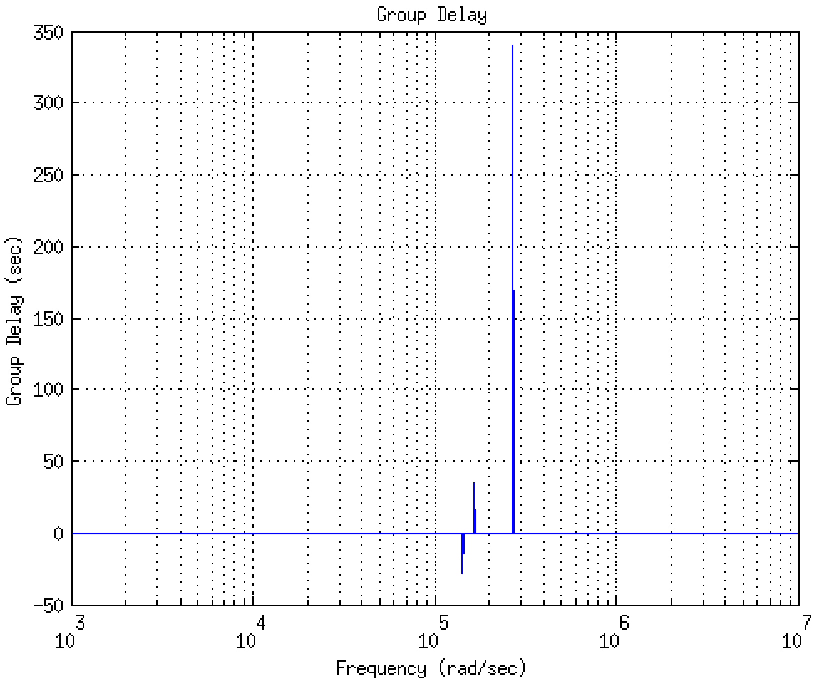


Figure 1: Bode Diagram for task **a**: You can see in the bode diagram, that the designed filter, meets the specification. The filter has a ripple in the stop-band and the pass-band. The ripple in the stop-band comes from the zero points of the filter. The drawback of this fast stop-passband transition is the ripple and a very bad group delay (non linear phase).



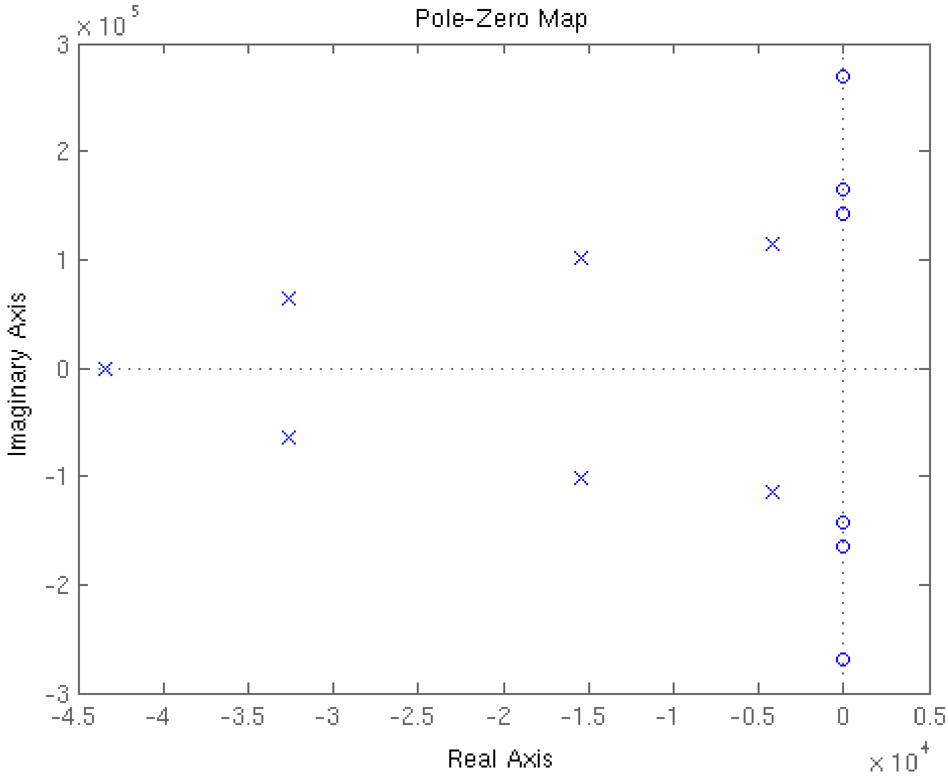


Figure 2: You can see that the filter consists of poles and zeros. All poles are on the left plane, which means the filter is stable.

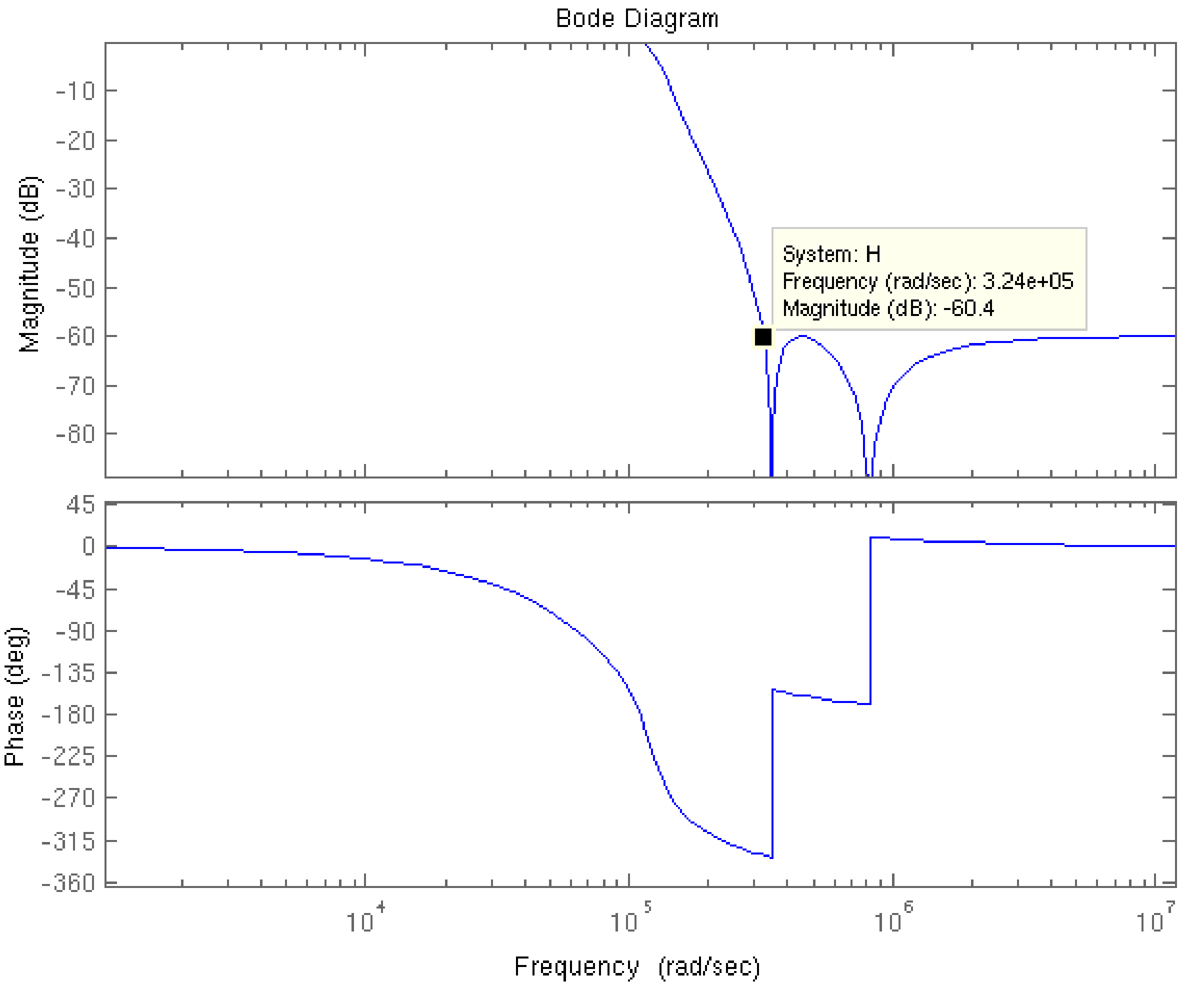


Figure 3: Bode Diagram for task **c**: You can see in the bode diagram, that the stop-band of the filter (60dB attenuation) starts at approximately $51.5kHz$.

2.3 Sampling and Reconstruction

a) Sampled Signal $x_s(t)$

The sampled signal is calculated by multiplying the continuous input signal with the sampling signal in the time domain. As the dirac delta is defined as 0 when its argument $\neq 0$, it has a masking characteristic.

$$\begin{aligned} x_s(t) &= x_c(t) \cdot s(t) \\ &= x_c(t) \cdot T \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &\overset{\text{masking}}{=} T \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t - nT) \end{aligned}$$

b) Sampled Signal in the Frequency Domain $X_s(j\Omega)$

As the sampling signal consisting of a dirac delta is hard to handle in a mathematical sense, it is quite a good idea to represent it as Fourier series and then transform it to the frequency domain.
The Fourier Series of $s(t)$ is determined by:

$$s(t) = \sum_{k=-\infty}^{\infty} A_k \cdot e^{j2\pi kt \cdot \frac{1}{T}}$$

with

$$\begin{aligned} A_k &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} T \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-j\frac{2\pi}{T}kt} dt \\ &= \frac{T}{T} \sum_{n=-\infty}^{\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \underbrace{\delta(t - nT) e^{-j\frac{2\pi}{T}k \cdot t}}_{\text{masking property}} dt \\ &= \sum_{n=-\infty}^{\infty} e^{-j\frac{2\pi}{T}k \cdot nT} \underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t - nT) dt}_{\text{1 for n=0, 0 else}} \\ &= e^{-j2\pi k0} = 1 \quad \forall k \end{aligned}$$

With $\Omega_s = \frac{2\pi}{T}$, $s(t)$ can be written as

$$s(t) = \sum_{k=-\infty}^{\infty} e^{j\Omega_s kt}$$

Now the representation as Fourier Series can be transformed to the frequency domain using the non-unitary Fourier transform with angular frequency.

$$
\begin{aligned}
S(j\Omega) &= \int_{-\infty}^{\infty} s(t) \cdot e^{-j\Omega t} dt \\
&= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{j\Omega_s kt} \cdot e^{-j\Omega t} dt \\
&= \sum_{k=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} e^{j\Omega_s kt} \cdot e^{-j\Omega t} dt}_{\text{Fourier transform of } e^{j\Omega_s kt}}
\end{aligned}
$$

According to the Fourier transformation table we know, that

$$
e^{j\Omega_s kt} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\Omega - k\Omega_s)
$$

Hence, the transformed sampling signal equals

$$
\begin{aligned}
S(j\Omega) &= \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - k\Omega_s) \\
&= 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)
\end{aligned}
$$

In order to get $x_c(t) \cdot s(t) = x_s(t)$ in the frequency domain (further denoted as $X_s(j\Omega)$), $X_c(j\Omega)$ and $S(j\Omega)$ are convolved and rescaled by $\frac{1}{2\pi}$ according to the definition for non-unitary Fourier transform with angular frequency

$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{2\pi} X_c(j\Omega) * 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \\
&= \frac{1}{2\pi} 2\pi \sum_{k=-\infty}^{\infty} X_c(j\Omega) * \delta(\Omega - k\Omega_s) \\
&\overset{\text{shifting prop.}}{=} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))
\end{aligned}
$$

c) Ideal Reconstruction Filter $H_r(j\Omega), h_r(t)$

An ideal reconstruction filter is supposed to recover exactly the input signal, when the sampled signal is convolved with the impulse response of this filter in the time domain. In the frequency domain, a convolution in the time domain corresponds to a multiplication. This leads to the following constraint on the ideal reconstruction filter:

$$
X_s(j\Omega) \cdot H_r(j\Omega) \overset{!}{=} X_c(j\Omega)
$$

With $X_s(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$ we get

$$
\begin{aligned}
X_c(j\Omega) &\stackrel{!}{=} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \cdot H_r(j\Omega) \\
&= X_c(j(\Omega - 0\cdot\Omega_s)) \cdot H_r(j\Omega) + \sum_{\substack{k=-\infty \\ \forall k \neq 0}}^{\infty} X_c(j(\Omega - k\Omega_s)) \cdot H_r(j\Omega) \\
&= X_c(j\Omega) \cdot H_r(j\Omega) + \underbrace{\sum_{\substack{k=-\infty \\ \forall k \neq 0}}^{\infty} X_c(j(\Omega - k\Omega_s)) \cdot H_r(j\Omega)}_{\stackrel{!}{=}0 \;\Rightarrow\; H_r(j\Omega)=0, |\Omega| \geq \frac{\Omega_s}{2}} \\
X_c(j\Omega) &= X_c(j\Omega) \cdot H_r(j\Omega) \\
&\Rightarrow H_r(j\Omega) \stackrel{!}{=} 1, |\Omega| < \frac{\Omega_s}{2} \\
&\Rightarrow H_r(j\Omega) = \begin{cases} 1 & \text{if } |\Omega| < \frac{\Omega_s}{2} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

To determine the impulse response $h_r(t)$ of the reconstruction filter in the time domain, the inverse Fourier transform is used

$$
\begin{aligned}
h_r(t) &= \mathcal{F}^{-1}(H_r(j\Omega)) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} H_r(j\Omega) \cdot e^{j\Omega t} d\Omega \\
&= \frac{1}{2\pi} \int_{-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}} 1 \cdot e^{j\Omega t} d\Omega \\
&= \frac{1}{2\pi} \left(\frac{1}{jt} \cdot e^{j\Omega t} \right) \Bigg|_{\Omega=-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}} \\
&= \frac{1}{2\pi} \cdot \frac{1}{jt} \underbrace{\left(e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t} \right)}_{2j \cdot sin(\frac{\Omega_s t}{2})} \\
&= \frac{2j}{2\pi jt} sin\left(\frac{\Omega_s t}{2}\right), \qquad \Omega_s = \frac{2\pi}{T} \\
&= \frac{1}{\pi t} sin\left(\frac{\pi}{T} \cdot t\right) \\
&= \frac{1}{T} \cdot \frac{1}{\frac{\pi}{T} \cdot t} sin\left(\frac{\pi}{T} \cdot t\right) \\
&= \frac{1}{T} \cdot sinc\left(\frac{\pi}{T} \cdot t\right)
\end{aligned}
$$

d) The Reconstructed Signal $x_r(t)$

To calculate the reconstructed signal in time domain, the convolution of the sampled signal and the impulse response of the reconstruction filter is evaluated.

$$
\begin{aligned}
x_r(t) &= x_s(t) * h_r(t) \\
&= \int_{-\infty}^{\infty} x_s(\tau) \cdot h_r(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} T \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(\tau - nT) \cdot \frac{1}{T} \cdot sinc\left(\frac{\pi}{T} \cdot \tau\right) d\tau \\
&= \sum_{n=-\infty}^{\infty} x_c(nT) \int_{-\infty}^{\infty} \delta(\tau - nT) \cdot sinc\left(\frac{\pi}{T} \cdot \tau\right) d\tau \\
&\overset{\text{masking prop.}}{=} \sum_{n=-\infty}^{\infty} x_c(nT) sinc\left(\frac{\pi}{T} \cdot (t - nT)\right) \\
&= \sum_{n=-\infty}^{\infty} x_c(nT) sinc\left(\pi \left(\frac{t}{T} - n\right)\right)
\end{aligned}
$$

e) Sampling Signal Correspondence

To determine if

$$
s(t) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - n\right)
$$

corresponds to

$$
s_1(t) = T \sum_{n=-\infty}^{\infty} \delta(t - nT)
$$

or

$$
s_2(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
$$

we take a closer look to the argument of the delta distribution. In the first place it is essential to analyse at which parameter t the argument equals zero. In the case of $s(t)$ its

$$
\begin{aligned}
\frac{t}{T} - n &\overset{!}{=} 0 \\
t &= n \cdot T
\end{aligned}
$$

In case of $s_1(t)$ and $s_2(t)$ we get

$$
\begin{aligned}
t - nT &\overset{!}{=} 0 \\
t &= n \cdot T
\end{aligned}
$$

So there is no difference between the behaviours of the delta distributions. Hence the first intuition is that $s(t)$ corresponds to $s_2(t)$, but it seems quite uncommon to claim $T \cdot \infty \neq \infty$ (at $n = 0$). To achieve a more mathematical reasoning, the Fourier Series Expansion can be investigated. For $s_1(t)$ it was already shown that the Fourier Series is

$$
s(t) = \sum_{k=-\infty}^{\infty} A_k \cdot e^{j2\pi kt \cdot \frac{1}{T}}
$$

with

$$A_k = 1 \quad \forall k$$

For $s(t)$ and $s_2(t)$ the Fourier Series is determined as

$$s_{(2)}(t) = \sum_{k=-\infty}^{\infty} A_k \cdot e^{j2\pi kt \cdot \frac{1}{T}}$$

with

$$\begin{aligned}
A_k &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-j\frac{2\pi}{T}kt} dt \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \underbrace{\delta(t - nT) e^{-j\frac{2\pi}{T}k \cdot t}}_{\text{masking property}} dt \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{-j\frac{2\pi}{T}k \cdot nT} \underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t - nT) dt}_{\text{1 for n=0, 0 else}} \\
&= \frac{1}{T} e^{-j2\pi k0} = \frac{1}{T} \quad \forall k
\end{aligned}$$

so $s(t)$ corresponds to $s_2(t)$ and $s_1(t)$ is really a different sampling signal.

If we would prefer $s_1(t) = T \sum_{n=-\infty}^{\infty} \delta(t - nT)$ or $s_2(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ as sampling signal, depends somehow on the application.
When using $s_1(t)$, $x_s(t)$ consists of values of $x_c(t)$ at $t = n \cdot T$ scaled by T, whereas $X_s(j\Omega)$ contains unscaled values of $X_c(j\Omega)$ at $\Omega = k \cdot \Omega_s$.
On the opposite, using $s_2(t)$ would force $x_s(t)$ to be an unscaled sampled version of $x_c(t)$ in the time domain, whereas in the frequency domain $X_s(j\Omega)$ would be a scaled sampled version of $X_c(j\Omega)$ by the factor $\frac{1}{T}$.