Bernstein Polynomials and Approximation

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(Joint work with Zhe Liu)

Definition. With f a real-valued function defined and bounded on the interval $[0,1]$, let $B_n(f)$ be the polynomial on $[0,1]$ that assigns to x the value

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

$B_n(f)$ is the nth Bernstein polynomial for f.

The following identities will be useful to us.

$$B_n(1) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1 \tag{1}$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k} = x \tag{2}$$

$$B_n(x^2) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} x^k (1-x)^{n-k} = \frac{(n-1)x^2}{n} + \frac{x}{n} \tag{3}$$

$$B_n(x^3) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^3}{n^3} x^k (1-x)^{n-k} = \frac{(n-1)(n-2)x^3}{n^2} + \frac{3(n-1)x^2}{n^2} + \frac{x}{n^2} \quad (4)$$

$$\begin{aligned} B_n(x^4) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^4}{n^4} x^k (1-x)^{n-k} &= \frac{(n-1)(n-2)(n-3)x^4}{n^3} \\ &+ \frac{6(n-1)(n-2)x^3}{n^3} + \frac{7(n-1)x^2}{n^3} + \frac{x}{n^3} \end{aligned} \quad (5)$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} = x(1-x)\frac{1}{n} \tag{6}$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^4 x^k (1-x)^{n-k} = x(1-x)\frac{(3n-6)x(1-x)+1}{n^3} \tag{7}$$

To prove these identities, first, from the binomial theorem,

$$B_n(1) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = [x + (1-x)]^n = 1.$$

Note that

$$\frac{d}{dp}\Big(\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \Big) = \frac{d}{dp}\big((p+q)^n\big) = n(p+q)^{n-1}.$$

Thus

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} p^k q^{n-k} = (p+q)^{n-1} p.$$

Replacing p by x and q by $1-x$ in the above expression, we have identity (2).

Now, differentiating this expression with respect to p three more times and each time multiplying both sides of the result by $\frac{p}{n}$, we have the following

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} p^k q^{n-k} = \frac{(n-1)(p+q)^{n-2}}{n} p^2 + \frac{(p+q)^{n-1}}{n} p$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k^3}{n^3} p^k q^{n-k} = \frac{(n-1)(n-2)(p+q)^{n-3}}{n^2} p^3 + \frac{3(n-1)(p+q)^{n-2}}{n^2} p^2 + \frac{(p+q)^{n-1}}{n^2} p$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k^4}{n^4} p^k q^{n-k} = \frac{(n-1)(n-2)(n-3)(p+q)^{n-4}}{n^3} p^4$$
$$+ \frac{6(n-1)(n-2)(p+q)^{n-3}}{n^3} p^3$$
$$+ \frac{7(n-1)(p+q)^{n-2}}{n^3} p^2 + \frac{(p+q)^{n-1}}{n^3} p.$$

Replacing p by x and q by $1-x$ in the above three identities, we obtain the identities (3), (4) and (5).

It follows that

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} = \left[\frac{(n-1)x^2}{n} + \frac{x}{n}\right] - 2x^2 + x^2$$

$$= x(1-x)\frac{1}{n}$$

and

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k}{n}-x\Big)^4 x^k(1-x)^{n-k}\\
=&\Big[\frac{(n-1)(n-2)(n-3)x^4}{n^3}+\frac{6(n-1)(n-2)x^3}{n^3}+\frac{7(n-1)x^2}{n^3}+\frac{x}{n^3}\Big]\\
&-4x\Big[\frac{(n-1)(n-2)x^3}{n^2}+\frac{3(n-1)x^2}{n^2}+\frac{x}{n^2}\Big]\\
&+6x^2\Big[\frac{(n-1)x^2}{n}+\frac{x}{n}\Big]-4x^4+x^4\\
=&x(1-x)\frac{(3n-6)x(1-x)+1}{n^3}.
\end{aligned}
$$

Theorem. Let f be a real-valued function defined, and bounded by M on the interval $[0,1]$. For each point x of continuity of f, $B_n(f)(x) \to f(x)$ as $n \to \infty$. If f is continuous on $[0,1]$, then the Bernstein polynomial $B_n(f)$ tends uniformly to f as $n \to \infty$. **With x a point of differentiability of f, $B_n'(f)(x) \to f'(x)$ as $n \to \infty$. If f is continuously differentiable on $[0,1]$, then $B_n'(f)$ tends to f' uniformly as $n \to \infty$.**

Proof. From

$$B_n(f)(x) - f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) - f(x) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \Big[f\left(\frac{k}{n}\right) - f(x) \Big],$$

it follows that, for each x in $[0, 1]$,

$$|B_n(f)(x) - f(x)| \leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \Big| f\left(\frac{k}{n}\right) - f(x) \Big|.$$

To estimate this last sum, we separate the terms into two sums $\sum'$ and $\sum''$, those where $|\frac{k}{n} - x|$ is less than a given positive δ and the remaining terms, those for which $\delta \leq |\frac{k}{n} - x|$.

Suppose that x is a point of continuity of f. Then for any $\varepsilon > 0$, there is a positive δ such that $|f(x') - f(x)| < \frac{\varepsilon}{2}$ when $|x' - x| < \delta$.

For the first sum,

$$\begin{aligned}
\sum{}' \binom{n}{k} x^k (1-x)^{n-k} \left| f\left(\frac{k}{n}\right) - f(x) \right| &< \sum{}' \binom{n}{k} x^k (1-x)^{n-k} \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

For the remaining terms, we have $\delta^2 \leq |\frac{k}{n} - x|^2$,

$$
\begin{aligned}
&\delta^2 \sum{}'' \binom{n}{k} x^k (1-x)^{n-k} \left| f\left(\frac{k}{n}\right) - f(x) \right| \\
\leq & \sum{}'' \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} \left| f\left(\frac{k}{n}\right) - f(x) \right| \\
\leq & \sum{}'' \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} 2M \\
\leq & 2M \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} \\
= & 2M \frac{x(1-x)}{n} \qquad\qquad \text{(from (6))} \\
\leq & \frac{2M}{n}.
\end{aligned}
$$

Thus

$$\sum{}'' \binom{n}{k} x^k (1-x)^{n-k} \Big| f\Big(\frac{k}{n}\Big) - f(x) \Big| \leq \frac{2M}{\delta^2 n}.$$

For this δ, we can choose n_0 large enough so that, when $n \geq n_0$, $\frac{2M}{\delta^2 n} < \frac{\varepsilon}{2}$.

For such an n and the given x

$$|B_n(f)(x) - f(x)| \leq \sum{}' + \sum{}'' < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $B_n(f)(x) \to f(x)$ as $n \to \infty$ for each point x of continuity of the function f.

If f is continuous at each point of $[0,1]$, then it is uniformly continuous on $[0,1]$, and for this given ε, we can choose δ so that $|f(x') - f(x)| < \frac{\varepsilon}{2}$ for each pair of points x' and x in $[0,1]$ such that $|x' - x| < \delta$. From the preceding argument, with n_0 chosen for this δ, and when $n \geq n_0$, $|B_n(f)(x) - f(x)| < \varepsilon$ for each x in $[0,1]$. Thus $\|B_n(f) - f\| \leq \varepsilon$, and $B_n(f)$ tends uniformly to f as $n \to \infty$.

Now, with x in $[0,1]$,

$$
\begin{aligned}
B_n'(f) &= \frac{d}{dx}\biggl(\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\Bigl(\frac{k}{n}\Bigr)\biggr)\\
&= \sum_{k=1}^{n}\binom{n}{k}kx^{k-1}(1-x)^{n-k}f\Bigl(\frac{k}{n}\Bigr)\\
&\quad -\sum_{k=0}^{n-1}\binom{n}{k}(n-k)x^k(1-x)^{n-k-1}f\Bigl(\frac{k}{n}\Bigr)+\Bigl[nx^{n-1}f(1)-n(1-x)^{n-1}f(0)\Bigr]\\
&= \sum_{k=0}^{n}\binom{n}{k}\Bigl[k(1-x)-(n-k)x\Bigr]x^{k-1}(1-x)^{n-k-1}f\Bigl(\frac{k}{n}\Bigr)\\
&= n\sum_{k=0}^{n}\binom{n}{k}\Bigl(\frac{k}{n}-x\Bigr)x^{k-1}(1-x)^{n-k-1}f\Bigl(\frac{k}{n}\Bigr).
\end{aligned}
$$

(Note that $\left(\frac{k}{n} - x\right)x^{k-1} = -1$ when $k = 0$ and $\left(\frac{k}{n} - x\right)(1-x)^{n-k-1} = 1$ when $k = n$.)

Also,

$$0 = f(x)\frac{d}{dx}(1) = f(x)\frac{d}{dx}\left(\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right)$$
$$= f(x)n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n} - x\right)x^{k-1}(1-x)^{n-k-1}.$$

Thus, for all x in $[0,1]$,

$$B_n'(f)(x) = n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n} - x\right)x^{k-1}(1-x)^{n-k-1}\Big[f\Big(\frac{k}{n}\Big) - f(x)\Big].$$

Suppose that x is a point of differentiability of f. Let a positive ε be given. We write

$$\frac{f\left(\frac{k}{n}\right) - f(x)}{\frac{k}{n} - x} = f'(x) + \xi_k.$$

From the assumption of differentiability of f at x, there is a positive δ such that, when

$$0 < |x' - x| < \delta,\ |\tfrac{f(x')-f(x)}{x'-x} - f'(x)| < \tfrac{\varepsilon}{2}.$$

Thus, when $0 < |\frac{k}{n} - x| < \delta$,

$$|\xi_k| = \left|\frac{f\left(\frac{k}{n}\right) - f(x)}{\frac{k}{n} - x} - f'(x)\right| < \frac{\varepsilon}{2}.$$

If $\frac{k}{n}$ happens to be x for some k, we define ξ_k to be 0 for that k and note that the inequality just stated, when $|\frac{k}{n} - x| > 0$, remains valid when $\frac{k}{n} = x$.

It follows that

$$
\begin{aligned}
B_n'(f)(x) &= n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)x^{k-1}(1-x)^{n-k-1}\left[f\left(\frac{k}{n}\right)-f(x)\right] \\
&= n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)x^{k-1}(1-x)^{n-k-1}\left[\left(\frac{k}{n}-x\right)f'(x)+\left(\frac{k}{n}-x\right)\xi_k\right] \\
&= f'(x)n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^{k-1}(1-x)^{n-k-1} \\
&\quad + n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^{k-1}(1-x)^{n-k-1}\xi_k \\
&= f'(x)+n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^{k-1}(1-x)^{n-k-1}\xi_k. \qquad (6)
\end{aligned}
$$

We estimate this last sum by separating it, again, into the two sums $\sum'$ and $\sum''$, those with the k for which $|\frac{k}{n} - x| < \delta$ and those for which $\delta \leq |\frac{k}{n} - x|$, respectively. For the first sum, we have

$$\begin{aligned}
\left|n \sum\nolimits' \right| &\leq n \sum\nolimits' \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1} |\xi_k| \\
&< n \sum\nolimits' \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1} \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1} \\
&= \frac{\varepsilon}{2}
\end{aligned}$$

from (6) and the choice of δ (that is, the differentiability of f at x).

For the second sum, we have that $\delta \le |\frac{k}{n} - x|$ so that

$$|\xi_k| \le \left|\frac{f\left(\frac{k}{n}\right) - f(x)}{\frac{k}{n} - x}\right| + |f'(x)| \le \frac{2M}{\delta} + |f'(x)|$$

and $(\delta^2 \leq |\frac{k}{n} - x|^2)$

$$
\begin{aligned}
\delta^2 \Big| n \sum{}'' \Big| &\leq \delta^2 n \sum{}'' \binom{n}{k} \Big(\frac{k}{n} - x\Big)^2 x^{k-1}(1-x)^{n-k-1} |\xi_k| \\
&\leq n \sum{}'' \binom{n}{k} \Big(\frac{k}{n} - x\Big)^4 x^{k-1}(1-x)^{n-k-1} |\xi_k| \\
&\leq n \sum{}'' \binom{n}{k} \Big(\frac{k}{n} - x\Big)^4 x^{k-1}(1-x)^{n-k-1} \Big(\frac{2M}{\delta} + |f'(x)|\Big) \\
&\leq n \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{k}{n} - x\Big)^4 x^{k-1}(1-x)^{n-k-1} \Big(\frac{2M}{\delta} + |f'(x)|\Big) \\
&= n \frac{(3n-6)x(1-x)+1}{n^3} \Big(\frac{2M}{\delta} + |f'(x)|\Big) \qquad \text{(from (7))} \\
&\leq n \frac{3}{n^2} \Big(\frac{2M}{\delta} + |f'(x)|\Big) = \frac{6M + 3\delta|f'(x)|}{n\delta}.
\end{aligned}
$$

Thus

$$\left|n\sum''\right| \leq \frac{6M + 3\delta|f'(x)|}{n\delta^3}.$$

For this δ, we can choose n_0 large enough so that, when $n \geq n_0$,

$$\frac{6M + 3\delta|f'(x)|}{n\delta^3} < \frac{\varepsilon}{2}.$$

For such n and the given x

$$\left|B_n'(f)(x) - f'(x)\right| \leq \left|n \sum{}'\right| + \left|n \sum{}''\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $B_n'(f)(x) \to f'(x)$ as $n \to \infty$ for each point x of differentiability of the function f.

We show, now, that if f is continuously differentiable on $[0,1]$, then the sequence $\{B_n'(f)\}$ tends to f' uniformly.

We intercept the proof for pointwise convergence at each point of differentiability of f at the formula:

$$B_n'(f)(x) = f'(x) + n\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k}{n} - x\Big)^2 x^{k-1}(1-x)^{n-k-1}\xi_k.$$

Assuming that f is everywhere differentiable on $[0,1]$ and f' is continuous on $[0,1]$, let M' be $\sup\{|f'(x)| : x \in [0,1]\}$. Choose δ positive and such that $|f'(x') - f'(x)| < \frac{\varepsilon}{2}$ when $|x' - x| < \delta$. Now, for any given x in $[0,1]$, recall that we had defined

$$\xi_k = \frac{f\left(\frac{k}{n}\right) - f(x)}{\frac{k}{n} - x} - f'(x) \quad \text{when } \frac{k}{n} \neq x, \quad \text{and } \xi_k = 0 \quad \text{when } \frac{k}{n} = x.$$

From the differentiability of f on $[0,1]$, the Mean Value Theorem applies, and

$$f\left(\frac{k}{n}\right) - f(x) = f'(x_k)\left(\frac{k}{n} - x\right)$$

where x_k is in the open interval with endpoints $\frac{k}{n}$ and x, when $\frac{k}{n} \neq x$. In case $\frac{k}{n} = x$, we may choose $f'(x_k)$ as we wish, and we choose x as x_k. With these choices, $\xi_k = f'(x_k) - f'(x)$. Our formula becomes

$$B_n'(f)(x) - f'(x) = n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1}\big(f'(x_k) - f'(x)\big).$$

In this case when we estimate the sum in the right-hand side of this equality by separating it into the two parts $\sum'$ and $\sum''$ exactly as we did before (for approximation of the derivatives at the single point x of differentiability), except that in this case, $|\xi_k|$ is replaced by $|f'(x_k) - f'(x)|$ and δ has been chosen by means of the uniform continuity of f' on $[0, 1]$ such that

$$|f'(x_k) - f'(x)| < \tfrac{\varepsilon}{2} \text{ when } |x_k - x| < \delta,$$

as is the case when $|\frac{k}{n} - x| < \delta$.

For the first sum $\sum'$, the sum over those k such that $|\frac{k}{n} - x| < \delta$,

$$
\begin{aligned}
& n \sum{}' \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1} |f'(x_k) - f'(x)| \\
<& \frac{\varepsilon}{2} n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1} \\
=& \frac{\varepsilon}{2}. \qquad \text{(from (6))}
\end{aligned}
$$

For the second sum $\sum''$, the sum over those k such that

$$\delta \leq |\tfrac{k}{n} - x|,$$

again, we have $\delta^2 \leq |\frac{k}{n} - x|^2$.

This time, $|f'(x_k) - f'(x)| \leq 2M'$ (and we really don't care that x_k may be very close to x as long as $|\frac{k}{n} - x| \geq \delta$ in this part of the estimate),

$$
\begin{aligned}
&\delta^2 n \sum\nolimits'' \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1}|f'(x_k) - f'(x)| \\
\leq& n \sum\nolimits'' \binom{n}{k} \left(\frac{k}{n} - x\right)^4 x^{k-1}(1-x)^{n-k-1} 2M' \\
\leq& 2M'n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^4 x^{k-1}(1-x)^{n-k-1} \\
=& 2M'n \frac{(3n-6)x(1-x)+1}{n^3} \qquad \text{(from (7))} \\
\leq& \frac{6M'}{n}.
\end{aligned}
$$

Again, for this δ, we can choose n_0 large enough so that, when $n > n_0$

$$n \sum'' \binom{n}{k} \Big(\frac{k}{n} - x\Big)^2 x^{k-1} (1-x)^{n-k-1} |f'(x_k) - f'(x)| \leq \frac{6M'}{\delta^2 n} < \frac{\varepsilon}{2},$$

and

$$\begin{aligned} \big|B_n'(f)(x) - f'(x)\big| &\leq n \sum' \binom{n}{k} \Big(\frac{k}{n} - x\Big)^2 x^{k-1} (1-x)^{n-k-1} |f'(x_k) - f'(x)| \\ &\quad + n \sum'' \binom{n}{k} \Big(\frac{k}{n} - x\Big)^2 x^{k-1} (1-x)^{n-k-1} |f'(x_k) - f'(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for each x in $[0, 1]$. Thus $\|B_n'(f) - f'\| \leq \varepsilon$, and $\{B_n'(f)\}$ tends to f' uniformly.