16.13 Let $\{a_n\}$ be any sequence of real numbers such that $\lim_{n\to\infty} na_n = 0$. Prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}+a_n\right)^n = e$$

Proof. Use factoring

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots + a^{n-1}) \le n(b-a)b^{n-1}.$$

Since $\lim_{n\to\infty} na_n = 0$, for sufficiently large n, $|na_n| < 1$. For efficiency, assume $a_n > 0$.

$$\begin{aligned}
\left|\left(1+\frac{1}{n}+a_n\right)^n - \left(1+\frac{1}{n}\right)^n\right| &\le |na_n|\left(1+\frac{1}{n}+|a_n|\right)^{n-1} \\
&\le na_n\left(1+\frac{1+1}{n}\right)^{n-1} \\
&\le na_n\left\{\left(1+\frac{1}{n}\right)^2\right\}^{n-1} \\
&\le na_n\left\{\left(1+\frac{1}{n}\right)^n\right\}^2 < e^2 na_n
\end{aligned}$$

This explains all.