Simulate 1D Focusing Nonlinear Schrödinger equation using Crank-Nicolson method

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What is Nonlinear Schrödinger Equation (NLS)?

Definition

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi$$

- $\psi(x,t)$: complex envelope of a wave

The nonlinear Schrödinger equation (NLSE) is a PDE that describes the dynamics of weakly nonlinear, monochromatic wave packets in a dispersive medium.

Applications: fiber optics, water waves

Types of Nonlinear Schrödinger Equations

Recall **Definition**

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + g|\psi|^2\psi$$

- $\psi(x,t)$: complex envelope of a wave

Focusing: g > 0

Defocusing: g < 0

In this project I will only focus on focusing type.

Using the Crank–Nicolson Method

$$i\frac{\psi_m^{n+1}-\psi_m^n}{\Delta t} =$$
$$-\frac{1}{4(\Delta x)^2}[(\psi_{m+1}^n - 2\psi_m^n + \psi_{m-1}^n) + (\psi_{m+1}^{n+1} - 2\psi_m^{n+1} + \psi_{m-1}^{n+1})]$$
$$+\frac{g}{2}(|\psi_m^n|^2\psi_m^n + |\psi_m^{n+1}|^2\psi_m^{n+1})$$

Since there are nonlinear terms, matrix solve is not available, must use root finder to step in time.

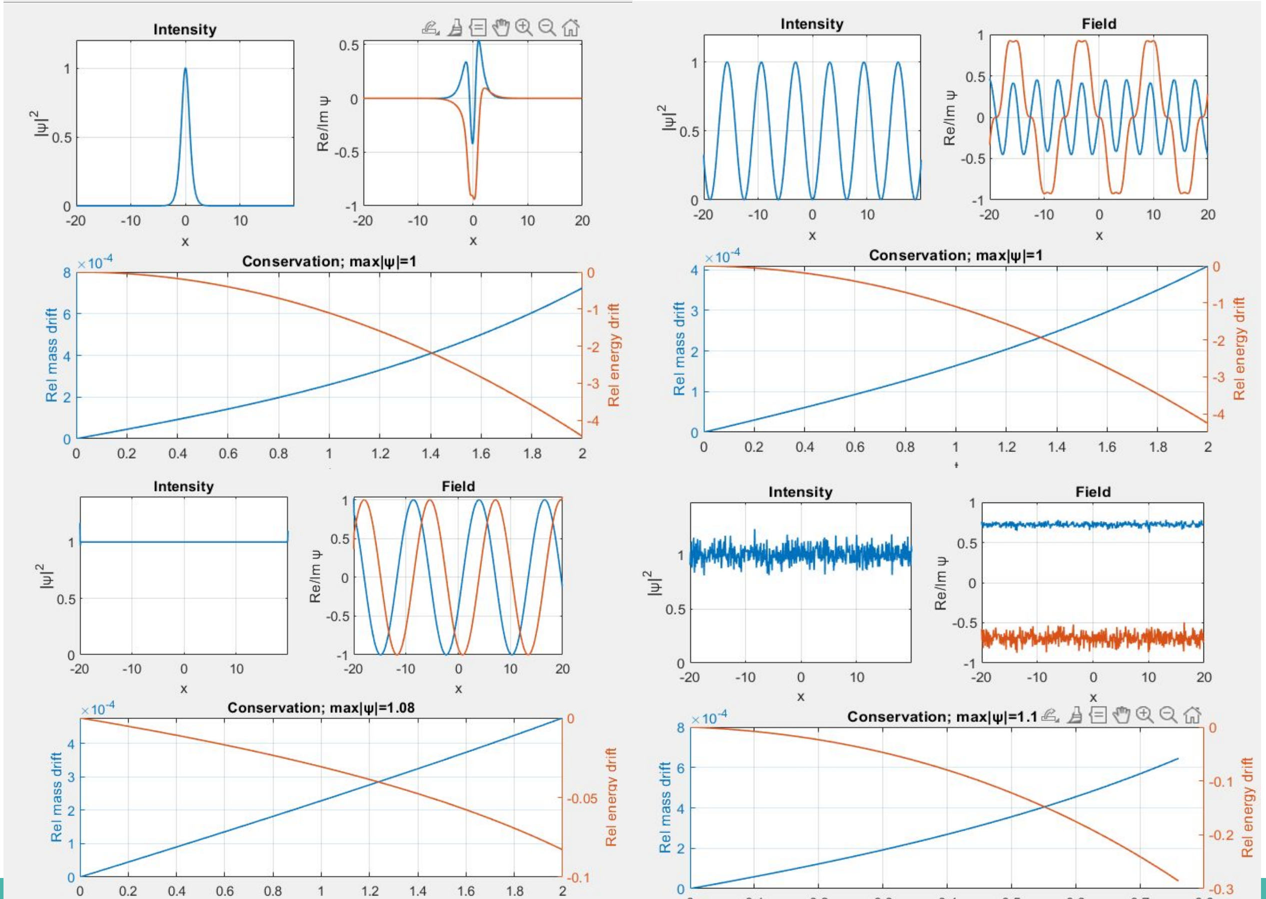
Boundary Conditions

To make it simple, here I will using periodic boundary condition:

$$\psi(-\frac{L}{2},t) = \psi(\frac{L}{2},t)$$
$$\frac{\partial\psi}{\partial x}(-\frac{L}{2},t) = \frac{\partial\psi}{\partial x}(\frac{L}{2},t)$$

No special treatment to boundary points is needed.

Simulation Results



Bright soliton
Sin wave
Plane wave
Noise wave

Stability Analysis

Suppose the solution is:

$$\psi(x_m, t_n) = g_n e^{ikx_m}$$

Recall the discretization:

$$i\frac{\psi_m^{n+1} - \psi_m^n}{\Delta t} =$$

$$-\frac{1}{4(\Delta x)^2}[(\psi_{m+1}^n - 2\psi_m^n + \psi_{m-1}^n) + (\psi_{m+1}^{n+1} - 2\psi_m^{n+1} + \psi_{m-1}^{n+1})]$$

$$+\frac{g}{2}(|\psi_m^n|^2\psi_m^n + |\psi_m^{n+1}|^2\psi_m^{n+1})$$

Plug in values, we have:

$$i\frac{g_{n+1}e^{ikx_m} - g_n e^{ikxm}}{\Delta t}$$

$$= -\frac{1}{4(\Delta x)^2}(g_n e^{ikx_{m+1}} - 2g_n e^{ikx_m} + g_n e^{ikx_{m-1}})$$

$$-\frac{1}{4(\Delta x)^2}(g_{n+1}e^{ikx_{m+1}} - 2g_{n+1}e^{ikx_m} + g_{n+1}e^{ikx_{m-1}})$$

$$+\frac{g}{2}(|g_{n+1}e^{ikx_{m+1}}|^2 g_{n+1}e^{ikx_{m+1}} + |g_n e^{ikx_m} g_n|^2 g_n e^{ikx_{m+1}})$$

After some nasty algebra, we get:

$$\frac{g_{n+1}}{g_n} = \frac{ia+\beta}{ia-\beta}$$

$$a = \frac{1}{\Delta t}, \beta \in \mathbb{R}$$

Note that:

$$G = \frac{g_{n+1}}{g_n} = \frac{ia+\beta}{ia-\beta}$$

$$|G^2| = \frac{-ia+\beta}{-ia-\beta}\frac{ia+\beta}{ia-\beta} = \frac{a^2+\beta^2}{a^2+\beta^2} = 1$$

Always stable in terms of dx and dt

Thank you